Title
COMBINATORIAL PROPERTIES OF NONARCHIMEDEAN CONVEX SETS

Journal
PACIFIC JOURNAL OF MATHEMATICS, 323(1)

ISSN
0030-8730

Authors
Chernikov, Artem
Mennen, Alex

Publication Date
2023

DOI
10.2140/pjm.2023.323.1

arXiv:2109.04591v2 [math.CO] 30 Mar 2023

COMBINATORIAL PROPERTIES OF NON-ARCHIMEDEAN CONVEX SETS

ARTEM CHERNIKOV AND ALEX MENNEN

Abstract. We study combinatorial properties of convex sets over arbitrary valued fields. We demonstrate analogs of some classical results for convex sets over the reals (e.g. the fractional Helly theorem and Bárány's theorem on points in many simplices), along with some additional properties not satisfied by convex sets over the reals, including finite breadth and VC-dimension. These results are deduced from a simple combinatorial description of modules over the valuation ring in a spherically complete valued field.

1. Introduction

Convexity in the context of non-archimedean valued fields was introduced in a series of papers by Monna in 1940's [Mon46], and has been extensively studied since then in non-archimedean functional analysis (see e.g. the monographs [PGS10, Sch13] on the subject). Convexity here is defined analogously to the real case, with the role of the unit interval played instead by a valuational unit ball (see Definition 2.1). Convex subsets of $\mathbb{R}^d$ admit rich combinatorial structure, including many classical results around the theorems of Helly, Radon, Carathéodory, Tverberg, etc. — we refer to e.g. [DLGMM19] for a recent survey of the subject. In the case of $\mathbb{R}$, or more generally a real closed field, there is a remarkable parallel between the combinatorial properties of convex and semi-algebraic sets (which correspond to definable sets from the point of view of model theory). They share many (but not all) properties in the form of various restrictions on the possible intersection patterns, including the fractional Helly theorem and existence of (weak) ε-nets. A well-studied phenomenon in model theory establishes strong parallels between definable sets in $\mathbb{R}$ and in many non-archimedean valued fields such as the p-adics $\mathbb{Q}_p$ or various fields of power series (see e.g. [vdD14]). In this paper we focus on the combinatorial study of convex sets over general valued fields, trying to understand if there is similarly a parallel theory. On the one hand, we demonstrate valued field analogs of some classical results for convex sets over the reals (e.g. the fractional Helly theorem and Bárány's theorem on points in many simplices). On the other, we establish some additional properties not satisfied by convex sets over the reals, including finite breadth and VC-dimension. This suggests that in a sense convex sets over valued fields are the best of both worlds combinatorially, and satisfy various properties enjoyed either by convex or by semialgebraic sets over the reals.

We give a quick outline of the paper. Section 2 covers some basics concerning convexity for subsets of K^d over an arbitrary valued field K, in particular discussing the connection to modules over the valuation ring. These results are mostly standard (or small variations of standard results), and can be found e.g. in [PGS10, Sch13] under the unnecessary assumption that K is spherically complete and $(\Gamma, +) \subseteq (\mathbb{R}_{>0}, \times)$; we provide some proofs for completeness. In Section 3 we give a simple combinatorial description of the submodules of K^d over the valuation ring $\mathcal{O}_K$ in the case of a spherically complete field K (Theorem 3.6 and Corollary 3.12), and an analog for finitely generated modules over arbitrary valued fields (Corollary 3.14). We also give an example of a convex set over the field of Puiseux series demonstrating that the assumption of spherical completeness is necessary for our presentation in the non-finitely generated case (Example 3.11). In Section 4 we use this description of modules to deduce various combinatorial properties of the family of convex subsets Conv_{K^d} of K^d over an arbitrary valued field K. First we show that Conv_{K^d} has breadth d (Theorem 4.3), VC-dimension $d+1$ (Theorem 4.8), dual VC-dimension d (Theorem 4.10) — in stark contrast, all of these are infinite for the family of convex subsets of $\mathbb{R}^d$ for $d \geq 2$. On the other hand, we obtain valued field analogs of the following classical results: the family Conv_{K^d} has Helly number $d+1$ (Theorem 4.5), fractional Helly number $d+1$ (Theorem 4.14), satisfies a strong form of Tverberg's theorem (Theorem 4.15) and Boros-Füredi/Bárány theorem on the existence of a common point in a positive fraction of all geometric simplices generated by an arbitrary finite set of points in K^d (Theorem 4.16). Some of the proofs here are adaptations of the classical arguments, and some rely crucially on the finite breadth property specific to the valued field context. Finally, in Section 5.1 we point out some further applications, e.g. a valued field analogue of the celebrated (p, q)-theorem of Alon and Kleitman [AK92] (Corollary 5.1), and that all convex sets over a spherically complete field are externally definable in the sense of model theory (Remark 5.7); as well as pose some questions and conjectures. We also discuss some other notions of convexity over non-archimedean fields appearing in the literature in Section 5.2, and place our work in the context of the study of abstract convexity spaces in discrete geometry and combinatorics in Section 5.3.

Acknowledgements. We thank the referees for many very helpful literature pointers and suggestions on improving the paper. In particular, Sections 5.2 and 5.3 were added following their suggestions. We thank Lou van den Dries for pointing out Monna's work to us, Dave Marker for pointing out Example 5.8, and Matthias Aschenbrenner for a helpful conversation. Both authors were partially supported by the NSF CAREER grant DMS-1651321, and Chernikov was additionally supported by a Simons fellowship.

2. Preliminaries on convexity over valued fields

Notation. For $n \in \mathbb{N}_{\geq 1}$, we write $[n] = \{1, \ldots, n\}$ and $\langle\rangle$ denotes the span in vector spaces. Throughout the paper, K will denote a valued field, with value group $\Gamma = \Gamma_K$, and valuation $\nu = \nu_K : K \to \Gamma_\infty := \Gamma \sqcup \{\infty\}$, valuation ring $\mathcal{O} = \mathcal{O}_K = \nu^{-1}([0, \infty])$, maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \nu^{-1}((0, \infty])$, and residue field[1] $k = \mathcal{O}/\mathfrak{m}$. The residue map $\mathcal{O} \to k$ will be denoted $\alpha \mapsto \bar{\alpha}$. For a ring R, $R^\times$ denotes its group of units.

The following definition of convexity is analogous to the usual one over $\mathbb{R}$, with the unit interval replaced by the (valuational) unit ball.

[1] Also commonly referred to as the "residue class field" in the literature.

Definition 2.1.
1. For $d \in \mathbb{N}_{\geq 1}$, a set $X \subseteq K^d$ is *convex* if, for any $n \in \mathbb{N}_{\geq 1}$, $x_1, \ldots, x_n \in X$, and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \ldots + \alpha_n = 1$ we have $\alpha_1 x_1 + \ldots + \alpha_n x_n \in X$ (in the vector space K^d).
2. The family of convex subsets of K^d will be denoted Conv_{K^d}.

It is immediate from the definition that the intersection of any collection of convex subsets of K^d is convex.

Definition 2.2. Given an arbitrary set $X \subseteq K^d$, its *convex hull* $\mathrm{conv}(X)$ is the convex set given by the intersection of all convex sets containing X, equivalently

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}, \alpha_i \in \mathcal{O}, x_i \in X, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Definition 2.3. A (valuational) *quasi-ball* is a set $B = \{x \in K : \nu(x - c) \in \Delta\}$ for some $c \in K$ and an upwards closed subset Δ of Γ_∞. In this case we say that B is *around* c, and refer to Δ as the quasi-radius of B. We say that B is a *closed* (respectively, *open*) *ball* if additionally $\Delta = \{\gamma \in \Gamma : \gamma \geq r\}$ (respectively, $\Delta = \{\gamma \in \Gamma : \gamma > r\}$) for some $r \in \Gamma$, and just *ball* if B is either an open or a closed ball (in which case we refer to r as its *radius*).

Remark 2.4.
1. If the value group Γ is Dedekind complete, then every quasi-ball is a ball (except for K itself, which is a quasi-ball of quasi-radius Γ_∞).
2. Note also that if B is a quasi-ball of quasi-radius Δ around c and $c' \in B$ is arbitrary, then B is also a quasi-ball of quasi-radius Δ around c'.
3. In particular, any two quasi-balls are either disjoint, or one of them contains the other.

Example 2.5.
1. The convex subsets of $K = K^1$ are exactly $\emptyset$ and the quasi-balls (see Proposition 2.10 and Example 2.11).
2. If $e_1, \ldots, e_d$ is the standard basis of the vector space K^d, then
$$\mathrm{conv}\left(\{0, e_1, \ldots, e_d\}\right) = \mathcal{O}^d.$$
3. The image and the preimage of a convex set under an affine map are convex. In particular, a translate of a convex set is convex, and a projection of a convex set is convex. (Recall that given two vector spaces V, W over the same field K, a map $f : V \to W$ is *affine* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in V, \alpha, \beta \in K, \alpha + \beta = 1$.)

One might expect, by analogy with real convexity, that the definition of a convex set could be simplified to: if $x, y \in X$, $\alpha, \beta \in \mathcal{O}$ such that $\alpha + \beta = 1$, then $\alpha x + \beta y \in X$. The following two propositions show that this is the case if and only if the residue field is not isomorphic to $\mathbb{F}_2$, and that in general we have to require closure under 3-element convex combinations.

Proposition 2.6. *Let K be a valued field and $X \subseteq K^d$. If X is closed under 3-element convex combinations (in the sense that if $x, y, z \in X$ and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$, then $\alpha x + \beta y + \gamma z \in X$), then X is convex.*

Proof. Suppose X is closed under 3-element convex combinations. We will show by induction on n that then X is closed under n-element convex combinations. Let

$n \geq 3$, $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \ldots + \alpha_n = 1$ be given. Then one of the following two cases holds.

Case 1: $\alpha_1 + \alpha_2 \in \mathcal{O}^\times$.
Then $\frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\frac{\alpha_2}{\alpha_1+\alpha_2}$ are elements of $\mathcal{O}$ that sum to 1, so

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2 \in X$$

by assumption. But then

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = (\alpha_1 + \alpha_2) \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2 \right) + \alpha_3 x_3 + \ldots + \alpha_n x_n \in X$$

by the induction hypothesis, as it is a convex combination of $n-1$ elements of X.

Case 2: $\alpha_1 + \alpha_2 \in \mathfrak{m}$.
Then, as $\nu \left(\sum_{i=1}^n \alpha_i \right) = 0$, there must exist some i with $3 \leq i \leq n$ such that $\alpha_i \in \mathcal{O}^\times$. Hence $\alpha_1 + \alpha_2 + \alpha_i \in \mathcal{O}^\times$, so $\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_i}$, $\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_i}$, and $\frac{\alpha_i}{\alpha_1+\alpha_2+\alpha_i}$ are elements of $\mathcal{O}$ that sum to 1. Thus

$$\left(\frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_i} \right) x_1 + \left(\frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_i} \right) x_2 + \left(\frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_i} \right) x_i \in X$$

by assumption, and so

$$\alpha_1 x_1 + \ldots + \alpha_n x_n =$$
$$(\alpha_1 + \alpha_2 + \alpha_i) \left(\frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_i} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_i} x_2 + \frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_i} x_i \right)$$
$$+ \alpha_3 x_3 + \ldots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \ldots + \alpha_n x_n \in X$$

by the induction hypothesis, as it is a convex combination of $n-2$ elements of X. □

Proposition 2.7. *For any valued field K, the following are equivalent:*

1. *(1) for every $d \geq 1$, every set in K^d that is closed under 2-element convex combinations is convex;*
2. *(2) the residue field k is not isomorphic to $\mathbb{F}_2$.*

Proof. (1) implies (2). If $k = \mathbb{F}_2$, consider the set

$$X := \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathcal{O}, \exists i\, a_i \in \mathfrak{m}\} \subseteq K^3.$$

We claim that X is closed under 2-element convex combinations. That is, given arbitrary $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X$ and $\alpha, \beta \in \mathcal{O}$ with $\alpha + \beta = 1$, we must show that $\alpha (a_1, a_2, a_3) + \beta (b_1, b_2, b_3) \in X$. We have $\bar{\alpha} + \bar{\beta} = 1$ in $k = \mathbb{F}_2$, so necessarily one of $\bar{\alpha}$ and $\bar{\beta}$ is 1 and the other is 0. Without loss of generality $\bar{\alpha} = 1$ and $\bar{\beta} = 0$. Then $\beta \in \mathfrak{m}$. By definition of X, $a_i \in \mathfrak{m}$ for some i. Then $\alpha a_i \in \mathfrak{m}$, and $\beta b_i \in \mathfrak{m}$ as $b_i \in \mathcal{O}$, so $\alpha a_i + \beta b_i \in \mathfrak{m}$. Thus $(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) \in X$. However X is not convex: for an arbitrary $a \in \mathfrak{m}$ we have $(0, 0, 0), (1, 0, 0), (0, 1, 1) \in X$, $1, -1 \in \mathcal{O}$, but $(-1)(0, 0, 0) + 1(1, 0, 0) + 1(0, 1, 1) = (1, 1, 1) \notin X$. (This example can be modified to work in K^2.)

(2) implies (1). If $k \ncong \mathbb{F}_2$, suppose X is closed under 2-element convex combinations. By Proposition 2.6, we only need to check that it is then closed under 3-element convex combinations. Let $x, y, z \in X$, and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$. Then one of the following two cases holds.

Case 1: At least one of $\alpha+\beta, \beta+\gamma, \alpha+\gamma$ is an element of $\mathcal{O}^\times$.
Without loss of generality, $\alpha + \beta \in \mathcal{O}^\times$. Then $\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in X$ by assumption, and thus

$$\alpha x + \beta y + \gamma z = (\alpha + \beta)\left(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y\right) + \gamma z \in X.$$

Case 2: $\alpha+\beta, \beta+\gamma, \alpha+\gamma \in \mathfrak{m}$.
In the residue field, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\gamma} = \bar{\alpha} + \bar{\gamma} = 0$, and $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 1$, hence necessarily $\bar{\alpha} = \bar{\beta} = \bar{\gamma} = 1$, and $\operatorname{char}(k) = 2$. Since $k \ncong \mathbb{F}_2$, there is $\delta \in \mathcal{O}$ such that $\bar{\delta} \notin \{0, 1\}$. Then $\bar{\alpha} + \bar{\delta} = 1 + \bar{\delta} \neq 0$ and $\bar{\beta} - \bar{\delta} + \bar{\gamma} = \bar{\delta} \neq 0$, so

$$\alpha x + \beta y + \gamma z =$$
$$(\alpha+\delta)\left(\frac{\alpha}{\alpha+\delta}x + \frac{\delta}{\alpha+\delta}y\right) + (\beta-\delta+\gamma)\left(\frac{\beta-\delta}{\beta-\delta+\gamma}y + \frac{\gamma}{\beta-\delta+\gamma}z\right) \in X.$$

□

The following proposition gives a very strong form of Radon's theorem (not only we obtain a partition into two sets with intersecting convex hulls, but moreover one of the points is in the convex hull of the other ones).

Proposition 2.8. *Let K be a valued field. For any $d+2$ points $x_1, \ldots, x_{d+2} \in K^d$, one of them is in the convex hull of the others.*

Proof. There exist $a_1, \ldots, a_{d+2} \in K$, *not all* 0, such that $\sum_{i=1}^{d+2} a_i x_i = 0$ and $\sum_{i=1}^{d+2} a_i = 0$ (because those are $d+1$ linear equations on $d+2$ variables, as we are working in K^d). Let $i \in [d+2]$ be such that $\nu(a_i)$ is minimal among $\nu(a_1), \ldots, \nu(a_{d+2})$, in particular $a_i \neq 0$. Then $x_i = \sum_{j \neq i} \frac{-a_j}{a_i} x_j$, and this is a convex combination: for $i \neq j$ we have $\frac{-a_j}{a_i} \in \mathcal{O}$ (as $\nu\left(\frac{-a_j}{a_i}\right) = \nu(a_j) - \nu(a_i) \geq 0$ by the choice of i) and $\sum_{j \neq i} \frac{-a_j}{a_i} = \frac{-\sum_{j \neq i} a_j}{a_i} = \frac{a_i}{a_i} = 1$. □

By a repeated application of Proposition 2.8 we immediately get a very strong form of Carathéodory's theorem:

Corollary 2.9. *Let K be a valued field. Then the convex hull of any finite set in K^d is already given by the convex hull of at most $d+1$ points from it.*

Convex sets over valued fields have a natural algebraic characterization.

Proposition 2.10. *(1) A subset $C \subseteq K^d$ is an $\mathcal{O}$-submodule of K^d if and only if it is convex and contains 0.*
(2) Nonempty convex subsets of K^d are precisely the translates of $\mathcal{O}$-submodules of K^d.

Proof. (1) First, $\mathcal{O}$-submodules of K^d are clearly convex and contain 0. Now suppose $C \subseteq K^d$ is convex and $0 \in C$. Then for any $\alpha \in \mathcal{O}$ and $x \in C$, $\alpha x = \alpha x + (1-\alpha)\,0 \in C$. And for any $x, y \in C$, $x + y = 1 \cdot x + 1 \cdot y - 1 \cdot 0 \in C$. Therefore C is an $\mathcal{O}$-submodule. (2) Given a non-empty convex $C \subseteq K^d$, we can choose $a \in K^d$ such that the translate $C + a$ contains 0, and it is still convex, hence $C + a$ is an $\mathcal{O}$-submodule of K^d by (1). ☐

Example 2.11. Let C be an $\mathcal{O}$-submodule of K, and take $\Delta := \nu(C)$. Then Δ is non-empty because it contains $\infty = \nu(0)$, and upward-closed because for $\gamma \in \Delta$ and $\delta > \gamma$, there is $x \in C$ with $\nu(x) = \gamma$, and $\alpha \in K$ with $\nu(\alpha) = \delta - \gamma$; then $\alpha x \in C$ and $\nu(\alpha x) = \delta$. Clearly $C \subseteq \{x \in K \mid \nu(x) \in \Delta\}$ by definition of Δ. To show $C \supseteq \{x \in K \mid \nu(x) \in \Delta\}$, given any $x \in K$ with $\nu(x) \in \Delta$, there is $y \neq 0 \in C$ with $\nu(y) = \nu(x)$, and $\frac{x}{y} \in \mathcal{O}$, so $x = \frac{x}{y} y \in C$. Thus $C = \{x \in K \mid \nu(x) \in \Delta\}$ is a quasi-ball around 0.

Corollary 2.12. *The convex hull of any finite set in K^d is the image of $\mathcal{O}^d$ under an affine map.*

Proof. By Corollary 2.9, the convex hull of a finite subset of K^d is the convex hull of some $d+1$ points $x_0, \ldots, x_d$ from it (possibly with $x_i = x_j$ for some i, j). Let $e_1, \ldots, e_d$ be the standard basis for K^d, and let f be an affine map $f : K^d \to K^d$ such that $f(0) = x_0$ and $f(e_i) = x_i$ for $1 \leq i \leq d$ (can take f to be the composition of two affine maps: the linear map sending e_i to $x_i - x_0$ for $1 \leq i \leq d$, and translation by x_0). Then we have $\operatorname{conv}(\{x_0, \ldots, x_d\}) = f(\operatorname{conv}\{0, e_1, \ldots, e_d\}) = f\left(\mathcal{O}^d\right)$ (by Example 2.5(2)). ☐

Proposition 2.13. *For any convex $C \subseteq K^d$ and $a \in K^d$, the translate $C + a := \{x + a \mid x \in C\}$ is either equal to or disjoint from C.*

Proof. If $x \in C \cap (C + a)$, then $\forall y \in C\ y + a = y + x - (x - a) \in C$, since that is a convex combination, and conversely, if $y + a \in C$ then $y = (y + a) - x + (x - a) \in C$. ☐

Definition 2.14. Given a valued field K, by a *valued K-vector space* we mean a K-vector space V equipped with a surjective map $\nu = \nu_V : V \to \Gamma_\infty = \Gamma \cup \{\infty\}$ such that $\nu(x) = \infty$ if and only if $x = 0$, $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ and $\nu(\alpha x) = \nu_K(\alpha) + \nu(x)$ for all $x, y \in V$ and $\alpha \in K$.

Remark 2.15. Here we restrict to the case when V has the same value group as K, and refer to [Fuc75] for a more general treatment (see also [Joh16, Section 6.1.3], [Hru14, Section 2.5] or [AvdDvdH17, Section 2.3]).

By a morphism of valued K-vector spaces we mean a morphism of vector spaces preserving valuation. If V and W are valued K-vector spaces, their direct sum $V \oplus W$ is the direct sum of the underlying vector spaces equipped with the valuation $\nu(x, y) := \min\{\nu_V(x), \nu_W(y)\}$. In particular, the vector space K^d is a valued K-vector space with respect to the valuation $\nu_{K^d} : K^d \to \Gamma_\infty$ given by

$$\nu_{K^d}(x_1, \ldots, x_d) := \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}.$$

Note that for any scalar $\alpha \in K$ and vector $v \in K^d$ we have $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v)$. By a *(valuational) ball* in K^d we mean a set of the form $\{x \in K^d :$

$\nu_{K^d}(x-c)\square r\}$ for some center $c \in K^d$, radius $r \in \Gamma \cup \{\infty\}$ and $\square \in \{>, \geq\}$ (corresponding to open or closed ball, respectively). The collection of all open balls forms a basis for the *valuation topology* on K^d turning it into a topological vector space. Note that due to the "ultra-metric" property of valuations, every open ball is also a closed ball, and vice versa. Equivalently, this topology on K^d is just the product topology induced from the valuation topology on K.

Recall that the *affine span* $\mathrm{aff}(X)$ of a set $X \subseteq K^d$ is the intersection of all affine sets (i.e. translates of vector subspaces of K^d) containing X, equivalently

$$\mathrm{aff}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}_{\geq 1}, \alpha_i \in K, x_i \in X, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

We have $\mathrm{conv}(X) \subseteq \mathrm{aff}(X)$ for any X.

Proposition 2.16. *Any convex set in K^d is open in its affine span.*

Proof. For $x \in C \subseteq K^d$, C convex, let $d' \leq d$ be the dimension of the affine span of C, and let $y_1, \ldots, y_{d'} \in C$ be such that $x, y_1, \ldots, y_{d'}$ are affinely independent, and thus have the same affine span as C. Then the map $(\alpha_1, \ldots, \alpha_{d'}) \mapsto x + \alpha_1 (y_1 - x) + \ldots + \alpha_{d'} (y_{d'} - x)$ is a homeomorphism from $K^{d'}$ to the affine span of C, and sends $\mathcal{O}^{d'}$ (which is open in $K^{d'}$) to a neighborhood of x contained in C. ☐

Corollary 2.17. *Convex sets in K^d are closed.*

Proof. For convex $C \subseteq K^d$ and $x \in \mathrm{aff}(C) \setminus C$, $C + x$ is an open subset of $\mathrm{aff}(C)$ that is disjoint from C, so C is a closed subset of its affine span, and hence closed in K^d, since affine subspaces are closed. ☐

3. Classification of $\mathcal{O}$-submodules of K^d

In this section we provide a simple description for the $\mathcal{O}$-submodules of K^d over a spherically complete valued field K (and over an arbitrary valued field K in the finitely generated case). Combined with the description of convex sets in terms of $\mathcal{O}$-submodules from Section 2, this will allow us to establish various combinatorial properties of convex sets over valued fields in the next section. In the following lemma, the construction of the valuation ν is a special case of the standard construction of the quotient norm, when modding out a normed space by a closed subspace, while the second part is more specific to our situation.

Lemma 3.1. *Let K be a valued field, and $V \subseteq K^d$ a subspace. Then the quotient vector space K^d/V is a valued K-vector space equipped with the valuation*

$$\nu(u) := \max\left\{\nu_{K^d}(v) \mid \pi(v) = u, v \in K^d\right\},$$

for $u \in K^d/V$, where $\pi : K^d \to K^d/V$ is the projection map (and the maximum is taken in Γ_∞). If $\dim(V) = n$, then $K^d/V \cong K^{d-n}$ as valued K-vector spaces, and there is a valuation preserving embedding of K-vector spaces $f : K^d/V \hookrightarrow K^d$ so that $\pi \circ f = \mathrm{id}_{K^d/V}$.

Proof. First we prove the lemma for $n = 1$. Let $V \subseteq K^d$ be one-dimensional. There exists $i \in [d]$ such that $\nu_{K^d}((x_1, \ldots, x_d)) = \nu_K(x_i)$ for all $(x_1, \ldots, x_d) \in V$ (indeed, if $\nu_K(x_i) = \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}$ for some $(x_1, \ldots, x_d) \in V$, then we also have $\nu_K(\alpha x_i) = \nu_K(\alpha) + \nu_K(x_i) = \nu_K(\alpha) + \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\} = \min\{\nu_K(\alpha x_1), \ldots, \nu_K(\alpha x_d)\}$ for any $\alpha \in K$). Given any $(x_1, \ldots, x_d) \in K^d$ with $x_i = 0$ and $(y_1, \ldots, y_d) \in V$, we have

$$\nu_{K^d}(x_1 + y_1, \ldots, x_d + y_d) = \min_{j \in [d]} \{\nu_K(x_j + y_j)\} = \tag{3.1}$$

$$\min\left\{\nu_K(y_i), \min_{j \neq i} \{\nu_K(x_j + y_j)\}\right\} \leq \nu_K(y_i) = \nu_{K^d}(y_1, \ldots, y_d).$$

Now consider an arbitrary affine translate $x + V$ of V, $x = (x_1, \ldots, x_d) \in K^d$. Then there exists $x' = (x'_1, \ldots, x'_d) \in x + V$ so that $x'_i = 0$. Indeed, fix any $0 \neq y' \in V$, then $V = \{\alpha y' : \alpha \in K\}$. Take $\alpha' := -\frac{x_i}{y'_i}$ (note that, by the choice of i, $y' \neq 0 \Rightarrow \nu_{K^d}(y') \neq \infty \Rightarrow \nu_K(y'_i) \neq \infty \Rightarrow y'_i \neq 0$), and let $x' := x + \alpha' y'$. We claim that $\nu_{K^d}(x') = \max\{\nu_{K^d}(z) : z \in x + V\}$, in particular the valuation ν on K^d/V is well-defined. Indeed, $x + V = x' + V$, so fix any $y \in V$. If $\nu_{K^d}(x') < \nu_{K^d}(x' + y)$, we must necessarily have $\nu_{K^d}(x') = \nu_{K^d}(y)$, but by (3.1) we have $\nu_{K^d}(x' + y) \leq \nu_{K^d}(y)$, so $\nu_{K^d}(y) < \nu_{K^d}(y)$ — a contradiction; thus $\nu_{K^d}(x') \geq \nu_{K^d}(x' + y)$.

Let $K' := \{(x_1, \ldots, x_d) \in K^d \mid x_i = 0\}$, then we have $K^d = V \oplus K'$ as vector spaces, hence the projection of K^d onto K' along V induces an isomorphism between K^d/V and K', which in turn is naturally isomorphic to K^{d-1}, and these isomorphisms preserve the valuation and give the desired embedding $f : K^d/V \to K^d$. The general case follows by induction on n using the vector space isomorphism theorems. □

We recall an appropriate notion of completeness for valued fields. Recall that a family $\{C_i : i \in I\}$ of subsets of a set X is *nested* if for any $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$.

Definition 3.2. A valued field K is *spherically complete* if every nested family of (closed or open) valuational balls has non-empty intersection.

For the following standard fact, see for example [Sch50, Theorem 5 in Section II.3 + Theorem 8 in section II.6].

Fact 3.3. *Every valued field K (with valuation ν_K, value group Γ_K and residue field k_K) admits a* spherical completion*, i.e. a valued field $\widetilde{K}$ (with valuation $\nu_{\widetilde{K}}$, value group $\Gamma_{\widetilde{K}}$ and residue field $k_{\widetilde{K}}$) so that:*

(1) $\widetilde{K}$ is an immediate extension *of K, i.e. $\widetilde{K}$ is a field extension of K, $\nu_{\widetilde{K}} \restriction_K = \nu_K$, $\Gamma_{\widetilde{K}} = \Gamma_K$ and $k_{\widetilde{K}} = k_K$;*
(2) $\widetilde{K}$ is spherically complete.

We remark that in general a valued field might have multiple non-isomorphic spherical completions.

Lemma 3.4. *If K is spherically complete, then every nested family of non-empty convex subsets of K^d has a non-empty intersection.*

Proof. By induction on d. For $d=1$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, so each C_i is a quasi-ball (see Example 2.5(1)). If there exists some $i\in I$ so that C_i is the smallest of these under inclusion then any element of C_i is in the intersection of the whole family. Hence we may assume that for each $i\in I$ there exists some $i'\in I$ such that $C_{i'}\subsetneq C_i$. Let Δ_i and $\Delta_{i'}$ be the quasi-radii of C_i and $C_{i'}$, respectively. We may assume that both quasi-balls are around the same point $x_i\in C_{i'}$ (by Remark 2.4), hence necessarily $\Delta_{i'}\subsetneq\Delta_i$. Let $r_i\in\Delta_i\setminus\Delta_{i'}$, and let C'_i be a (open or closed) ball of radius r_i around x_i. We have $C'_i\subseteq C_i$, so if $\bigcap_{i\in I}C'_i$ is nonempty, then so is $\bigcap_{i\in I}C_i$. Hence it is sufficient to show that $\{C'_i\}_{i\in I}$ is nested, and then the intersection is non-empty by spherical completeness of K. By construction for any $i,j\in I$ there exists some $\ell\in I$ such that $C_\ell\subseteq C'_i\cap C'_j$, so C'_i and C'_j have non-empty intersection, and are thus nested as they are balls.

For $d\geq 2$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, and let $\pi_1: K^d\to K$ be the projection onto the first coordinate. Then $\{\pi_1(C_i)\}_{i\in I}$ is a nested family of nonempty convex sets in K, hence has an intersection point x. Then $\left\{\pi_1^{-1}(x)\cap C_i\right\}_{i\in I}$ is a nested family of nonempty convex sets in $\pi_1^{-1}(x)\cong K^{d-1}$, which is nonempty by the induction hypothesis. ☐

Lemma 3.5. *If $C\subseteq K^d$ is an $\mathcal{O}$-module, and $\gamma\in\Gamma_\infty$, then the set*

$$X_\gamma=\left\{(x_1,\ldots,x_{d-1})\in\mathcal{O}^{d-1}\mid\exists\alpha\in K\ \nu(\alpha)=\gamma,\ (\alpha,\alpha x_1,\ldots,\alpha x_{d-1})\in C\right\}$$

is convex.

Proof. Let $x=(x_1,\ldots,x_{d-1}),y=(y_1,\ldots,y_{d-1}),z=(z_1,\ldots,z_{d-1})\in X_\gamma$ and $\beta_1,\beta_2,\beta_3\in\mathcal{O}$ with $\beta_1+\beta_2+\beta_3=1$ be arbitrary. Then there exist some $\alpha_1,\alpha_2,\alpha_3\in K$ with $\nu(\alpha_i)=\gamma$ so that

$$(\alpha_1,\alpha_1x_1,\ldots,\alpha_1x_{d-1}),(\alpha_2,\alpha_2y_1,\ldots,\alpha_2y_{d-1}),(\alpha_3,\alpha_3z_1,\ldots,\alpha_3z_{d-1})\in C.$$

Taking $\alpha:=\alpha_1$, we have

$$x':=(\alpha,\alpha x_1,\ldots,\alpha x_{d-1}),y':=(\alpha,\alpha y_1,\ldots,\alpha y_{d-1}),z':=(\alpha,\alpha z_1,\ldots,\alpha z_{d-1})\in C,$$

as for every $i\in[3]$, $\frac{\alpha}{\alpha_i}\in\mathcal{O}$, and hence $\frac{\alpha}{\alpha_i}v\in C$ for any $v\in C$ as C is an $\mathcal{O}$-module. Using this and convexity of C we thus have

$$\Big(\alpha,\alpha(\beta_1x_1+\beta_2y_1+\beta_3z_1),\ldots,\alpha(\beta_1x_{d-1}+\beta_2y_{d-1}+\beta_3z_{d-1})\Big)=$$
$$\beta_1(\alpha,\alpha x_1,\ldots,\alpha x_{d-1})+\beta_2(\alpha,\alpha y_1,\ldots,\alpha y_{d-1})+\beta_3(\alpha,\alpha z_1,\ldots,\alpha z_{d-1})=$$
$$\beta_1x'+\beta_2y'+\beta_3z'\in C.$$

This shows that $\beta_1x+\beta_2y+\beta_3z\in X_\gamma$, and hence that X_γ is convex by Proposition 2.6. ☐

Combining the lemmas, we obtain the following description of the $\mathcal{O}_K$-submodules of K^d for spherically complete K.

Theorem 3.6. *Suppose K is a spherically complete valued field, $d\in\mathbb{N}_{\geq 1}$, and let $C\subseteq K^d$ be an $\mathcal{O}$-submodule. Then there exists a complete flag of vector subspaces $\{0\}\subsetneq F_1\subsetneq\ldots\subsetneq F_d=K^d$ and a decreasing sequence of nonempty, upwards-closed subsets $\Delta_1\supseteq\Delta_2\supseteq\ldots\supseteq\Delta_d$ of Γ_∞ such that*

$$C=\{v_1+\ldots+v_d\mid v_i\in F_i,\ \nu(v_i)\in\Delta_i\}.$$

Remark 3.7. If F_i, Δ_i satisfy the conclusion of Theorem 3.6 for C, then $\nu_{K^d}(C \cap F_1) = \nu_{K^d}(C) = \Delta_1$.

Indeed, any $v \in C$ is of the form $v = v_1 + \ldots + v_d$ with $v_i \in F_i$, $\nu(v_i) \in \Delta_i$ and $\Delta_1 \supseteq \Delta_i$ for all $i \in [d]$, hence $\nu(v) \geq \min\{\nu(v_i) : i \in [d]\} \in \Delta_1$, hence $\nu(v) \in \Delta_1$ as Δ_1 is upwards closed, so $\nu(C) \subseteq \Delta_1$. Conversely, assume $\gamma \in \Delta_1$. If $\gamma = \infty$, then $\nu(0) = \infty$ and $0 \in F_1$. So assume $\gamma \in \Gamma$ and let v be any non-zero vector in F_1, in particular $\delta := \nu(v) \in \Gamma$. Taking $\alpha \in K$ so that $\nu_K(\alpha) = \gamma - \delta$, we have $\alpha v \in F_1$ and $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v) = \gamma$. Note also that $\alpha v = v_1 + \ldots + v_d$ with $v_1 := \alpha v, v_i := 0$ for $2 \leq i \leq d$, in particular $v_i \in F_i$ and $\nu(v_i) \in \Delta_i$, so $\alpha v \in C$, hence $\Delta_1 \subseteq \nu(F_1 \cap C)$.

Proof of Theorem 3.6. By induction on d. For $d = 1$, every $\mathcal{O}$-submodule of K is a quasi-ball $C = \{x \in K : \nu(x) \in \Delta\}$ for some upwards-closed $\Delta \subseteq \Gamma \cup \{\infty\}$ (see Example 2.11), hence we take $F_1 := K$ and $\Delta_1 := \Delta$.

For $d > 1$, let $\Delta_1 := \{\gamma \in \Gamma_\infty \mid \exists v \in C\ \nu_{K^d}(v) = \gamma\}$. Note that Δ_1 is nonempty because it contains $\infty = \nu(0)$. Then there is some $i \in [d]$ such that every $\gamma \in \Delta_1$ is the valuation of the ith coordinate of some element of C. To see this, note that for each $i \in [d]$, the set

$$S_i := \{\gamma \in \Gamma_\infty \mid \exists v = (v_1, \ldots, v_d) \in C\ \nu_{K^d}(v) = \nu(v_i) = \gamma\}$$

is upwards closed in Γ_∞. Indeed, assume $v = (v_1, \ldots, v_d) \in C$, $\gamma = \nu(v_i) = \min\{\nu(v_j) : j \in [d]\}$ and $\delta \geq \gamma$ in Γ_∞. Let $\alpha \in K$ be arbitrary with $\nu(\alpha) = \delta - \gamma$, then $\alpha \in \mathcal{O}$, hence $\alpha v \in C$, and so $\nu_{K^d}(\alpha v) = \min\{\nu(\alpha v_j) : j \in [d]\} = \nu(\alpha v_j) = \delta$. As we also have $\Delta_1 = \bigcup_{i \in [d]} S_i$, it follows that $\Delta_1 = S_i$ for some $i \in [d]$ as wanted (and in particular Δ_1 is upwards closed in Γ_∞).

Without loss of generality we may assume $i = 1$. Then, given any $\gamma \in \Delta_1$, there is some $(\alpha, y_1, \ldots, y_{d-1}) \in C$ such that $\gamma = \nu(\alpha) \leq \min\{\nu(y_j) : j \in [d-1]\}$. Taking $x_j := \frac{y_j}{\alpha} \in \mathcal{O}$, we thus have $(\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C$. Hence for any $\gamma \in \Delta_1$, the set

$$X_\gamma := \left\{(x_1, \ldots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists \alpha \in K\ \nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C\right\}$$

is nonempty, and convex (by Lemma 3.5). Note that for $\gamma < \delta \in \Gamma_\infty$ we have $X_\gamma \subseteq X_\delta$, hence $\bigcap_{\gamma \in \Delta_1} X_\gamma \neq \emptyset$ by Lemma 3.4. That is, there exists $(x_1, \ldots, x_{d-1}) \in \mathcal{O}^{d-1}$ such that $\forall \gamma \in \Delta_1\ \exists \alpha \in K\ \ (\nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C)$. Hence

$$\forall \alpha \in K,\ \nu(\alpha) \in \Delta_1 \implies (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C \tag{3.2}$$

(since we have $\exists \beta \in K\, \nu(\beta) = \nu(\alpha) \wedge (\beta, \beta x_1, \ldots, \beta x_{d-1}) \in C$, so $\frac{\alpha}{\beta} \in \mathcal{O}$ and multiplying by it we get $(\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C$).

Let $F_1 := \langle (1, x_1, \ldots, x_{d-1}) \rangle$. Let $\pi : K^d \twoheadrightarrow K^d / F_1$ be the projection map, $f : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, and $\pi' := f \circ \pi : K^d \to K^d$. Note that $K^d/F_1 \cong K^{d-1}$ as a valued K-vector space by Lemma 3.1, and that $\widetilde{C} := \pi(C)$ is still an $\mathcal{O}$-submodule of K^d/F_1. By induction hypothesis there is a full flag $\{0\} \subsetneq \widetilde{F}_2 \subsetneq \ldots \subsetneq \widetilde{F}_d = K^d/F_1$ and upwards-closed subsets $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of Γ_∞ satisfying the conclusion of the theorem with respect to $\widetilde{C}$ (the equality $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2$ is by Remark 3.7). Note that

$$\forall v \in K^d,\ \nu_{K^d}(\pi'(v)) = \nu_{K^d/F_1}(\pi(v)) \geq \nu_{K^d}(v). \tag{3.3}$$

In particular we have $\Delta_1 \supseteq \Delta_2$.

Let the subspaces $F_2, \ldots, F_d$ be the preimages of $\widetilde{F}_2, \ldots, \widetilde{F}_d$ in K^d. We let $W := f(K^d/F_1) \subseteq K^d$ be the image of the valuation preserving embedding $f : K^d/F_1 \hookrightarrow K^d$. Then we have

$$C = \{v_1 + w \mid v_1 \in F_1,\ \nu_{K^d}(v_1) \in \Delta_1,\ w \in C \cap W\}. \tag{3.4}$$

To see this, given an arbitrary $v \in C$, let $w := \pi'(v)$ and $v_1 := v - w$. As $\pi \circ f = \mathrm{id}_{K^d/F_1}$ by assumption, we have $\pi(w) = \pi(\pi'(v)) = \pi(f(\pi(v))) = \pi(v)$, hence $v_1 \in F_1$. By (3.3) we have $\nu_{K^d}(w) \geq \nu_{K^d}(v)$, and thus $\nu_{K^d}(v_1) \geq \min\{\nu_{K^d}(v), \nu_{K^d}(w)\} \geq \nu_{K^d}(v)$ as well. Thus $\nu_{K^d}(v_1) \in \Delta_1$, and $v_1 \in F_1$, which together with (3.2) and the definition of F_1 implies $v_1 \in C$; hence $w = v - v_1 \in C$ as well. The opposite inclusion is obvious.

Furthermore, applying the isomorphism $f : K^d/F_1 \to W$ to

$$\widetilde{C} = C/F_1 = \left\{v_2 + \ldots + v_d \mid v_i \in \widetilde{F}_i, \nu_{K^d/F_1}(v_i) \in \Delta_i\right\}$$

we get

$$C \cap W = \{v_2 + \ldots + v_d \mid v_i \in F_i \cap W,\ \nu_{K^d}(v_i) \in \Delta_i\},$$

which together with (3.4) implies

$$C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i,\ v_i \in W \text{ for } i \geq 2\}.$$

Now $C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$ follows because for any such vectors $v_1, \ldots, v_d$, the vector v_i (for $i \geq 2$) can be moved into W by subtracting an element of F_1 with valuation in Δ_1, and collecting the differences in with v_1. That is, given arbitrary $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$, let $w_i := \pi'(v_i) \in W$ for $i \geq 2$, and let $w_1 := v_1 + (v_2 - \pi'(v_2)) + \ldots + (v_d - \pi'(v_d))$. As above, using (3.3), for each $i \geq 2$ we have $\nu_{K^d}(v_i - \pi'(v_i)) \geq \min\{\nu_{K^d}(v_i), \nu_{K^d}(\pi'(v_i))\} \geq \nu_{K^d}(v_i) \in \Delta_i \subseteq \Delta_1$. Hence $\nu_{K^d}(w_1) \geq \min\{v_1, v_2 - \pi'(v_2), \ldots, v_d - \pi'(v_d)\} \in \Delta_1$. We also have $\nu_{K^d}(w_i) \geq \nu_{K^d}(v_i) \in \Delta_i$ for $i \geq 2$ by (3.3). Using that f is a one-sided inverse of π as above, we also have $v_i - \pi'(v_i) \in F_1 \subseteq F_i$ for $i \geq 2$. It follows that $w_i \in F_i$ for all $i \in [d]$. Putting all of this together, we get $w_1 + \ldots + w_d = v_1 + \ldots + v_d$, $w_i \in F_i$, $\nu(w_i) \in \Delta_i$, and $w_i \in W$ for $i \geq 2$. □

Remark 3.8. Note that as $F_d = K^d$ in Theorem 3.6, we have

$$\Delta_d = \left\{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C\right\}.$$

That is, Δ_d is the quasi-radius of the largest quasi-ball around 0 contained in C.

Remark 3.9. Given a convex set $0 \in C \subseteq K^d$ and any $F_i, \Delta_i, i \in [d]$ satisfying the conclusion of Theorem 3.6 with respect to it, for every $j \in [d]$ we have

$$C \cap F_j = \{v_1 + \ldots + v_j \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i \text{ for all } j \in [i]\}.$$

Indeed, if $x \in C \cap F_j$, then $x = v_1 + \ldots + v_d \in F_j$ for some $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$ for $i \in [d]$. Then, using that the F_i are increasing under inclusion and Δ_i are increasing under inclusion and upwards closed, $v_{j+1} + \ldots + v_d \in F_j$ and taking $v'_j := v_j + \ldots + v_d$ we have $v'_j \in F_j, \nu(v'_j) \geq \min\{\nu(v_i) : j \leq i \leq d\} \in \Delta_j$ and $x = v_1 + \ldots + v_{j-1} + v'_j$.

Conversely, any element $x = v_1 + \ldots + v_j$ with $v_i \in F_i, \nu(v_i) \in \Delta_i$ for $i \in [j]$ can be written as $x = v_1 + \ldots + v_d$ with $v_i := 0 \in F_i$ and $\nu(v_i) = \infty \in \Delta_i$ for $j+1 \le i \le d$. So $x \in C \cap F_j$.

Remark 3.10.
(1) It follows from the conclusion of Theorem 3.6 that the subspace F_{d-1} is a linear hyperplane in K^d, and every element of C differs from an element of F_{d-1} (and hence of $F_{d-1} \cap C$ in view of Remark 3.9) by a vector in K^d with valuation in Δ_d (with Δ_d as in Remark 3.8).
(2) Conversely, F_{d-1} can be chosen to be *any* linear hyperplane H in K^d such that every element of C differs from an element of H by a vector in K^d with valuation in Δ_d. To see this, let H be such a hyperplane in K^d. Then $C \cap H$ is a convex subset of $H \cong K^{d-1}$ containing 0, hence an $\mathcal{O}$-submodule of H by Proposition 2.10. Applying Theorem 3.6 to $C \cap H$ in H (with the induced valuation on H), there are $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_{d-1}$ and a full flag $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_{d-1} = H$, such that $C \cap H = \{v_1 + \ldots + v_{d-1} \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$. Then

$$\{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} = \{w + v_d \mid w \in C \cap H,\ \nu(v_d) \in \Delta_d\} = C.$$

Example 3.11. The assumption of spherical completeness of K is necessary in Theorem 3.6. For example, let $K := \bigcup_{n \ge 1} k\left(\left(t^{\frac{1}{n}}\right)\right)$ be the field of Puiseux series over a field k, and let $\widetilde{K} := k\left[\left[t^{\mathbb{Q}}\right]\right]$ be the field of Hahn series over k with rational exponents, it is the spherical completion of K (both fields have value group $\mathbb{Q}$ and valuation $\nu(x) = q$ where x has leading term t^q; see e.g. [AvdDvdH17, Example 3.3.23]). In particular $\sum_{n \ge 1} t^{1-\frac{1}{n}} \in \widetilde{K} \setminus K$, and let

$$\widetilde{C} := \left\{ \alpha \left(1, \sum_{n \ge 1} t^{1-\frac{1}{n}}\right) + v \mid \alpha \in \widetilde{K}, v \in \widetilde{K}^2, \nu_{\widetilde{K}}(\alpha) \ge 0,\ \nu_{\widetilde{K}^2}(v) \ge 1 \right\} \subseteq \widetilde{K}^2,$$

and let $C := \widetilde{C} \cap K^2$. Then $\widetilde{C}$ is convex in $\widetilde{K}^2$, and hence C is also convex as a subset of K^2. The basic idea behind why C is not of the form described in Theorem 3.6 is that C is close enough to $\widetilde{C}$, and the subspace F_1 appearing in the conclusion of Theorem 3.6 for $\widetilde{C}$ must be close to $\left\langle \left(1, \sum_{n \ge 1} t^{1-\frac{1}{n}}\right) \right\rangle$; specifically, it must be $\left\langle \left(1, x + \sum_{n \ge 1} t^{1-\frac{1}{n}}\right) \right\rangle$ for some $x \in K^2$ with $\nu(x) \ge 1$, but K^2 contains no such subspaces.

Indeed, by Remark 3.7, given any F_i, Δ_i satisfying the conclusion of Theorem 3.6 with respect to C, the valuation of every element of C must also be the valuation of some element of $F_1 \cap C$. So, to show that C is not of the form described in Theorem 3.6, it suffices to show that C contains elements of valuation arbitrarily close to 0, but that for every 1-dimensional subspace $F_1 \subset K^2$, there is some $q > 0$ in Γ such that every element of $F_1 \cap C$ has valuation at least q (and note that from the definition of C, every element in it has positive valuation).

Claim 1. *For every $n \in \mathbb{N}_{\ge 1}$, there is some $v \in C$ with $\nu_{K^2}(v) = \frac{1}{n}$.*

Proof. To see this, note that

$$t^{\frac{1}{n}}\left(1,\sum_{m=1}^{n-1}t^{1-\frac{1}{m}}\right)=t^{\frac{1}{n}}\left(1,\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)-t^{\frac{1}{n}}\left(0,\sum_{m\geq n}t^{1-\frac{1}{m}}\right)\in C$$

as $\nu_K\left(t^{\frac{1}{n}}\right)=\frac{1}{n}\geq 0$ and $\nu_{K^2}\left(t^{\frac{1}{n}}\left(0,\sum_{m\geq n}t^{1-\frac{1}{m}}\right)\right)=\frac{1}{n}+\left(1-\frac{1}{n}\right)\geq 1$. □

Claim 2. *For every* 1*-dimensional subspace* $F_1\subset K^2$*, there is some* $n\in\mathbb{N}_{n\geq 1}$ *such that every element of* $F_1\cap C$ *has valuation at least* $\frac{1}{n}$*.*

Proof. We prove this by breaking into two cases.

Case 1. $F_1=\langle(0,1)\rangle$.

Assume $x\in F_1\cap C$, then $x=(x_1,x_2)=\alpha\left(1,\sum_{n\geq 1}t^{1-\frac{1}{n}}\right)+v$ for some $\alpha\in K, v=(v_1,v_2)\in\widetilde{K}^2$ with $\nu_{\widetilde{K}}(\alpha)\geq 0,\nu_{\widetilde{K}^2}(v)\geq 1$, and $x_1=0$, so $\alpha=-v_1$. But $1\leq\nu_{\widetilde{K}^2}(v)=\min\{\nu_{\widetilde{K}}(v_1),\nu_{\widetilde{K}}(v_2)\}$, hence $\nu_{\widetilde{K}}(\alpha)\geq 1$ as well. Since $\nu_{\widetilde{K}}\left(\sum_{n\geq 1}t^{1-\frac{1}{n}}\right)=0$, it follows that $\nu_{\widetilde{K}^2}(x)=\min\left\{\nu_{\widetilde{K}}(0),\nu_{\widetilde{K}}\left(\alpha\left(\sum_{n\geq 1}t^{1-\frac{1}{n}}\right)\right)\right\}\geq 1$. Thus every element of $F_1\cap C$ has valuation at least 1.

Case 2. $F_1=\langle(1,x)\rangle$ for some $x\in K$.

Given any $x\in K$, there must exist some $n\in\mathbb{N}$ such that $\nu_{\widetilde{K}}\left(x-\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)\leq 1-\frac{1}{n}$. Given any $v\in F_1\cap C$, we have

$$v=\alpha(1,x)=\beta\left(1,\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)+w$$

for some $\alpha\in K$, some $\beta\in\widetilde{K}$ with $\nu_{\widetilde{K}}(\beta)\geq 0$ and $w=(w_1,w_2)\in\widetilde{K}^2$ with $\nu_{\widetilde{K}^2}(w)\geq 1$. Without loss of generality $\alpha\neq 0$, so we have

$$x=\frac{\alpha x}{\alpha}=\left(w_2+\beta\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)(w_1+\beta)^{-1}=\left(\frac{w_2}{\beta}+\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)\left(1+\frac{w_1}{\beta}\right)^{-1}.$$

If $\nu_{\widetilde{K}}(\beta)<\frac{1}{n}$, then

$$\nu_{\widetilde{K}}\left(\frac{w_1}{\beta}\right)>1-\frac{1}{n},\ \nu_{\widetilde{K}}\left(\frac{w_2}{\beta}\right)>1-\frac{1}{n},\ \nu_{\widetilde{K}}\left(\left(1+\frac{w_1}{\beta}\right)^{-1}\right)=0,\text{ and}$$

$$\nu_{\widetilde{K}}\left(\left(1+\frac{w_1}{\beta}\right)^{-1}-1\right)>1-\frac{1}{n},\text{ so}$$

$$\nu\left(x-\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)=\nu\left(\frac{w_2}{\beta}(w_1+\beta)^{-1}+\left(\sum_{m\geq 1}t^{1-\frac{1}{m}}\right)\left(\left(1+\frac{w_1}{\beta}\right)^{-1}-1\right)\right)$$
$$>1-\frac{1}{n},$$

a contradiction to the choice of n. Thus $\nu(\beta)\geq\frac{1}{n}$, and hence $\nu(v)\geq\frac{1}{n}$. □

Thus no 1-dimensional subspace F_1 of K^2 can fill its desired role in the presentation for C.

Theorem 3.6 implies the following simple description of convex sets over spherically complete valued fields.

Corollary 3.12. *If K is a spherically complete valued field and $d \in \mathbb{N}_{\geq 1}$, then the non-empty convex subsets of K^d are precisely the affine images of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$.*

Proof. Let $C \subseteq K^d$ be an affine image of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$. Note that $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ is convex, and an image of a convex set under an affine map is convex (Example 2.5), hence C is convex.

Conversely, let $\emptyset \neq C \subseteq K^d$ be convex. Since the affine images of $\mathcal{O}$-submodules of K^d give us all non-empty convex sets by Proposition 2.10, without loss of generality $0 \in C$ and C is an $\mathcal{O}$-submodule of K^d. Let $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and $\nu_{K^d}(C) = \Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ be as given by Theorem 3.6 for C. Using Lemma 3.1 we can choose $v_1, \ldots, v_d \in K^d$ such that for every $i \in [d]$ we have:

(1) $v_1, \ldots, v_i$ is a basis for F_i,
(2) $\nu(v_i) = 0$,
(3) $\nu(v_i + x) \leq 0$ for all $x \in F_{i-1}$.

Then C is the image of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ under the linear map $f : K^d \to K^d$ such that $f(e_i) = v_i$, where e_i is the ith standard basis vector. Indeed, if $x \in f(\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d))$ then $x = \sum_{i=1}^d c_i v_i$ for some c_i with $\nu(c_i) \in \Delta_i$. Using (2) this implies $\nu(c_i v_i) = \nu(c_i) \in \Delta_i$, and $c_i v_i \in F_i$, hence $x \in C$. Conversely, let x be an arbitrary element of C, then $x = w_1 + \ldots + w_d$ for some $w_i \in F_i$ with $\nu(w_i) \in \Delta_i$. Each w_i is a linear combination of $v_1, \ldots, v_i$, say $w_i = \sum_{j=1}^i c_{i,j} v_j$.

Now we claim that for any $i \in [d]$, $\alpha \in K$ and $v \in F_{i-1}$ we have $\nu(\alpha v_i + v) = \min\{\nu(\alpha v_i), \nu(v)\}$. Indeed, replacing v and α by $\alpha^{-1} v \in F_{i-1}$ and $\alpha^{-1}\alpha \in K$, respectively, changes both sides of the claimed equality by the same amount, hence we may assume that $\alpha = 0$ or $\alpha = 1$. The first case holds trivially, in the second case we need to show that $\nu(v_i + v) = \min\{\nu(v_i), \nu(v)\}$. If $\nu(v_i) \neq \nu(v)$ this holds by the ultrametric inequality, so we assume $\nu(v_i) = \nu(v) = 0$ (using (2)). Then, using (3), $0 \geq \nu(v_i + v) \geq \min\{\nu(v_i), \nu(v)\} = 0$, so $\nu(v_i + v) = 0$ as well.

Applying this claim by induction on $i \in [d]$, we get

$$\nu\left(\sum_{j=1}^{i} c_{i,j} v_j\right) = \min_j \{\nu(c_{i,j} v_j)\},$$

which using (2) implies $\nu(w_i) = \nu\left(\sum_{j=1}^i c_{i,j} v_j\right) = \min_j \{\nu(c_{i,j})\}$ for each $i \in [d]$. As for each $i \in [d]$ we have $\nu(w_i) \in \Delta_i$ and Δ_i is upwards closed, it follows that $\nu(c_{i,j}) \in \Delta_i$ for all $i \in [d], j \in [i]$. Regrouping the summands $c_{i,j} v_i$, it follows that $x = w_1 + \ldots + w_d$ is a linear combination of $v_1, \ldots, v_d$ where the coefficient of v_i has valuation in Δ_i, hence x belongs to $f(\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d))$. □

We can eliminate the assumption of spherical completeness of the field when only considering convex hulls of finite sets. We will say that a convex set is *finitely generated* if it is the convex hull of a finite set of points.

Lemma 3.13. *A subset $C \subseteq K^d$ is a finitely generated $\mathcal{O}$-module if and only if it is a finitely generated convex set and contains* 0.

Proof. If an $\mathcal{O}$-module $C \subseteq K^d$ is generated as an $\mathcal{O}$-module by some finite set X, then it is the convex hull of $X \cup \{0\}$. If a set C is the convex hull of some finite set X and contains 0, then it is an $\mathcal{O}$-module by Proposition 2.10, clearly generated as an $\mathcal{O}$-module by X. □

We have the following analog of Theorem 3.6 in the finitely generated case over an arbitrary valued field.

Corollary 3.14. *Let K be an* arbitrary *valued field and C a* finitely generated *convex set containing* 0. *Then there is a full flag $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and an increasing sequence $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_d \in \Gamma_\infty$ such that*

$$C = \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \geq \gamma_i\}.$$

Proof. Let $C \ni 0$ be the convex hull of some finite set $X \subseteq K^d$. By a repeated application of Proposition 2.8, C is the convex hull of some $d+1$ elements $v_0, \ldots, v_d$ from X (possibly with $x_i = x_j$ for some i, j). As $0 \in C$, we have $0 = \sum_{i=0}^d \alpha_i v_i$ for some $\alpha_i \in \mathcal{O}$ with $\sum_{i=0}^d \alpha_i = 1$. Let j be such that $\nu(\alpha_j)$ is minimal among $\{\nu(\alpha_i) : 0 \leq i \leq d\}$. In particular $\alpha_j \neq 0$, hence $v_j = \left(1 - \sum_{i \neq j} \frac{\alpha_i}{\alpha_j}\right) 0 + \sum_{i \neq j} \frac{\alpha_i}{\alpha_j} v_i$. By the choice of j we have $\frac{\alpha_i}{\alpha_j} \in \mathcal{O}$ for all $i \neq j$, hence also $1 - \sum_{i \neq j} \frac{\alpha_i}{\alpha_j} \in \mathcal{O}$, thus $v_j \in \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$, and so also $C = \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$. Reordering if necessary, we can thus assume that C is the convex hull of some $\{0, v_1, \ldots, v_d\} \subseteq C$ with $\nu(v_1) \leq \nu(v_i)$ for each $i \in [d]$.

Let $F_1 := \langle v_1 \rangle$ and $\gamma_1 := \nu(v_1)$. Let $\pi_1 : K^d \twoheadrightarrow K^d/F_1$ be the projection map, $f_1 : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, $V_1 := f_1\left(K^d/F_1\right)$ and $\pi_1' := f_1 \circ \pi_1 : K^d \to K^d$.

For $i \geq 2$, as explained after (3.4) in the proof of Theorem 3.6 we have $v_i - \pi_1'(v_i) \in F_1$; and by (3.3) there and assumption we have $\nu(\pi_1'(v_i)) \geq \nu(v_i) \geq \nu(v_1)$. So $v_i - \pi_1'(v_i) \in \mathcal{O}v_1$ for all $i \geq 2$, which implies

$$\operatorname{conv}(\{0, v_1, \pi_1'(v_2), \ldots, \pi_1'(v_d)\}) = \operatorname{conv}(\{0, v_1, \ldots, v_d\}) = C.$$

Without loss of generality we suppose $\nu(\pi_1'(v_2)) \leq \nu(\pi_1'(v_i))$ for $i \geq 3$, and let $F_2 := \langle v_1, \pi_1'(v_2) \rangle$ and $\gamma_2 := \nu(\pi_1'(v_2)) \geq \nu(v_1) = \gamma_1$ by assumption. By definition of the valuation on the quotient space, using the properties of f, we have

$$\nu_K(\pi_1'(v_i)) = \nu_{K^d/F_1}(\pi_1(v_i)) = \nu_{K^d/F_1}(\pi_1(\pi_1'(v_i))) \geq \nu_{K^d}(\pi_1'(v_i) + \alpha v_1)$$

for all $\alpha \in K$. As in the proof of Corollary 3.12, this implies $\nu(\beta\pi_1'(v_i) + \alpha v_1) = \min\{\beta\nu(\pi_1'(v_i)), \nu(\alpha v_1))\}$ for all $i \geq 2$ and $\alpha, \beta \in K$. It follows that

$$\{nv_1 + m\pi_1'(v_2) \mid n, m \in \mathcal{O}\} = \{w_1 + w_2 \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}.$$

To see that the set on the right is contained in the set on the left, assume $x = w_1 + w_2$ for some $w_i \in F_i, \nu(w_i) \geq \gamma_i$. Then $w_1 = \alpha_1 v_1$ and $w_2 = \alpha_2 v_1 +$

$\beta\pi_1'(v_2)$ for some $\alpha_1, \alpha_2, \beta \in K$, and by the observation above $\gamma_2 \leq \nu(w_2) = \min\{\nu(\alpha_2 v_1), \nu(\beta\pi_1'(v_2))\}$. So $x = (\alpha_1 + \alpha_2)v_1 + \beta\pi_1'(v_2)$, $\nu((\alpha_1 + \alpha_2)v_1) \geq \gamma_1 = \nu(v_1)$, so $(\alpha_1 + \alpha_2) \in \mathcal{O}$, and $\nu(\beta) \geq \gamma_2$, as wanted.

Now we replace v_i by $\pi_1'(v_i)$ for $i \geq 2$, and let $\pi_2 : K^d \twoheadrightarrow K^d/F_2$ be the projection map, $f_2 : K^d/F_2 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, $V_2 := f_2\left(K^d/F_2\right)$ and $\pi_2' := f_2 \circ \pi_2 : K^d \to K^d$. For $i \geq 3$, $v_i - \pi_2'(v_i) \in F_2$ and $v_i - \pi_2'(v_i) \in \mathcal{O}v_1 + \mathcal{O}v_2$, so again replacing v_i with $\pi_2'(v_i)$ for $i \geq 3$ does not change the convex hull. Again we may assume $\nu(\pi_2'(v_3)) \leq \nu(\pi_2'(v_i))$ for $i \geq 4$, and let $F_3 := \langle v_1, v_2, v_3\rangle$ and $\gamma_3 := \nu(\pi_2'(v_3))$. Repeating this argument as above d times, we have chosen vectors v_i, increasing spaces $F_i = \langle v_1, \ldots, v_i\rangle$ and increasing $\gamma_i = \nu(v_i) \in \Gamma$ for $i \in [d]$ so that

$$C = \operatorname{conv}(\{0, v_1, \ldots, v_d\}) =$$
$$\{n_1 v_1 + \ldots + n_d v_d \mid n_i \in \mathcal{O}\} = \{w_1 + \ldots + w_d \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}. \qquad \square$$

4. Combinatorial properties of convex sets

The following definition is from [ADH+16, Section 2.4].

Definition 4.1. Given a set X and $d \in \mathbb{N}_{\geq 1}$, a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *breadth* d if any nonempty intersection of finitely many sets in $\mathcal{F}$ is the intersection of at most d of them, and d is minimal with this property.

Lemma 4.2. *Let K be a valued field and S a convex subset of K^d.*

(1) If $0 \in S$ and S is finitely generated, then it is generated as an $\mathcal{O}$-module by a finite linearly independent set of vectors.
(2) Let $\widetilde{K}$ be a valued field extension of K and $\widetilde{S} := \operatorname{conv}_{\widetilde{K}^d}(S) \subseteq \widetilde{K}^d$. Then $\widetilde{S} \cap K^d = S$.

Proof. (1) By Lemma 3.13, S is generated as an $\mathcal{O}$-module by some finite set $v_1, \ldots, v_n \in S$. Assume these vectors are not linearly independent, then $0 = \sum_{i\in[n]} \alpha_i v_i$ for some $\alpha_i \in K$ not all 0. Let $i \in [n]$ be such that $\nu(\alpha_i) \leq \nu(\alpha_j)$ for all $j \in [n]$, in particular $\alpha_i \neq 0$. Then $v_i = \sum_{j\neq i} \frac{\alpha_j}{-\alpha_i} v_j$ and $\nu\left(\frac{\alpha_j}{-\alpha_i}\right) = \nu(\alpha_j) - \nu(\alpha_i) \geq 0$, hence $\frac{\alpha_j}{-\alpha_i} \in \mathcal{O}$ for all $j \neq i$, and S is still generated as an $\mathcal{O}$-module by the set $\{v_j : j \neq i\}$. Repeating this finitely many times, we arrive at a linearly independent set of generators.

(2) Since convexity is invariant under translates, we may assume $0 \in S$. Since every element in the convex hull of a set is in the convex hull of some finite subset, we may also assume that S is finitely generated as an $\mathcal{O}$-module, and by (1) let $v_1, \ldots, v_n \in S$ be a linearly independent (in the vector space K^d, so $n \leq d$) set of its generators. Let $v_{n+1}, \ldots, v_d \in K^d$ be so that $\{v_i : i \in [d]\}$ is a basis of K^d, and say $v_i = (v_{i,j} : j \in [d])$ with $v_{i,j} \in K$. Then the square matrix $A := (v_{i,j} : i, j \in [d]) \in M_{d\times d}(K)$ is invertible, so $A^{-1} \in M_{d\times d}(K) \subseteq M_{d\times d}(\widetilde{K})$, so A is also invertible in $M_{d\times d}(\widetilde{K})$, hence $\{v_i : i \in [d]\}$ are linearly independent vectors in $\widetilde{K}^d$ as well. But now if $\sum_{i\in[n]} \alpha_i v_i = u$ for some $\alpha_i \in \widetilde{K}$ and $u \in K^d$, then necessarily $\alpha_i \in K$ for all i (otherwise we would get a non-trivial linear combination of $v_1, \ldots, v_d$ in $\widetilde{K}^d$). In

particular, any element of the $\mathcal{O}_{\widetilde{K}}$-module generated by $v_1, \ldots, v_n$ which is in K^d already belongs to the $\mathcal{O}_K$-module generated by $v_1, \ldots, v_n$, hence $\widetilde{S} \cap K^d = S$. $\square$

We can now demonstrate an (optimal) finite bound on the breadth of the family of convex sets over valued fields. In sharp contrast, over the reals there is no finite bound on the breadth already for convex subsets of $\mathbb{R}^2$ (for any n, a convex n-gon in $\mathbb{R}^2$ is the intersection of n half-planes, but not the intersection of any fewer of them).

Theorem 4.3. *Let K be a valued field and $d \geq 1$. Then the family* Conv_{K^d} *has breadth d. That is, any nonempty intersection of finitely many convex subsets of K^d is the intersection of at most d of them.*

Proof. The family Conv_{K^d} cannot have breadth less than d because the d coordinate-aligned hyperplanes are convex, have common intersection $\{0\}$, but any $d-1$ of them intersect in a line.

We now show that Conv_{K^d} has breadth at most d, by induction on d. The case $d = 1$ is clear by Example 2.5(1) since for any two quasi-balls, they are either disjoint or one is contained in the other. For $d > 1$, assume $C_1, \ldots, C_n \in \mathrm{Conv}_{K^d}$ with $n \geq d$ are convex and satisfy $\bigcap_{i\in[n]} C_i \neq \emptyset$. Translating, we may assume $0 \in \bigcap_{i\in[n]} C_i$.

We may also assume that K is spherically complete. Indeed, let $\widetilde{K}$ be a spherical completion of K as in Fact 3.3, and let $\widetilde{C}_i := \mathrm{conv}_{\widetilde{K}^d}(C_i) \in \mathrm{Conv}_{\widetilde{K}^d}$. By Lemma 4.2(2), $\widetilde{C}_i \cap K^d = C_i$ for each $i \in [n]$. Hence $\bigcap_{i\in[n]} \widetilde{C}_i \neq \emptyset$, and if $\bigcap_{i\in[n]} \widetilde{C}_i = \bigcap_{i\in S} \widetilde{C}_i$ for some $S \subseteq [n]$ with $|S| \leq d$, then also $\bigcap_{i\in[n]} C_i = \bigcap_{i\in S} C_i$.

Then let the vector subspaces $\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d$ and the upwards closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of Γ_∞ be as given by Theorem 3.6 for the convex set $C := C_1 \cap \ldots \cap C_n$. By Remark 3.8 we have

$$\Delta_d = \left\{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C_1 \cap \ldots \cap C_n\right\}.$$

It follows that there is some $i_d \in [n]$ such that in fact

$$\Delta_d = \left\{\gamma \in \Gamma_\infty \mid\ \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C_{i_d}\right\} \tag{4.1}$$

(since these are finitely many upwards closed sets in Γ, their intersection is already given by one of them).

Let $\{0\} \subsetneq F'_1 \subsetneq \ldots \subsetneq F'_d = K^d$ and $\Delta'_1 \supseteq \Delta'_2 \supseteq \ldots \supseteq \Delta'_d$ be as given by Theorem 3.6 for C_{i_d}. By Remark 3.10(1), F'_{d-1} is a linear hyperplane so that every element of C_{i_d} differs from an element of $F'_{d-1} \cap C_{i_d}$ by a vector with valuation in Δ'_d. As $\Delta_d = \Delta'_d$ by (4.1) and $C \subseteq C_{i_d}$, by Remark 3.10(1) we may assume that $F_{d-1} = F'_{d-1}$, hence every element in C_{i_d} differs from an element of $F_{d-1} \cap C_{i_d}$ by a vector with valuation in Δ_d.

Consider $C \cap F_{d-1} = C_1 \cap \ldots \cap C_n \cap F_{d-1} = (C_1 \cap F_{d-1}) \cap \ldots \cap (C_n \cap F_{d-1})$. Note that each $C_i \cap F_{d-1}$ is a convex subset of $F_{d-1} \cong K^{d-1}$, so by induction hypothesis there exist $i_1, \ldots, i_{d-1} \in [n]$ such that

$$C_{i_1} \cap \ldots \cap C_{i_{d-1}} \cap F_{d-1} = C_1 \cap \ldots \cap C_n \cap F_{d-1} = C \cap F_{d-1}. \tag{4.2}$$

Let $x \in C_{i_1} \cap \ldots \cap C_{i_d}$ be arbitrary. As $x \in C_{i_d}$, by the choice of F_{d-1}, $x = w + v_d$ for some $w \in F_{d-1}$ and $v_d \in K^d$ with $\nu(v_d) \in \Delta_d$. By the choice of Δ_d we have in particular $v_d \in C_{i_1} \cap \ldots \cap C_{i_d}$. And as each C_i is a module, it follows that also $w \in C_{i_1} \cap \ldots \cap C_{i_d}$. Combining this with (4.2) and using Remark 3.9 (for $j = d-1$) we thus have

$$\begin{aligned} C_{i_1} \cap \ldots \cap C_{i_d} = \{w + v_d \mid w \in C_{i_1} \cap \ldots \cap C_{i_d} \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} = \\ \{w + v_d \mid w \in C \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} = \\ \{v_1 + \ldots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} = \\ C_1 \cap \ldots \cap C_n. \end{aligned}$$

□

Definition 4.4. (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *Helly number* $k \in \mathbb{N}_{\geq 1}$ if given any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$, if every k-subset of $\{S_1, \ldots, S_n\}$ has nonempty intersection, then $\bigcap_{i \in [n]} S_i \neq \emptyset$.
(2) The *Helly number of* $\mathcal{F}$ refers to the minimal k with this property (or ∞ if it does not exist).
(3) We say that $\mathcal{F}$ has the *Helly property* if it has a finite Helly number.

Theorem 4.5. *Let K be a valued field and $d \geq 1$. Then the Helly number of* Conv_{K^d} *is $d+1$.*

Proof. The Helly number is bounded by the Radon number minus 1 in an arbitrary convexity space (see Section 5.3), but we include a proof for completeness. Let n be arbitrary, and let $S_1, \ldots, S_n \subseteq K^d$ be convex sets so that any $d+1$ of them have a non-empty intersection. We will show by induction on n that $S_1 \cap \ldots \cap S_n \neq \emptyset$.

Base case: $n = d+2$.

By assumption for each $i \in [d+2]$ there exists some $x_i \in K^d$ so that $x_i \in \bigcap_{j \in [d+2] \setminus \{i\}} S_j$. By Proposition 2.8 there exists some $i^* \in [d+2]$ so that $x_{i^*} \in \mathrm{conv}(\{x_i \mid i \neq i^*\})$. By the choice of the x_i's we have $x_{i^*} \in S_i$ for all $i \neq i^*$. We also have $x_i \in S_{i^*}$ for all $i \neq i^*$, S_{i^*} is convex and $x_{i^*} \in \mathrm{conv}(\{x_i \mid i \neq i^*\})$, hence $x_{i^*} \in S_{i^*}$. Thus $x_{i^*} \in \bigcap_{i \in [d+2]} S_i$, as wanted.

Inductive step: $n > d+2$.

Let $\widetilde{S}_{n-1} := S_{n-1} \cap S_n$, in particular $\widetilde{S}_{n-1}$ is convex. By induction hypothesis, any $n-1$ sets from $\{S_1, \ldots, S_n\}$ have a non-empty intersection. Hence any $n-2$ sets from $\left\{S_1, \ldots, S_{n-2}, \widetilde{S}_{n-1}\right\}$ have a non-empty intersection. As $n-2 \geq d+1$ by assumption, applying the induction hypothesis again we get

$$S_1 \cap \ldots \cap S_n = S_1 \cap \ldots \cap S_{n-2} \cap \widetilde{S}_{n-1} \neq \emptyset.$$

This completes the induction, and shows that Conv_{K^d} has Helly number $d+1$.

It remains to show that Conv_{K^d} does not have Helly number d. Let $e_i \in K^d$ be the ith standard basis vector. In particular the set $E := \{0, e_1, \ldots, e_d\}$ is affinely independent, hence the intersection of the affine spans of its $d+1$ maximal proper subsets is empty. The convex hull of a subset of K^d is contained in its affine hull, hence the intersection of the $d+1$ convex hulls of its maximal proper subsets is also empty. But for any d among the $(d+1)$ maximal proper subsets of E, some element of E belongs to their intersection, and hence in particular the intersection of their convex hulls is non-empty. □

We recall some terminology around the *Vapnik-Chervonenkis dimension* (and refer to [ADH+16, Sections 1 and 2] for further details).

Definition 4.6. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X.

(1) For a subset $Y \subseteq X$, we let $\mathcal{F} \cap Y := \{S \cap Y : S \in Y\} \subseteq \mathcal{P}(Y)$.
(2) We say that $\mathcal{F}$ *shatters* a subset $Y \subseteq X$ if $\mathcal{F} \cap Y = \mathcal{P}(Y)$.
(3) The *VC-dimension* of $\mathcal{F}$, or $\mathrm{VC}(\mathcal{F})$, is the largest $k \in \mathbb{N}$ (if one exists) such that $\mathcal{F}$ shatters some subset of X size k. If $\mathcal{F}$ shatters arbitrarily large finite subsets of X, then it is said to have infinite VC-dimension.
(4) The *dual family* $\mathcal{F}^* \subseteq \mathcal{P}(\mathcal{F})$ is given by $\mathcal{F}^* = \{S_x \mid x \in X\}$, where $S_x = \{A \in \mathcal{F} \mid x \in A\}$.
(5) The *dual VC-dimension* of $\mathcal{F}$, or $\mathrm{VC}^*(\mathcal{F})$, is the VC-dimension of $\mathcal{F}^*$. Equivalently, it is the largest $k \in \mathbb{N}$ (or ∞ if no such k exists) such that there are sets $S_1, \ldots, S_k \in \mathcal{F}$ that generate a Boolean algebra with 2^k atoms (i.e. for any distinct $I, J \subseteq [k]$, $\bigcap_{i \in I} S_i \cap \bigcap_{i \in [k] \setminus I} (X \setminus S_i) \neq \bigcap_{i \in J} S_i \cap \bigcap_{i \in [k] \setminus J} (X \setminus S_i)$).
(6) The *shatter function* $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{F}$ is

$$\pi_{\mathcal{F}}(n) := \max \{|\mathcal{F} \cap Y| : Y \subseteq X, |Y| = n\}.$$

(7) By the Sauer-Shelah lemma (see e.g. [ADH+16, Lemma 2.1], if $\mathrm{VC}(\mathcal{F}) \leq d$, then $\pi_{\mathcal{F}}(n) \leq \left(\frac{e}{d}\right)^d n^d$ for all $n \geq d$ (and $\pi_{\mathcal{F}}(n) = 2^n$ for all n if $\mathrm{VC}(\mathcal{F}) = \infty$).
(8) The *VC-density* of $\mathcal{F}$, or $\mathrm{vc}(\mathcal{F})$, is the infimum of all $r \in \mathbb{R}_{>0}$ so that $\pi_{\mathcal{F}}(n) = O(n^r)$, and ∞ if there is no such r. (In particular $\mathrm{vc}(\mathcal{F}) \leq \mathrm{VC}(\mathcal{F})$.)
(9) Finally, we define the *dual shatter function* $\pi^*_{\mathcal{F}} := \pi_{\mathcal{F}^*}$ and the *dual VC-density* $\mathrm{vc}^*(\mathcal{F}) := \mathrm{vc}(\mathcal{F}^*)$ of the family $\mathcal{F}$.

Remark 4.7. Note that if $\mathcal{F} \subseteq \mathcal{P}(X)$ and $Y \subseteq X$, then $\mathrm{VC}(\mathcal{F} \cap Y) \leq \mathrm{VC}(\mathcal{F})$ and $\mathrm{VC}^*(\mathcal{F} \cap Y) \leq \mathrm{VC}^*(\mathcal{F})$.

The following results is in stark contrast with the situation for the family of convex sets over the reals, where already the family of convex subsets of $\mathbb{R}^2$ has infinite VC-dimension (e.g., any set of points on a circle is shattered by the family of convex hulls of its subsets).

Theorem 4.8. *Let K be a valued field and $d \geq 1$. Then the family Conv_{K^d} has VC-dimension $d+1$.*

Proof. We have $\mathrm{VC}(\mathrm{Conv}_{K^d}) \geq d+1$ since the set $E := \{0, e_1, \ldots, e_d\} \subseteq K^d$, with e_i the ith vector of the standard basis, is shattered by Conv_{K^d}. Indeed, the convex hull of any subset is contained in its affine span, and for any $S \subseteq E$, $\mathrm{aff}(S)$ does not contain any of the points in $E \setminus S$.

On the other hand, $\mathrm{VC}(\mathrm{Conv}_{K^d}) \leq d+1$ as no subset Y of K^d with $|Y| \geq d+2$ can be shattered by Conv_{K^d}. Indeed, by Proposition 2.8, at least one of the points of Y belongs to every convex set containing all the other points of Y. □

The dual VC-dimension of a family of sets is bounded by its breadth.

Fact 4.9. [ADH$^+$16, Lemma 2.9] *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X of breadth at most d. Then also* $\mathrm{VC}^*(\mathcal{F}) \leq d$.

Using it, we get the following:

Theorem 4.10. *For any valued field K and $d \geq 1$, the family* Conv_{K^d} *has dual VC-dimension d.*

Proof. The dual VC-dimension of Conv_{K^d} is at least d because the d coordinate-aligned (convex) hyperplanes in K^d generate a Boolean algebra with 2^d atoms.

Conversely, the breadth of Conv_{K^d} is d by Theorem 4.3, hence by Fact 4.9 its dual VC-dimension is also at most d. □

Definition 4.11. (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *fractional Helly number* $k \in \mathbb{N}_{\geq 1}$ if for every $\alpha \in \mathbb{R}_{>0}$ there exists $\beta \in \mathbb{R}_{>0}$ so that: for any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$ (possibly with repetitions), if there are $\geq \alpha\binom{n}{k}$ k-element subsets of the multiset $\{S_1, \ldots, S_n\}$ with a non-empty intersection, then there are $\geq \beta n$ sets from $\{S_1, \ldots, S_n\}$ with a non-empty intersection.
(2) The *fractional Helly number* of $\mathcal{F}$ refers to the minimal k with this property. Say that $\mathcal{F}$ has the *fractional Helly property* if it has a fractional Helly number.

Note that any finite family of sets trivially has fractional Helly number 1 by choosing β sufficiently small with respect to the size of $\mathcal{F}$. We will use the following theorem of Matoušek.

Fact 4.12. [Mat04, Theorem 2] *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a set system whose dual shatter function satisfies $\pi^*_{\mathcal{F}}(n) = o(n^k)$, i.e. $\lim_{n\to\infty} \pi^*_{\mathcal{F}}(n)/n^k = 0$, where k is a fixed integer. Then $\mathcal{F}$ has fractional Helly number k.*

Remark 4.13. Moreover, if $\mathrm{VC}^*(\mathcal{F}) = d < \infty$, then the fractional Helly number is $\leq d+1$, and the β witnessing this can be chosen depending only on d and α (and not on the family $\mathcal{F}$).

Indeed, by Definition 4.6, if $\mathrm{VC}^*(\mathcal{F}) \leq d$, then $\pi^*_{\mathcal{F}}(n) \leq \left(\frac{e}{d}\right)^d n^d$ for all $n \geq d$, hence $\pi^*_{\mathcal{F}}(n) \leq cn^d$ for all $n \in \mathbb{N}$, where $c = c(d) := \left(\frac{e}{d}\right)^d + 2^d$. In particular we can choose $m = m(d, \alpha)$ so that $\pi^*_{\mathcal{F}}(m) < \frac{1}{4}\alpha\binom{m}{d+1}$. Then it follows from the proof of [Mat04, Theorem 2] that $\beta = \frac{1}{2m}$ works for all $n \geq \frac{m}{\beta} = 2m^2$, and trivially $\beta = \frac{1}{2m^2}$ works for all $n \leq 2m^2$, hence $\beta := \beta(\alpha, d) := \frac{1}{2m^2}$ works for all $n \in \mathbb{N}$.

Using this, we get the following:

Theorem 4.14. *If K is a valued field, $d \geq 1$, and $X \subseteq K^d$ is an arbitrary subset, then the fractional Helly number of the family*

$$\mathrm{Conv}_{K^d} \cap X = \{C \cap X : C \in \mathrm{Conv}_{K^d}\} \subseteq \mathcal{P}(X)$$

is at most $d+1$. Moreover, β in Definition 4.11 can be chosen depending only on d and α (and not on the field K or set X). And if K is infinite, then the fractional Helly number of the family Conv_{K^d} *is exactly $d+1$.*

Proof. By Fact 4.12 we have that the fractional Helly number of a set system is at most the smallest integer larger than its dual VC-density. Dual VC-density is, in turn, at most its dual VC-dimension. Also for any set $X \subseteq K^d$ we have $\mathrm{VC}^*(\mathrm{Conv}_{K^d} \cap X) \leq \mathrm{VC}^*(\mathrm{Conv}_{K^d})$ by Remark 4.7. So $\mathrm{Conv}_{K^d} \cap X$ has dual VC-density at most d by Theorem 4.10, hence its fractional Helly number is at most $d+1$ by Fact 4.12. And an appropriate β can be chosen depending only on d and α by Remark 4.13.

To show that the fractional Helly number of Conv_{K^d} is at least $d+1$ when K is infinite, we can use the standard example with affine hyperplanes in general position. We include the details for completeness. First note that as the field K is infinite, for any K-vector space V of dimension k and $v \in V \setminus \{0\}$ there exists an infinite set $S \subseteq V$ so that $v \in S$ and any k vectors from S are linearly independent. This is clear for $k = 1$ by taking any infinite set of non-zero vectors, so assume that $k > 1$. By induction on $i \in \mathbb{N}_{\geq k}$ we can find sets S_i such that $v \in S_i$, $|S_i| \geq i$ and every k vectors from S_i are linearly independent, for all i. Let S_k be any basis of V containing v. Assume $i > k$ and S_i satisfies the assumption. Since K is infinite, V is not a union of finitely many proper subspaces, in particular there exists some

$$w \in V \setminus \bigcup_{s \subseteq S_i, |s| = k-1} \langle s \rangle.$$

Let $S_{i+1} := S_i \cup \{w\}$. Since in particular any $s \subseteq S_i$ with $|s| = k - 1$ is linearly independent by the inductive assumption, it follows that $s \cup \{w\}$ is also linearly independent, hence S_{i+1} satisfies the assumption. Finally, $S := \bigcup_{i \in \mathbb{N}_{\geq k}} S_i$ is as wanted.

In particular, we can find an infinite set of vectors S in $K^d \times K$ so that any $d+1$ of them are linearly independent and the standard basis vector $e_{d+1} \in S$. Then

$$X := \{\langle v, - \rangle : v \in S\} \subseteq \left(K^d \times K\right)^*$$

is an infinite set of dual vectors such that any $d+1$ of them are linearly independent, and it contains the projection map onto the last coordinate $\pi_{d+1} := \langle e_{d+1}, - \rangle : (x_1, \ldots, x_{d+1}) \mapsto x_{d+1}$. Consider the family

$$\mathcal{H} := \{\ker(f) \mid f \in X \setminus \{\pi_{d+1}\}\} \subseteq \mathcal{P}\left(K^d \times K\right)$$

of kernels of these dual vectors (excluding the projection map onto the last coordinate), and let

$$\mathcal{H}' := \left\{\left\{v \in K^d \mid (v, 1) \in H\right\} \mid H \in \mathcal{H}\right\} \subseteq \mathcal{P}\left(K^d\right).$$

Then $\mathcal{H}'$ is an infinite family of affine hyperplanes in K^d, and we wish to show that any d element of $\mathcal{H}'$ intersect in a point, and any $d+1$ elements of $\mathcal{H}'$ have empty intersection. For any pairwise distinct $f_1, \ldots, f_d \in X \setminus \{\pi_{d+1}\}$, by linear independence

$$\dim(\ker(f_1) \cap \ldots \cap \ker(f_d)) = d + 1 - \dim(\langle f_1, \ldots, f_d \rangle) = 1.$$

And by their linear independence with π_{d+1},

$$\dim(\ker(f_1) \cap \ldots \cap \ker(f_d) \cap \ker(\pi_{d+1})) = 0.$$

That is, $\ker(f_1) \cap \ldots \cap \ker(f_d)$ is a line in $K^d \times K$ that intersects $\ker(\pi_{d+1}) = K^d \times \{0\}$ only at the origin, and thus must also intersect $K^d \times \{1\}$ in a single point;

this shows that every d elements of $\mathcal{H}'$ intersect in a point. And any pairwise distinct $f_1, \ldots, f_{d+1} \in X \setminus \{\pi_{d+1}\}$ span $\left(K^d \times K\right)^*$ by linear independence, so $\ker(f_1) \cap \ldots \cap \ker(f_{d+1}) = \{0\}$, and thus has empty intersection with $K^d \times \{1\}$. This shows that every $d+1$ elements of $\mathcal{H}'$ have empty intersection.

Using $\alpha = 1$, for any $\beta > 0$, take an arbitrary $n \geq \frac{d+1}{\beta}$. Let $H_1, \ldots, H_n \in \mathcal{H}'$ be any distinct hyperplanes from this collection. All d-subsets (so, $\alpha\binom{n}{d}$ of them) of $\{H_1, \ldots, H_n\}$ have an intersection point, but there are no $\beta n \geq d+1$ of them with a common intersection point. Therefore Conv_{K^d} does not have fractional Helly number d. □

Note that Theorems 4.5 and 4.14 replicate results for real convex sets, while Theorems 4.3, 4.8, and 4.10 do not: as we have already remarked, $\mathrm{Conv}_{\mathbb{R}^2}$ has infinite breadth, VC-dimension, and dual VC-dimension. The following result is somewhere in between. The classical Tverberg theorem says that for any $X \subseteq \mathbb{R}^d$ with $|X| \geq (d+1)(r-1)+1$, X can be partitioned into r disjoint subsets $X_1, \ldots, X_r$ whose convex hulls intersect; that is, $\mathrm{conv}(X_1) \cap \ldots \cap \mathrm{conv}(X_r) \neq \emptyset$. Over valued fields, we obtain a much stronger version (note that any element of the non-empty set X_r in the statement of theorem 4.15 belongs to the convex hulls of each of the sets $X_i, i \in [r]$ — which gives the usual conclusion of Tverberg's theorem over the reals):

Theorem 4.15. *Let K be a valued field and $d, r \in \mathbb{N}_{\geq 1}$. Then any set $X \subseteq K^d$ with*

$$|X| \geq (d+1)(r-1)+1$$

points in K^d can be partitioned into subsets $X_1, \ldots, X_r$ such that $|X_i| = d+1$ for $i < r$, $|X_r| = |X| - (d+1)(r-1)$, and $\mathrm{conv}(X_i) \supseteq \mathrm{conv}(X_j)$ for all $i \leq j \in [r]$.

Proof. Since any finitely generated convex set is the convex hull of some $d+1$ points from it by Corollary 2.9, we can find $X_1 \subseteq X$ with $|X_1| = d+1$ and $\mathrm{conv}(X_1) = \mathrm{conv}(X)$, $X_2 \subseteq X \setminus X_1$ with $|X_2| = d+1$ and $\mathrm{conv}(X_2) = \mathrm{conv}(X \setminus X_1)$, and so on: once $X_1, \ldots, X_{i-1}$ have been chosen, pick $X_i \subseteq X \setminus \left(\bigcup_{j=1}^{i-1} X_j\right)$ such that $|X_i| = d+1$, $\mathrm{conv}(X_i) = \mathrm{conv}\left(X \setminus \bigcup_{j=1}^{i-1} X_j\right)$, and then let X_r consist of everything left over at the end. □

From this strong Tverberg theorem and the fractional Helly property, we finally get an analog of the result due to Boros-Füredi [BF84] and Bárány [Bár82] on the common points in the intersections of many "simplices" over valued fields (note that the conclusion is actually stronger than over the reals: the common point comes from the set X itself). This answers a question asked by Kobi Peterzil and Itay Kaplan. Our argument is an adaptation of the second proof in [Mat02, Theorem 9.1.1].

Theorem 4.16. *For each $d \geq 1$ there is a constant $c = c(d) > 0$ such that: for any valued field K and any finite $X \subseteq K^d$ (say $n := |X|$), there is some $a \in X$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ subsets of X of size $d+1$.*

Proof. Let $X \subseteq K^d$ with $|X| = n$ be given, and let

$$\mathcal{F} := \mathrm{Conv}_{K^d} \cap X = \{C \cap X : C \in \mathrm{Conv}_{K^d}\}$$

be the family of all subsets of X cut out by the convex subsets of K^d. Let $(S_i)_{i\in[N]}$ with $S_i \in \mathrm{Conv}_{K^d}$ be the sequence listing all convex hulls of subsets of X of size $d+1$ in an arbitrary order (possibly with repetitions). Then $N = \binom{n}{d+1}$, and for a $(d+1)$-element subset $Y \subseteq X$ we let $g(Y) \in [N]$ be the index at which $\mathrm{conv}(Y)$ appears in this sequence. For each $i \in [N]$ let $S'_i := S_i \cap X \in \mathcal{F}$. It is thus sufficient to show that there exists some $\alpha > 0$, depending only on d, such that at least $\alpha\binom{N}{d+1}$ of the $(d+1)$-element subsets $I \subseteq [N]$ satisfy $\bigcap_{i\in I} S'_i \neq \emptyset$ — as then Theorem 4.14 applied to $\mathcal{F} \subseteq \mathcal{P}(X)$ shows the existence of $c > 0$ depending only on α, d, and hence only on d, so that for some $I \subseteq [N]$ with $|I| \geq cN = c\binom{n}{d+1}$ there exists some $a \in \bigcap_{i\in I} S'_i \subseteq \bigcap_{i\in I} S_i$ (in particular $a \in X$).

Now we find an appropriate α. For any $(d+1)^2$-element subset $Y \subseteq X$, by Theorem 4.15 (with $r := d+1$), we can fix a partition of Y into $d+1$ disjoint parts $Y_1, \ldots, Y_{d+1}$, each of which has $d+1$ elements, and so that $\mathrm{conv}(Y_i) \supseteq \mathrm{conv}(Y_j)$ for all $i \leq j \in [d+1]$. In particular any element of the non-empty set $Y_{[d+1]} \subseteq X$ belongs to $\bigcap_{i\in[d+1]} (\mathrm{conv}(Y_i) \cap X) = \bigcap_{i\in[d+1]} \left(S'_{g(Y_i)}\right)$. As g is a bijection, $Y \mapsto \{g(Y_i) : i \in [d+1]\}$ gives a function f from $(d+1)^2$-element subsets of X to $(d+1)$-element subsets $I \subseteq [N]$ so that $\bigcap_{i\in I} S'_i \neq \emptyset$. Moreover, f is an injection. Indeed, given a set $\{j_i : i \in [d+1]\}$ in the image of f, as g is a bijection, there is a unique set $\{Y_1, \ldots, Y_{d+1}\}$ with $Y_i \subseteq X$ disjoint of size $d+1$ so that $g(Y_i) = j_i$ for all $i \in [d+1]$, and there can be only one set $Y \subseteq X$ of size $(d+1)^2$ for which it is a partition. If follows that the number of sets $I \subseteq [N]$ with $\bigcap_{i\in I} S'_i \neq \emptyset$ is at least

$$\binom{n}{(d+1)^2} = \Omega\left(n^{(d+1)^2}\right) \geq \alpha\binom{N}{d+1}$$

for some sufficiently small α depending only on d. □

5. Final remarks and questions

5.1. **Some further results and future directions.** The results of Section 4 imply the following analog of the celebrated (p,q)-*theorem* of Alon and Kleitman [AK92] for convex sets over valued fields.

Corollary 5.1. *For any $d, p, q \in \mathbb{N}_{\geq 1}$ with $p \geq q \geq d+1$ there exists $T = T(p,q,d) \in \mathbb{N}$ such that: if K is a valued field and $\mathcal{F}$ is a family of convex subsets of K^d such that among every p sets of $\mathcal{F}$, some q have a non-empty intersection, then there exists a T-element set $Y \subseteq K^d$ intersecting all sets of $\mathcal{F}$.*

Corollary 5.1 follows formally by applying [AKMM02, Theorem 8] since the family Conv_{K^d} has fractional Helly property (Theorem 4.14) and is closed under intersections. Alternatively, it follows with a slightly better bound on T by combining the fractional Helly property with the existence of ε-nets for families of bounded VC-dimension (Theorem 4.8), as outlined at the end of [Mat04, Section 1]. The problem of determining the optimal bound on $T(p,q,d)$ is widely open over the reals (see [BK22, Section 2.6]), and we expect that it might be easier in the valued fields setting.

Kalai [Kal84] and Eckhoff [Eck85] proved that in the fractional Helly property for convex sets over the reals, one can take $\beta(d, \alpha) = 1-(1-\alpha)^{\frac{1}{d+1}}$ (and this bound is sharp).

Problem 5.2. What is the optimal dependence of β on d, α in Theorem 4.14?

Over $\mathbb{R}$, Sierksma's Dutch cheese conjecture predicts a lower bound for the number of Tverberg partitions (see e.g. [DLGMM19, Conjecture 3.12] and the references there). We expect the same bound to holds over valued fields:

Conjecture 5.3. For any valued field K and $X \subset K^d$ with $|X| = (r-1)(d+1)+1$, there are at least $((r-1)!)^d$ partitions of X into parts whose convex hulls intersect.

Remark 5.4. In Theorem 4.15, we showed the existence of Tverberg partitions satisfying the stronger property that the convex hulls of the parts are linearly ordered by inclusion. It is not true that for $X \subseteq K^d$ with $|X| = (d+1)(r-1)+1$, there are at least $((r-1)!)^d$ different ways of partitioning X into $X_1, \dots, X_r$ such that $\mathrm{conv}(X_1) \supseteq \dots \supseteq \mathrm{conv}(X_r)$. Thus any attempt to prove Conjecture 5.3 would have to involve other Tverberg partitions that do not have this property. For an example in K^2 where this bound fails, let $x \in K$ with $\nu(X) \neq 0$, and let $X := \{(x^n, x^{-n}) | n \in [3(r-1)+1]\}$. For any partition of X into $X_1, \dots, X_r$ such that $\mathrm{conv}(X_1) \supseteq \dots \supseteq \mathrm{conv}(X_r)$, for each $i < r$, X_i must consist of the points corresponding to the lowest and highest values of n among all points not already in $X_1 \cup \dots \cup X_{i-1}$, together with one of the other $3(r-i)-1$ remaining points, and X_r must consist of whatever point is left over. So the number of partitions of X of this form is $\prod_{i=1}^{r-1}(3(r-i)-1) < \prod_{i=1}^{r-1} 3(r-i) = 3^{r-1}(r-1)! < ((r-1)!)^2$ for large enough r.

We expect that the *colorful* Tverberg theorem also holds over valued fields, however the proofs for convex sets over $\mathbb{R}$ rely on topological arguments not readily available in the valued field context:

Conjecture 5.5. For any integers $r, d \geq 2$ there exists $t \geq r$ such that: for any valued field K and $X \subseteq K^d$ with $|X| = t\,(d+1)$, partitioned into $d+1$ color classes $C_1, \dots, C_{d+1}$ each of size t, there exist pairwise disjoint $X_1, \dots, X_r \subseteq X$ with $|X_i \cap C_j| = 1$ for $i \in [r]$ and $j \in [d+1]$, and $\bigcap_{i \in [r]} \mathrm{conv}\,(X_i) \neq \emptyset$.

It would formally imply (see e.g. [Mat02, Section 9.2]) the "second selection lemma" over valued fields generalizing Theorem 4.16:

Conjecture 5.6. For each $d \in \mathbb{N}_{\geq 1}$ there exist $c, s > 0$ such that: for any valued field K, $\alpha \in (0,1]$ and $n \in \mathbb{N}$, for every $X \subseteq K^d$ with $|X| = n$, and every family $\mathcal{F}$ of $(d+1)$-element subsets of X with $|\mathcal{F}| \geq \alpha\binom{n}{d+1}$, there is a point contained in the convex hulls of at least $c\alpha^s\binom{n}{d+1}$ of the elements of $\mathcal{F}$.

Corollary 3.12 has the following immediate model-theoretic application.

Remark 5.7. If K is a spherically complete valued field, then every convex subset of K^d is definable in the expansion of the field K by a predicate for each Dedekind cut of the value group (so in particular definable in *Shelah expansion of* K by all externally definable sets [She09, CS13]). And conversely, every Dedekind cut of the value group is definable in the expansion of K by a predicate for each $\mathcal{O}$-submodule of K. In particular, if K has value group $\mathbb{Z}$, then all convex subsets of K^d form a definable family.

Example 5.8. In contrast, naming a single (bounded) convex subset of $\mathbb{R}^2$ in the field of reals allows to define the set of integers. Indeed, we can define a continuous and piecewise linear function $f : [0,1] \to [0,1]$ such that

$$C := \{(x, y) : x \in [0,1], 0 \leq y \leq f(x)\}$$

is convex but the set of points where f is not differentiable is exactly $\left\{\frac{1}{n} : n \in \mathbb{N}_{\geq 2}\right\}$. Now in the field of reals with a predicate for C we can define f and the set of points where it is not differentiable, hence $\mathbb{N}$ is also definable.

5.2. **Other notions of convexity over non-archimedean fields.** We briefly overview several other kinds of convexities over non-archimedean fields considered in the literature. The extension of Hilbert (projective) geometry to convex sets in a generalized sense is a topic of high current interest, see e.g. [Gui16]. In a different spirit, in tropical geometry, convex sets over real closed non-archimedean fields have been considered (unlike what is done here, this leads to a combinatorial convexity similar to the classical one, since by Tarski's completeness theorem, polyhedral properties of a combinatorial nature are the same over all real closed fields). Moreover, tropical polyhedra are obtained as images of such polyhedra by the nonarchimedean valuation, see e.g. [DY07]. Polytopes and simplexes in p-adic fields are introduced in [Dar17, Dar19], and demonstrated to play in p-adically closed fields the role played by real simplexes in the classical results of triangulation of semi-algebraic sets over real closed fields. Although we are not aware of any direct link of these results with the present work, we hope for some connections to be found in the future.

5.3. **Abstract convexity spaces.** Our results here can be naturally placed in the context of abstract convexity spaces, we refer to e.g. [vDV93] for an introduction to the subject. A *convexity space* is a pair $(X, \mathcal{C})$, where X is a set and $\mathcal{C} \subseteq 2^X$ is a family of subsets of X closed under intersection with $\emptyset, X \in \mathcal{C}$. The sets in $\mathcal{C}$ are called *convex*. Given a subset $Y \subseteq X$, the *convex hull* of Y, denoted $\operatorname{conv}(Y)$, is the smallest set in $\mathcal{C}$ containing Y (equivalently, the intersection of all sets in $\mathcal{C}$ containing Y). A convex set $C \in \mathcal{C}$ is called a *half-space* if its complement is also convex. The convexity space $(X, \mathcal{C})$ is *separable* if for every $C \in \mathcal{C}$ and $x \in X \setminus C$, there exists a half-space $H \in \mathcal{C}$ so that $C \subseteq H$ and $x \notin H$ (equivalently, if every convex set is the intersection of all half-spaces containing it). Separability is an abstraction of the hyperplane separation (and more generally Hahn-Banach) theorem. In particular, $\left(\mathbb{R}^d, \operatorname{Conv}_{\mathbb{R}^d}\right)$ is a separable convexity space (see e.g. [MY19, Section 1.1] or [vDV93] for many other examples). The *Radon number*[2] of a convexity space $(X, \mathcal{C})$ is the smallest $k \in \mathbb{N}_{\geq 1}$ (if it exists) such that every $Y \subseteq X$ with $|Y| > k$ can be partitioned into two parts Y_1, Y_2 such that $\operatorname{conv}(Y_1) \cap \operatorname{conv}(Y_2) \neq \emptyset$ (the classical Radon's theorem states that the Radon number of $\left(\mathbb{R}^d, \operatorname{Conv}_{\mathbb{R}^d}\right)$ equals $d+1$).

[2] Sometimes in the literature it is defined with "$\geq$" instead of "$>$" leading to the value off by 1, we are following the notation from [vDV93, Chapter II] here.

Given $\emptyset \neq Y \subseteq X$, a partition $Y_1, \ldots, Y_r$ of Y is *Tverberg* if $\bigcap_{i=1}^{r} \operatorname{conv}(Y_i) \neq \emptyset$. The *$r$th Tverberg number* of $(X, \mathcal{C})$ is the smallest k so that every $Y \subseteq X$ with $|Y| > k$ has a Tverberg partition in $r+1$ parts. Note that the first Tverberg number is the Radon number, and the classical theorem of Tverberg says that the rth Tverberg number of $(\mathbb{R}^d, \operatorname{Conv}_{\mathbb{R}^d})$ is $r(d+1)$.

Now let K be a valued field and $d \in \mathbb{N}_{\geq 1}$. Then $(K^d, \operatorname{Conv}_{K^d})$ is a convexity space, but we stress that it is *not separable*; in fact, $\emptyset$ and K^d are the only half-spaces. This is because for any non-empty proper convex set C, let $x \in C$, $y \in K^d \setminus C$, and $\alpha \in K \setminus \mathcal{O}$. Then $z := x + \alpha(y - x) \notin C$, since $y = \alpha^{-1}z + (1 - \alpha^{-1})x$ is a convex combination. But then $x = (1-\alpha)^{-1}(z - \alpha y)$ is a convex combination of elements of $K^d \setminus C$, so $K^d \setminus C$ is not convex.

Proposition 2.8 implies that the Radon number of $(K^d, \operatorname{Conv}_{K^d})$ is $d+1$. By the Levi inequality in an arbitrary convexity space ([vDV93, Chapter II(1.9)]), it follows that the Helly number of $\operatorname{Conv}_{K^d}$ (Definition 4.4) is $\leq d+1$ (we included a proof in Theorem 4.5 for completeness). It was also recently shown in [HL21] that in any convexity space $(X, \mathcal{C})$ with Radon number k, $\mathcal{C}$ has a fractional Helly number (Definition 4.11) bounded by some function of k. In the case of $(K^d, \operatorname{Conv}_{K^d})$ this general bound is much weaker than the optimal bound $d+1$ given in Theorem 4.14. Corollary 2.9 implies that the Carathéodory number of $(K^d, \operatorname{Conv}_{K^d})$ is $d+1$ (see [vDV93, Chapter II(1.5)] for the definition). Finally, Theorem 4.15 implies that the rth Tverberg number of $(K^d, \operatorname{Conv}_{K^d})$ is $r(d+1)$ (finiteness of the rth Tverberg numbers for all r follows from the finiteness of the Radon number in an arbitrary convexity space, with a much weaker bound [vDV93, Chapter II(5.2)]).

References

[ADH+16] Matthias Aschenbrenner, Alf Dolich, Deirdre Haskell, Dugald Macpherson, and Sergei Starchenko. Vapnik-Chervonenkis density in some theories without the independence property, I. *Transactions of the American Mathematical Society*, 368(8):5889–5949, 2016.

[AK92] Noga Alon and Daniel J Kleitman. Piercing convex sets and the Hadwiger-Debrunner (p, q)-problem. *Advances in Mathematics*, 96(1):103–112, 1992.

[AKMM02] Noga Alon, Gil Kalai, Jiří Matoušek, and Roy Meshulam. Transversal numbers for hypergraphs arising in geometry. *Advances in Applied Mathematics*, 29(1):79–101, 2002.

[AvdDvdH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. *Asymptotic differential algebra and model theory of transseries*. Princeton University Press, 2017.

[Bár82] Imre Bárány. A generalization of Carathéodory's theorem. *Discrete Mathematics*, 40(2-3):141–152, 1982.

[BF84] Endre Boros and Zoltán Füredi. The number of triangles covering the center of an n-set. *Geometriae Dedicata*, 17(1):69–77, 1984.

[BK22] Imre Bárány and Gil Kalai. Helly-type problems. *Bulletin of the American Mathematical Society*, 59(4):471–502, 2022.

[CS13] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs. *Israel Journal of Mathematics*, 194(1):409–425, 2013.

[Dar17] Luck Darniere. Polytopes and simplexes in p-adic fields. *Annals of Pure and Applied Logic*, 168(6):1284–1307, 2017.

[Dar19] Luck Darnière. Semi-algebraic triangulation over p-adically closed fields. *Proceedings of the London Mathematical Society*, 118(6):1501–1546, 2019.

[DLGMM19] Jesús De Loera, Xavier Goaoc, Frédéric Meunier, and Nabil Mustafa. The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg. *Bulletin of the American Mathematical Society*, 56(3):415–511, 2019.

[DY07] Mike Develin and Josephine Yu. Tropical polytopes and cellular resolutions. *Experimental Mathematics*, 16(3):277–291, 2007.

[Eck85] Jürgen Eckhoff. An upper-bound theorem for families of convex sets. *Geometriae Dedicata*, 19(2):217–227, 1985.

[Fuc75] László Fuchs. Vector spaces with valuations. *Journal of Algebra*, 35(1-3):23–38, 1975.

[Gui16] Antonin Guilloux. Yet another p-adic hyperbolic disc: Hilbert distance for p-adic fields. *Groups, Geometry, and Dynamics*, 10(1):9–43, 2016.

[HL21] Andreas F Holmsen and Donggyu Lee. Radon numbers and the fractional helly theorem. *Israel Journal of Mathematics*, 241(1):433–447, 2021.

[Hru14] Ehud Hrushovski. Imaginaries and definable types in algebraically closed valued fields. *Valuation Theory in Interaction*, pages 297–319, 2014.

[Joh16] William Andrew Johnson. *Fun with fields*. PhD thesis, UC Berkeley, 2016.

[Kal84] Gil Kalai. Intersection patterns of convex sets. *Israel Journal of Mathematics*, 48(2-3):161–174, 1984.

[Mat02] Jiří Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[Mat04] Jiří Matoušek. Bounded VC-dimension implies a fractional Helly theorem. *Discrete & Computational Geometry*, 31(2):251–255, 2004.

[Mon46] A. F. Monna. Sur les espaces linéaires normés. I, II, III, IV. *Nederl. Akad. Wetensch., Proc.*, 49, 1946.

[MY19] Shay Moran and Amir Yehudayoff. On weak ϵ-nets and the Radon number. In *35th International Symposium on Computational Geometry (SoCG 2019)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2019.

[PGS10] C. Pérez-Garciá and Wilhelmus Hendricus Schikhof. *Locally convex spaces over non-Archimedean valued fields*. Cambridge Univ. Press, 2010.

[Sch50] Otto Franz Georg Schilling. *The theory of valuations*, volume 4. American Mathematical Soc., 1950.

[Sch13] Peter Schneider. *Nonarchimedean functional analysis*. Springer Science & Business Media, 2013.

[She09] Saharon Shelah. Dependent first order theories, continued. *Israel Journal of Mathematics*, 173(1):1, 2009.

[vdD14] Lou van den Dries. Lectures on the model theory of valued fields. In *Model theory in algebra, analysis and arithmetic*, pages 55–157. Springer, 2014.

[vDV93] M.L.J. van De Vel. *Theory of convex structures*. Elsevier, 1993.

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095, USA

Email address: chernikov@math.ucla.edu

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095, USA

Email address: alexmennen@math.ucla.edu