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Diagrammatic Representations of Finitely-Generated Generalized Temperley-Lieb Algebras

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Diagrammatic Representations of Finitely-Generated Generalized Temperley-Lieb Algebras

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by

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Abstract

In this paper, we give an overview of the classical Temperley-Lieb algebra, reviewing some of the basic results about the algebra. Then we will discuss the construction of the generalized Temperley-Lieb algebras from Coxeter groups, and the connection between fully commutative elements of the Coxeter group and the reduced elements of the corresponding Temperley-Lieb algebra. Finally, we will look at the diagrammatic representations of some finite-dimensional Temperley-Lieb algebras, specifically those of type B_n, D_n and E_n, and describe their bases.

1 Introduction

The defining relations of the classical Temperley-Lieb algebra, TL_n, were first described in 1971 by Temperley and Lieb [1]. They occurred in the relationships of transfer matrices, which are used to solve problems in statistical mechanics. These relations were studied independently by Jones in 1983 [2]. He proved many of the well-known facts known about the algebra, and later used TL_n to define a knot invariant, the Jones polynomial [3]. In 1987, Kauffman found a faithful diagrammatic representation of TL_n [4].

Jones proved that TL_n could be defined as a quotient of the Hecke algebra of A_{n-1} by a subspace generated by certain elements of the Kazhdan-Lusztig basis [3]. In his thesis, Graham generalized this definition of $TL_n(\delta)$ to create Temperley-Lieb algebras associated with other Coxeter groups, Γ, denoted by $TL(\Gamma, \delta)$ to reflect their dependence on a parameter δ. We will survey and summarize some of the literature on the subject of these generalized Temperley-Lieb algebras, focusing especially on the diagrammatic representations of these algebras.

The elements of the Kazhdan-Lusztig basis that generate the kernel of the quotient map are those which correspond to braid moves. These are the maximal words generated by pairs of adjacent elements of the Coxeter graph. For example, for A_n, these elements of the KL-basis are indexed by the elements $a_i a_{i+1} a_i$ for $i = 1, \ldots, n-1$, where a_i denote the vertices of the Coxeter graph. These elements correspond to braid moves in the Coxeter group (e.g. $a_i a_{i+1} a_i \mapsto a_{i+1} a_i a_{i+1}$).

If no braid moves can be applied to any reduced expression for an element w of the Coxeter group, then we call w 'fully commutative' [5]. These 'fully commutative' elements of the Coxeter group index a basis for $TL(\Gamma, \delta)$ [6]. In 2017, Ernst et al. published an algorithm [7] to find the fully commutative element associated with any given diagram from the diagrammatic representation of $TL_n(\delta)$.

Graham was further able to determine which of these algebras are finite dimensional: those whose Coxeter groups are finite, as well as those corresponding to E_n, G_n and H_n. Since his thesis, faithful diagrammatic representations have also been created for some of these finite dimensional algebras, namely for those of type B_n, D_n and E_n ([8], [9], [10]), by adding decorations to the diagrammatic representations of TL_n.

We have extended the algorithm of Ernst et al. to the diagrams of type D_n, giving a fully commutative element corresponding to any diagram. In his proof that the diagrammatic representation of type D_n is faithful, Green created an algorithm to find the word in $TL(D_n)$ corresponding to a given diagram, which in turn corresponds to a word in D_n. We then used combinatorial properties of the word to show that it was indeed a reduced expression for a fully commutative element of D_n.

We also looked at the diagrammatic representation of $TL(E_n)$. The idea for the representation was first described by tom Dieck [8], who added 'pillars' to the diagrams for TL_{n-1} to represent the exceptional generator of E_n. He defined relations on these pillars that respect those of $TL(E_n)$, ensuring that there is a well-defined map from $TL(E_n)$ to the algebra of TL_{n-1} diagrams with pillars. However, he only showed that this map is faithful for $TL(E_6)$, and did not describe the image of the map. While constructing a knot invariant, Green indirectly showed that the map is faithful for all n [10]. This showed that there is a subalgebra of diagrams that is in bijection with $TL(E_n)$. He also gave a necessary (but not sufficient) condition for a diagram to be in the image. We extended this condition to a description of a basis for the diagrammatic representation, and an alternate proof that $TL(E_n)$ is finite.

2 The Temperley-Lieb Algebra (of A_n)

2.1 The Origins of the Temperley-Lieb Algebra

The classical Temperley-Lieb Algebra $TL_n(\delta)$, for a constant $\delta \in \mathbb{C}$, is the algebra over $\mathbb{C}$ generated by 1, e_1, ..., e_{n-1} with the relations

$$e_i^2 = \delta e_i \tag{2.1.1}$$

$$e_i e_{i\pm1} e_i = e_i \tag{2.1.2}$$

$$e_i e_j = e_j e_i \text{ for } |i-j| > 1 \tag{2.1.3}$$

H. N. V. Temperley and Elliott H. Lieb first described a version of this algebra indirectly in 1971 [1]. They used the algebra in the form of 'transfer matrices' to make calculations about planar lattices, which they were using to work on "ice"-type problems.

Similar relations were studied independently by Jones in 1983 [2], in the context of subalgebras of a von Neumann algebra with a faithful normal normalized trace. He used idempotent generators, which yielded slightly different relations than the now standard relations:

$$a_i^2 = a_i \tag{2.1.4}$$

$$a_i a_{i\pm1} a_i = \tau a_i \tag{2.1.5}$$

$$a_i a_j = a_j a_i \text{ for } |i-j| > 1 \tag{2.1.6}$$

He called the algebra generated by $1, a_1, \ldots, a_n$ and these relations $A_n(\tau)$.

Kauffman was the first to give the relations in their current form [4]. For Kauffman, the relations originated from a diagrammatic algebra obtained by removing the crossings from n-strand braid groups, which we will describe in Section 2.3.

We can convert between $TL_{n+1}(\delta)$ and $A_n(\tau)$ by

$$a_i = \delta^{-1} e_i \qquad \text{and} \qquad \tau = \delta^{-2} \tag{2.1.7}$$

Then

$$a_i^2 = (\delta^{-1} e_i)^2 = \delta^{-1} e_i = a_i$$

and

$$a_i a_j a_i = \delta^{-3} e_i e_j e_i = \tau \delta^{-1} e_i = \tau a_i$$

for $|i - j| = 1$. So we see that the two are equivalent.

This difference in relations was not relevant for Jones' combinatorial results, since he was working with words in $e_1, \ldots, e_{n-1}$ under the relations

$$\begin{cases} e_i^2 \sim e_i \\ e_i e_{i\pm 1} e_i \sim e_i \\ e_i e_j \sim e_j e_i \qquad \text{for } |i - j| > 1 \end{cases} . \tag{2.1.8}$$

Importantly, these results showed that $TL_n(\delta)$ is finite dimensional and gave a normal form for a basis of TL_n in terms of the generators $e_1, \ldots, e_{n-1}$.

Definition 2.1.1. A word in $e_1, \ldots, e_n$ is called *reduced* if it is of minimal length under the relations (2.1.8). A word of the form

$$(e_{j_1} e_{j_1 - 1} \cdots e_{k_1})(e_{j_2} e_{j_2 - 1} \cdots e_{k_2}) \cdots (e_{j_m} e_{j_m - 1} \cdots e_{k_m}) \tag{2.1.9}$$

where $j_i \geq k_i$, $j_{i+1} > j_i$ and $k_{i+1} > k_i$ is said to be in 'decreasing normal form.' There is an analogous form wherein the indices increase rather than decrease (so $j_i \leq k_i$), which we call 'increasing normal form.'

Lemma 2.1.1. *1. In a reduced word, the maximal (minimal) index occurs only once.*

2. *Words in decreasing (increasing) normal form are reduced, and each reduced word can be written uniquely in decreasing (increasing) normal form.*

3. *The number of unique reduced words is the $(n+1)$th Catalan number, $C_{n+1} = \frac{1}{n+2}\binom{2(n+1)}{n+1}$.*

Proof. These are ([2] Lemma 4.1.2), ([2] Aside 4.1.4), and ([2] Aside 5.1.1), respectively. A summary is as follows.

1. By induction on the length of the word.

   If the word has length one, then clearly the maximal index occurs only once.

   Suppose that the result holds for words of length less than or equal to k and we have a reduced word w of length $k+1$ in which the maximal index (m) occurs twice. Then

$$w = w_1 e_m w_2 e_m w_3$$

   such that w_2 does not contain e_m. Since w is reduced, w_2 is reduced as well and its length is less than k, so the maximal index of w_2 occurs at most once.

   By choice of m, this means that e_{m-1} occurs at most once in w_2, and e_{m+1} never occurs. If e_{m-1} doesn't occur in w_2, then w_2 commutes with e_m, so

$$w \sim w_1 e_m^2 w_2 w_3$$

is not reduced. If e_{m-1} occurs in w_2, then $w_2 = w_2' e_{m-1} w_2''$ such that w_2', w_2'' commute with e_m, so

$$w \sim w_1 w_2' e_m e_{m-1} e_m w_2'' w_3$$

is not reduced. This is a contradiction. Thus, no such w exists.

2. Push the largest letter with maximal index as far right (left) as possible.

3. The words in decreasing normal form are in bijection with the paths on the integer lattice between $(0, 0)$ and $(n+1, n+1)$ that stay on or below the diagonal. It is well known that the number of such paths is the $(n+1)$th Catalan number.

   In particular, we can associate to

$$(e_{j_1} e_{j_1-1} \cdots e_{k_1})(e_{j_2} e_{j_2-1} \cdots e_{k_2}) \cdots (e_{j_m} e_{j_m-1} \cdots e_{k_m})$$

   the path of $2m$ steps whose $(2i)$-th step is to (j_i, k_{i-1}) and $(2i+1)$-th step is to (j_i, k_i), letting $k_0 = j_0 = 0$ and $j_{m+1} = k_{m+1} = n+1$, and each path gives us a word in the decreasing form.

□

2.2 As a quotient of a Hecke algebra

Later, in 1987, Jones revisited the algebra in defining a knot invariant known as the Jones polynomial. He used it as a representation of a braid group, and showed that $A_n(\tau)$ is a quotient of the Hecke algebra of A_n [3].

Definition 2.2.1. For a scalar u and $n \geq 2$, let $H(n, u)$ be the algebra generated by

$g_1, \ldots, g_{n-1}$ with relations

$$g_i^2 = (u^2 - 1)g_i + q \tag{2.2.1}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{2.2.2}$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| > 1. \tag{2.2.3}$$

This algebra is called the *Hecke algebra of type* A_{n-1}.

In section 11 of [3], Jones showed that $A_n(\tau)$ is the quotient of the Hecke algebra $H(n+1, u)$ by the ideal generated by

$$g_i g_j g_i + g_i g_j + g_j g_i + g_i + g_j + 1 \tag{2.2.4}$$

for all $|i-j| = 1$ by choosing $\tau = (u+u^{-1})^{-2}$ and make the substitution $g_i = ua_i - (1-a_i)$. The following is the analogous result for $TL_n(\delta)$.

Proposition 2.2.1. *There is a scalar u such that $TL_n(\delta)$ is a quotient of the Hecke algebra $H(n, u)$ by the ideal generated by*

$$g_i g_j g_i + g_i g_j + g_j g_i + g_i + g_j + 1 \tag{2.2.5}$$

for all $|i-j| = 1$.

Proof. Choose u such that $\delta = u + u^{-1}$.

We want to show that $H(n, u)$ modulo the relations

$$g_i g_j g_i + g_i g_j + g_j g_i + g_i + g_j + 1 = 0 \tag{2.2.6}$$

for $|i-j|=1$ is isomorphic to $TL_n(\delta)$. Note that the relations (2.2.6) imply (2.2.3), since (2.2.6) implies that

$$g_ig_jg_i = -(g_ig_j + g_jg_i + g_i + g_j + 1) = g_jg_ig_j, \tag{2.2.7}$$

so we only need show that $\langle g_1, \ldots, g_{n-1} | (2.2.1), (2.2.2), (2.2.6)\rangle$

is the equivalent to $\langle e_1, \ldots, e_{n-1} | (2.1.1), (2.1.2), (2.1.3)\rangle$. To see this, make the substitution

$$g_i = ue_i - 1. \tag{2.2.8}$$

Then (2.2.1) is equivalent to (2.1.1):

$$g_i^2 = g_i(u^2-1) + u^2 \tag{2.2.9}$$

$$(ue_i - 1)^2 = (ue_i - 1)(u^2-1) + u^2 = u(u^2-1)e_i + 1 \tag{2.2.10}$$

$$e_i^2 = (u + u^{-1})e_i = \delta e_i, \tag{2.2.11}$$

Clearly, (2.1.3) and (2.2.3) are equivalent, since $g_i = ue_i - 1$ and $g_j = ue_j - 1$ will commute precisely when e_i and e_j commute.

Making the substitution in $g_ig_jg_i + g_ig_j + g_jg_i + g_i + g_j + 1 = 0$ and applying (2.1.2) yields

$$u^3e_ie_je_i - u^2e_i^2 + ue_i = 0 \tag{2.2.12}$$

$$u^3e_ie_je_i + (-u^2\delta + u)e_i = 0 \tag{2.2.13}$$

$$e_ie_je_i = e_i, \tag{2.2.14}$$

so (2.2.6) is equivalent to (2.1.3). Thus

$$\langle g_1, \ldots, g_{n-1} | (2.2.1), (2.2.2),\ (2.2.6) \rangle \cong \langle e_1, \ldots, e_{n-1} | (2.1.1), (2.1.2), (2.1.3) \rangle.$$

□

It is this conception of the Temperley-Lieb algebra, as the quotient of a Hecke algebra, that is generalized by Graham [11] to give us the generalized Temperley-Lieb algebras. To generalize the relations, note that $u^{-3}(g_i g_j g_i + g_i g_j + g_j g_i + g_i + g_j + 1)$ ($|i - j| = 1$) are elements of a Kazhdan-Lusztig basis for $H(q, n)$.

2.3 The Diagrammatic Representation of TL_n

Also in 1987, Kauffman [4] gave us another way to view the relations, as diagrams rather than quotients. He defined the diagrammatic algebra $\mathbb{D}TL_n(\delta)$ as the $\mathbb{C}$ algebra spanned by (n, n)-diagrams modulo the bubble-popping relation (fig. 1), which is where the constant δ arises. He asserted that this diagrammatic algebra $\mathbb{D}TL_n$ is isomorphic to the Temperley-Lieb algebra TL_n.

Westbury gave a proof of this isomorphism ([12] Section 3) by showing that the diagrammatic algebra is generated by the diagrams $E_1, \ldots, E_{n-1}$ and proving that the dimensions of the algebras are the same.

Definition 2.3.1. An (n, n) *diagram* is a box with n nodes on the top and bottom faces (labeled 1 through n), with non-intersecting edges such that each node is an endpoint of exactly one edge.

We define the product of two (n, n) diagrams to be the diagram that results from stacking the first diagram on top of the second (fig. 3).

We call an edge *non-propagating* if its endpoints are both on the same face of the diagram (top or bottom). We call an edge a *bubble* if it has no endpoints.

We define $\mathbb{D}TL_n(\delta)$ to be the algebra which is spanned by diagrams modulo the "bubble-popping relation" (fig. 1), which is that a diagram with a loop is equivalent to δ times the same diagram without the loop.

The element E_i (fig. 2) is the diagram with two non-propagating edges connecting nodes i to $i+1$ on the top and bottom, and all other edges just connecting the corresponding nodes on the top and bottom faces of the box. The identity diagram is the diagram whose only edges connect the corresponding nodes on the top and bottom faces.

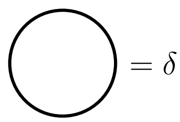


Figure 1: the bubble-popping relation

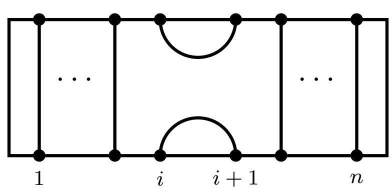


Figure 2: E_i, the i-th generator of $\mathbb{D}TL_n$

Lemma 2.3.1. *The diagrammatic algebra $\mathbb{D}TL_n$ is generated by the diagrams E_i.*

Proof by induction. If $n = 2$, then there is only one loop-free diagram besides the identity, which is E_1.

Let $n > 2$ and assume that the loop-free (m, m)-diagrams can be written as products of $E_1, \ldots, E_{m-1}$ for $m < n$.

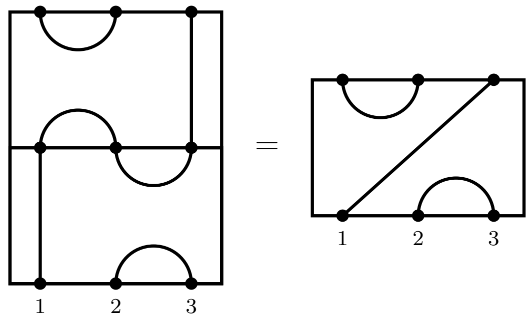


Figure 3: the product E_1E_2 in $\mathbb{D}TL_3$

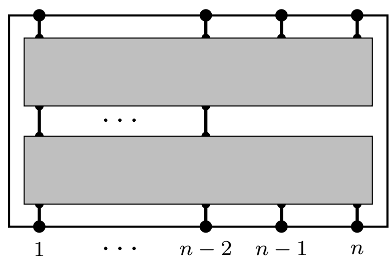


Figure 4: A path that crosses $n-2$ edges

Let D be a loop-free (n,n)-diagram besides the identity. Then D must have at least one non-propagating edge. In particular, there must be some edge that connects nodes i and $i+1$ on either the top or bottom face. This gives us a path from the left face of D to the right face that crosses $n-2$ edges (fig. 2.3).

Then we can deform D so that it is composed of two (loop-free) $(n-1,n-1)$-diagrams separated by one copy of E_{n-1} (fig. 2.3). Since we know that each of these $(n-1,n-1)$-diagrams can be decomposed into products of $E_1,\ldots,E_{n-2}$, we see that D is also a product of E_i. $\square$

Lemma 2.3.2. *The dimensions of TL_n and $\mathbb{D}TL_n$ over $\mathbb{C}$ are equal.*

Proof. ([12], Section 3) We already know from Lemma 2.1.1 that $\dim(TL_n) = C_n$, by putting the reduced words in correspondence with the lattice paths from $(0,0)$ to (n,n)

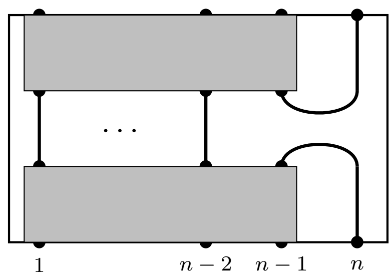


Figure 5: Deformed into two $(n-1, n-1)$-diagrams separated by one copy of E_{n-1}

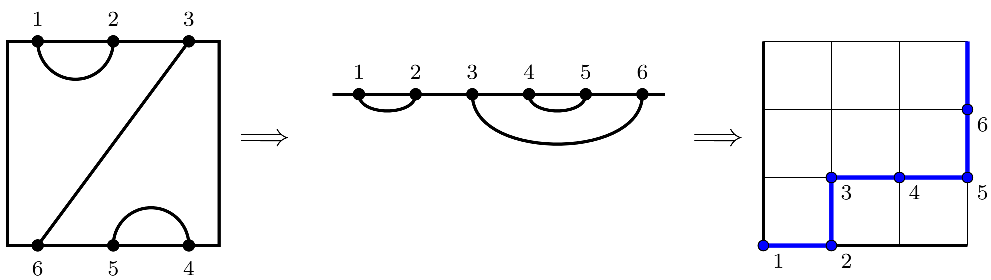


Figure 6: Example of correspondence between diagrams and paths

that stay below the diagonal.

Now, we will show that $\mathbb{D}TL_n = C_n$ as well, by putting the loop-free (n, n)-diagrams in correspondence with those same paths.

First, we unfold the diagram into what we will call a $2n$-cup diagram. This is a line with $2n$ nodes and arcs (cups) connecting pairs nodes such that each node is either the starting point or end point of exactly one arc. Then for the ith step in the corresponding path, go

- right one if the ith node is a starting point of an arc, or
- up one if it is an end point.

Note that the first step is always to the right, and the last step is always up.

□

Theorem 2.3.3. $TL_n \cong \mathbb{D}TL_n$

Proof. ([12], Section 3) We can see that homomorphism $TL_n \to \mathbb{D}TL_n$ generated by sending e_i to the diagram E_i is well defined, since the relations of TL_n hold true in $\mathbb{D}TL_n$ as well.

This homomorphism is a surjection, since each (non-identity) diagram is a product of the diagrams E_i. Since both algebras have the same dimension and the map is a surjection, it is an isomorphism. $\square$

Given a word in $e_1, \ldots, e_{n-1}$, we may easily find the corresponding a diagram in $\mathbb{D}TL_n$ by stacking the appropriate generating diagrams. Using the recursive method in the proof of Lemma 2.3.1, we can also find a word in $e_1, \ldots, e_{n-1}$ which corresponds to a given diagram. Note that this word is not necessarily reduced. Recently, Ernst et al. introduced a non-recursive algorithm which finds a word in $e_1, \ldots, e_{n-1}$ of minimal length (which is unique up to commutation) [7]. An example of this algorithm being applied can be seen in fig. 7. In that example, we see that the diagram corresponds to the word $e_4e_1e_3e_5e_2e_4e_6e_3e_5$ ($e_1e_4e_3e_2e_5e_4e_3e_6e_5$, in decreasing normal form).

To show that each resulting word has minimal length, they show that each diagram is associated to a "fully commutative" word in the associated Coxeter group. For that reason, we will describe their methods further in section 3.3, after defining fully commutative words in section 3.1.

3 Generalized Temperley-Lieb Algebras

In his 1995 PhD thesis [11], J.J. Graham extended the idea of presenting the classical Temperley-Lieb algebra as a quotient of the Hecke algebra of A_n to Coxeter groups of

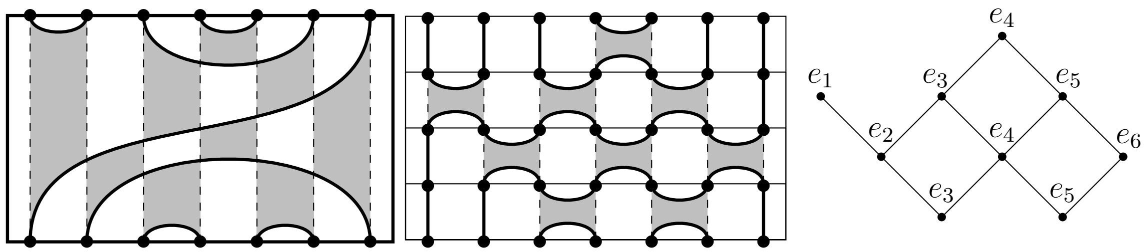


Figure 7: An example of the algorithm of Ernst et al. applied to an element of $\mathbb{D}TL_7$.

different types, giving us the generalized Temperley-Lieb algebras, $TL(\gamma)$. He further proved that the finite Temperley-Lieb algebras are those whose associated Coxeter group is finite or of type E_n, F_n or H_n and that these generalized Temperley-Lieb algebras have a basis indexed by the "fully commutative" elements of the corresponding Coxeter group. Before we describe the quotient, we will review some relevant terminology about Coxeter groups and define "fully commutative."

3.1 Coxeter Groups and Fully Commutative Elements

A Coxeter diagram γ is a simple, finite, undirected graph whose edges are labeled with integers greater than two. For our purposes, the edges are usually labeled 3. Thus, if the label of an edge is 3, we will omit the label.

Let S be the set of vertices of γ. For each $s, t \in S$, we define a constant $m(s,t) = m(t,s)$ such that

1. $m(s,t) = 1$ if and only if $s = t$,

2. $m(s,t) = 2$ if there is no edge connecting s and t, and otherwise

3. $m(s,t) = p$, where p is the label of the edge connecting s and t.

We can construct a Coxeter system $(W(\gamma), S)$, where $W(\gamma)$ is the group with presentation $\langle s \in S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S\rangle$, called the *Coxeter group of* γ. As in [7], we will refer to $W(\gamma)$ simply as W.

Note that all elements of S have order 2, since $m(s, s) = 1$. Hence, for any $s, t \in S$, $(st)^{m(s,t)} = 1$ implies that

$$\underbrace{sts\cdots}_{m(s,t)\text{ terms}} = \underbrace{tst\cdots}_{m(s,t)\text{ terms}}. \tag{3.1.1}$$

Call this element $w(s, t)$, the longest word in the subgroup generated by s and t.

Letting S^* be the free monoid on S, we say that an element $\mathbf{w} \in S^*$ is an *expression* for $w \in W$ if $\mathbf{w}$ and $\mathbf{w}'$ are equal when considered as elements of W. We call a subsequence of $\mathbf{w}$ a *subexpression*, or a *subword* if it is a consecutive subsequence.

Definition 3.1.1. An expression for w is *reduced* if it is of minimal length. We denote the length of this expression by $l(w)$.

This gives us a partial order on W, the *Bruhat order*: we say that $y \leq w$ if some subword of a reduced expression for w is a reduced expression for y.

Definition 3.1.2. If $m(s, t) = 2$, this simply means that s and t commute, $st = ts$, and so this is called a *commutation relation*. If $m(s, t) \geq 3$, then $\underbrace{sts\cdots}_{m(s,t)\text{ terms}} = \underbrace{tst\cdots}_{m(s,t)\text{ terms}}$ is called a *braid relation*. Replacing one side with the other is then called a *commutation* or *braid move*, respectively.

Suppose that $\mathbf{w}$ and $\mathbf{w}'$ are reduced expressions for an element $w \in W$. As described in Ernst's paper ([9] Section 2), we say that $\mathbf{w} \sim \mathbf{w}'$ if $\mathbf{w}'$ can be obtained from $\mathbf{w}$ with one commutation, and $\mathbf{w} \approx \mathbf{w}'$ if they can be obtained from each other by a series of commutations alone, no braid moves.

The equivalence classes of $\approx$ are called *commutation classes*.

Definition 3.1.3. An element w of the Coxeter group W is *fully commutative*, and we write $w \in FC(\gamma)$, if it has only one commutation class. If $FC(\gamma)$ is finite, we say that γ is FC-finite.

Proposition 3.1.1 ([13], Theorem 3)**.** *If* $\mathbf{w}$ *and* $\mathbf{w}'$ *are expressions for the same element of* W*, then there is a series of simplifications (commutations, braid moves and cancellations) which can be applied to* $\mathbf{w}$ *and* $\mathbf{w}'$ *that will take them to the same expression.*

In particular, this means that increasing the length of the word (by adding consecutive pairs of letters) is not necessary to decrease the length of the word. Note also that if $\mathbf{w}'$ is reduced, then no cancellations can be applied and the other moves can be reversed, so there is a series of simplifications that take $\mathbf{w}$ to w'.

Corollary 3.1.2 (Matsumoto's Theorem, [7], Proposition 2.1)**.** *If* $\mathbf{w}$ *and* $\mathbf{w}'$ *are both reduced expressions for* $w \in W$*, then there is a finite sequence of commutations and braid moves that take* $\mathbf{w}$ *to* $\mathbf{w}'$*.*

Since commutations and braid moves preserve length, each word in the sequence must be an expression for w. Hence, if no braid move can be applied to any reduced expression for w, then the sequences must consist only of commutations, so all the expressions for w must be in the same commutation class.

Corollary 3.1.3. *([7], Proposition 2.6) An element* $w \in W$ *is fully commutative if and only if no braid operation can be applied to any reduced expression for* w*. In other words,* w *is fully commutative if and only if* $\underbrace{sts\cdots}_{m(s,t)\ terms}$ *never occurs as a subword of a reduced expression for* w *when* $m(s,t) > 2$*.*

Thus $\mathbf{w} = s_{x_1} \cdots s_{x_n}$ is a reduced expression for $w \in FC(\gamma)$ iff $\mathbf{w}' \approx \mathbf{w}$ implies neither $\underbrace{s_x s_y s_x \cdots}_{m(s_x,s_y) \text{ terms}}$ (for $m(s_x, s_y) > 2$) nor s_x^2 occurs in $\mathbf{w}'$ for any x, y.

Proposition 3.1.4. *Suppose* $\mathbf{w} = s_1 \cdots s_k \in S^*$ *and that whenever* $i < j$ *and* $s_i = s_j$*, there exist* $i < k < k' < j$ *such that* $m(s_i, s_k), m(s_i, s_{k'}) > 2$*. Then* $\mathbf{w}$ *is a reduced expression for* $w \in FC(\gamma)$*.*

That is, between every pair of copies of the same letter in $\mathbf{w}$, there are two letters which do not commute with it.

Proof. Proof by contradiction. Suppose that this property holds but that $\mathbf{w}$ is not reduced, or is an expression for $w \in W$ which is not fully commutative.

Then there is an reduced expression for $\mathbf{w}'$ for w which either is shorter than $\mathbf{w}$, or to which a braid move can be applied.

By Proposition 3.1.1, since $\mathbf{w}$ and $\mathbf{w}'$ are expressions for the same element and $\mathbf{w}'$ is reduced, there is a series of commutations, braid moves and cancellations which take $\mathbf{w}$ to $\mathbf{w}'$.

Thus there is a series of commutations which take $\mathbf{w}$ to $\mathbf{w}''$, to which we can apply a move which is either a braid move or a cancellation. That is, either s^2 or $\underbrace{sts\cdots}_{m(s,t)>2 \text{ terms}}$ occur in $\mathbf{w}''$, so there are two copies of s in $\mathbf{w}''$ which are separated by at most one other term.

However, those copies of s in $\mathbf{w}$ are separated by at least two terms which don't commute with s. Since we've applied only commutations, these two terms must still be between the copies of s. This is a contradiction. □

3.1.1 Heaps

Labelled partially-ordered sets, "heaps", are a useful tool to understand fully commutative words. They were used by Stembridge to determine which Coxeter groups are FC-finite [5] and by Ernst et al. to prove that their algorithm produces a fully commutative word [7].

Definition 3.1.4. Let $\mathbf{w} = s_1 \cdots s_k \in S^*$. We can define a partial order on the set $\{1, \ldots, k\}$ by taking the transitive closure of the relation $i \prec j$ if $i < j$ and $m(s_i, j_i) \neq 2$. Label each element i of the poset s_i. This labelled partially ordered set $P_{\mathbf{w}}$ is called the *heap* of $\mathbf{w}$.

If multiple elements of $P_{\mathbf{w}}$ are labelled s, we call the occurrences $s^{(1)}, \ldots, s^{(m)}$ such that $s^{(i)} \prec s^{(i+1)}$.

Let $\mathcal{L}(P_{\mathbf{w}})$ be the set of possible extensions of $P_{\mathbf{w}}$ to a total order. For a total order $<_P \in \mathcal{L}(P_{\mathbf{w}})$, there exists π such that $i <_P j$ in this total order if $\pi(i) < \pi(j)$. Let

$$\mathbf{w}_{<_P} = s_{\pi(1)} \cdots s_{\pi(k)}. \tag{3.1.2}$$

Lemma 3.1.5 ([5], Proposition 2.2)**.** *Suppose* $\mathbf{w}$ *is a reduced expression.*

Then $\mathbf{w}' \approx \mathbf{w}$ *if and only if* $\mathbf{w}' = \mathbf{w}_<$ *for some* $< \in \mathcal{L}(P_{\mathbf{w}})$*.*

Note that every element of the commutation class has the same heap, so if w is a fully commutative element then all its reduced expressions have the same heap, which we call the heap of w and denote by P_w.

A *convex chain* of a partial order $\preceq$ is a subset C such that for any $x, y \in C$, x and y are comparable and if $x \preceq z \preceq y$ then $z \in C$.

Lemma 3.1.6 ([5], Proposition 3.3)**.** *For* $\mathbf{w} \in S^*$, $\mathbf{w}$ *is a reduced expression for a fully commutative element if and only if*

- *there is no covering relation* $i \prec i'$ *with* $s_i = s_{i'}$*, and*
- *whenever* $m(s,t) > 2$*, there is no convex chain* $i_1 \prec i_2 \prec \cdots \prec i_{m(s,t)}$ *such that* $s_{i_{2k+1}} = s$ *and* $s_{i_{2k}} = t$*.*

.

In particular, if $i \prec j$ is a covering relation ($i \preceq k \prec j$ implies $k = i$) and $s_i = s_j$, then $\mathbf{w}$ is not reduced.

Definition 3.1.5. A connected subgraph τ of a γ is called a *branch* of γ if

- τ is of type A,
- there exists a unique $t \in \tau$ and a unique $s \in \gamma \setminus \tau$ such that $m(s,t) > 2$, and
- t is an end node of τ, i.e. t either has order one or is the only vertex of τ.

In that case, s and t are called the *points of contact* of τ. We say that τ is a *simple branch* if $m(s,t) = 3$.

Lemma 3.1.7 ([5], Lemma 5.3)**.** *Suppose* $w \in FC(\gamma)$ *and* P_w *is its heap. Let* τ *be a simple branch of* γ *with points of contact* $s \in \tau$ *and* $t \in \tau \setminus \gamma$*.*

Let the copies of t *in* P_w *be* $t^{(1)} \prec \cdots \prec t^{(k)}$*. Then if* $t^{(i)} \prec j \prec t^{(i+1)}$ *and* $s_j = s$*, this is chain is unrefinable.*

Importantly, since τ is of type A and s is an end node, a word in τ has at most one copy of s.

Lemma 3.1.8 ([5], Lemma 5.4)**.** *Suppose $w \in FC(\gamma)$ and P_w is its heap. Let τ be a simple branch of γ with points of contact $s \in \tau$ and $t \in \tau \setminus \gamma$.*

If there is an unrefinable chain $i_1 \prec j_1 \prec \ldots \prec j_m \prec i_{m+1}$ such that $s_{i_k} = t$ and $s_{j_k} = s$ for each k, then $m \leq |\tau|$.

Proof. By induction on $|\tau|$.

Suppose $|\tau| = 1$. Then $\tau = \{s\}$, so s is commutes with any other element of γ. If $m \geq 2$, $j_1 \prec i_2 \prec j_2$ would be convex. Since $m(s,t) = 3$, this is impossible. Thus $m \leq 1 = |\tau|$.

Suppose $|\tau| > 1$ and $m \geq 2$. Notice that $\tau' = \tau \setminus \{s\}$ is also a simple branch of γ, with points of contact $s' \in \tau'$ and s.

We must have $j_k \prec l_k \prec j_{k+1}$ with $s_{l_k} = s'$. Otherwise, since $j_k \prec i_{k+1} \prec j_{k+1}$ is unrefinable and s commutes with everything but t and s', $j_k \prec i_{k+1} \prec j_{k+1}$ would be convex.

By Lemma 3.1.7, $j_k \prec l_k \prec j_{k+1}$ is unrefinable. Thus, we have another unrefinable chain, $j_1 \prec l_1 \prec \ldots \prec l_{m-1} \prec j_m$. Since $1 \leq |\tau \setminus \{s\}| < |\tau|$, by induction we know that $m - 1 \leq |\tau \setminus \{s\}| = |\tau| - 1$. Thus $m \leq |\tau|$. □

3.1.2 FC-finite Coxeter groups

In this section, we will give Stembridge's proof that the Coxeter groups with finitely many fully commutative elements are exactly those of types A_n, B_n, D_n, E_n, F_n, H_n and $I_2(n)$.

Lemma 3.1.9. *Coxeter groups of type H_n are FC-finite.*

To prove this, Stembridge showed that there are at most n copies of h_1 in a reduced expression for a fully commutative word. Between copies of h_1 are words from the finite group $H_n \setminus \{h_1\} \cong A_{n-1}$, hence there are only finitely many fully commutative words.

Proof. Suppose that P_w is a heap for $w \in FC(H_n)$ and $h_1^{(1)}, \ldots, h_1^{(m)}$ are the elements of P_w labelled h_1. To show that $m \leq n$, Stembridge noted that P_w must have an chain $h_1^{(1)} \prec i_1 \prec \cdots \prec i_{m-1} \prec h_1^{(m)}$, where $s_{i_k} = h_2$, since h_2 is the only element that does not commute with h_1. Because $H_n \setminus \{h_1\}$ is a branch, we know this chain is unrefinable by Lemma 3.1.7.

In a similar fashion, P_w must have another unrefinable chain $i_1 \prec j_1 \prec \cdots \prec j_{m-2} \prec i_{m-1}$, where $s_{j_k} = h_3$. If there were no $i_k \prec j \prec i_{k+1}$ with j labelled h_3 for some k, the unrefinable chain $h_1^{(k)} \prec i_k \prec h_1^{(k+1)} \prec i_{k+1} \prec h_1^{(k+2)}$ would be convex, since h_3 is the only other element that does not commute with h_2. That would contradict Lemma 3.1.6, since w is fully commutative.

Since $H_n \setminus \{h_1, h_2\}$ is a simple branch, we know that $m-2 \leq n-2$, by Lemma 3.1.8. Thus there are at most n copies of h_1 in P_w, so there at most n in a reduced expression for any $w \in FC(H_n)$. Thus H_n is FC-finite. □

Stembridge used the same methods to show there are at most $n-1$ copies of f_2 in a reduced expression for any fully commutative word. Since there are finitely many reduced words in $\{f_1, f_3, \cdots, f_n\}$, this shows that F_n is also FC-finite.

Lemma 3.1.10. *Coxeter groups of type F_n are FC-finite.*

Proof. Suppose that P_w is a heap for $w \in FC(F_n)$ and $f_2^{(1)}, \ldots, f_2^{(m)}$ are the elements labelled f_2.

Suppose $f_2^{(k)} \prec j \prec f_2^{(k+1)}$ such that j is labelled f_1. Because $\{f_1\}$ is a simple branch, this chain is unrefinable and there cannot be a copy of f_1 between $f_2^{(k+1)}$ and $f_2^{(k+2)}$ by

Lemmas 3.1.7 and 3.1.8, respectively. So P_w has at most one copy of f_1 between $f_2^{(k)}$ and $f_2^{(k+2)}$.

For each k, there must be i_k labelled f_3 with $f_2^{(k)} \prec i_k \prec f_2^{(k+1)}$. Otherwise, since f_1 is the only other element that commutes with f_2, either $f_2^{(k)} \prec f_2^{(k+1)}$ or $f_2^{(k)} \prec j \prec f_2^{(k+1)}$ (with j labelled f_1) is an convex chain. This is impossible, since w is fully commutative.

For each k, at least one of $f_2^{(k)} \prec i_k \prec f_2^{(k+1)}$ and $f_2^{(k+1)} \prec i_{k+1} \prec f_2^{(k+2)}$ must be convex, since there is at most one copy of f_1 between $f_2^{(k)}$ and $f_2^{(k+2)}$. Thus, if there is no copy of f_4 between i_k and i_{k+1}, then $f_2^{(k)} \prec i_k \prec f_2^{(k+1)} \prec i_{k+1}$ or $i_k \prec f_2^{(k+1)} \prec i_{k+1} \prec f_2^{(k+2)}$ is convex. As $m(f_2, f_3) = 4$, this is not possible, so we must have $i_k \prec j_k \prec i_{k+1}$ with j_k labelled f_4 for each k.

Now,

$$i_1 \prec j_1 \prec \cdots \prec j_{m-2} \prec i_{m-1}$$

is an unrefinable chain, since $F_4 \setminus \{f_1, f_2, f_3\}$ is a simple branch. Thus $m - 2 \leq n - 3$, so $m \leq n - 1$ by Lemma 3.1.8.

Hence there are at most $n - 1$ copies of f_2 in P_w and in any reduced expression for any $w \in FC(F_4)$. Thus F_4 is also FC-finite. $\square$

Lemma 3.1.11. *Every FC-finite graph is either of type $Y(p, q, r)$ or $I(p, q, m)$, as shown in fig. 8.*

Proof. Suppose γ is FC-finite. Then each of the following conditions must be met:

- There is (a) at most one vertex of order 3, and (b) no vertex of order 4 or greater.

  (a) Suppose there exist two vertices of order 3 or greater. Then there is a subgraph

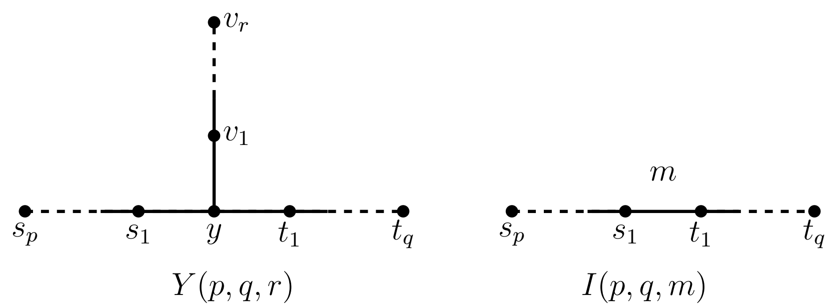


Figure 8: Graphs of Type $I(p,q,m)$ and $Y(p,q,r)$

which looks like fig. 9. Then

$$(s_3 s_1 s_2 s_3 \cdots s_p s_{p+1} s_{p+2} s_p s_{p-1} \cdots s_4)^k \tag{3.1.3}$$

is fully commutative for each k, since the only possible moves are to commute s_1 and s_2 or s_{p+1} and s_{p+2}.

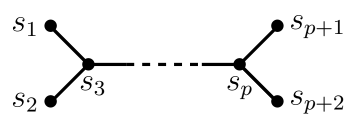


Figure 9: Subgraph of any graph with two branching points

(b) Suppose s_0 is a vertex of order 4 or greater and that $s_1, \ldots, s_4$ each share an edge with it, as shown in fig. 10. Since the graph is necessarily loop-free, $m(s_i, s_{i+1}) = 2$ and $m(s_0, s_i) > 2$ for $i > 0$. So

$$(s_0 s_1 s_2 s_0 s_3 s_4)^k \tag{3.1.4}$$

is fully commutative for each k.

- There is at most one edge with label 4 or greater.

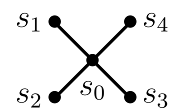


Figure 10: Subgraph of any graph with a vertex of order 4.

Suppose that there are two edges with label 4 or greater. Then there is a path $s_1, \ldots, s_{p+1}$ such that $m(s_1, s_2), m(s_{p+1}, s_p) > 3$ and $m(s_i, s_{i+1}) = 3$ for $1 < i < p$. Then the words

$$(s_2 s_1 s_2 s_3 \cdots s_{p-1} s_p s_{p+1} s_p s_{p-1} \cdots s_3)^k \tag{3.1.5}$$

are fully commutative for each k, since they are rigid.

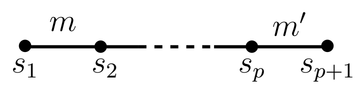


Figure 11: Subgraph of any graph with a branching point and an edge of label m.

- There cannot be a vertex of order 3 and an edge with label 4 or greater.

  If there is, it will have fig. 12 as a subgraph, where $m > 3$. Then the words

$$(s_1 s_2 s_3 \cdots s_{p-1} s_p s_{p+1} s_p s_{p-1} \cdots s_3)^k \tag{3.1.6}$$

  are fully commutative for each k, since the only move which can be applied to commute s_1 and s_2.

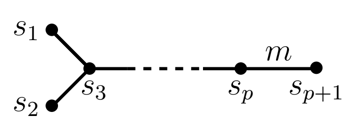


Figure 12: Subgraph of any graph with a branching point and an edge of label m.

Thus if γ is FC-finite, it either has exactly one vertex of order three and all edges labelled three, or no vertices of order 3 and all edges but one labelled three. These are types $Y(p,q,r)$ and $I(p,q,m)$, respectively.

□

Lemma 3.1.12. *The FC-finite Coxeter groups of type $Y(p,q,r)$ can only be of type D_n or E_n.*

Proof. Suppose that $Y(p,q,r)$ is FC-finite. Then

(i) $\min\{p,q,r\}=1$

If not, we can construct the following words:

$$(s_2 s_1 t_1\, y\, t_2 t_1 v_1\, y\, s_1 v_2 v_1\, y)^k. \tag{3.1.7}$$

Note that, for each k, between any pair of copies the same letter, there are two with which it does not commute. For example, between any pair of s_1, there are two copies of y or y and s_2. Hence these words are fully commutative by Proposition 3.1.4.

(ii) If $r=1$, then $\min\{p,q\}=2$.

If not, the following word is fully commutative for each k:

$$(ys_1t_1yv_1t_2t_1ys_1s_2s_3t_3t_2t_1yv_1s_1s_2)^k. \tag{3.1.8}$$

Again, between any copies of the same letter, there are two with which it does not commute.

Thus, FC-finite groups of type $Y(p,q,r)$ can only be of type D_n or E_n. $\square$

Lemma 3.1.13. *The FC-finite Coxeter groups of type $I(p,q,m)$ are of one of the following types: A_n, B_n, F_n, H_n or $I_2(n)$.*

Proof. Suppose $I(p,q,m)$ is FC-finite. Then the following restrictions apply:

(a) If $m > 5$, then $p = q = 1$.

Suppose $p \geq 2$ and $m > 5$. Then $t_1s_1t_1s_1t_1$ is fully commutative, so

$$(s_1t_1s_1t_1s_2)^k \tag{3.1.9}$$

is fully commutative, since the only move which can be applied is commuting t_1 and s_2.

(b) If $m = 5$, $\min\{p,q\} = 1$.

Suppose $p, q \geq 2$ and $m = 5$. For each k, the following word is fully commutative, since the only moves which can be applied are switching s_1 and t_2 or s_2 and t_1, so there are at most four terms in any string of s_1 and t_1:

$$(t_1t_2s_1t_1s_1t_1s_2s_1)^k. \tag{3.1.10}$$

(c) If $m = 4$, then $\min\{p,q\} = 2$.

For this case, an infinite set of fully commutative words used by Stembridge is most easily described using their heaps, as shown in fig. 13. We can see that these heaps will satisfy Lemma 3.1.6.

Thus the FC-finite graphs of type $I(p,q,m)$ can only be of type $I_2(m)$, B_n, F_n, or H_n. $\square$

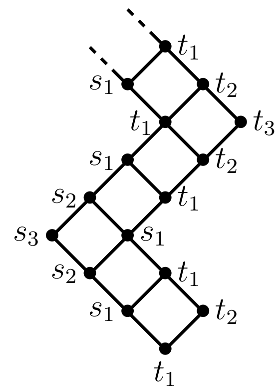
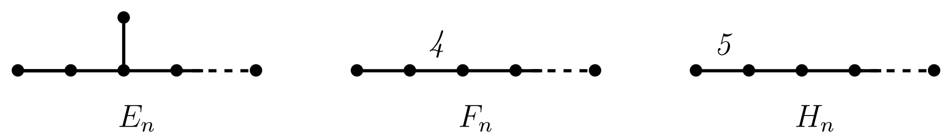


Figure 13: Infinite set of fully commutative words of Type $I(p, q, 4)$, $p, q \geq 3$

Theorem 3.1.14 ([5], Theorem 5.1)**.** *A Coxeter group is FC-finite if and only if it is finite or of type E_n, F_n, or H_n.*



Certainly, the finite Coxeter groups are necessarily also FC-finite. From Lemmas 3.1.9 and 3.1.10, we know that H_n and F_n are also FC-finite. Stembridge proved that E_n is FC-finite in a similar way, but we will provide a different proof in Section 3.4.3.

Combining Lemma 3.1.11 with Lemmas 3.1.13 and 3.1.12, we see that these are the only FC-finite Coxeter groups.

3.2 The Construction of the Generalized Temperley-Lieb Algebra

As discussed in section 2.2, $TL_n(\delta)$ can also be defined as a quotient of the Hecke algebra of A_{n-1} by the subspace generated by elements of the form $g_ig_jg_i + g_ig_j + g_jg_i + g_i + g_j + 1$, $|i - j| = 1$, which are elements of a Kazhdan-Lusztig basis for $H(n, q)$. In his thesis, Graham generalized this definition of $TL_n(\delta)$ to create Temperley-Lieb algebras associated with other Coxeter groups, $TL(\gamma, \delta)$.

Definition 3.2.1. For a Coxeter graph γ with Coxeter system (W, S), we define $H(\gamma, u)$ to be the algebra with basis T_w for $w \in W$ over $\mathbb{Z}[u, u^{-1}]$ and multiplication defined by the following relations:

$$(T_s + 1)(T_s - u^2) = 0 \tag{3.2.1}$$

$$T_{w_1 w_2} = T_{w_1} T_{w_2} \qquad \text{if } l(w_1 w_2) = l(w_1) + l(w_2) \tag{3.2.2}$$

for any $s \in S$ and $w_i \in W$.

Note that, in particular, (3.2.2) implies that

$$\underbrace{T_s T_t T_s \cdots}_{m(s,t) \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m(s,t) \text{ terms}} \tag{3.2.3}$$

for $s, t \in S$. Additionally, if there are two expressions of minimal length for w, then one can be reached from the other using only the commutation and braid relations by Matsumoto's Theorem. Thus, (3.2.3) implies (3.2.2). Hence

$$\langle T_s \mid (T_s + 1)(T_s - u^2) = 0,\ \underbrace{T_s T_t T_s \cdots}_{m(s,t) \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m(s,t) \text{ terms}} \rangle \tag{3.2.4}$$

is a presentation for $H(\gamma, u)$.

In particular, $H(A_{n-1}, u)$ is generated by $g_i = T_{a_i}$ and so (3.2.1) becomes $(g_i + 1)(g_i -$

$u^2) = 0$, which is equivalent to $g_i^2 = (u^2 - 1)g_i + u^2$ (2.2.1), and (3.2.3) becomes

$$g_i g_j g_i = T_{a_i a_j a_i} = T_{a_j a_i a_j} = g_j g_i g_j \qquad \text{for } |i-j| = 1 \qquad (2.2.2), \text{ or}$$

$$g_i g_j = T_{a_i a_j} = T_{a_i a_j} = g_j g_i \qquad \text{for } |i-j| > 1 \qquad (2.2.3).$$

So $H(A_{n-1}, u)$ is the same as $H(n, u)$ from section 2.2.

Note that, for $s \in S$, T_s has an inverse, $T_s^{-1} = u^{-2}(T_s - u^2 + 1)$. Thus we can define an involution D on $H(\gamma, u)$ such that, for p_w Laurent polynomials with coefficients in $\mathbb{Z}$,

$$D(\sum_{w \in \gamma} p_w(u) T_w) = \sum p_w(u^{-1}) T_{w^{-1}}^{-1}. \qquad (3.2.5)$$

In particular, D sends u to u^{-1} and T_s to T_s^{-1}. This involution is a ring automorphism of $H(\gamma, u)$.

Definition 3.2.2. The *Kazhdan-Lusztig basis* for $H(\gamma, u)$ is the unique set of elements

$$C'_w = u^{-l(w)} \sum_{y \leq w} P_{y,w}(u^2) T_y. \qquad (3.2.6)$$

such that

(1) $P_{w,w} = 1$ and $\deg(P_{y,w}) \leq [l(w) - l(y) - 1]/2$ if $y < w$

(2) $D(C'_w) = C'_w$

Note that $C'_s = u^{-1}(T_s + 1)$, which is the same substitution that we made for e_i in Lemma 2.2 to show that $TL_n \cong H(n,u)/\langle g_i g_j g_i + g_i g_j + g_j g_i + g_i + g_j + 1 = 0 \mid |i-j| = 1\rangle$,

Note also that

$$C'_{sts} = u^{-3}(T_{sts} + T_{st} + T_{ts} + T_t + T_s + 1) = C'_s C'_t C'_s - C'_s \qquad (3.2.7)$$

for $s, t \in S$. Note that this means that the relations from the quotient are equivalent to $C'_{sts} = 0$ for s and t adjacent.

Definition 3.2.3. Let $\delta = u + u^{-1}$ (also denoted [2] by some authors). Define $TL(\gamma, \delta)$ to be the quotient of $H(\gamma, u)$ by the ideal $\mathcal{K}$ generated by the elements $C'_{w(s,t)}$ for s, t adjacent in γ, where $w(s,t) = \underbrace{sts\cdots}_{m(s,t)\text{ terms}} = \underbrace{tst\cdots}_{m(s,t)\text{ terms}}$.

For $s \in S$, define $e_s := C'_s + \mathcal{K}$. These elements generate $TL(\gamma)$.

Note that $w(s,t)$ are the words that represent braid moves, so this quotient effectively removes the elements of $H(W, u)$ that correspond to words in W that contain braid moves.

Proposition 3.2.1 ([11], Chapter 6)**.** *For $s, t \in S$ and $1 \leq m \leq m(s,t)$, let $w_m(s,t) = \underbrace{sts\cdots}_{m\ terms}$. Then*

$$C'_{w_m(s,t)} = u^{-m} \sum_{y \leq w_m(s,t)} T_y = Q_m(s,t), \tag{3.2.8}$$

where

$$Q_m(s,t) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-1-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-2i\ terms}. \tag{3.2.9}$$

In particular, this means that the relations in $TL(\gamma)$ from the quotient are

$$\sum_{i=0}^{\lfloor m(s,t)/2 \rfloor} (-1)^i \binom{m-1-i}{i} \underbrace{(e_s e_t e_s \cdots)}_{m(s,t)-2i\text{ terms}} = 0. \tag{3.2.10}$$

Applying this to $m(s,t) = 3$ yields the familiar relation

$$e_s e_t e_s - e_s = 0. \tag{3.2.11}$$

Proof. Note that $Q_m(s,t)$ is fixed by the involution D because D is an automorphism and C'_x is fixed by D for every $x \in S$.

We can see that $P_{y,q}(u^2) = 1$ satisfies the first requirement for the Kahzdan-Lusztig basis and $Q_m(s,t)$ satisfies the second, so if we can show that

$$u^{-m} \sum_{y \le w} T_y = Q_m(s,t), \tag{3.2.12}$$

then $Q_m(s,t)$ must be the unique element $C'_{w_m(s,t)}$, where $w_m(s,t) = \underbrace{sts\cdots}_{m \text{ terms}}$.

In order to prove this, we will proceed by induction on m.

For $m = 1$, we see that

$$Q_m(s,t) = C'_s = u^{-1}(T_s + 1). \tag{3.2.13}$$

For $m = 2$,

$$Q_m(s,t) = C'_s C'_t = u^{-2}(T_s T_t + T_s + T_t + 1) = u^{-2} \sum_{y \le st} T_y. \tag{3.2.14}$$

Now suppose $m > 2$ and that $Q_k(s,t) = u^{-k} \sum_{y \le w_k(s,t)} T_y$ for $1 \le k < m$. Let $x = s$ if m is odd, or t if m is even.

Since $w_m(s,t) = \underbrace{sts\cdots}_{m \text{ terms}}$ and $m \le m(s,t)$, $l(w_m(s,t)) = m$. Thus, $y \le w_m(s,t)$ if and only if $y = w_m(s,t)$ or $y \in \langle s,t \rangle$ and $l(y) < m$. Then

$$\begin{aligned}
\sum_{y \leq w_m(s,t)} T_y - \sum_{y \leq w_{m-1}(s,t)} T_y \\
&= T_{w_m(s,t)} + T_{w_{m-1}(t,s)} = \underbrace{T_s T_t T_s \cdots}_{m \text{ terms}} + \underbrace{T_t T_s T_t \cdots}_{m-1 \text{ terms}} \\
&= \Bigg(\underbrace{T_s T_t T_s \cdots}_{m-1 \text{ terms}} + \underbrace{T_t T_s T_t \cdots}_{m-2 \text{ terms}} \Bigg) T_x \\
&= \Bigg(\sum_{y \leq w_{m-1}(s,t)} T_y - \sum_{y \leq w_{m-2}(s,t)} T_y \Bigg) T_x. \quad (3.2.15)
\end{aligned}$$

By our inductive hypothesis, $\sum_{y \leq w_k(s,t)} T_y = u^k Q_k(s,t)$ for $1 \leq k < m$, so we can rewrite this equation as

$$\sum_{y \leq w_m(s,t)} T_y - u^{m-1} Q_{m-1}(s,t) = \left(u^{m-1} Q_{m-1}(s,t) - u^{m-2} Q_{m-2}(s,t) \right) T_x, \qquad (3.2.16)$$

which we can rearrange to get

$$u^{-m} \sum_{y \leq w_m(s,t)} T_y = u^{-1} Q_{m-1}(1 + T_x) - u^{-2} Q_{m-2}(s,t) T_x. \qquad (3.2.17)$$

Since $C'_x = u^{-1}(T_x+1)$, we have that $C'_x T_x = u^{-1}(T_x^2+T_x) = u(T_x+1) = u^2 C'_x$. Thus

$$\begin{aligned}
Q_{m-2}(s,t)T_x &= \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} (-1)^i \binom{m-3-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-2-2i \text{ terms}} T_x \\
&= \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} (-1)^i \binom{m-3-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-3-2i \text{ terms}} (C'_x T_x) \\
&= \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} (-1)^i \binom{m-3-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-3-2i \text{ terms}} (u^2 C'_x) = u^2 Q_{m-2}(s,t).
\end{aligned} \tag{3.2.18}$$

By applying this to (3.2.17), we get

$$u^{-m} \sum_{y \le w_m(s,t)} T_y = Q_{m-1}(s,t)C'_x - Q_{m-2}(s,t). \tag{3.2.19}$$

Expanding $Q_{m-1}(s,t)$ and $Q_{m-2}(s,t)$, we get

$$\begin{aligned}
&u^{-m} \sum_{y \le w_m(s,t)} T_y \\
&= \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-2-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-1-2i \text{ terms}} C'_x - \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} (-1)^i \binom{m-3-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-2-2i \text{ terms}} \\
&= \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m-2-i}{i} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-2i \text{ terms}} + \sum_{i=1}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-2-i}{i-1} \underbrace{(C'_s C'_t C'_s \cdots)}_{m-2i \text{ terms}}.
\end{aligned} \tag{3.2.20}$$

Note that if m is even, the term for $i = m/2$ is zero for the second sum. Thus, in either case, this sum becomes

$$\underbrace{(C'_sC'_tC'_s\cdots)}_{m\text{ terms}}+\sum_{i=1}^{\lfloor(m-1)/2\rfloor}(-1)^i\left(\binom{m-2-i}{i}+\binom{m-2-i}{i-1}\right)\underbrace{(C'_sC'_tC'_s\cdots)}_{m-2i\text{ terms}}]$$
$$=\sum_{i=0}^{\lfloor(m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}\underbrace{(C'_sC'_tC'_s\cdots)}_{m-2i\text{ terms}}. \quad (3.2.21)$$

Again, if m is even, the term $i=m/2$ in the sum of $Q_m(s,t)$ would be zero, so

$$u^{-m}\sum_{y\le w_m(s,t)}T_y=\sum_{i=0}^{\lfloor(m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}\underbrace{(C'_sC'_tC'_s\cdots)}_{m-2i\text{ terms}}=Q_m(s,t). \quad (3.2.22)$$

By induction, we have that $Q_m(s,t)=u^{-m}\sum_{y\le w_m(s,t)}T_y$ for $1\le m\le m(s,t)$. □

Proposition 3.2.2. *The algebra $TL(\gamma,\delta)$ is generated by e_s for $s\in S$ modulo the relations*

$$e_s^2=\delta e_s \quad (3.2.23)$$
$$e_se_t=e_te_s \quad \textit{if } m(s,t)=2 \quad (3.2.24)$$
$$\sum_{i=0}^{\lfloor m(s,t)\rfloor}(-1)^i\binom{m(s,t)-1-i}{i}\underbrace{e_se_te_s\cdots}_{m(s,t)-2i\ \textit{terms}} \quad \textit{if } m(s,t)>2. \quad (3.2.25)$$

Proposition 3.2.1 shows us that $TL(\gamma,\delta)$ is the quotient of $H(W(\gamma),u)$ modulo the relations

$$\sum y\le w(s,t)T_y=0 \quad (3.2.26)$$

for $s,t\in S$ such that $m(s,t)>2$. This is how Green defines the generalized Temperley-

Lieb algebras in [9].

We could also write these relations as

$$\underbrace{T_sT_tT_s\cdots}_{m(s,t)\text{ terms}} = -\sum_{y<w(s,t)} T_y, \tag{3.2.27}$$

from which we can see that these new relations imply the relations (3.2.3) for $m(s,t) > 2$. Thus, if we use the alternative presentation for $H(\gamma, u)$ 3.2.4, we see that $TL(\gamma)$ is generated by T_s modulo the relations

$$(T_s + 1)(T_s - u^2) = 0, \tag{3.2.28}$$

$$T_sT_t = T_sT_t \quad \text{if } m(s,t) = 2, \tag{3.2.29}$$

$$Q_{m(s,t)}(s,t) = 0 \quad \text{if } (s,t) > 3. \tag{3.2.30}$$

By substituting $e_s = C'_s$ into these relations, we see that $TL(\gamma, \delta)$ is generated by $\{e_s \mid s \in S\}$ modulo the relations

$$e_s^2 = \delta e_s \tag{3.2.31}$$

$$e_se_t = e_te_s \quad \text{if } m(s,t) = 2 \tag{3.2.32}$$

$$\sum_{i=0}^{\lfloor m(s,t) \rfloor} (-1)^i \binom{m(s,t)-1-i}{i} \underbrace{e_se_te_s\cdots}_{m(s,t)-2i\text{ terms}} = 0 \quad \text{if } m(s,t) > 2. \tag{3.2.33}$$

We can see that this more closely resembles our definition of TL_n. In fact, applying this proposition to A_{n-1} gives us TL_n.

Notice that this means that, for $m(s,t) > 3$, $\underbrace{e_se_te_s\cdots}_{m(s,t)\text{ terms}}$ can be written as a linear combination of shorter words. Thus, it does not appear in any word of minimal length.

Theorem 3.2.3. *([11], Theorem 6.2) The fully commutative elements of γ index a basis for $TL(\gamma)$ over $\mathbb{Z}[u, u^{-1}]$. In particular, $\{T_w + \mathcal{K} \mid w \in FC(\gamma)\}$ is a basis for $TL(\gamma)$.*

Definition 3.2.4. Let w be a fully commutative element of γ. Let $\mathbf{w} = s_1 \cdots s_k$ be a reduced expression for w. Define $e_w = e_{s_1} \cdots e_{s_k}$.

To see that this is well defined, remember that if $\mathbf{w}$ and $\mathbf{w}'$ are two reduced expressions for w, then $\mathbf{w} \approx \mathbf{w}'$. That is, $\mathbf{w}'$ can be reached from $\mathbf{w}$ by commutation alone, and any commutation that can be applied to $\mathbf{w}$ can be applied to e_w without changing the word, since $e_s e_t = e_t e_s$ if $st = ts$.

Corollary 3.2.4. *([6], Proposition 2.4) $\{e_w \mid w \in FC(\gamma)\}$ is a basis for $TL(\gamma)$ over $\mathbb{Z}[u, u^{-1}]$.*

Proof. Suppose that $w \in FC(\gamma)$ and $s_1 \cdots s_m$ is a reduced expression for w. Then

$$e_w = e_1 \cdots e_m = C'_{s_1} \cdots C'_{s_m} + \mathcal{K} \tag{3.2.34}$$

Since $C'_s = u^{-1}(T_s + 1)$ and $T_w = T_{s_1} \cdots T_{s_m}$, we see that

$$e_w = u^{-m} T_w + \sum_{y < w} a_y T_y + \mathcal{K} \tag{3.2.35}$$

for some $a_y \in \mathbb{Z}[u, u^{-1}]$. Since $\{T_w \mid w \in FC(\gamma)\}$ is a basis for $TL(\gamma)$, $\{e_w \mid w \in FC(\gamma)\}$ must also be a basis.

□

Lemma 3.2.5. *A monomial $e_{s_1} \cdots e_{s_m}$ is of minimal length with respect to the relations on $TL(\gamma)$ if and only if $s_1 \cdots s_m$ is a reduced expression for $w \in FC(\gamma)$.*

Proof. Suppose that $e_{s_1} \cdots e_{s_m}$ is of minimal length with respect to the relations on $TL(\gamma)$. Then there is no way to commute the letters to get e_s^2 or $\underbrace{e_s e_t \cdots}_{m(s,t) \text{ terms}}$ for $m(s,t) > 3$. Since e_s and e_t commute if and only if $m(s,t) \leq 2$, this means that there is no way to commute the letters of $s_1 \cdots s_m$ to get s^2 or $\underbrace{sts \cdots}_{m(s,t) \text{ terms}}$ for $m(s,t) > 3$. By Proposition 3.1.1, since a reduced expression for $\mathbf{w} = s_1 \cdots s_m$ can be reached using commutation, braid moves and cancellation, $\mathbf{w}$ must be of minimal length. Since we cannot commute to get $\underbrace{sts \cdots}_{m(s,t) \text{ terms}}$ for $m(s,t) > 3$, we see that $\mathbf{w}$ has only one commutation class. Thus $\mathbf{w}$ is a reduced expression for some $w \in FC(\gamma)$.

Suppose that $s_1 \cdots s_m$ is a reduced expression for $w \in FC(\gamma)$ and $e_{s_1} \cdots e_{s_m}$ is not of minimal length. Then there are $s'_1, \ldots, s'_k$ such that $k < m$ and

$$e_w = \sum_{i=1}^{k} a_i e_{s_{i_1}} \cdots e_{s_{i_1}} \tag{3.2.36}$$

such that $a_i \in \mathbb{Z}[u, u^{-1}]$ and each $e_{s_{i_1}} \cdots e_{s_{i_1}}$ is of minimal length. Since they are of minimal length, each $s_{i_1} \cdots s_{i_1}$ must be a reduced expression for some $w_i \in FC(\gamma)$.

Then $e_w = \sum_{i=1}^{k} a_i e_{w_i}$, which contradicts Corollary 3.2.4. □

Theorem 3.2.6 ([11], Theorem 7.1)**.** *The algebra $TL(\gamma)$ is finite dimensional if the corresponding Coxeter group $W(\gamma)$ is finite or if γ is E_n, F_n, or H_n.*

Proof. By Corollary 3.2.4, $TL(\gamma)$ is finite dimensional if and only if γ has only finitely many fully commutative elements. By Theorem 3.1.14, the only irreducible Coxeter groups with finitely many fully-commutative elements are the finite Coxeter groups, as well as E_n, F_n, and H_n. □

3.3 Diagram Decomposition Algorithm

In this section, we will describe the algorithm developed by Ernst et al. in [7]. This algorithm takes a diagram in $\mathbb{D}TL_n$ and non-recursively returns a reduced word in TL_n.

Let D be a loop-free diagram in $\mathbb{D}TL_n$. The first step of the algorithm is to isotopically deform D so that each edge is either never crosses the same vertical line twice or is itself a vertical line. In other words, no edges in D double back on themselves.

Note that if we have a reduced word in the generators $e_1, \ldots, e_{n-1}$, then the corresponding stack of $E_1, \ldots, E_{n-1}$ is already in this form. By Theorem 2.3.1, each diagram can be deformed into such a stack.

The second step is to add edges from each node on the top face of D to the corresponding node on the bottom edge. These new edges divides D into n columns, and the original edges divide each columns into regions.

Since every E_i has an even number of edges crossing each column, and D can be deformed into a stack of E_i, we see that each column of D must also have an even number of edges.

Definition 3.3.1. The *depth* of each region is the number of regions between it and the top face of the diagram. The top region in each column has depth 0. Call a region with odd depth a *1-region* and a region with even depth a *0-region*.

We say that two regions are *horizontally adjacent* if they are in the same column and share an edge.

We say that two regions are *horizontally adjacent* if they are in adjacent columns and there is a path between the regions that does not cross any of the original edges of D. Note that if one of the edges of D is a vertical line, then two regions may be in adjacent columns and share an edge without being horizontally adjacent.

We say that two regions are *diagonally adjacent* if there is a third region that is horizontally adjacent to one region and vertically adjacent to the other.

We say $R_1 \prec R_2$ if they are diagonally adjacent such that a third region R_3 is horizontally adjacent to R_1 and vertically adjacent to and below R_2.

Note that these adjacency relations aren't changed by deforming the edges.

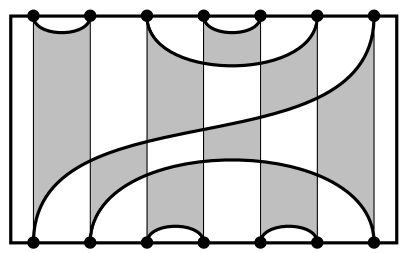


Figure 14: The first step of the algorithm with the regions of odd depth shaded

Lemma 3.3.1 ([7], Lemma 4.3)**.** *If R_1 and R_2 are horizontally adjacent, then*

$$|depth(R_1) - depth(R_2)| = 1. \tag{3.3.1}$$

Proof. Without loss of generality, suppose that depth$(R_1) \leq$ depth(R_2).

Proceed by induction on depth(R_1). Suppose depth$(R_1) = 0$, so R_1 is the top region in its column. Since R_2 is horizotally adjacent to R_2, there is a path from R_1 to R_2 that does not cross the original edges of D.

Consider the edge of D emanating from the node between R_1 and R_2. This edge must cross the column R_2 lies in, or else any path from R_1 (as the top region) to R_2 would cross the edge. Additionally, we know that R_2 must be below the edge, or we have the same issue. If R_2 lay below a second edge, that edge would necessarily end on the other side of the column R_1 lies in, any path to R_1 would have to cross that edge.

Thus R_2 lies below just one edge, so $\text{depth}(R_2) = 1$.

Suppose that $\text{depth}(R_1) > 0$. Then $\text{depth}(R_2) > 0$ as well, so there are regions R_1' and R_2' immediately above R_1 and R_2.

Since neither region has depth zero, R_1 and R_2 must share a top edge, hence R_1' and R_2' share a bottom edge which is an original edge of D. So there must be a path from R_1' to R_2' which doesn't cross the original edges of D. Since they are in adjacent columns, they must also be horizontally adjacent. Note that $\text{depth}(R_i') = \text{depth}(R_i) - 1$, so, by induction,

$$|\text{depth}(R_1) - \text{depth}(R_2)| = |\text{depth}(R_1') - \text{depth}(R_2')| = 1. \tag{3.3.2}$$

□

Lemma 3.3.2 ([7], Lemma 4.4)**.** *If $R_1 \prec R_2$, then R_1 is a 1-region if and only if R_2 is a 1-region.*

Proof. Suppose $R_1 \prec R_2$. Then there is a third region R_3 such that which is horizontally adjacent to R_1 and vertically adjacent to and below R_2.

Since R_3 and R_2 share an edge and are in the same column, we can see that there depth must differ by one. So R_3 is a 0-region if and only if R_2 is a 1-region. By Lemma 3.3.1, the depths of R_1 and R_3 also differ by one so R_3 is a 0-region if and only if R_1 is a 1-region.

Therefore R_1 is a 1-region if and only if R_2 is a 1-region. □

By taking the transitive closure of $\prec$, we can extend it to a partial order $\preceq$ on the set R of 1-regions (the shaded regions in fig. 14). Label the elements of this partially ordered set so that a region in column i is labelled a_i. This is a heap, which will call P_D. Since adjacency is preserved when the edges of D are deformed, there is a unique heap for each diagram.

Theorem 3.3.3 ([7])**.** *, Theorem The heap P_D corresponds to a fully commutative word w_D in the Coxeter group A_{n-1}.*

Proof. Suppose R_1 and R_2 are two of the 1-regions in the ith column of D, with

$$\operatorname{depth}(R_2) - \operatorname{depth}(R_1) = 2. \tag{3.3.3}$$

Then there is exactly one region between them, R_3. Note that R_3 is vertically adjacent to both, and below R_1.

Since R_1 is a 1-region, we know there is an edge above the top edge of R_3 in the ith column. Since there are an even number of edges in the column and an odd number above R_2, there must also be another edge below the bottom edge of R_3 (top edge of R_2).

Since there are edges above and below in the column, we know that neither the top nor bottom edges of R_3 can have i or $i+1$ as endpoints. Thus they must also cross the columns $i \pm 1$. Let S_1, S_2 be the regions in columns $i \pm 1$ which share their top and bottom edges with R_3. Then these regions are horizontally adjacent to R_3, so $S_j \prec R_1$.

Consider also the regions in columns $i \pm 1$ whose top edge is the bottom edge of R_3. These regions are horizontally adjacent to R_2, and below and vertically adjacent to S_j (depending on the column). Thus $R_2 \prec S_j$. Note also that $R_2 \prec R_1$.

Thus, if we label the 1-region in column i by 1 through m, from highest to lowest, we have that $R_m \prec \cdots \prec R_1$. Thus, we have that the elements of P_D which are labelled a_i

form a chain, $a_i^{(m)} \prec \cdots \prec a_i^{(1)}$. Additionally, for each $1 \leq k < m$, we have j_1, j_2 such that

$$a_i^{(k+1)} \prec a_{i-1}^{(j_1)}, a_{i+1}^{(j_2)} \prec a_i^{(k+1)} \tag{3.3.4}$$

So we see that

- there is no covering relation $i \prec i'$ where i and i' have the same label, and
- no convex chain $i_1 \prec i_2 \prec i_3$ such that i_1, i_3 are labelled a_i and i_2 is labelled $a_{i\pm 1}$.

By Lemma 3.1.6, this means that our heap, P_D, corresponds to a fully commutative word. □

Corollary 3.3.4 ([7], Theorem 4.7)**.** *The heap P_D corresponds to a reduced word in $e_1, \ldots, e_n$. The image of this word under the isomorphism $e_i \mapsto E_i$ is D.*

By the previous theorem, P_D corresponds to some $w_D \in FC(A_{n-1})$. Then let e_{w_D} be the corresponding word in $e_1, \ldots, e_n$. This word is reduced, by Theorem 3.2.4.

Remember that D can be deformed into a stack of the generators and that stack will still have the same associated heap. The occurrences of a_i in w_D correspond to the 1-regions in the ith column of that stack, which are the occurrences of E_i in the stack. So we see that the image of e_{w_D} under the isomorphism $e_i \mapsto E_i$ will be D.

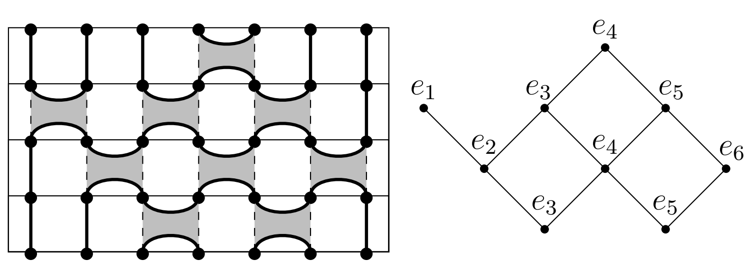


Figure 15: The decomposed diagram and associated heap

3.4 The Diagrammatic Representation of Finite Dimensional Generalized Temperley-Lieb Algebra

Since Graham introduced generalized Temperley-Lieb algebras, faithful diagrammatic representations have been created for some of these finite dimensional algebras, namely for those of type B_n, D_n and E_n ([8], [9], [10]) by adding blobs, which we call decorations, to the diagrams of $\mathbb{D}TL_n$.

While the relations on the undecorated (n, n)-diagrams are the same in every representation, these decorations have different relations in each representation. Accordingly, we will use different for blobs under different relations.

For B_n and D_n, Green used decorations that lie on the edges of the (n, n)-diagrams. Of the (n, n)-diagrams with decorated edges, the following are of particular interest.

Definition 3.4.1. Define E to be the identity diagram with a single decoration on its leftmost edge. Then $E_{\bar{1}}$ is defined to be EE_1E, which is E_1 with two decorations, one each on the top left and bottom left edges.

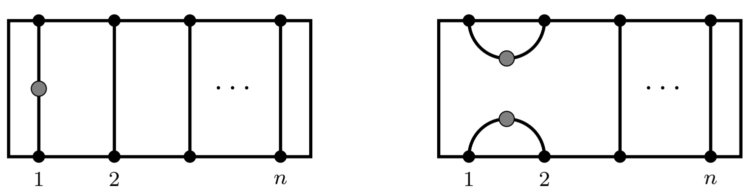


Figure 16: The decorated (n, n)-diagrams E and $E_{\bar{1}}$

We say that a decoration is *left-exposed* if there is a path from the decoration to the left edge which crosses no edges. If a bubble-free diagram has at most one decoration per edge and all decorations are left-exposed, then that diagram is called a *blob diagram.*

3.4.1 The Temperley-Lieb algebra of B_n

The Coxeter graph B_n differs from A_n in that $m(a_1, a_2) = 3$, while $m(b_1, b_2) = 4$. This is the only graph we are considering with $m(s,t) > 3$. This gives us a slightly different

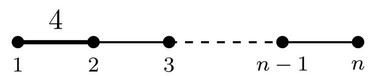


Figure 17: Dynkin diagram B_n

relation than usual. Applying Proposition 3.2.1, we have

$$C'_{stst} = Q_4(s,t) = \sum_{i=0}^{2} (-1)^i \binom{3-i}{i} \underbrace{T_s T_t \cdots}_{4-2i \text{ terms}} = C'_s C'_t C'_s C'_t - 2C'_s C'_t. \tag{3.4.1}$$

Let $b_i = C'_{b_i} + \mathcal{K}$. Then, by Proposition 3.2.2, $TL(B_n)$ is the algebra generated by $b_1, \ldots, b_n$ under the relations

$$b_i^2 = \delta b_i \tag{3.4.2}$$

$$b_i b_j = b_j b_i \qquad \text{if } |i-j| > 1 \tag{3.4.3}$$

$$b_i b_j b_i = b_i \qquad \text{if } |i-j| = 1 \text{ and } \{i,j\} \neq \{1,2\} \tag{3.4.4}$$

$$b_i b_j b_i b_j = 2 b_i b_j \qquad \{i,j\} = \{1,2\}. \tag{3.4.5}$$

To represent this new relation diagrammatically, Green used $2E_{\bar{1}}$ to represent b_1.

Definition 3.4.2. We define $\mathbb{D}TL(B_n)$ to be the algebra of decorated $(n+1, n+1)$-diagrams over $\mathbb{Z}[u, u^{-1}, \frac{1}{2}]$ generated by $E_{\bar{1}}, E_2, \ldots, E_n$ with, in addition to the usual bubble-popping relation, the following relations on the decorations:

(1) multiple decorations on the same line are equivalent to a single decoration on that line, and

(2) a bubble with a decoration is equivalent to $\delta' = \delta/2$.

Figure 18: The additional relations on the decorations

Proposition 3.4.1 ([9], Proposition 5.1)**.** *The blob (n, n)-diagrams are a basis for the algebra generated by $E, E_1, \ldots, E_{n-1}$ modulo the relations in fig. 18. This algebra is called the blob algebra, denote $b_n(\delta, \delta')$.*

In particular, for every blob (n, n)-diagram, there is a reduced word in E and E_i $(1 \leq i < n)$ which is equal to that diagram. Notice that copies of E and E_1 in such a reduced word must alternate, or else we could commute to get E^2, E_1^2, or $E_1E_2E_1$, which are not reduced.

We will now outline Green's proof that this representation is faithful, which proved that the map $E_1 \mapsto 2E_{\bar{1}}$ and $E_i \mapsto E_i$ $(1 < i \leq n)$ is an isomorphism. In order to do this, Green described a basis for $\mathbb{D}TL(B_n)$.

Lemma 3.4.2 ([9] Lemma 6.3)**.** *The algebra $\mathbb{D}TL(B_n)$ has as a basis those diagrams who are of one of the following types:*

(1) there are no decorations and there is an edge from the first top node to the first bottom node; or

(2) a blob diagram with at least one non-propagating edge such that the edge(s) with the first top node and the first bottom node as endpoints are decorated.

Proof. Note that E_i is of type 1 for $i > 1$ and $E_{\bar{1}}$ is of type 2. Additionally, this set of diagrams is closed under multiplication, so all diagrams of $\mathbb{D}TL(B_n)$ must be of one of these two types. It only remains to show that any diagram of one of these types is in $\mathbb{D}TL(B_n)$.

If a diagram is of type 1, then it is in $\mathbb{D}TL_{n+1}$. Since no edge crosses the first column, we can see from the decomposition algorithm of Section 3.3 that it is a product of $E_2, \ldots, E_n$ only, and thus is an element of $\mathbb{D}TL(B_n)$.

Suppose that D is a diagram of type 2. Since it is a blob diagram, there is a reduced word E_w in $E, E_1, \ldots, E_n$ which is equal to D.

Since D has at least one decoration, E_w must have a subsequence of alternating copies of E and E_1 in E_w. Because the edges with nodes 1 as endpoints are decorated, $DE = D = ED$, by the first relation. Thus $EE_w = E_w = E_wE$, so the subsequence must begin and end in E, since otherwise we could commute the letters of EE_w or E_wE to get $EE_1E = \delta' E_1$.

Now, $D = E_w$ is equal to a word in $E_{\bar{1}}, E_2, \ldots, E_n$ ([9], Lemma 5.4). To see this, note that we can expand E_w by replacing the copies of E between pairs of E_1 with E^2. Then there are two copies of E for each copy of E_1, and we can commute so that all copies of E and E_1 occur in subwords $EE_1E = E_{\bar{1}}$. Thus $D \in \mathbb{D}TL(B_n)$.

Therefore the diagram of type 1 and 2 form a basis for $\mathbb{D}TL(B_n)$. $\square$

Green then used the following lemma about the number of blob diagrams to show that the dimensions of $TL(B_n)$ and $\mathbb{D}TL(B_n)$ are the same.

Lemma 3.4.3 ([9], Lemma 5.7)**.** *There is an isomorphism ϕ from $b_n(\delta, 1)$ to the subalgebra of $\mathbb{D}TL_{2n}(\delta^2)$ spanned by symmetric diagrams such that $\phi(E) = E_n/\delta$ and $\phi(E_i) = E_{n-i}E_{n+i}$. Additionally, there are $(n+1)C_n$ distinct blob diagrams.*

Proof. We can see that this mapping is a homomorphism. In particular, $\phi(E)^2 = \phi(E^2)$

(decoration cancellation) and $\phi(E_1)\phi(E)\phi(E_1) = \phi(E_1) = \delta'\phi(E_1)$ (decorated bubble popping relation). Additionally, the symmetric diagrams of $\mathbb{D}TL_{2n}$ are spanned by E_n and $E_{n-i}E_{n+i}$ $(1 \leq i < n)$ ([9], Lemma 5.6). Thus ϕ is a surjective homomorphism.

We will show that the blob (n, n)-diagrams are in bijection with the symmetric $(2n, 2n)$-diagrams, which shows that ϕ is an isomorphism. Since there are $(n + 1)C_n$ symmetric $(2n, 2n)$-diagrams ([9], Lemma 5.6), the second result follows.

Suppose that d is a blob (n, n)-diagram. Since all the decorations of d are left-exposed, we can deform its edges so that all the decorations are on the left face of the diagram.

Then, break the edges where there is a decoration, so that the edge is divided into two distinct edges, both of which have an endpoint on the left face. Finally, add the reflection of this diagram across the left face. Now each of the broken edges is completed to an arc connecting the two halves of the diagram and there are $2n$ nodes on the top and bottom.

This new diagram is a symmetric $(2n, 2n)$-diagram. The process can be reversed to turn any symmetric $(2n, 2n)$-diagram into a blob (n, n)-diagram. Note that the symmetric form of a blob diagram is its image under ϕ, up to a constant.

This process is illustrated in fig. 19.

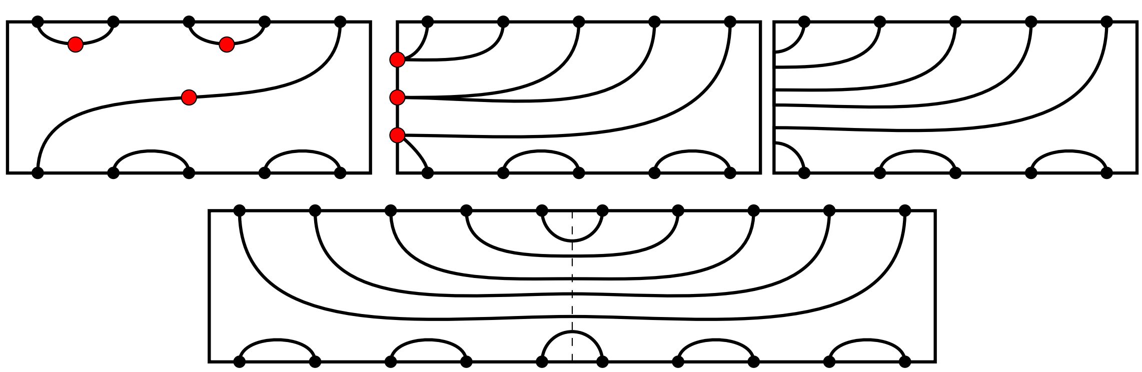


Figure 19: The bijection from blob diagrams to symmetric $\mathbb{D}TL_{2n}$ diagrams

□

Theorem 3.4.4 ([9] Theorem 4.1)**.** *The algebra $\mathbb{D}TL(B_n)$ over the ring $\mathbb{Z}[u, u^{-1}, \frac{1}{2}]$ is isomorphic to $TL(B_n)$. In particular, the homomorphism $\phi_B : TL(B_n) \mapsto \mathbb{D}TL(B_n)$ defined by $b_1 \mapsto 2E_{\bar{1}}$ and $b_i \mapsto E_i$ for $i > 1$ is an isomorphism.*

Proof. To see that ϕ_B is a homomorphism, we only need to check the relations with $E_{\bar{1}}$. We easily can see that $E_{\bar{1}}$ commutes with E_i for $i > 2$, that $(2E_{\bar{1}})^2 = \delta(2E_{\bar{1}})$ and $(2E_{\bar{1}})E_2(2E_{\bar{1}})E_2 = 2(2E_{\bar{1}}E_2)$. So the relations of $TL(B_n)$ are respected in $\mathbb{D}TL(B_n)$.

Since $\mathbb{D}TL(B_n)$ is generated by $\phi_B(b_i)$, ϕ_B is a necessarily a surjection. Hence, to prove that ϕ_B is an isomorphism, it suffices to show that their dimensions are equal. Fan ([16], Section 7.2) tells us that

$$|FC(B_n)| = \dim(TL(B_n)) = (n+2)C_n - 1. \tag{3.4.6}$$

Note that the number of diagrams of type 1 is $\dim(\mathbb{D}TL_n) = C_n$, since any diagram of $\mathbb{D}TL_n$ can be converted into a distinct diagram of type 1 by adding two nodes and non-propagating edge to the left side of the diagram.

The symmetric $(2n, 2n)$-diagrams are in bijection with the blob diagrams such that the edge(s) with nodes 1 as endpoints are decorated.

To see this, suppose D is such a diagram. Then we can unfold D into a symmetric $(2n+2, 2n+2)$-diagram. Because there are decorations on the edge(s) from nodes 1 (top and bottom) in D, in the unfolded diagram there are edges connecting nodes $n+1$ and $n+2$. If we remove the nodes $n+1$ and $n+2$ and the edges connecting them, the result is a symmetric $(2n, 2n)$-diagram.

If we reverse the process on any symmetric $(2n, 2n)$-diagram, we will get a blob-diagram with decorations on the edge(s) with nodes 1 as endpoints.

Note that any diagram of this sort is E or a diagram of type 2. Since there are $(n+1)C_n$ symmetric $(2n, 2n)$-diagrams in total, this means that there are $(n+1)C_n - 1$ diagrams of type 2.

Thus $\dim(\mathbb{D}TL(B_n)) = (n+2)C_n - 1 = \dim(TL(B_n))$, so ϕ_B is an isomorphism. $\square$

3.4.2 The Temperley-Lieb algebra of D_n

Since all the edges in the Coxeter graph D_n are labelled 3, the quotient gives us essentially the same relations as A_n.

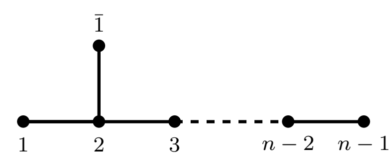


Figure 20: Dynkin diagram D_n

The algebra $TL(D_n)$ is generated by $d_{\bar{1}}, d_1, \ldots, d_n$ under the following relations, for $x, y \in \{\bar{1}, 1, \ldots, n\}$ (let $|\bar{1} - y| = |1 - y|$):

$$d_x^2 = \delta d_x \tag{3.4.7}$$

$$d_x d_y = d_x d_y \qquad \text{if } |x - y| \neq 1 \tag{3.4.8}$$

$$d_x d_y d_x = d_x \qquad \text{if } |x - y| = 1 \tag{3.4.9}$$

Remember that a monomial in $d_{\bar{1}}, d_1, \ldots, d_{n-1}$ is reduced if and only if it is indexed by a reduced expression for some $w \in FC(D_n)$ (Lemma 3.2.5). Let s_x denote the element of D_n corresponding to d_x, and let $d_{\mathbf{w}}$ be the monomial indexed by an expression $\mathbf{w}$.

Lemma 3.4.5. *If $d_{\mathbf{w}} \in TL(D_n)$ is reduced and we can commute its letters to get $d_1 d_{\bar{1}}$, then $d_{\mathbf{w}}$ contains only one copy each of d_1 and $d_{\bar{1}}$.*

Proof. Suppose we can commute $d_{\mathbf{w}}$ to get $d_1 d_{\bar{1}}$. Then the resulting word has a subword that contains $d_1 d_{\bar{1}}$ and one other copy of either d_1 or $d_{\bar{1}}$. Since $d_1 d_{\bar{1}} = d_{\bar{1}} d_1$, we can commute to get a monomial $d'_{\mathbf{w}}$ with a subword $d''_{\mathbf{w}}$ which has two copies of d_x and no copies of d_y, where $\{x, y\} = \{1, \bar{1}\}$.

Note that $\langle d_x, d_2, \ldots, d_{n-1} \rangle \cong TL_n$ ($d_x \mapsto e_1$, $d_i \mapsto e_i$ for $i > 1$), so $d''_{\mathbf{w}}$ is equivalent to a monomial in TL_n with two copies of e_1. This monomial is not reduced, by Lemma 2.1.1, since it has more than one term with minimal index. Since $d'_{\mathbf{w}}$ has a subword which is not reduced, $d'_{\mathbf{w}}$ cannot be either. As $d_{\mathbf{w}}$ and $d'_{\mathbf{w}}$ represent the same word and have the same length, $d_{\mathbf{w}}$ is not reduced. $\square$

Proposition 3.4.6. *Each nontrivial reduced monomial in $TL(D_n)$ can be written in one of two forms, which we call form 1 and form 2, respectively:*

$$[d_{j_1} d_{j_1 - 1} \cdots d_2 d_{\bar{1}} d_1 \cdots d_{k_1}][d_{j_2} d_{j_2 - 1} \cdots d_{k_2}] \cdots [d_{j_m} d_{j_m - 1} \cdots d_{k_m}] \tag{3.4.10}$$

or

$$[d_{j_1} \cdots d_2 d_{x_{-1}}] \cdots [d_{j_m} \cdots d_2 d_{x_{(-1)^m}}][d_{j_{m+1}} d_{j_{m+1} - 1} \cdots d_{k_{m+1}}] \cdots [d_{j_{m'}} d_{j_{m'} - 1} \cdots d_{k_{m'}}] \tag{3.4.11}$$

such that

(i) $j_i \geq k_i$, $j_i < j_{i+1}$, and $k_i < k_{i+1}$ for all i

(ii) $k_i > 1$ for $i > m$, and

(iii) $\{x_1, x_{-1}\} = \{1, \bar{1}\}$.

This is analogous to the decreasing normal form of TL_n.

(If $j_1 = 1$, then $d_{j_1} \cdots d_2 d_{\bar{1}} = d_{\bar{1}}$.)

Proof. Let $d_{\mathbf{w}}$ be a reduced monomial.

Suppose that $d_{\mathbf{w}}$ can be commuted to get $d_1 d_{\bar{1}}$ as a subword. Then there are no other copies of either d_1 or $d_{\bar{1}}$ in $d_{\mathbf{w}}$ (Lemma 3.4.5).

Now, we can commute the letters of $d_{\mathbf{w}}$ to get $d_{\bar{1}} d_1$ as a subword and commute this subword as far left as possible. The result is a monomial of the form

$$d_{\mathbf{w}_1} d_{\bar{1}} d_1 d_{\mathbf{w}_2} \tag{3.4.12}$$

where $d_{\mathbf{w}_i}$ are monomials in $d_2, \ldots, d_{n-1}$. Again, $d_{\mathbf{w}_i}$ are equivalent to words in $TL_T L_{n-1}$, so we can assume that they were commuted to be in decreasing normal form. Then

$$d_{\mathbf{w}_1} = [d_{j_1} d_{j_1-1} \cdots d_{k_1}][d_{j_2} d_{j_2-1} \cdots d_{k_2}] \cdots [d_{j_m} d_{j_m-1} \cdots d_{k_m}] \tag{3.4.13}$$

where $j_i \geq k_i \geq 2$, $j_i < j_{i+1}$, and $k_i < k_{i+1}$ for all i, and likewise

$$d_{\mathbf{w}_2} = [d_{j'_1} \cdots d_{k'_1}] \cdots [d_{j'_m} \cdots d_{k'_m}] \tag{3.4.14}$$

where $j'_i \geq k'_i \geq 2$, $j'_i < j'_{i+1}$, and $k'_i < k'_{i+1}$ for all i.

Note that any occurrences of d_2 in d_{w_1} occur in $[d_{j_1} d_{j_1-1} \cdots d_{k_1}]$. Thus either $m = 1$ and $k_1 = 2$, or d_{w_1} is empty. Otherwise, we could commute $d_{\bar{1}} d_1$ further. So we have

$$d_{\mathbf{w}_1} d_{\bar{1}} d_1 d_{\mathbf{w}_2} = d_{j_1} d_{j_1-1} \cdots d_2 d_{\bar{1}} d_1 d_{\mathbf{w}_2} \tag{3.4.15}$$

($d_{\mathbf{w}_1}$ is empty iff $j_1 = 1$).

We now have the following cases: $j_1 < j'_1$, $j_1 \geq j'_1 > 2$ and $j_1 \geq j'_1 = 2$.

In the first case, let $k_1 = 1$ and relabel j'_i, k'_i as j_{i+1}, k_{i+1}. The result is a word of form 1.

In the second case, note that $d_{j'_1}$ commutes with $d_{\bar{1}}$ and d_i for $1 \leq i < j'_1 - 1$. We have as a subword

$$d_{j'_1} \cdots d_2 d_1 d_{\bar{1}} d_{j'_1} = d_{j'_1} d_{j'_1 - 1} d_{j'_1} d_{j'_1 - 2} \cdots d_1 d_{\bar{1}}, \tag{3.4.16}$$

which is not reduced. Then d_w would not be reduced, so the second case is impossible.

In the third case, there is some m'' so that $j'_i \leq j_1$ for $i < m''$ and $j'_i > j_1$ for $i \geq m''$.

Suppose that $j'_i > j'_{i-1} + 1$ for some $1 < i < m''$. Then $d_{j'_i}$ commutes with d_l for $1 \leq l \leq j'_{i-1}$ and $d_{\bar{1}}$, so

$$\begin{aligned} &d_{j'_1} \cdots d_{j'_i} \cdots d_2 d_1 d_{\bar{1}} [d_{j'_1} \cdots d_{k'_1}] \cdots [d_{j'_{i-1}} \cdots d_{k'_{i-1}}] d_{j'_i} \\ &\qquad = d_{j'_1} \cdots d_{j'_i} d_{j'_i - 1} d_{j'_i} d_{j'_i - 2} \cdots d_2 d_1 d_{\bar{1}} [d_{j'_1} \cdots d_{k'_1}] \cdots [d_{j'_{i-1}} \cdots d_{k'_{i-1}}]. \end{aligned} \tag{3.4.17}$$

Since this subword is not reduced, d_w would not be reduced.

Thus for all $i < m''$, $j'_{i-1} + 1 \leq j'_i > j'_{i-1}$, so $j'_i = j'_{i-1} + 1$. Since $j'_1 = 2$, this implies $j'_i = i + 1$ for $i < m''$. Since $j'_i \geq k'_i > k'_{i-1} \geq 2$, we see that $k'_i = i + 1$ as well. So

$$d_{\mathbf{w}_1} d_{\bar{1}} d_1 d_{\mathbf{w}_2} = d_{j'_1} \cdots d_2 d_1 d_{\bar{1}} d_2 \cdots d_{m''} [d_{j'_{m''}} \cdots d_{k'_{m''}}] \cdots [d_{j'_m} \cdots d_{k'_m}] \tag{3.4.18}$$

which is a word of form 1, since $j'_{m''} > j_1$ and $k'_{m''} > k'_{m''-1} = m''$. Thus, if we can commute to get $d_1 d_{\bar{1}}$ as a subword, $d_{\mathbf{w}}$ can be written as a word of form 1.

Suppose that we cannot commute to get $d_1 d_{\bar{1}}$ as a subword of $d_{\mathbf{w}}$. Then

$$d_{\mathbf{w}} = d_{\mathbf{w}_1} d_{j_1} d_{\mathbf{w}_2} d_{i_2} \cdots d_{j_m} d_{\mathbf{w}_{m+1}}, \tag{3.4.19}$$

where $d_{\mathbf{w}_i} \in \langle d_2, \cdots d_{n-1} \rangle$ and $j_i \in \{1, \bar{1}\}$. Let $\{1, \bar{1}\} = \{x_1, x_{-1}\}$ so that $j_1 = x_{-1}$. Note that copies of d_1 and $d_{\bar{1}}$ alternate, or else we would have a subword equivalent to a word in TL_n with two elements of minimal index. Thus $j_i = x_{(-1)^i}$.

We can commute d_{j_1} as far left as possible, so that

$$d_{\mathbf{w}_1} d_{j_1} = d_{\mathbf{w}'_1} d_{j_1} d_{\mathbf{w}''_1}. \tag{3.4.20}$$

Note that $d_{\mathbf{w}'_1}, d_{\mathbf{w}''_1}, \in \langle d_2, \ldots, d_{n-1} \rangle \cong TL_{n-1}$ are still reduced, since we reached them from an reduced expression using commutation. Thus we can assume that they are in decreasing normal form. Repeat this process, commuting so that

$$d_{\mathbf{w}''_i} d_{\mathbf{w}_{i+1}} d_{j_{i+1}} = d_{\mathbf{w}'_{i+1}} d_{j_{i+1}} d_{\mathbf{w}''_{i+1}} \tag{3.4.21}$$

and $d_{\mathbf{w}'_{i+1}}, d_{\mathbf{w}''_{i+1}}$ are in decreasing normal form. Let $d_{\mathbf{w}'_{m+1}} = d_{\mathbf{w}''_m}$. So we can reach

$$d_{\mathbf{w}'_1} d_{j_1} d_{\mathbf{w}'_2} d_{i_2} \cdots d_{\mathbf{w}'_m} d_{j_m} d_{\mathbf{w}'_{m+1}}, \tag{3.4.22}$$

from $d_{\mathbf{w}}$ by commutation, and each $d_{\mathbf{w}'_i}$ is in decreasing normal form, and d_{i_j} is as far left as possible in the subword $d_{\mathbf{w}'_j} d_{i_j}$.

Suppose that $j_{i+1} > j_i$, $k_{i+1} > k_i$ and $j_i \geq k_i \geq 2$, and that d_x cannot be commuted any further left in the following word:

$$[d_{j_1} \cdots d_{k_1}] \cdots [d_{j_m} \cdots d_{k_m}] d_{x_{\pm 1}} \tag{3.4.23}$$

Since $d_{x_{\pm 1}}$ commutes with d_i for $i > 2$ and 2 occurs as an index at most once, when $k_1 = 2$, we see that $m = 1$ and $k_1 = 2$. Thus, for $i < m$, either $d_{\mathbf{w}'_i}$ is the identity or $d_{\mathbf{w}'_i} = d_{j_i} \cdots d_2$.

Since we cannot commute the letters of $d_{\mathbf{w}}$ to get $d_1 d_{\bar{1}}$, we see that $d_{\mathbf{w}_i}$ cannot be the identity for $1 < i < m + 1$. Thus we have

$$d_{\mathbf{w}} = [d_{j_1} \cdots d_2 d_{x_{-1}}] \cdots [d_{j_m} \cdots d_2 d_{x_{(-1)^m}}][d_{j_{m+1}} d_{j_{m+1}-1} \cdots d_{k_{m+1}}] \cdots [d_{j_{m'}} \cdots d_{k_{m'}}], \tag{3.4.24}$$

where $j_1 \geq 1$ and $j_i \geq k_i \geq 2$ for $i > 1$. The case where $j_1 = 1$ is when $d_{\mathbf{w}'_1}$ is the identity.

Notice that $k_i < k_{i+1}$ and $j_i < j_{i+1}$ for $i > m$, so it only remains to show that $j_1 < j_2 < \cdots < j_{m+1}$.

Suppose that $j_i \geq j_{i+1}$ for some $1 \leq i \leq m$. Then $j_{i+1} \geq 2$, so we have as a subword

$$d_{j_i} \cdots d_{j_{i+1}} \cdots d_2 d_{x_{(-1)^i}} d_{j_{i+1}} \tag{3.4.25}$$

If $j_{i+1} = 2$, then $d_2 d_{x_{(-1)^i}} d_2$ is not reduced. If $j_{i+1} > 2$, then

$$d_{j_i} \cdots d_{j_{i+1}} \cdots d_2 d_{x_{(-1)^i}} d_{j_{i+1}} = d_{j_i} \cdots d_{j_{i+1}} d_{j_{i+1}-1} d_{j_{i+1}} \cdots d_2 d_{x_{(-1)^i}} \tag{3.4.26}$$

which is also not reduced. Thus this is impossible, and $j_i < j_{i+1}$ for all i. Thus we can write $d_{\mathbf{w}}$ in form 2.

Therefore any reduced word in $TL(D_n)$ can be written in form 1 or 2. It can be written in form 1 if we can commute to get $d_1 d_{\bar{1}}$, or form 2 otherwise. □

Proposition 3.4.7. *All monomials of form 1 or 2 are reduced.*

Proof. To see that the words of this form are fully reduced, we use Lemma 3.2.5. These monomials are reduced if and only if they are indexed by reduced expressions for fully commutative words.

(Form 1) Consider an expression of form 1 and suppose that it is not a reduced expression for a fully commutative word,

$$[s_{j_1} s_{j_1-1} \cdots s_2 s_{\bar{1}} s_1 s_2 \cdots s_{k_1}][s_{j_2} s_{j_2-1} \cdots s_{k_2}] \cdots [s_{j_m} s_{j_m-1} \cdots s_{k_m}]. \tag{3.4.27}$$

If it is reduced but not fully commutative, we can commute to get $s_x s_y s_x$ with $m(s_x, s_y) = 3$. If it is not reduced, we can turn this expression into a reduced expression using commutation, elimination and braid moves, by Proposition 3.1.1. Thus, in either case, there must exist $x \in \{1, \bar{1}, 2, \ldots, n-1\}$ such that we can commute to get either s_x^2 or $s_x s_y s_x$.

In the subword

$$s_{j_1} s_{j_1-1} \cdots s_2 s_{\bar{1}} s_1 \tag{3.4.28}$$

that there is at most one letter of each index. Note that the other half of the expression,

$$s_2 \cdots s_{k_1}[s_{j_2} s_{j_2-1} \cdots s_{k_2}] \cdots [s_{j_m} s_{j_m-1} \cdots s_{k_m}], \tag{3.4.29}$$

is in decreasing normal form, so it indexes a reduced word and thus is itself reduced (Lemma 3.2.5). Consequently, between any pair of letters in this subword, we have two that do not commute with it (Lemma 3.1.4).

Thus, one copy of s_x must be in the subword (3.4.29), and the other must be in the subword (3.4.28).

All copies of $s_{\bar{1}}, s_1, \ldots, s_{k_1}$ occur in the subword

$$s_{k_1} \cdots s_2 s_{\bar{1}} s_1 s_2 \cdots s_{k_1}. \tag{3.4.30}$$

In this subword, however, between any two pairs of letters s_x, we have two that do not commute with it, either s_{x-1} or $s_1, s_{\bar{1}}$. Thus $k_1 < x$.

Then we must have a copy of s_x in $s_{j_1} \cdots s_{k_1+1}$, so $k_1 < s_x \leq j_1$. Note that a copy of s_{x-1} occurs to the left of s_x in the subword (3.4.28).

Since there is another copy of s_x in the subword (3.4.29) and $x > k_1$, it must be that $k_2 \leq s_x$. Since $j_2 > j_1 \geq s_x$, a copy of s_{x+1} must occur to the right of s_x in the subword (3.4.29). Thus, between any occurrence of s_x in the subword (3.4.28) and the subword (3.4.29), there are two letters that do not commute with it.

Between any pair of letters s_x in the expression, there are two letters that do not commute with it. Thus, it is impossible to reach s_x^2 or $s_x s_y s_x$ by commutation alone. Hence the expression is a reduced expression for a fully commutative word.

(Form 2) Consider an expression of form 2,

$$[s_{j_1} \cdots s_2 s_{x_{-1}}] \cdots [s_{j_m} \cdots s_2 s_{x_{(-1)^m}}][s_{j_{m+1}} \cdots s_{k_{m+1}}] \cdots [s_{j_{m'}} \cdots s_{k_{m'}}], \tag{3.4.31}$$

and suppose that it is not a reduced expression for a fully commutative word. Again, this implies that there exists an index x such that we can commute to get s_x^2 or $s_x s_y s_x$.

First, consider x_{-1}. Between any pair of $s_{x_{-1}}$, there is a copy of s_{x_1}. Between any pair of $s_{x_{-1}}$ and s_{x_1}, there is a copy of s_2. Thus there are two copies of s_2 between the copies of $s_{x_{-1}}$. Thus $x \neq x_{-1}$. The same holds for x_1.

So $x \geq 2$. Since the subword

$$[s_{j_{m+1}} s_{j_{m+1}-1} \cdots s_{k_{m+1}}] \cdots [s_{j_{m'}} s_{j_{m'}-1} \cdots s_{k_{m'}}] \tag{3.4.32}$$

is in decreasing normal form, we know that between any pairs of s_x, there are two letters that do no commute with it.

Thus the leftmost copy of s_x in the pair must occur in

$$[s_{j_1} s_{j_1-1} \cdots s_{x-1}] \cdots [s_{j_m} s_{j_m-1} \cdots s_2 s_{x_{(-1)^m}}]. \tag{3.4.33}$$

In particular, it occurs in the subword $[s_{j_i} s_{j_i-1} \cdots s_2 s_{x_{(-1)^i}}]$ for some $i \leq m$. Note that $j_{i+1} > j_i \geq x$.

If $i < m$, then we have as a subword

$$[s_{j_i} \cdots s_x \cdots s_2 s_{x_{(-1)^i}}][s_{j_{i+1}} \cdots s_{x+1} s_x \cdots s_{x_{(-1)^{i+1}}}] \tag{3.4.34}$$

If $i = m$, then we must have $k_{m+1} \leq x$ or this is the leftmost copy of s_x in the expression. Thus we have as a subword

$$[s_{j_m} \cdots s_x \cdots s_2 s_{x_{(-1)^m}}][s_{j_{m+1}} \cdots s_{x+1} s_x \cdots s_{k_{m+1}}]. \tag{3.4.35}$$

If $x = 2$, we have $s_{x_{(-1)^i}}$ and s_3 between this copy of s_x and the next. If $x > 2$, we have s_{x-1} and s_{x+1}. In either case, there are at least two letters that do not commute with s_x between the leftmost copy of the pair and any copy of s_x to the right of it.

Thus, between any pair of letters s_x in the expression, there are two letters that do not commute with s_x. Again, this implies that the expression is a reduced expression for a fully commutative word.

Therefore, since expressions of form 1 or 2 are reduced expressions for fully commutative words, any word of $TL(D_n)$ in form 1 or 2 is reduced. $\square$

To represent $TL(D_n)$ diagrammatically, we again use the diagram $E_{\bar{1}}$, but we have new relations on the decorations.

Definition 3.4.3. Call a bubble with a single decoration a *decorated bubble.*

We have the following relations for our decorations:

(1) any pair of decorations cancel, and

(2) a decoration bubble removes all other decorations.

There is no popping relation for decorated bubbles. We call the algebra generated by

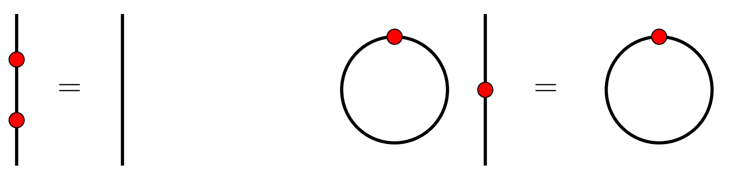


Figure 21: The new relations on the decorations

$E_{\bar{1}}, E_1, \ldots, E_{n-1}$ under these relations $\mathbb{D}TL(D_n)$.

Lemma 3.4.8 ([9], Lemma 6.6)**.** *The algebra $\mathbb{D}TL(D_n)$ has as a basis diagrams of one the following types*

(1) there is a decorated bubble and at least one non-propagating edge, and there are no other decorations, or

(2) a blob diagram whose decorations occur in pairs.

To prove this, Green used the following lemma.

Lemma 3.4.9 ([9], Lemma 5.8)**.** *The number of decorations in a blob diagram D is equal to the number of copies of E in a reduced decomposition of D.*

In particular, this means that neither of the relations on decorations are used to transform the decomposition into D. Thus this same decomposition will still yield D under the decoration relations of $\mathbb{D}TL(D_n)$.

Proof. Suppose D is a blob (n, n)-diagram with k decorations and let D' be the symmetric diagram which results from unfolding D.

Because D has k decorations, there are $2k$ edges crossing the nth column of D'. When we apply the decomposition algorithm of Section 3.3, this means that a reduced expression for D' will have k copies of E_n. Since the image of D under the isomorphism ϕ from Lemma 3.4.3 is a constant multiple of D' and $\phi(E) = E_n/\delta$, there is a reduced expression for D in $E, E_1, \ldots E_{n-1}$ which has k copies of E. □

Proof of Lemma 3.4.8.

(Type 1) It is clear that any diagram with a decorated bubble must have no other decorations, from the second defining relationship. Additionally, in order for a decorated bubble to form, there must have been at least one copy of $E_{\bar{1}}$. Thus, there must be at least one non-propagating edge. Hence any diagram of $\mathbb{D}TL(D_n)$ with a decorated bubble must be of type 1.

Now, suppose that we have a diagram D of type 1. Let D' be D without the decorated bubble. Then D' is a non-identity diagram in $\mathbb{D}TL_n$, and thus must have a nontrivial, reduced decomposition into $E_1, \ldots, E_{n-1}$, which can be found using the algorithm of Section 3.3. By Lemma 2.1.1, this decomposition has a unique term with minimal index,

E_k. Replace E_k with

$$\widetilde{E_k} = \begin{cases} E_{\bar{1}}E_1 & \text{if } k = 1 \\ E_k E_{k-1} \cdots E_2 E_{\bar{1}} E_1 E_2 \cdots E_{k-1} E_k & \text{if } k > 1 \end{cases} \tag{3.4.36}$$

Because k is minimal and $\tilde{E_k}$ and E_k both commute with E_j for $j > k+1$ and do not commute with E_{k+1}, that making this replacement in any reduced decomposition for D' will produce a decomposition which is equal to this one (up to commutation).

Note that the diagram $\widetilde{E_k}$ is E_k with a decorated bubble, since $E_{\bar{1}}E_1$ is E_1 with decorated bubble. Thus, our new decomposition will yield D. Since D can be decomposed into $E_{\bar{1}}, E_1, \ldots, E_n$, $D \in \mathbb{D}TL(D_n)$.

(Type 2) Note that there is a path from each decoration of $E_{\bar{1}}$ to the left side that crosses no edges. The bubble popping and decoration deleting relations don't create any additional edges, nor does deforming the edges, so this path must still exist for any decoration in a diagram of $\mathbb{D}TL(D_n)$, since all decorations come from copies of $E_{\bar{1}}$. Thus the decorations must all be left-exposed. From the cancellation property and bubble-popping property, any diagram of $\mathbb{D}TL(D_n)$ without a decorated bubble is equivalent to a blob diagram.

Because there is no popping relation for decorated bubbles, no diagram without a decorated bubble is equivalent to any diagram with one. So for diagrams without decorated bubbles, the second relation is never used. Since decorations are introduced in pairs ($E_{\bar{1}}$) and removed in pairs (the first relation), any decorations in the diagram must occur in equal numbers. Hence any diagram of $\mathbb{D}TL(D_n)$ without a decorated bubble is equivalent to a diagram of type 2.

By Lemma 3.4.9, each diagram of type 2 can be decomposed into a reduced word in $E, E_1 \ldots, E_{n-1}$ with an even number of copies of E. Since this word is reduced (with respect to the relations of $b_n(\delta, 1)$), the copies of E and E_1 must alternate. Thus, if we commute the $(2i-1)-$ and $(2i)$th copies of E together for each i, all our copies of E will be in subwords $EE_1E = E_{\bar{1}}$. Thus the diagram can be decomposed into a word in the generators, $E_{\bar{1}}, E_1, \ldots, E_{n-1}$, so it is an element of $\mathbb{D}TL(D_n)$.

Thus every diagram of type 1 or 2 is an element of $\mathbb{D}TL(D_n)$, and every diagram of $\mathbb{D}TL(D_n)$ is equivalent to a diagram of type 1 or 2. Therefore these diagrams form a basis for $\mathbb{D}TL(D_n)$. □

Theorem 3.4.10 ([9], Theorem 4.2)**.** *The map $\phi_D : TL(D_n) \to \mathbb{D}TL(D_n)$ such that $d_x \mapsto E_x$ is an isomorphism. Consequently, $\mathbb{D}TL(D_n)$ is a faithful representation of $TL(D_n)$.*

Proof. Since $\mathbb{D}TL(D_n)$ is generated by $\phi_D(e_x)$ and the relations of $TL(D_n)$ are respected in $\mathbb{D}TL(D_n)$, ϕ_D is a surjective homomorphism. To prove that ϕ_D is an isomorphism, it suffices to show that $\dim(TL(D_n)) = \dim(\mathbb{D}TL(D_n))$.

From Fan ([16], Section (6.2)), we know that there are $\frac{n+3}{2}C_n - 1$ fully commutative elements in D_n. Since $FC(D_n)$ indexes a basis for $TL(D_n)$, $\dim(TL(D_n)) = \frac{n+3}{2}C_n - 1$.

The diagrams of type 1 are in bijection with the non-identity diagrams of $\mathbb{D}TL_n$, so there are $C_n - 1$ diagrams of type 1.

Exactly half of the blob diagrams are of type 2. To see this, take a blob diagram D and toggle the decoration on the top left edge (the edge which has node 1 on the top face as an endpoint). That is, remove the decoration if it has one, or add it if it does not. Call the result D'. If you toggle the decoration on D', the result is D. Both D and D' are blob diagrams, and exactly one of them has an even number of decorations. Since we can partition the blob diagrams into such pairs, exactly half of them must be have an even number of decorations. By Lemma 3.4.3, the total number of blob n-diagrams is $(n+1)C_n$, so there are $\frac{n+1}{2}C_n$ diagrams of type 2.

Thus the basis of $\mathbb{D}TL_n$ has $\frac{n+3}{2}C_n - 1 = \dim(TL(D_n))$ elements. Since their dimensions are equal, the surjective homomorphism ϕ_D is an isomorphism.

Therefore $TL(D_n) \cong \mathbb{D}TL_n$. $\square$

Proposition 3.4.11. *The decompositions in Lemma 3.4.8 of diagrams into words in the generators $E_{\bar{1}}, E_1, \ldots, E_{n-1}$ yield reduced words.*

Proof.

(Type 1) A reduced expression in $E_1, \ldots, E_{n-1}$ is equal, up to commutation, to an expression in decreasing normal form. Thus, the decomposition of D that we get by replacing E_k with $\widetilde{E_k}$ is equal, up to commutation, to a word of form 1 in $E_{\bar{1}}, E_1, \ldots, E_{n-1}$. Since words of form 1 are reduced, this decomposition of D must also be reduced.

(Type 2) Suppose that we have a reduced expression for the symmetric form of D in $\mathbb{D}TL_{2n}$. Then its image under ϕ is reduced in $E, E_1, \ldots, E_{n-1}$ under the relations of $b_n(\delta, \delta')$, since ϕ is an isomorphism.

Under the relations of $\mathbb{D}TL(D_n)$, a word in $E_{\bar{1}}, E_1, \ldots, E_{n-1}$ is reduced if and only if we cannot commute to get any word of the form $E_{\bar{1}}^2$, E_i^2, or $E_i E_{i\pm 1} E_i$, $E_2 E_{\bar{1}} E_2$, or $E_{\bar{1}} E_2 E_{\bar{1}}$.

Notice that

$$E_{\bar{1}}^2 = EE_1E^2E_1E, \tag{3.4.37}$$

$$E_2E_{\bar{1}}E_2 = E_2EE_1EE_2 = E(E_2E_1E_2)E, \tag{3.4.38}$$

$$E_{\bar{1}}E_2E_{\bar{1}} = EE_1EE_2EE_1E = EE_1E^2E_2E_1E, \tag{3.4.39}$$

so none of these are reduced under the relations of $b_n(\delta, \delta')$. Thus if a decomposition of the symmetric form of D is reduced in $\mathbb{D}TL_{2n}$, the corresponding decomposition of D in $\mathbb{D}TL(D_n)$ is reduced.

If we use the algorithm of Section 3.3 to find a decomposition for the symmetric form of D, then the resulting word in $E_1, \ldots, E_{2n-1}$ is reduced. Thus the algorithm will give us a reduced word in $\mathbb{D}TL(D_n)$, which we can find by choosing a total order on the heap such that $E_n^{(2i-1)} < E_{n-1} < E_{n+1} < E_n^{(2i)}$ and this chain is convex. It is possible to choose such a chain because we know that we can commute to get a word in $E_{\bar{1}}, E_1, \ldots, E_{n-1}$, and commutations correspond to different total orders. □

3.4.3 The Temperley-Lieb algebra of E_n

Green also showed that $TL(E_n)$ had a faithful diagrammatic representation ([10] 2007), by elaborating on tom Dieck's 'bridges with pillars' representation ([8] 1997). The re-

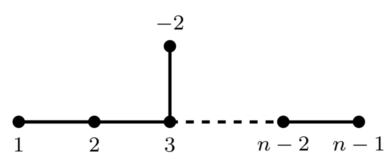


Figure 22: Dynkin diagram E_n

lations of $TL(E_n)$ are very similar to those of $TL(D_n)$. The algebra is generated by

$e_{-2}, e_1, \ldots, e_n$ under the following relations:

$$e_i^2 = \delta e_i \tag{3.4.40}$$

$$e_i e_j = e_j e_i \qquad \text{if } |i-j| \neq 1 \text{ and } \{i,j\} \neq \{-2,3\} \tag{3.4.41}$$

$$e_i e_j e_i = e_i \qquad \text{if } |x-y| = 1 \text{ or } \{i,j\} = \{-2,3\}. \tag{3.4.42}$$

Remember that a monomial in $e_{-2}, e_1, \ldots, e_{n-1}$ is reduced if and only if it is indexed by a reduced expression for some $w \in FC(E_n)$ (Lemma 3.2.5). Let s_x denote the element of E_n corresponding to e_x, and let $e_{\mathbf{w}}$ be the monomial indexed by an expression $\mathbf{w}$.

We have also have a form similar to that of Proposition 3.4.6 for the reduced monomials in $e_{-2}, e_1, \ldots, e_n$, which form a basis for $TL(B_n)$.

Lemma 3.4.12. *A nontrivial reduced monomial in $e_{-2}, e_1, \ldots, e_n$ can be written in of the following forms. If it contains no copies of e_1, it can be written in form 1 or 2 (analogous to those of Proposition 3.4.6)*

$$[e_{j_1} e_{j_1-1} \cdots e_3 e_{-2} e_2 \cdots e_{k_1}][e_{j_2} e_{j_2-1} \cdots e_{k_2}] \cdots [e_{j_m} e_{j_m-1} \cdots e_{k_m}] \tag{3.4.43}$$

or

$$[e_{j_1} e_{j_1-1} \cdots e_{-x}] \cdots [e_{j_m} e_{j_m-1} \cdots e_{(-1)^m x}][e_{j_{m+1}} e_{j_{m+1}-1} \cdots e_{k_{m+1}}] \cdots [e_{j_{m'}} e_{j_{m'}-1} \cdots e_{k_{m'}}] \tag{3.4.44}$$

such that

(i) $j_i \geq k_i$, $j_i < j_{i+1}$, *and* $k_i < k_{i+1}$ *for all* i

(ii) $k_i > 2$ *for* $i > m$, *and*

(iii) $x = \pm 2$.

If $j_1 = 1$, *let* $e_{j_1} \cdots e_3 e_{-2} = e_{-2}$.

If it does contain any copies of e_1, *then it can be written in the following form:*

$$e_{\mathbf{w}_1} \cdots e_{\mathbf{w}_m} \tag{3.4.45}$$

where $e_{\mathbf{w}_m}$ *is the identity or in form 1 or 2 and, for* $i < m$,

$$e_{\mathbf{w}_i} = [e_{l_1^{(i)}} e_{l_1^{(i)}-1} \cdots e_{(-1)^{k_i-1}2}] \cdots [e_{l_{k_i-1}^{(i)}} e_{l_{k_i-1}^{(i)}-1} \cdots e_{-2}][e_{l_{k_i}^{(i)}} e_{l_{k_i}^{(i)}-1} \cdots e_2 e_1] \tag{3.4.46}$$

such that

(i) $k_i \geq 3$ *and* $l_1^{(i)} \geq 2$ *for* $1 < i \leq m$,

(ii) $l_j^{(i)} < l_{j'}^{(i)}$ *if* $j < j'$, *and*

(iii) For $1 < i < m - 1$, $l_{k_i}^{(i)} < l_1^{(i+1)}$

Proof. The first part is clear from the fact that $\langle e_{-2}, e_2, \ldots, e_{n-1} \rangle \cong TL(D_{n-1})$ and Proposition 3.4.6.

For the second, suppose that $e_{\mathbf{w}}$ is a reduced word and contains a copy of e_1. Then

$$e_{\mathbf{w}} = e_{\mathbf{w}_1} e_{j_1} e_{\mathbf{w}_2} \cdots e_{\mathbf{w}_{m-1}} e_{j_{m-1}} e_{\mathbf{w}_m} \tag{3.4.47}$$

where $j_i = 1$ and $\mathbf{w}_i \in \langle e_{-1}, e_2, \ldots, e_{n-1} \rangle \cong TL(D_{n-1})$. Commute e_{j_1} as far left as possible, so that we get

$$e_{\mathbf{w}_1} e_{j_1} = e_{\mathbf{w}_1'} e_{j_1} e_{\mathbf{w}_1''} \tag{3.4.48}$$

where $e_{\mathbf{w}_1'}, e_{\mathbf{w}_1''} \in \langle e_{-1}, e_2, \ldots, e_{n-1} \rangle$. Since $e_{\mathbf{w}}$ is reduced, $e_{\mathbf{w}_1'}, e_{\mathbf{w}_1''}$ are reduced and so they are the identity or we can assume that they have been commuted to be in form 1 or 2. Repeat this process so that

$$e_{\mathbf{w}''_i} e_{\mathbf{w}_{i+1}} e_{j_{i+1}} = e_{\mathbf{w}'_{i+1}} e_{j_i} e_{\mathbf{w}''_{i+1}} \tag{3.4.49}$$

where $e_{\mathbf{w}'_{i+1}}, e_{\mathbf{w}''_{i+1}}$ are the identity or in form 1 or 2 and e_{j_i} is as far left as possible. Let $e_{\mathbf{w}'_m} = e_{\mathbf{w}''_{m-1}}$. So we can commute to get

$$e_{\mathbf{w}} = e_{\mathbf{w}'_1} e_{j_1} e_{\mathbf{w}'_2} \cdots e_{\mathbf{w}'_{m-1}} e_{j_{m-1}} e_{\mathbf{w}'_m}. \tag{3.4.50}$$

If we have a word of form 1 in $e_{-2}, e_2, \ldots, e_{n-1}$,

$$[e_{j_1} e_{j_1-1} \cdots e_3 e_{-2} e_2 \cdots e_{k_1}][e_{j_2} e_{j_2-1} \cdots e_{k_2}] \cdots [e_{j_m} e_{j_m-1} \cdots e_{k_m}], \tag{3.4.51}$$

then we can commute e_{-2} and e_2 to get the following subword:

$$e_{-2} e_3 \cdots e_{k_1} [e_{j_2} e_{j_2-1} \cdots e_{k_2}] \cdots [e_{j_m} e_{j_m-1} \cdots e_{k_m}] \in \langle e_{-2} e_3, \ldots, e_{n-1} \rangle. \tag{3.4.52}$$

For a word of form 2,

$$[e_{j_1} e_{j_1-1} \cdots e_{-x}] \cdots [e_{j_m} e_{j_m-1} \cdots e_{(-1)^m x}][e_{j_{m+1}} e_{j_{m+1}-1} \cdots e_{k_{m+1}}] \cdots [e_{j_{m'}} e_{j_{m'}-1} \cdots e_{k_{m'}}], \tag{3.4.53}$$

if $(-1)^m x = -2$, then we have the following subword

$$[e_{j_m} e_{j_m-1} \cdots e_{(-1)^m x}][e_{j_{m+1}} e_{j_{m+1}-1} \cdots e_{k_{m+1}}] \cdots [e_{j_{m'}} e_{j_{m'}-1} \cdots e_{k_{m'}}] \in \langle e_{-2} e_3, \ldots, e_{n-1} \rangle. \tag{3.4.54}$$

Otherwise, $(-1)^m x = 2$, and we have the subword

$$[e_{j_{m+1}} e_{j_{m+1}-1} \cdots e_{k_{m+1}}] \cdots [e_{j_{m'}} e_{j_{m'}-1} \cdots e_{k_{m'}}] \in \langle e_{-2}e_3, \ldots, e_{n-1} \rangle. \tag{3.4.55}$$

If a word is in $\langle e_{-2}e_3, \ldots, e_{n-1} \rangle$, then it commutes with e_1. Thus, if $e_{j_i} = e_1$ is as far left as possible, $e_{\mathbf{w}'_i}$ cannot have any of these subwords, so it must be that

$$e_{\mathbf{w}'_i} e_{j_i} = [e_{l_1^{(i)}} e_{l_1^{(i)}-1} \cdots e_{(-1)^{k_i-1}2}] \cdots [e_{l_{k_i-1}^{(i)}} e_{l_{k_i-1}^{(i)}-1} \cdots e_{-2}][e_{l_{k_i}^{(i)}} e_{l_{k_i}^{(i)}-1} \cdots e_2 e_1] \tag{3.4.56}$$

for $i < m$. It remains to show that

(i) $k_i \geq 3$ and $l_1^{(i)} \geq 2$ for $1 < i \leq m$,

(ii) $l_j^{(i)} < l_{j'}^{(i)}$ if $j < j'$, and

(iii) For $1 < i < m-1$, $l_{k_i}^{(i)} < l_1^{(i+1)}$.

If (i) does not hold for some i, then $e_{j_{i-1}} e_{\mathbf{w}'_i} e_{j_i}$ would have only one copy of e_2, and we could commute to get $e_1e_2e_1$, showing that $e_{\mathbf{w}}$ is not reduced. Consequently, $e_{\mathbf{w}'_i}$ is not the identity, so $l_1^{(i)} \geq 2$. Thus (i) holds, and we see that

By our choice of $e_{\mathbf{w}'_i}$, we know that $l_j^{(i)} < l_{j'}^{(i)}$ if $j < j'$, so (ii) holds as well.

Suppose that $1 < i < m-1$, $l_{k_i}^{(i)} \geq l_1^{(i+1)}$. By choice of i, we have as a subword

$$e_{l_{k_i-1}^{(i)}} \cdots e_{-2} e_{l_{k_i}^{(i)}} \cdots e_2 e_1 e_{l_1^{(i+1)}} \cdots e_{(-1)^{(k_{i+1}-1)}2} \tag{3.4.57}$$

If $l_1^{(i+1)} = -2$, then we have

$$e_{-2} e_{l_{k_i}^{(i)}} \cdots e_3 e_2 e_1 e_{-2} = e_{l_{k_i}^{(i)}} \cdots e_4 e_{-2} e_3 e_{-2} e_2 e_1,$$

which is not reduced, since e_{-2} commutes with all letters but e_3. Hence $l_1^{(i+1)} \geq 2$. But then we have the subword

$$e_{l_1^{(i+1)}} \cdots e_2 e_1 e_{l_1^{(i+1)}} \in \langle e_1, \ldots, e_{n-1} \rangle \cong TL_n \tag{3.4.58}$$

which has two elements of maximal index, so this word is not reduced either. Thus (iii) must hold as well. $\square$

Corollary 3.4.13. *The Coxeter group E_n is FC-finite.*

Proof. We know from Lemma 3.2.5 that the reduced words of $TL(E_n)$ are indexed by the fully commutative words of E_n. Thus, it suffices to show that there are only finitely many reduced words of $TL(E_n)$.

Since $TL(D_{n-1})$ has finite dimension, there are finitely many reduced words which are the identity or type 1 or 2. By Proposition 3.4.12, we see that any other reduced word can be written as

$$e_{\mathbf{w}_1} \cdots e_{\mathbf{w}_m} \tag{3.4.59}$$

where $e_{\mathbf{w}_1}, e_{\mathbf{w}_m}$ is the identity or in form 1 or 2 and, for $1 < i < m$,

$$e_{\mathbf{w}_i} = [e_{l_1^{(i)}} e_{l_1^{(i)}-1} \cdots e_{(-1)^{k_i-1}2}] \cdots [e_{l_{k_i-1}^{(i)}} e_{l_{k_i-1}^{(i)}-1} \cdots e_{-2}][e_{l_{k_i}^{(i)}} e_{l_{k_i}^{(i)}-1} \cdots e_2 e_1] \tag{3.4.60}$$

such that $l_j^{(i)} < l_{j'}^{(i)}$, if $j < j'$, and $l_{k_i}^{(i)} < l_1^{(i+1)}$. Thus $l_1^{(2)}, \ldots, l_{k_2}^{(2)}, \ldots, l_1^{(m-1)}, \ldots, l_{k_{m-1}}^{(m-1)}$ is an increasing sequence in $2, \ldots, n$. Additionally, since $k_i \geq 3$, we see that $m \leq \lfloor \frac{n-1}{3} \rfloor + 1$.

Since m is the number of copies of e_1 in the word, we see that there are fewer than n of e_1, which intersperse reduced monomials in $\langle e_{-2}, e_2, \ldots, e_{n-1} \rangle \cong TL(D_n)$. Since there are finitely many such monomials (because $TL(D_n)$ has finite dimension), we see that there are finitely many words of this form as well.

Therefore $TL(E_n)$ is finite dimensional and E_n is FC-finite. $\square$

Proposition 3.4.14. *A nontrivial reduced monomial in $e_{-2}, e_1, \ldots, e_n$ which contains a copy of e_1 can be written in the form described in Proposition 3.4.12. Labelling $e_{\mathbf{w}_m}$ so that*

$$e_{\mathbf{w}_m} = [e_{j_1^{(m)}} \cdots e_3 e_{-2} e_2 \cdots e_{l_1}][e_{j_2^{(m)}} e_{j_2^{(m)}-1} \cdots e_{l_2}] \cdots [e_{j_{k_m}^{(m)}} e_{j_{k_m}^{(m)}-1} \cdots e_{l_{k_m}}] \tag{3.4.61}$$

or

$$[e_{j_1^{(m)}} \cdots e_{-x}] \cdots [e_{j_{k_m}^{(m)}} \cdots e_{(-1)^m x}][e_{j_{k_m+1}^{(m)}} e_{j_{k_m+1}^{(m)}-1} \cdots e_{l_{k_m+1}}] \cdots [e_{j_{k'_m}^{(m)}} e_{j_{k'_m}^{(m)}-1} \cdots e_{l_{k'_m}}], \tag{3.4.62}$$

we can make the additional requirements that

(iv) $j_1^{(i+1)} > j_{k_i}^{(i)}$ *for* $1 < i < m$

(v) If $k_1 > 1$*, then* $j_1^{(2)} > j_{k_1}^{(1)}$

(vi) If $k_1 = 1$*, then* $j_k^{(i)} > {l_1}^{(1)}$ *or*

$$j_k^{(i)} = \begin{cases} -2 & \text{if } i = 2, k = 1 \\ j_{k-1}^{(i)} + 1 & \text{if } k > 1 \\ j_{k_{i-1}}^{(i-1)} + 1 & \text{if } k = 1, i > 2 \end{cases}$$

Additionally, if $m = 1$ *and* $e_{\mathbf{w}_m}$ *is in form 1, then* $j_1^{(m)} > {l_1}^{(1)}$.

If $j_k^{(i)} < j_{k'}^{(i')}$ *whenever* $i < i'$ *or* $i = i'$ *and* $k < k'$*, we say that* $e_{\mathbf{w}}$ *is in form 3. Otherwise, we say that* $e_{\mathbf{w}}$ *is in form 4.*

Proof. We already know that we can write $e_{\mathbf{w}_m}$ in one of those two forms.

The first new requirement, (iv), follows from the same argument as (iii). The second (v) does as well, since in that case $e_{\mathbf{w}_1}$ has a copy of -2.

Suppose that $k_1 = 1$. Then $e_{\mathbf{w}_1} = e_{j_1^{(1)}} \cdots e_2 e_1$.

Suppose $m = 2$ and $e_{\mathbf{w}_m}$ is in form 1. If $j_1^{(m)} \leq {l_1}^{(1)}$, then we have a subword

$$e_{j_1^{(m)}} \cdots e_2 e_1 e_{j_1^{(m)}} \in \langle e_1, \ldots, e_{n-1} \rangle \cong TL_n$$

(unless $e_{j_1^{(m)}} = 2$, in which case we can commute e_{-2} and e_2 to get this subword), which has two terms with maximal index and thus is not reduced. Thus, in this case, $j_1^{(m)} > {l_1}^{(1)}$.

Suppose that $j_1^{(2)} \leq {l_1}^{(1)}$ and $j_1^{(2)} \neq -2$. Then we have as a subword

$$e_{j_1^{(2)}} \cdots e_2 e_1 e_{j_1^{(2)}} \in \langle e_1, \ldots, e_{n-1} \rangle \cong TL_n$$

which has two elements of maximal index. Thus this is impossible.

Suppose that $k > 1$ or $i > 2$, $j_k^{(i)} \leq {l_1}^{(1)}$ and $j_k^{(i)} \neq j_{k'}^{(i')} + 1$ (where $i' = i$ and $k' = k - 1$ if $k > 1$ or else $i' = i - 1$ and $k' = k_{i-1}$).

Notice that $j_k^{(i)} > j_{k''}^{(i'')}$ if $1 < i'' < i$ or $i'' = i$ and $k'' < k$. Thus $j_k^{(i)} > j_{k'}^{(i')} + 1$, and in fact will commute with any such $j_{k''}^{(i'')}$. Thus we can commute $e_{\mathbf{w}}$ to get the following subword:

$$j_k^{(i)} \cdots e_2 e_1 j_k^{(i)}$$

which is not reduced. Thus, (vi) must hold as well.

$\square$

Proposition 3.4.15. *Any word in form 1-4 is reduced. Thus the words of these types form a basis for $TL(E_n)$.*

Proof. By 3.4.7, words of form 1 or 2 are reduced, since $TL(E_n)/\langle e_1\rangle \cong TL(D_{n-1})$.

For forms 3 and 4, we use the same method as Proposition 3.4.7. That is, we show that between any pair of copies of e_x, there are two elements it does not commute with. This shows that the word is reduced.

First, suppose that a word is in form 3. Since $k_i \geq 3$ for $1 < i < m$, we see that between any two copies of e_1, there are at least two copies of e_2.

Since $j_k^{(i)} < j_{k'}^{(i')}$ whenever $i < i'$ or $i = i'$ and $k < k'$, if we have a copy of e_2, then there is at least one copy of e_3 between it and the next copy of e_2. If both copies are in $e_{\mathbf{w}_i}$ for some i, then there are at least two copies of e_3 between them, since we will have

$$e_2 e_{k_1} \cdots e_3 e_{-2} e_{k_2} \cdots e_4 e_3 e_2$$

for $2 < k_1 < k_2$. If they are in different copies of $e_{\mathbf{w}_i}$, then there is a copy of e_1 between them. In either case, there are at least two elements e_2 does not commute with.

In a similar way, between any two pairs of e_{-2} there are two copies of e_3.

Suppose $i > 2$. If both copies occur in $e_{\mathbf{w}_m}$, we are done because $e_{\mathbf{w}_m}$ is in form 1 or 2. So there is at least one copy in $e_{\mathbf{w}_j}$ for $j < m$. Then either the copy is followed immediately by e_{i-1} or e_{-2} (if $i = 3$). The next occurrence will be preceded by e_{i+1}, since the indices are increasing from group to group.

Thus any pair of e_x are separated by two elements with which they do not commute, so words of form 3 are reduced.

Suppose that a word is of form 4. This only occurs when $k_1 = 1$. Notice that $e_{\mathbf{w}_1} = e_{j_1^{(1)}} \cdots e_2 e_1$ and $e_{\mathbf{w}_2} \cdots e_{\mathbf{w}_m}$ are in form 3 and thus are both reduced. Suppose that one copy of e_x is in $e_{\mathbf{w}_1}$ and the other is in $e_{\mathbf{w}_2} \cdots e_{\mathbf{w}_m}$.

If $x = 1$, then since $k_2 \geq 3$, there are two copies of e_2 between the first e_1 and any other. If $x = 2$, then since $j_1^{(2)} = -2$, $j_2^{(2)} \geq 3$, so we have the subword

$$e_2 e_1 e_{-2} e_{j_2^{(2)}} \cdots e_3 e_2$$

and thus we have e_1 and e_3 between this first copy of e_2 and the next.

If $x > 2$, then e_x is immediately followed by e_{x-1}. The next copy of e_x either comes immediately after e_{x+1}, or else $x = j_k^{(i)}$ for some $i > 1$. Note that if $i = 2$, then $k \neq 1$. Then either $j_k^{(i-1)} = x - 1$ or $j_{k-1}^{(i)} = x - 1$, so there are two copies of e_{x-1} between the copies of e_x.

Thus, between any pairs of copies of e_x, there are two elements it does not commute with, so words of form 4 are also reduced.

The second half follows from the fact that the reduced monomials form a basis for $TL(E_n)$ and a monomial is reduced if and only if it is in one of the four forms. $\square$

The placement of the branching point creates some new issues in a diagrammatic representation, since the element corresponding with e_{-2} must commute with both E_2 and E_4 and not E_3. In order to represent this, tom Dieck replaced the on-edge decorations with 'pillars' [8], blobs which occur inside the regions bordered by edges. To represent e_{-2}, we add a new generator, $\tilde{E}$, which is the identity diagram with a decoration between the 3rd and 4th edge. To get the desired relations on the diagrammatic algebra, we require that

(1) two pillars in the same region are equivalent to one decoration with δ,

(2) a pillar in a bubble pops with no scalar multiple, and

(3) if there is a pair of decorations separated by only one region, the two decorations and regions containing the decorations will become a single region with a single

decoration, splitting the intermediary region into two.

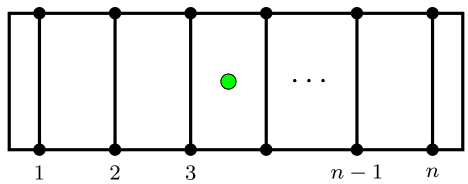


Figure 23: The decorated tangle $\tilde{E}$

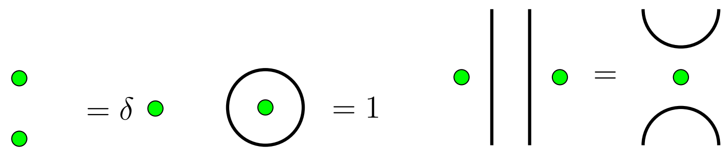


Figure 24: The relations on 'pillars'

Theorem 3.4.16 ([10], Theorem 1.1)**.** *The diagrammatic algebra $\mathbb{D}TL(E_n)$ which is generated by $\tilde{E}, E_1, \ldots, E_{n-1}$ under the given relations is a faithful representation of $TL(E_n)$. In particular, there is an isomorphism such that $e_i \mapsto E_i$ for $i > 0$ and $e_{-1} \mapsto \tilde{E}$.*

Green proved this by showing that the diagrams create a non-degenerate Markov trace.

Tammo tom Dieck noted that it is a necessary condition that each pillar be an odd number of regions from the left edge. Green then formalized this by giving orientations to the edges.

Lemma 3.4.17 ([10])**.** *Let the odd nodes be oriented up, and the even down. This gives each edge an orientation.*

Under this orientation, any pillar in a diagram of $\mathbb{D}TL(E_n)$ must occur in a region that is oriented clockwise.

Proof. Notice that $\tilde{E}$ has this property, and that regions combined when composing diagrams must have the same orientation. $\square$

However, this condition is not sufficient. Consider the identity diagram with a single decoration in the first clockwise region.

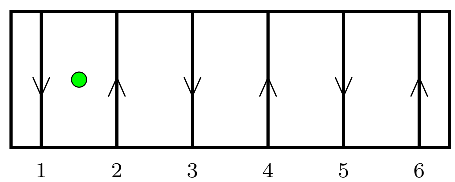


Figure 25: The condition 3.4.3 is not sufficient

Lemma 3.4.18. *Suppose D is an (n, n)-diagram such that non-intersecting lines can be drawn across the diagram for each pillar of a diagram D, and that each line passes through at most 3 edges to the left of the diagram and $n-3$ to the right. Then D is in $\mathbb{D}TL(E_n)$.*

Proof. If such lines exist, we can deform the edges of the diagram and the lines so that the lines are straight and cross exactly 3 edges to the left and $n-3$ to the right.

To see this, first not that since any pillar is an odd number of regions away from the left side, a line crosses either one or three edges. In the first case, we can deform the edge it does cross so that that edge crosses it three times.

If n is odd, then the rightmost region, which has the right face as an edge, is oriented counterclockwise. Then the line must pass through an even number of edges to get to it, $n-3-2k$ for some k. Likewise for n even. Thus we can deform the edges which already cross the line to cross it $2k$ more times, since there is at least one edge that already crosses it, on the left side of the decoration.

Once all the lines are straight and cross 3 edges to the left and $n-3$ to the right, we can replace these lines (and their pillars) with a copy of $\tilde{E}$, and the result is the same diagram. Between copies of $\tilde{E}$ will be undecorated (n,n)-diagrams, which can be decomposed into words in $E_1,\ldots,E_{n-1}$, using the algorithm of Section 3.3. Note that not all such decompositions will give us a reduced word. $\square$

This condition is not necessary, see the example in fig. 26 below. The diagrams are all equivalent under our relations (applying relation 2), but one does not satisfy the condition. They are all equal to

$$(E_6E_5E_4E_3E_2E_1)\tilde{E}(E_3E_2)(E_4E_3)\tilde{E}(E_5E_4E_3E_2)(E_6E_5E_4E_3)\tilde{E}(E_1E_2E_3E_4E_5E_6) \tag{3.4.63}$$

Note that this counterexample has the maximum number of copies of E_1 for a word in $\langle \tilde{E}, E_1, \ldots, E_{n-1}\rangle$. In order to fully describe a basis of $\mathbb{D}TL(D_n)$, we require some new terminology.

Definition 3.4.4. An edge that connecting nodes i and $i+1$ on the top (bottom) face of the diagram is called a *cup* (*cap*).

A cup or cap is *empty* if it doesn't contain a pillar.

A pillar is in the *top (bottom) left region* if there are paths from the pillar to nodes 1 and 2 on the top (bottom) side of the diagram that do not cross any edges. In other words, the top (bottom) left region has the segment of the top (bottom) face of the diagram connecting nodes 1 and 2 as an edge.

Theorem 3.4.19. *There is a basis for $\mathbb{D}TL(E_n)$ which consists of diagrams which are reduced under the relations (bubble-free and no pillars distance two apart) of the following two types:*

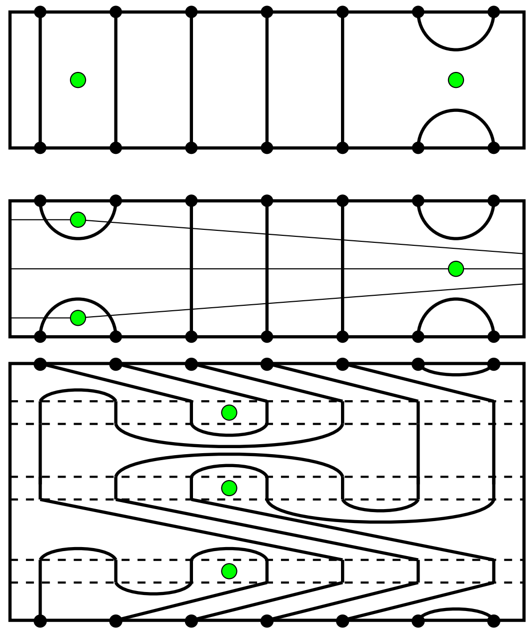


Figure 26: The condition of Lemma 3.4.18 is not necessary.

(1) Every decoration is distance exactly 3 from the left.

(2) There is one decoration which is distance 1 from the left and all other decorations are distance 4 from this decoration, and

(i) if the top left region has a decoration, there is an empty cup,

(ii) if the bottom left region has a decoration, there is an empty cap.

Notice that the top diagram in fig. 26 is an example of a word of type 2.

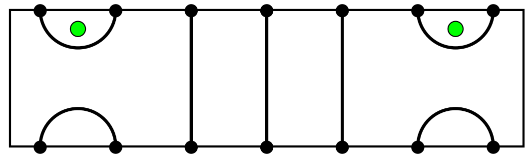


Figure 27: Non-example of a basis element

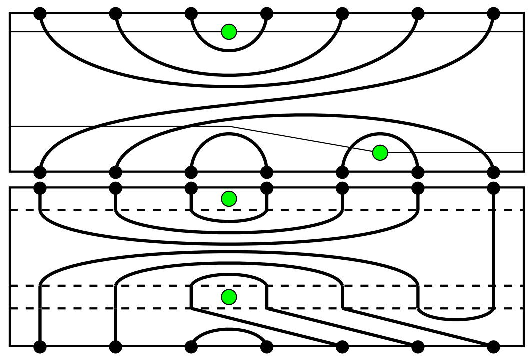


Figure 28: Example of a basis element of type 1

Proof. We can see that $\tilde{E}, E_1, \ldots, E_{n-1}$ satisfy the first condition.

If we compose two diagrams of this type and reduce it using only the relations on E_i and the first or second relations for pillars ($\tilde{E}^2 = \delta\tilde{E}$, $E_3 E E_3 = E_3$), then the remaining pillars in the resulting diagram cannot be further away than they were. Since they are necessarily in clockwise-oriented regions, this means that they are either distance 1 or 3 from the left face, or else distance 2 or 4 from a pillar which is distance one from the left. If applying the second relation removes the distance one pillar, then the pillars which were distance 2 or 4 from that pillar are now distance 1 or 3 from the edge.

If there are no pillars which are distance 1 from the left, then all remaining pillars are distance 3 from the edge. Suppose we combine one pillar distance three from the edge to another pillar (at least 3 from the edge) using the third relation. The combined pillar will be distance 3 from the edge. If this combination creates edges between that move a pillar from distance 3 to distance 5, then there is a path that pillar to the edge which passes 5 edges and through the combined pillar. Thus, either the combined pillar is distance 1 (impossible) from the edge or distance 2 from the distance 5 pillar. So we can combine all the pillars which are now distance 5 with the combined pillar which is distance 3, and continue until all the pillars are distance three from the edge.

Now, we can combine all pillars which are distance 1 from the left, since they are all distance 2 from each other. So this word has at most one pillar which is distance 1 from the left. Combining the pillars of distance 1 from the left does not change their distance from the left, or their distance from the other pillars. The result is a word of type 1.

Suppose that, after applying the first two relations, a diagram has a pillar distance 1, call it the primary pillar. Notice that any pillars distance 3 from the left are distance 2 or 4 from the primary pillar, and any other pillar of distance 1 are distance 2 from the primary pillar. Thus all pillars are distance 2 or 4 from a pillar which is distance 1 from the left. Combining pillars will not increase their distance from other pillars, so if we combine the pillars of distance 1 with the primary pillar, all pillars are now distance 2 or 4 from the primary pillar. Again, we can combine with the primary pillar so that all remaining pillars are distance 4 from the primary pillar.

If we combine two pillars which are distance 4 from the primary pillar, and this causes a pillar to be distance 6 from the primary pillar, then there is a path which crosses 6 edges and the combined pillar from the distance 6 pillar to primary pillar. Since the combined pillar is still distance 4 from the primary pillar, the distance 6 pillar must be distance two from the combined pillar. In the same way as with words of type 1, we can continue combining until all remaining pillars are distance 4 from the primary pillar, which is distance 1 from the edge.

To see that (i) and (ii) also hold, suppose that we have a pillar in the top left region. If the top diagram in the composition was the identity or had a pillar in its top left region, then we have our empty cup. Otherwise, the top diagram has some non-propagating edge and the edge connected to node 1 is propagating. Therefore the top diagram has a cup. Again, if this cup is empty, we are done.

If it is not empty, the pillar within it is either distance 3 from the left face or distance 4 from a pillar from the which is distance 1 from the edge in the top diagram. In the second case, since the edge from node 1 is propagating and there is no pillar in the top left region, we can see that the path to the distance 1 pillar must cross that edge. Consequently, it must be distance 2 from the top left region. Likewise, in the first case, any path from it to the left face must cross the edge, so in both cases it is distance 2 from the top left region. Thus, we can use the third relation to combine this pillar with the one which will be in the top left region. Combining them creates a non-propagating edge which does not contain any pillars, so there will be some empty cup underneath it.

Thus, if there is a pillar in the top left region, there is an empty cup. The case for the bottom left region is analogous.

Thus every diagram of $\mathbb{D}TL(E_n)$ is of form 1 or 2.

All such diagrams are in $\mathbb{D}TL(E_n)$.

Suppose that we have a diagram of type 1. Then there are non-intersecting lines from each pillar to the left face which cross three edges. That such lines exist comes from the fact that they are distance 3 from the left, and that they do not intersect comes from the fact that if they do intersect, they can both go the same, shortest way.

Notice that the pillars must be in regions attached to the top or bottom face, and that they must then have a path to a node i which does not cross any edges (or else we could apply the second relation), for $i \geq 3$ (if $i < 3$, they must be less than distance 3 from the edge). Then there is a path from the node to the right edge which crosses at most $n - 3$ edges, so there is such a path from the pillar. In the same way, we can ensure that the paths do not cross.

Suppose that we have a diagram of type 2. Call the pillar which is distance one from the left the primary pillar. Consider the pillar which are distance 4 from the primary pillar. They are necessarily distance 3 or 5 from the left. If a pillar is distance 5 from the left, then the primary pillar must be between two edges which separate the pillar from the left. Thus, if we apply the third relation in reverse, we can get a path to the edge which is distance 3 from the left. Notice that after applying the relation, all pillars will be distance 4 (or 0) from a pillar which is distance one from the left. Additionally, it will not increase the distance from any pillar to the left. Thus, we can apply this relation until all pillars are distance one or three from the left.

Again, for any pillars distance three from the left, we can find non-crossing paths which cross three edges to the left and at most $n-3$ to the right.

For the pillars which are distance one, we use the fact that there is a path from them to a nodes on the top or bottom edge (or else we could apply the second relation). In particular (because it is in an clockwise region) it will have paths to nodes $2i-1$ and $2i$ for some i. If $i=1$, then it is in the top or bottom left region, in which case we know that there must be an empty cup or a cap (respectively). Then there is a path through the pillar that follows along the top edge, except to avoid the cup or cap. This path crosses through 1 edge to the left and exactly $n-3$ to the right, and does not cross the path of any other pillar. If it is not in the top or bottom left region, then $i>1$, and there is a path to the right which follows the edge and crosses at most $n-(2i-1) \leq n-3$ edges, and does not cross the path of any other pillar. We can achieve this by layering the paths so that the path of the leftmost distance one pillar in a region is closest to the face, as illustrated in fig. 29, and each path only avoids empty cups (or caps) and the regions of the distance 1 pillars to the left of it.

Thus we can construct paths of the sort in Lemma 3.4.18 for any diagram of type 1

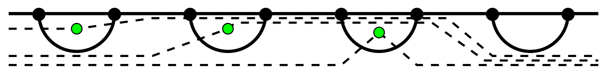


Figure 29: The paths of regions distance one from the left

or 2. Consequently, all such diagrams are in $\mathbb{D}TL(E_n)$. □

Let $E_n(k)$ be the Coxeter graph which contain A_{n-1} as a subgraph, and has an additional vertex and edge connected to the kth vertex, shown in fig. 30. Notice that $E_n(1) = A_n$, $E_n(2) = D_n$, and $E_n(3) = E_n$.

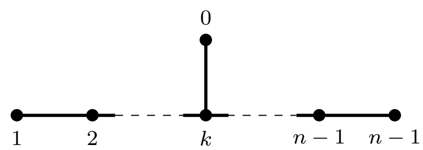


Figure 30: The Coxeter graph $E_n(k)$

In the paper where he introduced the diagrammatic representation for $TL(E_n)$, Tammo tom Dieck suggested that a similar representation could be used for $TL(E_n(k))$, by representing e_0 by a pillar in the kth column. Call this diagram $\tilde{E}_k$ and left $\mathbb{D}(n,k)$ be the diagrammatic algebra generated by E_i and $\tilde{E}_k$, modulo the same relations as $\mathbb{D}TL(E_n)$. He showed that there was a surjective homomorphism from $TL(E_n(k))$ to $\mathbb{D}(n,k)$ ([8], Theorem 2.5), and conjectured that this homomorphism might be an isomorphism, as it is for $k \leq 3$.

Proposition 3.4.20. *If* $\min\{n-k,k\} > 3$*, then* $TL(E_n(k))$ *is not isomorphic to* $\mathbb{D}(n,k)$*.*

Proof. If $\min\{n-k,k\} > 3$, then $E_n(k) = Y(k-1, n-k-1, 1)$ and $\min\{k-1, n-k-1\} > 2$, so $E_n(k)$ is not FC-finite by Lemma 3.1.12. Since $FC(E_n(k))$ indexes a basis for $TL(E_n(k))$, this implies that $TL(E_n(k))$ is infinite dimensional.

Under the relations on $\mathbb{D}TL(E_n)$, there is at most one pillar per region. Each (n,n)-diagram has finitely many regions, so there are finitely many ways to decorated a given (n,n)-diagram. Since there are finitely many (n,n)-diagrams, there are finitely many decorated (n,n)-diagrams. Since some subset of these decorated (n,n)-diagrams form a basis for $\mathbb{D}(n,k)$, it is necessarily finite dimensional.

Since $\mathbb{D}(n,k)$ is finite dimensional and $TL(E_n(k))$ is not, they cannot be isomorphic.

$\square$

For example, consider $E_8(4)$. This has an infinite set of fully commutative words,

$$\{(s_0s_4s_3s_5s_4s_0s_6s_5s_4s_3s_2s_1s_7s_6s_5s_4s_3s_2)^i \mid i \geq 1\}, \tag{3.4.64}$$

all of which yield the same diagram (up to a constant multiple) under the mapping $s_0 \mapsto \tilde{E}_k$, $s_i \mapsto E_i$, shown in fig. 31. These fully commutative words index linearly independent elements of $TL(E_n(k))$, but their images are linearly dependent.

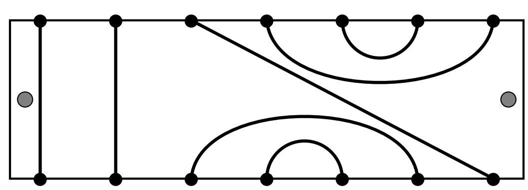


Figure 31: The image of the words in equation 3.4.64

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