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Narrow Feshbach Bound State of an Ultracold Atom Pair with an Extrapolation to a Many Body Phase

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Physics

by

Nicolas Lopez

December 2013

Dissertation Committee:
Dr. Shan-Wen Tsai, Chairperson
Dr.Vivek Aji
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The Dissertation of Nicolas Lopez is approved:

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Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

Narrow Feshbach Bound State of an Ultracold Atom Pair with an Extrapolation to a Many Body Phase

by

Nicolas Lopez

Doctor of Philosophy, Graduate Program in Physics
University of California, Riverside, December 2013
Dr. Shan-Wen Tsai, Chairperson

Trapped cold atoms at very low temperatures, now readily accessible experimentally, can be controlled and serve as a testing ground for many theoretical physical properties which are paralleled in condensed matter systems. Ultracold trapped atom experiments show promise to recreate analogous condensed matter systems in a controlled environment and to create novel many-body quantum states of matter. The tunability of trapping laser intesities and magnetically controlled Feshbach interactions are the main tools for varying lattice hopping strengths and inter-particle interactions. In this thesis, we search for the phase of a many-body system of cold fermion atoms that can convert to boson molecules and vice a versa. We then look at the microscopic details of two atoms that interact to form a quasibound molecule. Our two-atom model explains the mechanism behind the magnetically tunable interparticle interaction known as the Feshbach resonance. We find a dependence of the

bound pair formation on the two-particle energy. Our results show that if a magnetic field can be fine tuned to sufficient precision, of order 10 mG, a narrow resonance can be achieved where longer lived bound states can be created. This narrow resonance reconciliates the two-body physics with the effective Hamiltonian that describes the many-body system.

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Chapter 1

Two Particle Feshbach Resonance

1.1 Introduction

Cold atom experiments that employ optical lattices use lasers to confine the particles at the lattice points. The separation between lattice points can be large enough and the laser intensities can be strong enough to suppress lattice site hopping and inter-site interactions. In this chapter we consider two neutral alkali atoms of the same species trapped at a lattice site by a harmonic confining potential field provided by a laser. We assume this lattice site has no interaction with it's environment so it is considered an isolated system. We allow the two trapped atoms to interact through a tunable short ranged magnetic interaction referred to as the magnetic Feshbach resonance [4, 17].

A system of two neutral alkali atoms in a magnetic field has an infinite number of Hilbert spaces given the relative-distance dependent magnetic interactions. The reason for this is

that the spin arrangement of the atoms depends on their relative distance. For this reason we choose a relative distance dependent spin basis that diagonalizes the inter-atomic potential. This in turn causes the kinetic energy part of the Hamiltonian to have off-diagonal terms. These off-diagonal terms will give rise to a pair of Schrodinger-like coupled equations that describe a two-atom scattering potential known as the "open channel" and a potential that supports a bound state known as the "closed channel". The closed channel equation has the inter-atomic potential that a pair of oppositely spin-aligned atoms experience. The open channel equation has a potential for a pair of spin-aligned atoms. These two-atom states (bound and scattering) are completely uncoupled at far relative distances. A scattering sate can become coupled to a bound state at short distances (in the order of nanometers) by a spin flip where both atoms become oppositely spin-aligned. This situation is facilitated by an external magnetic field that tunes the the triplet potential so that the two-atom relative energy is near the energy of a supported bound state. When the two-atom state becomes a bound state or a quasi-bound state (close to a bound state) as a result of the inter-atomic potential tuning, it is called a Feshbach resonance.

How can we determine that a two-atom state is in a Feshbach resonance? Atomic resonances are usually observed when the atomic scattering length diverges $a \to \infty$ (See the Appendix for a background on scattering lengths). The boundary conditions are that at far relative distances $r \to \infty$, the two-atom state is a scattering state and non-existent at $r = 0$. As a result, we solve the coupled equations for an energy dependent scattering length $a(E)$.

At this point we solve the energy spectrum of the two interacting neutral atoms placed in a harmonic trap by matching the energy dependent scattering length derived from the coupled equations to the scattering length of two delta-interaction particles in a harmonic trap as solved by Thomas Busch, et. al. [9, 10]. Our result is a magnetic field dependent energy spectrum of the two-neutral atoms in the harmonic trap.

Though Feshbach resonances have been used extensively in cold atom experiments [12, 13, 5, 6, 2, 3, 14]and have had a sufficient theoretical explanation [4, 16, 17], the energy dependent scattering length makes a distinction between the traditional Feshbach resonance and the sensitive narrow Feshbach resonance. The energy dependence allows for a fined tuned magnetic field (in the order of 10 mGauss) to pinpoint a quasi bound state with a higher probability than the broad Feshbach resonance (resolution $> 100mGauss$).

Perhaps the most interesting result of this research is that the bound state amplitude expression shows a dependence on the external magnetic field resolution. The higher the resolution, the larger the bound state amplitude. We show that magnetic fields of broad Feshbach resonances give low bound state occupation numbers for the harmonically trapped two-atom system.

1.2 Two-Atom System

We consider two neutral alkali atoms at relative distance $r = r_1 - r_2$, with masses $m_1 = m_2 = m$ placed in a magnetic field. Not considering the trapping harmonic potential, the Hamiltonian is

$$H = KE + V(r)$$

where KE is the kinetic energy operator for the two particles

$$KE = -\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2$$

and the interaction potential $V(r)$ depends on two-atom spinor components (electronic and nuclear spins).

1.3 Magnetic Effects

We consider three magnetic effects when studying the two atom system. One effect is the hyperfine interaction which is the result of each valence electron being in the presence of the magnetic field of its own atom's nucleus. A second effect is the Zeeman effect which is the interaction of each atom with an external magnetic field B. The third magnetic effect is the inter-atom electron spin-spin (exchange) interaction (each valence electron influenced by the other valence electron's magnetic field).

1.3.1 The hyperfine energy

The hyperfine energy in units of $\hbar$

$$H_{hf} = a_{hf}(\vec{s}_1 \cdot \vec{i}_1 + \vec{s}_2 \cdot \vec{i}_2)$$

is the dot product of each electron's spin $\vec{s}$ with it's nuclear spin $\vec{i}$ with a_{hf} as the hyperfine constant.

1.3.2 Zeeman Effect

The energy of the interaction of the nuclei and valence electrons of each atom with an external magnetic field $\vec{B}$ is

$$H_Z = 2\mu_B(\vec{s}_1 + \vec{s}_2) \cdot \vec{B} - \mu_N(\vec{i}_1 + \vec{i}_2) \cdot \vec{B}$$

and μ_B and μ_N as the Bohr and nuclear magneton.

1.3.3 Exchange Interaction

The atom-atom interaction will be a Van der Waals potential with an electron spin dependent part. We can write this as

$$H_{exch} = V_{S=0}(r)\hat{\Pi}_0 + V_{S=1}(r)\hat{\Pi}_1$$

where $\hat{\Pi}_1$ and $\hat{\Pi}_0$ are projection operators of triplet and singlet two-atom spin states due to the electron spin configuration. $V_S(r)$ is the space dependent potential. The spin projection operators are $\hat{\Pi}_1 = \frac{3}{4} + \vec{s}_1 \cdot \vec{s}_2$ and $\hat{\Pi}_0 = \frac{1}{4} - \vec{s}_1 \cdot \vec{s}_2$, therefore

$$H_{exch} = \frac{V_{S=0}(r) + 3V_{S=1}(r)}{4} + (V_{S=1}(r) - V_{S=0}(r))\vec{s}_1 \cdot \vec{s}_2$$

We refer to the total of these magnetic effects as $V(r)$

$$V(r) = H_{hf} + H_Z + H_{exch}$$

1.4 Spin Basis

At large distances $r \gg R$, where the Van der Waals potential dies off, the hyperfine interaction will be the dominant magnetic effect. In this case, it would be useful to choose a total atom spin basis $|f_1, m_{f1}; f_2, m_{f2}\rangle$ where the total spin of each atom is just the electron spin plus nuclear spin $\vec{F}_i = \vec{s}_i + \vec{i}_i$.

$V(r)$ is simplified

$$V(r) = a_{hf}(\vec{s}_1 \cdot \vec{i}_1 + \vec{s}_2 \cdot \vec{i}_2) + 2\mu_B(\vec{s}_1 + \vec{s}_2) \cdot \vec{B} - \mu_N(\vec{i}_1 + \vec{i}_2) \cdot \vec{B}$$

$$V(r) = a_{hf}(\vec{s}_1 \cdot \vec{i}_1 + \vec{s}_2 \cdot \vec{i}_2) + 2\mu_B B_z(s_{1z} + s_{2z}) - \mu_N B_z(i_1 + i_2)$$

Where $\vec{s}_1, \vec{s}_2$ and $\vec{i}_1, \vec{i}_2$ refers to the single valence electron of each atom in the ground state ($L = 0$) and the nucleus of each atom respectively. Good quantum numbers for each atom (Operators that commute with $\vec{s}_j \cdot \vec{i}_j$ for i, j=1,2) in this asymptotic limit are the total atom spin $F = s + i$ and its projection onto the z-axis m_F. The binary-atom basis is $|f_1, m_{f1}; f_2, m_{f2}\rangle$. Note that this basis does not diagonalize H_{exch} since F does not commute with S^2. Therefore we cannot adopt this basis for shorter distances.

At shorter distances where the Van der Waals interaction becomes larger than the hyperfine interaction, it would be useful to choose an operator that commutes with S^2 like the total electron spin operator $\vec{S} = \vec{s}_1 + \vec{s}_2$ with basis $|s_1 + s_2\rangle$. But at these distances, the hyperfine interaction would not be diagonalized.

The interaction of the two atoms occurs through a spin dependent and space dependent potential $V(r)$ where $r = |r_1 - r_2|$. In order to diagonalize this spin dependent potential matrix, we switch to an adiabatic basis $|\alpha(r)\rangle$ which is a distance dependent basis that diagonalizes the spin dependent potential at every relative distance r. The adiabatic basis is such that

$$V(r)|\alpha(r)\rangle = \varepsilon_\alpha|\alpha(r)\rangle$$

where ε_α is the eigen energy and $|\alpha(r)\rangle$is a spinor that diagonalizes the potential $V(r)$ at all distances and specifies the "channel". For our description of the narrow Feshbach resonance we will only use two channels–the open and closed channels which refer to a scattering state and a bound state respectively.

The kinetic energy part of the Hamiltonian is not diagonalized by the adiabatic basis but will serve to couple different spinor $|\alpha(r)\rangle$ states. The radial part of the Hamiltonian leads to wavefunction solution of the usual form $\frac{U_\alpha(r)}{r}$ multiplied by a space dependent spin wavefunction $|\alpha(r)\rangle$as given above. The two-atom states are superpositions of these wavefunctions

$$\Psi(r) = \sum_\alpha \frac{\chi_\alpha(r)}{r\sqrt{4\pi}}|\alpha(r)\rangle$$

In general there may be many channels, but we will restrict the channels to only 2. Using the open $\chi_o(r)$ and closed $\chi_c(r)$ channel wavefunctions and spinors.

$$\Psi(r) = \frac{1}{r}\frac{1}{\sqrt{4\pi}}[\chi_o(r)|o(r)\rangle + \chi_c(r)|c(r)\rangle]$$

1.5 Average Kinetic Energy Operator

1.5.1 General Expression for Kinetic Energy

Given the two-atom state as superposition of spatial wavefunctions times spinor wavefunctions $\Psi(r) = \sum_\alpha \frac{\chi_\alpha(r)}{r\sqrt{4\pi}}|\alpha(r)\rangle$, we can find an expression for the average kinetic energy.

$$\langle\hat{K}\rangle = -\frac{\hbar^2}{2\mu}\int d^3r\Psi^*(r)\nabla_r^2\Psi(r)$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{4\pi}\sum_{\alpha,\beta}\int d^3r\langle\alpha(r)|\frac{\chi^*_\alpha(r)}{r}\nabla^2_r\frac{\chi_\beta(r)}{r}|\beta(r)\rangle$$

We integrate the angular part and replace the Laplacian with only its radial part $\nabla^2_r \to \frac{1}{r}\frac{\partial^2}{\partial r^2}r$

$$= -\frac{\hbar^2}{2\mu}\sum_{\alpha,\beta}\int_0^\infty drr^2\langle\alpha(r)|\frac{\chi^*_\alpha(r)}{r}\frac{1}{r}\frac{\partial^2}{\partial r^2}r\frac{\chi_\beta(r)}{r}|\beta(r)\rangle$$

$$\langle\hat{K}\rangle = -\frac{\hbar^2}{2\mu}\sum_{\alpha,\beta}\int_0^\infty dr\langle\alpha(r)|\chi^*_\alpha(r)\frac{\partial^2}{\partial r^2}\chi_\beta(r)|\beta(r)\rangle$$

Then, integrating by parts

$$\langle\hat{K}\rangle = -\frac{\hbar^2}{2\mu}\sum_{\alpha,\beta}[\int_0^\infty dr\frac{d}{dr}(\langle\alpha(r)|\chi^*_\alpha(r)\frac{d}{dr}\chi_\beta(r)|\beta(r)\rangle) -$$

$$\int_0^\infty dr\frac{d}{dr}(\langle\alpha(r)|\chi^*_\alpha(r))\frac{d}{dr}(\chi_\beta(r)|\beta(r)\rangle)]$$

but $\chi_\alpha(0) = \chi_\alpha(\infty) = 0$ since we assume the two particles cannot be at the same location ($r = 0$) and the wavefunction $\chi_\alpha(r)$ is normalizable. Therefore

$$\langle\hat{K}\rangle = \frac{\hbar^2}{2\mu}\sum_{\alpha,\beta}\int_0^\infty dr\frac{d}{dr}(\langle\alpha(r)|\chi^*_\alpha(r))\frac{d}{dr}(\chi_\beta(r)|\beta(r)\rangle)$$

Keeping in mind orthogonality of spin states,

$$\langle \alpha(r) | \beta(r) \rangle = \delta_{\alpha,\beta}$$

We expand the derivative

$$\langle \hat{K} \rangle = \frac{\hbar^2}{2\mu} \sum_{\alpha} \int_0^{\infty} dr \frac{d\chi_{\alpha}^*(r)}{dr} \frac{d\chi_{\alpha}(r)}{dr} +$$

$$\frac{\hbar^2}{2\mu} \sum_{\alpha,\beta} \int_0^{\infty} dr \chi_{\alpha}^*(r) \frac{d\chi_{\beta}(r)}{dr} \langle \frac{d\alpha(r)}{dr} | \beta(r) \rangle +$$

$$\frac{\hbar^2}{2\mu} \sum_{\alpha,\beta} \int_0^{\infty} dr \frac{d\chi_{\alpha}^*(r)}{dr} \chi_{\beta}(r) \langle \alpha(r) | \frac{d\beta(r)}{dr} \rangle +$$

$$\frac{\hbar^2}{2\mu} \sum_{\alpha,\beta} \int_0^{\infty} dr \chi_{\alpha}^*(r) \chi_{\beta}(r) \langle \frac{d\alpha(r)}{dr} | \frac{d\beta(r)}{dr} \rangle$$

1.5.2 Average Kinetic Energy with Two Basis Wavefunctions–The Open and Closed Channels

Let $|o\rangle$ denote the open channel spinor and let $|c\rangle$ denote the closed channel spinor. The kinetic energy operator in this case is

$$\langle \hat{K} \rangle = \frac{\hbar^2}{2\mu} \int_0^\infty dr \frac{d}{dr}(\Psi^*(r)) \frac{d}{dr}(\Psi(r))$$

We assume the wavefunction is a superposition of only two basis functions

$$\Psi(r) = \frac{1}{\sqrt{4\pi}}(\frac{\chi_o(r)}{r}|o(r)\rangle + \frac{\chi_c(r)}{r}|c(r)\rangle)$$

The kinetic energy is

$$\langle \hat{K} \rangle = \frac{\hbar^2}{2\mu} \int_0^\infty dr[\langle \frac{do(r)}{dr}|\chi_o^*(r) + \langle o(r)|\frac{d\chi_o^*(r)}{dr} + \langle \frac{dc(r)}{dr}|\chi_c^*(r) + \langle c(r)|\frac{d\chi_c^*(r)}{dr}]\times$$

$$[\chi_o(r)|\frac{do(r)}{dr}\rangle + \frac{d\chi_o(r)}{dr}|o(r)\rangle + \chi_c(r)|\frac{dc(r)}{dr}\rangle + \frac{d\chi_c(r)}{dr}|c(r)\rangle]$$

$$\langle \hat{K} \rangle = \frac{\hbar^2}{2\mu} \int_0^\infty dr\{\frac{d\chi_o^*(r)}{dr}\frac{d\chi_o(r)}{dr} + \frac{d\chi_c^*(r)}{dr}\frac{d\chi_c(r)}{dr}\}+$$

$$\frac{\hbar^2}{2\mu} \int_0^\infty dr\{\chi_o^*(r)\frac{d\chi_o(r)}{dr}\langle \frac{do(r)}{dr}|o(r)\rangle + \chi_o^*(r)\frac{d\chi_c(r)}{dr}\langle \frac{do(r)}{dr}|c(r)\rangle+$$

$$\chi_c^*(r)\frac{d\chi_o(r)}{dr}\langle \frac{dc(r)}{dr}|o(r)\rangle + \chi_c^*(r)\frac{d\chi_c(r)}{dr}\langle \frac{dc(r)}{dr}|c(r)\rangle\}+$$

$$\frac{\hbar^2}{2\mu}\int_0^\infty dr\{\frac{d\chi_o^*(r)}{dr}\chi_o(r)\langle o(r)|\frac{do(r)}{dr}\rangle+\frac{d\chi_o^*(r)}{dr}\chi_c(r)\langle o(r)|\frac{dc(r)}{dr}\rangle+$$

$$\frac{d\chi_c^*(r)}{dr}\chi_o(r)\langle c(r)|\frac{do(r)}{dr}\rangle+\frac{d\chi_c^*(r)}{dr}\chi_c(r)\langle c(r)|\frac{dc(r)}{dr}\rangle\}$$

Since $\langle\frac{d\alpha(r)}{dr}|\alpha(r)\rangle=-\langle\alpha(r)|\frac{d\alpha(r)}{dr}\rangle$and $\langle\frac{d\alpha(r)}{dr}|\alpha(r)\rangle=\langle\alpha(r)|\frac{d\alpha(r)}{dr}\rangle$implies that $\langle\frac{d\alpha(r)}{dr}|\alpha(r)\rangle=0$, we can simplify the expression for $\langle K\rangle$,

$$\langle\hat{K}\rangle=\frac{\hbar^2}{2\mu}\int_0^\infty dr\{\frac{d\chi_o^*(r)}{dr}\frac{d\chi_o(r)}{dr}+\frac{d\chi_c^*(r)}{dr}\frac{d\chi_c(r)}{dr}\}+$$

$$\frac{\hbar^2}{2\mu}\int_0^\infty dr\{\chi_o^*(r)\frac{d\chi_c(r)}{dr}\langle\frac{do(r)}{dr}|c(r)\rangle+\chi_c^*(r)\frac{d\chi_o(r)}{dr}\langle\frac{dc(r)}{dr}|o(r)\rangle$$

$$\frac{\hbar^2}{2\mu}\int_0^\infty dr\{\frac{d\chi_o^*(r)}{dr}\chi_c(r)\langle o(r)|\frac{dc(r)}{dr}\rangle+\frac{d\chi_c^*(r)}{dr}\chi_o(r)\langle c(r)|\frac{do(r)}{dr}\rangle\}$$

+terms with two spinor function derivatives which will be omitted

1.5.3 Kinetic Energy Matrix

Let's define the derivative coupling

$$a(r)=\langle o(r)|\frac{dc(r)}{dr}\rangle$$

which is equivalent to

$$a(r) = -\langle \frac{do(r)}{dr} | c(r) \rangle$$

We establish properties of the conjugate of $a(r)$

$$a(r)* = a(r)$$

by

$$a(r) = \langle o(r) | \frac{dc(r)}{dr} \rangle = \langle \frac{dc(r)}{dr} | o(r) \rangle$$

$$-a(r) = \langle \frac{do(r)}{dr} | c(r) \rangle = \langle c(r) | \frac{do(r)}{dr} \rangle$$

$$\langle \hat{K} \rangle = \frac{\hbar^2}{2\mu} \int_0^\infty dr \{ \frac{d\chi_o^*(r)}{dr} \frac{d\chi_o(r)}{dr} + \frac{d\chi_c^*(r)}{dr} \frac{d\chi_c(r)}{dr} \} +$$

$$+ \frac{\hbar^2}{2\mu} \int_0^\infty dr \{ \chi_o^*(r) \frac{d\chi_c(r)}{dr} \langle \frac{do(r)}{dr} | c(r) \rangle + \chi_c^*(r) \frac{d\chi_o(r)}{dr} \langle \frac{dc(r)}{dr} | o(r) \rangle$$

$$+ \frac{\hbar^2}{2\mu} \int_0^\infty dr \{ \frac{d\chi_o^*(r)}{dr} \chi_c(r) \langle o(r) | \frac{dc(r)}{dr} \rangle + \frac{d\chi_c^*(r)}{dr} \chi_o(r) \langle c(r) | \frac{do(r)}{dr} \rangle \}$$

The average kinetic energy becomes

$$\langle \hat{K} \rangle = \frac{\hbar^2}{2\mu} \int_0^\infty dr \{ \frac{d\chi_o^*(r)}{dr} \frac{d\chi_o(r)}{dr} + \frac{d\chi_c^*(r)}{dr} \frac{d\chi_c(r)}{dr} \} +$$

$$\frac{\hbar^2}{2\mu} \int_0^\infty dr \{ -\chi_o^*(r) a(r) \frac{d\chi_c(r)}{dr} + \chi_c^*(r) a(r) \frac{d\chi_o(r)}{dr} + \frac{d\chi_o^*(r)}{dr} a(r) \chi_c(r) - \frac{d\chi_c^*(r)}{dr} a(r) \chi_o(r) \}$$

We can try to evaluate and rearrange these integrals in order to extract a matrix that operates on the vector $\begin{bmatrix} \chi_o \\ \\ \chi_c \end{bmatrix}$ and its conjugate $\begin{bmatrix} \chi_o^* & \chi_c^* \end{bmatrix}$. We integrate the first two and the last two integrals by parts so that χ_o^* and χ_c^* are not operated by derivatives

$$\langle \hat{K} \rangle = -\frac{\hbar^2}{2\mu} \int_0^\infty dr \{ \chi_o^*(r) \frac{d^2\chi_o(r)}{dr^2} + \chi_c^*(r) \frac{d^2\chi_c(r)}{dr^2} \} +$$

$$\frac{\hbar^2}{2\mu} \int_0^\infty dr \{ -\chi_o^*(r) a(r) \frac{d\chi_c(r)}{dr} + \chi_c^*(r) a(r) \frac{d\chi_o(r)}{dr} +$$

$$-\chi_o^*(r) \frac{d[a(r)\chi_c(r)]}{dr} + \chi_c^*(r) \frac{d[a(r)\chi_c(r)]}{dr} \}$$

and then rearrange

$$\langle \hat{K} \rangle = -\frac{\hbar^2}{2\mu} \int_0^\infty dr \{ \chi_o^*(r) \frac{d^2\chi_o(r)}{dr^2} + \chi_c^*(r) \frac{d^2\chi_c(r)}{dr^2} \} +$$

$$-\frac{\hbar^2}{2\mu}\int_0^\infty dr\{\chi_o^*(r)\frac{d[a(r)\chi_c(r)]}{dr}+\chi_o^*(r)a(r)\frac{d\chi_c(r)}{dr}\}+$$

$$+\frac{\hbar^2}{2\mu}\int_0^\infty dr\{\chi_c^*(r)\frac{d[a(r)\chi_o(r)]}{dr}+\chi_c^*(r)a(r)\frac{d\chi_o(r)}{dr}\}$$

We can write the average kinetic energy in matrix form

$$\langle\hat{K}\rangle=-\frac{\hbar^2}{2\mu}\int_0^\infty dr\begin{bmatrix}\chi_o^* & \chi_c^*\end{bmatrix}\begin{bmatrix}\frac{d}{dr} & a\\ -a & \frac{d}{dr}\end{bmatrix}\begin{bmatrix}\frac{d}{dr} & a\\ -a & \frac{d}{dr}\end{bmatrix}\begin{bmatrix}\chi_o\\ \chi_c\end{bmatrix}+$$

$$-\frac{\hbar^2}{2\mu}\int_0^\infty dr\begin{bmatrix}\chi_o^* & \chi_c^*\end{bmatrix}\begin{bmatrix}a^2 & 0\\ 0 & a^2\end{bmatrix}\begin{bmatrix}\chi_o\\ \chi_c\end{bmatrix}$$

1.6 Open and Closed Channel Coupled Equations

In the adiabatic basis , there are two $|\alpha(r)\rangle$ states that become important near a Feshbach resonance. These two states are called the open channel $\chi_o(r)$ and the closed (bound state) channel $\chi_c(r)$ with the condition that the two-particle system is in a superposition of scattering and bound states$\int dr[|\chi_o(r)|^2+|\chi_c(r)|^2]=1$. One of our goals is to determine the amplitude of the bound state $\chi_c(r)$ (it is called the quasi-bound state because the collision energy is positive. In a true bound state the binding energy is negative). The Feshbach res-

onance will occur when the eigenenergy of the the two-atom system in the continuum (limit $r \to \infty$) is equal to or very near the energy of a closed channel bound state.

The average energy of the two-atom system is

$$\langle H \rangle = \int_0^\infty dr e$$

$$e = \frac{\hbar^2}{2\mu}\{\frac{d\chi_o^*(r)}{dr}\frac{d\chi_o(r)}{dr} + \frac{d\chi_c^*(r)}{dr}\frac{d\chi_c(r)}{dr}\}+$$

$$+\frac{\hbar^2}{2\mu}\{\chi_c^*(r)\frac{d[a(r)\chi_o(r)]}{dr} + \chi_c^*(r)a(r)\frac{d\chi_o(r)}{dr}\}+$$

$$-\{\chi_o^*(r)\frac{d[a(r)\chi_c(r)]}{dr} + \chi_o^*(r)a(r)\frac{d\chi_c(r)}{dr}\}+$$

$$+v_o\chi_o^*(r)\chi_o(r) + v_c\chi_c^*(r)\chi_c(r)$$

with v_o and v_c as the open and closed potentials. We use the method of lagrange multipliers.

$$\nabla\langle H \rangle = \lambda \nabla g$$

where $\lambda = E$, and $g = \int_0^\infty [\chi_o^*(r)\chi_o(r)+\chi_c^*(r)\chi_c(r)]-1$ is the constraint function(normalization condition).

We minimize the average energy with respect to $\chi_o^*(r)$ and $\chi_c^*(r)$.

$$\frac{\delta}{\delta\chi_o^*}(\langle H\rangle - E(\int_0^\infty [\chi_o^*(r)\chi_o(r) + \chi_c^*(r)\chi_c(r)] - 1) = 0$$

$$\frac{\delta}{\delta\chi_c^*}(\langle H\rangle - E(\int_0^\infty [\chi_o^*(r)\chi_o(r) + \chi_c^*(r)\chi_o(r)] - 1]) = 0$$

The Euler equation gives the minimization

$$\frac{\delta H}{\delta\chi_0^*} = \frac{de}{d\chi_o^*} - \frac{d}{dr}(\frac{de}{d\frac{d\chi_o^*}{dr}})$$

gives the open channel equation

$$E\chi_o(r) = \frac{\hbar^2}{2\mu}\{-\frac{d[a(r)\chi_c(r)]}{dr} - a(r)\frac{d\chi_c(r)}{dr} + v_o\chi_o(r) - \frac{d^2\chi_o(r)}{d^2r}\}$$

$$E\chi_o(r) = -\frac{\hbar^2}{2\mu}\frac{d^2\chi_o(r)}{d^2r} + v_o\chi_o(r) - \frac{\hbar^2}{2\mu}\frac{da(r)}{dr}\chi_c(r) - \frac{\hbar^2}{\mu}a(r)\frac{d\chi_c(r)}{dr}$$

and

$$\frac{\delta H}{\delta\chi_c^*} = \frac{de}{d\chi_c^*} - \frac{d}{dr}(\frac{de}{d\frac{d\chi_c^*}{dr}})$$

gives the closed channel equation

$$E\chi_c(r) = -\frac{\hbar^2}{2\mu}\frac{d^2\chi_c(r)}{d^2r} + v_c\chi_c(r) + \frac{\hbar^2}{2\mu}\frac{da(r)}{dr}\chi_o(r) + \frac{\hbar^2}{\mu}a(r)\frac{d\chi_o(r)}{dr}$$

Notice that the closed and open channel equations are coupled by the term $g(r) = \frac{\hbar^2}{2\mu}\frac{da(r)}{dr} + \frac{\hbar^2}{\mu}a(r)\frac{d}{dr}$.

1.7 Delta Coupling Approximation to Coupled Equations

The coupled Schrodinger equations are

$$Eu_o(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_o(r)]u_o(r) + g(r)u_c(r) \tag{1.1}$$

and

$$Eu_c(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_c]u_c(r) + g^*(r)u_o(r) \tag{1.2}$$

with open channel potential V_o, closed channel potential V_c, and the coupling function $g(r)$. These coupled equations give an amplitude $\langle u_c(r)|g(r)|u_0(r)\rangle \neq 0$. In other words, $g(r)$ determines the strength of the free colliding atoms to go into a bound state.

Let's assume there exists a bound wavefunction supported by the closed channel potential V_c. We denote the energy of this quasi-bound state by ε . The Schrodinger equation is

$$\varepsilon\chi_c(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_c]\chi_c(r) \tag{1.3}$$

with the normalization requirement

$$\int_0^\infty dr|\chi(r)|^2 = 1$$

We propose that the binary atom closed channel state is a superposition of quasi-bound states. To simplify matters, we choose the binary atom bound state u_c to be a super position of bound states. For simplicity we choose it to be a superposition of only one quasi-bound state $u_c(r) = C\chi_c(r)$.

We make the proper substitution into the closed channel equation

$$CE\chi_c(r) = C[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_c(r)]\chi_c(r) + g^*(r)u_o(r)$$

$$CE\chi_c(r) = C\varepsilon\chi_c(r) + g^*(r)u_o(r)$$

$$C(E-\varepsilon)\chi_c(r) = g^*(r)u_o(r)$$

$$C(E-\varepsilon)\int_0^\infty dr\chi_c^*(r)\chi_c(r) = \int_0^\infty dr\chi_c^*(r)g^*(r)u_o(r)$$

to get an expression for the constant C, which is the amplitude of the bound state,

$$C = \frac{\int_0^\infty dr\chi_c^*(r)g^*(r)u_o(r)}{E-\varepsilon}$$

So now we return to the open channel equation and make the substitution for C

$$Eu_o(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_o(r)]u_o(r) + g(r)C\chi_c(r)$$

$$Eu_o(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_o(r)]u_o(r) + g(r)\chi_c(r)\frac{\int_0^\infty dr\chi_c^*(r)g^*(r)u_o(r)}{E-\varepsilon}$$

Since coupling between the two channels is strongest in the region around $r = R$, when the inter-atomic potential (exchange interaction) becomes comparable to the hyperfine interaction of each atom, we now propose that the coupling function is a delta function $g^*(r) = g^*\delta(r-R)$. The bound state amplitude becomes

$$C = \frac{\chi_c^*(R)g^*u_o(R)}{E-\varepsilon}$$

To simplify, we let $\gamma^* = \chi_c^*(R)g^*$ so that

$$C = \frac{\gamma^* u_o(R)}{E - \varepsilon}$$

The equation for the open channel becomes

$$Eu_o(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_o(r)]u_o(r) + \frac{|g|^2\delta(r-R)\chi_c(r)\chi_c^*(R)u_o(R)}{E-\varepsilon}$$

$$Eu_o(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_o(r)]u_o(r) + \delta(r-R)\frac{|g|^2|\chi_c(R)|^2u_o(R)}{E-\varepsilon}$$

Therefore,

$$Eu_o(r) = [-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_o(r)]u_o(r) + \delta(r-R)\frac{\gamma^2}{E-\varepsilon}u_o(R)$$

$$\frac{d^2}{dr^2}u_o(r) = [-\frac{2\mu}{\hbar^2}(E - V_o(r))]u_o(r) + \delta(r-R)\frac{2\mu\gamma^2}{\hbar^2(E-\varepsilon)}u_o(R) \tag{1.4}$$

This is the Schrodinger equation with a delta function potential which serves to modify the derivative at the boundary $r = R$. We can show this in the following way: to solve the equation above, we choose a square well potential for $V_o(r) = -V_o\Theta(R-r)$. For the region $0 < r < R$,which we will call region I ,

$$\frac{d^2}{dr^2}u_{oI}(r) = [-\frac{2\mu}{\hbar^2}(E + V_o)]u_{oI}(r) + \delta(r-R)\frac{2\mu\gamma^2}{\hbar^2(E-\varepsilon)}u_{oI}(R)$$

simplifying

$$\frac{d^2}{dr^2}u_{oI}(r) = -\kappa^2 u_{oI}(r) + \frac{\beta}{E-\varepsilon}\delta(r-R)u_{oI}(r) \tag{1.5}$$

where

$$\kappa = \frac{\sqrt{2\mu(E+V_o)}}{\hbar}$$

and

$$\beta = \frac{2\mu\gamma^2}{\hbar^2}$$

We integrate from $R-\epsilon$ to $R+\epsilon$

$$\int_{R-\epsilon}^{R+\epsilon} \frac{d^2u_o}{dr^2} = -\int_{R-\epsilon}^{R+\epsilon} \kappa^2 u_{oI}(r) + \frac{\beta}{E-\varepsilon}\int_{R-\epsilon}^{R+\epsilon} \delta(r-R)u_o(r)$$

$$\frac{du_o}{dr}|_{r=R+\epsilon} - \frac{du_o}{dr}|_{r=R-\epsilon} = -\int_{R-\epsilon}^{R+\epsilon} \kappa^2 u_{oI}(r) + \frac{\beta}{E-\varepsilon}\int_{R-\epsilon}^{R+\epsilon} \delta(r-R)u_o(r)$$

and taking the limit $\epsilon \to 0$

$$\frac{du_o}{dr}|_{r=R_+} = \frac{du_o}{dr}|_{r=R_-} + \frac{\beta}{E-\varepsilon}u_o(R)$$

Thus, the derivative changes discontinuously at the boundary.

1.8 Boundary Conditions Considering the Harmonic Trap

Let's divide space into an inner region called region I and an outer region called region II. Region I will be in the range of the inter-atomic interaction with a range $0 < r < R$ where $R \ll l_T$ (R$\sim nm$ and $l_T \sim \mu m$) . Region II will be in the range $R < r < \infty$.

In region II, the wavefunction solutions will be denoted as $U^{II}(r)$. We will denote the region I solutions as $U^I(r)$ with the boundary condition that $U^I(0) = 0$. The outer and inner wavefunctions and their derivatives need to match at the boundary $r = R$. In short, we say that their logarithmic derivatives must be equal

$$\frac{\frac{dU^I(E,r)}{dr}}{U^I(E,r)}|_{r=R} = \frac{\frac{dU^{II}(E,r)}{dr}}{U^{II}(E,r)}|_{r=R}$$

The two wavefunctions $U^{II}(r)$, $U^I(r)$ are not normalized. To have a normalized wavefunction, we choose a normalized solution $U^N(E,r)$ to the harmonic potential which has the range of all space but it is primarily in region II since $l_T \gg R$. $U^N(E,r)$ which will be in the form of Hermite polynomials, though the details are not important yet. The normalization condition

$$\int_0^\infty dr|U^N(E,r)|^2 = 1$$

$U^N(E,r) = U^{II}(E,r)$ in region II with the boundary condition that $U^N(\infty) = 0$ but not does satisfy the boundary for region I that $U^I(0) = 0$. More importantly we will require that the wavefunctions match at $r = R$ $U^N(E,r)|_{r=R} = U^I(E,r)|_{r=R}$.

By choosing these conditions on $U^N(E,r)$ we are approximately normalizing U^I and U^{II} in the limit $R \ll l_T$ where the region $0 < r < R$ can be neglected in the integral

$$\int_0^R dr|U^I(E,r)|^2 + \int_R^\infty dr|U^{II}(E,r)|^2 \approx \int_R^\infty dr|U^N(E,r)|^2$$

$$\approx \int_0^\infty dr|U^{II}(E,r)|^2 = 1$$

1.9 Scattering Physics

At low energies, the scattering wavefunction U_{sc} outside of the inter atomic distance $(r > R)$, is of the form

$$U_{sc}(E,r) = \mathcal{N}\sin(kr + \delta_0)$$

where δ_0 is the energy dependent s-wave scattering phase shift, and $\mathcal{N}$ is the normalization constant. To understand the conditions for Feshbach resonance, we need to match boundary conditions with the inner wavefunction $U^I(E,r)$.

$$\frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} = \frac{U_{sc}(E,r)}{dU_{sc}(E,r)/dr}|_{r=R} = \frac{\tan(kR+\delta_0)}{k}$$

In the short range and low energy limit, $kR \ll 1$, we can expand the tangent function

$$\frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} = \frac{\tan(\delta_0)}{k} + \frac{\sec^2(\delta_0)}{k}(kR)$$

$$\frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} = \frac{\tan(\delta_0)}{k} + R[1+\tan^2(\delta_0)]$$

For shorthand notation, and to make a connection with the scattering amplitude, we introduce the energy dependent effective scattering length $a_{eff}(E)$.

$$a_{eff}(E) = -\frac{\tan(\delta_0)}{k}$$

With this notation, the boundary conditions give

$$\frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} = -a_{eff}(E) + R[1+k^2a_{eff}^2(E)]$$

$$= -a_{eff}(E) + R + ka_{eff}^2(E)(kR)$$

and in the limit $kR \ll 1$,

$$\frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} \approx R - a_{eff}(E)$$

Recall that the coupled equations introduce a discontinuity of the inner wavefunction's derivative due to the coupling between the open and closed channels of the form

$$\frac{dU_o}{dr}|_{r=R_+} = \frac{dU_o}{dr}|_{r=R_-} + \frac{\beta}{E-\varepsilon}U_o$$

where $\beta = \frac{2\mu\alpha^2}{\hbar^2}$.

In the absence of coupling ($\beta = 0$), the derivative of the inner wavefunction is continuous across the boundary. In this case, we will denote the wavefunction as $U_A(E, r)$ (absence of coupling).

The inner wavefunction with coupling is denoted as $U^I(E, r)$. Therefore, the equation above can be written as an equation for the inner wavefunction under the presence of coupling in terms of the wavefunction in the absence of coupling

$$\frac{dU^I}{dr}|_{r=R} = \frac{dU_A}{dr}|_{r=R} + \frac{\beta}{E-\varepsilon}U_A$$

The scattering length in the absence of coupling is denoted as $a_A(E)$ and in the same approximation $r \ll R$,

$$\frac{U_A(E,r)}{dU_A(E,r)/dr}|_{r=R} \approx R - a_A(E)$$

while the scattering length in the presence of coupling $a_{eff}(E)$ is

$$\frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} \approx R - a_{eff}(E)$$

.

1.10 Effective Scattering Length Due to Coupled Channels

Now we will find an expression for $a_{eff}(E)$ in terms of $a_A(E)$

$$R - a_{eff}(E) \approx \frac{U^I(E,r)}{dU^I(E,r)/dr}|_{r=R} = \frac{U_A(E,r)}{\frac{dU_A}{dr}|_{r=R} + \frac{\beta}{E-\varepsilon}U_A}$$

Recall that $U_A(E,r) = U^I(E,r)$. Inverting both sides,

$$\frac{1}{R - a_{eff}(E)} = \frac{\frac{dU_A}{dr}|_{r=R} + \frac{\beta}{E-\varepsilon}U_A}{U_A(E,r)}$$

$$\frac{1}{R - a_{eff}(E)} = \frac{1}{R - a_A(E)} + \frac{\beta}{E-\varepsilon}$$

The effective scattering length diverges $a_{eff}(E) \to \infty$ when $\varepsilon = \varepsilon_{res}$ the bound state energy, or resonant energy.

Solving for ε in the limits $a_{eff}(E) \to \infty, E \to 0$,

$$0 = \frac{1}{R - a_A} - \frac{\beta}{\varepsilon_{res}}$$

$$R - a_A = \frac{\varepsilon_{res}}{\beta}$$

We write the equation for the effective scattering length as

$$\frac{1}{R - a_{eff}(E)} = \frac{1}{R - a_A(E)} + \frac{1}{\frac{E-(\varepsilon-\varepsilon_{res})}{\beta} - \frac{\varepsilon_{res}}{\beta}}$$

$$= \frac{1}{R - a_A(E)} + \frac{1}{\frac{\varepsilon_{res}}{\beta}[\frac{E-(\varepsilon-\varepsilon_{res})}{\varepsilon_{res}} - 1]}$$

$$\frac{1}{R - a_{eff}(E)} = \frac{1}{R - a_A(E)} + \frac{1}{R - a_A[\frac{E-(\varepsilon-\varepsilon_{res})}{\varepsilon_{res}} - 1]}$$

At very low collision energies $E \to 0$, $a_A(E)$ becomes the background scattering length a_0. In our low energy approximation, we can express the scattering length in the absence of coupling $a_A(E)$ in the usual effective range expansion

$$\frac{1}{a_A(E)} \approx \frac{1}{a_0} + \frac{r_A}{2}k^2$$

$$\frac{1}{a_A(E)} \approx \frac{1}{a_0}[1 + \frac{r_A a_0}{2}k^2]$$

$$a_A(E) \approx a_0(1 + \frac{r_A a_0}{2}k^2)^{-1}$$

where r_A is the effective range in the absence of a Feshbach resonance. At low energies, or the limit $k \to 0$,

$$a_A(E) \approx a_0[1 - \frac{r_A a_0}{2} k^2]$$

r_A and a_0 are in the nm range while the energy scale of $a_A(E)$ is in the $10^6 \hbar\omega$range. Therefore, the second factor above will be much smaller than the energy scales in which we are interested. The resonant state ε_{res} is in the $\hbar\omega$ range. Thus in this range, $a_A(E) \approx a_0$.

The expression for the effective scattering length is now

$$\frac{1}{R - a_{eff}(E)} = \frac{1}{R - a_0}[1 + \frac{1}{\frac{E-(\varepsilon-\varepsilon_{res})}{\varepsilon_{res}} - 1}]$$

$$= \frac{1}{R - a_0}[\frac{\frac{E-(\varepsilon-\varepsilon_{res})}{\varepsilon_{res}}}{\frac{E-(\varepsilon-\varepsilon_{res})}{\varepsilon_{res}} - 1}]$$

$$\frac{1}{R - a_{eff}(E)} = \frac{1}{R - a_0}[\frac{1}{1 - \frac{\varepsilon_{res}}{E-(\varepsilon-\varepsilon_{res})}}]$$

Taking the inverse of both sides gives

$$a_{eff}(E) = R - (R - a_0)[1 - \frac{\varepsilon_{res}}{E - (\varepsilon - \varepsilon_{res})}]$$

$$a_{eff}(E) = a_0[1 + (\frac{R - a_0}{a_0})\frac{\varepsilon_{res}}{E - (\varepsilon - \varepsilon_{res})}]$$

Dividing the numerator and denominator by $\frac{\partial \varepsilon}{\partial B}$ gives

$$a_{eff}(E) = a_0[1 + (\frac{R - a_0}{a_0})\frac{\frac{\varepsilon_{res}}{\frac{\partial \varepsilon}{\partial B}}}{\frac{E}{\frac{\partial \varepsilon}{\partial B}} - \frac{(\varepsilon - \varepsilon_{res})}{\frac{\partial \varepsilon}{\partial B}}}]$$

and then casting it in a form that is more familiar

$$a_{eff}(E) = a_0[1 + \frac{\Delta}{\frac{E}{\frac{\partial \varepsilon}{\partial B}} - (B - B_0)}]$$

gives $\Delta = (\frac{R-a_0}{a_0})\frac{\varepsilon_{res}}{\frac{\partial \varepsilon}{\partial B}}$.

We have used the fact that the quasi bound energy ε has a Taylor series expansion with dependence on B to first order around the resonance energy

$$\varepsilon - \varepsilon_{res} \approx \frac{\partial \varepsilon}{\partial B}(B - B_0)$$

.

We define a quantity called the resonance length scale denoted by l_δ

$$\Delta \frac{\partial \varepsilon}{\partial B} = \frac{\hbar^2}{2\mu l_\delta^2}$$

Substituting for Δas found above,

$$\Delta = \frac{\hbar^2}{2\mu l_\delta^2} / \frac{\partial \varepsilon}{\partial B} = (\frac{R - a_0}{a_0})\varepsilon_{res} / \frac{\partial \varepsilon}{\partial B}$$

Recall that $\beta = \frac{\varepsilon_{res}}{R-a_0}$ from the resonance condition, so that we can substitute for ε_{res}.

$$\frac{\hbar^2}{2\mu l_\delta^2} / \frac{\partial \varepsilon}{\partial B} = (\frac{R-a_0}{a_0})\beta(R-a_0) / \frac{\partial \varepsilon}{\partial B}$$

$$\frac{\hbar^2}{2\mu l_\delta^2} = \frac{(R-a_0)^2}{a_0^2} a_0 \beta$$

which gives

$$\beta = \frac{\hbar^2}{2\mu a_0 (\frac{R}{a_0}-1)^2} \frac{1}{l_\delta^2}$$

but we also know that

$$\beta = \frac{2\mu}{\hbar^2} |\gamma|^2$$

Therefore ,

$$|\gamma| = \frac{\hbar^2}{2\mu} \frac{1}{l_\delta \sqrt{|a_0|} (\frac{R}{a_0}-1)}$$

1.11 The Narrow Resonance Eigenvalue Equation

The expression derived by Th. Busch, et al. [9] gives the explicit energy dependence of the two-particle scattering length as the two particles interact through contact interaction in a harmonic trap potential. The logarithmic derivative for the outer wavefunction (outside the

inter atomic potential) is given as a ratio of two Gamma functions

$$\frac{dU^N(\frac{E}{\hbar\omega_T},x)/dx}{U^N(\frac{E}{\hbar\omega_T},x)}|_{x=0} = 2\frac{\Gamma(-\frac{1}{2}\frac{E}{\hbar\omega_T}+\frac{3}{4})}{\Gamma(-\frac{1}{2}\frac{E}{\hbar\omega_T}+\frac{1}{4})} = \frac{1}{a}$$

where $x = \frac{r}{l_T}$ is the distance measured in harmonic oscillator units of length $l_T = \sqrt{\frac{\hbar^2}{\mu\omega_T}}$. a is the scattering length for a delta contact potential as derived by Busch [9]. We now suppose that the two-atom system interacting through the short length spin open and closed potentials is confined by a harmonic trap. The range of the atom-atom potentials is in nanometers while typical optical harmonic traps are in the micrometer range. This gives the limit $R \ll l_T$. The continuity of the wavefunctions across the edge$r = R$ of the interatomic potentials allow us to match the effective scattering length derived in the previous section to the scattering length derived by Busch [9]. Measured in oscillator length units, the effective scattering length outside the spin interaction potential (atom-atom) is

$$\frac{l_T}{a_{eff}(E)} = 2\frac{\Gamma(-\frac{1}{2}\frac{E}{\hbar\omega_T}+\frac{3}{4})}{\Gamma(-\frac{1}{2}\frac{E}{\hbar\omega_T}+\frac{1}{4})}$$

We introduce harmonic oscillator-scaled quantities

$$\eta = \frac{E}{\hbar\omega_T}$$

$$\delta = \frac{\Delta\partial\varepsilon/\partial B}{\hbar\omega_T}$$

$$\alpha = \frac{a_0}{l_T}$$

and the magnetic width-scaled magnetic detuning

$$b = \frac{B - B_0}{\Delta}$$

Now, we recall the effective scattering length inside the atom-atom potential and scale it with the units above

$$\frac{l_T}{a_{eff}(E)} = \frac{l_T}{a_0} \frac{1}{[1 + \frac{\Delta \frac{\partial \varepsilon}{\partial B}/\omega_T}{\frac{E}{\omega_T} - (B - B_0)\frac{\partial \varepsilon}{\partial B}/\omega_T}]}$$

$$\frac{l_T}{a_{eff}(E)} = \frac{1}{\alpha} \frac{1}{[1 + \frac{\delta}{\eta - (B - B_0)\frac{\delta}{\Delta}}]}$$

$$= \frac{1}{\alpha} \frac{1}{[1 + \frac{\delta}{\eta - b\delta}]}$$

$$\frac{l_T}{a_{eff}(E)} = \frac{1}{\alpha} \frac{1}{[1 + \frac{1}{\frac{\eta}{\delta} - b}]}$$

inverting gives

$$\frac{a_{eff}(E)}{l_T} = \alpha[1 + \frac{1}{\frac{\eta}{\delta} - b}]$$

We equate this expression to Busch's effective scattering length. This is the same as matching inner and outer wavefunctions and their respective derivatives at the boundary. The scattering

length for two particles in a harmonic trap with only contact interaction as derived by Th. Busch, et. al. [9] is

$$a = \frac{1}{\sqrt{2}} \frac{\Gamma(-\eta + \frac{1}{4})}{\Gamma(-\eta + \frac{3}{4})}$$

Equating the two scattering lengths

$$\alpha[1 + \frac{1}{\frac{\eta}{\delta} - b}] = \frac{1}{\sqrt{2}} \frac{\Gamma(-\frac{\eta}{2} + \frac{1}{4})}{\Gamma(-\frac{\eta}{2} + \frac{3}{4})}$$

It is easier to solve for b in this case

$$\frac{1}{\frac{\eta}{\delta} - b} = -1 + \frac{1}{2\alpha} \frac{\Gamma(-\frac{\eta}{2} + \frac{1}{4})}{\Gamma(-\frac{\eta}{2} + \frac{3}{4})}$$

$$\frac{\eta}{\delta} - b = \frac{1}{-1 + \frac{1}{2\alpha} \frac{\Gamma(-\frac{\eta}{2} + \frac{1}{4})}{\Gamma(-\frac{\eta}{2} + \frac{3}{4})}}$$

Therefore, we have an expression for the magnetic field detuning as a function of eigenenergies

$$b = \frac{\eta}{\delta} + \frac{1}{1 - \frac{1}{2\alpha} \frac{\Gamma(-\frac{\eta}{2} + \frac{1}{4})}{\Gamma(-\frac{\eta}{2} + \frac{3}{4})}}$$

We plot the two-particle energy spectrum as a function of magnetic field for two different magnetic widths Δ.

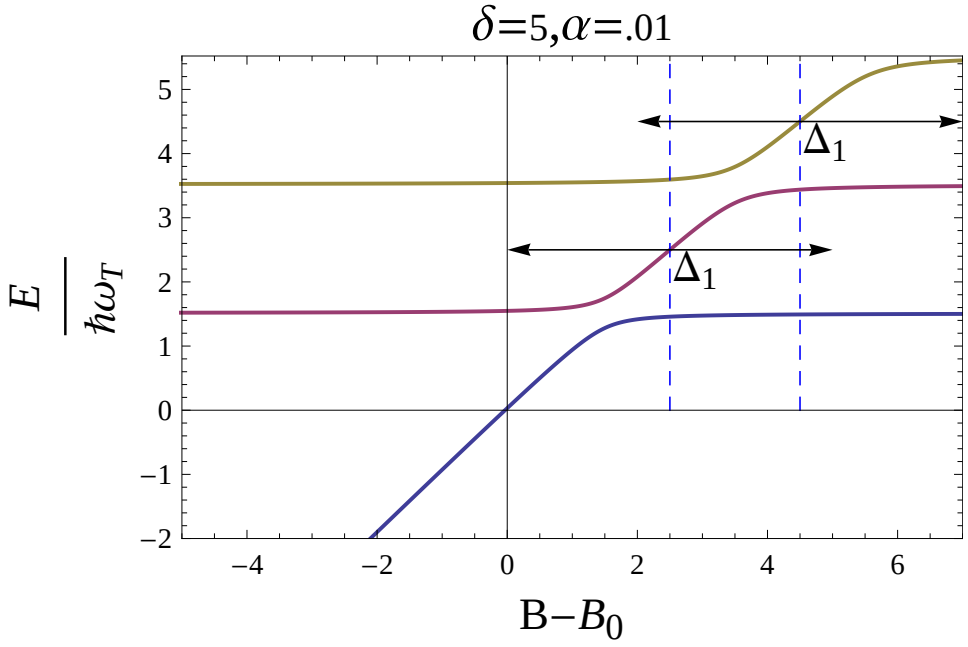

Figure 1.1: Plot of the eigenvalues of the harmonically confined binary atom system as a function of the magnetic field detuning for a broad resonance $\delta = 5$. The vertical dotted lines indicate the magnetic field values for which a level reaches resonance, i. e. its effective scattering length diverges. Note that Δ_1 spreads around the resonance fields, indicated by the horizontal arrows, and overlap with another resonance.

1.12 The Limit of Short Range Interactions

In the limit $R \ll l_T$, we will use the short range limit of $U^N(E, r)$. This means that we will evaluate $U^N(E, r)$ at $r = 0$. The boundary condition is

$$R - a_{eff}(E) = \frac{U^N(E, r)}{dU^N(E, r)/dr}|_{r=R}$$

Therefore, $\frac{U^N(E,r)}{dU^N(E,r)/dr}$ is determined by the boundary condition. The higher order ratios $\frac{d^m U(E,r)/dr^m}{d^{m+1}U(E,r)/dr^{m+1}}$ will have to $\to 0$ in the short range limit.

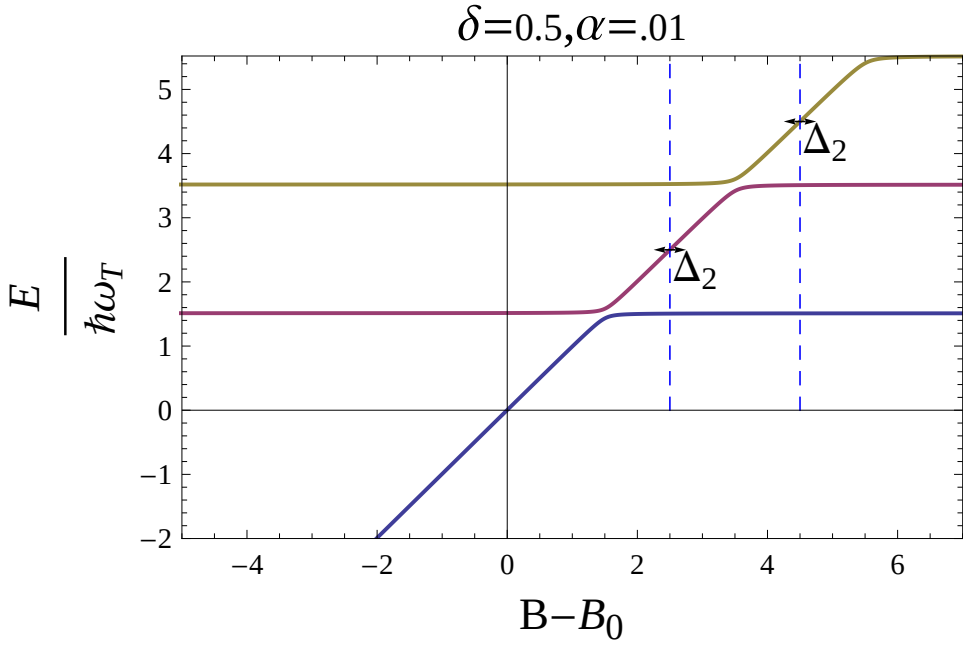

Figure 1.2: Plot of the eigenvalues of the harmonically confined binary atom system as a function of the magnetic field detuning for a narrow resonance $\delta = 0.5$. The vertical dotted lines indicate the magnetic field values for which a level reaches resonance, i. e. its effective scattering length diverges. Note that Δ_2-spreads are much narrower than the magnetic field distance between adjacent resonances indicated by vertical lines.

We can make an expansion of $\frac{U^N(E,r)}{dU^N(E,r)/dr}$ in R around $r = 0$, by expanding the numerator and denominator separately (we drop the superscript N for convenience)

$$\frac{U(E,r)}{dU(E,r)/dr}|_{r=R} \approx \frac{U(E,0) + RdU(E,0)/dr + \frac{R^2}{2}d^2U(E,0)/dr^2 + ...}{dU(E,0)/dr + Rd^2U(E,0)/dr^2 + \frac{R^2}{2}d^3U(E,0)/dr^2...}$$

$$\frac{U(E,r)}{dU(E,r)/dr}|_{r=R} \approx \frac{U(E,0) + RdU(E,0)/dr}{dU(E,0)/dr} \times [\frac{1 + \frac{\frac{R^2}{2}d^2U(E,0)/dr^2}{U(E,0)+RdU(E,0)/dr} + ...}{1 + \frac{Rd^2U(E,0)/dr^2}{dU(E,0)/dr} + ...}]$$

Let's look at the first term on the right,

$$\frac{U(E,0)+RdU(E,0)/dr}{dU(E,0)/dr}=\frac{U(E,0)}{dU(E,0)/dr}+R$$

if we equate it to the boundary condition

$$R-a_{eff}(E)=\frac{U(E,0)}{dU(E,0)/dr}+R$$

then

$$\frac{U(E,0)}{dU(E,0)/dr}=-a_{eff}(E)$$

which is consistent with the fact that we are in the short range limit (evaluate $U^N(E,r)$ and all it's derivatives at $r=0$). In other words, we have the expression for $a_{eff}(E)$ by just looking at logarithmic derivative at $r=0$. For the above equation to be true, then the term in the bracket of the expansion must $\to 1$. This results in $\frac{d^mU(E,r)/dr^m}{d^{m+1}U(E,r)/dr^{m+1}}\to 0$. We can show this. Looking at the numerator in brackets and using $U(E,0)+R\frac{dU(E,0)}{dr}=\frac{dU(E,0)}{dr}[R-a_{eff}(E)]$,

$$1+\frac{\frac{R^2}{2}d^2U(E,0)/dr^2}{U(E,0)+RdU(E,0)/dr}+...\approx 1+\frac{\frac{R^2}{2}d^2U(E,0)/dr^2}{\frac{dU(E,0)}{dr}[R-a_{eff}(E)]}=1+\frac{R^2}{2[R-a_{eff}(E)]}\frac{d^2U(E,0)/dr^2}{\frac{dU(E,0)}{dr}}$$

which means that $\frac{d^2U(E,0)/dr^2}{\frac{dU(E,0)}{dr}}$ must go to 0 in this short range limit $R/l_T\to 0$.

Looking at the denominator in brackets,

$$1 + \frac{Rd^2U(E,0)/dr^2}{dU(E,0)/dr} + ... \approx 1 + R\frac{d^2U(E,0)/dr^2}{dU(E,0)/dr}$$

is also consistent with the above result.

1.13 Normalizing the Trapped Two-Atom State

The two-particle relative coordinate wavefunction is

$$|\Psi_{r.c.}(r)\rangle = \mathcal{N}U^N(r)|o\rangle + C\chi(r)|c\rangle$$

where $|o\rangle$and $|c\rangle$ are open and closed spin wavefunctions respectively, $U^N(r)$is the normalized wavefunction in the open channel, and $\chi(r)$ is the bound state wavefunction in the closed channel.$\mathcal{N}$and C are coefficients. The normalization condition

$$\int_0^\infty dr\langle\Psi_{r.c.}(r)|\Psi_{r.c.}(r)\rangle = 1$$

gives

$$\mathcal{N}^2 + C^2 = 1$$

since $U^N(r)$ and $\chi(r)$ are each normalized over all space and orthogonal.

From the coupled equations, we have determined that

$$C = \frac{\gamma^* u_o(R)}{E - \varepsilon}$$

and

$$u_0(R) = \mathcal{N} U^N(R)$$

so that

$$C = \frac{\gamma^* \mathcal{N} U^N(R)}{E - \varepsilon}$$

and the normalization gives

$$\mathcal{N}^2[1 + \frac{|\gamma|^2 |U^N(R)|^2}{(E-\varepsilon)^2}] = 1$$

$$\mathcal{N}^2 = \frac{1}{1 + \frac{|\gamma|^2 |U^N(R)|^2}{(E-\varepsilon)^2}}$$

Inserting this back into $C = \frac{\gamma^* \mathcal{N} U^N(R)}{E-\varepsilon}$,

$$C^2 = \frac{|\gamma|^2 |U^N(R)|^2}{(E-\varepsilon)^2[1 + \frac{|\gamma|^2 |U^N(R)|^2}{(E-\varepsilon)^2}]}$$

$$C^2 = \frac{|\gamma|^2 |U^N(R)|^2}{(E-\varepsilon)^2[1 + \frac{|\gamma|^2 |U^N(R)|^2}{(E-\varepsilon)^2}]}$$

$$C^2 = \frac{\varphi^2}{1+\varphi^2}$$

where

$$\varphi = \frac{\gamma}{E-\varepsilon} U^N(R)$$

1.14 The φ in Observable Quantities

According to the short range limit, $R/l_T \to 0$,

$$U^N(E,r)|_{r=R} \approx U^N(E,0) + R\frac{dU^N(E,0)}{dr}$$

$$= U^N(E,0)[1 + R\frac{dU^N(E,0)/dr}{U^N(E,0)}]$$

$$U^N(E,r)|_{r=R} = U^N(E,0)[1 - \frac{R}{a_{eff}(E)}]$$

Therefore,

$$\varphi = \frac{\gamma}{E-\varepsilon} U^N(E,0)[1 - \frac{R}{a_{eff}(E)}]$$

$$\varphi = \gamma U^N(E,0)[1 - \frac{R}{a_{eff}(E)}]\frac{\gamma^2 \frac{2\mu}{\hbar^2}}{E-\varepsilon}\frac{\hbar^2}{2\mu\gamma^2}$$

and we make the substitution for $\beta = \frac{2\mu\gamma^2}{\hbar^2}$

$$\varphi = \gamma U^N(E,0)[1 - \frac{R}{a_{eff}(E)}]\frac{\beta}{E-\varepsilon}\frac{\hbar^2}{2\mu\gamma^2}$$

and we recall the relation from the coupled equations

$$\frac{1}{R - a_{eff}(E)} = \frac{1}{R - a_0} + \frac{\beta}{E - \varepsilon}$$

$$\frac{\beta}{E-\varepsilon} = \frac{R - a_0 - R - a_{eff}(E)}{(R - a_{eff}(E))(R - a_0)}$$

$$\frac{\beta}{E-\varepsilon} = \frac{a_{eff}(E) - a_0}{(R - a_{eff}(E))(R - a_0)}$$

Therefore,

$$\varphi = \gamma U^N(E,0)[\frac{a_{eff}(E) - R}{a_{eff}(E)}]\frac{a_{eff}(E) - a_0}{(R - a_{eff}(E))(R - a_0)}\frac{\hbar^2}{2\mu\gamma^2}$$

$$\varphi = U^N(E,0)[\frac{1}{a_{eff}(E)}]\frac{a_{eff}(E) - a_0}{(R - a_0)}\frac{\hbar^2}{2\mu\gamma}$$

$$|\gamma| = \frac{\hbar^2}{2\mu}\frac{1}{l_\delta\sqrt{|a_0|}(\frac{R}{a_0} - 1)}$$

$$\varphi = U^N(E,0)[\frac{1}{a_{eff}(E)}]\frac{a_{eff}(E)-a_0}{(R-a_0)}l_\delta\frac{\sqrt{|a_0|}}{a_0}(R-a_0)$$

$$\varphi = U^N(E,0)[\frac{a_{eff}(E)-a_0}{a_{eff}(E)}]\frac{l_\delta}{\sqrt{|a_0|}}$$

In harmonic oscillator units,

$$\varphi = [\frac{a_{eff}(E)-a_0}{a_{eff}(E)}]\frac{l_\delta}{\sqrt{|a_0|}\sqrt{l_T}}\tilde{U}^N(E,0)$$

We can use the expression derived by Th. Busch

$$\frac{1}{a_{eff}(E)} = \frac{2}{l_T}\frac{\Gamma(-\frac{E}{\hbar\omega_T}+\frac{3}{4})}{\Gamma(-\frac{E}{\hbar\omega_T}+\frac{1}{4})}$$

to write

$$\frac{a_{eff}(E)-a_0}{a_{eff}(E)} = 1 - \frac{a_0}{a_{eff}(E)}$$

$$\frac{a_{eff}(E)-a_0}{a_{eff}(E)} = 1 - 2\alpha\frac{\Gamma(-\frac{E}{\hbar\omega_T}+\frac{3}{4})}{\Gamma(-\frac{E}{\hbar\omega_T}+\frac{1}{4})}$$

so that

$$\varphi = \tilde{U}^N(E,0)[1-2\alpha\frac{\Gamma(-\frac{E}{\hbar\omega_T}+\frac{3}{4})}{\Gamma(-\frac{E}{\hbar\omega_T}+\frac{1}{4})}]\frac{l_\delta/l_T}{\sqrt{|\alpha|}}$$

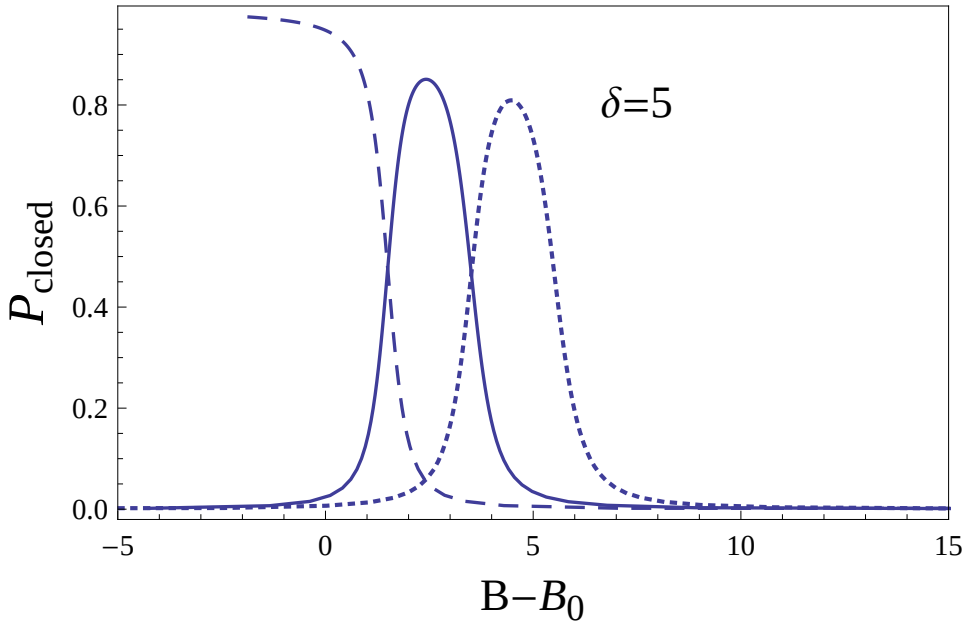

Figure 1.3: Plot of the closed channel occupation number or the probability of finding the binary atom system in the closed channel quasi-bound state. The figures shows curves for the 3 lowest energy eigenstates as a function of the magnetic field at relatively broad resonance $\delta = 5$. The long dotted line represents the closed channel occupation number for the state that has the lowest energy in the off resonant $\varepsilon < 0$ $(B - B_0)$ limit. The full and short dashed lines show the closed channel occupation numbers of the first and second excited states.

with $\frac{l_\delta}{l_T} = \sqrt{\frac{1}{2\delta}}$

$$\varphi = \frac{1}{\sqrt{2|\alpha|\delta}}[1 - 2\alpha\frac{\Gamma(-\frac{E}{\hbar\omega_T} + \frac{3}{4})}{\Gamma(-\frac{E}{\hbar\omega_T} + \frac{1}{4})}]\tilde{U}^N(E, 0)$$

1.15 Level Resonance

Resonance occurs when the effective scattering length diverges. From the definition of the scattering length, $\frac{dU^N(E,r)}{dr}|_{r=0} = 0$ implies $a_{eff}(E) \to \infty$. The energy when this occurs is $E_n^{res} = (\frac{3}{2} + n)\hbar\omega$ for $n = 1, 3, 5, ...$ Or when $\eta = \frac{5}{2}, \frac{9}{2}$,...

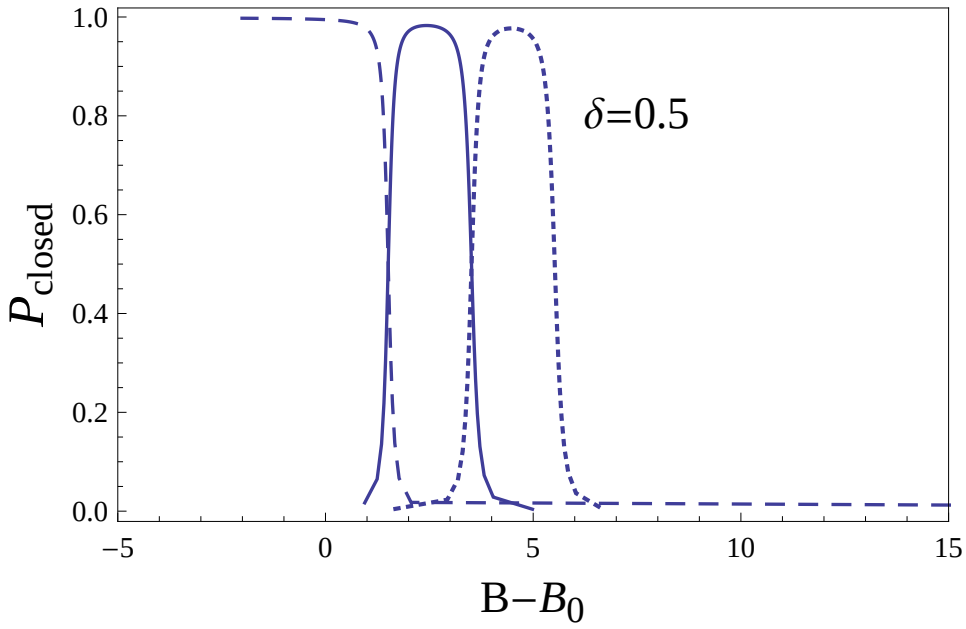

Figure 1.4: Plot of the closed channel occupation number or the probability of finding the binary atom system in the closed channel quasi-bound state. The figures shows curves for the 3 lowest energy eigenstates as a function of the magnetic field at relatively narrow resonance $\delta = 0.5$. The long dotted line represents the closed channel occupation number for the state that has the lowest energy in the off resonant $\varepsilon < 0$ $(B - B_0)$ limit. The full and short dashed lines show the closed channel occupation numbers of the first and second excited states. Note that at this narrow resonance, the closed channel probabilities are more narrow around the resonant values (the peaks of the curves).

For $\delta \sim 1$, the magnetic field offset can be determined for on resonance energies from

$$b = \frac{\eta}{\delta} + \frac{1}{1 - \frac{1}{2\alpha}\frac{\Gamma(-\eta+\frac{1}{4})}{\Gamma(-\eta+\frac{3}{4})}}$$

We can attain an analytical expression for the quasi-bound state amplitude for the resonant energies. The amplitude is given by the φ_n factor which we denote as the factor for the nth energy level

$$|C|^2 = \frac{\varphi_n^2}{1 + \varphi_n^2}$$

$$\varphi_n = \sqrt{2}h(0)\frac{l_\delta/l_T}{\sqrt{\alpha}}$$

Here, $h(y)$ is the normalized harmonic potential wavefunction evaluated at $x = 0$.

We need to relate the normalized wavefunction for the spherical 3D potential $\tilde{U^N}$ to the normalized Hermite polynomial solution in 1D $h(y)$. The values of this function is given by Hermite polynomials

$$h(x) = (\frac{1}{\pi l_T})^{\frac{1}{4}}\frac{1}{\sqrt{2^{2n}(2n)!}}H_{2n}(y)e^{-\frac{y^2}{2}}$$

where

$$y = \frac{x}{l_T}$$

and

$$l_T = \sqrt{\frac{\hbar}{\mu\omega_T}}$$

The recursion relation for the nth Hermite polynomial is

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y)$$

We need the recursion for even n evaluated at $y = 0$

$$H_{n+1}(0) = -2nH_{n-1}(0)$$

The first Hermite polynomial is $H_0(y) = 1$

From the recursion above we can find the pattern, so for $n = 1$,

$$H_2(0) = -2(1)H_0(0) = -2$$

for $n = 2$, $H_3 = 0$, for $n = 3$

$$H_4(0) = -2(3)H_2(0) = -2(3)(-2) = 12$$

$$H_6(0) = -2(5)H_4(0) = -2(5)(-2)(3)(-2)(1) = -120$$

The pattern is

$$H_{2n}(0) = -2(2n-1)(-2)(2n-3)(-2)(2n-5)...(-2)(1)$$

rearranging

$$H_{2n}(0) = (-2)(-2)...(-2)(2n-1)!!$$

$$H_{2n}(0) = (-2)^n(2n-1)!!$$

for $n = 0, 1, 2, ...$

Now we plug this into

$$h(0) = (\frac{1}{\pi l_T})^{\frac{1}{4}} \frac{1}{\sqrt{2^{2n}(2n)!}} H_{2n}(0)$$

$$h(0) = (\frac{1}{\pi l_T})^{\frac{1}{4}} \frac{1}{\sqrt{2^{2n}(2n)!}} (-2)^n (2n-1)!!$$

we can simplify

$$h(0) = (\frac{1}{\pi l_T})^{\frac{1}{4}} \frac{(-2)^n}{2^n} \frac{(2n-1)!!}{\sqrt{(2n)!}}$$

$$= (\frac{1}{\pi l_T})^{\frac{1}{4}} (-1)^n \frac{(2n-1)!!}{\sqrt{(2n)!}}$$

We can use a relationship between double and single factorials to have an expression of only single factorials. This relationship is

$$(2n-1)!! = \frac{(2n)!}{2^n n!}$$

Then we plug this into $U^N(0)$ so that

$$h(0) = (\frac{1}{\pi l_T})^{\frac{1}{4}} (-1)^n \frac{(2n)!}{2^n n!} \frac{1}{\sqrt{(2n)!}}$$

We can simplify this expression

$$= (\frac{1}{\pi l_T})^{\frac{1}{4}}(-1)^n \frac{(2n)!}{2^n n!} \frac{1}{\sqrt{(2n)!}}$$

Now we need to plug this into φ_n which is related to the bound state amplitude

$$\varphi_n = 2h(0)\frac{l_\delta/l_T}{\sqrt{\alpha}}$$

$$\varphi_n = 2(\frac{1}{\pi l_T})^{\frac{1}{4}}(-1)^n \frac{(2n)!}{2^n n!} \frac{1}{\sqrt{(2n)!}} \frac{l_\delta/l_T}{\sqrt{\alpha}}$$

Then squaring both sides

$$|\varphi_n|^2 = (\frac{1}{\pi\alpha^2 l_T})^{\frac{1}{2}} \frac{[(2n)!]^2}{(2n)!} \frac{4}{2^{2n}(n!)^2} \frac{l_\delta^2}{l_T^2}$$

$$|\varphi_n|^2 = (\frac{1}{\pi\alpha^2 l_T})^{\frac{1}{2}} \frac{(2n)!}{(n!)^2} \frac{4}{2^{2n}} \frac{l_\delta^2}{l_T^2}$$

$$|C_{2n}|^2 = \frac{1}{\frac{1}{\varphi_n^2} + 1}$$

$$|C_{2n}|^2 = \frac{1}{(\pi\alpha^2 l_T)^{\frac{1}{2}} \frac{(n!)^2 2^{2n}}{(2n)!4} \frac{l_T^2}{l_\delta^2} + 1}$$

$$|C_{2n}|^2 = \frac{1}{(\pi\alpha^2 l_T)^{\frac{1}{2}} \frac{(n!)^2 2^{2n}}{(2n)!4} \frac{l_T^2}{l_\delta^2} + \frac{(2n)!2\frac{l_\delta^2}{l_T^2}}{(2n)!2\frac{l_\delta^2}{l_T^2}}}$$

$$|C_{2n}|^2 = \frac{4(2n)!(l_\delta^2/l_T^2)}{(\pi\alpha^2 l_T^5)^{\frac{1}{2}}(n!)^2 2^{2n} + 2(2n)!(l_\delta^2/l_T^2)}$$

for $n = 1, 2, 3, ...$

1.16 Semiclassical Approximation of Bound State Amplitude

In a semiclassical view, we consider high energy levels of the harmonic oscillator. These correspond to n level states being much higher than the ground state $\hbar\omega(n + \frac{3}{2}) \gg \hbar\omega\frac{3}{2}$. In that case, we can use an approximation for $n!$ in order to have a simple expression for C_{2n}.

For large n,

$$n! \approx \sqrt{2\pi n} n^n e^{-n}$$

$$h(0) = (\frac{1}{\pi l_T})^{\frac{1}{4}}(-1)^n \frac{(2n)!}{2^n n!} \frac{1}{\sqrt{(2n)!}}$$

We can use the Sterling approximation for large n so approximate

$$|h(0)|^2 = (\frac{1}{\pi l_T})^{\frac{1}{2}} \frac{(2n)!}{2^{2n}(n!)^2}$$

to obtain

$$|h(0)|^2 \approx (\frac{1}{\pi l_T})^{\frac{1}{2}} \frac{\sqrt{2\pi 2n}(2n)^{2n}e^{-2n}}{2^{2n}2\pi n(n)^{2n}e^{-2n}}$$

$$h(0)|^2 \approx (\frac{1}{\pi l_T})^{\frac{1}{2}} \frac{2\sqrt{\pi n}}{2\pi n}$$

Canceling similar numerator and denominator terms leaves

$$h(0)|^2 \approx (\frac{1}{\pi l_T})^{\frac{1}{2}} \frac{\sqrt{\pi n}}{\pi n} \frac{\sqrt{\pi n}}{\sqrt{\pi n}}$$

$$h(0)|^2 \approx (\frac{1}{\pi l_T})^{\frac{1}{2}} \frac{1}{\sqrt{\pi n}}$$

$$h(0)|^2 \approx (\frac{1}{l_T})^{\frac{1}{2}} \frac{1}{\pi\sqrt{n}}$$

$$|\varphi_n|^2 \approx (\frac{1}{\alpha^2 l_T})^{\frac{1}{2}} \frac{4}{\pi} \frac{1}{\sqrt{n}} \frac{l_\delta^2}{l_T^2}$$

$$|C_{2n}|^2 \approx \frac{1}{(\frac{1}{\alpha^2 l_T})^{\frac{1}{2}} \frac{4}{\pi} \frac{1}{\sqrt{n}} \frac{l_\delta^2}{l_T^2} + 1}$$

Chapter 2

From Microscopic Physics to the Effective Lattice Model

2.1 Effective Hamiltonian

Effective Hamiltonians that describe interactions on a lattice distill the essential physics of the microscopic interactions. The simplified models can allow an extrapolation to the macroscopics of the many body physics. We consider an effective single-site Hamiltonian

$$H_i = eb_i^\dagger b_i + un_{i,\uparrow}n_{i,\downarrow} + g(b^\dagger f_{i,\uparrow}f_{i,\downarrow} + f_{i,\uparrow}^\dagger f_{i,\downarrow}^\dagger b_i)$$

where u is the on site fermion repulsion energy, ε is the boson molecule binding energy, and g is the coupling parameter that converts two oppositely spin aligned fermions on the same site into a boson molecule.

We can diagonalize this Hamiltonian if we choose the basis where the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ denotes a boson and no fermions at a lattice site and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is two fermions and no bosons at a lattice site. In this basis, the Hamiltonian can be expressed as a matrix

$$H_i = \begin{bmatrix} e & g \\ g & u \end{bmatrix}$$

We find the eigen energies by diagonalizng the matrix

$$\det \begin{bmatrix} e-E & g \\ g & u-E \end{bmatrix} = (e-E)(u-E) - g^2 = 0$$

$$E^2 - E(e+u) = -eu + g^2$$

$$E^2 - E(e+u) + \frac{(e+u)^2}{4} = +\frac{(e+u)^2}{4} - eu + g^2$$

$$(E - \frac{e+u}{2})^2 = \frac{e^2+2eu+u^2}{4} - eu + g^2$$

$$(E - \frac{e+u}{2})^2 = \frac{e^2 - 2eu + u^2}{4} + g^2$$

$$E_{u,l} = \frac{1}{2}[e + u \pm \sqrt{(e-u)^2 + 4g^2}]$$

with E_u as the "upper" branch and E_l as the "lower" branch. What is the eigenvector of the upper branch? Let's denote it as $\begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}$. One can see that v_2 gives the amplitude of a boson and no fermions at a single site and v_2 gives the amplitude of two fermions and no bosons at a single site. We are interested in a boson (molecule) at a single site or $\langle b_i^\dagger b_i \rangle$,which gives the closed channel probability. More explitly,

$$\langle b_i^\dagger b_i \rangle = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = v_1^2$$

We start by solving for v_2 in terms of v_1.

$$\begin{bmatrix} e - E_u & g \\ \\ g & u - E_u \end{bmatrix} \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

gives

$$(e - E_u)v_1 = -gv_2$$

$$v_2 = \frac{E_u - e}{g} v_1$$

We normalize the eigenvector $|v_1|^2 + |v_2|^2 = 1$ and interpret the normalization condition as the sum of the probabilities of finding the site with 1 boson or 2 fermions at energy E_u. Now we can find the eigenvector by subsitution into the normalization condition

$$|v_1|^2 + \frac{(E_u - e)^2}{g^2} |v_1|^2 = 1$$

so that the closed channel probability or occupation number is given by

$$\langle b_i^\dagger b_i \rangle = v_1^2 = \frac{\phi^2}{1 + \phi^2}$$

with

$$\phi = \frac{g}{E_u - e}$$

2.2 Boson Occupation Number and Binding Energy

To test the validity of the effective on-site Hamiltonian, we would like to compare the value $\phi = \frac{g}{E-e}$, which is related to the bound state probability in the on-site approximation, to the value $\phi(\eta)$, which is related to the bound state amplitude of two atoms in a harmonic trap as calculated in the previous chapter.

We also compare the binding energy ε (quasibound molecule energy) of the effective Hamiltonian to the detuning

$$\nu(\eta) = \frac{\varepsilon - \varepsilon_{res}}{\hbar\omega_T} = \eta - \frac{\sqrt{2}\alpha\delta\frac{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}}{1-\sqrt{2}\alpha\frac{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}}$$

(magnetically controlled) resonant energy in the two-atom trap problem.

We begin by finding an expression for $\phi = \frac{g}{E_u - e}$ only as a function of E , $\phi(E)$. The eigen energy solution to the on-site effective Hamiltonian is $E = \frac{1}{2}[e+u+\sqrt{(e-u)^2+4g^2}]$. We can solve for e

$$E = \frac{1}{2}[e + u + \sqrt{(e-u)^2 + 4g^2}]$$

$$2E - (e+u) = \sqrt{(e-u)^2 + 4g^2}$$

$$4E^2 - 4E(e+u) + (e+u)^2 = (e-u)^2 + 4g^2$$

$$4E^2 - 4E(e+u) + e^2 + 2eu + u^2 = e^2 - 2eu + u^2 + 4g^2$$

$$4E^2 - 4Ee + -4Eu = -4eu + 4g^2$$

$$4E^2 - 4g^2 - 4Eu = -4eu + 4Ee$$

$$E^2 - Eu - g^2 = e(E - u)$$

$$e = E - \frac{g^2}{E - u}$$

and we say that the binding energy is a function of the eigen energy,

$$e(E) = E - \frac{g^2}{E - u}$$

Now we can plug this result into $\phi = \frac{g}{E-e}$ and we have a function of E only,

$$\phi(E) = \frac{E - u}{g}$$

2.3 Relating the Effective Single-Site Hamiltonian Physics to the Two-Atom Harmonic Trap

How does the boson occupation number as given by the effective Hamiltonian agree with the bound state probability of the two atoms in the harmonic trap? To answer this question we

need to find a relationship between the effective Hamiltonian parameters and the parameters of the two-atom system.

2.3.1 Far From Resonance

Given the single-site Hamiltonian

$$H_i = eb_i^\dagger b_i + u(f_{i,\uparrow}^\dagger f_{i,\uparrow})(f_{i,\downarrow}^\dagger f_{i,\downarrow}) + g(b_i^\dagger f_{i,\uparrow} f_{i,\downarrow} + f_{i,\uparrow}^\dagger f_{i,\downarrow}^\dagger b_i)$$

We consider the valididity of the solutions at the far from resonance limit $e \to -\infty$ where there is little or no coupling to and from the quasibound state $g \approx 0$. We also take the limit where the on-site fermion-fermion repulsion is negligible compared to the binding energy $u \ll e$. In other words, two opposite spin aligned fermions are allowed to be on the same site without converting to a quasibound state but also a quasibound fermion pair will not dissociate to two free fermions. We compare the respective energy spectra. A feature that is significant when comparing spectra is the avoided crosssing point.

The avoided crossing refers to the point in the energy spectrum where two energy curves (levels) meet and have the same energy (degeneracy) in an uncoupled system $g = 0$, but avoid this point in a coupled system $g \neq 0$. In other words, at an avoided crossing, energy levels don't overlap or meet but avoid each other. We refer to the crossing point as the point at which the energy levels meet, or the point at which the energy is denenerate.

The energy spectrum of the single-site Hamiltonian is

$$E_{u,l} = \frac{1}{2}[e + u \pm \sqrt{(e-u)^2 + 4g^2}]$$

In the absence of coupling $g = 0$ and in the limit $e \gg u$, the eigen energy of the on-site Hamiltonian becomes

$$E_{u,l} = \frac{1}{2}[e + u \pm \sqrt{(e-u)^2}]$$

$$E_{u,l} = e$$

Or in other words, the upper energy curve and lower energy curve meet or are denenerate. $E = 0, \varepsilon = 0$ is the crossing point.

2.3.2 Effetive Detuning And Effective Boson Occupation Number

We need to relate the boson binding energy ε of the single-site effective Hamiltonian to the magnetic detuning $\nu = \frac{\partial \varepsilon}{\partial B}\frac{B-B_0}{\hbar\omega_T} = \frac{\varepsilon - \varepsilon_{res}}{\hbar\omega_T}$ of the harmonically confined two-atom system. We also want to relate the the eigen energy E in the single-site Hamiltonian to the eigen energy $\eta = \frac{E}{\hbar\omega_T}$ of the two-atom trap.

In chapter 1, we derived the the eigenvalue equation of the two-atom system in the Harmonic trap potential

$$\nu(\eta) = \eta - \frac{\sqrt{2}\alpha\delta\frac{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}}{1-\sqrt{2}\alpha\frac{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}}$$

The crossing point is $\eta = \frac{3}{2}$, $\nu = \frac{3}{2}$ in the absensce of coupling. Recall $\delta = \frac{\Delta\frac{\partial\varepsilon}{\partial B}}{\hbar\omega_T}$, and $\alpha = \frac{a_0}{l_T}$. The eigen energy also behaves linearly as

$$\eta = \nu$$

though the crossing points are different. η is shifted up $\frac{3}{2}$, and ν is also shitfed $\frac{3}{2}$ to the right.

We find the relations between E and η, and ε and ν

$$E = \eta - \frac{3}{2}$$

and

$$e = \nu - \frac{3}{2}$$

We recall single-site Hamiltonian binding energy $e(E) = E - \frac{g^2}{E-u}$ from the previous section and substitute $E = \eta - \frac{3}{2}$

$$e = \eta - \frac{3}{2} - \frac{g^2}{\eta - \frac{3}{2} - u}$$

We define the "effective magnetic detuning" ν_{eff} for the effective Hamiltonian. Substituting $\varepsilon = \nu_{eff} - \frac{3}{2}$ into the left side of the equation above

$$\nu_{eff} - \frac{3}{2} = \eta - \frac{3}{2} - \frac{g^2}{\eta - \frac{3}{2} - u}$$

$$\nu_{eff}(\eta) = \eta - \frac{g^2}{\eta - \frac{3}{2} - u}$$

And we also define the "effective ϕ" from $\phi(E) = \frac{E-u}{g}$,

$$\phi_{eff}(\eta) = \frac{\eta - \frac{3}{2} - u}{g}$$

2.3.3 Relationship Between Effective Hamiltonian Parameters and Microscopic Parameters

Far from resonance $e \to -\infty$, the effective Hamiltonian agrees well with the microscopic description of the two-atom system in a harmonic trap if we identify the lattice model parameters with the trapped two-atom system parameters as

$$e = E$$

$$\frac{u}{\hbar\omega_T} = \frac{2}{\sqrt{\frac{\pi}{2}}}\frac{a_0}{l_T}$$

$$\frac{g}{\hbar\omega_T} = \sqrt{\frac{2}{\sqrt{\frac{\pi}{2}}}\frac{a_0}{l_T}\frac{\Delta}{\hbar\omega_T}}$$

These relations are substituted into ν_{eff} and ϕ_{eff} from the previous section to give

$$\nu_{eff}(\eta) = \eta + \frac{\delta}{(1 - \frac{\eta - \frac{3}{2}}{\frac{2\alpha}{\sqrt{\pi}}})}$$

$$\phi_{eff}(\eta) = \sqrt{\frac{2}{\sqrt{\pi}}\frac{\alpha}{\delta}}(1 - \frac{\eta - \frac{3}{2}}{\frac{2\alpha}{\sqrt{\pi}}})$$

These functions are compared to the analogous microscopic two-atom system functions derived in the previous chapter

$$\nu(\eta) = \eta - \frac{\sqrt{2}\alpha\delta\frac{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}}{1 - \sqrt{2}\alpha\frac{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}}$$

$$\phi(\eta) = \frac{1}{\sqrt{2\alpha\delta}}[1 - \alpha\sqrt{2}\frac{\Gamma(-\frac{\eta}{2}+\frac{1}{4})}{\Gamma(-\frac{\eta}{2}+\frac{3}{4})}]U^N(\eta, x = 0)$$

We notice, as seen in the figure below, that as magnetic field approaches the resonant value ($\eta = \frac{5}{2}$), the effective Hamiltonian boson occupation number disagrees with the trapped

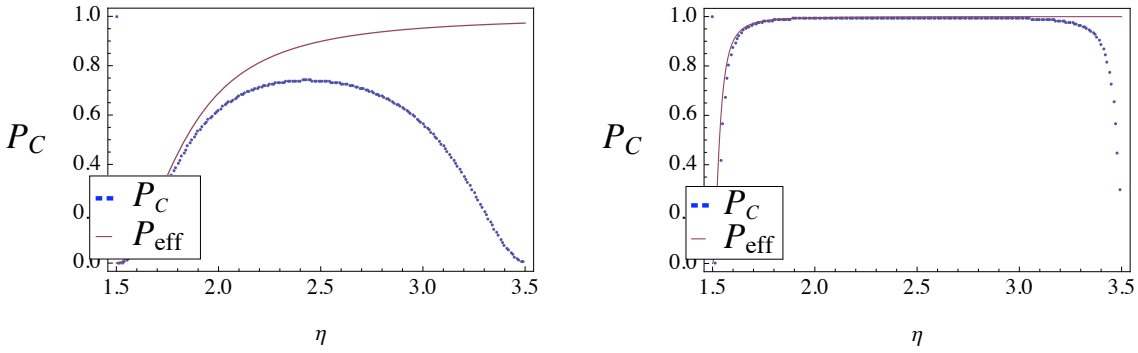

Figure 2.1: The two figures above show the closed channel occupation number of the first excited state as a function of the eigen energy. The exactly calculated value P_c is in blue dots and the effective Hamiltonian predictions in red $P = \frac{\phi^2}{1+\phi^2}$ with $\phi = \frac{2g}{(e+u)+\sqrt{(e-u)^2+4g^2}}$. The left graph is at relatively broad resonance $\delta = 10$ and the graph on the right at relatively narrow resonance $\delta = 0.1$. Note that the red and blue curves approach each other as the resonance becomes narrower (δ becomes smaller). But as the system approaches the 2nd energy level $E = 3.5\hbar\omega_T$, the effective Hamiltonian breaks down. In an optical lattice, this 2nd energy level corresponds to a higher energy band.

two-atom system bound state probability for broad resonances. For narrow resonances, the breakdown happens closer to $\eta = \frac{7}{2}$.

To quantify the success of the effective Hamiltonian approximation, we plot the detuning value $\nu = (B - B_0)\frac{\frac{\partial\varepsilon}{\partial B}}{\hbar\omega_T}$ at which the approximation breaks down as a function of the resonance width $\delta = \Delta\frac{\frac{\partial\varepsilon}{\partial B}}{\hbar\omega_T}$. The breakdown value we choose is $\frac{P_c - P_{eff}}{P_c} = 0.1$

We conclude, as seen from the figure below, that at very narrow resonances $\frac{\Delta}{\hbar\omega_T} \ll 1$, the effective Hamiltonian occupation number agrees very well with the bound state probabiliy of the two-atom system up to $\frac{7}{2}\hbar\omega_T$. For broad resonances, large Δ, the effective Hamiltonian approximation breaksdown further from resonance. This may be an explanation for observations seen in some experiments [2].

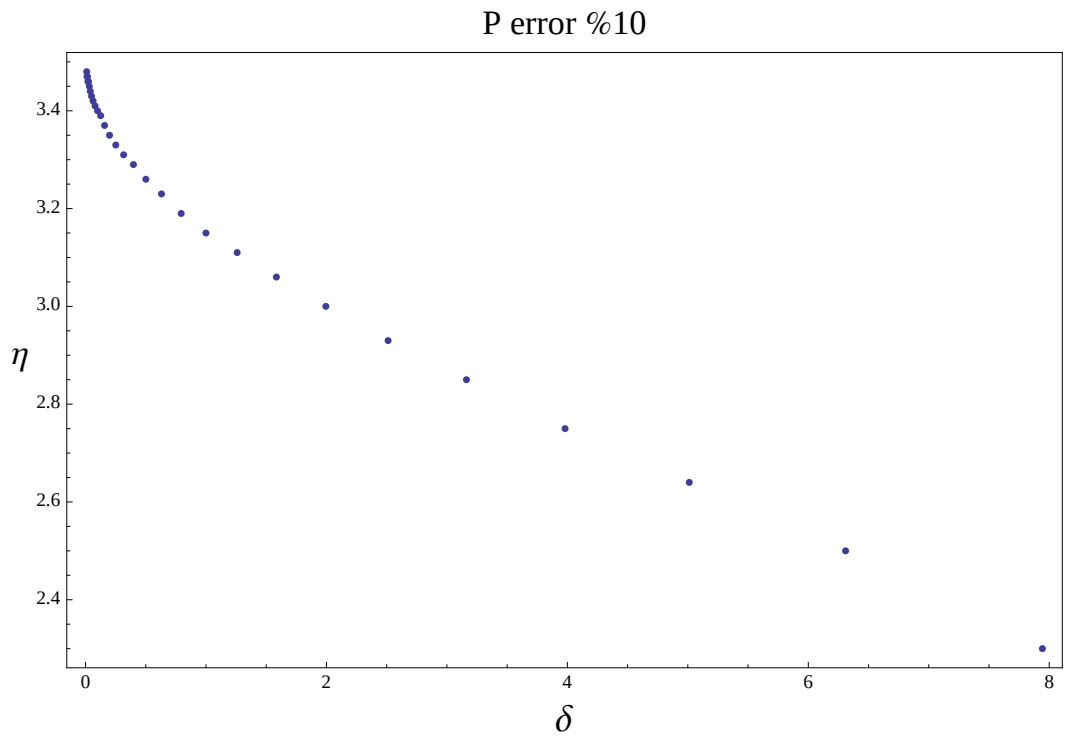

Figure 2.2: The detuning ν (magnetic distance from resonance) when the effective boson occupation number differs by 10 percent from the trapped two-atom bound state probability plotted as a function of δ

Chapter 3

FRG Theory of Dilute Gas Mixture of Ultra Cold Bosons and Fermions

3.1 Introduction

Cold atom experiments have realized a variety of multicomponent quantum mixtures, including Bose-Fermi atomic mixtures. Mixtures of fermionic atoms and diatomic molecules, which are bosons, have also been obtained by tuning of the interactions with external fields[12, 13, 5, 6]. We study many-body correlations in such a system where the molecules are weakly bound and therefore pairs of fermionic atoms easily convert into and dissociate from the bound molecule state. This exchange mediates a long-range interaction between the fermions. We consider a simple many-body Hamiltonian that includes the destruction of fermionic atom pairs to form single bosonic molecules and vice versa[15, 25]. We em-

ploy a functional renormalization-group[18, 23] approach and calculate the renormalized frequency-dependent interaction vertices and fermion self-energies. We find an instability from the disordered quantum liquid phase to a BCS phase and calculate the energy scale for the transition. The unusual frequency-dependence of this mediated interaction leads to strong renormalization of the self-energy, and also affects the couplings in the BCS channel.

3.2 The Model

We consider a two-dimensional square lattice of fermions and bosons [15, 25] with a second-quantized Hamiltonian

$$H = H_b + H_f + H_{bf}$$

The fermion and boson single particle energies are given by

$$H_b = -t_b \sum_{i,j} (b^\dagger_{i,j} b_{i+1,j} + b^\dagger_{i,j} b_{i,,j+1} + h.c.) + (\mu_b - \varepsilon_0) \sum_{i,j} b^\dagger_{i,j} b_{i,j}$$

$$H_f = -t_f \sum_{i,j} (f^\dagger_{i,j} f_{i+1,j} + f^\dagger_{i,j} f_{i,j+1} + h.c.) + \mu_f \sum_{i,j} f^\dagger_{i,j} f_{i,j}$$

where t_b and t_f are the nearest neighbor boson and fermion hopping strengths respectively. The chemical potential μ_f sets the high energy limit or Fermi energy for the total number of fermions. The quantity $(\mu_b - \varepsilon_0)$ is the boson binding energy ε_0 minus the boson chemical potential μ_b. This quantity will play an important role in the RG analysis.

The fermion-boson interaction is given by

$$H_{bf} = -\alpha \sum_{i,j} b^{\dagger}_{i,j} f_{i,j,\sigma+} f_{i,j,\sigma-} + h.c.$$

where a boson is formed or dissociated by the pairing or the annihilation of two opposite spin σ^{+}, σ^{-} fermions in the same lattice site with a coupling strength set by the parameterα.

The creation and annihilation operators are expanded as single-particle plane waves with momentum $\vec{k}$ states in 2-D as

$$b_{i,j} = \frac{1}{2\pi} \int d\vec{k} e^{-i\vec{k}\cdot\vec{x}_{ij}} \phi(\vec{k})$$

and

$$f_{i,j} = \frac{1}{2\pi} \int d\vec{k} e^{-i\vec{k}\cdot\vec{x}_{ij}} \psi(\vec{k})$$

Expanding the boson part of the Hamiltonian in plane-wave states gives

$$\tilde{H}_b = \frac{1}{(2\pi)^2} \int d\vec{k} \bar{\phi}(\vec{k}) \phi(\vec{k}) \varepsilon_b(\vec{k})$$

Where we define

$$\varepsilon_b(\vec{k}) \equiv -2t_b[\cos(k_x) + \cos(k_y)] + \mu_b - \varepsilon_0$$

as the boson dispersion relation shifted by the boson chemical potential and the boson binding energy.

In the same manner we expand the fermion part of the Hamiltonian in single-particle plane waves

$$\tilde{H}_f = \frac{1}{(2\pi)^2} \int d\vec{k} \bar{\psi}(\vec{k}) \psi(\vec{k}) \varepsilon_f(\vec{k})$$

defining

$$\varepsilon_f(\vec{k}) \equiv -2t_f[\cos(k_x) + \cos(k_y)] + \mu_f$$

as the dispersion relation shifted by the chemical potential.

We consider a regime where the square lattice can be approximated by a dilute two-dimensional gas. In that case the quantities $\varepsilon_b(\vec{k})$ and $\varepsilon_f(\vec{k})$ are modified to have quadratic dispersions

$$\varepsilon_b(\vec{k}) \equiv -t_b[k_x^2 + k_y^2] + \mu_b - \varepsilon_0$$

$$\varepsilon_f(\vec{k}) \equiv -t_f[k_x^2 + k_y^2] + \mu_f$$

At ultra cold temperatures $T \to 0$, fermion excitations are confined around the Fermi surface. In this approximation, the fermion energy dispersion is approximated as linear in $\vec{k}$ around

the fermi momentum $\vec{k}_F$

$$\varepsilon_f(\vec{k}) \approx -2\vec{k_F}t_f(\vec{k}-\vec{k}_F)] + \mu_f$$

3.2.1 Matsubara Frequencies And Action

We consider that the boson and fermion fields have a time variable, τ , and Fourier expand the fields into the conjugate variable called Matsubara frequencies.

$$\psi(\vec{k},\tau) = \sum_n \frac{e^{-i\omega_n\tau}}{\beta}\psi(\vec{k},\omega_n)$$

and

$$\phi(\vec{k},\tau) = \sum_n \frac{e^{-i\omega_n\tau}}{\beta}\phi(\vec{k},\omega_n)$$

where

$$\beta = \frac{1}{T}$$

$$\omega_n = \begin{cases} \frac{\pi(2n+1)}{\beta} & \text{fermions} \\ \frac{\pi 2n}{\beta} & \text{bosons} \end{cases}$$

so that

$$S_b = \sum_{n,m}\frac{1}{\beta^2}\int_0^\beta d\tau\bar{\phi}(\omega_n)[\frac{\partial}{\partial\tau} - \varepsilon_b]\phi(\omega)e^{i(\omega_n-\omega_m)\tau}$$

Since the system is at zero temperature, we take the limit as $\beta \to \infty$

$$\lim_{\beta\to\infty} \sum_n \frac{e^{-i\omega_n \tau}}{\beta} \to \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega$$

We also use the orthogonality relation

$$\int_0^\beta e^{i\omega_n \tau} e^{-i\omega_m \tau} d\tau = \delta_{mn}\beta$$

We write the partition function of this system in the Feynman path integral form

$$Z = \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi] \int [\mathcal{D}\bar{\phi}\mathcal{D}\phi] e^{-S_b - S_f - S_{bf}}$$

where the Hamiltonian is now written as part of the boson, fermion, and boson-fermion action S_b, S_f, and S_{bf}.

$$S_b = \int_0^\beta d\tau \bar{\phi}(\vec{k},\tau)[\frac{\partial}{\partial\tau} - \varepsilon_b]\phi(\vec{k},\tau)$$

$$S_f = \int_0^\beta d\tau \bar{\psi}(\vec{k},\tau)[\frac{\partial}{\partial\tau} - \varepsilon_f]\psi(\vec{k},\tau)$$

$$\begin{aligned} S_{bf} = \alpha \prod_{i=1}^{3} \int d\vec{k}_i \int_0^\beta \quad d\tau \quad & [\bar{\phi}(\vec{k}_3,\tau)\psi(\vec{k}_2,\tau)\psi(\vec{k}_1,\tau) \\ + \quad & \bar{\psi}(\vec{k}_3,\tau)\bar{\psi}(\vec{k}_2,\tau)\phi(\vec{k}_1,\tau)] \end{aligned}$$

3.3 The Partition Function

3.3.1 Fermion Scattering Channel

At this point we consider the important channels for the fermion-fermion interaction. Since we are in a circular Fermi surface, we consider only BCS channel scattering where a fermion with k momentum can scatter only with another fermion of $-k$ momentum. Other types of scattering will have a much smaller phase space in the RG process and will be insignificant [18]. From now on, we drop the k dependence of the fields and only keep the Matsubara variable.

3.3.2 Action and Partition Function

We can organize the different parts of the action to expose a boson Gaussian term in the partition function,

$$S_f = \int_{-\infty}^{\infty} d\omega \bar{\psi}(\omega)[i\omega - \varepsilon_f]\psi(\omega)$$

$$S_b = \int_{-\infty}^{\infty} d\omega \bar{\phi}(\omega)[i\omega - \varepsilon_b]\phi(\omega)$$

$$S_{bf} = \alpha \sum_{n,m,l} \frac{1}{\beta^3} \int_0^{\beta} d\tau \bar{\phi}(\omega)\psi(\omega)\psi(\omega)e^{i(\omega_n - \omega_m - \omega_l)\tau} +$$

$$\alpha \sum_{n,m,l} \frac{1}{\beta^3} \int_0^\beta d\tau \bar{\psi}(\omega)\bar{\psi}(\omega)\phi(\omega) e^{i(\omega_n - \omega_m - \omega_l)\tau}$$

$$\int_0^\beta d\tau e^{i(\omega_n - \omega_m - \omega_l)\tau} = \frac{e^{i(\omega_n - \omega_m - \omega_l)\beta} - 1}{i(\omega_n - \omega_m - \omega_l} = \delta_{\omega_n, \omega_m + \omega_l}\beta$$

$\delta_{\omega_n, \omega_m + \omega_l} \rightarrow \omega_n = \omega_m + \omega_l$.

$$\sum_{m,n,l} \rightarrow \sum_{m,l} \rightarrow \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2$$

$$S_{bf} = \frac{\alpha}{\beta^2} \int_{-\infty}^{\infty} d\omega [\bar{\phi}(\omega)\psi(\omega)\psi(\omega) + \bar{\psi}(\omega)\bar{\psi}(\omega)\phi(\omega)]$$

Combining the boson and interaction part of the action,

$$\begin{aligned} S_b + S_{bf} = \int d\omega [\bar{\phi}(\omega) + \frac{\alpha}{\beta^2} \frac{\bar{\psi}(\omega)\bar{\psi}(\omega)}{i\omega - \varepsilon_b}][i\omega - \varepsilon_b(k)][\phi(\omega) \\ + \frac{\alpha}{\beta^2} \frac{\psi(\omega)\psi(\omega)}{i\omega - \varepsilon_b(k)}] + -\alpha^2 \int d\omega \frac{\bar{\psi}(\omega)\bar{\psi}(\omega)\psi(\omega)\psi(\omega)}{i\omega - \varepsilon_b(k)} \end{aligned}$$

Redefining or transforming the bosonic fields

$$\bar{\phi}(\omega) + \alpha \frac{\bar{\psi}(\omega)\bar{\psi}(\omega)}{i\omega - \varepsilon_b(k)} \rightarrow \bar{\phi}'$$

$$\phi(\omega) + \alpha \frac{\psi(\omega_m)\psi(\omega_n)}{i\omega - \varepsilon_b(k)} \rightarrow \phi'$$

which leaves the Jacobian of the path integral unchanged.

$$Z_0 = \int [\mathcal{D}\bar{\phi}\mathcal{D}\phi] \exp\{-\int d\omega [\bar{\phi}'(\omega)[i\omega - \varepsilon_b(k)][\phi'(\omega)]\}$$

$$Z_0 = \prod_{\omega} \frac{1}{\sqrt{i\omega - \varepsilon_b(k)}}$$

We make new definitions of the parts of the action

$$S_0 = \int_{-\infty}^{\infty} d\omega \bar{\psi}(\omega)[i\omega - \varepsilon_f(k)]\psi(\omega)$$

$$S_I = -\alpha^2 \int d\omega \frac{\bar{\psi}(\omega)\bar{\psi}(\omega)\psi(\omega)\psi(\omega)}{i\omega - \varepsilon_b(k)}$$

and the partition function is now

$$\mathcal{Z} = Z_0 \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi] e^{-S_0 - S_I}$$

$$S_I = -\alpha^2 \int d\omega \frac{\bar{\psi}_4\bar{\psi}_3\psi_2\psi_1}{i\omega - \varepsilon_b(k)}$$

with an independent fermion action S_0 and a fermion-boson interacting action S_I.

3.3.3 Fast and Slow Field Decomposition

The next step is to decompose the fermionic fields into slow and fast modes (low energy and high energy excitations) and integrate the fast fields of the partition function. As a result , we will write an effective partition function with new terms that will modify the initial interaction potential $U_0(\omega)$. By continuously doing the integration of the fast modes, the energy window that confines the particle scattering is shrunk towards the Fermi surface. This procedure can be thought of as looking for the temperature at which the phase transition occurs. Note that we are really not lowering the temperature since we have formulated this theory as being at $T = 0$. Another way to view the energy rescaling is that we are restraining the fermion excitations around the fermi surface. And this is analogous to lowering the temperature in a real system.

At lower energy scales, the system of fermions has less degrees of freedom and high energy interactions are less likely to occur. There may exist an energy scale at which we can observe the phase of the ground state. This phase of course would be a BCS ground state since we are only considering this type of scattering. The question we would like to answer is what are the conditions or constraints on the system parameters to realize the BCS phase.

To answer this question, we need to follow the RG flow of interaction potential as the energy scale is reduced. This can be done numerically by solving the RG differential equations that we will set up. Then we will look for a divergent correlation function. In this system, we expect that the interaction potential will diverge at some critical energy scale. This would imply that the interaction potential is relevant or in other words, that the BCS phase that we

are considering exists given that the fermions are interacting through the formation and dissociation of a boson. If the interaction potential RG flow were to decay, then it would mean that the BCS phase does not exist given this type of fermion interaction.

The first step in the Renormalization Group procedure is to split the fields into high energy modes (large fermion k momentum scattering with the Fermi momentum as the reference point set at $k_f = 0$) and low energy modes (small fermion k momentum scattering)

The fermion fields will be $\psi_<$ for $0 < k < \Lambda/s$ and $\psi_>$ for $\Lambda/s < k < \Lambda$ where Λ is called the momentum cutoff (also called the energy cutoff due to some energy-momentum relation for fermions) and s is a parameter(probably just a number, but symbolic here). The exponential in the partition function will be

$$e^{-\bar{\psi}\psi} = e^{-(\bar{\psi}_< + \bar{\psi}_>)(\psi_< + \psi_>)}$$

$$e^{-\bar{\psi}\psi} = e^{-(\bar{\psi}_<\psi_< + \bar{\psi}_<\psi_> + \bar{\psi}_>\psi_< + \bar{\psi}_>\psi_>)}$$

which create new parts of the action S.

The partition function for fermions is now

$$\mathcal{Z} = Z_0 \int e^{S_0(\psi_<) + S_0(\psi_>) + S_I(\psi_<, \psi_>)} [\mathcal{D}\bar{\psi}_< \mathcal{D}\psi_<][\mathcal{D}\bar{\psi}_> \mathcal{D}\psi_>]$$

with the action S composed of slow and fast fields

$$S(\psi_<, \psi_>) = S_0(\psi_<) + S_0(\psi_>) + S_I(\psi_<, \psi_>)$$

We integrate the high energy modes $S_0(\psi_>)$ that leaves a partition function of an effective fermion-fermion interaction. We organize the partition function by separating the high energy mode integral from the low energy mode

$$\mathcal{Z}_0 = Z_0 \int [\mathcal{D}\bar{\psi}_< \mathcal{D}\psi_<] e^{S_0(\psi_<)} \int [\mathcal{D}\bar{\psi}_> \mathcal{D}\psi_>] e^{-S_0(\psi_>) - S_I(\psi_<, \psi_>)}$$

Now we manipulate the partition function in the following way

$$\begin{aligned}
\mathcal{Z} &= Z_0 \int [\mathcal{D}\bar{\psi}_< \mathcal{D}\psi_<] e^{-S_0(\psi_<)} \frac{\int [\mathcal{D}\bar{\psi}_> \mathcal{D}\psi_>] e^{-S_0(\psi_>)} e^{-S_I(\psi_<, \psi_>)}}{\int [\mathcal{D}\bar{\psi}_> \mathcal{D}\psi_>] e^{-S_0(\psi_>)}} \\
&\quad \times \int [\mathcal{D}\bar{\psi}_> \mathcal{D}\psi_>] e^{-S_0(\psi_>)} \\
&= Z_0 \int [\mathcal{D}\bar{\psi}_< \mathcal{D}\psi_<] e^{-S_0(\psi_<)} \int [\mathcal{D}\bar{\psi}_> \mathcal{D}\psi_>] e^{-S_0(\psi_>)} \langle e^{-S_I(\psi_<, \psi_>)} \rangle_{\psi_>} \\
&= Z_0 \int [\mathcal{D}\bar{\psi} \mathcal{D}\psi] e^{-S_0(\psi_<)} \langle e^{-S_I(\psi_<, \psi_>)} \rangle_{\psi_>}
\end{aligned}$$

where $\langle \rangle_{\psi_>}$ denotes an average with respect to the fast modes.

We can deal with the interaction term $S_I(\psi_<, \psi_>)$ by performing a cumulant expansion, assuming the interaction parameter α^2 is small . We identify the interaction potential, or coupling function $U_0(\omega)$, as $-\frac{\alpha^2}{i\omega - \omega_b(k)}$

$$\langle e^{-S_I} \rangle = \langle 1 - S_I + \frac{1}{2!} S_I^2 + \ldots \rangle$$

for small α^2 we can keep the first three terms only

$$\langle e^{-S_I} \rangle \approx 1 - \langle S_I \rangle + \frac{1}{2!} \langle S_I^2 \rangle$$

$$\approx e^{\ln[1 - \langle S_I \rangle + \frac{1}{2!} \langle S_I^2 \rangle]}$$

using the expansion of $\ln(1+\epsilon)$ for small ϵ

$$\begin{aligned} \ln[1 - \langle S_I \rangle + \frac{1}{2!} \langle S_I^2 \rangle] &\approx -\langle S_I \rangle + \frac{1}{2!} \langle S_I^2 \rangle \\ &\quad -\frac{1}{2!} [\langle S_I \rangle^2 - 2 \langle S_I \rangle \langle S_I^2 \rangle + \frac{1}{4} \langle S_I^2 \rangle^2] \end{aligned}$$

then cut off terms which are not linear or quadratic (which we can do for small α^2)

$$\approx -\langle S_I \rangle + \frac{1}{2} [\langle S_I^2 \rangle - \langle S_I \rangle^2]$$

so that the average of an exponential is an exponential of averages up to quadratic terms

$$\langle e^{-S_I} \rangle \approx e^{-\langle S_I \rangle + \frac{1}{2} [\langle S_I^2 \rangle - \langle S_I \rangle^2]}$$

which will modify the interaction potential $U(\omega)$. We can approximate the partition function of the system as

$$\begin{aligned} \mathcal{Z} &= Z_0 \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi] e^{-S_0} \langle e^{-S_I} \rangle_{\psi_>} \\ &\approx Z_0 \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi] e^{-S_0} \langle e^{-\langle S_I \rangle + \frac{1}{2}[\langle S_I^2 \rangle - \langle S_I \rangle^2]} \rangle_{\psi_>}] \end{aligned}$$

$$\begin{aligned} S_I(\psi_<, \psi_>) = \int dk \int_{-\infty}^{\infty} d\omega_1 d\omega_2 U_0(\omega_1 + \omega_2)(\bar{\psi}_{4<} + \bar{\psi}_{4>})(\bar{\psi}_{3<} + \bar{\psi}_{3>}) \\ \times(\psi_{2<} + \psi_{2>})(\psi_{1<} + \psi_{1>}) \end{aligned}$$

$$S_I(\psi_<, \psi_>) = S_I(\psi_>) \times \int_{-d\Lambda}^{d\Lambda} dk \int_{-\infty}^{\infty} d\omega_1 d\omega_2 U_0(\omega_1 + \omega_2) \bar{\psi}_{4<} \bar{\psi}_{3<} \psi_{2<} \psi_{1<}$$

with $U_0(\omega_1, \omega_2) = -\frac{\alpha^2}{i(\omega_1+\omega_2)-\omega_b(k)}$.

3.4 Renormalization-Group Flow Equations and Results

3.4.1 Self-Energy Corrections

We refer to [18] for setting up the RG equations. The RG equation for the self energy is

$$\frac{d\Sigma_l(\omega,k)}{dl} = -\Lambda_l \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{N(\Lambda_l)}{i\omega' - \Lambda - \Sigma_l(\omega)} \times \frac{-\alpha^2}{i(\omega+\omega') - \omega_{k+k'}}$$

In the above equation Σ is the self energy which depends on the fermion Matsubara frequency ω, the fermion momentum k, and the energy scale parameter l such that $\Lambda = \Lambda_0 e^{-l}$ (The energy scale is decreased slowly as high energy modes are integrated).

The self energy consists of a real part and an imaginary part. $\Sigma = \Sigma^{'} + \imath\Sigma^{"}$. The real part is related to the chemical potential. Since, in the case of fermions, there is no change in the number of particles, we set the real part of the self energy to be irrelevant under renormalization. The imaginary part of Σ is related to the lifetime of the fermion quasi particle. $\Sigma^{"} \sim \tau^{-1}$, where τ is the quasi particle lifetime.

The differential equation $\frac{d\Sigma_l(\omega,k)}{dl}$ can be separated into real and imaginary parts. Omitting the integral for clarity,

$$\frac{d\Sigma_l(\omega,k)}{dl} = \frac{N(\Lambda_l)}{i\omega' - \Lambda - \imath\Sigma_l^{"}(\omega')} \times \frac{-i\omega' - \Lambda + \imath\Sigma_l^{"}(\omega')}{-i\omega' - \Lambda + \imath\Sigma_l^{"}(\omega')} \times \frac{-\alpha^2}{i(\omega+\omega') - \omega_{k+k'}} \times \frac{-i(\omega+\omega') - \omega_{k+k'}}{-i(\omega+\omega') - \omega_{k+k'}}$$

$$= \frac{N(\Lambda_l)}{-\Lambda + i[\omega' - \Sigma_l^{"}(\omega)]} \times \frac{-\Lambda - i[\omega' - \Sigma^{"}(\omega')]}{-\Lambda - i[\omega' - \Sigma^{"}(\omega')]} \times \frac{-\alpha^2}{-\omega_{k+k'} + i(\omega + \omega')} \times \frac{-\omega_{k+k'} - i(\omega + \omega')}{-\omega_{k+k'} - i(\omega + \omega')}$$

$$= \frac{-\alpha^2 N(\Lambda_l)(-\Lambda - i[\omega' - \Sigma^{"}(\omega')])}{\Lambda^2 + [\omega' - \Sigma^{"}(\omega')]^2} \times \frac{-\omega_{k+k'} - i(\omega + \omega')}{\omega_{k+k'}^2 + (\omega + \omega')^2}$$

The imaginary part of the above is

$$\frac{-\alpha^2 N(\Lambda_l)[\Lambda(\omega + \omega') + (\omega' - \Sigma^{"}(\omega'))\omega_{k+k'}]}{[\Lambda^2 + (\omega' - \Sigma^{"}(\omega'))^2][\omega_{k+k'}^2 + (\omega + \omega')^2]}$$

Therefore the RG flow equation for the imaginary part of the self energy is

$$\frac{d\Sigma_l^{"}(\omega, k)}{dl} = N(\Lambda_l)\alpha^2 \Lambda_{l'} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{[\Lambda_{l'}(\omega + \omega') + (\omega' - \Sigma^{"}(\omega'))\omega_{k+k'}]}{[\Lambda_{l'}^2 + (\omega' - \Sigma_{l'}^{"}(\omega'))^2][\omega_{k+k'}^2 + (\omega + \omega')^2]}$$

Since we are interested in low energy excitations with respect to the Fermi surface, we can approximate the density of states $N(\Lambda_l)$ to be constant N. For now, we will also make the assumptions that the dispersion relation $\omega_{k+k'}$ does not depend on l.

$$\frac{d\Sigma_l^{"}(\omega, k)}{dl} = N\alpha^2 \Lambda_{l'} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{[\Lambda_{l'}(\omega + \omega') + (\omega' - \Sigma^{"}(\omega'))\omega_{k+k'}]}{[\Lambda_{l'}^2 + (\omega' - \Sigma_{l'}^{"}(\omega'))^2][\omega_{k+k'}^2 + (\omega + \omega')^2]}$$

We have solved this equation numerically with the initial condition at $l = 0$, $\Sigma_{l=0}^{"}(\omega') = 0$. Our results show that $\Sigma_l^{"}(\omega)$ is a very slowly varying function of ω with small values that render the function as an insignificant contribution to the renormalization of the fermion-

fermion interaction potential $U(\omega_1 + \omega_2)$. For $\omega = 0$, we found that $\Sigma_l^{"}(\omega) = 0$ for all l values. Therefore, from now on, we will use $\Sigma_l^{"}(\omega) \approx 0$ for all ω and l .

3.4.2 Vertex Corrections

The RG equation for the effective fermion-fermion interaction potential is

$$\frac{dU}{dl} = \int d\omega' G_1(\omega', \Sigma^{"}(\omega)) G_2(\omega', \Sigma^{"}(\omega)) U(\omega_1, \omega_2, l) U(\omega_1, \omega_2, l)$$

where $G_1(\omega', \Sigma_1(\omega)) = \frac{1}{i\omega' - \Lambda_l - i\Sigma_l^{"}(\omega')}$ and $G_2(\omega', \Sigma(\omega)) = \frac{1}{i(\omega_1+\omega_2-\omega') - \Lambda_l - i\Sigma_l^{"}(\omega_1+\omega_2-\omega')}$.

The initial fermion-fermion Matsubara frequency dependent interaction $U_0(\omega_1, \omega_2) = -\frac{\alpha^2}{i(\omega_1+\omega_2) - \omega_b(k)}$ is complex.

The numerical solution to the RG equation gives trajectories in the complex plane that flow with energy scale l for each fermion frequencies combination ω_1 and ω_2.

Our numerical and analytic results show that $U_l(\omega_1, \omega_2)$ diverges for frequencies $\omega_1 = -\omega_2$ at some critical energy scale Λ_c.

3.4.3 Semi-Analytic Solution to RG Equation for $\omega_1 = -\omega_2$

Only frequencies that satisfy $\omega_1 + \omega_2 = \omega = 0$, show a divergent interaction potential $U_l(\omega_1, \omega_2)$. Using $\Sigma_l(\omega = 0) = 0$, the RG equation for this frequency combination is

$$\frac{dU_{l,\omega=0}}{dl} = U^2 \int d\omega' \frac{1}{i\omega' - \Lambda_l} \times \frac{1}{-i\omega' - \Lambda_l}$$

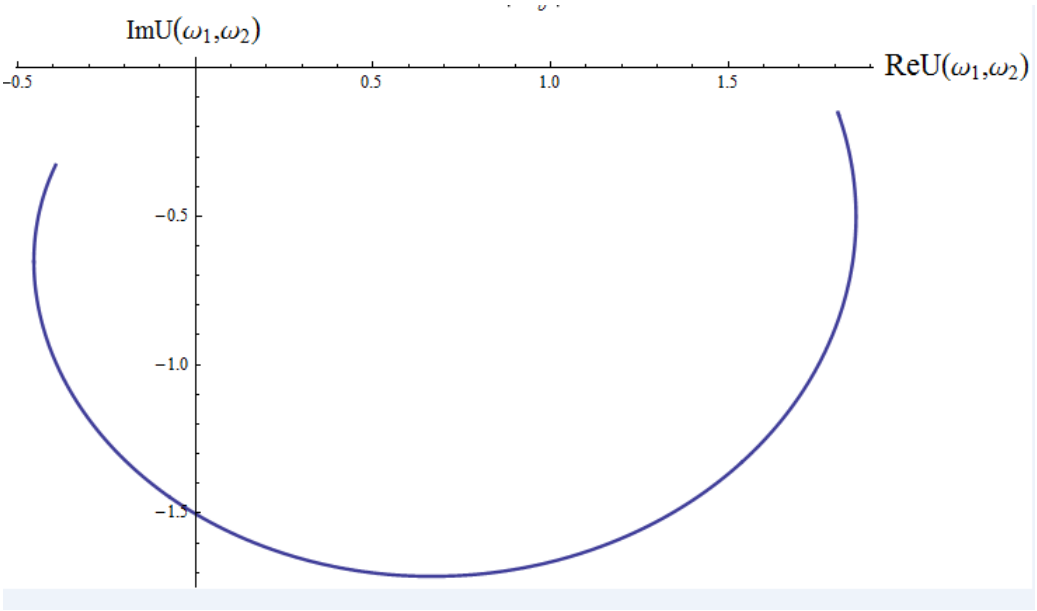

Figure 3.1: The complex interaction potential for a particular frequency combination where $\omega_1+\omega_2 \neq 0$ flows counterclockwise. The potential flows past the critical point to illustrate its asymmetry. A symetric flow for the frequency independent situation is shown in Appendix D.

$$= U^2 \int d\omega' \frac{1}{\omega'^2 + \Lambda_l^2}$$

$$\frac{dU_{l,\omega=0}}{dl} = U^2 \frac{2}{\Lambda_l} \arctan(\frac{\omega_{max}}{\Lambda_l})$$

which is solved easily

$$\int_{U_0}^{U} \frac{dU}{U} = 2 \int_{\Lambda_0}^{\Lambda_c} \frac{d\Lambda_l}{\Lambda_l} \arctan(\frac{\omega_{max}}{\Lambda_l})$$

$$-\frac{1}{U} + \frac{1}{U_0} = 2 \int_{\Lambda_0}^{\Lambda_c} \frac{d\Lambda_l}{\Lambda_l} \arctan(\frac{\omega_{max}}{\Lambda_l})$$

$$\frac{1}{U} = \frac{1}{U_0} - 2 \int_{\Lambda_0}^{\Lambda_c} \frac{d\Lambda_l}{\Lambda_l} \arctan(\frac{\omega_{max}}{\Lambda_l})$$

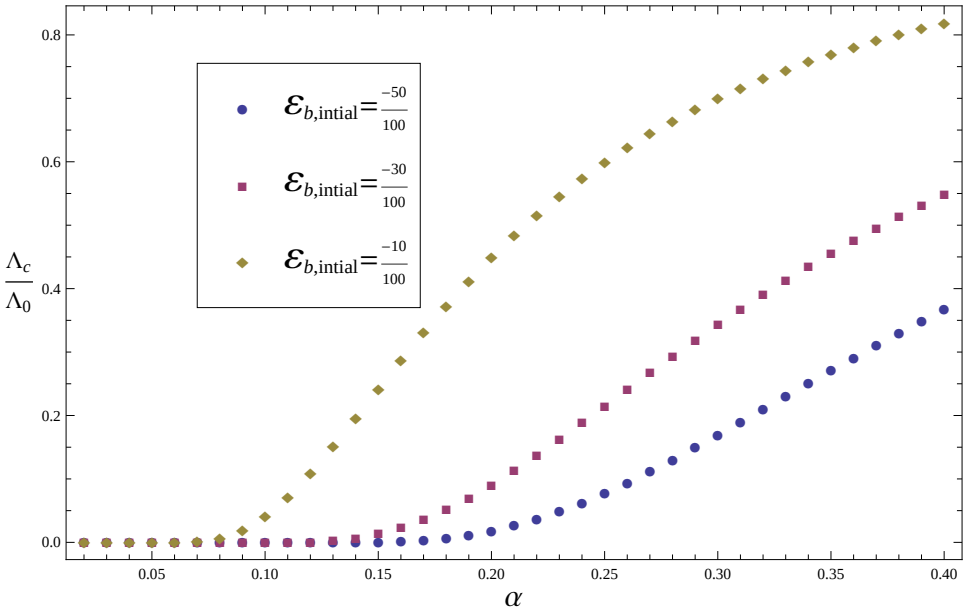

Figure 3.2: The figure shows a plot of the energy scale, measured in terms of the initial energy window Λ_0 of allowed fermion excitations, at which the fermion interaction potential diverges as a function of the coupling parameter. This energy scale can be interpreted loosely as the phase transition temperature from a Fermi liquid to a BCS phase.

$$\frac{1}{U} = \frac{U_0}{1 - 2U_0 \int_{\Lambda_0}^{\Lambda_c} \frac{d\Lambda_l}{\Lambda_l} \arctan(\frac{\omega_{max}}{\Lambda_l})}$$

The integral over Λ_l can be done numerically and the solution can be written as

$$\frac{1}{U} = \frac{U_0}{1 - Re[(-iPolyLog[2, \frac{i\omega_{max}}{\Lambda_l}])|_{\Lambda_0}^{\Lambda_c}}$$

We solve for the critical energy scale Λ_c by setting the denominator equal to 0 . The results agree very well with our numerical solutions.

Method of Integration

An integral of the form $\int \frac{d\omega}{-i\omega-\Lambda} \times \frac{1}{i(\omega+x)-\Lambda}$ can be split into two integrals

$$\frac{1}{ix-2\Lambda}[\int \frac{d\omega}{-i\omega-\Lambda} + \int \frac{d\omega}{i(\omega+x)-\Lambda}]$$

The first integral is solved by taking imaginary and real parts

$$\int \frac{d\omega}{-i\omega-\Lambda} = \int \frac{d\omega}{-i\omega-\Lambda} \times \frac{i\omega-\Lambda}{i\omega-\Lambda}$$

$$= \int \frac{i\omega d\omega}{\omega^2+\Lambda^2} - \Lambda \int \frac{d\omega}{\omega^2+\Lambda^2}$$

$$= \frac{i}{2}\ln(\omega^2+\Lambda^2) - \frac{1}{\Lambda}\int \frac{d\omega}{(\frac{\omega}{\Lambda})^2+1}$$

$$\int \frac{d\omega}{-i\omega-\Lambda} = \frac{i}{2}\ln(\omega^2+\Lambda^2) - \tan^{-1}(\frac{\omega}{\Lambda})$$

The second integral is solved similarly

$$\int \frac{d\omega}{i(\omega+x)-\Lambda} = -\frac{i}{2}\ln((\omega+x)^2+\Lambda^2) - \tan^{-1}(\frac{\omega+x}{\Lambda})$$

The solution of the total integral, including the symmetric limits of the Matsubara frequency ω from $\omega = -\omega_{max}$ to $\omega = \omega_{max}$

$$\int_{-\omega_{max}}^{\omega_{max}} \frac{d\omega}{-i\omega-\Lambda} \times \frac{1}{i(\omega+x)-\Lambda} = \frac{1}{ix-2\Lambda}[\frac{i}{2}\ln(\omega^2+\Lambda^2)|_{-\omega_{max}}^{\omega_{max}} - \tan^{-1}(\frac{\omega}{\Lambda})|_{-\omega_{max}}^{\omega_{max}} +$$

$$-\frac{i}{2}\ln((\omega+x)^2+\Lambda^2)|_{-\omega_{max}}^{\omega_{max}} - \tan^{-1}(\frac{\omega+x}{\Lambda})|_{-\omega_{max}}^{\omega_{max}}]$$

$$= \frac{1}{ix-2\Lambda}[\frac{i}{2}\ln[\frac{\omega_{max}^2+\Lambda^2}{\omega_{max}^2+\Lambda^2}] - 2\tan^{-1}(\frac{\omega_{max}}{\Lambda}) +$$

$$-\frac{i}{2}\ln[\frac{(\omega_{max}+x)^2+\Lambda^2}{(-\omega_{max}+x)^2+\Lambda^2}] - \tan^{-1}(\frac{\omega_{max}+x}{\Lambda}) + \tan^{-1}(\frac{-\omega_{max}+x}{\Lambda})]$$

We are interested in the case when $x = 0 = \omega_1 + \omega_2$. The solution is simply

$$\lim_{x\to 0}\int_{-\omega_{max}}^{\omega_{max}} \frac{d\omega}{-i\omega-\Lambda} \times \frac{1}{i(\omega+x)-\Lambda} = \frac{2}{\Lambda}\tan^{-1}(\frac{\omega_{max}}{\Lambda})$$

3.5 Green's Function Corrections

The Greens function for the boson molecule is

$$G_l^b(\omega) = \frac{1}{\omega - \varepsilon_b(l) + i\delta}$$

Initially at $l = 0$, after integrating the bosonic degrees of freedom, the effective fermion-fermion interaction in the fermi sea is

$$U_{l=0}(\omega_1, \omega_2) = -\frac{\alpha^2}{i(\omega_1+\omega_2) - \varepsilon_b(l=0)}$$

After analytic continuation, this interaction is the same a the boson molecule Green's function

$$U_{l=0}(\omega_1, \omega_2) = -\alpha^2 G_l^b(\omega_1 + \omega_2)$$

.

At any RG step (step in the numerical integration) before reaching the critical energy scale Λ_c, we can extract the bosonic molecule Green's function. This is done by numerically extracting the correlation $U(\omega_1, \omega_2)$ as a function of Matsubara frequencies. The numerical procedure involves solving the RG flow equations for all combinations of frequencies. The $U(\omega_1, \omega_2)$ function is then constructed by at any RG step . The numerical data is then fit to a function of the form $\frac{a}{i\omega - \varepsilon_b(l)}$. The function is analytically continued $i\omega \to \omega + i\delta$ and value of $\varepsilon_b(l)$ is extracted.

At the critical scale l_c, we found that $\varepsilon_b(l_c) = \mu_b - \varepsilon_0 = 0$. Note that we have set the kinetic energy of the boson molecule to zero since we are at $T = 0$ and assume that the molecule is at its zero point state. At chemical equilibrium it is required that $\mu_b = 2\mu_f$. This in turn dictates that

$$\varepsilon_0 = 2\mu_f$$

The binding energy of the boson should equals twice the chemical potential of its constituent Fermi particles. In the usual cold atom experiments, the binding energies can be tuned via magnetic fields (Feshbach resonance). In a real system, this can be realized if the there exists a molecular resonance at $2\mu_f$. Then tuning the external magnetic field to the resonant value

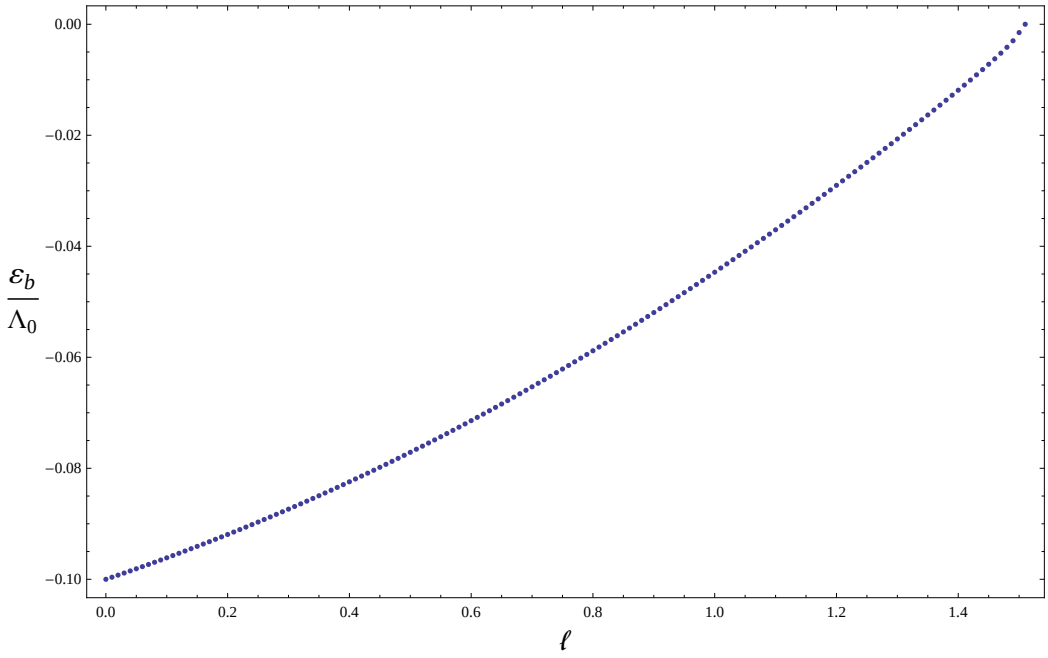

Figure 3.3: The boson energy is numerically extracted from the boson Green's function. When the boson energy equals 0, the fermion interaction potential diverges.

would establish chemical equilibrium necessary for a BCS phase. On the other hand, could we forget about an external magnetic field and simply tune the fermion chemical potential to reach the same results? The consequences of these results need to be explored further.

3.6 Conclusion

We applied the method of functional renormalization group to a dilute gas mixture of ultra cold bosons and fermions. The fermion-fermion interactions are mediated by the formation of a bosonic molecule with the conservation of the total number of particles. After integrating the bosonic degrees of freedom, the dominant BCS fermion-fermion scattering in the disordered Fermi liquid implies an instability to a BCS superfluid phase. We found this

phase to exist independent of the coupling parameters but strongly dependent on the boson and fermion chemical potentials. Our calculations suggest that finding a molecular resonant energy equal to twice the fermion chemical potential may be the condition necessary for the many body BCS ground state. Furthermore as a mathematical interest, we found that the imaginary coupling function results in Renormalization Group flows that are complex (see appendix) as seen in high energy theory [24].

Chapter 4

Conclusions

Taking a microscopic view point, we proposed a low temperature two-atom model that takes into account magnetic interaction effects dependent on the relative distance between the two atoms. The adiabatic basis that diagonalizes the relative potential energy in the two-body Hamiltonian leads to a pair of coupled equations between the closed and open channels. These in turn give an energy dependent s-wave scattering length that is used to find the energy spectrum in an ideal case of two atoms placed in a confining harmonic potential. Our results show that narrow Feshbach resonances shape the two-atom bound state probability curves to select a single resonant energies. In contrast, broad Feshbach resonances cannot pinpoint single resonances but instead mix different two-atom energy levels due to the wide spread in the probability curves. This case may be realized in an optical lattice with sufficient hopping suppression.

We have compared the harmonically confined two-atom bound state probability to the boson occupation number in an effective single-site lattice Hamiltonian where the two possible states are a boson or two fermions. The bound state amplitude of the trapped two atom system agrees very well with the boson occupancy number provided the applied magnetic field is of relatively high resolution. For broader Feshbach resonances, the occupancy numbers agree far from the resonant values but the agreement breaks down as the applied magnetic field approaches the magnetic field resonant value.

Given the validity of the effective Hamiltonian in the narrow Feshbach resonance regime, we have considered a cold atom mixture of fermions interacting through boson conversion. We proposed a two dimensional lattice model where fermions are allowed to hop to nearest neighbor sites and oppositely spin-aligned fermions can bind into a boson particle. Using Functional Renormalization Group theory we concluded that the many body ground state is BCS with s-wave pairing. The conditions to reach this ground state are simply chemical equilibrium and by definition the limit of zero temperature. Furthermore, the renormalization of the boson molecule energy dictates that the binding energy should be tuned to twice the value of the fermion chemical potential in order to realize the BCS ground state.

The effective single site Hamiltonian validates the many-body Hamiltonian that leads to a BCS ground state with s-wave paring. This implies that, an experimentalist could create a BCS phase given a dilute gas of fermions or an optical lattice of fermions if the fermion-fermion interactions are controlled by a narrow Feshbach resonance. The fermion

to molecule conversion is more likely when the applied magnetic field is of high resolution (in the oder of 10 mGauss). Experiments have yet to verify this prediction.

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Appendix A

Cross Section and Scattering Amplitude

(The information in this section was compiled from several sources [19, 20, 21, 22]) An important quantity to calculate is the cross section (the probability of an observed event). The cross section is an experimentally measurable probability of scattered particles. We can formulate the cross section in the following way. Suppose you know the differential probability of an incoming particle traveling in the z direction with velocity v in 3 dimensions and the particle is in a beam of area A. The differential probability is simply related to the square modulus of it's wavefunction times the differential volume $dV = Avdt$.

$$dP = |\psi_{in}|^2 Avdt$$

If the particle scatters off some potential, it can go in any direction in 3 dimensions but it's probability must be conserved in time. There's no reason why the probability should change

(i.e. the probability current must be constant similar to the flow of water through pipes). So the outgoing differential probability in spherical coordinates

$$dP = |\psi_{out}|^2 Avdt = |\psi_{out}|^2 r \sin\theta d\theta d\phi v dt$$

where the differential volume in spherical coordinates is $dV = Avdt = rd\theta r \sin\theta d\phi dr = r^2 d\theta \sin\theta d\phi v dt$.

Therefore,

$$dP = |\psi_{out}|^2 Avdt = |\psi_{out}|^2 r^2 \sin\theta d\theta d\phi v dt$$

If the incoming particle is a free particle it's wavefunction is $\psi_{in} = e^{ikz}$ in the z-direction (suppose the experimenter chooses the direction of the particles) and the outgoing wavefunction must have a free particle form with the possibility of an angle dependence as$\psi_{out} = \frac{e^{ikr}}{r} f(\theta)$. This form describes the particle could go in any combination of x,y, and z directions. The $\frac{1}{r}$ dependence serves to cancel out the r^2 in the differential volume. It must be canceled out because the probability should not have an r dependence. As a particle travels in the r direction, it's probability of it's location should not change. If that were the case, then it would matter how far the detector could be placed from the scatterer.But there can be a θdependence since the outgoing wave is not traveling in the $\hat{\theta}$direction. So the scattering

probability on a unit area, more commonly called the differential cross section, is

$$d\sigma = |f(\theta)|^2 \sin\theta d\theta d\phi$$

and is related to the differential probability by

$$dP = d\sigma v dt$$

But since experiments place stationary detectors to observe scattering, it is much easier to detect the scattered particles with only space dependence. The total cross section is the probability of scattering in all angular directions.

$$\sigma = \int_0^{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \sin\theta d\theta d\phi \tag{A.1}$$

$f(\theta)$ is called the scattering amplitude and it is the quantity we must determine to calculate the total probability of observing a scattered particle at the detector. For low temperature physics, we can consider low energy scattering. How does the scattering amplitude depend on energy at low temperatures?

A.0.1 Connection to Phase Shift

Since we don't know much about $f(\theta)$, let's try to make a connection between the scattering amplitude and something more intuitive, the wavefunction phase shift. We can begin by

solving for the scattered wavefunction from the 3 dimensional Schrodinger equation. By separation of variables (the r, θ, ϕ), the full 3 dimensional Schrodinger equation can be split into three equations. The radial equation is reduced to a 1 dimensional equation $\chi(r) = r\psi(r)$ with angular momentum dependence l given by

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(r)}{dr} + [V(r) - E]\chi(r) = \frac{l(l+1)}{r^2}\chi(r) \tag{A.2}$$

The solution to the angular part is Legendre polynomials $P_l \cos(\theta)$. At very large r, where experimentalists take measurements (placing the detector far from the scatterer), we can neglect the scattering potential $V(r)$, since it is short ranged, and the centrifugal term $\frac{l(l+1)}{r^2}\chi(r)$. We are left with

$$\frac{d^2\chi(r)}{dr} = -\frac{2mE}{\hbar^2}\chi(r) \tag{A.3}$$

which has a solution

$$\chi(r) = A_l \sin(kr - \frac{\pi l}{2} + \delta_l) \tag{A.4}$$

with $k = \frac{\sqrt{2mE}}{\hbar}$, the $\frac{\pi l}{2}$ term is placed for convenience, and δ_l is an angular momentum dependent phase shift due to the scattering potential $V(r)$. This phase shift is the only change that can occur to the radial part of incoming wavefunction as a result of interacting with a

potential $V(r)$. The solution to the 3 dimensional Schrodinger equation at large r is

$$\psi_{total}(r) = \sum_{l=0}^{\infty} A_l \frac{\sin(kr - \frac{l\pi}{2} + \delta_l)}{kr} P_l \cos(\theta) \tag{A.5}$$

(remember that $\psi(r) = \frac{\chi(r)}{r}$).

Now we would like to make a connection between the intuitive notion of a wave phase shift δ_l and a scattering amplitude $f(\theta)$ of which we know nothing. To make a connection we can begin by expanding the incoming wave of the total wavefunction in terms of Legendre polynomials at a large distance r

$$\Psi_{total}(r) = e^{ikz} + \frac{f(\theta)}{r} e^{ikr} \tag{A.6}$$

$$\psi_{in} = e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l \cos(\theta) \tag{A.7}$$

where $j_l(kr)$ is the Bessel function. The $\lim_{r\to\infty} j_l(kr) = \frac{\sin(kr - \frac{l\pi}{2})}{kr}$, so that

$$\psi_{in} = e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) \frac{\sin(kr - \frac{l\pi}{2})}{kr} P_l \cos(\theta) \tag{A.8}$$

The description of a wave with phase shift must be equivalent to the description of the wave with a scattering amplitude,

$$\sum_{l=0}^{\infty} A_l \frac{\sin(kr - \frac{l\pi}{2} + \delta_l)}{kr} P_l \cos(\theta) = \sum_{l=0}^{\infty} i^l (2l+1) \frac{\sin(kr - \frac{l\pi}{2})}{kr} P_l \cos(\theta) + \frac{f(\theta)}{r} e^{ikr} \tag{A.9}$$

Our goal is to make a connection between the scattering amplitude $f(\theta)$ and the phase shift δ_l. We can solve for the coefficients A_l by expanding the $\sin$function in terms of exponential

$$\sum_{l=0}^{\infty} A_l \frac{e^{ikr}e^{-i\frac{l\pi}{2}}e^{i\delta_l} - e^{-ikr}e^{i\frac{l\pi}{2}}e^{-i\delta_l}}{2ikr} P_l\cos(\theta) = \sum_{l=0}^{\infty} i^l(2l+1)\frac{e^{ikr}e^{-i\frac{l\pi}{2}} - e^{-ikr}e^{i\frac{l\pi}{2}}}{2ikr} P_l\cos(\theta) + \frac{f(\theta)}{r}e^{ikr}$$

and comparing e^{-ikr} terms

$$A_l \frac{e^{-ikr}e^{i\frac{l\pi}{2}}e^{-i\delta_l}}{2ikr} = i^l(2l+1)\frac{e^{-ikr}e^{i\frac{l\pi}{2}}}{2ikr} \tag{A.10}$$

Note first that $\pm i^l = (e^{\pm i\frac{\pi}{2}})^l$ since $e^{\pm i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) \pm i\sin(\frac{\pi}{2}) = \pm i$ so that eqtn. 3.9 becomes

$$A_l e^{i\frac{l\pi}{2}}e^{-i\delta_l} = e^{i\frac{l\pi}{2}}(2l+1)e^{i\frac{l\pi}{2}}$$

and

$$A_l = (2l+1)e^{i(\frac{l\pi}{2}+\delta_l)}$$

Now we can solve for the scattering amplitude $f(\theta)$ by matching the terms with e^{ikr}

$$A_l \frac{e^{ikr}e^{-i\frac{l\pi}{2}}e^{i\delta_l}}{2ikr} P_l\cos(\theta) = e^{i\frac{l\pi}{2}}(2l+1)\frac{e^{ikr}e^{-i\frac{l\pi}{2}}}{2ikr} P_l\cos(\theta) + \frac{f(\theta)}{r}e^{ikr}$$

$$(2l+1)\frac{e^{i(\frac{l\pi}{2}+\delta_l)}}{2ik}e^{-i\frac{l\pi}{2}}e^{i\delta_l}P_l\cos(\theta) = (2l+1)\frac{1}{2ik}P_l\cos(\theta) + f(\theta)$$

$$(2l+1)e^{i2\delta_l}\frac{1}{2ik}P_l\cos(\theta) = (2l+1)\frac{1}{2ik}P_l\cos(\theta) + f(\theta)$$

which gives

$$f(\theta) = \frac{1}{2ik}(2l+1)(e^{i2\delta_l}-1)P_l\cos(\theta) \tag{A.11}$$

or

$$f(\theta) = (2l+1)e^{i\delta_l}\frac{\sin(\delta_l)}{k}P_l\cos(\theta) \tag{A.12}$$

So we have an expression for the scattering amplitude in terms of the scattering angle and the wavefuntion phase shift, which we have assumed to have an implicit and not yet known angular momentum dependence.

The differential scattering cross section is

$$d\sigma = \sum_{l=0}^{\infty}|f(\theta)|^2$$

$$d\sigma = \sum_{l=0}^{\infty}|\frac{(2l+1)}{k}e^{i\delta_l}\sin\delta_l interatomic P_l\cos(\theta)|^2 \tag{A.13}$$

and total cross section is

$$\sigma = \sum_{l=0}^{\infty}\frac{(2l+1)^2}{k^2}\sin^2\delta_l\int d\Omega P_l\cos(\theta)P_{l'}\cos(\theta)$$

$$\sigma = 4\pi^2 \sum_{l=0}^{\infty} \frac{(2l+1)^2}{k^2} \sin^2 \delta_l \tag{A.14}$$

due the orthogonality of the Legendre polynomials. Note that the total cross section has a maximum of $\sigma_{max} = 4\pi^2 \sum_{l=0}^{\infty} \frac{(2l+1)^2}{k^2}$ at $\delta_l = \frac{n\pi}{2}$. A question that arises now is what is the energy dependence of the phase shift δ_l on the energies of the scattering particles? A relationship between the δ_l and E would allow us to construct a probability distribution of detecting scattered particles of given energies.

A.0.2 Low Energy Scattering

We would like to calculate the phase shift δ_l for a low energy scattering of a potential $V(r)$ that extends to a range of r_0. To calculate δ_l, we need to solve the 1-dimensional radial part of the Schrodinger equation

$$\frac{d^2\chi(r)}{dr} - \frac{2m}{\hbar^2}[V(r) - E]\chi(r) = \frac{l(l+1)}{r^2}\chi(r) \tag{A.15}$$

We can easily solve this equation at low energies $E \ll V(r)$ and look at the case where $l = 0$ (referred to as S-wave scattering). Thus eqtn. 2.1 becomes

$$\frac{d^2\chi(r)}{dr} - \frac{2m}{\hbar^2}V(r)\chi(r) = 0 \tag{A.16}$$

Since the potential $V(r)$ has a limited range of r_0, the solution to 3.2 must match the solution at larger distances $r \gg r_0$ but not too large $r \ll \infty$where $V(r) = 0$ and eqtn. 2.2 becomes

$$\frac{d^2\chi(r)}{dr} = 0 \tag{A.17}$$

with a linear solution

$$\chi(r) = c(1 + \kappa r)$$

where we have allowed for the constants c and κ we can compare to a low energy solution of the asymptotic wavefunction $\chi(r) = A_l \sin(kr - \frac{\pi l}{2} + \delta_l)$ in the same region $r_0 \ll r \ll \infty$. We are interested in low energies where $kr \ll 1$.

Therefore, expanding in terms of kr, $\chi(r) = A_l \sin(kr - \frac{\pi l}{2} + \delta_l) \approx (2l+1)e^{i(\frac{l\pi}{2}+\delta_l)}[\sin(-\frac{\pi l}{2} + \delta_l) + \cos(-\frac{\pi l}{2} + \delta_l)kr + \ldots]$ for low energies ($kr \ll 1$) and looking the $l = 0$ term. We compare the linear solution to the expansion to find a relationship between the phase and particle energy by first solving for the linear coefficients C and κ

$$c = \chi_{k_0} = e^{i\delta_0} \sin(\delta_0)$$

and

$$c\kappa = \frac{d\chi_{k_0}}{dr} = k \cos(\delta_0)$$

Substituting C into the second equation,

$$e^{i\delta_0}\sin(\delta_0)\kappa = k\cos(\delta_0)$$

and more generally

$$\kappa = \frac{1}{\chi_{k_0}}\frac{d\chi_{k_0}}{dr}|_{r_0}$$

where the function χ and its derivative are evaluated at the boundaryr_0. The relationship between δ_0 and k

$$e^{i\delta_0}\tan(\delta_0)\kappa = k$$

Obviously this relationship is not valid for $\delta_0 = n\frac{\pi}{2}$ or any other values not close to 0. So for small phase shifts $\delta_0 \ll 1$, we expand the left side,

$$[1 + i\delta_0 + \ldots][\delta_0 + \ldots]\kappa = k$$

and to first order

$$\delta_0 \sim k$$

with proportionality constant $\frac{1}{\kappa}$

$$\delta_0 = \frac{k}{\kappa} \tag{A.18}$$

$\frac{1}{\kappa}$ is commonly referred to as the scattering length

$$\frac{1}{\kappa} = a$$

so that for low energy scattering we have the famous relation between the phase shift and scattering length

$$\delta_0 = ka$$

According to [11] this formula is valid for only $\delta_0 < 1$. So they've set an upper limit on the phase shift, which seems reasonable since going higher than 1 is approaching the divergence at $\frac{\pi}{2}$.

To calculate the total cross section, we expand it for small δ_0,

$$\sigma_{l=0} = \frac{4\pi^2}{k^2} \sin^2 \delta_0 \approx \frac{4\pi^2}{k^2} \delta_0^2$$

and then using eqtn. 2.4

$$\sigma_{l=0} \approx \frac{4\pi^2}{k^2} \frac{k^2}{\kappa^2}$$

$$\sigma_{l=0} \approx \frac{4\pi^2}{\kappa^2}$$

which is more commonly written as

$$\sigma_{l=0} \approx 4\pi^2 a^2$$

with $a = \frac{1}{\kappa}$ and a is called the scattering length. A common technique to find the scattering length is to find it through boundary conditions

$$a = \frac{\chi_{k_0}}{\frac{d\chi_{k_0}}{dr}}|_{r_0}$$

Graphically a can be seen as the x-intercept of the asymptotic wavefunction in the long wavelength limit (low k).

$$\chi(kr) = C(1 - \frac{1}{a}r)$$

$$0 = C - \frac{C}{a}r_{int}$$

$$a = r_{int}$$

If we look again at the linear solution

$$\chi(r) = C(1 + \kappa r) = \chi(r) = e^{i\delta_0}\sin(\delta_0)(1 + \frac{1}{a}r)$$

This means that at Feshbach resonances, where $\sigma \to \infty$, $a \to \infty$ and the slope of $\chi(r) \to 0$. At these resonances, the probability of observing a scattered particle has a peak with an energy width. In the case of Feshbach resonances, they also occur at low energies but with

applied magnetic fields dependence. As a result, the scattering cross section has a magnetic field dependence $\sigma \sim a(B)$.

Higher l terms contribute very little to the total cross section at low energies. The complete solution to the the radial part of the Schrodinger equation eqtn. 1.2

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(r)}{dr} + [V(r) - E]\chi(r) = \frac{l(l+1)}{r^2}\chi(r) \tag{A.19}$$

consists of spherical Bessel functions $j_l(r)$ and spherical Neumann functions $\eta_l(r)$ and is given by

$$\frac{\chi(kr)}{kr} = A_l j_l(kr) + B_l \eta_l(kr)$$

but we would like to look at the intermediate region $r_0 \ll r \ll \infty$ and low energies $E \ll 1$ which amounts to

$$kr \ll 1$$

In this limit

$$\frac{\chi(kr)}{kr} \to A_l \frac{(kr)^l}{(2l+1)!} + B_l (kr)^{-(l+1)}(2l-1)!$$

We need to make a connection with phase shifts δ_l so we need to match this solution to the asymptotic solution

$$\chi(kr) = C_l \sin(kr - \frac{\pi l}{2} + \delta_l)$$

$$\chi(kr) = C_l[\sin(kr - \frac{\pi l}{2})\cos(\delta_l) + \sin(\delta_l)\cos(kr - \frac{\pi l}{2})]$$

$$A_l \frac{(kr)^l}{(2l+1)!} + B_l(kr)^{-(l+1)}(2l-1)! = C_l[\sin(kr - \frac{\pi l}{2})\cos(\delta_l) + \sin(\delta_l)\cos(kr - \frac{\pi l}{2})]$$

This analysis seems to be easy considering the repulsive potential (hard sphere problem) and it turns out that

$$\delta_l \sim (kr)^{2l+1}$$

meaning that momenta with $l > 0$, give negligible weight to the phase shift when $kr \ll 1$.

An intuitive argument of why higher l terms are negligible in scattering is that a low energy particle does not have enough energy to penetrate the centrifugal term $\frac{l(l+1)}{r^2}$ in the Schrodinger equation. As a result, at low energies the scattering amplitude is dominated by the S-wave ($l = 0$) scattering.

A.0.3 Low Energy Scattering (alternative view)

The scattering amplitude at zero angular momentum $l = 0$ is given by

$$f(k) = \frac{e^{2i\delta_0} - 1}{2ik}$$

This can be rearranged in the following way

$$f(k) = \frac{e^{i\delta_0}(e^{i\delta_0} - e^{-i\delta_0})}{2ik}$$

$$f(k) = \frac{e^{i\delta_0}\sin(\delta_0)}{k}$$

$$f(k) = \frac{1}{\frac{k}{e^{i\delta_0}\sin(\delta_0)}}$$

$$f(k) = \frac{1}{\frac{k\cos(\delta_0) - ik\sin(\delta_0)}{\sin(\delta_0)}}$$

$$f(k) = \frac{1}{k\cot(\delta_0) - ik}$$

So we know that the general form of the scattering amplitude is

$$f(k) = \frac{1}{g_k - ik}$$

Now in the limit of $k \to 0$, g_k should be a constant. We can now make the connection to the scattering length. The scattering length a is defined as

$$\lim_{k\to 0} f(k) = -a$$

g_k is commonly (Ref H.A. Bethe, "Theory of the Effective Range in Nuclear Scattering", Phys. Rev. 76 (1), 1949) expanded in even powers of k (even powers because it is actually

an expansion in the kinetic energy $E = \frac{\hbar^2 k^2}{2m}$)

$$g_k = a^{-1} + \frac{1}{2} r_e k^2 + \dots$$

and for low energies we can approximate

$$\lim_{k \to 0} g_k^{-1} = -a$$

$$Re[f(k)] \approx -a$$

Appendix B

Relative Coordinates Hamiltonian

We introduce a center of mass coordinate $\vec{R} = \frac{m_2 r_1 + m_2 r_2}{m_1 + m_2}$ which $= \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ for $m_1 = m_2$ and relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ so that $\vec{r}_1 = \vec{R} + \frac{1}{2}\vec{r}$ and $\vec{r}_2 = \vec{R} - \frac{1}{2}\vec{r}$. To transform the Laplacian of each atom

$$\vec{\nabla}_1 = \frac{\partial}{\partial R}\frac{\partial R}{\partial r_1} + \frac{\partial}{\partial r}\frac{\partial r}{\partial r_1} = \frac{1}{2}\vec{\nabla}_R + \vec{\nabla}_r$$

$$\vec{\nabla}_1^2 = (\frac{\partial}{\partial R}\frac{\partial R}{\partial r_1} + \frac{\partial}{\partial r}\frac{\partial r}{\partial r_1})(\frac{1}{2}\vec{\nabla}_R + \vec{\nabla}_r)$$

$$= (\frac{1}{2}\vec{\nabla}_{\vec{R}} + \vec{\nabla}_r)(\frac{1}{2}\vec{\nabla}_R + \vec{\nabla}_r)$$

$$\vec{\nabla}_1^2 = \frac{1}{4}\vec{\nabla}_R^2 + \vec{\nabla}_R\vec{\nabla}_r + \vec{\nabla}_r^2$$

and similarly

$$\vec{\nabla}_2^2 = \frac{1}{4}\vec{\nabla}_R^2 - \vec{\nabla}_R\vec{\nabla}_r + \vec{\nabla}_r^2$$

so that

$$\vec{\nabla}_1^2 + \vec{\nabla}_2^2 \rightarrow \frac{1}{2}\vec{\nabla}_R^2 + 2\vec{\nabla}_r^2$$

and

$$-\frac{\hbar}{2m}(\vec{\nabla}_1^2 + \vec{\nabla}_2^2) = -\frac{1}{2}(\frac{\hbar}{2m})\vec{\nabla}_R^2 + -2(\frac{\hbar}{2m})\vec{\nabla}_r^2$$

Another quantity we need to compute is $r_1^2 + r_2^2$ which becomes

$$r_1^2 + r_2^2 = (\vec{R} + \frac{1}{2}\vec{r})^2 + (\vec{R} - \frac{1}{2}\vec{r})^2$$

$$= 2\vec{R}^2 + \frac{1}{2}\vec{r}^2$$

The Hamiltonian of the two-atom system in relative coordinates is

$$H = -\frac{\hbar^2}{2(2m)}\vec{\nabla}_R^2 - \frac{\hbar^2}{2(\frac{m}{2})}\vec{\nabla}_r^2 + V(r)$$

We will use the definition of the reduced mass

$$\frac{1}{\mu} = \frac{2}{m}$$

and renormalized mass

$$m* = 2m$$

to simplify

$$H = -\frac{\hbar^2}{2m*}\vec{\nabla}_R^2 - \frac{\hbar^2}{2\mu}\vec{\nabla}_r^2 + V(r)$$

noting that the Hamiltonian consists of a center-of-mass coordinate Hamiltonian with renormalized mass and a relative-coordinate Hamiltonian with reduced mass . The solution to the relative coordinate Hamiltonian with an atom-atom interaction potential will be the subject of this section.

Appendix C

Renormalization Procedure

This section of the appendix is an expansion on the methods described in [18].

C.0.4 Fermionic Operators

Consider a fermion field operator Ψ with the following properties

1. $[\Psi, \Psi]_+ = 0$
2. $[\Psi^\dagger, \Psi^\dagger]_+ = 0$
3. $[\Psi^\dagger, \Psi]_+ = 1$

We can expand Property 1

$$\Psi\Psi = -\Psi\Psi$$

and expand Property 2

$$\Psi^\dagger\Psi^\dagger = -\Psi^\dagger\Psi^\dagger$$

which implies that $\Psi\Psi = 0$ and $\Psi^\dagger\Psi^\dagger = 0$.

Rewriting Property 3

$$\Psi\Psi^\dagger = 1 - \Psi^\dagger\Psi$$

Let $N = \Psi^\dagger\Psi$.

We can show that N is the number operator. Let's compute N^2.

$$N^2 = \Psi^\dagger\Psi\Psi^\dagger\Psi$$

$$N^2 = \Psi^\dagger(1 - \Psi^\dagger\Psi)\Psi$$

$$N^2 = \Psi^\dagger\Psi - \Psi^\dagger\Psi^\dagger\Psi\Psi$$

using the fact that $\Psi^\dagger\Psi^\dagger = \Psi\Psi = 0$

$$N^2 = \Psi^\dagger\Psi = N$$

Thus the eigenvalues of N can only 0 or 1. We can make up a state $|0\rangle$ and a state $|1\rangle$ associated with the eigenvalues 0 and 1 respectively.

$$N|0\rangle = 0|0\rangle$$

$$N|1\rangle = 1|1\rangle$$

$\Psi^\dagger|0\rangle = 1|1\rangle$

$$N\Psi^\dagger|0\rangle = \Psi^\dagger\Psi\Psi^\dagger|0\rangle$$

$$= \Psi^\dagger(1 - \Psi^\dagger\Psi)|0\rangle$$

$$= \Psi^\dagger|0\rangle - 0|0\rangle$$

$$N\Psi^\dagger|0\rangle = \Psi^\dagger|0\rangle$$

$$therefore N = 1$$

which implies that $\Psi^\dagger|0\rangle$refers to the $|1\rangle$ state.

Now we can show that the norm of $\Psi^\dagger|0\rangle$ is 1.

$$|\Psi^\dagger|0\rangle|^2 = (\Psi^\dagger|0\rangle)^*(\Psi^\dagger|0\rangle)$$

$$= \langle 0|\Psi\Psi^{\dagger}|0\rangle$$

$$= \langle 0|1 - \Psi^{\dagger}\Psi|0\rangle$$

$$= \langle 0|0\rangle - N|0\rangle$$

Using the assumption that $|0\rangle$ and $|1\rangle$are orthonormal states, and the fact that $N|0\rangle = 0$,

$$|\Psi^{\dagger}|0\rangle|^2 = \langle 0|0\rangle = 1$$

$$therefore \Psi^{\dagger}|0\rangle = 1|1\rangle$$

$\Psi|1\rangle = 1|0\rangle$

$$N\Psi|1\rangle = \Psi^{\dagger}\Psi\Psi|1\rangle$$

$$= 0|1\rangle = 0\Psi|1\rangle$$

$$N\Psi|1\rangle = 0\Psi|1\rangle$$

$$therefore N = 0$$

So this means that the state $\Psi|1\rangle$ refers to the $|0\rangle$state.

Now we can show that the norm of $\Psi|1\rangle$ is 1.

$$|\Psi|1\rangle|^2 = (\Psi|1\rangle)^*(\Psi|1\rangle)$$

$$= \langle 1|\Psi^\dagger\Psi|1\rangle$$

$$= \langle 1|N|1\rangle$$

$$= \langle 1|1\rangle = 1$$

$$therefore |\Psi|1\rangle|^2 = 1$$

Now what remains unknown is $\Psi|0\rangle$and $\Psi^\dagger|1\rangle$. Let's consider $N\Psi|0\rangle$.

$$N\Psi|0\rangle = \Psi^\dagger\Psi\Psi|0\rangle$$

$$= 0|0\rangle$$

implying that $\Psi|0\rangle$refers to the state $|0\rangle$. What is the norm of $\Psi|0\rangle$?

$$|\Psi|0\rangle|^2 = \langle 0|\Psi^\dagger\Psi|0\rangle$$

$$= \langle 0|N|0\rangle$$

$$= 0\langle 0|0\rangle = 0$$

Therefore the norm of $\Psi|0\rangle$ is equal to 0 and as a result

$$\Psi|0\rangle = 0|0\rangle = 0$$

If we refer to the state $|0\rangle$ as the vaccuum state and the operator Ψ as the fermion destruction operator , then the result above says that one cannot destroy a particle if there is nothing there.

To see what $\Psi^\dagger|1\rangle$ is , we can expand $N\Psi^\dagger|1\rangle$.

$$N\Psi^\dagger|1\rangle = \Psi^\dagger\Psi\Psi^\dagger|1\rangle$$

$$= (1 - \Psi\Psi^\dagger)\Psi^\dagger|1\rangle$$

$$= 1|1\rangle - \Psi\Psi^\dagger\Psi^\dagger|1\rangle = 1|1\rangle - 0$$

$$= 1|1\rangle$$

implying that $\Psi^\dagger|1\rangle$refers to the state $|1\rangle$. What is the norm of $\Psi^\dagger|1\rangle$?

$$|\Psi^\dagger|1\rangle|^2 = \langle 1|\Psi\Psi^\dagger|1\rangle$$

$$= \langle 1|1 - \Psi^\dagger\Psi|1\rangle$$

$$= \langle 1|1\rangle - \langle 1|N|1\rangle$$

$$= \langle 1|1\rangle - 1\langle 1|1\rangle = 0$$

Therefore the norm of $\Psi^\dagger|1\rangle$ is equal to 0 and as a result

$$\Psi^\dagger|1\rangle = 0|1\rangle = 0$$

If we refer to the state $|1\rangle$ as the state of a fermion and the operator $\Psi^\dagger$ as the fermion creation operator , then the result above says that one cannot create a fermion in a state that is occupied by another fermion.

C.0.5 Grassman Numbers

Consider a type of numbers called Grassman numbers denoted by ψ and $\bar{\psi}$. These numbers posses the following properties

1. $[\psi, \psi]_+ = 0$
2. $[\bar{\psi}, \bar{\psi}]_+ = 0$
3. $[\bar{\psi}, \psi]_+ = 0$
4. $[\psi, \Psi]_+ = [\bar{\psi}, \Psi]_+ = 0$ where Ψdenotes fermionic operators.
5. $\psi|state\rangle = -|state\rangle\psi$
6. $\langle state|\psi = -\psi\langle state|$

The above properties state that all Grassman numbers anticommute with themselves, with each other, and with fermionic operators. (Note: $\bar{\psi}$ cannot be thought of as the complex conjugate of ψ even though it may appear to have similar properties as a conjugate. It may be better to think of $\bar{\psi}$ as just a different Grassman number when compared to ψ.

Properties 1 and 2 imply that $\psi^2 = \bar{\psi}^2 = 0$ just like fermionic operators. $|\psi\rangle = |0\rangle - \psi|1\rangle$ is called a fermionic coherent state .

A coherent state is an eigenstate of Ψ with eigenvalue ψ

$$\Psi|\psi\rangle = \Psi(|0\rangle - \psi|1\rangle)$$

$$= \Psi|0\rangle - \Psi\psi|1\rangle$$

$$= 0 + \psi\Psi|1\rangle$$

$$= \psi|0\rangle$$

$$= \psi|0\rangle - 0|1\rangle$$

$$= \psi|0\rangle - \psi\psi|1\rangle$$

$$\Psi|\psi\rangle = \psi(|0\rangle - \psi|1\rangle)$$

$$therefore \Psi|\psi\rangle = \varphi|\psi\rangle$$

Similarly,

$\langle\bar{\psi}| = \langle 0| - \langle 1|\bar{\psi}\,.$

$\langle\bar{\psi}|\Psi^\dagger = \langle\bar{\psi}|\bar{\psi}$

$$\langle\bar{\psi}|\Psi^\dagger = (\langle 0| - \langle 1|\bar{\psi})\Psi^\dagger$$

$$= \langle 0|\Psi^\dagger - \langle 1|\bar{\psi}\Psi^\dagger$$

$$= 0 + \langle 1|\Psi^\dagger\bar{\psi}$$

$$= \langle 0|\bar{\psi}$$

$$= \langle 0|\bar{\psi} - \langle 1|0$$

$$= \langle 0|\bar{\psi} - \langle 1|\bar{\psi}\bar{\psi}$$

$$= (\langle 0| - \langle 1|\bar{\psi})\bar{\psi}$$

$$\langle\bar{\psi}|\Psi^\dagger = \langle\bar{\psi}|\bar{\psi}$$

Now we can compute the inner product of two coherent states.

$$\langle\bar{\psi}|\varphi\rangle = (\langle 0| - \langle 1|\bar{\psi})(|0\rangle - \psi|1\rangle)$$

$$= \langle 0|0\rangle - \langle 0|\psi|1\rangle - \langle 1|\bar{\psi}|0\rangle + \langle 1|\bar{\psi}\psi|1\rangle$$

$$= 1 + \psi\langle 0||1\rangle + \bar{\psi}\langle 1||0\rangle + \bar{\psi}\psi\langle 1|1\rangle$$

$$= 1 + \bar{\psi}\psi$$

$$\langle\bar{\psi}|\varphi\rangle = e^{\bar{\psi}\psi}$$

The above can be justified because the higher order terms in the expansion of $e^{\bar{\psi}\psi}$ are all equal to 0 since the square of any two Grassman numbers is 0.

$$e^{\bar{\psi}\psi} = 1 + \bar{\psi}\psi + \frac{\bar{\psi}\psi\bar{\psi}\psi}{2!} + \dots$$

$$e^{\bar{\psi}\psi} = 1 + \bar{\psi}\psi - \frac{\bar{\psi}\bar{\psi}\psi\psi}{2!} + \dots$$

$$e^{\bar{\psi}\psi} = 1 + \bar{\psi}\psi + \frac{0}{2!} + \frac{0}{3!} \dots$$

$\int \psi d\psi = 1$

The integral is symbolic. It has no limits and you don't compute it.

$\int 1 d\psi = 0$

We can state the rule that an integral with an odd number of Grassman variables will always be zero. (Note: $d\psi$ and $d\bar{\psi}$ are also counted as Grassman variables)

$$\int \bar{\psi}\psi d\psi d\bar{\psi} = 1$$

Property 3 implies that

$$\int \psi\bar{\psi} d\psi d\bar{\psi} = -1$$

and any other interchange would give the same result. But two interchanges would give back 1.

$$\int \bar{\psi}\psi d\bar{\psi} d\psi = -1$$

$$\int \psi\bar{\psi} d\bar{\psi} d\psi = 1$$

$\int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi = a$

$$\int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi = \int (1 - a\bar{\psi}\psi) d\bar{\psi} d\psi$$

$$= \int d\bar{\psi} d\psi - a \int \bar{\psi}\psi d\bar{\psi} d\psi$$

$$= 0 - a(-1) = a$$

$$therefore \int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi = a$$

$\int e^{-\bar{\psi} M \psi} d\bar{\psi} d\psi = detM$

We can prove this for a 2×2 matrix and two component Grassman vector. Let $\psi = \begin{matrix} \psi_1 \\ \psi_2 \end{matrix}$, $\bar{\psi} = \begin{matrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{matrix}$, and $d\bar{\psi} d\psi = d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$.

$$\begin{bmatrix} \bar{\psi}_1 & & \\ & & \\ \bar{\psi}_2 & & \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ & \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \bar{\psi}_1 \\ \\ \bar{\psi}_2 \end{bmatrix} \begin{bmatrix} M_{11}\psi_1 + M_{12}\psi_2 \\ \\ M_{21}\psi_1 + M_{22}\psi_2 \end{bmatrix}$$

$$\bar{\psi} M \psi = \bar{\psi}_1 M_{11} \psi_1 + \bar{\psi}_1 M_{12} \psi_2 + \bar{\psi}_2 M_{21} \psi_1 + \bar{\psi}_2 M_{22} \psi_2$$

To do this integral $e^{-\bar{\psi}M\psi}$, we need to expand the exponential. Since the integral is over four variables $d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$, we need only to keep terms of the expansion that contain four Grassman numbers. These will be the 2nd order terms of the form

$$(\bar{\psi}_1 M_{11}\psi_1+\bar{\psi}_1 M_{12}\psi_2+\bar{\psi}_2 M_{21}\psi_1+\bar{\psi}_2 M_{22}\psi_2)\times(\bar{\psi}_1 M_{11}\psi_1+\bar{\psi}_1 M_{12}\psi_2+\bar{\psi}_2 M_{21}\psi_1+\bar{\psi}_2 M_{22}\psi_2) =$$

and any terms that have the same Grassman number more than once will be zero so we only keep these terms

$$\bar{\psi}_1 M_{11}\psi_1\bar{\psi}_2 M_{22}\psi_2 + \bar{\psi}_1 M_{12}\psi_2\bar{\psi}_2 M_{21}\psi_1 + \bar{\psi}_2 M_{21}\psi_1\bar{\psi}_1 M_{12}\psi_2 + \bar{\psi}_2 M_{22}\psi_2\bar{\psi}_1 M_{11}\psi_1 =$$

$$2(M_{11}M_{22}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 + M_{12}M_{21}\bar{\psi}_1\psi_2\bar{\psi}_2\psi_1)$$

$$\int e^{-\bar{\psi}M\psi} d\bar{\psi}d\psi = \int (M_{11}M_{22}\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 + M_{12}M_{21}\bar{\psi}_1\psi_2\bar{\psi}_2\psi_1) d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$\int e^{-\bar{\psi}M\psi} d\bar{\psi}d\psi = M_{11}M_{22} - M_{12}M_{21}$$

To construct averages,

$$\langle\bar{\psi}\psi\rangle = \frac{\int \bar{\psi}\psi e^{a\bar{\psi}\psi} d\bar{\psi}d\psi}{\int e^{a\bar{\psi}\psi} d\bar{\psi}d\psi}$$

$\langle\bar{\psi}\psi\rangle = \frac{1}{a} = -\langle\psi\bar{\psi}\rangle$

$$\langle\bar{\psi}\psi\rangle = \frac{\int \bar{\psi}\psi e^{a\bar{\psi}\psi} d\bar{\psi}d\psi}{\int e^{a\bar{\psi}\psi} d\bar{\psi}d\psi}$$

$$= \frac{\int \bar{\psi}\psi(1 + a\bar{\psi}\psi)d\bar{\psi}d\psi}{\int (1 + a\bar{\psi}\psi)d\bar{\psi}d\psi}$$

$$= \frac{\int \bar{\psi}\psi d\bar{\psi}d\psi + a\int \bar{\psi}\psi\bar{\psi}\psi d\bar{\psi}d\psi}{\int d\bar{\psi}d\psi + a\int \bar{\psi}\psi d\bar{\psi}d\psi}$$

$$= \frac{-1 + a(0)}{0 + a(-1)}$$

$$= \frac{1}{a}$$

and we prove the right side

$$-\langle\psi\bar{\psi}\rangle = -\frac{\int \psi\bar{\psi}e^{a\bar{\psi}\psi}d\bar{\psi}d\psi}{\int e^{a\bar{\psi}\psi}d\bar{\psi}d\psi}$$

$$= -\frac{\int \psi\bar{\psi}(1 + a\bar{\psi}\psi)d\bar{\psi}d\psi}{\int (1 + a\bar{\psi}\psi)d\bar{\psi}d\psi}$$

$$= -\frac{\int \psi\bar{\psi}d\bar{\psi}d\psi + a\int \psi\bar{\psi}\bar{\psi}\psi d\bar{\psi}d\psi}{\int d\bar{\psi}d\psi + a\int \bar{\psi}\psi d\bar{\psi}d\psi}$$

$$= -\frac{1 + 0}{0 + a(-1)}$$

$$-\langle\psi\bar{\psi}\rangle = \frac{1}{a}$$

$$therefore \langle\bar{\psi}\psi\rangle = \frac{1}{a} = -\langle\psi\bar{\psi}\rangle$$

If considering two sets of Grassman numbers, the gaussian measure is defined this way

$$\int e^{a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$\langle\bar{\psi}_i\psi_j\rangle = \frac{\delta_{ij}}{a_i} \equiv \langle\bar{i}j\rangle$ To prove this propostion, let us establish a few facts concerning integrals over two sets of Grassman numbers.

1. $e^{a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2} = 1 + a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2 + \frac{(a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2)^2}{2!}$

$$= 1 + a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2 + (a_1\bar{\psi}_1\psi_1)^2 + (a_2\bar{\psi}_2\psi_2)^2 + \frac{a_1a_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 + a_1a_2\bar{\psi}_2\psi_2\bar{\psi}_1\psi_1}{2!}$$

$$= 1 + a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2 + 0 + 0 + a_1a_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$$

$$= 1 + a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2 + a_1a_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$$

2. $\int 1\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = 0.$

3. $\int a\bar{\psi}_i\psi_i\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = 0$ for $i = 1$ or 2

4. $a_1 a_2 \int \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = -a_1 a_2 \int \psi_1\bar{\psi}_2\psi_2\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$

$$= -a_1 a_2 \int \bar{\psi}_2\psi_2\psi_1\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= a_1 a_2 \int \psi_2\bar{\psi}_2\psi_1\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= a_1 a_2$$

5. $\int \bar{\psi}_i\psi_j d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = 0$

6. $\int \bar{\psi}_i\psi_j[a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2] d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 =$

$$a_1 \int \bar{\psi}_i\psi_j\bar{\psi}_1\psi_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 + a_2 \int \bar{\psi}_i\psi_j\bar{\psi}_2\psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= \begin{cases} a_1 \text{if} & i = j = 2 \\ a_2 \text{if} & i = j = 1 \end{cases}$$

$$\langle \bar{\psi}_i\psi_j \rangle = \frac{\int \bar{\psi}_i\psi_j e^{a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{\int e^{a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}$$

$$= \frac{\int \bar{\psi}_i\psi_j[1 + a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2 + a_1 a_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2] d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{\int [1 + a_1\bar{\psi}_1\psi_1 + a_2\bar{\psi}_2\psi_2 + a_1 a_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2] d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}$$

$$= \frac{0 + \begin{cases} a_1 \text{if} \quad i = j = 2 \\ \\ a_2 \text{if} \quad i = j = 1 \end{cases} + a_1 a_2 \int \bar{\psi}_i \psi_j \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{0 + 0 + 0 + a_1 a_2}$$

$$= \frac{\begin{cases} a_1 \text{if} \quad i = j = 2 \\ \\ a_2 \text{if} \quad i = j = 1 \end{cases} + 0}{a_1 a_2}$$

$$= \begin{cases} \frac{a_1}{a_1 a_2} \text{if} \quad i = j = 2 \\ \\ \frac{a_2}{a_1 a_2} \text{if} \quad i = j = 1 \end{cases}$$

$$= \frac{\delta_{ij}}{a_i}$$

$$therefore \langle \bar{\psi}_i \psi_j \rangle = \frac{\delta_{ij}}{a_i}$$

Wick's Theorem for fermions is

$$\langle \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l \rangle = \frac{\delta_{il}}{a_i} \frac{\delta_{jk}}{a_j} - \frac{\delta_{ik}}{a_i} \frac{\delta_{jl}}{a_j}$$

where $i \neq j$ and $k \neq l$

$$\langle \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l \rangle = \frac{\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l e^{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{\int e^{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}$$

$$= \frac{\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l [1 + a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2 + a_1 a_2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2] d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{\int [1 + a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2 + a_1 a_2 \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2] d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}$$

$$= \frac{\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{a_1 a_2 \int \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}$$

$$= \frac{\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{a_1 a_2}$$

For the numerator, let's consider four cases.

If $i = k = 1$ and $j = l = 2$

$$\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \int \bar{\psi}_1 \bar{\psi}_2 \psi_1 \psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= - \int \bar{\psi}_2 \psi_1 \psi_2 \bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= \int \bar{\psi}_2 \psi_2 \psi_1 \bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= -\int \psi_2\bar{\psi}_2\psi_1\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$\int \bar{\psi}_i\bar{\psi}_j\psi_k\psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = -1$$

If $i = k = 2$ and $j = l = 1$

$$\int \bar{\psi}_i\bar{\psi}_j\psi_k\psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \int \bar{\psi}_2\bar{\psi}_1\psi_2\psi_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= \int \bar{\psi}_2\psi_2\psi_1\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= -\int \psi_2\bar{\psi}_2\psi_1\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$\int \bar{\psi}_i\bar{\psi}_j\psi_k\psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = -1$$

If $i = l = 1$ and $j = k = 2$

$$\int \bar{\psi}_i\bar{\psi}_j\psi_k\psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \int \bar{\psi}_1\bar{\psi}_2\psi_2\psi_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= -\int \bar{\psi}_2\psi_2\psi_1\bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= \int \psi_2 \bar{\psi}_2 \psi_1 \bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = 1$$

If $i = l = 2$ and $j = k = 1$

$$\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \int \bar{\psi}_2 \bar{\psi}_1 \psi_1 \psi_2 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= \int \bar{\psi}_2 \psi_1 \psi_2 \bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= - \int \bar{\psi}_2 \psi_2 \psi_1 \bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$= \int \psi_2 \bar{\psi}_2 \psi_1 \bar{\psi}_1 d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2$$

$$\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = 1$$

Therefore we conclude that when $i = k$ and $j = l$, the numerator is -1. And when $i = l$ and $j = k$, the numerator is 1.

$$\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \frac{\delta_{il}\delta_{jk}}{a_1 a_2} - \frac{\delta_{ik}\delta_{jl}}{a_1 a_2}$$

In either case, $i \neq j$, so we can write $a_1 a_2 = a_i a_j$

$$therefore \int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \frac{\delta_{il}\delta_{jk}}{a_i a_j} - \frac{\delta_{ik}\delta_{jl}}{a_i a_j}$$

Or in the notation used for the previous proposition

$$\langle \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l \rangle = \langle \bar{i}l \rangle \langle \bar{j}k \rangle - \langle \bar{i}k \rangle \langle \bar{j}l \rangle$$

C.0.6 Alternative Representation For The Partition Function

The partiton function of a statistical system is $Z = \sum_n e^{-\beta E_n}$

We can show that $Z = Tr(e^{-\beta H})$

$$Tr(e^{-\beta H}) = \sum_n \langle n | e^{-\beta H} | n \rangle$$

$$= \sum_n \langle n | 1 - \beta H + \frac{(-\beta H)^2}{2!} + \ldots | n \rangle$$

Using the assumption that $H|n\rangle = E_n|n\rangle$, where $E_n$are the eigenvalues of the hamiltonian H. Then,

$$Tr(e^{-\beta H}) = \sum_n \langle n|1 - \beta E_n + \frac{(-\beta E_n)^2}{2!} + \ldots |n\rangle$$

$$= \sum_n e^{-\beta E_n}\langle n|n\rangle = \sum_n e^{-\beta E_n}$$

$$therefore Z = Tr(e^{-\beta H})$$

C.0.7 Partition Function For Fermions

We would like to formulate a path integral representation of the partition function for fermions. To begin, we need to establish a few properties which will build upon Grassman numbers and fermion operators. $I = \int |\psi\rangle\langle\bar{\psi}|e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$ where the identity $I = |0\rangle\langle 0| + |1\rangle\langle 1|$.

$$\int |\psi\rangle\langle\bar{\psi}|e^{-\bar{\psi}\psi}d\bar{\psi}d\psi = \int (|0\rangle - \psi|1\rangle)(\langle 0| - \langle 1|\bar{\psi})(1 - \bar{\psi}\psi)d\bar{\psi}d\psi$$

$$= \int (|0\rangle\langle 0| - |0\rangle\langle 1|\bar{\psi} - \psi|1\rangle\langle 0| + \psi|1\rangle\langle 1|\bar{\psi})d\bar{\psi}d\psi - \int (|0\rangle - \psi|1\rangle)(\langle 0| - \langle 1|\bar{\psi})\bar{\psi}\psi d\bar{\psi}d\psi$$

$$= 0 + 0 + 0 + |1\rangle\langle 1| - \int (|0\rangle\langle 0| - \psi|1\rangle\langle 0| - \psi|1\rangle\langle 0| + \psi|1\rangle\langle 1|\bar{\psi})\bar{\psi}\psi d\bar{\psi}d\psi$$

$$= |1\rangle\langle 1| + |0\rangle\langle 0| + 0 + 0 + 0$$

$$therefore \int |\psi\rangle\langle\bar{\psi}|e^{-\bar{\psi}\psi}d\bar{\psi}d\psi = |1\rangle\langle 1| + |0\rangle\langle 0|$$

Now we would like to formulate the Trace of an operator. Luckily someone has already done it, and we just have to prove it. $Tr\Omega = \int\langle -\bar{\psi}|\Omega|\psi\rangle e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$ where Ω is a bosonic operator which is made up of an even number of fermionic operators. (Note that the $-$sign in $\langle -\bar{\psi}|$ is merely a symbolic symbol. It does not change the coherent state, i.e. $\langle -\bar{\psi}| = \langle\bar{\psi}|$.)

$$\int\langle -\bar{\psi}|\Omega|\psi\rangle e^{-\bar{\psi}\psi}d\bar{\psi}d\psi = \int(\langle 0| - \langle 1|\bar{\psi})\Omega(|0\rangle - \psi|1\rangle)e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= \int(\langle 0|\Omega|0\rangle + \langle 1|\bar{\psi}\Omega\psi|1\rangle)e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= \int\langle 0|\Omega|0\rangle e^{-\bar{\psi}\psi}d\bar{\psi}d\psi + \int\langle 1|\bar{\psi}\Omega\psi|1\rangle e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= \langle 0|\Omega|0\rangle - \langle 1|\Omega|1\rangle\int\bar{\psi}\psi e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= \langle 0|\Omega|0\rangle - \langle 1|\Omega|1\rangle\int\bar{\psi}\psi(1 - \bar{\psi}\psi)d\bar{\psi}d\psi$$

$$= \langle 0|\Omega|0\rangle + \langle 1|\Omega|1\rangle$$

$$therefore \int \langle -\bar{\psi}|\Omega|\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi} d\psi = Tr\Omega$$

Now that we have established how to represent the Tr of a bosonic operator, we can form the partition funtion of the single harmonic fermion oscillator.

Consider the following hamiltonian, which represents the energy for a fermion.

$$H_0 = \Omega_0 \Psi^\dagger \Psi$$

To consider the energy of a fermion in a fermi sea, we need a hamiltonian which measures the energy with respect to the chemical potential μ, assuming no interaction with other fermions. The chemical potential is the amount of energy required to add one more fermion to a system of fermions in the ground state(T=0). So to measure the relative energy, we suggest using the following Hamiltonian

$$H = H_0 - \mu N = (\Omega_0 - \mu) N$$

Now the partition function of this fermionic oscillator is

$$Z = Tr(e^{-\beta H}) = Tr(e^{-\beta(\Omega_0 - \mu)\Psi^\dagger \Psi})$$

$$Z = \int \langle -\bar{\psi}| e^{-\beta(\Omega_0 - \mu)\Psi^\dagger \Psi} |\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi} d\psi$$

We can do yet another check on this expression for the partition function to see if it yields the expected partition function for a fermion. We can begin by expanding $e^{-\beta(\Omega_0 - \mu)\Psi^\dagger \Psi}$.

$$e^{-\beta(\Omega_0 - \mu)\Psi^\dagger \Psi} = 1 - \beta(\Omega_0 - \mu)\Psi^\dagger \Psi + \frac{\beta^2(\Omega_0 - \mu)^2}{2!}\Psi^\dagger \Psi \Psi^\dagger \Psi + \ldots$$

But $\Psi^\dagger \Psi \Psi^\dagger \Psi = \Psi^\dagger (1 - \Psi^\dagger \Psi)\Psi = \Psi^\dagger \Psi$ and in general $(\Psi^\dagger \Psi)^n = \Psi^\dagger \Psi$. Thus,

$$e^{-\beta(\Omega_0 - \mu)\Psi^\dagger \Psi} = 1 - \beta(\Omega_0 - \mu)\Psi^\dagger \Psi + \frac{\beta^2(\Omega_0 - \mu)^2}{2!}\Psi^\dagger \Psi - \frac{\beta^3(\Omega_0 - \mu)^3}{3!}\Psi^\dagger \Psi + \ldots$$

$$e^{-\beta(\Omega_0 - \mu)\Psi^\dagger \Psi} = 1 + (e^{-\beta(\Omega_0 - \mu)} - 1)\Psi^\dagger \Psi$$

So we can now write the partition function as

$$Z = \int \langle -\bar{\psi}| 1 + (e^{-\beta(\Omega_0 - \mu)} - 1)\Psi^\dagger \Psi |\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi} d\psi$$

$$= \int \langle -\bar{\psi}| 1 + (e^{-\beta(\Omega_0 - \mu)} - 1)(-\bar{\psi}\psi) |\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi} d\psi$$

$$= \int e^{-\bar{\psi}\psi}[1 + (e^{-\beta(\Omega_0-\mu)} - 1)(-\bar{\psi}\psi)]e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= \int [1 - (e^{-\beta(\Omega_0-\mu)} - 1)\bar{\psi}\psi]e^{-2\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= \int 1e^{-2\bar{\psi}\psi}d\bar{\psi}d\psi - \int (e^{-\beta(\Omega_0-\mu)} - 1)\bar{\psi}\psi e^{-2\bar{\psi}\psi}d\bar{\psi}d\psi$$

$$= 2 + e^{-\beta(\Omega_0-\mu)} - 1$$

$$Z = 1 + e^{-\beta(\Omega_0-\mu)}$$

Thus we have shown that our expression $Z = \int \langle -\bar{\psi}|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|\psi\rangle e^{-\bar{\psi}\psi}d\bar{\psi}d\psi$ checks to be correct for a single fermion oscillator.

C.0.8 Path Integral Representation of the Partition Function

We would like to construct the partition function $Z = Tre^{-\beta H}$ in terms of the path integral. We will assume that H is bosonic operator. i.e. H is made up of an even number of fermionic operators. $H = H(\Psi^\dagger, \Psi)$ Let us first write

$$e^{-\beta H} = [e^{-(\beta/N)H}]^N$$

Let $\epsilon = \frac{\beta}{N}$.

$$e^{-\beta H} = [e^{-\epsilon H}]^N$$

Now we let $N \to \infty$, which means that $\epsilon \to 0$. Since ϵ is so small, we can keep only the first term in the expansion of $e^{-\epsilon H}$.

$$\lim_{\epsilon \to 0} e^{-\epsilon H} = 1 - \epsilon H$$

Therefore,

$$e^{-\beta H} = (e^{-\epsilon H})^N = \underbrace{(1-\epsilon H)(1-\epsilon H)\dots(1-\epsilon H)}_{N}$$

and the partition function

$$Z = Tr\, \underbrace{(1-\epsilon H)(1-\epsilon H)\dots(1-\epsilon H)}_{N}$$

$$Z = \int \langle -\bar{\psi}| \underbrace{(1-\epsilon H)(1-\epsilon H)\dots(1-\epsilon H)}_{N} |\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi} d\psi$$

Now, we will insert an identity $N-1$ times, while also giving an index enumeration to the ψ's. ψ_1 are the original ψ in the integral.

$$\begin{aligned} Z &= \int \langle -\bar{\psi}_1|(1-\epsilon H)|\psi_{N-1}\rangle e^{-\bar{\psi}_{N-1}\psi_{N-1}} \langle \bar{\psi}_{N-1}|(1-\epsilon H)|\psi_{N-2}\rangle e^{-\bar{\psi}_{N-2}\psi_{N-2}} \langle \bar{\psi}_{N-2}| \dots \\ &\quad |\psi_2\rangle e^{-\bar{\psi}_2\psi_2} \langle \bar{\psi}_2|(1-\epsilon H|\psi_1\rangle e^{-\bar{\psi}_1\psi_1} \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i \end{aligned}$$

Writing H as $H(\Psi^\dagger, \Psi)$ and since $\epsilon \to 0$, we can claim that

$$\langle -\bar{\psi}_{i+1}|1 - \epsilon H(\Psi^\dagger, \Psi)|\psi_i\rangle e^{-\bar{\psi}_i\psi_i} = \langle -\bar{\psi}_{i+1}|e^{-\epsilon H(\Psi^\dagger, \Psi)}|\psi_i\rangle e^{-\bar{\psi}_i\psi_i}$$

Since $H(\Psi^\dagger, \Psi)$ consists of an even number of fermion operators, it will commute with coherent states $\langle -\bar{\psi}|$ and $|\psi\rangle$.

$$\langle -\bar{\psi}_{i+1}|e^{-\epsilon H(\Psi^\dagger, \Psi)}|\psi_i\rangle e^{-\bar{\psi}_i\psi_i} = \langle -\bar{\psi}_{i+1}|\psi_i\rangle e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_i\psi_i}$$

Let $\bar{\psi}_N = -\bar{\psi}_1$ and $\psi_N = -\psi_1$, so that

$$\begin{aligned} Z &= \int \langle \bar{\psi}_N|\psi_{N-1}\rangle e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_{N-1}\psi_{N-1}} \langle \bar{\psi}_{N-1}|\psi_{N-2}\rangle e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_{N-2}\psi_{N-2}} \langle \bar{\psi}_{N-2}| \dots \\ &\quad |\psi_2\rangle e^{-\bar{\psi}_2\psi_2} \langle \bar{\psi}_2| - \psi_N\rangle e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{\bar{\psi}_N\psi_N} \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i \end{aligned}$$

We can also deal with the inner products in this way,

$$\langle \bar{\psi}_{i+1}|\psi_i\rangle e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_{i+1}\psi_i} = e^{\bar{\psi}_{i+1}\psi_i} e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_{i+1}\psi_i}$$

$$= e^{\bar{\psi}_{i+1}\psi_i - \bar{\psi}_{i+1}\psi_i} e^{-\epsilon H(\bar{\psi}_{i+1}, \psi_i)}$$

$$= e^{(\bar{\psi}_{i+1}-\bar{\psi}_i)\psi_i} e^{-\epsilon H(\bar{\psi}_{i+1},\psi_i)} = e^{\frac{(\bar{\psi}_{i+1}-\bar{\psi}_i)}{\epsilon}\epsilon\psi_{i-1}} e^{-\epsilon H(\bar{\psi}_{i+1},\psi_i)}$$

$$= e^{[\frac{(\bar{\psi}_{i+1}-\bar{\psi}_i)}{\epsilon}\psi_i - H(\bar{\psi}_{i+1},\psi_i)]\epsilon}$$

So now the partition function is

$$Z = \lim_{\epsilon\to 0} \int e^{[\frac{(\bar{\psi}_{i+1}-\bar{\psi}_i)}{\epsilon}\psi_i - H(\bar{\psi}_{i+1},\psi_i)]\epsilon} \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i$$

C.0.9 Imaginary Time

If we were to study the dynamics of a system such as that of a single Fermion, we need to adopt the Heisenberg picture. In this view, from the time-dependent Schrodinger equation, the wavefunction of the fermion has a time dependence and can be determined by

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$$

where $U(t) = e^{-\frac{iH}{\hbar}t}$ is called the unitary time operator. In order to compute the dynamics of an observable, we need to compute its average given its value at time $t = 0$.

$$\langle A(t)\rangle = \langle\Psi(t)|A(0)|\Psi(t)\rangle$$

$$\langle A(t)\rangle = \langle \Psi(0)|e^{\frac{iH}{\hbar}t}A(0)e^{-\frac{iH}{\hbar}t}|\Psi(0)\rangle$$

We can define the middle term on the right hand side as the time operator $A(t) = e^{\frac{iH}{\hbar}t}A(0)e^{-\frac{iH}{\hbar}t}$ so that its average is given as

$$\langle A(t)\rangle = \langle \Psi(0)|A(t)|\Psi(0)\rangle$$

For the case of the single fermion in a fermion sea, $H = (\Omega_0 - \mu)\Psi^\dagger\Psi$. In this case, we are intersted in the average of the time-dependent fermi fields $\Psi(t)$ and $\Psi^\dagger(t)$, which can be written in terms of the fermi fields at $t = 0$,

$$\Psi(t) = e^{i(\Omega_0-\mu)\Psi^\dagger\Psi t}\Psi(0)e^{-(\Omega_0-\mu)\Psi^\dagger\Psi t}$$

We can drop the $t = 0$ on $\Psi(0)$ since wavefuntions are time independent

$$\Psi(t) = [1 + i(\Omega_0 - \mu)\Psi^\dagger\Psi t + i^2(\Omega_0 - \mu)^2(\Psi^\dagger\Psi)^2t^2 + \ldots]\Psi\times$$

$$[1 - i(\Omega_0 - \mu)\Psi^\dagger\Psi t - i^2(\Omega_0 - \mu)^2(\Psi^\dagger\Psi)^2t^2 + \ldots]$$

$$= \Psi - i(\Omega_0 - \mu)t\Psi\Psi^\dagger\Psi + -i^2(\Omega_0 - \mu)^2\Psi(\Psi^\dagger\Psi)^2 + \ldots$$

$$\Psi(t) = [1 - i(\Omega_0 - \mu)t(1 - \Psi^\dagger\Psi) + -i^2(\Omega_0 - \mu)^2(1 - \Psi^\dagger\Psi)\Psi\Psi^\dagger + \ldots]\Psi$$

using $\Psi\Psi = 0$,

$$= \Psi - i(\Omega_0 - \mu)t\Psi + -i^2(\Omega_0 - \mu)^2\Psi\Psi^\dagger\Psi + \ldots$$

$$= \Psi[1 - i(\Omega_0 - \mu)t + -i^2(\Omega_0 - \mu)^2(1 - \Psi\Psi^\dagger) + \ldots]$$

$$= \Psi[1 - i(\Omega_0 - \mu)t + -i^2(\Omega_0 - \mu)^2 + \ldots]$$

$$\Psi(t) = \Psi(0)e^{-i(\Omega_0 - \mu)t}$$

And similarly,

$$\Psi^\dagger(t) = \Psi^\dagger(0)e^{i(\Omega_0 - \mu)t}$$

Since we are interested in averages, we can cast these operators in a form that will be appropriate in a partition function. That is, we introduce an imaginary time variable

$$\tau = it$$

such that the time dependent fermion operators are

$$\Psi(\tau) = \Psi(0)e^{-(\Omega_0-\mu)\tau}$$

and

$$\Psi^\dagger(\tau) = \Psi^\dagger(0)e^{(\Omega_0-\mu)\tau}$$

The imaginary time Green's function for a fermion in a fermi sea is analogous to a real time Green's function

$$G(\tau) = \langle T(\Psi(\tau)\Psi^\dagger(0))\rangle$$

where T is the imaginary time ordering operator and $\langle\rangle$ denotes the average with respect to the partition function $Z = Tr[e^{-\beta H}]$ with β as the inverse temperature

$$G(\tau) = \frac{1}{Z}Tr[e^{-\beta H}\Psi(\tau)\Psi^\dagger(0)]$$

In our example of the single fermion in a fermi sea, there are only two states for the fermion $|0\rangle$ and $|1\rangle$ so the Green's function is

$$G(\tau) = \frac{\langle 0|e^{-\beta H}T\Psi(\tau)\Psi^\dagger(0)|0\rangle + \langle 1|e^{-\beta H}T\Psi(\tau)\Psi^\dagger(0)|1\rangle}{\langle 0|e^{-\beta H})|0\rangle + \langle 1|e^{-\beta H}|1\rangle}$$

The only way to have non-zero terms in the numerator is to have $\tau > 0$ in the first term and $\tau < 0$ in the second term

$$G(\tau) = \frac{\Theta(\tau)\langle 0|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}\Psi(\tau)\Psi^\dagger(0)|0\rangle - \Theta(-\tau)\langle 1|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}\Psi^\dagger(0)\Psi(\tau)|1\rangle}{\langle 0|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|0\rangle + \langle 1|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|1\rangle}$$

$$= \frac{\Theta(\tau)\langle 0|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}\Psi(0)e^{-(\Omega_0-\mu)\tau}|1\rangle - \Theta(-\tau)\langle 1|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}\Psi^\dagger(0)e^{-(\Omega_0-\mu)\tau}|0\rangle}{\langle 0|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|0\rangle + \langle 1|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|1\rangle}$$

$$= \frac{\Theta(\tau)\langle 0|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|0\rangle e^{-(\Omega_0-\mu)\tau} - \Theta(-\tau)\langle 1|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|1\rangle e^{-(\Omega_0-\mu)\tau}}{1+e^{-\beta(\Omega_0-\mu)}}$$

$$= \frac{\Theta(\tau)e^{-(\Omega_0-\mu)\tau} - \Theta(-\tau)e^{-\beta(\Omega_0-\mu)}e^{-(\Omega_0-\mu)\tau}}{1+e^{-\beta(\Omega_0-\mu)}}$$

$$G(\tau) = \frac{\Theta(\tau)e^{-(\Omega_0-\mu)\tau} - \Theta(-\tau)e^{-(\Omega_0-\mu)(\tau+\beta)}}{1+e^{-\beta(\Omega_0-\mu)}}$$

We will be taking the zero temperature limit, $\beta \to \infty$. Then the imaginary time Green's function becomes

$$G(\tau) = \begin{cases} \Theta(\tau)e^{-(\Omega_0-\mu)\tau} & \Omega_0 > \mu \\ -\Theta(-\tau)e^{-(\Omega_0-\mu)\tau} & \Omega_0 < \mu \end{cases}$$

C.0.10 Matsubura Frequencies

$\omega_n = \frac{\pi(2n+1)}{\beta}$ for n all positive integers and 0. ω_n is called the fermion Matsubura frequency. (Just as a side note, Matsubura frequencies posses anti-periodicity.

$$e^{i\omega_n(\tau+\beta)} = e^{\frac{i\pi(2n+1)\tau}{\beta}}e^{i\pi(2n+1)}$$

$$= e^{\frac{i\pi(2n+1)\tau}{\beta}}(-1)$$

$$e^{i\omega_n(\tau+\beta)} = -e^{i\omega_n\tau}$$

)

We can expand Grassman numbers as a Fourier sum of Matsubura frequencies.

$$\psi(\tau) = \sum_n \frac{e^{-i\omega_n\tau}}{\beta}\psi(\omega_n)$$

and

$$\bar{\psi}(\tau) = \sum_n \frac{e^{i\omega_n\tau}}{\beta}\bar{\psi}(\omega_n)$$

More explicitly,

$$\psi = \frac{1}{\beta}(e^{-i\pi\tau} + e^{-3i\pi\tau} + e^{-5i\pi\tau} + \ldots)\psi(\omega)$$

The inverse transforms are

$$\psi(\omega_n) = \int_0^\beta \psi(\tau)e^{i\omega_n\tau}d\tau$$

$$\bar{\psi}(\omega_n) = \int_0^\beta \bar{\psi}(\tau)e^{-i\omega_n\tau}d\tau$$

We may need the orthogonality property,

$$\int_0^\beta e^{i\omega_n \tau} e^{-i\omega_m \tau} d\tau = \int_0^\beta e^{i(\omega_n - \omega_m)\tau} d\tau = \delta_{mn}$$

C.0.11 Final Expression For The Partition Function

We left the partition function in this form

$$Z = \lim_{\epsilon \to 0} \int e^{[\frac{(\bar{\psi}_{i+1} - \bar{\psi}_i)}{\epsilon} \psi_i - H(\bar{\psi}_{i+1}, \psi_i)]\epsilon} \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i$$

Since we are taking the limit as $\epsilon \to 0$, then we can claim that $\bar{\psi}_i = \bar{\psi}(\tau)$ and $\bar{\psi}_{i+1} = \bar{\psi}(\tau + \epsilon)$. So Z becomes

$$Z = \lim_{\epsilon \to 0} \int e^{[\frac{(\bar{\psi}(\tau+\epsilon) - \bar{\psi}(\tau))}{\epsilon} \psi(\tau) - H(\bar{\psi}_{i+1}, \psi_i)]\epsilon} \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i$$

and

$$\lim_{\epsilon \to 0} \frac{(\bar{\psi}(\tau + \epsilon) - \bar{\psi}(\tau))}{\epsilon} \psi(\tau) = \frac{\partial \bar{\psi}(\tau)}{\partial \tau} \psi(\tau)$$

Also notice that the partition function is of this form

$$\lim_{\epsilon \to 0} \int e^{f(\bar{\psi}, \psi)\epsilon} \prod_{i=1}^{N-1} d\bar{\psi}(\tau) d\psi(\tau)$$

which will become (in the limit of small ϵ)

$$= \int e^{\int f(\bar{\psi},\psi)} \prod_{i=1}^{N-1} d\bar{\psi}(\tau) d\psi(\tau)$$

The partition function is now

$$Z = \lim_{\epsilon \to 0} \int e^{\int [\frac{\partial \bar{\psi}(\tau)}{\partial \tau} \psi(\tau) - H(\bar{\psi}(\tau+\epsilon), \psi(\tau)] d\tau} \prod_{i=1}^{N-1} d\bar{\psi}(\tau) d\psi(\tau)$$

We can transform the Grassman numbers ($\psi(\tau) = \sum_n e^{i\omega_n \tau} \psi(\omega_n)$) in the partition function so that

$$\int [\frac{\partial \bar{\psi}(\tau)}{\partial \tau} \psi(\tau) - H(\bar{\psi}(\tau+\epsilon), \psi(\tau)] d\tau \quad =$$
$$\int [\frac{\partial \sum_n \frac{e^{i\omega_n \tau}}{\beta} \bar{\psi}(\omega_n)}{\partial \tau} \sum_m \frac{e^{-i\omega_m \tau}}{\beta} \psi(\omega_m) - H(\bar{\psi}(\tau+\epsilon), \psi(\tau)] d\tau$$

$$= \int [i\omega_n \sum_n \frac{e^{i\omega_n \tau}}{\beta} \bar{\psi}(\omega_n) \sum_m \frac{e^{-i\omega_m \tau}}{\beta} \psi(\omega_m) - H(\bar{\psi}(\tau+\epsilon), \psi(\tau)] d\tau$$

Let's also take the limit as $\beta \to \infty$ so that

$$\lim_{\beta \to \infty} \sum_n e^{i\omega_n \tau} \to \int e^{i\omega\tau} d\omega$$

So what we are doing is taking $\lim_{\beta\to\infty}\omega_n = \lim_{\beta\to\infty}\frac{\pi(2n+1)}{\beta}$, which is letting the Matsubura frequencies ω_n become continuous. The partition function now becomes

$$Z = \int e^{\int[i\omega\int\frac{e^{i\omega\tau}}{\beta}\bar{\psi}(\omega)d\omega\int\frac{e^{-i\omega'\tau}}{\beta}\psi(\omega')d\omega' - H(\bar{\psi}(\tau+\epsilon),\psi(\tau))]d\tau}\prod_{i=1}^{N-1}d\bar{\psi}(\omega)d\psi(\omega)$$

Then, looking at only the term of the exponent,

$$\begin{aligned}\int[i\omega\int\frac{e^{i\omega\tau}}{\beta}\bar{\psi}(\omega)d\omega\int\frac{e^{-i\omega'\tau}}{\beta}\psi(\omega')d\omega' - H(\bar{\psi}(\tau+\epsilon),\psi(\tau))]d\tau &= \\ \int i\omega\frac{1}{\beta^2}[\int\bar{\psi}(\omega)d\omega\int e^{-i\omega'\tau}e^{i\omega\tau}\psi(\omega)d\omega' - H(\bar{\psi}(\tau+\epsilon),\psi(\tau))]d\tau &= \\ \frac{1}{\beta^2}\int\int\bar{\psi}(\omega)i\omega\psi(\omega')\int e^{-i\omega\tau}e^{i\omega'\tau}d\tau d\omega d\omega' - \int H(\bar{\psi}(\tau+\epsilon),\psi(\tau))d\tau\end{aligned}$$

Looking at only the first integral,

$$\frac{1}{\beta^2}\int\int\bar{\psi}(\omega)i\omega\psi(\omega')\int e^{-i\omega\tau}e^{i\omega'\tau}d\tau d\omega d\omega' = \frac{i\omega}{\beta^2}\int\int\bar{\psi}(\omega)\psi(\omega')\delta(\omega-\omega')d\omega d\omega'$$

$$= \frac{1}{\beta^2}\int\bar{\psi}(\omega)i\omega\psi(\omega)d\omega$$

$$Z = \int e^{\frac{1}{\beta^2}\int\bar{\psi}(\omega)i\omega\psi(\omega)d\omega - \int H(\bar{\psi}(\tau+\epsilon),\psi(\tau))d\tau}$$

Now we consider the right integral, and rembering that $\epsilon \to 0$, we can claim that

$$H(\bar{\psi}(\tau+\epsilon), \psi(\tau)) = H(\bar{\psi}(\tau), \psi(\tau))$$

Since H originally consist of an even number of fermion operators, it is of the form

$$H(\Psi^\dagger, \Psi) = g(t,\mu)\Psi^\dagger\Psi$$

where g is a function not of $\Psi's$ but of other parameters such as the chemical potential μ or the hopping strength t. As H was carried though our formulation of the partition function, it had to act on a set of coherent states in this way

$$\int \langle -\bar{\psi}|1-\epsilon H|\psi\rangle d\tau = \langle -\bar{\psi}|1-\epsilon g(t,\mu)\Psi^\dagger\Psi|\psi\rangle$$

$$= \int \langle -\bar{\psi}|1-\epsilon g(t,\mu)\bar{\psi}(\tau)\psi(\tau)|\psi(\tau)\rangle d\tau$$

$$= \int \langle -\bar{\psi}|e^{-\epsilon g(t,\mu)\bar{\psi}(\tau)\psi(\tau)}|\psi\rangle d\tau$$

so $H(\Psi^\dagger, \Psi)$ has been replaced by $H(\bar{\psi}, \psi) = g(t,\mu)\bar{\psi}(\tau)\psi(\tau)$ in the exponent. So the exponent is now

$$\int e^{-\int g(t,\mu)\bar{\psi}(\tau)\psi(\tau)d\tau}$$

The after doing a Fourier Transform and taking the limit as $\beta \to \infty$,

$$\int e^{-\int g(t,\mu)\bar{\psi}(\tau)\psi(\tau)d\tau} = \int e^{-\int \frac{g(t,\mu)}{\beta^2} \int e^{i\omega\tau}\bar{\psi}(\omega) \int e^{-i\omega'\tau}\psi(\omega)d\tau d\omega d\omega'}$$

$$= \int e^{-\int \frac{g(t,\mu)}{\beta^2}\bar{\psi}(\omega)\psi(\omega)d\omega}$$

Therfore our final partition function will be

$$Z = \int e^{\frac{1}{\beta^2}\int \bar{\psi}(\omega)i\omega\psi(\omega)d\omega - \int \frac{g(t,\mu)}{\beta^2}\bar{\psi}(\omega)\psi(\omega)d\omega}$$

or more compactly

$$Z = \int e^{\frac{1}{\beta^2}\int \bar{\psi}(\omega)[i\omega - g(t,\mu)]\psi(\omega)d\omega}$$

taking note that $g(t,\mu)$ is a function specific to the system's Hamiltonian.

For the the single fermion problem, the hamiltonian $H = (\Omega_0 - \mu)\Psi^\dagger\Psi$, so $g(t,\mu) = \Omega_0 - \mu$ and the partition function for this system is

$$Z = \int e^{\frac{1}{\beta^2}\int \bar{\psi}(\omega)[i\omega - \Omega_0 + \mu]\psi(\omega)d\omega} D\bar{\psi}D\psi$$

The term in the exponent of the partition function is called the action and is defined this way.

$S = \int \bar{\psi}(\omega)[i\omega - g(t,\mu)]\psi(\omega)d\omega$ is called the action.

C.0.12 The Green's Function

$G(\omega_1, \omega_2) = \langle \bar{\psi}(\omega_1)\psi(\omega_2)\rangle$

For the case of the fermion oscillator,

$$\langle \bar{\psi}(\omega_1)\psi(\omega_2)\rangle = \frac{\int \bar{\psi}(\omega_1)\psi(\omega_2)d\bar{\psi}(\omega)d\psi(\omega)exp[\int_{-\infty}^{\infty}\bar{\psi}(\omega)[i\omega - \Omega_0 + \mu]\psi(\omega)\frac{d\omega}{2\pi}]}{\int_{-\infty}^{\infty} d\bar{\psi}(\omega)d\psi(\omega)exp[\int \bar{\psi}(\omega)[i\omega - \Omega_0 + \mu]\psi(\omega)\frac{d\omega}{2\pi}]}$$

We can compute this Green's function by recalling that,

$$\langle \bar{\psi}_i\psi_j\rangle = \int \bar{\psi}_i\psi_j e^{-a_1\bar{\psi}_1\psi_1 - a_2\bar{\psi}_2\psi_2}d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 = \frac{\delta_{ij}}{a_i}$$

To use the above property, we will have to discretize the integral in the exponent of $\langle \bar{\psi}(\omega_1)\psi(\omega_2)\rangle$.

$$\langle \bar{\psi}(\omega_1)\psi(\omega_2)\rangle = \frac{\int \bar{\psi}(\omega_1)\psi(\omega_2)exp[\sum_\omega \bar{\psi}(\omega)\psi(\omega)\frac{[i\omega-\Omega_0+\mu]}{2\pi}]\prod_{i=1}^{N-1}d\bar{\psi}_i d\psi_i}{\int_{-\infty}^{\infty} exp[\sum_\omega \bar{\psi}(\omega)\frac{[i\omega-\Omega_0+\mu]}{2\pi}\psi(\omega)]\prod_{i=1}^{N-1}d\bar{\psi}_i d\psi_i}$$

Upon expanding $exp[\sum_\omega \bar{\psi}(\omega)\psi(\omega)\frac{[i\omega-\Omega_0+\mu]}{2\pi}]$ in the numerator, all terms will vanish except the $(N-2)^{th}$ term because there needs to be $N-1$ pairs of Grassman numbers $\bar{\psi}(\omega)\psi(\omega)$. One pair is already given by $\bar{\psi}(\omega_1)\psi(\omega_2)\delta(\omega_1-\omega_2)$.

In the denominator, all terms will vanish except the $(N-1)^{th}$ term because there needs to be $N-1$ pairs of Grassman numbers also.

$$\langle\bar{\psi}(\omega_1)\psi(\omega_2)\rangle = \frac{(N-2)\frac{[i\omega-\Omega_0+\mu]}{2\pi}]\delta(\omega_1-\omega_2)}{(N-1)\frac{[i\omega-\Omega_0+\mu]}{2\pi}]}$$

As a result,

$$\langle\bar{\psi}(\omega_1)\psi(\omega_2)\rangle = \frac{2\pi\delta(\omega_1-\omega_2)}{i\omega-\Omega_0+\mu}$$

This is an important result which we will use extensively. It is known as the Green's Function for fermions.

C.0.13 Perturbation Of The Partition Function

In Renormalization Group Theory one would like to predict macroscopic properties of a system. In condendensed matter, this can be done if we look at physics of long wavelengths, i.e. when $\lambda \gg a$ where a is the inter-lattice space distance. When we are working in momentum space, we want $k \ll \frac{1}{a}$. To integrate high energy modes, or short wavelength physics, we split the fields into high energy modes (large k) and low energy modes (small k)

$$\psi_< for 0 < k < \Lambda/s$$

$$\psi_> for \Lambda/s < k < \Lambda$$

where Λ is called energy cutoff and s is a parameter(probably just a number, but symbolic here).

The action S will be composed of different modes

$$S(\psi_<, \psi_>) = S_0(\psi_<) + S_0(\psi_>) + S_I(\psi_<, \psi_>)$$

The partition function for fermions, which will be similar for bosons, is now

$$Z = \int e^{S_0(\psi_<)+S_0(\psi_>)+S_I(\psi_<,\psi_>)}[d\bar{\psi}_< d\psi_<][d\bar{\psi}_> d\psi_>]$$

We choose to integrate the high energy modes $S_0(\psi_>)$ and leave low energy, since that is our interest. We can integrate high energy modes using Wick's Theorem

$$Z = \int e^{S_0(\psi_>)} \int e^{S_0(\psi_<)+S_I(\psi_<,\psi_>)}[d\bar{\psi}_< d\psi_<][d\bar{\psi}_> d\psi_>]$$

After integrating high energy modes,

$$Z = \int e^{S_0(\psi_<)+S_I(\psi_<,\psi_>)}[d\bar{\psi}_< d\psi_<]$$

We assume that the action can be expanded in a series this way

$$S = S_0 + S_2 + S_4 + \ldots$$

where we have ommitted odd subscripts since they imply that the action has an odd number of fields. And remember that an odd number of fields in the path integral will give zero. We will let $S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}[i\omega - H(\bar{\psi}, \psi)]\psi$ and $Z_0 = \int e^{S_0}[d\bar{\psi}_<][d\psi_<]$. We can get a general idea about what the other terms in the expansion are if we expand the exponent in the path integral.

$$e^{-\bar{\psi}\psi} = e^{-(\bar{\psi}_< + \bar{\psi}_>)(\psi_< + \psi_>)}$$

$$e^{-\bar{\psi}\psi} = e^{-(\bar{\psi}_<\psi_< + \bar{\psi}_<\psi_> + \bar{\psi}_>\psi_< + \bar{\psi}_>\psi_>)}$$

Fields that do not have pairs, like the two middle terms in the exponent, will not contribute to the path integral. As a result,

$$e^{-\bar{\psi}\psi} = e^{-(\bar{\psi}_<\psi_< + \bar{\psi}_>\psi_>)}$$

After integrating the high order modes $>$, we are left with $e^{-\bar{\psi}\psi} = e^{-\bar{\psi}_<\psi_<}$. Therefore, we claim that the next order in the expansion of the action is

$$S_2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}\mu(k,\omega)\psi$$

and

$$Z_2 = \int e^{S_2}[d\bar{\psi}_<][d\psi_<]$$

where μ is a called a coupling function to be determined for the specific system.

Expanding to the next order,

$$e^{-\bar{\psi}\psi} = 1 - \bar{\psi}\psi + \bar{\psi}\psi\bar{\psi}\psi$$

Since we are considering fermionic operators, all powers of the same field are equal to 0 ($(\bar{\psi}\psi)^n = 0$). Therefore, we must label them with a number that refers to the particles destroyed (1 and 2) and particles created (3 and 4). These terms will be in the exponential part of the partition function so that any which do not come in pairs or repeat will cancel.

$$\bar{\psi}_4\bar{\psi}_3\psi_2\psi_1 = [\bar{\psi}_{<4} + \bar{\psi}_{>4}][\bar{\psi}_{<3} + \bar{\psi}_{>3}][\psi_{<2} + \psi_{>2}][\psi_{<1} + \psi_{>1}])$$

$$
\begin{aligned}
= \quad & \bar{\psi}_{<4}\bar{\psi}_{<3}\psi_{<2}\psi_{<1} + \bar{\psi}_{>4}\bar{\psi}_{>3}\psi_{>2}\psi_{>1} + \\
& \bar{\psi}_{>4}\bar{\psi}_{<3}\psi_{<2}\psi_{>1} + \bar{\psi}_{>4}\bar{\psi}_{<3}\psi_{>2}\psi_{<1} + \\
& \bar{\psi}_{<4}\bar{\psi}_{>3}\psi_{<2}\psi_{>1} + \bar{\psi}_{<4}\bar{\psi}_{>3}\psi_{>2}\psi_{<1}
\end{aligned}
$$

where we have ommitted writing the fields that don't pair up such as $\bar{\psi}_{<4}\bar{\psi}_{<3}\psi_{>2}\psi_{<1}$ or $\bar{\psi}_{<4}\bar{\psi}_{<3}\psi_{>2}\psi_{>1}$. The term $\bar{\psi}_{>4}\bar{\psi}_{>3}\psi_{>2}\psi_{>1}$ will be integrated explicitly, and we are left with 5 terms. We will keep the first term since it consist of only slow modes $\bar{\psi}_{<4}\bar{\psi}_{<3}\psi_{<2}\psi_{<1}$. As for the rest of the terms, they are a mixture of slow $<$ and fast $>$ modes. The 3rd term in our S series is

$$S_4 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} u(k,\omega)\bar{\psi}_4\bar{\psi}_3\psi_2\psi_1$$

and

$$Z_4 = \int e^{S_2}[d\bar{\psi}_<][d\psi_<][d\bar{\psi}_>][d\psi_>]$$

We would like to integrate the high modes, and we can do that using the properties of Grassman numbers. So that we end up with a partition function for only low energy modes

$$Z_4 = \int e^{S_2}[d\bar{\psi}_<][d\psi_<]$$

C.0.14 Fourier Transform of Spinless Fermions on a 1-D Lattice

We begin by looking at the hamiltonian of 1-D fermions considering the hopping term, the chemical potential (energy cost of particle existing), and the interaction term (potential energy between nearest neigbor particles).

$$H = -t_f \sum_j (C_j^\dagger C_{j+1} + C_{j+1}^\dagger C_j) + \mu \sum_j C_j^\dagger C_j - u \sum_j C_j^\dagger C_{j+1}^\dagger C_j C_{j+1}$$

where j is the denotes the position of a particle on a 1-D lattice. t is a parameter that denotes the hopping strength,μ is the chemical potential, and u is the interaction strength parameter (also called the coupling constant), which is a measure of how likely the scattering of two fermion can result in their anihilation and creation of two new fermions. We would like to work in momentum space. We can transform the hamiltonian above by using the following relation

$$C_j = \int dk \Psi(k) e^{-ikj}$$

Therefore, the hopping term is

$$\sum_j C_j^\dagger C_{j+1} = \sum_j \int dk \Psi^\dagger(k) e^{ikj} \int dk' \Psi^\dagger(k') e^{-ik'(j+1)}$$

$$= \int dk \Psi^{\dagger}(k) \int dk^{'} \Psi(k^{'}) \sum_{j} e^{ikj} e^{-ik^{'}(j+1)}$$

$$= \int dk \Psi^{\dagger}(k) \int dk^{'} \Psi(k^{'}) \sum_{j} e^{i(k-k^{'})j} e^{-ik^{'}}$$

$$= \int dk \Psi^{\dagger}(k) \int dk^{'} \Psi(k^{'}) \delta(k-k^{'}) e^{-ik^{'}}$$

$$= \int dk \Psi^{\dagger}(k) \Psi(k) e^{-ik}$$

In the same way, the hermitian conjugate

$$\sum_{j} C_{j+1}^{\dagger} C_{j} = \int dk \Psi^{\dagger}(k) \Psi(k) e^{ik}$$

Therefore,

$$\sum_{j} C_{j}^{\dagger} C_{j+1} + C_{j+1}^{\dagger} C_{j} = t_{f} \int dk \Psi^{\dagger}(k) \Psi(k) (e^{-ik} + e^{ik})$$

$$\sum_{j} C_{j}^{\dagger} C_{j+1} + C_{j+1}^{\dagger} C_{j} = -2t_{f} \int dk \Psi^{\dagger}(k) \Psi(k) \cos(k)$$

We transform the chemical potential term

$$\mu \sum_j C_j^\dagger C_j = \int dk \Psi^\dagger(k) \int dk' \Psi(k') \sum_j e^{ikj} e^{-ik'j}$$

$$= \mu \int dk \Psi^\dagger(k) \int dk' \Psi(k') \delta(k - k')$$

$$\mu \sum_j C_j^\dagger C_j = \mu \int dk \Psi^\dagger(k) \Psi(k)$$

We define the dispersion relation as

$$\varepsilon(k) \equiv -2t_f \cos(k) + \mu(k)$$

so that the bare Hamiltonian in momentum space is

$$H_0(k) = \int dk \Psi^\dagger(k) \Psi(k) \varepsilon(k)$$

We also perform the Fourier transform of the interaction term which gives the strength of attraction or repulsion between two particles.

$$H_I = -u \sum_j C_j^\dagger C_{j+1}^\dagger C_j C_{j+1}$$

It describes an anihilation of two existing particles that have scattered by any method followed by the creation of two new particles. We can label the 4 particles which take part in

the process. The particles labeled 1 and 2 will be the original particles which are destroyed through scattering. The others, labeled 3 and 4, are the particles created. We consider both cases of scattering.

First Case: $1 \to 3$ and $2 \to 4$.

Second Case: $1 \to 4$ and $2 \to 3$.

In our Fourier transform, we will use the second case, though we could just as well use the first.

$$H_I(k_4, k_3, k_1, k_2) = -u \sum_j \int dk_4 \Psi^\dagger(k_4) \int dk_3 \Psi^\dagger(k_3) \int dk_1 \Psi(k_1) \int dk_2 \Psi(k_2) e^{ik_4 j} e^{ik_3 j} e^{-ik_1 j} e^{-ik_2 j}$$

$$= -u \int dk_4 \Psi^\dagger(k_4) \int dk_3 \Psi^\dagger(k_3) \int dk_1 \Psi(k_1) \int dk_2 \Psi(k_2) \sum_j e^{i(k_4 + k_3 - k_1 - k_2)j}$$

$$= -u \int dk_4 \Psi^\dagger(k_4) \int dk_3 \Psi^\dagger(k_3) \int dk_1 \Psi(k_1) \int dk_2 \Psi(k_2) 2\pi\delta(k_4 + k_3 - k_1 - k_2)$$

$$H_I(k_4, k_3, k_2) = -u[\prod_{i=1}^{4} \int \frac{dk_i}{2\pi}] 2\pi\delta(k_4 + k_3 - k_1 - k_2) \Psi^\dagger(k_4) \Psi^\dagger(k_3) \Psi(k_1) \Psi(k_2)$$

In general, H_I must consider both cases. So

$$\begin{aligned} H_I(k_l, k_m) &= -u[\prod_{i=1}^{4} \int \frac{dk_i}{2\pi}] 2\pi\delta(k_4 + k_3 - k_1 - k_2) e^{i(k_l - k_m)} \Psi^\dagger(k_4)\Psi^\dagger(k_3)\Psi(k_1)\Psi(k_2) \\ &\times \quad [\delta_{l,3}\delta_{m,2} + \delta_{l,4}\delta_{m,1}] \end{aligned}$$

The complete transformed hamiltonian $H = H_0 + H_I$ is

$$\begin{aligned} H &= \int dk \Psi^\dagger(k)\Psi(k)\varepsilon(k) + \\ &\quad -u[\prod_{i=1}^{4} \int \frac{dk_i}{2\pi}] 2\pi\delta(k_4 + k_3 - k_1 - k_2) e^{i(k_3 - k_2)} \Psi^\dagger(k_4)\Psi^\dagger(k_3)\Psi(k_2)\Psi(k_1) \end{aligned}$$

Now that we have formed the Hamiltonian in momentum space, we will form the partition function Z.

$$Z = \int [D\bar{\psi}][D\psi] e^S$$

where $\bar{\psi}$and ψ are Grassman variables and

$$\begin{aligned} S &= \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \bar{\psi}[i\omega - \varepsilon(k)]\psi + \\ &\quad -u \prod_{i=1}^{4} \int \frac{dk_i}{2\pi} \int \frac{d\omega_i}{2\pi} \bar{\psi}_i [e^{i(k_3 - k_2)}] \psi_i \delta(k_4 + k_3 - k_1 - k_2) \delta(\omega_4 + \omega_3 - \omega_1 - \omega_2) \end{aligned}$$

To show this is true for the interaction term, let's perform the necessary steps to go from $\psi(\tau)$ to $\psi(\omega)$. Remembering that the action $S = \int \bar{\psi}(\tau)[i\omega - H(\Psi^\dagger, \Psi)]\psi(\tau)d\omega = \int \bar{\psi}(\tau)[i\omega - H(\bar{\psi}(\tau,k), \psi(\tau,k)]\psi(\tau)d\omega$, we split S into parts as

$$S = S_0 + S_4$$

where S_4 is the term related to H_I and S_0 is related to H_0.

We must also point out that since we are working in momentum space, $\psi(\tau)$ is now $\psi(k,\tau)$ because the field ψ must carry all the information about the particles. When it is transformed, it will be a function of momentum k and frequency ω . Therefore

$$S_0 = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int \bar{\psi}(\omega)[i\omega - \varepsilon(k)]\psi(\omega)\frac{d\omega}{2\pi}$$

where we are integrating in momentum space with a cutoff Λ . This Λ , by the dispersion relation, is also an energy width extending from the Fermi surface.

Transforming S_0, and taking the limit of 0 temperature lim $\beta \to \infty$,

$$S_0 = -\int \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \bar{\psi}(k,\tau) H_0(k,\tau)\psi(k,\tau)d\tau$$

$$= \int \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int e^{i\omega\tau}\bar{\psi}(\omega,k)[i\omega - \varepsilon(k)] \int e^{-i\omega'\tau}\psi(\omega',k)d\tau \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}$$

$$= \int_{-\Lambda}^{\Lambda} \int \frac{dk}{2\pi} \int \bar{\psi}(\omega, k)[i\omega - \varepsilon(k)]\psi(\omega', k) \int e^{i\omega\tau} e^{-i\omega'\tau} d\tau \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi}$$

$$= \int_{-\Lambda}^{\Lambda} \int \frac{dk}{2\pi} \mu \int \bar{\psi}(\omega, k)[i\omega - \varepsilon(k)]\psi(\omega', k) 2\pi\delta(\omega - \omega') \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi}$$

$$= \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \bar{\psi}(\omega, k)[i\omega - \varepsilon(k)]\psi(\omega, k) \frac{d\omega}{2\pi}$$

$$S_0 = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(\omega, k)[i\omega - \varepsilon(k)]\psi(\omega, k)$$

Therefore, after transformation

$$S_0 = \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \bar{\psi}(\omega, k)[i\omega - \varepsilon(k)]\psi(\omega, k)$$

Now let's consider the S_I and do the same. Take note that we must integrate over k for all four particles that are involved in the process.

$$S_4 = -u \int d\tau [\prod_{i=1}^{4} \int \frac{dk_i}{2\pi}] 2\pi\delta(k_4 + k_3 - k_1 - k_2) e^{i(k_3 - k_2)} \Psi^\dagger(k_4) \Psi^\dagger(k_3) \Psi(k_1) \Psi(k_2)$$

$$
\begin{aligned}
&= -u[\prod_{i=1}^{4}\int \frac{dk_i}{2\pi}]2\pi\delta(k_4+k_3-k_1-k_2)e^{i(k_3-k_2)} \\
&\times \ [\int\prod_{i=1}^{4}\int \frac{d\omega_i}{2\pi}]\bar{\psi}(k_4,\omega_4)\bar{\psi}(k_3,\omega_4)\psi(k_1,\omega_1)\psi(k_2,\omega_2)\int d\tau e^{i\omega_4\tau}e^{i\omega_3\tau}e^{-i\omega_1\tau}e^{-i\omega_2\tau}
\end{aligned}
$$

$$
\begin{aligned}
&= -u[\prod_{i=1}^{4}\int \frac{dk_i}{2\pi}]2\pi\delta(k_4+k_3-k_1-k_2)e^{i(k_3-k_2)} \\
&\times \ [\int\prod_{i=1}^{4}\int \frac{d\omega_i}{2\pi}]\bar{\psi}(k_4,\omega_4)\bar{\psi}(k_3,\omega_4)\psi(k_1,\omega_1)\psi(k_2,\omega_2)2\pi\delta(\omega_4+\omega_3-\omega_1-\omega_2)
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= -u[\prod_{i=1}^{4}\int \frac{dk_i}{2\pi}][\int\prod_{i=1}^{4}\int \frac{d\omega_i}{2\pi}](2\pi)^2\delta(k_4+k_3-k_1-k_2)\delta(\omega_4+\omega_3-\omega_1-\omega_2) \\
&\times \ e^{i(k_3-k_2)}\bar{\psi}(k_4,\omega_4)\bar{\psi}(k_3,\omega_4)\psi(k_1,\omega_1)\psi(k_2,\omega_2)
\end{aligned}
$$

Therefore, the partition function for a system of fermions on a lattice with on-site contact interaction is

$$
Z = \int [D\bar{\psi}][D\psi]e^{S_0+S_4}
$$

C.0.15 Renormalization Group Theory Applied to 1-D Spinless Fermions

We formulated the partition function for the 1-D spinless fermions in the previous chapter. The dispersion realtion we acquired is $\varepsilon(k) = -2t_f \cos(k) + \mu(k)$. The ground state for this system is where all the states k are filled between $k = -\frac{\pi}{2}$ to $k = \frac{\pi}{2}$. We will consider only trasitions nearby these 2 points. That will allow us to linearize the dispersion relation so that $\varepsilon(k) \approx -2t_f k + \mu(k)$. Now we would like to integrate out the fast modes in the action S where $\Lambda/s < k < \Lambda$.

Recall that the partition function can be written as

$$Z = \int e^{S_0(\psi_<)+S_0(\psi_>)+S_I(\psi_<,\psi_>)}[d\bar{\psi}_< d\psi_<][d\bar{\psi}_> d\psi_>]$$

but we would like to end up with a partition funciton which is a function of only the slow modes $\psi_<$. We can do this by writing

$$Z = \int e^{S_0(\psi_<)}[d\bar{\psi}_< d\psi_<] \int e^{S_0(\psi_>)} e^{S_I(\psi_<,\psi_>)}[d\bar{\psi}_> d\psi_>]$$

$$= Z_< \int e^{S_0(\psi_>)} e^{S_I(\psi_<,\psi_>)}[d\bar{\psi}_> d\psi_>]$$

where $Z_< = \int e^{S_0(\psi_<)}[d\bar{\psi}_< d\psi_<]$. Then we can divide by and multiply by $\int e^{S_0(\psi_>)}[d\bar{\psi}_> d\psi_>]$ so that

$$Z = Z_< \frac{\int e^{S_0(\psi_>)} e^{S_I(\psi_<,\psi_>)}[d\bar{\psi}_> d\psi_>]}{\int e^{S_0(\psi_>)}[d\bar{\psi}_> d\psi_>]} \int e^{S_0(\psi_>)}[d\bar{\psi}_> d\psi_>]$$

$$= Z_< \langle e^{S_I(\psi_<,\psi_>)} \rangle \int e^{S_0(\psi_>)} [d\bar{\psi}_> d\psi_>]$$

$$Z = Z_< Z_> \langle e^{S_I(\psi_<,\psi_>)} \rangle_{s_{0>}}$$

where the average is over the fast modes. More explicitly, the partition functon is

$$Z = \int e^{S_0(\psi_<)} [d\bar{\psi}_< d\psi_<] \int e^{S_0(\psi_>)} [d\bar{\psi}_> d\psi_>] \langle e^{S_I(\psi_<,\psi_>)} \rangle$$

In order to evaluate $\langle e^{S_I(\psi_<,\psi_>)} \rangle$, we can use the cumulant expansion $\langle e^{\Omega} \rangle = e^{[\langle \Omega \rangle + (\langle \Omega^2 \rangle - \langle \Omega \rangle^2) + \ldots]}$. Therefore, we just have to evaluate $\langle S_I \rangle + \langle S_I^2 \rangle - \langle S_I \rangle^2 + \ldots$.

Let us apply the RG transformations only to S_0 where we take into account the dispersion relation (not considering the interaction potential u between particles). The question is, what will happen to the terms in the Hamiltonian associated with S_0 such as the chemical potential μ? Begin with

$$S_0 = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\omega)[i\omega - \varepsilon(k)]\psi(k,\omega)$$

which can be split into $S_0(\psi_<) + S_0(\psi_>)$.The original integral $\int_{-\Lambda}^{\Lambda}$ becomes $\int_{-\Lambda/s}^{\Lambda/s} + \int_{\Lambda/s<|k|<\Lambda}$, which are a slow modes integration and a fast modes integration respectively.

Explicitly, the fast modes and slow modes partition functions are

$$Z_> = \int [d\bar{\psi}_> d\psi_>] exp[\int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}'_>(k,\omega)[i\omega - \varepsilon(k)]\psi'_>(k,\omega)]$$

$$Z_< = \int [d\bar{\psi}_< d\psi_<] exp[\int_{-\Lambda/s}^{\Lambda/s} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}'_<(k,\omega)[i\omega - \varepsilon(k)]\psi'_<(k,\omega)]$$

The partition function for the fast modes $Z_>$ can be integrated and will just add a constant to the overall partition function. For an RG transformation we must rewrite the slow modes partition function $Z_<$ by rescaling the momenta k and the fields $\psi_<(k,\omega)$ so that the action S_0 remains fixed. We define a new momentum

$$k^{'} = sk$$

and a new frequency

$$\omega' = s\omega$$

where s is called the renormalization factor. We also transform the fermion fields by an unknown variable ξ to be determined

$$\bar{\psi}' = \xi^{-1}\bar{\psi}$$

$$\psi^{'} = \xi^{-1}\psi$$

Note that $dk = \frac{dk'}{s}$ and $d\omega = \frac{d\omega'}{s}$.

$$S_0(\psi_<) = \xi^2 \int_{-\Lambda/s}^{\Lambda/s} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}'_<(k', \omega')[i\omega - \varepsilon(k)]\psi'_<(k', \omega')$$

Since $\int_{-\Lambda/s}^{\Lambda/s} dk$ is the same as $\int_{-\Lambda}^{\Lambda} dk'$, then

$$S_0(\psi'_<) = \xi^2 s^{-3} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}'_<(k', \omega')[i\omega' - \varepsilon(k')]\psi'_<(k', \omega')$$

We want to fix the action under Renormalization Group transformation. In other words, we want to Hamiltonian to be of the same for as the original Hamiltonian as we look at the system at longer wavelengths. We want the action S_0 to go to a fix point $S^\star$ as we transform.

The action can be fixed if $\xi = s^{3/2}$. As a result

$$S_0(\psi'_<) = \frac{1}{s^3} s^3 \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}'_<(k', \omega')[i\omega' - \varepsilon(k')]\psi'_<(k', \omega')$$

and we can drop the $'$ symbol to see that the action now has the same form as the original action with rescaled momenta k and frequency ω. Since the fast modes have been integrated, we are now working in a new momentum space that lacks those fast modes. We refer to the trasformed field as a field of the new fast and slow modes in the new phase space.

$$S_0(\psi) = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k, \omega)[i\omega - \varepsilon(k)]\psi(k, \omega)$$

with the fields $\psi^{'} = s^{-3/2}\psi$. This choice of ξ allows $S_0(\psi'_<)$ to look the same as $S_0(\psi)$, meaning that under RG transformation, the action S_0 will not change. Therefore, we have fixed the action

$$S_0(\psi'_<) = S^\star = S_0(\psi)$$

Now we would like to know what happens to the dispersion relation $\varepsilon(k)$ as transformation occurs, or more importantly what happens to the chemical potential. We assume that the chemical potential has a Taylor expansion as a function of momentum and frequency. If it is not a function of k and ω, then we would only keep the 0th order term.

$$\mu(k,\omega) = \mu_{00} + \mu_{10}k + \mu_{01}\omega + \dots$$

We defined the rescaled chemical potential μ' as

$$\mu'(k',\omega') = \mu'_{00} + \mu'_{10}k' + \mu'_{01}\omega' + \dots$$

so that

$$\mu'(k,\omega) = s\mu(k,\omega)$$

Therefore, the rescaled dispersion relation around the Fermi point is

$$\varepsilon(k') = -2t_f k' + \mu'_{00} + \mu'_{10}k' + \mu'_{01}\omega' + \dots$$

We transform $S_0(\psi_<) \rightarrow S_0'$ as

$$S_0 = \xi^2 \int_{-\Lambda/s}^{\Lambda/s} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}'_<(k,\omega)[i\omega - \varepsilon(k)]\psi'_<(k,\omega)$$

$$\bar{\psi}' = \xi^{-1}\bar{\psi}_<$$

$$\psi' = \xi^{-1}\psi_<$$

then subsituting $\omega' = s\omega$ and $k^{'} = sk$ in the action S_0 to get the transformed action

$$S_0^{'} = \xi^2 s^{-3} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}'_<(k,\omega)[i\omega' - \varepsilon(k')]\psi'_<(k,\omega)$$

$$S_0^{'} = \xi^2 s^{-3} \int_{-\Lambda/s}^{\Lambda/s} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}'_<(k,\omega)[i\omega' - \varepsilon(k')]\psi'_<(k,\omega)$$

$$S_0^{'} = \xi^2 s^{-3} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}'_<(k,\omega)[i\omega' - 2t_f k' + \mu'(k',\omega')]\psi'_<(k,\omega)$$

with $\xi = s^{\frac{3}{2}}$,

$$S_0 = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\omega)[i\omega - 2t_f k + \mu'(k',\omega')]\psi(k,\omega)$$

Therefore the change in the action is

$$\delta S_0 = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\omega) s\mu(k,\omega) \psi(k,\omega)$$

Recalling that

$$\mu'(k,\omega) = s\mu(k,\omega)$$

gives

$$\mu_{00}^{'} = s\mu_{00}$$

and the linear terms

$$\mu'_{10} k' = s\mu_{10} k = s\mu_{10} \frac{k'}{s}$$

$$\mu'_{10} = \mu_{10}$$

and in exactly the same way,

$$\mu'_{01} = \mu_{01}$$

This brings us to defining some terminology. Terms such as μ_{00} are called relevant because they become larger under RG transformations. So if the chemical potential is independent of k and ω, and if we did not take into account interaction , then the chemical potential would become larger. Other terms such as μ_{10} are called marginal because they become smaller

under RG transformations. And terms such as μ_{20} which transform as

$$\mu'_{20}k'^2 = s\mu_{20}k^2 = s\mu_{20}\frac{k'^2}{s^2} = \mu_{20}\frac{k'^2}{s}$$

$$\mu'_{20} = \frac{\mu_{20}}{s}$$

are called irrelevant because they become smaller under RG transformations.

C.0.16 Four Point Vertex

Let's consider the a general S_4 term of the action and apply the parameters that we used to fix the S_0 order term.

$$\begin{aligned} S_4 &= -u[\prod_{i=1}^{4}\int\frac{dk_i}{2\pi}][\int\prod_{i=1}^{4}\int\frac{d\omega_i}{2\pi}](2\pi)^2\delta(k_4+k_3-k_1-k_2)\delta(\omega_4+\omega_3-\omega_1-\omega_2) \\ &\times \bar{\psi}(k_4,\omega_4)\bar{\psi}(k_3,\omega_4)\psi(k_2,\omega_2)\psi(k_1,\omega_1) \end{aligned}$$

Let's write this in shorthand notation by setting

$$\int = [\prod_{i=1}^{4}\int\frac{dk_i}{2\pi}][\int\prod_{i=1}^{4}\int\frac{d\omega_i}{2\pi}](2\pi)^2\delta(k_4+k_3-k_1-k_2)\delta(\omega_4+\omega_3-\omega_1-\omega_2)$$

so that now

$$S_4 = -\int \bar{\psi}(k_4,\omega_4)\bar{\psi}(k_3,\omega_3)\psi(k_2,\omega_2)\psi(k_1,\omega_1)u$$

Let's separate S_4 into fast and slow modes. We have already done this in Chapter 4 and we ended up with 6 valid terms. 4 of those terms had mixtures of slow and fast modes.

$$\begin{aligned} S_4 &= -\int [\bar{\psi}_<(k_4,\omega_4)\bar{\psi}_>(k_3,\omega_4) - \bar{\psi}_>(k_4,\omega_4)\bar{\psi}_<(k_3,\omega_4)] \\ &\times \quad [\psi_<(k_1,\omega_1)\psi_>(k_2,\omega_2) - \psi_>(k_1,\omega_1)\psi_<(k_2,\omega_2)]u \end{aligned}$$

$$\begin{aligned} S_4 &= -u\int [\bar{\psi}_<(k_4,\omega_4)\bar{\psi}_>(k_3,\omega_4)\psi_<(k_1,\omega_1)\psi_>(k_2,\omega_2) - \\ &\quad \bar{\psi}_<(k_4,\omega_4)\bar{\psi}_>(k_3,\omega_4)\psi_>(k_1,\omega_1)\psi_<(k_2,\omega_2) - \\ &\quad \bar{\psi}_>(k_4,\omega_4)\bar{\psi}_<(k_3,\omega_4)\psi_<(k_1,\omega_1)\psi_>(k_2,\omega_2) + \\ &\quad \bar{\psi}_>(k_4,\omega_4)\bar{\psi}_<(k_3,\omega_4)\psi_>(k_1,\omega_1)\psi_<(k_2,\omega_2)] \end{aligned}$$

Notice that all four terms contribute the same, so we can write

$$S_4 = \int \bar{\psi}_<(4)\bar{\psi}_>(3)\psi_<(2)\psi_>(1)u$$

We have written S_4 as a mixture of fast and slow modes, and the numbers identify the particles. In this case 4 and 3 are particles created and 1 and 2 are particles destroyed. We can use our trick to evaluate the partition function. Recall

$$Z = Z_< Z_> \langle e^{S_I(\psi_<,\psi_>)} \rangle_{s_{0>}}$$

where the average is over the 0th order action $S_{0>} = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}_>(k,\omega)[i\omega - \varepsilon(k)]\psi_>(k,\omega)$and we are interested in renormalizing S_I. In our case S_I is S_4 . We also recall the cumulant expansion $\langle e^{\Omega} \rangle = e^{[\langle\Omega\rangle + (\langle\Omega^2\rangle - \langle\Omega\rangle^2) + \ldots]}$. So that

$$\langle e^{S_4(\psi_<,\psi_>)} \rangle = e^{[\langle S_4(\psi_<,\psi_>)\rangle + (\langle S_4(\psi_<,\psi_>)^2\rangle - \langle S_4(\psi_<,\psi_>)\rangle^2) + \ldots]}$$

$$\begin{aligned} \langle S_4(\psi_<,\psi_>)\rangle &= \langle \int \bar{\psi}_<(4)\bar{\psi}_>(3)\psi_<(2)\psi_>(1)u\rangle \\ &= \frac{\int [d\bar{\psi}_> d\psi_>] e^{S_{0>}} \int \bar{\psi}_<(4)\bar{\psi}_>(3)\psi_<(2)\psi_>(1)u}{\int [d\bar{\psi}_> d\psi_>] e^{S_{0>}}} \end{aligned}$$

keeping in mind that $\int$ means $[\prod_{i=1}^{4} \int \frac{dk_i}{2\pi}][\int \prod_{i=1}^{4} \int \frac{d\omega_i}{2\pi}](2\pi)^2 \delta(k_4 + k_3 - k_1 - k_2)\delta(\omega_4 + \omega_3 - \omega_1 - \omega_2)$

We can use Wick's theorem for fermions to evaluate this average.

$$\begin{aligned}\langle S_4(\psi_<,\psi_>)\rangle &= \int \bar\psi_<(4)\psi_<(2)\delta(k_4+k_3-k_1-k_2)\delta(\omega_4+\omega_3-\omega_1-\omega_2)\\ &\times \frac{\int[d\bar\psi_> d\psi_>]e^{S_{0>}}\bar\psi_>(3)\psi_>(1)u}{\int[d\bar\psi_> d\psi_>]e^{S_{0>}}}\end{aligned}$$

$$\begin{aligned}&= u\int_{-\Lambda/s}^{\Lambda/s}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar\psi_<(4)\psi_<(2)\delta(k_4+k_3-k_1-k_2)\delta(\omega_4+\omega_3-\omega_1-\omega_2)\\ &\times \int_{\Lambda/s<|k|<\Lambda}\frac{dk_3}{2\pi}\int_{\Lambda/s<|k|<\Lambda}\frac{dk_1}{2\pi}2\pi\delta(k_3-k_1)\int_{-\infty}^{\infty}\frac{d\omega_3}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{2\pi\delta(\omega_3-\omega_1)}{[i\omega-\varepsilon(k)]}\end{aligned}$$

$$\begin{aligned}\langle S_4(\psi_<,\psi_>)\rangle &= u\int_{-\Lambda/s}^{\Lambda/s}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar\psi_<(4)\psi_<(2)\delta(k_4-k_2)\delta(\omega_4-\omega_2)\\ &\times \int_{\Lambda/s<|k|<\Lambda}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{[i\omega-\varepsilon(k)]}\end{aligned}$$

$$\langle S_4(\psi_<,\psi_>)\rangle = u\int_{-\Lambda/s}^{\Lambda/s}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar\psi_<(k,\omega)\psi_<(k,\omega)$$

$$\times\int_{\Lambda/s<|k|<\Lambda}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{[i\omega-\varepsilon(k)]}$$

where the first term on the right can be rescaled and the second term is the change in $\langle S_4 \rangle$ that we call

$$\delta\langle S_4\rangle = \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{[i\omega - \varepsilon(k)]}$$

$k^{'} = sk$,$\omega' = s\omega$,$\bar{\psi}' = s^{-3/2}\bar{\psi}$, $\psi^{'} = s^{-3/2}\psi$

$$u = u_{00} + u_{10}k + u_{01}\omega + \ldots$$

$$u' = u'_{00} + u'_{10}k' + u'_{01}\omega' + \ldots$$

$$u' = su$$

$$\langle S_4(\psi,\psi)\rangle \approx s^3 s^{-3} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \bar{\psi}'(k',\omega')(u'_{00} + u'_{10}k' + u'_{01}\omega' + \ldots)\psi'(k',\omega')$$

$$\times \delta\langle S_4\rangle$$

To first order in u , we can write this as

$$\langle S_4(\psi_<,\psi_>)\rangle \approx \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\psi}(k,\omega) u' \psi(k,\omega)$$

where

$$u' = u'_{00} \times \delta\langle S_4\rangle$$

and the change in u is

$$du = u_{00} \times \delta\langle S_4 \rangle$$

$$du = u_{00} \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{[i\omega - \varepsilon(k)]}$$

We can set up a differential equation to determine u. The integral of the fermi states $\int dk$ is converted to an integral over energy by the fact that the number of states equals the number of states per energy (density of states) times the energy. In differential form,

$$dk = N(\xi)d\xi$$

Let's assume that near the fermi points, around which we have expanded the dispersion relation, the density of state is constant $N(\xi) = N$. If instead of using the discrete parameter s, we change the parameter to a continuous one t and we let $\Lambda = \Lambda_0 e^{-t}$ be equivalent to the fermion energy. Therefore the momentum space $\Lambda/s < |k| < \Lambda = 2|d\Lambda|$ will be an energy space $\Lambda/s < |k| < \Lambda = 2|d\Lambda|$ and

$$du = u_{00} 2N \int_{d\Lambda} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{[i\omega - \varepsilon(k)]}$$

$$du = u_{00} N d\Lambda \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{1}{[i\omega - \varepsilon(k)]}$$

We then differentiate both sides by $d\Lambda$,

$$\frac{du}{d\Lambda} = Nu_{00} \int \frac{d\omega}{\pi} \frac{1}{[i\omega - \varepsilon(k)]}$$

or since $d\Lambda = -\Lambda$ we can also write the differential equation in terms of t ,

$$\frac{du}{dt} = -\Lambda u_{00} \int \frac{d\omega}{2\pi} \frac{N}{[i\omega - \varepsilon(k)]}$$

C.0.17 Quartic Perturbation

Now we need to calculate the next order averages of

$$\left\langle e^{S_4(\psi_<,\psi_>)} \right\rangle = e^{[\langle S_4(\psi_<,\psi_>)\rangle + (\langle S_4(\psi_<,\psi_>)^2\rangle - \langle S_4(\psi_<,\psi_>)\rangle^2) + \ldots]}$$

.

$$\langle S_4^2 \rangle = \int \bar{\psi}(k,\omega)\psi(k,\omega)\bar{\psi}(k,\omega)\psi(k,\omega)u^2 \int \bar{\psi}(k,\omega)\psi(k,\omega)\bar{\psi}(k,\omega)\psi(k,\omega)$$

$$= \int \bar{\psi}(8)\bar{\psi}(7)\psi(6)\psi(5)u^2\bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)$$

$$
\begin{aligned}
&= \int [\bar{\psi}_>(8)+\bar{\psi}_<(8)][\bar{\psi}_>(7)+\bar{\psi}_<(7)][\psi_>(6)+\psi_<(6)][\psi_>(5)+\psi_<(5)]u^2 \\
&\times \quad [\bar{\psi}_>(4)+\bar{\psi}_<(4)][\bar{\psi}_>(3)+\bar{\psi}_<(3)][\psi_>(2)+\psi_<(2)][\psi_>(1)+\psi_<(1)]
\end{aligned}
$$

There are 2^8 terms in this expansion, but not all will survive. Only 3 types will contribute. The rest are either not paired evently, one constitutes all fast fields which adds a constant to the overall partition function, one is all slow fields which we can renormalize, and there are 70 terms that have equal number of slow and fast modes. We need to integrate those that will survive in the term $\langle S_4(\psi_<,\psi_>)^2\rangle - \langle S_4(\psi_<,\psi_>)\rangle^2)$.

$$
\bar{\psi}_<(8)\bar{\psi}_>(7)\psi_<(6)\psi_>(5)u^2\bar{\psi}_<(4)\bar{\psi}_>(3)\psi_<(2)\psi_>(1)
$$

where

$$
\begin{aligned}
\int &= [\prod_{i=1}^{8}\int\frac{dk_i}{2\pi}\int\frac{d\omega_i}{2\pi}](2\pi)^2\delta(k_4+k_3-k_1-k_2)\delta(\omega_4+\omega_3-\omega_1-\omega_2) \\
&\times \quad \delta(k_8+k_7-k_6-k_5)\delta(\omega_8+\omega_7-\omega_6-\omega_5)
\end{aligned}
$$

$$
du = \int_{-\infty}^{\infty}\int_{\Lambda/s<|k|<\Lambda}\frac{d\omega dk}{4\pi^2}\int_0^{2\pi}\frac{u(P-k,k,2,1)u(4,3,P-k,k)}{[i\omega-\varepsilon(k)][-i\omega-\epsilon(p-k)]}
$$

$$du = \int_{\Lambda/s<|k|<\Lambda} \frac{dk}{4\pi^2} \int_0^{2\pi} \frac{u(P-k,k,2,1)u(4,3,P-k,k)}{[i\omega - \varepsilon(k)][-i\omega - \epsilon(p-k)]}$$

$$\frac{du}{dl} = u^2$$

$$u(l)^{-1} - u(0)^{-1} = l$$

$$u(l) = \frac{1}{l + u(0)^{-1}}$$

$$u(l) = \frac{u(0)}{lu(0) + 1}$$

$$u(l) = \frac{1}{1 + u(0)l}$$

C.0.18 Fourier Transform of 2 Dimensional Non-Interacting Spinless Fermions

Consider a 2 dimensional lattice of fermions that are allowed to hop to nearest neighbor sites.

The many body Hamiltonian in second quantized form is

$$H = -t_f \sum_{i,j} (C^{\dagger}_{i,j} C_{i+1,j} + C^{\dagger}_{i+1,j} C_{i,j} + C^{\dagger}_{i,j} C_{i,j+1} + C^{\dagger}_{i,j+1} C_{i,j}) + \mu \sum_{j} C^{\dagger}_{i,j} C_{i,j}$$

The letters i,j the denotes the position of a particle on the 2-D lattice. t is a parameter that denotes the hopping strength,μ is the chemical potential. We transform the Hamiltonian above by expanding the fermion operators as a superposition of plane waves

$$C_{i,j} = \int dk \Psi(\vec{k}) e^{-i\vec{k}\cdot(x+y)}$$

Therefore, the first hopping term is transformed as

$$\sum_{i,j} C^{\dagger}_{i,j} C_{i+1,j} = \sum_{x,y} \int d\vec{k} \Psi^{\dagger}(\vec{k}) e^{ik_x x} e^{ik_y y} \int d\vec{k'} \Psi^{\dagger}(\vec{k'}) e^{-ik'_x(x+1)} e^{-ik'_y y}$$

where we have changed the notation from i, j to x and y as the lattice site denotation.

$$\sum_{i,j} C^{\dagger}_{i,j} C_{i+1,j} = \int d\vec{k} \Psi^{\dagger}(\vec{k}) \int d\vec{k'} \Psi(\vec{k'}) \sum_{x,y} e^{ik_x x} e^{ik_y y} e^{-ik'_x(x+1)} e^{-ik'_y y}$$

$$= \int d\vec{k} \Psi^{\dagger}(\vec{k}) \int d\vec{k'} \Psi(\vec{k'}) \sum_{x,y} e^{i(k_x - k'_x)x} e^{i(k_y - k'_y)y} e^{-ik'_x}$$

$$= \int d\vec{k} \Psi^{\dagger}(\vec{k}) \int d\vec{k'} \Psi(\vec{k'}) \delta(k_x - k'_x) \delta(k_y - k'_y) e^{-ik'_x}$$

$$= \int d\vec{k}\Psi^{\dagger}(\vec{k})\Psi(\vec{k})e^{-ik_x}$$

and its conjugate is transformed as

$$\sum_j C^{\dagger}_{i+1,j}C_{i,j} = \sum_{x,y}\int d\vec{k}\Psi^{\dagger}(\vec{k})e^{ik_x(x+1)}e^{ik_y y}\int d\vec{k'}\Psi^{\dagger}(\vec{k'})e^{-ik'_x x}e^{-ik'_y y}$$

$$= \int d\vec{k}\Psi^{\dagger}(\vec{k})\int d\vec{k'}\Psi(\vec{k'})\sum_{x,y} e^{ik_x(x+1)}e^{ik_y y}e^{-ik'_x x}e^{-ik'_y y}$$

$$= \int d\vec{k}\Psi^{\dagger}(\vec{k})\int d\vec{k'}\Psi(\vec{k'})\sum_{x,y} e^{i(k_x-k'_x)x}e^{i(k_y-k'_y)y}e^{ik_x}$$

$$= \int d\vec{k}\Psi^{\dagger}(\vec{k})\int d\vec{k'}\Psi(\vec{k'})\delta(k_x-k'_x)\delta(k_y-k'_y)e^{ik_x}$$

$$= \int d\vec{k}\Psi^{\dagger}(\vec{k})\Psi(\vec{k})e^{ik_x}$$

$$\sum_j C^{\dagger}_{j+1}C_j = \int dk\Psi^{\dagger}(k)\Psi(k)e^{ik_x}$$

Therefore the hopping term in the x-direction transforms as

$$-t_f(\sum_j C^\dagger_{i,j}C_{i+1,j} + \sum_j C^\dagger_{i+1,j}C_{i,j}) = \int d\vec{k}\Psi^\dagger(\vec{k})\Psi(\vec{k})e^{-ik_x} + \int dk\Psi^\dagger(k)\Psi(k)e^{ik_x}$$

$$= \int d\vec{k}\Psi^\dagger(\vec{k})\Psi(\vec{k})(e^{-ik_x} + e^{ik_x})$$

$$= -2t_f \int d\vec{k}\Psi^\dagger(\vec{k})\Psi(\vec{k})\cos(k_x)$$

The next hopping terms (in the j or y direction) are transformed in the same way. As a result, the Hamiltonian in k space is transformed as

$$H(\vec{k}) = \int dk\Psi^\dagger(\vec{k})\Psi(\vec{k})\varepsilon(\vec{k})$$

with

$$\varepsilon(\vec{k}) = -2t_f[\cos(k_x) + \cos(k_y)] + \mu$$

Appendix D

General Complex RG Differential Equation

This section refers to work done by [26].

The frequency independent RG differential equation is

$$\frac{dU}{dl} = -U^2$$

We can solve this easily by integrating both sides

$$-\int \frac{dU}{U^2} = l$$

$$\frac{1}{U} - \frac{1}{U_0} = l$$

$$U = (l + \frac{1}{U_0})^{-1}$$

$$U = \frac{U_0}{U_0 l + 1}$$

If U , the interaction potential, is complex then the differential equation can be broken up into two coupled differential equations

$$U_0 = x_0 + iy_0$$

$$x_0 = ReU_0$$

$$y_0 = ImU_0$$

We can substitute this into the solution for U

$$U = \frac{x_0 + iy_0}{(x_0 + iy_0)l + 1}$$

and then solve for ReU and ImU

$$U = \frac{x_0 + iy_0}{x_0 l + 1 + iy_0 l} \times \frac{x_0 l + 1 - iy_0 l}{x_0 l + 1 - iy_0 l}$$

$$\frac{x_0(x_0l+1)+iy_0x_0l+iy_0-ix_0y_0l+y_0^2l}{(x_0l+1)^2+y_0^2l^2}$$

$$U=\frac{x_0(x_0l+1)+y_0^2l+iy_0}{(x_0l+1)^2+y_0^2l^2}$$

Therefore the real and imaginary parts are

$$ReU=\frac{x_0+(x_0^2+y_0^2)l}{(x_0l+1)^2+y_0^2l^2}$$

$$ImU=\frac{y_0}{(x_0l+1)^2+y_0^2l^2}$$

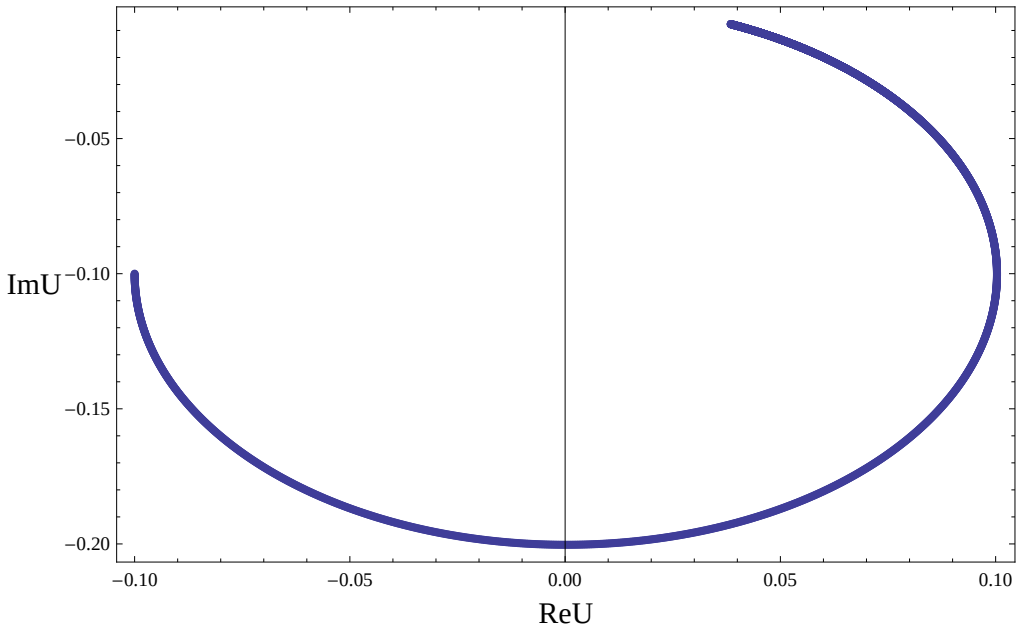

Figure D.1: The frequency independent complex interaction potential flows counterclockwise for intial values $x_0 = 0.1$ and $y_0 = 0.1$. It is symmetric since the imaginary and real part of the potential compete equally. In the case where one is larger than the other, as in Chapter 3, the flow would be asymetric.