## Title

BICUBIC SPLINE SOLUTIONS FOR SECOND ORDER HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS ON A RECTANGLE



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## Publication Date

1972-11-01

Submitted to Nuclear Science
and Engineering



LBL-1516
Preprint



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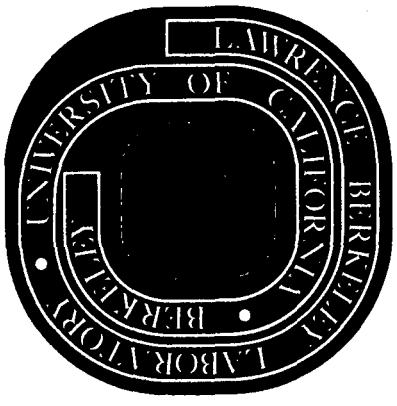
November 1972

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48





## DISCLAIMER

# BICUBIC SPLINE SOLUTIONS FOR SECOND ORDER HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS ON A RECTANGLE

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November 1972

## ABSTRACT

Approximate eigen values and corresponding eigen-functions are found for a rather general homogeneous linear second-order partial differential equation in two variables with homogeneous boundary conditions on a rectangle. The bicubic spline formulation for the matrix method which is used is analogous to the one-dimensional procedure previously described.[1] The construction of the linear system to be treated parallels that used for solving the non-homogeneous problem.[2]

Numerical examples (including some with known analytic solutions) are solved to illustrate the method. In particular, the Helmholtz equation (so-called spatial wave equation) is solved and the results are compared with the exact solution.

## INTRODUCTION

We consider the following differential equation

$$au_{xx} + bu_{yy} + cu_x + du_y + eu + \lambda(fu_x + gu_y + hu) = 0 \qquad (1)$$

defined on the rectangle

$$R = [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \qquad (2)$$

where a, b, c, d, e, f, g and h are known functions of x and y.

The following boundary conditions are imposed

$$\underline{g}u + \underline{q}u_x = 0 \quad \text{for} \quad x = \underline{x} \qquad (3)$$

$$\bar{g}u + \bar{q}u_x = 0 \quad \text{for} \quad x = \bar{x} \qquad (4)$$

$$\underline{f}u + \underline{p}u_y = 0 \quad \text{for} \quad y = \underline{y} \qquad (5)$$

$$\bar{f}u + \bar{p}u_y = 0 \quad \text{for} \quad y = \bar{y} \qquad (6)$$

where the "coefficients" are known functions of the pertinent single variable (x or y).

Obviously, Equations (1), (3), (4), (5) and (6) are satisfied by the trivial solution u=0, however we seek values for $\lambda$ (eigenvalues) such that non-trivial solutions (eigenfunctions) exist. We approximate such solutions by use of bicubic splines.

## FORMULATION

As in reference 2, the interval $[\underline{x}, \bar{x}]$ is partioned into m-1 subintervals of equal length

$$\delta_x = (\bar{x}-\underline{x})/(m-1)$$

obtaining

$$\underline{x} = x_1, \ldots, x_m = \bar{x}$$

Similarly for $[\underline{y}, \overline{y}]$ with n-1 subintervals

$$\delta y = (\overline{y}-\underline{y})/(n-1)$$

$$\underline{y} = y_1, \ \cdots \ y_n = \overline{y}.$$

We thus obtain on the rectangle (including its boundaries) a set of grid-points

$$(x_i, y_j) \quad \text{for} \quad i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n.$$

Then as in reference 2, convenient basic cubic splines in x and y are chosen and a linear system constructed by applying the differential equation and boundary conditions at the grid-points. As in reference 1 we obtain this form:

$$A\vec{\alpha} + \lambda B\vec{\alpha} = 0 \tag{7}$$

where the vector $\vec{\alpha}$ has as components, approximate normal derivatives at boundary grid-points, approximate solution values at all grid-points and four dummy values (arbitrarily zero) included for convenience in indexing.

Except in very pathological cases the matrix A will be non-singular and Equation (7) can be written in the form

$$(C-\mu I)\ \vec{\alpha} = 0 \tag{8}$$

where $C = -A^{-1}B$ and $\mu = 1/\lambda$. We need then only find eigen values, $\mu$, for the matrix and set

$$\lambda = 1/\mu \quad \text{for} \quad \mu \neq 0.$$

thereby obtaining approximate eigen values for the differential system.

## AN APPROXIMATE SOLUTION

For any real distinct approximate eigenvalue $\lambda^*$, obtained by the process outlined above, we may (except in rare circumstances) obtain a

corresponding approximate eigenfunction. We must solve the homogeneous linear system

$$D\vec{\alpha} = 0 \tag{9}$$

where

$$D = A + \lambda^* B$$

Remembering that the components of $\vec{\alpha}$ are normal derivative values and solution values except for four dummy components, we can usually find a non-trivial solution for Equation (9) by arbitrarily assigning a value 1 (with discretion) to either a normal derivative or to the solution at some grid-point. (Obviously we cannot assign the value 1 to any of the dummy variables since they are arbitrarily zero.) This solution consists of approximate values for normal derivatives at boundary grid-points and approximate solution values at all grid-points. In accordance with reference 3, there is an optimal bicubic spline on R which assumes the above values. This bicubic spline can be used to approximate the solution at <u>any</u> point in R. Obviously the bicubic spline s thus obtained depends on the eigenvalue $\lambda^*$ used. Note also that s has continuous first and second derivatives on R.

## INTEGRAL RATIO PROCESS

As in reference (1), the approximation $\lambda^*$ may usually improved by an integral ratio process. We let

$$Q = - \frac{\int_R\int s(as_{xx} + bs_{yy} + cs_x + ds_y + es)dxdy}{\int_R\int s(fs_x + gs_y + hs)dxdy}$$

and use the value Q thus obtained as a new (usually improved) approximation for the eigenvalue.

Although the above integrations can be performed analytically on subrectangles and summed for the whole (s is a cubic on any subinterval of any gridline), it is usually more convenient when using computers to perform the integrations numerically first in one direction (say x) and then in the other. For this process, the six point closed Newton-Cotes quadrature applied to each subinterval seems adequate.

## COMPUTER CODE

A computer code, YOVALU, has been written in FORTRAN for the CDC 7600 to perform all the computation necessary relating to the previous sections. The code requires m=6 and n=6 which results in matrices A, B, C and D consisting of 64 rows and 64 columns each. Because the dummy variables (4 of these) and boundary conditions (24 of these) produce only zero eigenvalues we obtain at most 36 non-zero eigenvalues, $\mu$. The code uses only real distinct eigenvalues for computing approximate solutions and for integrating to obtain new eigenvalues.

The computer code also provides approximation for eigenfunctions and their derivatives on a fine (26 X 26) mesh.

The numerical results given in a later section were obtained by use of this code. A listing and description of the code may be obtained from the authors.

## ERROR ANALYSIS

The bicubic spline approximate solution satisfies the differential equation at the grid-points when $\lambda^*$ is used. However in general the integral ratio process gives us an improved approximate eigenvalue; $\lambda^{**}$. This suggests a very simple measurement for the error of the approximation:

$$e_{\lambda^*} = |\lambda^* - \lambda^{**}| / |\lambda^{**}|$$

However the numerical integration requires interpolation for values of the bicubic spline and its derivatives on a finer mesh than the original grid. If the solution were exact the fine solution, v, should satisfy the differential equation. This fact suggests defining the error by

$$e \equiv av_{xx} + bv_{yy} + cv_x + dv_y + ev + \lambda^{**}(fv_x + gv_y + hv)$$

At each fine grid-point $(x_k, y_\ell)$ we define $e_{k\ell} = e(x_k, y_\ell)$. The following error measures can be computed:

$$e_{max} = \max_{k,\ell} |e_{k,\ell}|$$

$$e_{av} = [\sum_{\ell} \sum_{k} e_{k\ell}]/(\text{no. of fine mesh points})$$

$$e_{rel.max.} = \max_{k,\ell} |e_{k\ell}/[\lambda^{**}(fu_x + gu_y + hu)]|$$

where the denominator is evaluated at $(x_k, y_\ell)$ and the computation is not performed if the denominator is zero.

$$e_{rel.av.} = \left[\iint_R e^2 \Big/ \iint_R [\lambda^{**}(fv_x + gv_y + hv)]^2\right]^{1/2}$$

Each of these error measures is computed by the code described in the previous section and is given for the numerical examples in the next section.

## NUMERICAL EXAMPLES

Example 1. The Helmholtz Equation

$$u_{xx} + u_{yy} + \lambda u = 0 \qquad \text{on } [0, \pi/2] \times [0, 1]$$

with boundary conditions:

$$u = 0 \quad \text{on all boundaries}$$

The system can be solved analytically to obtain eigenvalues

$$\lambda = (2j)^2 + (k\pi)^2 \quad j = 1, 2\ldots \quad k = 1, 2\ldots$$

and eigenfunctions

$$u = \sin(2jx) \sin(k\pi y)$$

Tabulated Results

| j | k | $\lambda$ | $\lambda^*$ | $\lambda^{**}$ | $e_{\lambda^*}$ | $e_{max}$ | $e_{av}$ | $e_{rel.max}$ | $e_{rel.av.}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 13.8696 | 14.3316 | 13.8696 | .0333 | .2265 | .0039 | .3498 | .0120 |
| 2 | 1 | 25.8696 | 28.3909 | 25.8738 | .0973 | .7627 | .0000 | 1.3343 | .0468 |
| 1 | 2 | 43.4784 | 49.0214 | 43.4878 | .1272 | 1.3811 | .0000 | .7169 | .0617 |
| 3 | 1 | 45.8696 | 57.2590 | 46.0723 | .2428 | 3.5853 | .0518 | 5.5570 | .1551 |
| 2 | 2 | 55.4784 | 63.0806 | 55.4964 | .1367 | 1.1675 | .0000 | 1.0205 | .0605 |
| 4 | 1 | 73.8696 | 91.9487 | 75.7803 | .2134 | 4.1891 | .0000 | 4.0702 | .1408 |
| 3 | 2 | 75.4784 | 102.5381 | 76.8600 | .3341 | 1.2904 | .0000 | 3.1028 | .2471 |

Example 2. Resonant Cavity

$$u_{xx}/x^{1/2} + u_{yy}/x^{1/2} - u_x/x^{3/2} + \lambda u/x^{1/2} = 0. \quad \text{on } [0, 1] \times [0, 1]$$

with boundary conditions:

$$u_x(0, y) = 0. \qquad u_x(1, y) = 0.$$

$$u_y(x, 0) = 0. \qquad u_y(x, 1) = 0.$$

and

$u(0, y) = 0$. (This equation is used instead of the differential equation for $x = 0$).

The system can be solved analytically to obtain eigenvalues:

$$\lambda = (r_j)^2 + (k\pi)^2 \quad \text{for } j = 1, 2\ldots \text{ and } k = 0, 1, 2\ldots$$

where the $r_j$ are roots of the Bessel function, Jo, and eigenfunctions:

$$u = J_1(r_j x) \cos(k\pi y)$$

Tabulated Results

| j | k | $\lambda$ | $\lambda^*$ | $\lambda^{**}$ | $e_{\lambda^*}$ | $e_{max}$ | $e_{av}$ | $e_{rel.max}$ | $e_{rel.av}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 5.7832 | 5.8088 | 5.7841 | .0043 | .4956- | .0846 | 3.7283 | .0441 |
| 1 | 1 | 15.6528 | 16.0072 | 15.6538 | .0226 | .5158 | .0000 | 1.3986 | .0189 |
| 2 | 0 | 30.4713 | 32.8040 | 30.5656 | .0732 | 16.0625 | 1.0429 | 2.9752 | .1371 |
| 2 | 1 | 40.3409 | 43.0024 | 40.4352 | .0635 | 16.2160 | .0000 | 2.2571 | .1037 |
| 1 | 2 | 45.2616 | 50.6969 | 45.2715 | .1198 | 6.0658 | .1449 | 0.8959 | .0586 |
| 2 | 2 | 69.9497 | 77.6921 | 70.0530 | .1090 | 18.5846 | .1386 | 1.5693 | .0707 |
| 3 | 0 | 74.8870 | 90.8570 | 76.2650 | .1913 | 376.4620 | 33.1866 | 2.0250 | .2429 |

Example 3. Sample General Problem

$$(y + 1)u_{xx} + (x + 1)u_{yy} + (\sin y)u_x + (\cos x)u_y + e^x u + \lambda(u_x + u_y + (x+1)^2 u) = 0$$

on $[0, 1] \times [0, 1]$

with boundary conditions

$$u(0, y) - u_x(0, y) = 0.$$

$$u(1, y) - u_x(1, y) = 0.$$

$$u(x, 0) - u_y(x, 0) = 0.$$

$$u(x, 1) - u_y(x, 1) = 0.$$

Analytic solution unknown.

Tabulated Results

| $\ell$ | $\lambda^*$ | $\lambda^{**}$ | $e_{\lambda^*}$ | $e_{max}$ | $e_{av}$ | $e_{rel.max}$ | $e_{rel.av}$ |
|---|---|---|---|---|---|---|---|
| 1* | -527259. | 6.6732 | | | | | |
| 2* | - 18656. | 6.7393 | | | | | |
| 3 | -1.3910 | -1.3957 | .0033 | .0514 | -.0008 | .0055 | .0021 |
| 4 | 3.8230 | 3.7540 | .0183 | 2.4926 | .4946 | 9.529 | .0310 |
| 5 | 23.0599 | 23.0586 | .00006 | 28634. | -1894. | 144.659 | .5821 |
| 6* | 133.33 | 2.9818 | | | | | |
| 7* | 165.53 | 1.4847 | | | | | |

*These values appear to be spurious

## CONCLUSION

The bicubic spline formulation for the matrix method[4] appears to be a useful approach to the problem of finding differential eigenvalues for a rectangular domain. It possesses essentially the advantages over the finite-difference formulation as were outlined in Reference 1.

The advantages of the matrix method over minimization techniques is that several instead of one approximate eigenvalue are obtained and no iteration is required and that the integral ratio process can be employed for possible improvement in the eigenvalue approximation. The latter also can be used to detect spurious values (See Example 3).

In contrast to the one-dimensional case boundary conditions can not be incorporated in the basic cubic splines, thus in each variable the dimension of the cubic spline space for m points is m + 2 instead of m. Coefficient matrices are correspondingly increased in size. For example if m points in each variable are used we require three matrices

of dimension M x M where $M = (m+2)^2$. Even for the most modern (large memory) computers, one cannot use a very fine mesh. Even for m=6, the large size of the matrix for which eigenvalues are sought and the presence of many zero values (arising from boundary condition equations) may lead to spurious results and affect the accuracy of those found acceptable.

The integral ratio process is most effective if the differential equation is in self-adjoint form (Examples 1 and 2). If not, it should be put in this form if at all feasible. If this results in singularity special handling may be necessary (Example 2).

The list of error measures defined is suggestive rather than exhaustive. Others may be devised and the use of those presented is somewhat discretionary. In our opinion, the most useful is $e_{\lambda^*}$, although from results in Examples 1 and 2 ( where actual eigenvalues are known) this measure may be fairly high with a reasonably good approximation for $\lambda$. Where this measure is very high (say greater than 1) as in Example 3 then the approximation is at least suspect if not actually spurious. (Another suspicious circumstance is $\lambda^{**}$ greater than $\lambda^*$). The other error measures relate more to the approximation of the eigenfunction on the fine mesh. The improvement of this approximation while retaining the eigenvalue approximation may be considered in a later paper.

ACKNOWLEDGEMENT

This work was done in part under the auspices of the Atomic Energy Commission.

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