

# Inferring Probability Comparisons

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## Abstract

The problem of inferring probability comparisons between events from an initial set of comparisons arises in several contexts, ranging from decision theory to artificial intelligence to formal semantics. In this paper, we treat the problem as follows: beginning with a binary relation  $\succsim$  on events that does not preclude a probabilistic interpretation, in the sense that  $\succsim$  has extensions that are probabilistically representable, we characterize the extension  $\succsim^+$  of  $\succsim$  that is exactly the *intersection* of all probabilistically representable extensions of  $\succsim$ . This extension  $\succsim^+$  gives us all the additional comparisons that we are entitled to infer from  $\succsim$ , based on the assumption that there is some probability measure of which  $\succsim$  gives us partial qualitative information. We pay special attention to the problem of extending an order on states to an order on events. In addition to the probabilistic interpretation, this problem has a more general interpretation involving measurement of any additive quantity: e.g., given comparisons between the weights of individual objects, what comparisons between the weights of groups of objects can we infer?

*Keywords:* qualitative probability, comparative probability, imprecise representability, sets of probability measures, additive measurement

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## 1. Introduction

The problem of inferring probability comparisons between events from an initial set of comparisons arises in several contexts, ranging from decision theory to artificial intelligence to formal semantics. In the context of normative decision theory, Gilboa et al. (2010) propose a notion of rationality applicable to probability comparisons as follows: a judgment of the form “ $E$  is at least as likely as  $F$ ” is *objectively rational* if it can be proven to follow from some accepted judgments—given by statistical analysis of evidence, scientific facts, the decision maker’s judgments, etc.—using principles of logic, mathematics, and decision theory as inference rules. We may view the initially accepted comparisons as encoded by a binary relation  $\succsim$  on events. This relation may be very incomplete, leaving undecided many comparisons between events. Then the objectively rational judgments that are provable from  $\succsim$  using the inference rules may constitute a proper extension  $\succsim^+$  of  $\succsim$ .

There are illuminating special cases of the general question of how to rationally extend a partial qualitative probability relation. One is the special case in which our initial comparisons are all between disjoint events  $E_1, \dots, E_n$  (disjoint subsets of a state space  $S$ ), and we are trying to infer comparisons between events in the algebra generated by  $E_1, \dots, E_n$ . For example,

suppose that our decision maker (DM) knows that a friend has just landed at the local airport. The DM does not know which airline the friend flew, but the DM has a likelihood ordering over the airlines: United is more likely than Delta, while Delta is more likely than Southwest, etc. Since the friend cannot have flown on both United and Delta, or both United and Southwest, etc., the DM has a likelihood ordering over disjoint events. Now suppose that the DM is driving toward a fork in the road near the airport, with one fork leading to a terminal A that services airlines  $A_1, \dots, A_n$  and the other fork leading to a terminal B that services airlines  $B_1, \dots, B_m$ . In order to decide which way to turn, the DM needs to decide whether it is more likely that the friend flew on  $A_1$  or  $A_2$  or  $\dots$  or  $A_n$  or that the friend flew on  $B_1$  or  $B_2$  or  $\dots$  or  $B_m$ . Then the question is whether the DM’s initial likelihood ordering over individual airlines is enough to deduce a rational likelihood comparison between the two disjunctions.

The special case of inferring comparisons in an algebra generated by disjoint events  $E_1, \dots, E_n$  can be reduced to another special case: we start with a basic ordering on states of the world and then try to infer comparisons between events. (Simply take the atoms of the algebra to be the states of a new state space.) In a decision-theoretic context, Kelsey (1993) proposes to represent a notion of “partial uncertainty” by a likelihood ordering on states: for states  $s_i$  and  $s_j$ , the relation  $s_i \geq s_j$  is interpreted to mean that “the agent believes that state  $s_i$  is at least as likely as state  $s_j$ .” For a given decision problem, the relevant states might be quite coarse-grained. In the example above, we may take one state of the world to be that the friend flew United, another that the friend flew Delta, and so on. Given the DM’s

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likelihood ordering on these states of the world, a natural question is how to extend this ordering to a likelihood ordering on events built up from those states—in order to deal with the kind of decision problem posed by the fork in the road.

Precisely this question of how to extend a likelihood ordering on states of the world to a likelihood ordering on events built up from those states has been studied in the context of artificial intelligence. In the well-known textbook, *Reasoning about Uncertainty*, Halpern (2003) writes in the chapter on extending likelihood relations on states to events: “Unfortunately, there are many ways of doing this; it is not clear which is ‘best’” (46). The same problem appears in the linguistics literature on the formal semantics of statements like “ $E$  is at least as likely as  $F$ .” One influential tradition gives the truth conditions for such statements in terms of an ordering on states of the world, lifted to an ordering on sets of states (Kratzer, 1991). Unfortunately, the traditional way of doing so supports intuitively invalid inferences, e.g., the inference from “ $E$  is at least as likely as  $F$ ” and “ $E$  is at least as likely as  $G$ ” to the conclusion “ $E$  is at least as likely as  $F$  or  $G$ ” (for discussion, see Yalcin 2010). Some better method of extending likelihood orderings is needed.

In this paper, starting with a binary relation  $\succsim$  on the powerset of a finite set, we study the following extension:

$$\succsim^+ = \bigcap \{ \succsim^p \mid \succsim^p \supseteq \succsim, \succsim^p \text{ probabilistically representable} \}.$$

Equivalently,  $\succsim^+$  is given by the unanimity rule:  $E \succsim^+ F$  if and only if  $\mu(E) \geq \mu(F)$  for all probability measures  $\mu$  that *almost agree* with  $\succsim$ , i.e., such that  $A \succsim B$  implies  $\mu(A) \geq \mu(B)$ . Intuitively,  $\succsim^+$  gives us all the additional comparisons that we are entitled to infer from  $\succsim$ , assuming that there is some probability measure of which  $\succsim$  gives us partial qualitative information.

One way to characterize the extension  $\succsim^+$  is to say that  $E \succsim^+ F$  if and only if this comparison is derivable using certain axioms for incomplete qualitative probability, including a generalized cancellation axiom (Ríos Insua, 1992; Alon and Lehrer, 2014). By contrast, we will characterize the extension by saying that  $E \succsim^+ F$  if and only if there exists a certain kind of injective mapping between partitions of the state space. Our characterization has a constructive character, and it is easy to visualize. In the special case where we start with an ordering on states, our characterization is simpler than the characterization in terms of derivability using generalized cancellation axioms. In the general case, we believe our characterization provides a complementary perspective to generalized cancellation axioms. In addition, our constructive characterization can be broken down into three simple inference rules.

After some preliminaries in §2, we begin in §3 with the special case of extending a likelihood ordering on states to one on events built up from those states. The analogous problem of extending a preference ordering on a set to one on its powerset (Gärdenfors, 1979; Packard, 1979) leads to well-known impossibility results (Kannai and Peleg, 1984; Barberá and Pattanaik, 1984), but in the case of relative likelihood, we have positive results. Following Holliday and Icard 2013, given a preorder  $\geq$  on a finite set  $S$ , we define the *injection extension*  $\geq^i$  on  $\mathcal{P}(S)$  as follows:  $E \geq^i F$  if and only if there is an injection  $g: F \rightarrow E$

such that for all  $x \in F$ ,  $g(x) \geq x$ . (Such a function is called *inflationary* with respect to  $\geq$ .) In other words, for every way that event  $F$  could obtain, there is a matching way that event  $E$  could obtain that is at least as likely (and this assignment is one-to-one). We prove that this  $\geq^i$  is precisely the desired extension  $\succsim^+$  above, where  $\succsim$  is the ordering of singleton events induced by  $\geq$ , i.e.,  $\{s\} \succsim \{s'\}$  if and only if  $s \geq s'$ .

The idea of the injection extension applies not only in the context of probability, but also in the context of additive measurement in general. For example, suppose we have used a balance scale to compare the weights of individual objects from a finite set  $S$ , giving us a preorder  $\geq$  on  $S$  with  $a \geq b$  meaning that  $a$  is at least as heavy as  $b$ . We may then wish to infer comparisons between the weights of groups of objects, e.g., to decide whether  $a$  and  $c$  together are at least as heavy as  $b$  and  $d$  together without making new measurements. For example, if we have measured that  $a$  is at least as heavy as  $b$ , and  $c$  is at least as heavy as  $d$ , then we may infer that  $a$  and  $c$  together are at least as heavy as  $b$  and  $d$  together. Our results in §3 show that the comparisons between groups of objects that one is entitled to infer are exactly those obtained by this type of inference.

In §4, we generalize the idea of the injection extension to the case where we start with an order on subsets of a finite set  $S$ , which in the probabilistic interpretation is a partial likelihood ordering on events. In this case, instead of asking for an inflationary injection that maps states to states, we ask for an inflationary injection that maps cells in a partition to cells in another partition. Returning to the example of the balance scale, we now begin with comparisons between the weights of groups of objects, and we wish to infer further comparisons of this kind. As before, if a group  $A$  is at least as heavy as a group  $B$ , and a group  $C$  is at least as heavy as a group  $D$ , then all of  $A$  and  $C$  together are at least as heavy as all of  $B$  and  $D$  together. (We do not require that  $A$  and  $C$  are disjoint, so “all of  $A$  and  $C$  together” may include multiples of a single object, and similarly for  $B$  and  $D$ .)<sup>2</sup> But now two additional types of inference are required to obtain all of the comparisons that follow from the given ones. First, if  $A$  and  $C$  together are at least as heavy as  $B$  and  $C$  together, then  $A$  is at least as heavy as  $B$ . Second, if  $n$  copies of a group  $A$  are together at least as heavy as  $n$  copies of a group  $B$ , then  $A$  is at least as heavy as  $B$ .

Orders on states and orders on events can be seen as special cases of orders on real-valued random variables. In the decision theory literature, there has been interest in when an order on real-valued random variables is representable by a set of probability measures (possibly together with a utility function) (Giron and Rios, 1980; Bewley, 2002). In §5, we relate our results to the representation theorems of Girotto and Holzer 2003 for real-valued random variables. The axioms to which our analysis leads, unlike generalized cancellation axioms, are special cases of the axioms used in the representation theorem for

<sup>2</sup>For many kinds of objects, it is natural to consider groups that contain multiple instances, e.g., two cans of soup. For others, it is less natural, in which case we may regard comparisons of the form *two copies of Thomas Jefferson are together at least as heavy as Benjamin Franklin* as “virtual” or “imaginary” comparisons that may be useful in intermediate steps of a deduction that begins and ends with comparisons involving no duplicates of objects.

real-valued random variables. Thus, our results can be viewed as showing which axioms, among those needed in the case of real-valued random variables, and which instances of those axioms, are sufficient when dealing with special kinds of random variables, e.g.,  $\{0, 1\}$ -valued random variables (events).

## 2. Preliminaries

### 2.1. Probabilistic Representability

In the literature on foundations of probability, there has been much discussion of the conditions under which a collection of judgments of the form “event  $E$  is at least as likely as event  $F$ ” can be represented by a numerical probability function (for surveys, see Fishburn 1986, Fine 1973, and Narens 2007). Often it is assumed that such a collection of judgments is *complete* in the sense that for every pair of events  $E$  and  $F$ , either  $E$  is judged to be at least as likely as  $F$ , or  $F$  is judged to be at least as likely as  $E$ . Yet since at least Keynes 1921, there has also been interest in contexts where an agent is not able to compare the likelihood of every two events, in which case the agent’s judgments may only be “imprecisely” representable by a *set* of probability functions (Ríos Insua 1992, Nehring 2009, Alon and Lehrer 2014). In this section, we review this relevant background on representability and imprecise representability.

Given a finite state space  $S$ , let  $\succsim$  be a reflexive binary relation on the powerset  $\mathcal{P}(S)$ .<sup>3</sup> We call elements  $E, F \in \mathcal{P}(S)$  *events*. The intended interpretation of  $E \succsim F$  is that event  $E$  is *at least as likely as* event  $F$ . As usual, a probability measure on  $\mathcal{P}(S)$  is a function  $\mu: \mathcal{P}(S) \rightarrow [0, 1]$  such that  $\mu(S) = 1$  and  $\mu(E \cup F) = \mu(E) + \mu(F)$  when  $E \cap F = \emptyset$ .<sup>4</sup>

**Definition 2.1.** For a set  $\Phi$  of probability measures on  $\mathcal{P}(S)$ :

1.  $\Phi$  *almost agrees* with  $\succsim$  if and only if for all  $E, F \in \mathcal{P}(S)$ ,

$$E \succsim F \Rightarrow \text{for all } \mu \in \Phi, \mu(E) \geq \mu(F).$$

2.  $\Phi$  *fully agrees* with  $\succsim$  if and only if for all  $E, F \in \mathcal{P}(S)$ ,<sup>5</sup>

$$E \succsim F \Leftrightarrow \text{for all } \mu \in \Phi, \mu(E) \geq \mu(F).$$

A single probability measure  $\mu$  *almost/fully agrees* with  $\succsim$  if and only if  $\{\mu\}$  almost/fully agrees with  $\succsim$ .

<sup>3</sup>The results of this paper easily generalize to an arbitrary algebra over  $S$ . However, we deal only with the case of a finite  $S$  (though see Remark 3.8), as the case of an infinite  $S$  presents complications. In particular, we know of no exact analogue to Theorem 2.8 below for the infinite case, though there are related results. See, e.g., §2.3 of Alon and Lehrer 2014. We leave for future research the extension of ideas in the present paper to the infinite case.

<sup>4</sup>As suggested in §1, (after rescaling) we may view such measure functions on a finite set  $S$  as representing measurement of any additive quantity, e.g., weight. For this interpretation, see, e.g., Krantz et al. 1971.

<sup>5</sup>Ríos Insua (1992) uses the term ‘quasi represents’ instead of ‘fully agrees’. An alternative notion of agreement, with  $E \succsim F \Leftrightarrow$  there exists  $\mu \in \Phi$  such that  $\mu(E) \geq \mu(F)$ , is suggested by the “justifiable preference” approach of Lehrer and Teper (2011); that approach also suggests a notion of agreement between  $\succsim$  and a set  $\{\Phi_i\}_{i \in I}$  of sets of measures, with  $E \succsim F \Leftrightarrow$  there exists  $i \in I$  such that for all  $\mu \in \Phi_i$ ,  $\mu(E) \geq \mu(F)$ .

We say that  $\succsim$  is *almost representable* if there is a probability measure  $\mu$  that almost agrees with  $\succsim$ ; that  $\succsim$  is *representable* if there is a probability measure  $\mu$  that fully agrees with  $\succsim$ ; and that  $\succsim$  is *imprecisely representable* if there is a set  $\Phi$  of probability measures that fully agrees with  $\succsim$ .

In §3, we will consider starting with an order  $\geq$  on  $S$ , instead of an order  $\succsim$  on  $\mathcal{P}(S)$ . In this context, we identify an order  $\geq$  on  $S$  with the corresponding order on singleton subsets of  $S$ . We will use the symbol  $\geq$  for an order on states and the symbol  $\succsim$  for an order on events.

To state conditions on  $\succsim$  that guarantee almost representability, representability, and imprecise representability, we need the following definition with notation from Fishburn 1986.

**Definition 2.2.** For any two sequences  $\langle E_1, \dots, E_k \rangle$  and  $\langle F_1, \dots, F_k \rangle$  of events from  $\mathcal{P}(S)$ ,

$$\langle E_1, \dots, E_k \rangle =_0 \langle F_1, \dots, F_k \rangle$$

if and only if for all  $s \in S$ , the cardinality of  $\{i \mid s \in E_i\}$  is equal to the cardinality of  $\{i \mid s \in F_i\}$ ; and

$$\langle E_1, \dots, E_k \rangle <_0 \langle F_1, \dots, F_k \rangle$$

if and only if for all  $s \in S$ , the cardinality of  $\{i \mid s \in E_i\}$  is strictly less than the cardinality of  $\{i \mid s \in F_i\}$ .

If  $\langle E_1, \dots, E_k \rangle =_0 \langle F_1, \dots, F_k \rangle$ , then we will say that the two sequences are *balanced*; every state appears the same number of times on the left side as on the right side. Note that  $\langle E_1, \dots, E_k \rangle <_0 \langle F_1, \dots, F_k \rangle$  means that *every* state appears *strictly* more times on the right side than on the left side.

Definition 2.3 below gives necessary and sufficient conditions in the finite case for  $\succsim$  to be representable. To motivate the definition, note the following fact about probability measures. If  $\langle E_1, \dots, E_k \rangle =_0 \langle F_1, \dots, F_k \rangle$  and for each  $i < k$ ,  $\mu(E_i) \geq \mu(F_i)$ , then in order to “keep the balance,” we must have  $\mu(F_k) \geq \mu(E_k)$ . This is the idea behind the **Finite Cancellation** axiom. Note that this axiom implies **Transitivity** of  $\succsim$ .

**Definition 2.3.** A relation  $\succsim$  on  $\mathcal{P}(S)$  is an *FC order* if and only if the following conditions hold:

1. **Finite Cancellation.** For all sequences

$$\langle E_1, \dots, E_n, A \rangle =_0 \langle F_1, \dots, F_n, B \rangle$$

of events from  $\mathcal{P}(S)$ , if  $E_i \succsim F_i$  for all  $i$ , then  $B \succsim A$ .

2. **Positivity.** For all  $E \in \mathcal{P}(S)$ ,  $E \succsim \emptyset$ .
3. **Non-Triviality.** It is not the case that  $\emptyset \succsim S$ .
4. **Completeness.** For all  $E, F \in \mathcal{P}(S)$ ,  $E \succsim F$  or  $F \succsim E$ .

**Theorem 2.4 (Kraft et al. 1959, Scott 1964).**  $\succsim$  is *representable* if and only if  $\succsim$  is an *FC order*.

For our later results, we will use the necessary and sufficient conditions for almost representability and imprecise representability, rather than representability. The following definition gives the conditions for almost representability.

**Definition 2.5.** A relation  $\succsim$  on  $\mathcal{P}(S)$  is an *AFC order* if and only if the following condition holds:

1. **Almost Finite Cancellation.** For all sequences

$$\langle E_1, \dots, E_n, A \rangle <_0 \langle F_1, \dots, F_n, B \rangle$$

of events from  $\mathcal{P}(S)$ , if  $E_i \succsim F_i$  for all  $i$ , then *not*  $A \succsim B$ .

**Theorem 2.6 (Kraft et al. 1959).**  $\succsim$  is almost representable if and only if  $\succsim$  is an AFC order.

While Theorems 2.4 and 2.6 are classic results, the following definition and theorem for imprecise representability are more recent and less well known.

**Definition 2.7.** A relation  $\succsim$  on  $\mathcal{P}(S)$  is a *GFC order* if and only if it satisfies **Reflexivity** (for all  $E \in \mathcal{P}(S)$ ,  $E \succsim E$ ), **Positivity**, **Non-Triviality**, and the following condition:

1. **Generalized Finite Cancellation (GFC).** For all sequences

$$\langle E_1, \dots, E_n, \underbrace{A, \dots, A}_{r \text{ times}} \rangle =_0 \langle F_1, \dots, F_n, \underbrace{B, \dots, B}_{r \text{ times}} \rangle$$

of events from  $\mathcal{P}(S)$ , if  $E_i \succsim F_i$  for all  $i$ , then  $B \succsim A$ .

**Theorem 2.8 (Ríos Insua 1992, Alon and Lehrer 2014).**  $\succsim$  is imprecisely representable if and only if  $\succsim$  is a GFC order.

It is shown in Harrison-Trainor et al. 2016 that **Generalized Finite Cancellation** is stronger than **Finite Cancellation** relative to **Reflexivity**, **Positivity**, and **Non-Triviality**.

From the normative perspective with which we began in §1, one could argue that a rational agent's comparative probability judgments should be at least imprecisely representable. In Icard 2016, it is shown that an agent avoids strict dominance in a canonical decision problem if and only if the agent's comparative probability judgements are imprecisely representable.

## 2.2. Random Variables

We will find it useful to deal with real-valued random variables on the state space  $S$ . A *random variable*  $X$  is a function  $X: S \rightarrow \mathbb{R}$ . We say that a random variable  $X$  dominates a random variable  $Y$ , and write  $X \geq Y$ , if for all  $s$ ,  $X(s) \geq Y(s)$ . The space of random variables naturally forms a vector space with addition  $X + Y$  and scalar multiplication  $rX$ .

We can view events as random variables taking values in  $\{0, 1\}$ . Random variables taking values in  $\mathbb{N}$  can be viewed as *multisets*. A multiset is, simply put, a collection in which objects may appear more than once. For example,  $\{0, 0, 1, 1, 1, 2\}$  and  $\{0, 0, 1, 2\}$  are distinct multisets (see Syropoulos 2001).

A *partition* of a random variable  $X$  is a collection of random variables  $Y_1, \dots, Y_n$  such that  $X = Y_1 + \dots + Y_n$ . We call  $Y_1, \dots, Y_n$  the cells of the partition. The reader can verify that, when  $X$  and  $Y_1, \dots, Y_n$  are events (i.e.,  $\{0, 1\}$ -valued random variables), this agrees with the standard notion of a partition of a set. We call such a partition a *set partition*. When  $X$  and  $Y_1, \dots, Y_n$  are  $\mathbb{N}$ -valued random variables, or multisets,

then this is the standard notion of a partition of a multiset. We call such a partition a *multiset partition* (or a *set partition* if  $X$  is a multiset but  $Y_1, \dots, Y_n$  are events). If  $X$  and  $Y$  are disjoint events, then  $X + Y$  is just the union of  $X$  and  $Y$ .

To define the notion of a function between multisets (here we follow Abramsky 1983), it helps to represent multisets as ordinary sets. Given a multiset  $X$ , define the set

$$\text{set}(X) = \{\langle k, a \rangle \in \mathbb{N} \times S \mid 1 \leq k \leq X(a)\}.$$

A function  $\alpha$  between multisets  $X$  and  $Y$  is an ordinary function  $\alpha: \text{set}(X) \rightarrow \text{set}(Y)$ . A function  $\alpha$  between multisets is injective if and only if it is injective as an ordinary function. The intuition is that  $\alpha$  assigns  $X(a)$  images from  $Y$  to each  $a \in S$ , possibly allowing repetitions, and if  $\alpha$  is injective then each  $b \in S$  can only appear as an image at most  $Y(b)$  times.

To distinguish these different types of random variables, we will use  $A, B, E, F$ , etc. for events, bold letters such as  $\mathbf{M}$  and  $\mathbf{N}$  for multisets, and  $U, V, X, Y$  for arbitrary random variables.

## 3. Extending an Order on States to an Order on Events

In this section, we begin with a preorder  $\geq$  on the state space  $S$ . (Note that since  $S$  is assumed to be finite, every preorder on  $S$  is almost representable by a probability measure.) As in Holliday and Icard 2013, we consider the following method of extending  $\geq$  to a preorder  $\succeq^i$  on  $\mathcal{P}(S)$ .

**Definition 3.1.** Given a preorder  $\geq$  on  $S$ , the *injection extension*  $\succeq^i$  of  $\geq$  is the binary relation on  $\mathcal{P}(S)$  defined by:  $E \succeq^i F$  if and only if there is an injection  $g: F \rightarrow E$  such that for all  $x \in F$ ,  $g(x) \geq x$ . Such an injection is called an *inflationary injection*.

Intuitively,  $E \succeq^i F$  means that we can pair off each state from  $F$  with a state from  $E$  which is at least as likely. Theorem 3.2 below shows that the comparisons between events entailed by  $\succeq$  are exactly those given by  $\succeq^i$ , and these comparison can be explicitly demonstrated by exhibiting an inflationary injection.

As in §1, given a binary relation  $\geq$  on  $S$ , which we regard as a relation on singleton subsets of  $S$ , we define the extension

$$\succeq^+ = \bigcap \{ \{ \succsim^p \mid \succsim^p \supseteq \geq, \succsim^p \text{ probabilistically representable} \},$$

where each  $\succsim^p$  is an order on  $\mathcal{P}(S)$ . Recall that  $\succeq^+$  gives us all the comparisons that can be inferred from  $\geq$ , assuming there is some probability measure of which  $\geq$  gives us partial information. There are several equivalent characterization of  $\succeq^+$ :

1.  $\succeq^+$  is the maximal relation extended by every representable extension of  $\geq$ ;
2.  $\succeq^+$  is the minimal extension of  $\geq$  that is imprecisely representable;
3. the set of all probability measures that almost agree with  $\geq$  fully agrees with  $\succeq^+$ .

We will show that  $\succeq^+$  is exactly the injection extension  $\succeq^i$  of Definition 3.1.

**Theorem 3.2.** *If  $\geq$  is a preorder on finite set  $S$ , then  $\geq^+ = \geq^i$ .*

Incidentally, Theorem 3.2 answers an open question posed by Yalcin (2010) in the literature on the formal semantics of English statements like “ $E$  is at least as likely as  $F$ .” Yalcin asks whether there is a method of extending an order on states to an order on events such that the class of extensions has certain desirable properties, which the class of GFC orders possesses. He observes that the method of extension used by previous authors—Lewis (1973), Kratzer (1991), and Halpern (1997) ( $E \geq^l F$  if and only if there is an inflationary function  $g: E \rightarrow F$ , not necessarily injective)—does not yield the desired properties (see Example 3.3 below for an illustration). By contrast, Theorem 3.2 shows that the injection extension  $\geq^i$  yields exactly the desired properties of GFC orders.

To see how the injection extension applies in a concrete example, recall the decision problem with the airlines from §1.

**Example 3.3.** The set of states is:

$$S = \{\text{United, Delta, Southwest, American, Frontier}\}.$$

Initially, the DM knows that United is more likely than Delta, that Delta is more likely than Southwest, and that American is more likely than Frontier. The corresponding order  $\geq$  on states is reflexive, transitive, and has

$$\text{United} \geq \text{Delta} \geq \text{Southwest} \quad \text{American} \geq \text{Frontier}.$$

Suppose first that terminal  $A$  services United and American, terminal  $B$  services Delta, and terminal  $C$  services Southwest and Frontier. (We identify  $A$  with the event {United, American} and so on.) Then we get  $A \geq^i B$  via the map  $g: B \rightarrow A$  which maps Delta to United. We also get  $A \geq^i C$  via the map  $g': C \rightarrow A$  which maps Southwest to United and Frontier to American. So the DM should choose to go to terminal  $A$  (assuming equal cost of going to each terminal), since it is more likely that the friend will land in terminal  $A$  than either of the other two.

Now suppose that instead there were only two terminals, terminal  $A$  which services United and American, and terminal  $B$  which services Delta, Southwest, and Frontier. The relation  $\geq^l$  mentioned above, used by Lewis (1973), Kratzer (1991), and Halpern (1997), would have  $A \geq^l B$  via the inflationary map  $g: B \rightarrow A$  which maps Delta and Southwest to United, and which maps Frontier to American. Thus  $\geq^l$  would say that the DM’s friend is most likely to arrive at terminal  $A$ . This is not a valid conclusion: maybe the friend in fact had a 40% chance of flying United, a 30% chance of flying Delta, a 20% chance of flying Southwest, a 7% chance of flying American, and a 3% chance of flying Frontier. Then the friend has a 47% chance of landing at terminal  $A$ , and a 53% chance of landing at terminal  $B$ . (Of course, there are other probabilities that are consistent with  $\geq$  and which make it more likely for the friend to arrive at terminal  $A$  than terminal  $B$ ; the DM cannot conclude from  $\geq$  which terminal is more likely.) The problem is that the map  $g$  described above is not injective, and in fact there is no injective inflationary map  $B \rightarrow A$ .  $A$  and  $B$  are  $\geq^i$ -incomparable.

To prove Theorem 3.2, we begin by recalling Hall’s Marriage Theorem. Let  $G = \langle X, Y, E \rangle$  be a bipartite graph with partite sets  $X$  and  $Y$  and edge relation  $E$ . An  $X$ -saturated matching for  $G$  is an injection  $m: X \rightarrow Y$  such that for all  $x \in X$ ,  $\langle x, m(x) \rangle \in E$ . Given  $A \subseteq X$ , let  $E[A]$  be the image of  $A$  under  $E$ . Hall’s Marriage Theorem gives a necessary and sufficient condition for an  $X$ -saturated matching to exist.

**Theorem 3.4 (Hall 1935).** *Let  $G = \langle X, Y, E \rangle$  be a bipartite graph. There is an  $X$ -saturated matching for  $G$  if and only if for each  $A \subseteq X$ ,  $|E[A]| \geq |A|$ .*

To show that  $\geq^i$  is a GFC order, we need to show that if  $\langle E_1, \dots, E_n, X \rangle$  and  $\langle F_1, \dots, F_n, Y \rangle$  are balanced and there are inflationary injections  $f_i: F_i \rightarrow E_i$  for  $i \leq n$ , then there is an inflationary injection  $h: X \rightarrow Y$ . Given the  $r$  repetitions of  $X$  and  $Y$  in the GFC axiom, we will show that there is an inflationary injection from the multiset  $rX$  to the multiset  $rY$ . Then the following lemma will give us the desired injection from  $X$  to  $Y$ . The proof of the lemma applies Hall’s Marriage Theorem.

**Lemma 3.5.** *Given a preorder  $\geq$  on  $S$  and  $X, Y \subseteq S$ , if there is an inflationary injection  $f: rX \rightarrow rY$ , then there is an inflationary injection  $h: X \rightarrow Y$ .*

PROOF. View  $rX$  as  $\{\langle k, x \rangle \mid 1 \leq k \leq r, x \in X\}$  and view  $rY$  as  $\{\langle k, y \rangle \mid 1 \leq k \leq r, y \in Y\}$ . Define a bipartite graph  $\langle X, Y, E \rangle$  by  $xEy$  if and only if for some  $k_1$  and  $k_2$ ,  $f(\langle k_1, x \rangle) = \langle k_2, y \rangle$ . We will use Hall’s Marriage Theorem to construct an  $X$ -saturated matching  $h: X \rightarrow Y$ . Then by definition of  $E$ , we will have that for all  $x \in X$ ,  $h(x) \geq x$ , so  $h$  is the desired inflationary injection.

To apply Hall’s Marriage Theorem, we must show that for every  $A \subseteq X$  we have  $|E[A]| \geq |A|$ . For  $A \subseteq X$ , observe that

$$\{\langle k, a \rangle \mid a \in A\} \subseteq \{\langle k, x \rangle \mid \exists b \in E[A] \exists k': f(k, x) = \langle k', b \rangle\},$$

so

$$\begin{aligned} r|A| &= |\{\langle k, a \rangle \mid a \in A\}| \\ &\leq |\{\langle k, a \rangle \mid \exists b \in E[A] \exists k': f(k, a) = \langle k', b \rangle\}| \\ &\leq r|E[A]| \end{aligned}$$

and hence  $|A| \leq |E[A]|$ , as desired.  $\square$

The following lemma is the key to showing that  $\geq^i$  is a GFC order and hence imprecisely representable. In the statement of the lemma, we extend the definition of  $=_0$  to multisets in the obvious way:

$$\langle \mathbf{M}_1, \dots, \mathbf{M}_n \rangle =_0 \langle \mathbf{N}_1, \dots, \mathbf{N}_n \rangle$$

if and only if

$$\sum_{i=1}^n \mathbf{M}_i = \sum_{i=1}^n \mathbf{N}_i.$$

We extend the definition of  $<_0$  in the same way.

**Lemma 3.6.** *If  $\langle E_1, \dots, E_n, \mathbf{M} \rangle =_0 \langle F_1, \dots, F_n, \mathbf{N} \rangle$  and  $E_i \geq^i F_i$  for all  $i$ , then there is an inflationary injection  $g: \mathbf{M} \rightarrow \mathbf{N}$ .*

PROOF. By assumption, we have an inflationary injection  $g_i: F_i \rightarrow E_i$  for each  $i$ . We show there is an inflationary injection  $h: \mathbf{M} \rightarrow \mathbf{N}$  by induction on the size of  $\mathbf{M}$ . The base case has  $\mathbf{M}$  empty, in which case the empty function is the desired  $h$ . Otherwise, pick  $a = a_0 \in \mathbf{M}$ . We will build a sequence  $\langle (a_0, i_0), (a_1, i_1), \dots, (a_{n-1}, i_{n-1}), a_n \rangle$  where  $a_n \in \mathbf{N}$ .

Given the *balancing assumption*

$$\langle E_1, \dots, E_n, \mathbf{M} \rangle =_0 \langle F_1, \dots, F_n, \mathbf{N} \rangle,$$

we have  $a_0 \in F_i$  for some  $i$  or  $a_0 \in \mathbf{N}$ . If  $a_0 \in \mathbf{N}$ , then terminate the sequence with  $a_n = a_0$ . If  $a_0 \notin \mathbf{N}$ , so  $a_0 \in F_i$  for some  $i = i_0$ , then let  $a_1 = g_{i_0}(a_0)$ , so  $a_1 \in E_{i_0}$ . Now either  $a_1 \in F_i$  for some  $i$  or  $a_1 \in \mathbf{N}$ . If  $a_1 \in \mathbf{N}$ , then terminate the sequence with  $a_n = a_1$ . So suppose  $a_1 \notin \mathbf{N}$ . If  $a_1 \neq a_0$ , then let  $i_1$  be any  $i$  such that  $a_1 \in F_i$ . If  $a_1 = a_0$ , then  $a_1 \in \mathbf{M}$ . Given  $a_1 \in \mathbf{M}$ ,  $a_1 \in E_{i_0}$ , and  $a_1 \notin \mathbf{N}$ , the balancing assumption implies there are  $j, j'$  with  $j \neq j'$  such that  $a_1 \in F_j$  and  $a_1 \in F_{j'}$ . Thus, we can choose  $i_1 \neq i_0$  so that  $a_1 \in F_{i_1}$ . Set  $a_2 = g_{i_1}(a_1)$ , so  $a_2 \in E_{i_1}$ .

In general, given  $a_k \in E_{i_{k-1}}$ , if  $a_k \in \mathbf{N}$ , then terminate the sequence with  $a_n = a_k$ . If  $a_k \notin \mathbf{N}$ , let  $A = \{l \mid a_k = a_l, l \leq k\}$ . Thus, for each  $l \in A$ ,  $a_k \in E_{i_{l-1}}$  (or  $a_k \in \mathbf{M}$  if  $l = 0$ ). It follows from the balancing assumption that for each  $l \in A$ , there is a  $F_{j_l}$  with  $a_k \in F_{j_l}$ . Thus, we can choose  $i_k$  such that  $a_k \in F_{i_k}$  and for all  $l \in A - \{k\}$ ,  $i_k \neq i_l$ . Set  $a_{k+1} = g_{i_k}(a_k)$ .

Continue building the sequence in this way, using the balancing assumption to ensure that each  $(a_k, i_k)$  appears only once. Since each  $g_i$  is inflationary, we have  $a_0 \leq a_1 \leq a_2 \dots$ . Now if our sequence were infinite, then  $\langle a_0, a_1, a_2, \dots \rangle$  would have an infinite subsequence with no repetitions, contradicting the finiteness of  $S$ . So our sequence terminates with  $a_n \in \mathbf{N}$ .

Now let

$$\begin{aligned} E'_i &= E_i - \{a_k \mid i = i_{k-1}\}; \\ F'_i &= F_i - \{a_k \mid i = i_k\}; \\ \mathbf{M}' &= \mathbf{M} - \{a_0\}; \\ \mathbf{N}' &= \mathbf{N} - \{a_n\}. \end{aligned}$$

Observe that

$$\langle E'_1, \dots, E'_n, \mathbf{M}' \rangle =_0 \langle F'_1, \dots, F'_n, \mathbf{N}' \rangle$$

since we removed  $a_0$  from  $\mathbf{M}$  and  $F_{i_0}$ , removed  $a_1$  from  $E_{i_0}$  and  $F_{i_1}$ , and so on. Also observe that  $E'_i \geq^i F'_i$  for all  $i$ , since the restriction of  $g_i$  to  $F'_i$  is an inflationary injection into  $E'_i$ . Thus, by the inductive hypothesis there is an injection  $h': \mathbf{M}' \rightarrow \mathbf{N}'$ . Now extend  $h'$  to  $h: \mathbf{M} \rightarrow \mathbf{N}$  by setting  $h(a_0) = a_n$ .  $\square$

**Lemma 3.7.** *The relation  $\geq^i$  is a GFC order.*

PROOF. For GFC, suppose

$$\langle E_1, \dots, E_n, \underbrace{A, \dots, A}_{r \text{ times}} \rangle =_0 \langle F_1, \dots, F_n, \underbrace{B, \dots, B}_{r \text{ times}} \rangle$$

and  $E_i \geq^i F_i$  for all  $i$ . By Lemma 3.6 with  $\mathbf{M} = rA$  and  $\mathbf{N} = rB$ , we have an inflationary injection  $h: rA \rightarrow rB$ , so by Lemma 3.5, we have an inflationary injection  $h: A \rightarrow B$ , so  $B \geq^i A$ . The other conditions are obvious.  $\square$

We are now prepared to prove Theorem 3.2: if  $\geq$  is a preorder on finite set  $S$ , then  $\geq^+ = \geq^i$ .

PROOF (THEOREM 3.2). As noted before Theorem 3.2,  $\geq^+$  is the minimal extension of  $\geq$  that is imprecisely representable. By Lemma 3.7 and Theorem 2.8,  $\geq^i$  is imprecisely representable. To see that it is the minimal such extension of  $\geq$ , consider an extension  $\succsim$  of  $\geq$  which is imprecisely represented by a set  $\Phi$  of probability measures. Now note that if  $A \geq^i B$ , so there is an inflationary injection  $f: B \rightarrow A$ , then for any measure  $\mu \in \Phi$ ,

$$\mu(A) \geq \sum_{b \in B} \mu\{f(b)\} \geq \sum_{b \in B} \mu\{b\} = \mu(B),$$

so  $A \succsim B$ . Thus  $\succsim$  extends  $\geq^i$ .  $\square$

**Remark 3.8.** Theorem 3.2 holds not only for finite preorders but more generally for any Noetherian preorder, i.e., any preorder  $\geq$  for which there is no infinite sequence  $x_1, x_2, x_3, \dots$  of distinct elements such that  $x_j \geq x_i$  for  $j \geq i$ . The proof of Theorem 3.2 above uses finiteness only in the penultimate paragraph of Lemma 3.6, where it is evident that the Noetherian condition is sufficient for the proof. Note that if  $\geq$  almost agrees with a probability measure  $\mu$ , then any sequence  $x_1, x_2, x_3, \dots$  as above must have  $\mu(\{x_n\}) = 0$  for each  $n$ ; for otherwise the sums  $\sum_{n=1}^m \mu(\{x_n\})$  grow without bound as  $m$  increases, contradicting the requirement that  $\sum_{n=1}^m \mu(\{x_n\}) \leq \mu(\{x_n \mid n \in \mathbb{N}\}) \leq 1$ .

It is interesting to note that an analogue of the Schröder-Bernstein theorem holds for  $\geq^i$ . Let  $a \simeq b$  if and only if  $a \geq b$  and  $b \geq a$ . If  $A \geq^i B$  and  $B \geq^i A$ , then there are injections from  $A$  to  $B$ , and from  $B$  to  $A$ , but for all we know so far, these two injections may not be inverses. In fact, in this situation we can always find injections that are inverses of each other.

**Proposition 3.9.** *For any preorder  $\geq$  on a finite set  $S$ , both  $A \geq^i B$  and  $B \geq^i A$  if and only if there is a bijection  $f: A \rightarrow B$  such that for all  $a \in A$ ,  $a \simeq f(a)$ .*

PROOF. Let  $g: A \rightarrow B$  and  $h: B \rightarrow A$  be such that  $g(a) \geq a$  for all  $a \in A$  and  $h(b) \geq b$  for all  $b \in B$ . We define  $f$  by recursion. We will give the main idea of the induction step.

Fix  $a = a_0 \in A$ . Then  $g(a_0) \geq a_0$ . Let  $a_1 = h(g(a_0))$ ,  $a_2 = h(g(a_1))$ , and in general,  $a_{i+1} = h(g(a_i))$ . Then we have

$$a_0 \leq g(a_0) \leq a_1 \leq g(a_1) \leq a_2 \leq \dots$$

Thus, since  $S$  is finite, for some  $m < n$  we have  $a_m = a_n$ . By injectivity of  $h$  and  $g$ ,  $a_{m-1} = a_{n-1}$ . Continuing in this way, we see that we may assume that  $m = 0$ ; i.e.,  $a_0 = a_\ell$  for some  $\ell$ . Choose  $\ell$  to be the least such. Then  $a_0, a_1, \dots, a_\ell$  are all distinct, for otherwise we could find some  $\ell' < \ell$  such that  $a_0 = a_{\ell'}$ . Since

$$a_0 \leq a_1 \leq a_2 \leq \dots \leq a_\ell = a_0,$$

we must have

$$a_0 \simeq g(a_0) \simeq a_1 \simeq g(a_1) \simeq \dots \simeq a_\ell = a_0.$$

Set  $f(a_i) = g(a_i)$  for each  $i$ .

Note that  $g$  and  $h$  can be restricted to the sets  $A - \{a_0, \dots, a_{\ell-1}\}$  and  $B - \{g(a_0), \dots, g(a_{\ell-1})\}$ . These restricted injections witness that

$$A - \{a_0, \dots, a_{\ell-1}\} \succeq^i B - \{g(a_0), \dots, g(a_{\ell-1})\}$$

and

$$B - \{g(a_0), \dots, g(a_{\ell-1})\} \succeq^i A - \{a_0, \dots, a_{\ell-1}\}.$$

We continue by recursion.  $\square$

Observe that the proof of Proposition 3.9 shows that if  $A \succeq^i B$  and  $B \succeq^i A$ , then every inflationary injection  $g: A \rightarrow B$  is a bijection with the property that for all  $a \in A$ ,  $a \approx g(a)$ .

#### 4. Extending an Order on Events

We will now move to the more general setting where we begin with an ordering  $\succeq$  on events. In this setting, the construction from §3 is no longer adequate. Of course, by Theorem 2.8, given a reflexive AFC order  $\succeq$  on  $\mathcal{P}(S)$ , the minimal extension of  $\succeq$  that is imprecisely representable can be characterized as the minimal extension of  $\succeq$  which is a GFC order. In this section, we will search for a characterization that is more similar in flavor to the extension in §3. The results in this section are not technically difficult but should be viewed as a different perspective on GFC with its own advantages.

Consider the following examples, the import of which will be explained below.

**Example 4.1.** Let  $S = \{a, b, c, d, e\}$ . Let  $\succeq$  be an imprecisely representable order with:

$$\{c, d\} \succeq \{a\} \quad \{e\} \succeq \{b, c\}.$$

Then for any probability measure  $\mu$  with  $\mu\{c, d\} \geq \mu\{a\}$  and  $\mu\{e\} \geq \mu\{b, c\}$ , we have

$$\mu\{d, e\} + \mu\{c\} = \mu\{c, d\} + \mu\{e\} \geq \mu\{a\} + \mu\{b, c\} = \mu\{a, b\} + \mu\{c\}$$

and hence  $\mu\{d, e\} \geq \mu\{a, b\}$ . So  $\{d, e\} \succeq \{a, b\}$ .

**Example 4.2.** Let  $S = \{a, b, c, d, e, f\}$ . Let  $\succeq$  be an imprecisely representable order with:

$$\{d, e\} \succeq \{a, b\} \quad \{e, f\} \succeq \{b, c\} \quad \{d, f\} \succeq \{a, c\}.$$

For any probability measure  $\mu$  which almost agrees with  $\succeq$ , we have

$$\begin{aligned} 2\mu\{d, e, f\} &= \mu\{d, e\} + \mu\{e, f\} + \mu\{d, f\} \\ &\geq \mu\{a, b\} + \mu\{b, c\} + \mu\{a, c\} \\ &= 2\mu\{a, b, c\} \end{aligned}$$

and hence  $\mu\{d, e, f\} \geq \mu\{a, b, c\}$ . So  $\{d, e, f\} \succeq \{a, b, c\}$ .

Examples 4.1-4.2 show that the injection extension of Definition 3.1 is not sufficient when we start with an order on events. Example 4.1 shows that in order to see that  $\{d, e\} \succeq \{a, b\}$ , we may need to introduce copies of a new element  $c$  on both sides. Example 4.2 shows that in order to see that  $\{d, e, f\} \succeq \{a, b, c\}$ , we may need to duplicate elements from both sides (i.e., to show that two copies of each of  $d, e$ , and  $f$  add up to as much as two copies of each of  $a, b$ , and  $c$ ). Definition 4.3 will modify Definition 3.1 to allow for both of these techniques.

**Definition 4.3.** Given a binary relation  $\succeq$  on  $\mathcal{P}(S)$ , the *injection extension*  $\succeq^i$  of  $\succeq$  is the binary relation on  $\mathcal{P}(S)$  defined by:  $E \succeq^i F$  if and only if there is a  $n \in \mathbb{N}$ , a multiset  $\mathbf{M}$  of elements of  $S$ , a set partition  $\mathcal{E}$  of  $nE + \mathbf{M}$ , a set partition  $\mathcal{F}$  of  $nF + \mathbf{M}$ , and an injection  $g: \mathcal{F} \rightarrow \mathcal{E}$  such that for all  $D \in \mathcal{F}$ ,  $g(D) \succeq D$ .<sup>6</sup>

In §5, we will give another characterization of  $\succeq^i$  in terms of three simple axioms.

According to Definition 4.3, the way we see that  $E \succeq^i F$  is intuitively as follows. We take  $n$  copies of  $E$  on the left hand side (this is  $nE$ ) and  $n$  copies of  $F$  on the right hand side (this is  $nF$ ). Then we add any number of states we want to both sides (this is  $\mathbf{M}$ ). Finally, we divide the right hand side up into groups of elements (this gives us the partition  $\mathcal{F}$ ), and for each such group we find a group of elements on the left hand side (in the partition  $\mathcal{E}$ ). If each group on the right hand side is less than or equal to (according to  $\succeq$ ) the corresponding group on the left hand side, then we set  $E \succeq^i F$ . Once again, the comparison is demonstrated in a constructive way.

In Theorem 4.4 below, we will prove that if  $\succeq$  is a reflexive AFC order on  $\mathcal{P}(S)$ , then  $\succeq^i$  is exactly  $\succeq^+$ , the intersection of all probabilistically representable extensions of  $\succeq$ , or equivalently, the least extension of  $\succeq$  that is imprecisely representable. By Theorem 2.8,  $\succeq^i$  is also the minimal GFC order extending  $\succeq$ .

Let us first apply Definition 4.3 to Examples 4.1 and 4.2. For Example 4.1, let  $\succeq$  be an AFC order on  $S = \{a, b, c, d, e\}$  with

$$\{c, d\} \succeq \{a\} \quad \{e\} \succeq \{b, c\}.$$

We claim that  $\{d, e\} \succeq^i \{a, b\}$ , as demonstrated in Figure 1. Let  $n = 1$  and  $\mathbf{M} = \{c\}$ . Then we partition  $\{a, b, c\}$  into  $\{a\}$  and  $\{b, c\}$ , partition  $\{c, d, e\}$  into  $\{c, d\}$  and  $\{e\}$ , and take

$$\{c, d\} \leftarrow \{a\} \quad \{e\} \leftarrow \{b, c\}$$

as our inflationary injection.

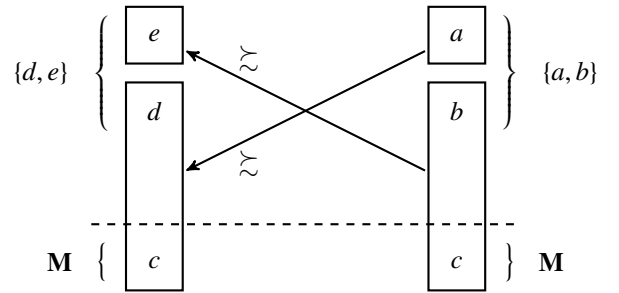


Figure 1: diagram witnessing  $\{d, e\} \succeq^i \{a, b\}$

Now consider Example 4.2. Let  $\succeq$  be an AFC order on  $S = \{a, b, c, d, e, f\}$  with:

$$\{d, e\} \succeq \{a, b\} \quad \{e, f\} \succeq \{b, c\} \quad \{d, f\} \succeq \{a, c\}.$$

<sup>6</sup>Recall that a set partition is a partition where all of the cells are sets.

We claim that  $\{d, e, f\} \succsim^i \{a, b, c\}$ , as demonstrated in Figure 2. Let  $n = 2$  and  $\mathbf{M} = \emptyset$ . Partition the multiset  $2\{a, b, c\}$  into  $\{a, b\}$ ,  $\{b, c\}$ , and  $\{a, c\}$ , and partition  $2\{d, e, f\}$  into  $\{d, e\}$ ,  $\{e, f\}$ , and  $\{d, f\}$ . Then our inflationary injection maps

$$\{d, e\} \leftarrow \{a, b\} \quad \{e, f\} \leftarrow \{b, c\} \quad \{d, f\} \leftarrow \{a, c\}.$$

So we can see that the definition of  $\succsim^i$  on events is exactly what was required for Examples 4.1 and 4.2.

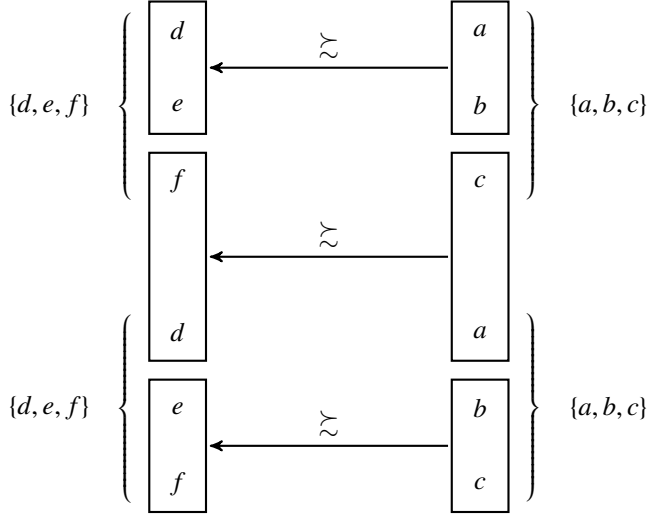


Figure 2: diagram witnessing  $\{d, e, f\} \succsim^i \{a, b, c\}$

It is a virtue of the injection extension that its very definition clearly separates the two different phenomena exhibited by Examples 4.1 and 4.2: the multiset  $\mathbf{M}$  deals with Example 4.1 and the  $n$  copies of  $E$  and  $F$  deal with Example 4.2. This distinction does not appear as clearly in the definition of a GFC order.

We can now generalize Theorem 3.2 to the present setting, where we start with orders on events.

**Theorem 4.4.** *If  $\succsim$  is a reflexive AFC order on  $\mathcal{P}(S)$ , then  $\succsim^+ = \succsim^i$ .*

The key lemma used to prove Theorem 4.4 is the following.

**Lemma 4.5.** *If  $\succsim$  is a reflexive AFC order, then  $\succsim^i$  is a GFC order.*

**PROOF.** It is easy to see that  $\succsim^i$  satisfies **Reflexivity** and **Positivity**. Now we will show that it satisfies **GFC**. Suppose that we have two sequences

$$\langle E_1, \dots, E_n, \underbrace{A, \dots, A}_{r \text{ times}} \rangle =_0 \langle F_1, \dots, F_n, \underbrace{B, \dots, B}_{r \text{ times}} \rangle$$

of events from  $\mathcal{P}(S)$  and that  $E_i \succsim^i F_i$  for all  $i$ . This means that for each  $i$ , there is some  $\ell_i$ , a multiset  $\mathbf{A}_i$ , partitions  $\mathcal{E}_i$  of  $\ell_i E_i + \mathbf{A}_i$  and  $\mathcal{F}_i$  of  $\ell_i F_i + \mathbf{A}_i$ , and an injection  $g_i: \mathcal{F}_i \rightarrow \mathcal{E}_i$  such that for all  $D \in \mathcal{F}_i$ ,  $g_i(D) \succsim D$ .

Let  $\mathbf{M}$  be the multiset of elements in common between  $E_1 + \dots + E_n$  and  $F_1 + \dots + F_n$ , and let  $C = A \cap B$ . Then from

$$E_1 + \dots + E_n + rA = F_1 + \dots + F_n + rB$$

we get  $rA + \mathbf{M} = F_1 + \dots + F_n + rC$  and  $rB + \mathbf{M} = E_1 + \dots + E_n + rC$ .

Let  $\ell = \ell_1 \dots \ell_n$ . We claim that  $B \succsim^i A$  as witnessed by the number  $\ell \cdot r$  and the multiset

$$\mathbf{N} = \ell \mathbf{M} + (\ell/\ell_1) \mathbf{A}_1 + \dots + (\ell/\ell_n) \mathbf{A}_n.$$

Observe that

$$\begin{aligned} & (\ell r)A + \mathbf{N} \\ &= \ell(rA + \mathbf{M}) + (\ell/\ell_1) \mathbf{A}_1 + \dots + (\ell/\ell_n) \mathbf{A}_n \\ &= \ell(F_1 + \dots + F_n + rC) + (\ell/\ell_1) \mathbf{A}_1 + \dots + (\ell/\ell_n) \mathbf{A}_n \\ &= (\ell/\ell_1)(\ell_1 F_1 + \mathbf{A}_1) + \dots + (\ell/\ell_n)(\ell_n F_n + \mathbf{A}_n) + (\ell r)C. \end{aligned}$$

Similarly,

$$(\ell r)B + \mathbf{N} = (\ell/\ell_1)(\ell_1 E_1 + \mathbf{A}_1) + \dots + (\ell/\ell_n)(\ell_n E_n + \mathbf{A}_n) + (\ell r)C.$$

First, partition  $(\ell r)A + \mathbf{N}$  into the multiset union of  $\ell r$  copies of  $C$  and  $\ell/\ell_i$  copies of  $\ell_i E_i + \mathbf{A}_i$  for each  $i$ , and partition  $(\ell r)B + \mathbf{N}$  into the multiset union of  $\ell r$  copies of  $C$  and  $\ell/\ell_i$  copies of  $\ell_i F_i + \mathbf{A}_i$  for each  $i$ . Next use the partitions  $\mathcal{E}_i$  of  $\ell_i E_i + \mathbf{A}_i$  and  $\mathcal{F}_i$  of  $\ell_i F_i + \mathbf{A}_i$  to create partitions of  $(\ell r)A + \mathbf{N}$  and  $(\ell r)B + \mathbf{N}$ . Then, piecing together the injections  $g_i$ , and mapping the copies of  $C$  to each other, we get the required injection witnessing  $B \succsim^i A$ .

Finally, we have to show that  $\succsim^i$  satisfies **Non-Triviality**. Suppose that  $\emptyset \succsim^i S$ . Then there is a number  $r$ , a multiset  $\mathbf{A}$ , partitions  $\mathcal{A}$  of  $\mathbf{A}$  and  $\mathcal{S}$  of  $rS + \mathbf{A}$ , and an inflationary injection  $g: \mathcal{S} \rightarrow \mathcal{A}$ . Let  $F_1, \dots, F_n$  be the cells of  $\mathcal{S}$ , and  $E_1, \dots, E_n$  the cells of  $\mathcal{A}$ , so that  $E_i \succsim F_i$ . Then we have

$$\langle E_1, E_2, \dots, E_n \rangle <_0 \langle F_1, F_2, \dots, F_n \rangle.$$

Since we assumed  $\succsim$  satisfies **AFC**, and since for each  $i < n$ ,  $E_i \succsim F_i$ , we have *not*  $E_n \succsim F_n$ . This is a contradiction.  $\square$

We are now prepared to prove Theorem 4.4 by showing that  $\succsim^i$  is the minimal GFC order extending  $\succsim$ .

**PROOF (THEOREM 4.4).** By Lemma 4.5,  $\succsim^i$  is a GFC order. We must show that it is the minimal GFC order extending  $\succsim$ .

Suppose that  $A \succsim^i B$ . Then there is a number  $r$ , a multiset  $\mathbf{M}$ , cells  $E_1, \dots, E_n$  partitioning  $rA + \mathbf{M}$  and cells  $F_1, \dots, F_n$  partitioning  $rB + \mathbf{M}$ , such that  $E_i \succsim F_i$ . Then

$$\langle E_1, \dots, E_n, \underbrace{B, \dots, B}_{r \text{ times}} \rangle =_0 \langle F_1, \dots, F_n, \underbrace{A, \dots, A}_{r \text{ times}} \rangle$$

is balanced and has  $E_i \succsim F_i$  for each  $i$ . Thus, any GFC order  $\succsim^*$  extending  $\succsim$  has  $A \succsim^* B$ .  $\square$

When we began with an order on states in §3, we were able to show that if  $E \succsim F$  and  $F \succsim E$ , then this was witnessed by a bijection (Proposition 3.9). The equivalent result in the context of this section would be to have it witnessed by a  $n \in \mathbb{N}$ , a multiset  $\mathbf{M}$  of elements of  $S$ , a partition  $\mathcal{E} \subseteq \mathcal{P}(S)$  of  $nE + \mathbf{M}$ , a partition  $\mathcal{F} \subseteq \mathcal{P}(S)$  of  $nF + \mathbf{M}$ , and a bijection  $g: \mathcal{F} \rightarrow \mathcal{E}$  such that for all  $D \in \mathcal{F}$ ,  $g(D) \succsim D$  and  $D \succsim g(D)$ . The following example shows that this does not necessarily happen.



**Example 4.6.** Let  $S = \{a, b, c, d, e, f, g, h\}$ . Let  $\succsim$  be an imprecisely representable order with:

$$\begin{aligned} \{a, c\} &\succsim \{e, g\} & \{b, d\} &\succsim \{f, h\} \\ \{e, f\} &\succsim \{a, b\} & \{g, h\} &\succsim \{c, d\}. \end{aligned}$$

Then for any probability measure  $\mu$  that almost agrees with  $\succsim$ , we have

$$\mu\{a, b, c, d\} = \mu\{a, c\} + \mu\{b, d\} \geq \mu\{e, g\} + \mu\{f, h\} = \mu\{e, f, g, h\}$$

and

$$\mu\{e, f, g, h\} = \mu\{e, f\} + \mu\{g, h\} \geq \mu\{a, b\} + \mu\{c, d\} = \mu\{a, b, c, d\}.$$

So we have  $\{a, b, c, d\} \succsim \{e, f, g, h\}$  and  $\{e, f, g, h\} \succsim \{a, b, c, d\}$ . However, one can check that there is no bijection as described above. The key fact is that in the four comparisons above,  $a$  does not appear with  $b$  on the left side of any comparison, and  $a$  does not appear with  $c$  on the right side of any comparison.

## 5. Discussion

An order on subsets of  $S$  is a special case of an order on real-valued random variables on  $S$ , namely a  $\{0, 1\}$ -valued random variable. Let  $\mathcal{R}(S)$  be the set of all real-valued random variables on our finite set  $S$ . Where  $X \in \mathcal{R}(S)$  and  $\mu$  is a probability measure on  $\mathcal{P}(S)$ , let

$$\mu(X) = \sum_{s \in S} X(s) \cdot \mu\{s\}.$$

Let us say that an order  $\succsim$  on  $\mathcal{R}(S)$  is *imprecisely representable* by a set  $\Phi$  of probability measures on  $\mathcal{P}(S)$  when for all  $X, Y \in \mathcal{R}(S)$ ,  $X \succsim Y$  if and only if for all  $\mu \in \Phi$ ,  $\mu(X) \geq \mu(Y)$ , i.e., according to  $\mu$  the *expected value* of  $X$  is at least that of  $Y$ .

For the following definition, recall that we view sets such as  $\emptyset$  and  $S$  as random variables, taking their values in  $\{0, 1\}$ , by identifying a set with its characteristic function.

**Definition 5.1.** A binary relation  $\succsim$  on  $\mathcal{R}(S)$  is an *EV order* if and only if it satisfies:

1. **Reflexivity.** For all  $X$ ,  $X \succsim X$ .
2. **Positivity.** If  $X \geq \emptyset$ , then  $X \succsim \emptyset$ .
3. **Non-triviality.** It is not the case that  $\emptyset \succsim S$ .
4. **Additivity.** If  $U \succsim X$  and  $V \succsim Y$ , then  $U + V \succsim X + Y$ .
5. **Cancellation.** If  $X + Z \succsim Y + Z$ , then  $X \succsim Y$ .
6. **Scaling.** If  $X \succsim Y$  and  $r \in \mathbb{R}_{>0}$ , then  $rX \succsim rY$ .
7. **Continuity.** If  $X_1, X_2, \dots$  is a sequence with  $\lim X_i = X$ , and  $Y_1, Y_2, \dots$  is a sequence with  $\lim Y_i = Y$ , and  $X_i \succsim Y_i$  for each  $i$ , then  $X \succsim Y$ .

These conditions are a reformulation of those in §2 of Girotto and Holzer 2003. The following representation theorem is essentially a special case of their Theorem 4.1 (cf. Theorem 3 of Rumbos 2001 for representability by a single measure).

**Theorem 5.2.** Suppose that  $S$  is finite. A relation  $\succsim$  on  $\mathcal{R}(S)$  is *imprecisely representable* if and only if  $\succsim$  is an EV order.

Our results in §4 can be viewed as showing which of the axioms from Definition 5.1 suffice for the case of  $\{0, 1\}$ -valued random variables. First, recall the perspective of Gilboa et al. (2010): we are interested in a method of deriving from certain accepted judgments, i.e., from an initial (incomplete) order  $\succsim$ , all of those further judgments that necessarily follow. One way to do this is to apply the **GFC** axiom; indeed, one can show that any judgment which follows from  $\succsim$  follows from a single application of **GFC**.<sup>7</sup> Our Theorem 4.4 shows that one can instead use three simpler axioms. Though  $\succsim$  begins as a relation on sets, and all of the judgments we want to infer are comparisons between sets, we temporarily consider  $\succsim$  as a relation on multisets. One can think of this detour through multisets as similar to the detour through imaginary numbers for solving cubic equations (though like imaginaries, multisets are also interesting to consider in their own right). The three axioms are:

1. **Additivity.** If  $A \succsim B$  and  $C \succsim D$ , then  $A + C \succsim B + D$ .
2. **Cancellation.** If  $A + M \succsim B + M$ , then  $A \succsim B$ .
3. **Discrete Scaling.** If  $mA \succsim mB$  ( $m \in \mathbb{N}$ ), then  $A \succsim B$ .

We may regard an order on the powerset  $\mathcal{P}(S)$  as an order on the set  $\mathcal{M}(S)$  of multisets. Given an order  $\succsim$  on  $\mathcal{M}(S)$ , let  $\succsim^*$  be the least extension of  $\succsim$  satisfying the three axioms above, plus **Reflexivity**, **Positivity**, and **Non-triviality** from §2.1. Note that **Transitivity** follows from **Additivity** and **Cancellation**. Also note that  $\succsim^*$  exists only if  $\succsim$  is an AFC order.

The relation  $\succsim^*$ , restricted to sets, is exactly our injection extension  $\succsim^i$  from §4.

**Lemma 5.3.** For any AFC order  $\succsim$  on  $\mathcal{P}(S)$  and  $E, F \in \mathcal{P}(S)$ ,  $E \succsim^i F$  if and only if  $E \succsim^* F$ .

**PROOF.** From left to right, if  $E \succsim^i F$ , then there is an  $n \in \mathbb{N}$ , a multiset  $\mathbf{M}$  of elements of  $S$ , a set partition  $\{E_1, \dots, E_\ell, \dots, E_m\}$  of  $nE + \mathbf{M}$  and a set partition  $\{F_1, \dots, F_\ell\}$  of  $nF + \mathbf{M}$  such that  $E_j \succsim F_j$  for  $1 \leq j \leq \ell$ . By **Positivity**,  $E_j \succsim^* \emptyset$  for  $\ell < j \leq m$ . Then by **Additivity**,

$$\begin{aligned} nE + \mathbf{M} &= E_1 + \dots + E_\ell + \dots + E_m \\ \succsim^* & F_1 + \dots + F_\ell \\ &= nF + \mathbf{M}. \end{aligned}$$

By **Cancellation**,  $nE \succsim^* nF$ , so by **Discrete Scaling**,  $E \succsim^* F$ .

In the other direction, suppose that  $E \succsim^* F$ . By Theorem 4.4,  $\succsim^i$  is imprecisely representable. Let  $\Phi$  be a set of probability measures that fully agrees with  $\succsim^i$ . Define the ordering  $\succsim^\Phi$  on  $\mathcal{M}(S)$  by  $A \succsim^\Phi B$  if and only if  $\mu(A) \geq \mu(B)$  for all  $\mu \in \Phi$ . It is easy to see that  $\succsim^\Phi$  satisfies **Additivity**, **Cancellation**, and **Discrete Scaling** (as well as **Reflexivity**, **Positivity**, and **Non-triviality**). Then by definition of  $\succsim^*$ ,  $\succsim^\Phi$  is an extension of  $\succsim^*$ , and so  $E \succsim^\Phi F$ . Since  $\Phi$  fully agrees with  $\succsim^i$ ,  $E \succsim^i F$ .

From this lemma, we obtain the following representation theorem.

<sup>7</sup>This follows from the proof of our Lemma 4.5.

**Theorem 5.4.** For any order  $\succsim$  on  $\mathcal{P}(S)$ , the following are equivalent:

1.  $\succsim$  is imprecisely representable;
2. for all  $E, F \in \mathcal{P}(S)$ ,  $E \succsim F$  if and only if  $E \succsim^* F$ .

**PROOF.** First, recall from §4 that  $\succsim^+$  is the least extension of  $\succsim$  that is imprecisely representable.

From 1 to 2, since  $\succsim$  is imprecisely representable, it is an AFC order for which  $\succsim = \succsim^+$ , so by Theorem 4.4,  $\succsim = \succsim^i$ . By Lemma 5.3, we also have  $\succsim^i = \succsim^*$ . Thus, part 2 holds.

From 2 to 1, first we claim that  $\succsim$  is an AFC order. Assume to the contrary that for sequences

$$\langle E_1, \dots, E_n, A \rangle <_0 \langle F_1, \dots, F_n, B \rangle$$

of events from  $\mathcal{P}(S)$ , we have  $E_i \succsim F_i$  for all  $i$ , but  $A \not\succsim B$ . Then by **Additivity**, we have

$$E_1 + \dots + E_n + A \succsim^* F_1 + \dots + F_n + B.$$

By our initial assumption, the right-hand side includes  $E_1 + \dots + E_n + A + S$ . By applying **Cancellation** with  $\mathbf{M} = E_1 + \dots + E_n + A$ , we obtain  $\emptyset \succsim^* S + \mathbf{C}$  for a multiset  $\mathbf{C}$ . By **Positivity**,  $\mathbf{C} \succsim^* \emptyset$ , and by **Reflexivity**,  $S \succsim^* S$ . Thus, by **Additivity**,  $S + \mathbf{C} \succsim^* S$ , so by **Transitivity**,  $\emptyset \succsim^* S$ , contradicting **Non-triviality**.

Since  $\succsim$  is an AFC order that agrees with  $\succsim^*$  on  $\mathcal{P}(S)$ , by Lemma 5.3 we have  $\succsim = \succsim^i$ , so by Theorem 4.4,  $\succsim = \succsim^+$ . Hence  $\succsim$  is imprecisely representable.

For the setting of §3, where we extended an order  $\geq$  on singletons of  $S$  to an injection extension  $\geq^i$  on  $\mathcal{P}(S)$ , we can characterize  $\geq^i$  as the least extension  $\geq^\#$  of  $\geq$  satisfying **Reflexivity**, **Positivity**, **Non-triviality**, **Transitivity**, and:

**Disjoint Additivity.** If  $E_1 \geq^\# F_1$ ,  $E_2 \geq^\# F_2$ , and  $E_1 \cap E_2 = \emptyset$ , then  $E_1 \cup E_2 \geq^\# F_1 \cup F_2$ .

**Transitivity** and **Disjoint Additivity** are exactly the part of **GFC** that is needed in the setting of states.

Thus, we obtain a hierarchy of three easily comparable sets of axioms for the three domains: one for orders on states, one for order on events, and one for orders on random variables.

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