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Decoupling of second-order linear systems by isospectral transformation

Daniel T. Kawano, Rubens G. Salsa Jr. and Fai Ma

Abstract. We consider the class of real second-order linear dynamical systems that admit real diagonal forms with the same eigenvalues and partial multiplicities. The nonzero leading coefficient is allowed to be singular, and the associated quadratic matrix polynomial is assumed to be regular. We present a method and algorithm for converting *any* such n-dimensional system into a set of n mutually independent second-, first-, and zeroth-order equations. The solutions of these two systems are related by a real, time-dependent, and nonlinear n-dimensional transformation. Explicit formulas for computing the $2n \times 2n$ real and time-invariant equivalence transformation that enables this conversion are provided. This paper constitutes a *complete* solution to the problem of diagonalizing a second-order linear system while preserving its associated Jordan canonical form.

Mathematics Subject Classification. 15A22, 34A30, 70J10.

Keywords. Second-order linear differential equations, Quadratic matrix polynomials, Diagonalization, Isospectral systems.

1. Introduction

Consider the real second-order linear dynamical system

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t) \tag{1.1}$$

for which the coefficients $M, C, K \in \mathbb{R}^{n\times n}$ and a solution $x(t) \in \mathbb{R}^n$ exists, where the independent time variable $t \geq 0$. The leading coefficient $M \neq 0$ (to maintain the second-order nature of (1.1)) is allowed to be singular. The inhomogeneity $f(t) \in \mathbb{R}^n$ is given and continuously differentiable. Initial values $x(0) \in \mathbb{R}^n$ and $\dot{x}(0) \in \mathbb{R}^n$ are also provided. When M is invertible, $x(0)$ and $\dot{x}(0)$ are arbitrary. For the case of singular M, we take $x(0)$ and $\dot{x}(0)$ to be *consistent* initial values; that is, they satisfy all constraints on their components such that $M\ddot{x}(0) + C\dot{x}(0) + Kx(0) = f(0)$ holds. Associated with the homogeneous form of (1.1) is the $n \times n$ *quadratic matrix polynomial* (or *matrix pencil*) $Q(\lambda) = M\lambda^2 + C\lambda + K$ in the scalar parameter $\lambda \in \mathbb{C}$. We assume $Q(\lambda)$ is *regular*, i.e., $Q(\lambda)$ does not have zero determinant for all values of λ. Equation (1.1) arises in various scientific and engineering applications. For example, (1.1) models the small-amplitude vibration of a lumped-parameter mechanical system (e.g., see [1]).

In general, the scalar component equations of (1.1) are mutually dependent because they cannot be arranged to make M, C, and K all diagonal. System (1.1) is said to be *coupled* in this case. We are concerned with the transformation of (1.1) into the real second-order system

$$A_2\ddot{p}(t) + A_1\dot{p}(t) + A_0p(t) = g(t), \tag{1.2}$$

where the coefficients $A_2, A_1, A_0 \in \mathbb{R}^{n\times n}$ are diagonal. The solution $p(t) \in \mathbb{R}^n$, the inhomogeneity $g(t) \in \mathbb{R}^n$, and $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ is the regular quadratic pencil that corresponds to the homogeneous form of (1.2). We say system (1.2) is *decoupled* because it comprises n mutually independent scalar equations. The process of converting (1.1) into (1.2) is referred to as *decoupling*.

If M is symmetric and positive definite and K is symmetric and at least positive semidefinite, then M and K can be simultaneously diagonalized by a real congruence transformation [2]. This transformation will also diagonalize a symmetric C that is at least positive semidefinite if and only if $CM^{-1}K = KM^{-1}C$ [3]. Recently, much progress has been made in decoupling system (1.1) free of this restriction and similar ones (e.g., see [4]) by a real transformation into (1.2) that preserves the eigenvalues and *partial multiplicities* (i.e., the sizes of the corresponding Jordan blocks) of the quadratic matrix polynomial $Q(\lambda)$—that is, $Q(\lambda)$ and $\widetilde{Q}(\lambda)$, or, loosely speaking, systems (1.1) and (1.2), have the same Jordan canonical form. Such systems are said to be *isospectral*; we refer to the decoupling transformation that relates these systems as an *isospectral transformation*.

The typical approach to diagonalization in the literature concerns the homogeneous form of (1.1) and expresses the problem of decoupling it via isospectral transformation as the conversion of $Q(\lambda)$ into $\widetilde{Q}(\lambda)$ by relating their *linearizations* (i.e., $2n \times 2n$ linear pencils with the same Jordan structure as $Q(\lambda)$ and $\widetilde{Q}(\lambda)$) through an equivalence transformation that preserves the block structure of the linearizations' coefficients, or a *diagonalizing structure-preserving transformation* (DSPT). (See [5] and [6] for recent examples.) When M is invertible but the coefficients M, C, and K are otherwise arbitrary, the conditions for which $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ are isospectral are well understood (see Lancaster and Zaballa's work in [7]), and algorithms for generating a corresponding DSPT are available (see [5] in particular). Thus, isospectral transformation of a homogeneous (1.1), with M nonsingular, into the homogeneous form of (1.2) is a settled matter. For the nonhomogeneous case, [8] and [9] offer insight into the role of the inhomogeneity $f(t)$ in decoupling. However, both presentations are problematic because it is never shown that $x(t)$ being a solution of (1.1) is a necessary condition for $p(t)$ to be a solution of (1.2) and the proposed mapping from $p(t)$ to $x(t)$ to hold; a reformulation of the given arguments is needed. When $M \neq 0$ is singular, Zúñiga Anaya [10] determined the necessary and sufficient conditions for $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ to be isospectral, but a method for constructing a corresponding DSPT is not pursued in [10] and has yet to be offered in the literature. In this paper, we show that a DSPT for the case when M is not invertible does indeed exist, develop an accompanying algorithm for generating this transformation using spectral data, and demonstrate that a solution $x(t)$ of coupled system (1.1) can be recovered from a solution $p(t)$ of decoupled system (1.2).

To summarize, the objective of our paper is to address the following open problems regarding decoupling: given *any* second-order system (1.1) that can be decoupled into (1.2) by a real isospectral transformation, what is the form of the corresponding transformation, and how does the solution $p(t)$ of the decoupled system map to the coupled system's solution $x(t)$? By answering these questions, we provide a *complete* solution to the problem of converting (1.1) into (1.2) by an isospectral transformation. Systems that can be decoupled in this manner are of considerable practical interest (for modal analysis, model reduction, damping characterization, etc.), but our focus here is on the theory of decoupling instead of its many applications. Our paper begins with a summary of background information in Sect. 2 that will prove useful in later developments. In Sect. 3, we discuss the conditions for which the quadratic matrix polynomial $Q(\lambda)$ for the coupled system is isospectral to the decoupled system's diagonal quadratic pencil $\widetilde{Q}(\lambda)$ and propose an indexing scheme for and an arrangement of the spectral data to construct a convenient form for $\widetilde{Q}(\lambda)$. Section 4 details the development of a DSPT that converts $Q(\lambda)$ into $\widetilde{Q}(\lambda)$, and this transformation is subsequently used in Sect. 5 to connect the solutions of the coupled and decoupled systems. We provide an algorithm for isospectral decoupling in Sect. 6 that we then demonstrate in two illustrative examples in Sect. 7. We close our paper with some concluding remarks in Sect. 8.

First, a few remarks on our notation. We generally use capital letters to denote matrices, lowercase Roman letters for column vectors, and lowercase Greek letters for scalars. There are some exceptions to these conventions: t for the independent time variable, r for the rank of a matrix, the imaginary unit $\mathrm{i} = \sqrt{-1}$, and n for the dimension of (1.1) and (1.2). We also use a subscripted or ornamented n for quantifying certain scalars. The letters j, k, and ℓ are reserved for indexing scalars, vectors, and matrices. We use the notation v_j ($j = 1, 2, \ldots, n$) to represent the sequence of vectors v_1, v_2, …, v_n. The same notation is adopted for a sequence of scalars α_j; these scalars are sometimes arranged in a vector $v = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, where the superscript T indicates the transpose. The complex conjugates of a vector v and a scalar α are denoted by $\overline{v}$ and $\overline{\alpha}$, respectively. We use I and 0 to denote, respectively, the identity matrix and a matrix of zeros, the sizes of which can usually be readily inferred from the context by compatibility. When clarity is needed, we will write I_α to signify the $\alpha \times \alpha$ identity matrix and $0_{\alpha\times\beta}$ to denote an $\alpha \times \beta$ matrix of zeros. Lastly, we construct a block diagonal matrix B from a sequence of arbitrary matrices A_j ($j = 1, 2, \ldots, n$) using direct sum notation: $B = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ or, more compactly, $B = \oplus_{j=1}^{n} A_j$.

2. Background

As a prelude to our extension of decoupling by an isospectral transformation to the case when $M \neq 0$ is singular, we briefly review relevant details regarding eigenvalues, eigenvectors, generalized eigenvectors, and linearization of a quadratic matrix polynomial $Q(\lambda) = M\lambda^2 + C\lambda + K$.

2.1. Eigenvalues, eigenvectors, and generalized eigenvectors

Solution of the scalar polynomial $\det(Q(\lambda)) = 0$ in λ yields the $2n$ eigenvalues λ_j ($j = 1, 2, \ldots, 2n$) of $Q(\lambda)$ [11–13], where the set of eigenvalues $\{\lambda_j\}$ is termed the *spectrum* of $Q(\lambda)$. If M is invertible, then all eigenvalues of $Q(\lambda)$ are finite; some eigenvalues are infinite when M is singular. We denote the *reversal* of $Q(\lambda)$ as $Q_{\text{rev}}(\lambda) = K\lambda^2 + C\lambda + M$ [14]. The finite eigenvalues of $Q(\lambda)$ and $Q_{\text{rev}}(\lambda)$ are reciprocals, and the eigenvalue at infinity for $Q(\lambda)$ when M is not invertible corresponds to the zero eigenvalue of $Q_{\text{rev}}(\lambda)$ [15].

An eigenvalue might have several partial multiplicities. The number of occurrences of an eigenvalue is its *algebraic multiplicity*, which is the sum of its partial multiplicities. The number of partial multiplicities is the *geometric multiplicity*. An eigenvalue is *simple* if it occurs only once; such an eigenvalue has unit partial, algebraic, and geometric multiplicities. A repeated eigenvalue is *semisimple* when its algebraic and geometric multiplicities coincide. Otherwise, the repeated eigenvalue is *defective*.

Associated with a semisimple eigenvalue λ_0 of $Q(\lambda)$ with algebraic multiplicity $\alpha \leq n$ are α eigenvectors $v_j \neq 0$ ($j = 1, 2, \ldots, \alpha$) that are the linearly independent column vectors in the null space of $Q(\lambda_0)$. (Consequently, a simple eigenvalue λ_0 has a single eigenvector $v_1 \neq 0$ that is the solution of $Q(\lambda_0)v_1 = 0$.) A defective eigenvalue λ_0 with algebraic multiplicity α and geometric multiplicity $\gamma < \alpha$ has γ eigenvectors $v_j \neq 0$ ($j = 1, 2, \ldots, \gamma$). Associated with each partial multiplicity $\mu_k > 1$ of the defective eigenvalue is a *Jordan chain* v_k^ℓ ($\ell = 1, 2, \ldots, \mu_k$) of length μ_k, where $v_k^1 \neq 0$ is an eigenvector and v_k^{j+1} ($j = 1, 2, \ldots, \mu_k - 1$) are *generalized eigenvectors* that satisfy [12]

$$\begin{bmatrix} Q(\lambda_0) & 0 & \cdots & 0 \\ Q^{(1)}(\lambda_0) & Q(\lambda_0) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{Q^{(\mu_k-1)}(\lambda_0)}{(\mu_k-1)!} & \frac{Q^{(\mu_k-2)}(\lambda_0)}{(\mu_k-2)!} & \cdots & Q(\lambda_0) \end{bmatrix} \begin{bmatrix} v_k^1 \\ v_k^2 \\ \vdots \\ v_k^{\mu_k} \end{bmatrix} = 0. \tag{2.1}$$

In (2.1), $Q^{(j)}(\lambda_0)$ denotes the jth derivative of $Q(\lambda)$ with respect to λ that is evaluated at $\lambda = \lambda_0$. For a matrix polynomial with degree greater than 1, the vectors in each Jordan chain need not be linearly independent, and the zero vector is an admissible generalized eigenvector [12].

Analogous results pertain to the reverse polynomial $Q_{\text{rev}}(\lambda)$. Of particular interest is the zero eigenvalue of $Q_{\text{rev}}(\lambda)$ that corresponds to the infinite eigenvalue of $Q(\lambda)$. If the zero eigenvalue of $Q_{\text{rev}}(\lambda)$ is semisimple with algebraic multiplicity α, then the associated eigenvectors $v_j \neq 0$ ($j = 1, 2, \ldots, \alpha$) are the

α linearly independent column vectors in the null space of $Q_{\text{rev}}(0)$. When the zero eigenvalue of $Q_{\text{rev}}(\lambda)$ is defective, the Jordan chain v_k^ℓ $(\ell = 1, 2, \ldots, \mu_k)$ of length μ_k for a $\mu_k \times \mu_k$ Jordan block, with $v_k^1 \neq 0$, satisfies

$$\begin{bmatrix} Q_{\text{rev}}(0) & 0 & \cdots & 0 \\ Q_{\text{rev}}^{(1)}(0) & Q_{\text{rev}}(0) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{Q_{\text{rev}}^{(\mu_k-1)}(0)}{(\mu_k-1)!} & \frac{Q_{\text{rev}}^{(\mu_k-2)}(0)}{(\mu_k-2)!} & \cdots & Q_{\text{rev}}(0) \end{bmatrix} \begin{bmatrix} v_k^1 \\ v_k^2 \\ \vdots \\ v_k^{\mu_k} \end{bmatrix} = 0. \tag{2.2}$$

Lastly, we note that because M, C, and K are real, any finite and nonreal eigenvalues, eigenvectors, and generalized eigenvectors must occur in conjugate pairs. Also, the eigenvectors and generalized eigenvectors associated with real eigenvalues can be, and are, taken to be real. Though the term *nonreal* is more precise than *complex*, we will subsequently use these terms interchangeably; that is, by *complex* we specifically mean a quantity with nonzero imaginary part.

2.2. A strong linearization

In generating an isospectral transformation that decouples (1.1) when M is singular, we must carefully select a $2n \times 2n$ linear pencil $L(\lambda)$ associated with $Q(\lambda) = M\lambda^2 + C\lambda + K$ such that both the finite and infinite eigenvalues of $L(\lambda)$ and their partial multiplicities are the same as those for $Q(\lambda)$. In this case, $L(\lambda)$ is said to be a *strong linearization* of $Q(\lambda)$. Gohberg, Kaashoek, and Lancaster [14] showed that the so-called *first companion form*

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \lambda + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} = L(\lambda)$$

is always a strong linearization of $Q(\lambda)$, though the qualifier *strong* did not appear until later in [16]. While there exist other options for a strong linearization of $Q(\lambda)$ to choose from (see [15]), we adopt the first companion form in our paper because, conveniently, it is associated with the homogeneous form of the $2n$-dimensional first-order realization

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

of coupled system (1.1) that later plays a role in relating a solution $p(t)$ of decoupled system (1.2) to a solution $x(t)$ of (1.1).

3. Isospectrality and spectral data

When can a quadratic pencil $Q(\lambda) = M\lambda^2 + C\lambda + K$ be converted into the diagonal form $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ by a transformation that preserves the eigenvalues (both finite and infinite) and their partial multiplicities? Here, we discuss the conditions for which decoupling via isospectral transformation is possible and introduce a particular indexing and arrangement of the allowable spectral data for $Q(\lambda)$ to generate an explicit and attractive form for $\widetilde{Q}(\lambda)$ that will be convenient as part of a decoupling algorithm.

3.1. Conditions for isospectrality

Lancaster and Zaballa [7] determined the necessary and sufficient conditions for which $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ are isospectral when M is invertible (see Theorems 6 and 7 in [7] and conditions (10), (13), and (15) that they reference), and Zúñiga Anaya [10] later extended these results to consider when M is singular.

Specifically, let λ_j $(j = 1, 2, \ldots, n_d)$ be the distinct finite eigenvalues of $Q(\lambda)$ with partial multiplicities μ_{jk} $(k = 1, 2, \ldots, \gamma_j)$, where $\gamma_j \leq n$ is the geometric multiplicity of each λ_j. The algebraic multiplicity of λ_j is then $\alpha_j = \sum_{k=1}^{\gamma_j} \mu_{jk} \leq 2n$. Let the eigenvalue at infinity have geometric multiplicity $\gamma_\infty \leq n$ and partial multiplicities $\mu_{\infty k}$ $(k = 1, 2, \ldots, \gamma_\infty)$ so that its algebraic multiplicity $\alpha_\infty = \sum_{k=1}^{\gamma_\infty} \mu_{\infty k} \leq 2n$. Zúñiga Anaya then established the following result.

Lemma 3.1. (Theorem 2 of [10]) *Let $Q(\lambda) = M\lambda^2 + C\lambda + K$, with $M, C, K \in \mathbb{C}^{n\times n}$, be a regular quadratic matrix polynomial. There exists a diagonal quadratic pencil $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$, where $A_2, A_1, A_0 \in \mathbb{C}^{n\times n}$, that is isospectral to $Q(\lambda)$ if and only if the following conditions hold:*

$$\sum_{j=1}^{n_d} \alpha_j + \alpha_\infty = 2n,$$

$$\mu_{jk} \in \{1, 2\} \quad (j = 1, 2, \ldots, n_d, \quad k = 1, 2, \ldots, \gamma_j), \tag{3.1}$$

$$\mu_{\infty k} \in \{1, 2\} \quad (k = 1, 2, \ldots, \gamma_\infty), \tag{3.2}$$

$$\frac{\alpha_j}{2} \leq \gamma_j \leq n + \beta_j - \beta_\infty - \hat{\beta} \quad (j = 1, 2, \ldots, n_d), \tag{3.3}$$

$$\frac{\alpha_\infty}{2} \leq \gamma_\infty \leq n - \hat{\beta}, \tag{3.4}$$

where β_j is the number of partial multiplicities $\mu_{jk} = 2$ for the jth distinct eigenvalue λ_j, β_∞ is the number of partial multiplicities $\mu_{\infty k} = 2$ for the eigenvalue at infinity, and $\hat{\beta} = \sum_{j=1}^{n_d} \beta_j$.

Note that the result in Lemma 3.1 concerns $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ with complex coefficients. By employing the same arguments as in the proof of Theorem 7 in [7], Lemma 3.1 can be applied specifically to quadratic matrix polynomials with real coefficients by imposing the additional requirement that all nonreal eigenvalues be semisimple and occur in conjugate pairs. Conditions (3.1)–(3.4) of Lemma 3.1 then pertain to the real eigenvalues and to the eigenvalue at infinity, so it would be prudent to separate the nonreal and real eigenvalues. Let λ_j^c $(j = 1, 2, \ldots, n_c)$ and $\overline{\lambda}_j^c$ be the distinct nonreal eigenvalues of $Q(\lambda)$ and their complex conjugates, respectively, both having partial multiplicities μ_{jk}^c $(k = 1, 2, \ldots, \gamma_j^c)$ and geometric multiplicities $\gamma_j^c \leq n$. The algebraic multiplicity of λ_j^c is then $\alpha_j^c = \sum_{k=1}^{\gamma_j^c} \mu_{jk}^c \leq n$, and the same is true of the complex conjugate. Denote the distinct real eigenvalues of $Q(\lambda)$ as λ_j^r $(j = 1, 2, \ldots, n_r)$. Each λ_j^r has partial multiplicities μ_{jk}^r $(k = 1, 2, \ldots, \gamma_j^r)$, geometric multiplicity $\gamma_j^r \leq n$, and algebraic multiplicity $\alpha_j^r = \sum_{k=1}^{\gamma_j^r} \mu_{jk}^r \leq 2n$. We then arrive at the following modification to Lemma 3.1 by the proof of Theorem 7 in [7].

Theorem 3.2. *Let $Q(\lambda) = M\lambda^2 + C\lambda + K$, with $M, C, K \in \mathbb{R}^{n\times n}$, be a regular quadratic matrix polynomial. There exists a diagonal quadratic pencil $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$, where $A_2, A_1, A_0 \in \mathbb{R}^{n\times n}$, that is isospectral to $Q(\lambda)$ if and only if the following conditions hold:*

$$2\sum_{j=1}^{n_c} \alpha_j^c + \sum_{j=1}^{n_r} \alpha_j^r + \alpha_\infty = 2n,$$

$$\mu_{jk}^c = 1 \quad (j = 1, 2, \ldots, n_c, \quad k = 1, 2, \ldots, \gamma_j^c),$$

$$\mu_{jk}^r \in \{1, 2\} \quad (j = 1, 2, \ldots, n_r, \quad k = 1, 2, \ldots, \gamma_j^r),$$

$$\mu_{\infty k} \in \{1, 2\} \quad (k = 1, 2, \ldots, \gamma_\infty),$$

$$\frac{\alpha_j^r}{2} \leq \gamma_j^r \leq n - \sum_{\ell=1}^{n_c} \alpha_\ell^c + \beta_j^r - \beta_\infty - \hat{\beta}^r \quad (j = 1, 2, \ldots, n_r),$$

$$\frac{\alpha_\infty}{2} \leq \gamma_\infty \leq n - \sum_{\ell=1}^{n_c} \alpha_\ell^c - \hat{\beta}^r,$$

where β_j^r *is the number of partial multiplicities* $\mu_{jk}^r = 2$ *for the* j*th distinct real eigenvalue* λ_j^r, β_∞ *is the number of partial multiplicities* $\mu_{\infty k} = 2$ *for the eigenvalue at infinity, and* $\hat{\beta}^r = \sum_{j=1}^{n_r} \beta_j^r$.

Based on Lancaster and Zaballa's explanations of their proofs of Theorems 6 and 7 in [7] and Zúñiga Anaya's discussion of his proof of Theorem 2 in [10], we can express the conditions of our Theorem 3.2 more conveniently as follows.

Corollary 3.3. *Let* $Q(\lambda) = M\lambda^2 + C\lambda + K$, *with* $M, C, K \in \mathbb{R}^{n\times n}$, *be a regular quadratic matrix polynomial. There exists a diagonal quadratic pencil* $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$, *where* $A_2, A_1, A_0 \in \mathbb{R}^{n\times n}$, *that is isospectral to* $Q(\lambda)$ *if and only if the following conditions hold:*

(i) *all nonreal eigenvalues are semisimple and occur in conjugate pairs;*
(ii) *all Jordan blocks associated with a defective eigenvalue, either real or infinite, are no larger than* 2×2*; and*
(iii) *excluding the nonreal eigenvalues and the* 2×2 *Jordan blocks, all remaining real eigenvalues and infinite eigenvalues, which have unit partial multiplicities, form pairs of distinct eigenvalues.*

3.2. Indexing and arrangement of the spectral data

As expressed in Corollary 3.3, the conditions for isospectrality of $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ impose restrictions on their shared spectrum and partial multiplicities and imply a particular scheme for pairing the eigenvalues. From condition (i), complex conjugate eigenvalues (which must be semisimple) are necessarily paired. By condition (ii), a 2×2 Jordan block of a finite and real eigenvalue or an infinite eigenvalue suggests a natural pairing of the defective eigenvalue with itself. Lastly, condition (iii) results in a pairing of a real eigenvalue of unit partial multiplicity with another, but different, real eigenvalue or an eigenvalue at infinity with a partial multiplicity of 1. This interpretation allows us to characterize the spectrum of a diagonalizable $Q(\lambda)$ as follows.

Let $r = \mathrm{rank}(M) \le n$ so that, in general, $Q(\lambda)$ admits n_f finite eigenvalues and $n_\infty \ge n - r$ eigenvalues at infinity such that $n_f + n_\infty = 2n$. Thus, $n_f \le n + r$. If there are n_1 pairs of complex conjugate eigenvalues, n_2 real Jordan blocks of size 2×2, and n_3 pairs of distinct real eigenvalues with unit partial multiplicities, then we can form at most $r = n_1 + n_2 + n_3$ total pairs for $n_f \le n + r$, leaving $n_f - 2r$ real eigenvalues to pair with $n_f - 2r$ eigenvalues at infinity with partial multiplicity 1. The remaining $(2n - n_f) - (n_f - 2r) = 2(n - n_f + r)$ eigenvalues must correspond to the 2×2 Jordan blocks of the infinite eigenvalue, and thus there are $n - n_f + r$ such blocks.

Now, for the finite eigenvalues, denote the n_1 semisimple complex eigenvalues with positive imaginary part as λ_j $(j = 1, 2, \ldots, n_1)$ and their corresponding eigenvectors as v_j. Specify λ_{r+j} and v_{r+j} as the complex conjugates: $\lambda_{r+j} = \overline{\lambda}_j$ and $v_{r+j} = \overline{v}_j$. Let λ_{n_1+j} $(j = 1, 2, \ldots, n_2)$ be the n_2 repeated and defective real eigenvalues with eigenvectors v_{n_1+j}; the matching eigenvalues $\lambda_{r+n_1+j} = \lambda_{n_1+j}$ have generalized eigenvectors v_{r+n_1+j}. From (2.1), these eigenvectors and generalized eigenvectors satisfy

$$\begin{bmatrix} Q(\lambda_{n_1+j}) & 0 \\ Q^{(1)}(\lambda_{n_1+j}) & Q(\lambda_{n_1+j}) \end{bmatrix} \begin{bmatrix} v_{n_1+j} \\ v_{r+n_1+j} \end{bmatrix} = 0 \tag{3.5}$$

because the Jordan blocks are 2×2. For the n_3 pairs of distinct real eigenvalues with unit partial multiplicities, denote the pairs' larger eigenvalues and the associated eigenvectors as $\lambda_{n_1+n_2+j}$ $(j = 1, 2, \ldots, n_3)$ and $v_{n_1+n_2+j}$, respectively; $\lambda_{r+n_1+n_2+j}$ are the smaller eigenvalues with eigenvectors $v_{r+n_1+n_2+j}$. Let λ_{2r+j} $(j = 1, 2, \ldots, n_f - 2r)$ be the $n_f - 2r$ real eigenvalues paired with infinite eigenvalues and v_{2r+j} be their associated eigenvectors. We can then construct an $n_f \times n_f$ Jordan matrix

$$\begin{aligned} J_{x,f} = &\left[\bigoplus_{j=1}^{n_1}\begin{bmatrix}\lambda_j & 0\\ 0 & \lambda_{r+j}\end{bmatrix}\right] \oplus \left[\bigoplus_{j=1}^{n_2}\begin{bmatrix}\lambda_{n_1+j} & 1\\ 0 & \lambda_{r+n_1+j}\end{bmatrix}\right] \\ &\oplus \left[\bigoplus_{j=1}^{n_3}\begin{bmatrix}\lambda_{n_1+n_2+j} & 0\\ 0 & \lambda_{r+n_1+n_2+j}\end{bmatrix}\right] \oplus \left[\bigoplus_{j=1}^{n_f-2r}\lambda_{2r+j}\right] \end{aligned} \tag{3.6}$$

and, conformable to (3.6), an $n \times n_f$ matrix

$$V_{x,f} = \begin{bmatrix} v_1 & v_{r+1} & v_2 & v_{r+2} & \cdots & v_r & v_{2r} & v_{2r+1} & \cdots & v_{n_f}\end{bmatrix} \tag{3.7}$$

of the corresponding eigenvectors and generalized eigenvectors, where the couple $(V_{x,f}, J_{x,f})$ constitutes a *finite Jordan pair* for $Q(\lambda)$ [11].

For the n_∞ eigenvalues of $Q(\lambda)$ at infinity, which correspond to the n_∞ zero eigenvalues of the reverse polynomial $Q_{\text{rev}}(\lambda) = K\lambda^2 + C\lambda + M$, $n_f - 2r$ of them have unit partial multiplicity, so their eigenvectors v_{n_f+j} $(j = 1, 2, \ldots, n_f - 2r)$ are in the null space of $Q_{\text{rev}}(0) = M$. The remaining $2(n - n_f + r)$ infinite eigenvalues are associated with $n - n_f + r$ Jordan blocks of size 2×2 with eigenvalue 0, and their eigenvectors $v_{2(n_f-r+j)-1}$ $(j = 1, 2, \ldots, n - n_f + r)$ and generalized eigenvectors $v_{2(n_f-r+j)}$ satisfy, from (2.2),

$$\begin{bmatrix} Q_{\text{rev}}(0) & 0\\ Q_{\text{rev}}^{(1)}(0) & Q_{\text{rev}}(0)\end{bmatrix}\begin{bmatrix} v_{2(n_f-r+j)-1}\\ v_{2(n_f-r+j)}\end{bmatrix} = \begin{bmatrix} M & 0\\ C & M\end{bmatrix}\begin{bmatrix} v_{2(n_f-r+j)-1}\\ v_{2(n_f-r+j)}\end{bmatrix} = 0. \tag{3.8}$$

Defining an $n_\infty \times n_\infty$ Jordan matrix

$$J_{x,\infty} = \left[\bigoplus_{j=1}^{n_f-2r} 0\right] \oplus \left[\bigoplus_{j=1}^{n-n_f+r}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right] \tag{3.9}$$

and an $n \times n_\infty$ matrix

$$V_{x,\infty} = \begin{bmatrix} v_{n_f+1}\ v_{n_f+2} \cdots v_{2n}\end{bmatrix} \tag{3.10}$$

of the corresponding eigenvectors and generalized eigenvectors, conformable to (3.9), we then have an *infinite Jordan pair* $(V_{x,\infty}, J_{x,\infty})$ for $Q(\lambda)$ [11].

3.3. The diagonal quadratic matrix polynomial and its spectral data

The eigenvalue pairing scheme associated with the isospectrality conditions of Theorem 3.2 (or, equivalently, Corollary 3.3) plays a critical role in constructing the diagonal quadratic pencil $\widetilde{Q}(\lambda)$ isospectral to $Q(\lambda)$. As explained by Lancaster and Zaballa in [7] and Zúñiga Anaya in [10], the eigenvalue pairs specified by Corollary 3.3 populate the diagonal of $\widetilde{Q}(\lambda)$ in the following manner:

(i) Each of the n_1 pairs of semisimple complex conjugate eigenvalues, λ_j $(j = 1, 2, \ldots, n_1)$ and $\lambda_{r+j} = \overline{\lambda}_j$, corresponds to a quadratic term $(\lambda - \lambda_j)(\lambda - \lambda_{r+j})$. (Note that this quadratic term is the product of two linear *elementary divisors* of $Q(\lambda)$.)

(ii) The n_2 Jordan blocks of size 2×2 for the real and defective eigenvalues $\lambda_{n_1+j} = \lambda_{r+n_1+j}$ $(j = 1, 2, \ldots, n_2)$ are associated with the quadratic terms $(\lambda - \lambda_{n_1+j})(\lambda - \lambda_{r+n_1+j}) = (\lambda - \lambda_{n_1+j})^2$. (These terms are quadratic elementary divisors of $Q(\lambda)$ for λ_{n_1+j}.)

(iii) As with the complex conjugate eigenvalues, the n_3 pairs of distinct real eigenvalues with unit partial multiplicities, $\lambda_{n_1+n_2+j}$ $(j = 1, 2, \ldots, n_3)$ and $\lambda_{r+n_1+n_2+j} \neq \lambda_{n_1+n_2+j}$, correspond to the quadratic terms $(\lambda - \lambda_{n_1+n_2+j})(\lambda - \lambda_{r+n_1+n_2+j})$.

(iv) The $n_f - 2r$ pairings of the real eigenvalues λ_{2r+j} $(j = 1, 2, \ldots, n_f - 2r)$ and an eigenvalue at infinity, both with unit partial multiplicity, are associated with a linear term $\lambda - \lambda_{2r+j}$.

(v) Each of the $n-n_f+r$ Jordan blocks of size 2×2 for the defective infinite eigenvalue corresponds to a constant entry of 1.

Using our particular indexing of the eigenvalues from Sect. 3.2, we can then construct an explicit form for $\widetilde{Q}(\lambda)$ and extract from it expressions for the real coefficients A_2, A_1, and A_0. To do so, first define the following diagonal matrices of the finite eigenvalues:

$$\Lambda_1 = \left[\bigoplus_{j=1}^{n_1}\lambda_j\right] \oplus \left[\bigoplus_{j=1}^{n_2}\lambda_{n_1+j}\right] \oplus \left[\bigoplus_{j=1}^{n_3}\lambda_{n_1+n_2+j}\right] = \bigoplus_{k=1}^{r}\lambda_k, \tag{3.11}$$

$$\Lambda_2 = \left[\bigoplus_{j=1}^{n_1}\lambda_{r+j}\right] \oplus \left[\bigoplus_{j=1}^{n_2}\lambda_{r+n_1+j}\right] \oplus \left[\bigoplus_{j=1}^{n_3}\lambda_{r+n_1+n_2+j}\right] = \bigoplus_{k=1}^{r}\lambda_{r+k}, \tag{3.12}$$

$$\Lambda_3 = \bigoplus_{j=1}^{n_f-2r}\lambda_{2r+j}. \tag{3.13}$$

Consequently, $\widetilde{Q}(\lambda)$ can be written in an attractive form that groups second-, first-, and zeroth-order terms in λ on the diagonal:

$$\begin{aligned}\widetilde{Q}(\lambda) &= \left[\bigoplus_{j=1}^{r}(\lambda-\lambda_j)(\lambda-\lambda_{r+j})\right] \oplus \left[\bigoplus_{j=1}^{n_f-2r}(\lambda-\lambda_{2r+j})\right] \oplus \left[\bigoplus_{j=1}^{n-n_f+r} 1\right] \\ &= (I_r\lambda-\Lambda_1)(I_r\lambda-\Lambda_2) \oplus \left(I_{n_f-2r}\lambda-\Lambda_3\right) \oplus I_{n-n_f+r} \\ &= \left(I_r\lambda^2-(\Lambda_1+\Lambda_2)\lambda+\Lambda_1\Lambda_2\right) \oplus \left(I_{n_f-2r}\lambda-\Lambda_3\right) \oplus I_{n-n_f+r} \\ &= A_2\lambda^2+A_1\lambda+A_0,\end{aligned}$$

and so the coefficients

$$\begin{aligned}A_2 &= I_r \oplus 0_{n-r}, \\ A_1 &= -(\Lambda_1+\Lambda_2) \oplus I_{n_f-2r} \oplus 0_{n-n_f+r}, \\ A_0 &= \Lambda_1\Lambda_2 \oplus -\Lambda_3 \oplus I_{n-n_f+r}.\end{aligned} \tag{3.14}$$

As a result, decoupled system (1.2) is conveniently partitioned such that the first r rows, the next n_f-2r rows, and the last $n-n_f+r$ rows contain second-, first-, and zeroth-order mutually independent equations, respectively.

We are now in a position to generate a finite Jordan pair $(V_{p,f}, J_{p,f})$ and an infinite Jordan pair $(V_{p,\infty}, J_{p,\infty})$ for $\widetilde{Q}(\lambda)$. First, we note that $J_{p,f}=J_{x,f}$ of (3.6) and $J_{p,\infty}=J_{x,\infty}$ in (3.9) because $\widetilde{Q}(\lambda)$ is isospectral to $Q(\lambda)$. Next, let e_j $(j=1,2,\ldots,n)$ be an n-dimensional column vector of zeros except for a 1 in the jth row. Because $\widetilde{Q}(\lambda)$ is diagonal, we can take e_j as an eigenvector of an eigenvalue that appears in the jth diagonal entry of $\widetilde{Q}(\lambda)$. Moreover, when a diagonal entry corresponds to a 2×2 Jordan block of a real eigenvalue, it is straightforward to confirm that we can always choose 0 as a generalized eigenvector of the defective eigenvalue. Using the notation $\operatorname{row}(X_j)_{j=1}^{\alpha} = \left[X_1\ X_2\ \cdots\ X_\alpha\right]$ to represent a block-row matrix, the $n\times n_f$ matrix $V_{p,f}$ of eigenvectors and generalized eigenvectors conformable to the Jordan matrix $J_{p,f}$ is then

$$\begin{aligned}V_{p,f} = \Big[&\operatorname{row}\left(\left[e_j \quad e_j\right]\right)_{j=1}^{n_1}, \quad \operatorname{row}\left(\left[e_{n_1+j} \quad 0\right]\right)_{j=1}^{n_2}, \\ &\operatorname{row}\left(\left[e_{n_1+n_2+j} \quad e_{n_1+n_2+j}\right]\right)_{j=1}^{n_3}, \quad \operatorname{row}\left(e_{r+j}\right)_{j=1}^{n_f-2r}\Big]\end{aligned}$$

or, equivalently,

$$V_{p,f} = \begin{bmatrix} \left[\bigoplus_{j=1}^{n_1}\begin{bmatrix}1 & 1\end{bmatrix}\right] \oplus \left[\bigoplus_{j=1}^{n_2}\begin{bmatrix}1 & 0\end{bmatrix}\right] \oplus \left[\bigoplus_{j=1}^{n_3}\begin{bmatrix}1 & 1\end{bmatrix}\right] \oplus I_{n_f-2r} \\ 0_{(n-n_f+r)\times n_f} \end{bmatrix}. \tag{3.15}$$

Now, the n_∞ eigenvalues of $\widetilde{Q}(\lambda)$ at infinity, which correspond to the n_∞ zero eigenvalues of the reversal $\widetilde{Q}_{\text{rev}}(\lambda) = A_0\lambda^2 + A_1\lambda + A_0$, have $n-r$ eigenvectors that are in the null space of $\widetilde{Q}_{\text{rev}}(0) = A_2$. From the structure of A_2 in (3.14), it is clear that these eigenvectors can be taken as e_{r+j} $(j = 1, 2, \ldots, n-r)$. Of these eigenvectors, $n - n_f + r$ are associated with the infinite eigenvalue's $n - n_f + r$ Jordan blocks of size 2×2. Analogous to (3.8), these $n - n_f + r$ eigenvectors w_{2k-1} $(k = 1, 2, \ldots, n - n_f + r)$ and the corresponding generalized eigenvectors w_{2k} satisfy

$$\begin{bmatrix} \widetilde{Q}_{\text{rev}}(0) & 0 \\ \widetilde{Q}^{(1)}_{\text{rev}}(0) & \widetilde{Q}_{\text{rev}}(0) \end{bmatrix} \begin{bmatrix} w_{2k-1} \\ w_{2k} \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} w_{2k-1} \\ w_{2k} \end{bmatrix} = 0. \tag{3.16}$$

Because of the form of A_1 given in (3.14), w_{2k-1} cannot be any of the first $n_f - 2r$ eigenvectors $e_{r+\ell}$ $(\ell = 1, 2, \ldots, n_f - 2r)$ for $A_1 w_{2k-1} + A_2 w_{2k} = 0$ of (3.16) to have finite solutions w_{2k}, so we must have $w_{2k-1} = e_{n_f - r + k}$. It is simple to verify that we can then choose $w_{2k} = 0$ as generalized eigenvectors so that (3.16) holds. Therefore, conformable to the Jordan matrix $J_{p,\infty}$, the $n \times n_\infty$ matrix $V_{p,\infty}$ of the associated eigenvectors and generalized eigenvectors is

$$V_{p,\infty} = \left[\text{row}\,(e_{r+j})_{j=1}^{n_f-2r}, \quad \text{row}\left(\begin{bmatrix} e_{n_f-r+j} & 0 \end{bmatrix}\right)_{j=1}^{n-n_f+r}\right].$$

Alternatively,

$$V_{p,\infty} = \begin{bmatrix} 0_{r\times n_\infty} \\ I_{n_f-2r} \oplus \left[\bigoplus_{j=1}^{n-n_f+r} \begin{bmatrix}1 & 0\end{bmatrix}\right] \end{bmatrix}. \tag{3.17}$$

4. A diagonalizing structure-preserving transformation

We now establish a transformation that converts an $n \times n$ quadratic matrix polynomial $Q(\lambda) = M\lambda^2 + C\lambda + K$ into the diagonal pencil $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ isospectral to $Q(\lambda)$ through their first companion forms. We define this particular type of diagonalizing structure-preserving transformation (DSPT) as follows.

Definition 4.1. Let the $n \times n$ quadratic matrix polynomials $Q(\lambda) = M\lambda^2 + C\lambda + K$ and $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ be regular and isospectral. If

$$L(\lambda) = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\lambda + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} \quad \text{and} \quad \widetilde{L}(\lambda) = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}\lambda + \begin{bmatrix} 0 & -I \\ A_0 & A_1 \end{bmatrix}$$

are the first companion forms of $Q(\lambda)$ and $\widetilde{Q}(\lambda)$, respectively, and (R, S) is a pair of real and invertible $2n \times 2n$ matrices satisfying

$$R\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}S = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \tag{4.1}$$

and

$$R\begin{bmatrix} 0 & -I \\ K & C \end{bmatrix}S = \begin{bmatrix} 0 & -I \\ A_0 & A_1 \end{bmatrix}, \tag{4.2}$$

then (R, S) will be called a *diagonalizing first companion structure-preserving transformation* from $Q(\lambda)$ to $\widetilde{Q}(\lambda)$.

Because this is the only type of DSPT we consider, we will often omit reference to the first companion form for convenience. To determine the transformation matrices R and S that comprise a DSPT, we utilize the notion of a *decomposable pair* of a regular pencil. Based on Section 7.3 of [11], we have the following definition pertaining to an $n \times n$ quadratic pencil.

Definition 4.2. Let $Q(\lambda) = M\lambda^2 + C\lambda + K$, where $M, C, K \in \mathbb{C}^{n\times n}$, be a regular quadratic matrix polynomial. The couple $\left(\begin{bmatrix} X_1 \; X_2 \end{bmatrix}, T_1 \oplus T_2\right)$ is called a *decomposable pair* of $Q(\lambda)$ if the following conditions hold:

(i) the matrices $X_1 \in \mathbb{C}^{n\times\hat{n}}$, $T_1 \in \mathbb{C}^{\hat{n}\times\hat{n}}$, $X_2 \in \mathbb{C}^{n\times(2n-\hat{n})}$, and $T_2 \in \mathbb{C}^{(2n-\hat{n})\times(2n-\hat{n})}$, with $0 \le \hat{n} \le 2n$, such that $\begin{bmatrix} X_1 \; X_2 \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ and $(T_1 \oplus T_2) \in \mathbb{C}^{2n\times 2n}$;

(ii) the matrix $\begin{bmatrix} X_1 & X_2T_2 \\ X_1T_1 & X_2 \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ is invertible; and

(iii) $MX_1T_1^2 + CX_1T_1 + KX_1 = 0$ and $KX_2T_2^2 + CX_2T_2 + MX_2 = 0$.

Conveniently, we can use spectral data to generate a decomposable pair for $Q(\lambda)$. Specifically, by Theorem 7.3 in [11], if $(V_{x,f}, J_{x,f})$ and $(V_{x,\infty}, J_{x,\infty})$ are, respectively, finite and infinite Jordan pairs for $Q(\lambda)$, then $\left(\begin{bmatrix} V_{x,f} \; V_{x,\infty} \end{bmatrix}, J_{x,f} \oplus J_{x,\infty}\right)$ is a decomposable pair for $Q(\lambda)$. We next make use of the following result.

Lemma 4.3. *Let $Q(\lambda) = M\lambda^2 + C\lambda + K$ be a regular quadratic matrix polynomial with finite Jordan pair $(V_{x,f}, J_{x,f})$, infinite Jordan pair $(V_{x,\infty}, J_{x,\infty})$, and first companion form $L(\lambda)$. Then $J(\lambda) = (I\lambda - J_{x,f}) \oplus (J_{x,\infty}\lambda - I)$ is a strong linearization of $Q(\lambda)$ and $J(\lambda) = R_x^{-1}L(\lambda)S_x$, where*

$$R_x = \begin{bmatrix} V_{x,f} & V_{x,\infty} \\ MV_{x,f}J_{x,f} & -KV_{x,\infty}J_{x,\infty} - CV_{x,\infty} \end{bmatrix} \quad \textit{and} \quad S_x = \begin{bmatrix} V_{x,f} & V_{x,\infty}J_{x,\infty} \\ V_{x,f}J_{x,f} & V_{x,\infty} \end{bmatrix}.$$

Proof. Lemma 4.3 follows immediately from specializing Theorem 7.6 in [11] to the case of a quadratic pencil and then applying Theorem 7.3 of [11]. □

Of course, Lemma 4.3 also applies to the diagonal pencil $\widetilde{Q}(\lambda)$. We are now ready to prove the following statement, which is one of the main results of this paper.

Theorem 4.4. *Let $Q(\lambda) = M\lambda^2 + C\lambda + K$ and $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ be regular and isospectral quadratic matrix polynomials with first companion forms $L(\lambda)$ and $\widetilde{L}(\lambda)$, respectively, where $(V_{x,f}, J_{x,f})$ and $(V_{x,\infty}, J_{x,\infty})$ are finite and infinite Jordan pairs for $Q(\lambda)$, and $(V_{p,f}, J_{p,f})$ and $(V_{p,\infty}, J_{p,\infty})$ are finite and infinite Jordan pairs for $\widetilde{Q}(\lambda)$. Then (R, S), where*

$$R = \begin{bmatrix} V_{p,f} & V_{p,\infty} \\ A_2V_{p,f}J_{p,f} & -A_0V_{p,\infty}J_{p,\infty} - A_1V_{p,\infty} \end{bmatrix} \begin{bmatrix} V_{x,f} & V_{x,\infty} \\ MV_{x,f}J_{x,f} & -KV_{x,\infty}J_{x,\infty} - CV_{x,\infty} \end{bmatrix}^{-1} \tag{4.3}$$

and

$$S = \begin{bmatrix} V_{x,f} & V_{x,\infty}J_{x,\infty} \\ V_{x,f}J_{x,f} & V_{x,\infty} \end{bmatrix} \begin{bmatrix} V_{p,f} & V_{p,\infty}J_{p,\infty} \\ V_{p,f}J_{p,f} & V_{p,\infty} \end{bmatrix}^{-1}, \tag{4.4}$$

is a diagonalizing first companion structure-preserving transformation from $Q(\lambda)$ to $\widetilde{Q}(\lambda)$.

Proof. By Lemma 4.3, for $Q(\lambda)$, we have $J(\lambda) = R_x^{-1}L(\lambda)S_x$, where

$$R_x = \begin{bmatrix} V_{x,f} & V_{x,\infty} \\ MV_{x,f}J_{x,f} & -KV_{x,\infty}J_{x,\infty} - CV_{x,\infty} \end{bmatrix} \quad \text{and} \quad S_x = \begin{bmatrix} V_{x,f} & V_{x,\infty}J_{x,\infty} \\ V_{x,f}J_{x,f} & V_{x,\infty} \end{bmatrix}$$

are invertible. Likewise, $\widetilde{J}(\lambda) = R_p^{-1}\widetilde{L}(\lambda)S_p$ for $\widetilde{Q}(\lambda)$, where the invertible matrices

$$R_p = \begin{bmatrix} V_{p,f} & V_{p,\infty} \\ A_2V_{p,f}J_{p,f} & -A_0V_{p,\infty}J_{p,\infty} - A_1V_{p,\infty} \end{bmatrix} \quad \text{and} \quad S_p = \begin{bmatrix} V_{p,f} & V_{p,\infty}J_{p,\infty} \\ V_{p,f}J_{p,f} & V_{p,\infty} \end{bmatrix}.$$

Because $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ are isospectral, the Jordan matrices $J_{x,f} = J_{p,f}$ and $J_{x,\infty} = J_{p,\infty}$, so $J(\lambda) = \widetilde{J}(\lambda)$. Consequently, $R_x^{-1}L(\lambda)S_x = R_p^{-1}\widetilde{L}(\lambda)S_p$. Equivalently, $RL(\lambda)S = \widetilde{L}(\lambda)$ with $R = R_pR_x^{-1}$ and $S = S_xS_p^{-1}$ as in (4.3) and (4.4), respectively. Invertibility of R and S is clear. Moreover, R and S must be real because the coefficients of $L(\lambda)$ and $\widetilde{L}(\lambda)$ are real. Thus, by Definition 4.1, (R, S) constitutes a diagonalizing first companion structure-preserving transformation from $Q(\lambda)$ to $\widetilde{Q}(\lambda)$. □

We should note that Theorem 4.4 is consistent with previously established results for isospectral transformation of a quadratic pencil $Q(\lambda)$ with M nonsingular. Specifically, when M is invertible (and thus A_2 is nonsingular), the linear pencils

$$L_1(\lambda) = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}^{-1} L(\lambda) = \lambda I + \begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}$$

and

$$\widetilde{L}_1(\lambda) = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}^{-1} \widetilde{L}(\lambda) = \lambda I + \begin{bmatrix} 0 & -I \\ A_2^{-1}A_0 & A_2^{-1}A_1 \end{bmatrix}$$

are strictly equivalent to the first companion forms $L(\lambda)$ and $\widetilde{L}(\lambda)$, respectively. Therefore, $L_1(\lambda)$ is a linearization of $Q(\lambda)$, and $\widetilde{L}_1(\lambda)$ is a linearization of $\widetilde{Q}(\lambda)$. In addition, the transformation matrices R and S in (4.3) and (4.4), respectively, reduce to

$$R = \begin{bmatrix} V_{p,f} \\ A_2V_{p,f}J_{p,f} \end{bmatrix} \begin{bmatrix} V_{x,f} \\ MV_{x,f}J_{x,f} \end{bmatrix}^{-1} \quad \text{and} \quad S = \begin{bmatrix} V_{x,f} \\ V_{x,f}J_{x,f} \end{bmatrix} \begin{bmatrix} V_{p,f} \\ V_{p,f}J_{p,f} \end{bmatrix}^{-1}$$

because the shared eigenvalues of $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ are all finite. Now, notice that

$$R = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} V_{p,f} \\ V_{p,f}J_{p,f} \end{bmatrix} \begin{bmatrix} V_{x,f} \\ V_{x,f}J_{x,f} \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} S^{-1} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}^{-1},$$

in which case condition (4.1) in Definition 4.1 for a DSPT is identically satisfied. Also,

$$R \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} S = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} S^{-1} \begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix} S$$

so that condition (4.2) of Definition 4.1 becomes

$$\begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix} S = S \begin{bmatrix} 0 & -I \\ A_2^{-1}A_0 & A_2^{-1}A_1 \end{bmatrix},$$

at which point we obtain what Garvey et al. call a *right block-companion structure-preserving transformation* in [6], and this transformation is also diagonalizing in this case.

5. Connecting the solutions of the coupled and decoupled systems

By Theorem 4.4, we now have a means of demonstrating that it is possible to recover a solution $x(t)$ of coupled system (1.1) from a solution $p(t)$ of decoupled system (1.2), when the inhomogeneity

$$g(t) = \left(A_1R_2 + R_4 + A_2R_2 \frac{\mathrm{d}}{\mathrm{d}t} \right) f(t), \tag{5.1}$$

by the real, time-dependent, and nonlinear mapping

$$x(t) = \left(S_1 + S_2 \frac{\mathrm{d}}{\mathrm{d}t} \right) p(t) - S_2R_2 f(t). \tag{5.2}$$

In (5.1) and (5.2), R_j $(j = 1, 2, 3, 4)$ and S_j represent the $n \times n$ blocks of the transformation matrices R and S defined in (4.3) and (4.4), respectively:

$$R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}.$$

We note that hypothesized inhomogeneity (5.1) and mapping (5.2) from $p(t)$ to $x(t)$ are inspired by our previous work in decoupling when M is invertible that uses a different method than the one we present here (see [9] in particular). For our forthcoming proof, it is useful to first observe that if

$$R^{-1} = P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

where P_j $(j = 1, 2, 3, 4)$ are $n \times n$ blocks, then the statement $PR = I$ contains the identities

$$P_1 R_2 + P_2 R_4 = 0 \quad \text{and} \quad P_3 R_2 + P_4 R_4 = I. \tag{5.3}$$

Theorem 5.1. *Let $Q(\lambda) = M\lambda^2 + C\lambda + K$ and $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ be regular and isospectral quadratic matrix polynomials with first companion forms $L(\lambda)$ and $\widetilde{L}(\lambda)$, respectively, where $(V_{x,f}, J_{x,f})$ and $(V_{x,\infty}, J_{x,\infty})$ are finite and infinite Jordan pairs for $Q(\lambda)$, and $(V_{p,f}, J_{p,f})$ and $(V_{p,\infty}, J_{p,\infty})$ are finite and infinite Jordan pairs for $\widetilde{Q}(\lambda)$. If $p(t)$ is a solution of the decoupled system $A_2\ddot{p}(t) + A_1\dot{p}(t) + A_0 p(t) = \left(A_1R_2 + R_4 + A_2R_2 \frac{\mathrm{d}}{\mathrm{d}t}\right) f(t)$ and $x(t) = \left(S_1 + S_2 \frac{\mathrm{d}}{\mathrm{d}t}\right) p(t) - S_2R_2 f(t)$, then $x(t)$ is a solution of the coupled system $M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)$.*

Proof. If $p(t)$ is a solution of $A_2\ddot{p}(t) + A_1\dot{p}(t) + A_0 p(t) = \left(A_1R_2 + R_4 + A_2R_2 \frac{\mathrm{d}}{\mathrm{d}t}\right) f(t)$, then it must also satisfy the first-order realization

$$\begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ A_0 & A_1 \end{bmatrix} \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \left(A_1R_2 + R_4 + A_2R_2 \frac{\mathrm{d}}{\mathrm{d}t}\right) f(t) \end{bmatrix}. \tag{5.4}$$

Define a function $y(t) = \left(S_3 + S_4 \frac{\mathrm{d}}{\mathrm{d}t}\right) p(t) - S_4R_2 f(t)$ so that it and $x(t) = \left(S_1 + S_2 \frac{\mathrm{d}}{\mathrm{d}t}\right) p(t) - S_2R_2 f(t)$ are jointly expressed as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} p(t) \\ \dot{p}(t) - R_2 f(t) \end{bmatrix}. \tag{5.5}$$

Apply the inverse of transformation (5.5) to (5.4) and multiply the resulting equation on the left by $R^{-1} = P$ to obtain, with some manipulation,

$$R^{-1} \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} S^{-1} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + R^{-1} \begin{bmatrix} 0 & -I \\ A_0 & A_1 \end{bmatrix} S^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (P_1R_2 + P_2R_4) f(t) \\ (P_3R_2 + P_4R_4) f(t) \end{bmatrix}. \tag{5.6}$$

By Theorem 4.4, (R, S) is a DSPT, so (5.6) becomes

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \tag{5.7}$$

where we have used the identities in (5.3). The upper and lower halves of (5.7) give $y(t) = \dot{x}(t)$ and $M\dot{y}(t) + Cy(t) + Kx(t) = f(t)$, respectively. Therefore, $x(t)$ is a solution of $M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)$. □

An immediate consequence of the proof of Theorem 5.1 is that, from (5.5), the relationship between solutions $p(t)$ and $x(t)$ of (1.2) and (1.1), respectively, and their derivatives can be expressed simultaneously and compactly as

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = S \begin{bmatrix} p(t) \\ \dot{p}(t) - R_2 f(t) \end{bmatrix},$$

which allows us to conveniently obtain the decoupled system's initial values $p(0)$ and $\dot{p}(0)$ from those of the coupled system, $x(0)$ and $\dot{x}(0)$:

$$\begin{bmatrix} p(0) \\ \dot{p}(0) \end{bmatrix} = S^{-1} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} + \begin{bmatrix} 0 \\ R_2 f(0) \end{bmatrix}. \tag{5.8}$$

6. A decoupling algorithm

Using our earlier developments, we present here an algorithm for converting coupled system (1.1) into diagonal form (1.2) by an isospectral transformation, when possible, and then recovering a solution $x(t)$ of (1.1) from a solution $p(t)$ of (1.2). Our algorithm is as follows:

1. We must first verify that (1.1) can in fact be decoupled by an isospectral transformation. To do so, determine the eigenvalues of the quadratic matrix polynomial $Q(\lambda) = M\lambda^2 + C\lambda + K$ and their partial multiplicities, and then check that the isospectrality conditions of Theorem 3.2 (or, alternatively, Corollary 3.3) are satisfied. If not, then isospectral decoupling of (1.1) is not possible.
2. After confirming that decoupling of (1.1) by isospectral transformation is possible, compute the eigenvectors of $Q(\lambda)$ and any Jordan chains using (3.5) and (3.8).
3. Next, index the eigenvalues, eigenvectors, and generalized eigenvectors as described in Sect. 3.2. Form the diagonal matrices Λ_1, Λ_2, and Λ_3 in (3.11)–(3.13) to obtain the decoupled system's real and diagonal coefficients A_2, A_1, and A_0 given in (3.14), and construct the finite and infinite Jordan pairs for both $Q(\lambda)$ and the diagonal pencil $\widetilde{Q}(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$ using (3.6), (3.7), (3.9), (3.10), (3.15), and (3.17).
4. If the coupled system's inhomogeneity $f(t) = 0$, then $g(t) = 0$ for the decoupled system, so decoupling is complete. To obtain the solution $x(t)$ of the homogeneous coupled system, calculate the transformation matrix S in (4.4), use (5.8) to determine the initial values of the homogeneous decoupled system, compute the solution $p(t)$, and then recover $x(t)$ from $p(t)$ using transformation (5.2).
5. If $f(t) \neq 0$, then compute the transformation matrix R in (4.3) to form the decoupled system's inhomogeneity $g(t)$ from (5.1), at which point decoupling of coupled system (1.1) is complete. If the solution $x(t)$ of (1.1) is desired, first calculate S in (4.4) to obtain the initial values of decoupled system (1.2) using (5.8), then solve for $p(t)$, and use transformation (5.2) to map $p(t)$ to $x(t)$.

This algorithm is summarized diagrammatically as a flowchart in Fig. 1.

We should note that for a particular system (1.1), decoupled system (1.2) isospectral to it is not necessarily unique because there might be multiple options for generating pairs of distinct real eigenvalues or pairs of real eigenvalues and eigenvalues at infinity, all with unit partial multiplicities. Different pairing schemes result in different forms for (1.2), but they are all members of an equivalence class because (1.1) and (1.2) are isospectral. Put another way, the different solutions of the different decoupled systems all yield the same solution to (1.1). For a particular choice of pairing, (1.2) is unique up to an arbitrary nonzero scaling of the eigenvectors and a permutation of the mutually independent component equations: scaling the eigenvectors has no effect on the homogeneous part of (1.2)—which, as shown in (3.14), is constructed solely from the eigenvalues—and the order in which the component equations appear is ultimately unimportant, though the partitioning of (1.2) into second-, first-, and zeroth-order equations by our decoupling algorithm is attractive. Theoretically, the choice of pairing scheme and eigenvector scaling is irrelevant to the decoupling process, but some choices might be better than others from a computational standpoint or simply for convenience. For the sake of generality, we do not advocate any particular pairing or normalization scheme here, but we refer the reader interested in suggested strategies to [17]. A related discussion of this nonuniqueness in decoupling can be found in Lancaster and Zaballa's work [5] on parameterizing structure-preserving transformations when M is invertible.

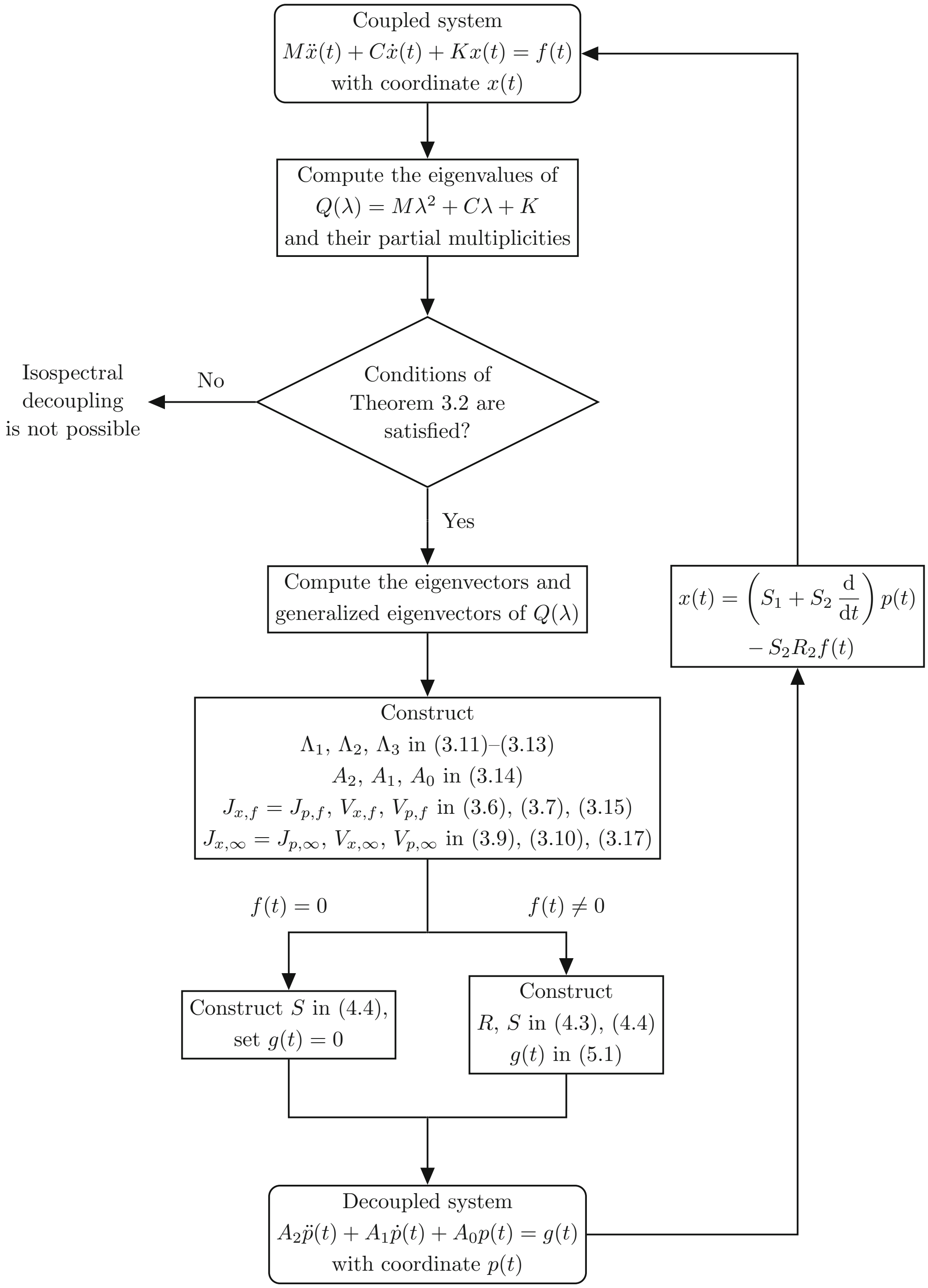

Fig. 1. Flowchart for decoupling a second-order linear system by an isospectral transformation and recovering its solution from that of the decoupled system

7. Illustrative examples

We now provide two examples to illustrate the use of our decoupling algorithm formulated in Sect. 6 and to further discuss the issue of nonuniqueness in decoupling.

Example 1. In this example, we demonstrate the entire process of solving (1.1) by decoupling. Consider the system

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & -1 & 1\end{bmatrix}\ddot{x}(t)+\begin{bmatrix}2 & 1 & 0\\-1 & 1 & 0\\0 & -1 & 2\end{bmatrix}\dot{x}(t)+\begin{bmatrix}1 & 2 & 0\\-1 & 2 & 0\\0 & 0 & 2\end{bmatrix}x(t)=\begin{bmatrix}2\cos t\\\sin 3t\\0\end{bmatrix} \tag{7.1}$$

with consistent initial values $x(0)=[1,0,-1]^T$ and $\dot{x}(0)=[1,2,-1]^T$. We find that the spectrum of its associated quadratic matrix polynomial consists of a pair of complex conjugate eigenvalues, $-1+\mathrm{i}$ and $-1-\mathrm{i}$; a simple real eigenvalue, -1; a repeated real eigenvalue, -2, that occurs twice and has one eigenvector (i.e., it has a partial multiplicity of 2); and a simple eigenvalue at infinity. It is clear via Corollary 3.3 that the conditions of Theorem 3.2 are satisfied, so system (7.1) can be decoupled by an isospectral transformation. In fact, there is only one set of admissible pairings for decoupling. Specifically, the decoupled system must contain one second-order independent equation that corresponds to the complex conjugate pair $(-1+\mathrm{i},-1-\mathrm{i})$; another second-order equation that is associated with the 2×2 Jordan block for the real eigenvalue -2 (that is, a pairing of -2 with itself); and one first-order equation that arises from pairing the simple real eigenvalue, -1, with the infinite eigenvalue. Consequently, upon indexing the eigenvalues as described in Sect. 3.2, we have

$$\Lambda_1=\begin{bmatrix}-1+\mathrm{i} & 0\\0 & -2\end{bmatrix},\quad \Lambda_2=\begin{bmatrix}-1-\mathrm{i} & 0\\0 & -2\end{bmatrix},\quad \text{and}\quad \Lambda_3=-1$$

from (3.11)–(3.13), and so, by (3.14), the decoupled system's coefficients are

$$A_2=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix},\quad A_1=\begin{bmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 1\end{bmatrix},\quad \text{and}\quad A_0=\begin{bmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 1\end{bmatrix}.$$

Using (3.6) and (3.9), the Jordan matrices

$$J_{x,f}=\begin{bmatrix}-1+\mathrm{i} & 0 & 0 & 0 & 0\\0 & -1-\mathrm{i} & 0 & 0 & 0\\0 & 0 & -2 & 1 & 0\\0 & 0 & 0 & -2 & 0\\0 & 0 & 0 & 0 & -1\end{bmatrix}\quad \text{and}\quad J_{x,\infty}=0,$$

and the corresponding eigenvectors and Jordan chain are such that, from (3.7) and (3.10),

$$V_{x,f}=\begin{bmatrix}0 & 0 & 0 & -1 & 1\\0 & 0 & 1 & 0 & 0\\1 & 1 & 1 & -1/2 & 0\end{bmatrix}\quad \text{and}\quad V_{x,\infty}=\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Associated with the decoupled system, $J_{p,f}=J_{x,f}$ and $J_{p,\infty}=J_{x,\infty}$, and

$$V_{p,f}=\begin{bmatrix}1 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}\quad \text{and}\quad V_{p,\infty}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

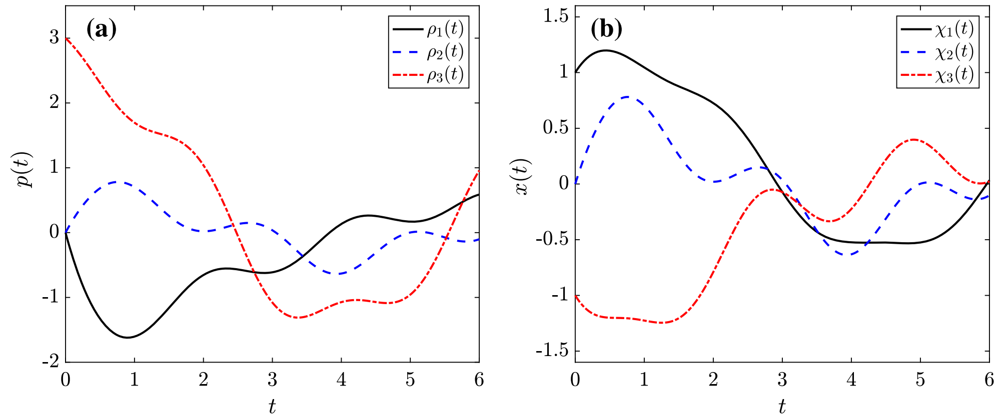

FIG. 2. Solution of **a** decoupled system (7.2) and **b** coupled system (7.1) in Example 1

by (3.15) and (3.17). Consequently, from (4.3) and (4.4), we have the DSPT

$$R = \begin{bmatrix} 1/2 & -1 & 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & -2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 4 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 & 1 \end{bmatrix}$$

that decouples system (7.1) into

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{p}(t) + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{p}(t) + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} p(t) = \begin{bmatrix} -\sin t - \sin 3t - (3/2)\cos 3t \\ 2\cos t + \sin 3t + 3\cos 3t \\ 2\cos t - \sin 3t \end{bmatrix}, \tag{7.2}$$

for which the consistent initial values are $p(0) = [0, 0, 3]^T$ and $\dot{p}(0) = [-4, 2, -1]^T$ using (5.8) and the inhomogeneity was generated by (5.1). Figure 2a depicts the solution $p(t)$ of (7.2), where we have denoted the components of $p(t)$ as $\rho_j(t)$ $(j = 1, 2, 3)$. The solution $x(t)$ of (7.1), with components $\chi_j(t)$, is recovered using (5.2) and illustrated in Fig. 2b.

Example 2. Here, we elaborate on the nonuniqueness of decoupled system (1.2) when there are multiple options for pairing distinct real eigenvalues and pairing real eigenvalues with eigenvalues at infinity. We also demonstrate how scaling of the eigenvectors affects the form of (1.2). Suppose we have the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} x(t) = \begin{bmatrix} \cos 3t \\ -\sin t \\ \sin 2t \end{bmatrix}, \tag{7.3}$$

where $x(0) = [1, 0, -1]^T$ and $\dot{x}(0) = [1, 1, 0]^T$ are consistent initial values. The associated quadratic pencil's spectrum comprises three simple and real eigenvalues (0, -1, and -2) and a defective infinite eigenvalue that occurs three times and has partial multiplicities 1 and 2. Using Corollary 3.3, it is straightforward to see that this spectrum satisfies the conditions of Theorem 3.2, and thus it is possible to decouple system (7.3) by an isospectral transformation. In this case, the decoupled system must consist of one second-order independent equation that corresponds to a pairing of two of the real eigenvalues; one first-order equation that arises from pairing the remaining real eigenvalue with the eigenvalue at infinity that has unit partial multiplicity; and one zeroth-order equation associated with the infinite eigenvalue's

2×2 Jordan block. However, the decoupled system isospectral to (7.3) is not unique because there are three possible pairing schemes involving the real eigenvalues and the infinite eigenvalue with unit partial multiplicity: $(0,-1)$ and $(-2,\infty)$; $(0,-2)$ and $(-1,\infty)$; and $(-1,-2)$ and $(0,\infty)$. We will concentrate on the first two pairing schemes.

For the first option, $(0,-1)$ and $(-2,\infty)$, if we index the eigenvalues per Sect. 3.2 and take the corresponding eigenvectors and generalized eigenvectors such that

$$J_{x,f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad V_{x,f} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & -3 \end{bmatrix},$$

$$J_{x,\infty} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad V_{x,\infty} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

then we arrive at the following decoupled system after a series of calculations from our decoupling algorithm in Sect. 6:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{p}(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{p}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} p(t) = \frac{1}{2} \begin{bmatrix} \sin t + \cos t - 4\cos 2t - 9\sin 3t + \cos 3t \\ -\sin t - \cos 3t \\ 2\sin 2t \end{bmatrix} \tag{7.4}$$

with consistent initial values $p(0) = [5/2, -1/2, 0]^T$ and $\dot{p}(0) = [1/2, 1/2, 2]^T$. The DSPT that relates systems (7.3) and (7.4) is governed by

$$R = \begin{bmatrix} 1 & 1 & 0 & 3/2 & -1/2 & -1 \\ 0 & 0 & 0 & -1/2 & -1/2 & 1 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & -1 & 0 & 1 \end{bmatrix}. \tag{7.5}$$

Alternatively, using the second pairing option, $(0,-2)$ and $(-1,\infty)$, with the same eigenvectors (but reordered to reflect the change in pairing scheme so that

$$J_{x,f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad V_{x,f} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & -3 & -1 \end{bmatrix}$$

instead) gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{p}(t) + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{p}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} p(t) = \begin{bmatrix} \sin t + \cos 3t \\ -\sin 2t + \cos 3t \\ \sin 2t \end{bmatrix}, \tag{7.6}$$

where $p(0) = [1, 1, 0]^T$ and $\dot{p}(0) = [1, 0, 2]^T$, and the associated DSPT is now

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{bmatrix}. \tag{7.7}$$

Therefore, the choice of pairing scheme affects all aspects of the decoupled system: the coefficient matrices, the inhomogeneity, and the initial values. Although the solutions $p(t)$ of the two decoupled systems, (7.4) and (7.6), are different, they both generate the same solution $x(t)$ to (7.3) through (5.2) because the transformation matrices R and S change accordingly with the choice of pairing, as evidenced by (7.5) and (7.7).

Sticking with the second pairing scheme, suppose we now scale some of the eigenvectors for the finite real eigenvalues differently:

$$V_{x,f} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ -2 & -3 & 1 \end{bmatrix},$$

in which case the decoupled system takes the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{p}(t) + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{p}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} p(t) = \frac{1}{4} \begin{bmatrix} 2\sin t - \cos t + 3\sin 3t + 2\cos 3t \\ 4\sin 2t - 4\cos 3t \\ 4\sin 2t \end{bmatrix} \tag{7.8}$$

with initial values $p(0) = [1/4, -1, 0]^T$ and $\dot{p}(0) = [3/4, 0, 2]^T$. Instead of (7.7), the DSPT relating systems (7.3) and (7.8) is

$$R = \begin{bmatrix} 1/2 & 0 & 0 & -1/4 & 1/4 & 0 \\ 0 & -1 & 0 & -1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -2 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{bmatrix}. \tag{7.9}$$

Thus, for the particular eigenvalue pairing scheme we used, the choice of eigenvector scaling affects the decoupled system's inhomogeneity and initial values but not the coefficient matrices, which depend only on the eigenvalues. Because the homogeneous parts of (7.6) and (7.8) are identical, we regard the two systems as being equivalent. Their solutions are different because the inhomogeneity and initial values differ, but they both yield the same solution to (7.3) via (5.2) because the transformation matrices R and S change to compensate for the difference in scaling of the eigenvectors, which (7.7) and (7.9) demonstrate.

8. Concluding remarks

We have shown that *any* real second-order linear system (1.1) with nonzero leading coefficient and whose associated spectrum satisfies the conditions of Theorem 3.2 can be converted into a real diagonal form (1.2) of the same dimension by a transformation that preserves the eigenvalues and their partial multiplicities. In general, the isospectral decoupled system contains a mixture of second-, first-, and zeroth-order independent scalar equations. Decoupling is made possible by a real DSPT, which can be constructed from spectral data, that relates the first companion linearizations associated with isospectral systems (1.1) and (1.2); this DSPT reduces to a previously established decoupling transformation when the leading coefficient of (1.1) is invertible. We have offered an algorithm for decoupling that features a convenient structure for the decoupled system whose solution recovers the response of the coupled system through a real, time-dependent, and nonlinear mapping. Thus, we have provided a *complete* solution to the problem of converting (1.1) into (1.2) by a real isospectral transformation.

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References

[1] Meirovitch, L.: Principles and Techniques of Vibrations. Prentice Hall, Upper Saddle River (1997)
[2] Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
[3] Caughey, T.K., O'Kelly, M.E.J.: Classical normal modes in damped linear dynamic systems. ASME J. Appl. Mech. **32**(3), 583–588 (1965)
[4] Ma, F., Caughey, T.K.: Analysis of linear nonconservative vibrations. ASME J. Appl. Mech. **62**(3), 685–691 (1995)

[5] Lancaster, P., Zaballa, I.: Parameterizing structure preserving transformations of matrix polynomials. Oper. Theory Adv. Appl. **218**, 403–424 (2012)
[6] Garvey, S.D., Lancaster, P., Popov, A.A., Prells, U., Zaballa, I.: Filters connecting isospectral quadratic systems. Linear Algebra Appl. **438**(4), 1497–1516 (2013)
[7] Lancaster, P., Zaballa, I.: Diagonalizable quadratic eigenvalue problems. Mech. Syst. Signal Process. **23**(4), 1134–1144 (2009)
[8] Garvey, S.D., Friswell, M.I., Prells, U.: Co-ordinate transformations for second order systems. Part I: general transformations. J. Sound Vib. **258**(5), 885–909 (2002)
[9] Morzfeld, M., Ma, F., Parlett, B.N.: The transformation of second-order linear systems into independent equations. SIAM J. Appl. Math. **71**(4), 1026–1043 (2011)
[10] Zúñiga Anaya, J.C.: Diagonalization of quadratic matrix polynomials. Syst. Control Lett. **59**(2), 105–113 (2010)
[11] Gohberg, I., Lancaster, P., Rodman, L.: Matrix Polynomials. Academic Press, New York (1982)
[12] Lancaster, P., Tismenetsky, M.: The Theory of Matrices, 2nd edn. Academic Press, San Diego (1985)
[13] Tisseur, F., Meerbergen, K.: The quadratic eigenvalue problem. SIAM Rev. **43**(2), 235–286 (2001)
[14] Gohberg, I., Kaashoek, M.A., Lancaster, P.: General theory of regular matrix polynomials and band Toeplitz operators. Integral Equ. Oper. Theory **11**(6), 776–882 (1988)
[15] Mackey, D.S., Mackey, N., Mehl, C., Mehrmann, M.: Vector spaces of linearizations for matrix polynomials. SIAM J. Matrix Anal. Appl. **28**(4), 971–1004 (2006)
[16] Lancaster, P., Psarrakos, P.: A note on weak and strong linearizations of regular matrix polynomials. Numerical analysis report 470, Manchester Centre for Computational Mathematics, Manchester (2005)
[17] Ma, F., Morzfeld, M., Imam, A.: The decoupling of damped linear systems in free or forced vibration. J. Sound Vib. **329**(15), 3182–3202 (2010)

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