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The Economic Record versus AER*:* Twenty Six Years Ahead on the Money-Goods Model*

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The Economic Record versus *AER*: Twenty Six Years Ahead on the Money-Goods Model

Abstract

We prove that the symmetric and negative semidefinite modified Slutsky matrix derived by Samuelson and Sato (1984) for the money-goods model of the consumer, is identical to that derived by Pearce (1958) a quarter century before and restated sixteen years later by Berglas and Razin (1974). We also prove that these conditions are only sufficient for the problem at hand and are encompassed by a more general, modified Slutsky matrix that is necessary and sufficient as derived by Paris and Caputo (2001). These results have crucial relevance for testing the implications of consumer behavior.

Keywords: Money-goods models; Slutsky Matrix; Comparative statics

JEL Classification Numbers: D11, E41

I Historical Introduction

The problem of how to model cash money balances within the theory of consumer behavior and how to obtain empirically verifiable conditions has had a long gestation period during which several distinguished economists have contributed their expertise. The most significant papers can be listed, in chronological order of appearance, as those of Leser (1943), Samuelson (1947), Patinkin (1948), Morishima (1952), Pearce (1958), Lloyd (1964, 1971), Berglas and Razin (1974), and Samuelson and Sato (1984).

The problem of dealing directly with money balances is of great relevance from both a theoretical and empirical viewpoint because it is the conduit for generalizing the specification of the prototype model of consumer behavior via the inclusion of prices of all real goods into the direct utility function itself. This generalized specification of consumer theory provides a distinct alternative hypothesis for testing the traditional consumer model and, thus, for increasing the power of the test. It is surprising therefore that, to this day, this general specification has not taken hold in the theoretical and empirical toolbox of research economists.

In evaluating–with hindsight's wisdom–the contributions listed above, it is crucial to keep in mind two criteria: (i) the relevant and most general specification of the money-price-dependent utility function and, (ii) the empirical verifiability of the conditions implied by it. We anticipate that, to date, the most general specification of the money and price-dependent direct utility function consistent with verifiable Slutsky-type equations requires the assumptions of weak separability between real goods and money and prices of real goods in the direct utility function, and the absence of the interest rate from the direct utility function.

The first paper clearly stating the need for a model of consumer behavior that includes cash balances was written by Leser (1943), who presented the following specification:

$$\max_{x_1,\ldots,x_n,M} \left\{ U\left(x_1,\ldots,x_n,\frac{M}{p_1},\ldots,\frac{M}{p_n}\right) \quad \text{s.t.} \quad p_1x_1+\cdots+p_nx_n+M=Y \right\} \tag{1}$$

where $(x_1,...,x_n)$ is a vector of real goods, M is the nominal money balance, $(p_1,...,p_n)$ is the vector of prices of real goods, and Y is the consumer's income. The structure of this model, however, is too general and Leser could not derive any empirically verifiable relations.

Samuelson (1947, chapter V) wrote a short note in the appendix to his chapter on consumer behavior in which he correctly spelled out the role of the interest rate but, otherwise, simply reproduced Leser's specification. Samuelson's model was stated as the maximization with respect to $(x_1,...,x_n)$ and M of:

$$U(x_1,...,x_n,Mp_m,p_1,...,p_n)=F\left(x_1,...,x_n,\frac{M}{p_1},...,\frac{M}{p_n}\right) \quad \text{s.t.} \quad p_1x_1+\cdots+p_nx_n+rp_mM=Y, \qquad (2)$$

where r is the interest rate and p_m is the price of money. The interest rate does not enter the utility function and this detail will be crucial in subsequent analyses. As with Leser's model, Samuelson's specification is too general and yields no empirically verifiable relation.

Patinkin (1948) made a vigorous attempt to convince the audience of his time that money in the form of real cash balances ought to enter the consumer's direct utility function. He even indicted Walras and Pareto for errors leading to the omission of money into the utility function. His specification of the consumer model, however, did not produce any empirically verifiable relation.

Morishima (1952) presented a model of consumer behavior integrated with the demand for cash and bonds. His most important contribution consists of the introduction of the assumption of weak separability between real goods and all the other variables entering the direct utility function, including cash balances, bonds, the prices of real goods, and the bond price. Although Morishima was unable to produce empirically verifiable relations, the notion of weak separability turned out to be a crucial assumption for the eventual and complete solution of the money-goods problem. The Morishima model takes on the following specification

$$\max_{x_1,...,x_n,M,B}\{U[x_1,...,x_n,q(M,B,p_1,...,p_n,p_b)] \quad \text{s.t.} \quad p_1x_1+\cdots+p_nx_n+M+p_bB=Y\}, \qquad (3)$$

where B represents bonds and p_b is the bond price. The interest rate is not included in this model. It is interesting to note that Morishima presents three relations (1952, p. 230), namely the compensated partial derivatives of real goods, cash balances and bonds with respect to the *k*th real good price, and regards them as "generalized Slutsky equations." These derivatives, however, include second derivatives of the utility function that render these relations unobservable. But observability is an essential trademark of a Slutsky equation and, thus, Morishima's characterization of his derivatives is improper and his "Slutsky equations" cannot form the basis for an empirically verifiable test of consumer theory as stated in problem (3). Morishima references the works of Leser, Samuelson, and Patinkin in his paper.

Pearce's contribution (1958) represents the first paper that came close to solving the problem of finding verifiable hypotheses for real goods and money demands, although his methodology lacks transparency. Actually, Pearce wrote a paper about demand analysis and savings, not cash balances, but his model can easily be re-interpreted and adapted to the problem at hand. Indeed, Pearce's model was re-interpreted by Berglas and Razin (1974), as we report further on. A close observation of Pearce's development reveals a rather tortuous line of reasoning from his specification of the utility function to his derivation of a Slutsky-type equation involving real goods. He begins with a consumer model that is formulated as follows

$$\max_{x_1,\ldots,x_n,s} \{U[x_1,\ldots,x_n,q_1,\ldots,q_n,r,s,w] \quad \text{s.t.} \quad p_1x_1+\cdots+p_nx_n+s=Y\}, \tag{4}$$

where $q_1,\ldots,q_n$ are expected future prices of real goods, s is current savings and w is wealth. Pearce states that expected future prices are related to current prices. Notice that the interest rate r enters, surprisingly, the utility function but not the budget constraint. Furthermore, there is no evidence, at this stage, that Pearce assumed weak separability between real goods and savings and expected future prices. These two facts would, in general, prevent a derivation of verifiable relations using the traditional methodology of equilibrium displacement And yet, Pearce offers a line of reasoning that, although difficult to follow, somehow produces a relation which is a veri-

fiable comparative statics condition under weak separability and the absence of the interest rate from the utility function.

It is interesting (and curious) to retrace in some detail Pearce's logic. First, he states (1958, p. 56) "the total rate of change" of the *i*th real good x_i with respect to changes in the price p_i

$$\frac{dx_i}{dp_i} = \frac{\partial x_i}{\partial p_i} - \frac{\partial x_i}{\partial c}\frac{\partial s}{\partial p_i} \tag{5}$$

where $c = Y - s$ is current expenditure on real goods. Secondly, he writes (1958, p. 57): "*An unjustified step now follows*. … the partial $\partial x_i/\partial p_i$ can be split up (as in a traditional Slutsky equation, our addition) into income and substitution effects as before, as long as we take care only to put c for Y whenever it occurs. That is, it is supposed that we can write

$$\frac{dx_i}{dp_i} = -x_i\frac{\partial x_i}{\partial c} + \sigma_{ii} - \frac{\partial x_i}{\partial c}\frac{\partial s}{\partial p_i} \tag{6}$$

$$= -\left(x_i + \frac{\partial s}{\partial p_i}\right)\frac{\partial x_i}{\partial c} + \sigma_{ii}"$$

where σ_{ii} is the traditional Slutsky substitution term of the prototype consumer model with $c = \sum_{i=1}^{n} p_i x_i$ as the portion of income to spend on real goods. Pearce continues (1958, pp.57-58): "using the fact that $Y = c + s$ or $\partial Y/\partial c = 1/(1 - \partial s/\partial Y)$ we can also write

$$\frac{dx_i}{dp_i} = -\frac{1}{1 - \partial s/\partial Y}\left(x_i + \frac{\partial s}{\partial p_i}\right)\frac{\partial x_i}{\partial Y} + \sigma_{ii} \tag{7}$$

Unfortunately the splitting $\partial x_i/\partial p_i$ into income and substitution effect is invalid unless a further assumption is made. … To validate our formula we require to impose some condition on the form of our new utility function U. … If it can be accepted that an increase in wealth will not affect the relative marginal utilities of goods in current consumption then the result (7) still holds." This last statement is the source of the weak separability assumption required to derive empiri-

cally verifiable relations. Relation (7) is cryptic because the statement and meaning of the total derivative on the left-hand-side of the equality sign is unclear, and because the demand functions, and hence the form of the utility maximization problem from which they derive, are obscured by the rather loose and unstructured differential development. Pearce did not reference Morishima's paper. Relation (7), properly restated, integrated, and re-interpreted, represents a modified Slutsky equation that, in principle, is empirically verifiable, as we demontrate below.

Lloyd's contribution, spelled out over two papers (1964, 1971), consists in refocusing the reader's attention on Patinkin's model dealing with the real-balance effect and on the form of the utility function under the assumption of weak separability (called "partial separability" by Lloyd) between real goods and money and the prices of real goods. In his 1971 paper, Lloyd referenced Morishima's and Pearce's papers both of whom had clearly identified the weak separability assumption as a crucial step toward empirically verifiable relations. Lloyd's development of the comparative statics conditions is not satisfactory and, in spite of his reading of Pearce's results, he failed to obtain empirically verifiable relations.

Berglas and Razin (1974), in a short comment on Lloyd's second paper, re-interpreted Pearce's results of his savings model and reproduced his development of Eq. (17), but added a two-stage maximization explanation. Berglas and Razin apparently recognized that they essentially copied Pearce's development since they wrote that (1974, p. 200): "…what we have done is implied by Pearce (1958, 1964, chapter 3)."

The evaluation of the literature about money-goods models presented so far reveals the difficult path mapped out by various contributions in search of a fruitful and intelligible framework toward the objective of finding verifiable conditions. Forty years after the original proposal by Leser, the money-goods problem had not yet received a rigorous, complete and satisfactory solution. It remained for Samuelson and Sato (1984) to make a quantum leap toward an elegant and almost complete analysis. Samuelson and Sato referenced neither Pearce's work (1958) (which by 1964 was also reproduced in a book on demand analysis) nor Berglas and Razin's (1974) paper. While it is beyond our scope to speculate about the reason for such an omission,

we acknowledge two facts: Samuelson and Sato clarified the money-goods model in such an intelligible way so as to provide us with the possibility of demonstrating that their result is identical to that obtained by Pearce twentysix years before.

Hence, we now show that before Samuelson and Sato's (1984) analysis of money-goods models was published, Pearce (1958), reinterpreted by Berglas and Razin (1974), derived an identical form of Samuelson and Sato's modified Slutsky matrix. We also demonstrate that, when a two-stage maximization framework is adopted, there are at least six identical forms of the modified Slutsky matrix for the money-goods model, and discuss the implications of this finding for empirical testing. Finally, we relate the results obtained to the more general results of Paris and Caputo (2001). Because the work of Samuelson and Sato (1984) is integral to our results, we begin by reviewing their basic setup and central result.

II Samuelson and Sato

The money-goods utility maximization problem of Samuelson and Sato [1984, Eq.(23)] under consideration is given by

$$V(r,\mathbf{P},Y) \stackrel{\text{def}}{=} \max_{M,\mathbf{X}} \{U[M, g(\mathbf{X}); \mathbf{P}] \quad \text{s.t.} \quad rM + \mathbf{P}'\mathbf{X} = Y\}, \tag{8}$$

where $M > 0$ is the nominal money balance, $r > 0$ is the interest rate, $Y > 0$ is the consumer's income, $\mathbf{X} \stackrel{\text{def}}{=} (x_1, x_2, \quad, x_n) \quad \in \mathbb{R}^n_+$ is the vector of real goods, $\mathbf{P} \stackrel{\text{def}}{=} (p_1, p_2, \quad, p_n) \quad \in \mathbb{R}^n_{++}$ is the vector of prices of the real goods, and $'$ denotes transposition. Note that we are following Samuelson and Sato's (1984) notation closely, diverging only in minor ways for the purpose of clarity. In model (8) the real goods $\mathbf{X}$ are assumed to be weakly separable with respect to both M and $\mathbf{P}$. To eliminate money illusion, the utility function is further assumed to be homogeneous of degree zero in the nominal money balance M and the prices $\mathbf{P}$. It is also assumed that the $C^{(2)}$ utility function is strictly increasing and strictly quasi-concave in $(M,\mathbf{X})$ for given $\mathbf{P}$. In addition, an interior solution to the money-goods utility maximization problem (8) is assumed.

Samuelson and Sato (1984) used a two-stage maximization process to obtain a modified Slutsky matrix involving the uncompensated demand functions for money and real goods, which we now briefly outline. Their first-stage maximization problem is

$$\max_{\mathbf{X}} \left\{ g(\mathbf{X}) \quad \text{s.t.} \quad \mathbf{P}'\mathbf{X} = \hat{Y} \right\}, \tag{9}$$

where $\hat{Y}$ is an arbitrary allocation of income for the real goods. The solution of problem (9) yields the *conditional* demand functions $\mathbf{H}[\,]$, with values $\mathbf{H}[\mathbf{P}, \hat{Y}]$. These demand functions obey all the prototypical properties of demand functions derived from the archetype model. Samuelson and Sato's second-stage maximization problem consists of

$$\max_{M,\hat{Y}} \left\{ U\left[M, g\left(\mathbf{H}[\mathbf{P}, \hat{Y}] \right); \mathbf{P} \right] \quad \text{s.t.} \quad rM + \hat{Y} = Y \right\}. \tag{10}$$

The solution of problem (10) yields the Marshallian *uncompensated* demand function for money, to wit $M(\,)$, with value $M(r, \mathbf{P}, Y)$, and the optimal allocation of income for the purchase of the real goods, namely $\hat{Y} = Y - rM(r, \mathbf{P}, Y)$. The Marshallian *uncompensated* demand functions $\mathbf{X}(\,)$ for the real goods, with values $\mathbf{X}(r, \mathbf{P}, Y)$, are the solution to problem (8) and the ultimate objects of interest, along with $M(r, \mathbf{P}, Y)$. Samuelson and Sato [1984, Eq. (31a)] showed that the values of $\mathbf{X}(\,)$ and $\mathbf{H}[\,]$ are related by the identity

$$\mathbf{X}(r, \mathbf{P}, Y) \equiv \mathbf{H}[\mathbf{P}, Y - rM(r, \mathbf{P}, Y)]. \tag{11}$$

Using identity (11) and the fact that the conditional demand functions $\mathbf{H}[\,]$ obey the archetype Slutsky properties, Samuelson and Sato [1984, Eq. (32b)] derived a modified Slutsky matrix $\mathbf{S}^{SS}$, with typical element given by

$$S_{ij}^{SS} \stackrel{\text{def}}{=} \left(\frac{\partial X^i}{\partial p_j} + \frac{\partial X^i}{\partial Y} X^j \right) - \frac{\left(\frac{\partial X^i}{\partial r} + \frac{\partial X^i}{\partial Y} M \right) \left(\frac{\partial M}{\partial p_j} + \frac{\partial M}{\partial Y} X^j \right)}{\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M}, \; i, j = 1, 2, \quad , n, \tag{12}$$

and showed that it is symmetric and negative semidefinite almost everywhere. Samuelson and Sato [1984, p. 595] also showed that the compensated slope of the money demand function is strictly negative almost everywhere, that is, $\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M < 0$ almost everywhere. Notice that the first squared bracket on the right-hand-side of Eq. (12) has the form of the traditional Slutsky

term in the archetypical model of the consumer. In Samuelson and Sato's model, however, this term is neither symmetric nor the matrix of which it is part is negative semidefinite. The second complex term on the right-hand-side of Eq. (12) is interpreted as the cash balances effect. Neither of the two terms have properties of their own but their combination as in Eq. (12) produces a symmetric and negative semidefinite matrix that can form the basis for an empirical verification of the money-goods model (8).

We now show that Pearce (1958), re-interpreted by Berglas and Razin (1974), anticipated and discussed precisely Samuelson and Sato's (1984) model (8). Moreover, using essentially the same two-stage maximization approach as Samuelson and Sato (1984), we demonstrate that Pearce (1958) and Berglas and Razin (1974) obtained empirically verifiable hypotheses that are *identical* to Samuelson and Sato's (1984) modified Slutsky matrix $\mathbf{S}^{SS}$. Pearce's (1958) and Berglas and Razin's (1974) comparative statics results differ only in appearance from that presented by Samuelson and Sato (1984) and given in in Eq. (12) here.

III Identical Forms of the Modified Slutsky Matrices

The goal of this section is to show that the modified Slutsky matrices of Pearce (1958, p. 57, eq. vi), Berglas and Razin (1974, p. 200, eq. 4), and Samuelson and Sato (1984, Eq. (32b) and our Eq. (12)) are identical. To this end, we begin by differentiating identity (11) with respect to $(r,\mathbf{P},Y)$ to get

$$\frac{\partial X^i}{\partial r} = -MH^i_{n+1} - rH^i_{n+1}\frac{\partial M}{\partial r},\ i=1,2,\quad ,n, \tag{13}$$

$$\frac{\partial X^i}{\partial p_j} = H^i_j - rH^i_{n+1}\frac{\partial M}{\partial p_j},\ i,j=1,2,\quad ,n, \tag{14}$$

$$\frac{\partial X^i}{\partial Y} = H^i_{n+1} - rH^i_{n+1}\frac{\partial M}{\partial Y},\ i=1,2,\quad ,n. \tag{15}$$

If $\left[1-r\frac{\partial M}{\partial Y}\right]\neq 0$, then we can solve Eq. (15) for H^i_{n+1} to obtain

$$H^i_{n+1} = \frac{\partial X^i}{\partial Y}\Big/\left[1-r\frac{\partial M}{\partial Y}\right],\ i=1,2,\quad ,n. \tag{16}$$

Next, compensate Eq. (13) with Eq. (15), that is compensate $\frac{\partial X^i}{\partial r}$ with $(\frac{\partial X^i}{\partial Y} M)$ and solve the resulting equation for H^i_{n+1}

$$H^i_{n+1} = -\left[\frac{\partial X^i}{\partial r} + \frac{\partial X^i}{\partial Y} M\right] \Big/ r\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M\right], \; i = 1,2, \quad ,n, \tag{17}$$

since $\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M < 0$ almost everywhere. Then substitute Eq. (17) into Eq. (15) to get

$$\begin{aligned} H^i_{n+1} &= \frac{\partial X^i}{\partial Y} + r H^i_{n+1} \frac{\partial M}{\partial Y} \\ &= \frac{\partial X^i}{\partial Y} - \left[\left[\frac{\partial X^i}{\partial r} + \frac{\partial X^i}{\partial Y} M\right] \frac{\partial M}{\partial Y}\right] \Big/ \left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M\right], \; i = 1,2, \quad ,n. \end{aligned} \tag{18}$$

We are now in a position to prove that the modified Slutsky matrix of Pearce (1958), restated by Berglas and Razin (1974), and that of Samuelson and Sato (1984, Eq. (32b)) corresponding to money-goods model (8) are identical. First, rearrange Eq. (14) to read

$$H^i_j = \frac{\partial X^i}{\partial p_j} + r H^i_{n+1} \frac{\partial M}{\partial p_j}, \; i, j = 1,2, \quad ,n, \tag{19}$$

$$H^i_j = \frac{\partial X^i}{\partial p_j} - \frac{\partial M}{\partial p_j}\left[\frac{\partial X^i}{\partial r} + \frac{\partial X^i}{\partial Y} M\right] \Big/ \left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M\right], \; i, j = 1,2, \quad ,n, \tag{20}$$

where Eq. (17) was used in deriving Eq. (20). Next, compensate Eq. (19) and Eq. (20) with the conditional income term $H^j H^i_{n+1}$ and use identity (11) to obtain the symmetric and negative semidefinite matrices

$$H^i_j + H^j H^i_{n+1} = \frac{\partial X^i}{\partial p_j} + H^i_{n+1}\left[X^j + r \frac{\partial M}{\partial p_j}\right], \; i, j = 1,2, \quad ,n, \tag{21}$$

$$H^i_j + H^j H^i_{n+1} = \frac{\partial X^i}{\partial p_j} - \frac{\partial M}{\partial p_j}\left[\frac{\partial X^i}{\partial r} + \frac{\partial X^i}{\partial Y} M\right] \Big/ \left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y} M\right] + X^j H^i_{n+1}, \quad i, j = 1,2, \quad ,n. \tag{22}$$

Finally, replace the conditional income effect H^i_{n+1} on the right-hand-side of Eq. (21) and Eq. (22) with the two forms given in Eqs. (16) and (18), respectively. This process yields, respectively, the symmetric and negative semidefinite modified Slutsky matrices

$$S_{ij}^{PBR} \stackrel{\text{def}}{=} \frac{\partial X^i}{\partial p_j} + \frac{\partial X^i}{\partial Y}\frac{\left[X^j + r\dfrac{\partial M}{\partial p_j}\right]}{\left[1 - r\dfrac{\partial M}{\partial Y}\right]}, \; i, j = 1,2, \quad ,n, \tag{23}$$

$$S_{ij}^{SS} \stackrel{\text{def}}{=} \left[\frac{\partial X^i}{\partial p_j} + \frac{\partial X^i}{\partial Y} X^j\right] - \frac{\left[\dfrac{\partial X^i}{\partial r} + \dfrac{\partial X^i}{\partial Y} M\right]\left[\dfrac{\partial M}{\partial p_j} + \dfrac{\partial M}{\partial Y} X^j\right]}{\left[\dfrac{\partial M}{\partial r} + \dfrac{\partial M}{\partial Y} M\right]}, \; i, j = 1,2, \quad ,n. \tag{24}$$

The modified Slutsky matrix $\mathbf{S}^{SS}$ is that derived by Samuelson and Sato [1984, Eq. (32b)] and given earlier in Eq. (12), while the modified Slutsky matrix $\mathbf{S}^{PBR}$ is that derived by Pearce (1958, p. 57, Eq. vi) and restated by Berglas and Razin [1974, Eq. (4)], who implicitly assumed a unitary interest rate. The failure of Samuelson and Sato (1984) to cite these two papers may be due to the ostensibly improbable equivalence between $\mathbf{S}^{PBR}$ and $\mathbf{S}^{SS}$.

Since the two expressions used for the conditional income effect H_{n+1}^i are identical, so too are the modified Slutsky matrices $\mathbf{S}^{SS}$ and $\mathbf{S}^{PBR}$. In the appendix, we provide a more detailed demonstration of this identity. Hence, we have established our main result that we summarize in

Proposition 1. *For the money-goods model* (8) *with two-stage maximization, the symmetric and negative semidefinite modified Slutsky matrices* $\mathbf{S}^{SS}$ *and* $\mathbf{S}^{PBR}$ *are identical if* $\left[1 - r\frac{\partial M}{\partial Y}\right] \neq 0$.

The lack of symmetry in $\mathbf{S}^{PBR}$ contrasts starkly with the elegance of $\mathbf{S}^{SS}$, which isolates a term corresponding to the traditional Slutsky matrix and relegates the money-balances effects to a second term. Moreover, in contrast to $\mathbf{S}^{PBR}$ the structure of $\mathbf{S}^{SS}$ makes it visually clear that the qualitative properties of the archetype consumer problem are contained as a special case of model (8). The symmetry and negative semidefiniteness of $\mathbf{S}^{SS}$ and $\mathbf{S}^{PBR}$ are only sufficient for the solution of problem (8). Their sufficiency stems from the two-stage maximization procedure

selected by all the authors. By solving problem (8) in one step, as done by Paris and Caputo (2001), it is possible to obtain necessary and sufficient conditions.

Because the two-stage optimization process yields only a sufficient condition for the solution of problem (8), another noteworthy feature of the proof of Proposition 1 is that if $\left[M + r\frac{\partial M}{\partial r}\right] \neq 0$, then one could solve Eq. (13) for the conditional income effect H^i_{n+1}, or use the expression for H^i_{n+1} given in Eq. (17), and substitute these into Eq. (21) to produce two more identical modified Slutsky matrices. In fact, the astute reader will notice that there are at least two more identical modified Slutsky matrices. They can be derived by solving Eq. (13) for the conditional income effect H^i_{n+1} (assuming that $\left[M + r\frac{\partial M}{\partial r}\right] \neq 0$) and substituting it into Eq. (15) to derive a new expression for H^i_{n+1}, or substituting Eq. (16) into Eq. (15) to derive another expression for H^i_{n+1}. These two expressions for H^i_{n+1} can then be substituted into Eq. (21) to produce two more identical modified Slutsky matrices. In fact, the reader will undoubtedly notice that there are several other expressions one can construct from Eqs. (13)-(15), which when substituted in Eqs. (21) and (22) will yield several other identical expressions for the modified Slutsky matrix. We leave the algebraic details of these derivations to the interested readers but wish to emphasize that these numerous and identical modified Slutsky matrices resulting from the two-stage maximization process have important implications for the empirical testing of model (8), which we address below.

IV Discussion and Conclusion

We have shown that the modified Slutsky matrix of Samuelson and Sato (1984) is identical to that derived twenty six years before by Pearce (1958) and restated sixteen years later by Berglas and Razin (1974) even though, superficially, the two matrices appear to be very different. Moreover, we have shown that there are at least six identical forms of the modified Slutsky matrix for the money-goods problem (8). This conclusion carries serious practical consequences if one is interested in carrying out a legitimate empirical test of problem (8). For example, not only would one have to test for the symmetry and negative semidefiniteness of each of the six

modified Slutsky matrices, but one would also have to test for the equality of all six matrices. Given the complexity of the six modified Slutsky matrices, this would be a cumbersome econometric task indeed. Fortunately, the recent results of Paris and Caputo (2001) permit one to overcome this practical difficulty. Their main result is obtained by solving problem (8) in one step yielding the following symmetric negative semidefinite matrix

$$\mathbf{S}_{22} = \left[\frac{\partial \mathbf{X}}{\partial \mathbf{P}} + \frac{\partial \mathbf{X}}{\partial Y}\mathbf{X}'\right] - \frac{\left[\frac{\partial \mathbf{X}}{\partial r} + \frac{\partial \mathbf{X}}{\partial Y}M\right]\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y}M\right]} + \frac{\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y}\mathbf{X}'\right]'\left[\frac{\partial M}{\partial \mathbf{P}} + \frac{\partial M}{\partial Y}\mathbf{X}'\right]}{\left[\frac{\partial M}{\partial r} + \frac{\partial M}{\partial Y}M\right]}. \quad (25)$$

Paris and Caputo (2001) prove that the modified Slutsky matrix of Samuelson and Sato (24) is a *special case* of their generalized Slutsky matrix (25) for money-goods problem (8). In other words, negative semidefiniteness of Samuelson and Sato's (1984) modified Slutsky matrix is sufficient, but not necessary, for negative semidefiniteness of Paris and Caputo's (2001) generalized Slutsky matrix. Hence, rather than adopt the aforementioned procedure involving six identical matrices for testing the empirical validity of the money-goods model (8), which is just a set of sufficient conditions anyway, one can test for the symmetry and negative semidefiniteness of Paris and Caputo's (2001) generalized Slutsky matrix, which is the most general *necessary* implication of the model (8) available at this time.

A further important conclusion of this analysis is that, from the perspective of comparative statics, a two-stage maximization approach to the consumer problem (8) is not, in general, equivalent to a one-step solution of the problem. The two-stage approach, although equivalent to a one-step solution as far as reaching the same optimal value of the decision variables and objective function, misses crucial information dealing with the symmetry of compensated derivatives that appear only in the one-step methodology. Hence, the multiplicity of forms of the modified Slutsky matrix discussed above has its origin precisely in the two-stage maximization approach. For this reason, this approach must be regarded as an inferior methodology for analyzing issues of comparative statics.

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Appendix

Two functional forms are identical if they can be transformed into each other by mathematical manipulations. In this appendix, we will show that the matrix derived by Pearce (1958) and given in this paper by Eq. (23) can be transformed into the matrix derived by Samuelson and Sato (1984) and given in this paper by Eq. (24).

We start by equating Eq. (16) with Eq. (17) and cross multiply the denominators to produce

$$r\frac{\partial X^i}{\partial Y}\left[\frac{\partial M}{\partial r}+\frac{\partial M}{\partial Y}M\right]=-\left[\frac{\partial X^i}{\partial r}+\frac{\partial X^i}{\partial Y}M\right]\left[1-r\frac{\partial M}{\partial Y}\right] \tag{26}$$

and

$$r\frac{\partial X^i}{\partial Y}=\frac{-\left[\frac{\partial X^i}{\partial r}+\frac{\partial X^i}{\partial Y}M\right]\left[1-r\frac{\partial M}{\partial Y}\right]}{\left[\frac{\partial M}{\partial r}+\frac{\partial M}{\partial Y}M\right]}. \tag{27}$$

Consider now Pearce's matrix (23)

$$S_{ij}^{PBR}\overset{\text{def}}{=}\frac{\partial X^i}{\partial p_j}+\frac{\partial X^i}{\partial Y}\frac{\left[X^j+r\frac{\partial M}{\partial p_j}\right]}{\left[1-r\frac{\partial M}{\partial Y}\right]}, \tag{28}$$

add and subtract the term $\dfrac{\partial X^i}{\partial Y}X^j$

$$S_{ij}^{PBR}=\frac{\partial X^i}{\partial p_j}+\frac{\partial X^i}{\partial Y}X^j-\frac{\partial X^i}{\partial Y}X^j+\frac{\partial X^i}{\partial Y}\frac{\left[X^j+r\frac{\partial M}{\partial p_j}\right]}{\left[1-r\frac{\partial M}{\partial Y}\right]}, \tag{29}$$

rearrange and cancel terms

$$S_{ij}^{PBR}=\frac{\partial X^i}{\partial p_j}+\frac{\partial X^i}{\partial Y}X^j+r\frac{\partial X^i}{\partial Y}\frac{\left[\frac{\partial M}{\partial p_j}+\frac{\partial M}{\partial Y}X^j\right]}{\left[1-r\frac{\partial M}{\partial Y}\right]}, \tag{30}$$

and, finally, replace $r\dfrac{\partial X^i}{\partial Y}$ with its equivalent expression given in Eq. (27) to obtain Samuelson and Sato's (1984) equation (24):

$$S_{ij}^{PBR} = \frac{\partial X^i}{\partial p_j} + \frac{\partial X^i}{\partial Y} X^j - \frac{\left[\dfrac{\partial X^i}{\partial r} + \dfrac{\partial X^i}{\partial Y} M\right]\left[\dfrac{\partial M}{\partial p_j} + \dfrac{\partial M}{\partial Y} X^j\right]}{\left[\dfrac{\partial M}{\partial r} + \dfrac{\partial M}{\partial Y} M\right]} = S_{ij}^{SS}. \tag{31}$$