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Stabilization of an Inverted Pendulum with 2 Degrees of Freedom, using a Five Bar Linkage Mechanism.

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Stabilization of an Inverted Pendulum with 2 Degrees of Freedom, using a Five Bar Linkage Mechanism.

A Thesis submitted in partial satisfaction of the requirements for the degree of Master of Science

in

Engineering Sciences (Mechanical Engineering)

by

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ABSTRACT OF THE THESIS

Stabilization of an Inverted Pendulum with 2 Degrees of Freedom, using a Five Bar Linkage Mechanism.

by

Sina Kouchaki

Master of Science in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2017

Professor Mauricio de Oliveira, Chair

A pendulum which has its center of mass above its pivot point, is an inverted pendulum. Inverted pendulum is an unstable system, and without applying an acceleration at the bottom of it, it will fall over. The dynamics of the inverted pendulum are non-linear. In this paper, utilizing linear control design technique, specifically a Linear Quadratic

Gaussian (LQG) controller is applied to stabilize two degrees of freedom inverted pendulum on a five-bar linkage mechanism. The linkage mechanism converts the rotational motion of two direct current (DC) motors into translational motion on the x-y plane, and it will provide the required acceleration that is needed in both x and y direction to stabilize the pendulum.

The equation of motion is obtained by the Lagrangian method instead of the Newtonian method. This method allows to neglect the reaction forces in the system and develops the equation of motion using the energy of the system. The other key aspect of this research paper is to simulate the non-linear model of the inverted pendulum rather than linearized version which makes controller more robust.

Chapter 1

Introduction

1.1. Scope of Thesis

Inverted pendulum is one of the classic problems used in control theory. Its non-linear dynamics as well as the unstable nature of it, makes it one of the most popular problems in which a wide range number of feedback systems can be used to stabilize the system. To balance the pendulum, one can apply acceleration at the bottom of the pendulum. There are many different instruments which are capable of providing the required transitional acceleration. Combining the inverted pendulum which is a popular system to control with a linkage mechanisms, which are conceivably the most fundamental class of machines, an interesting problem can be studied. Since the inverted pendulum is a system with 2 degrees of freedom, the required accelerations to stabilize the pendulum must be provided by the linkage mechanism in both x and y direction. Linkages are used to translate one type of motion into another, since the linkage system must be able to provide acceleration in two directions, two DC motors are used to translate a rotational motion into a translational motion. One of the key objective of this project was to simulate the non-linear system and capture most of its dynamics. Furthermore, to study the behavior of this non-linear system, we looked at the response of the non-linear system in both open and close loop, rather than using the linearized system. In the next few sections of chapter 1, the theories and approaches that are used in the project are explored in detail.

1.2. The Lagrangian Method

Newton's law governs the motion of particles and rigid bodies. It can be used to describe all mechanical system, and drive equation of motions. Newtonian method develops equation of motion based on forces and acceleration as describe by Newton's three laws. Lagrangian method is a replacement for Newtonian method. It offers a systematic way to for formulate the equations of motion of a mechanical system or a (flexible) structural system with multiple degrees of freedom. [1]. Lagrange methods avoid dealing with some constraints, such as reaction forces at a joint. It is a more convenient method for complex problems with high degree of freedom. Lagrangian method develops equations of motion using potential and kinetic energy. Some key notes are that both Lagrangian and Newtonian method are equivalent, however one big difference is that the Lagrangian method does not capture the friction of the system. Lagrangian equation avoids some constraints. In the next section, the Lagrangian method and its equation is shown in detail.

1.2.1. Lagrange's Equation

The Lagrange equation uses the kinetic and potential energy to solve the equation of motion. Unlike Newtonian method, there is no need to solve for accelerations of the system. In the kinetic energy term in the Lagrange equation, there is a velocity term which cause the acceleration to be found.

Let's define

$$L = T - V \tag{1.1}$$

Equation (1.1) is called the Lagrangian, where T is the kinetic energy of the system, and V represent the potential energy. For conservative systems, the Lagrange's equation is defined as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{1.2}$$

Equation (1.2) is the Euler-Lagrange equation, which can be used to drive the equation of motion of the inverted pendulum and the linkage mechanism.

1.3. Five Bar Linkage Mechanism

1.3.1. Definitions

A linkage, or kinematic chain, is an assembly of links and joints that provide a desired output motion in response to a specified input motion [3]. A node is defined as a point which is used to attach other links together. Links are frequently rigid bodies that have of two nodes (binary), however they may contain three or four nods (ternary and quaternary). Motion is allowed in a linkage mechanism at the joints between the links, the joints are also referred to as pivots. A kinematic chain in which at least one link is connected to a frame of reference is called a mechanism. The linkage is a system of links connected at pivot points. 4 bar linkage is the most common type of linkages which is consists of four links. The links are attached to one another at the pivot points to form a close kinematics. 4 bar linkage has 1 degrees of freedom, which for this project is not enough; therefore, 5 bar linkage was used which has 2 degrees of freedom.

1.3.2. Degree of Freedom

A linkage mechanism can be characterized by its number of degrees of freedom. DOF can be defined as the number of input motions that must be provided to provide the desired output, or the number of independent coordinates required to define the position and orientation of an object [3]. Equation (1.3) can be used to find the number of degree of freedom , where l is the number of linkages, and j is the total number of joints.

$$DOF = 3(l - 1) - 2j \tag{1.3}$$

Equation (1.3) is called the Gruebler's Equation, which is used for a planar mechanism. Planar mechanism is defined where all the motions of the links are in one plane or in parallel planes. Spherical mechanism is when there is a motion which is not in the same plane or in parallel planes.

Typically, there are four types of linkages:

1) Revolute: One degree of freedom (Rotational)
2) Prismatic: One degree of freedom (Translational)
3) Cylindrical: Two degree of freedom
4) Spherical: Three degree of freedom

1.4. State Space Models

The idea of state space model comes from the state variable method of describing differential equations. In this method, the differential equations describing a dynamic system are organized as a set of first order differential equations in the vector valued state of system, and the solution is visualized as a trajectory of this state vector in space [4].

A continuous time state space model for linear system is defined by the following equations:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1.4}$$

In equation (1.4), A, B, C, and D are vectors and matrices which we will discuss later. x is called the state vector, u is the input or control vector of the system, and y is the output vector of the system. In equation (1.4), the relationship between the input and output of the system is shown. To determine the output of the system, we need to solve the state equation at a particular initial condition $x(0)$ for a given input of the system and then substitute it into the output equation to obtain the system output.

Once we have a mathematical model for the system in a form of differential equation, we can write this model as a state space model. Let's consider the following nth order LTI system:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = u \tag{1.5}$$

with some initial conditions.

Now let's define the states as the following vector:

$$x = \begin{bmatrix} y^{(n-1)} \\ y^{(n-2)} \\ \vdots \\ y' \\ y \end{bmatrix} \tag{1.6}$$

Using our original ODE shown in (1.5) we can write the state equation as:

$$\dot{x} = \begin{bmatrix} y^{(n)} \\ y^{(n-1)} \\ \vdots \\ y'' \\ y' \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u \tag{1.7}$$

$$y = [b_1 \quad b_2 \quad \cdots \quad b_{n-1} \quad b_n]\, x$$

Equation (1.7) is now in a special form which is called space state equations. It is in form of equation (1.4) where D matrix is zero in this example. In the next section, we will examine the method of converting a non-linear differential equation to space state model via linearization.

1.5. Non-Linear System and Linearization

Nonlinear systems can be written as:

$$\dot{x} = f(t, x, u)$$
$$y = h(t, x, u) \tag{1.8}$$

System (1.8) is time variant system. When f and h are continuous and differentiable functions and the state vector and input are within a small neighborhood of a point $(\bar{\bar{x}}, \bar{u})$ or trajectory $(\bar{\bar{x}}(t), \bar{u}(t))$, it is natural to expect that the behavior of the non-linear system can be approximated by that of properly define linear system [6]. The method that is used to convert a non-linear system to space state model which is for linear systems, is called linearization.

To obtain the linear system lets take the point $(\bar{\bar{x}}, \bar{u})$ such that $f(\bar{\bar{x}}, \bar{u}) = 0$, which is the equilibrium of the system. In dynamic system in which there are no external forces acting on the system, the system that starts at equilibrium stays around that point. When we linearize a system, it is done around a certain point, where usually it will make it simpler to linearize the system around the its equilibrium point. Using partial differential equation, we can linearize the system around its equilibrium point and write it in space state form where:

$$A = \frac{\partial f}{\partial x}\Big|_{\substack{x = \bar{x} \\ u = \bar{u}}} \quad , \quad B = \frac{\partial f}{\partial u}\Big|_{\substack{x = \bar{x} \\ u = \bar{u}}} \quad , \quad C = \frac{\partial h}{\partial x}\Big|_{\substack{x = \bar{x} \\ u = \bar{u}}} \quad , \quad D = \frac{\partial h}{\partial u}\Big|_{\substack{x = \bar{x} \\ u = \bar{u}}} \qquad (1.9)$$

In the next and last section of this introduction chapter we will introduce the control design method that was used, which is called Linear Quadratic Regulator and Linear Quadratic Gaussian Control (LQR and LQG).

1.6. Linear Quadratic Gaussian Control

Let's look at the linear time-invariant (LTI) system of our state space model equation (1.4) with the initial condition of $x(0) = x_0$. Controllability describes the ability of an external input to move the internal state of a system from any initial state to any other final state in a finite time interval. A system with an initial state, $x(0) = x_0$ is observable if and only if the value of the initial state can be determined from the system output y(t) that has been observed through the time interval $t_0 < t < t_f$ [7].

The system is said to be controllable if the controllability matrix is full rank, and it is observable if the observability matrix is full rank.

The controllability and observability matrices are defined as:

$$C(A, B) = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

$$\mathrm{O}(\mathrm{A}, \mathrm{C}) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad (1.10)$$

Controllability and observability are the two important property of any dynamical system. They are used in Linear Quadratic Gaussian (LQG) control design and they are one of the key characteristic in those types of control design.

LQG concerns uncertain linear systems which involve having incomplete state information and undergoing control subject to quadratic cost. Furthermore, the solution is unique and constitutes a linear dynamic feedback control law that is easily computed and implemented. Finally, the LQG controller is also fundamental to the optimal control of perturbed non-linear systems [12].

Chapter 2
System Modeling

2.1. Kinematics

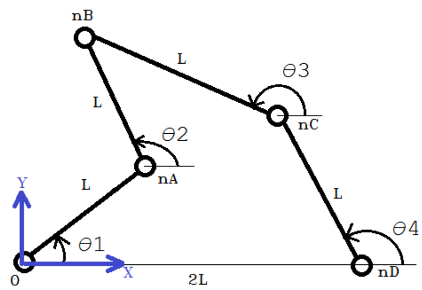

Figure 1, Five bar linkage mechanism

Kinematics is the motion of object without reference to the forces that cause the motion. In order to develop a model for the system, first we need to consider the kinematics of the five-bar linkage. In figure 1, the five-barlinkage mechanism is shown, using this figure the coordinate of each linkage can be written as:

$$Rod1 = (\cos(\theta 1(t)\,, \sin(\theta 1(t))$$
$$Rod2 = (\cos(\theta 2(t)\,, \sin(\theta 2(t))$$
$$Rod3 = (\cos(\theta 3(t)\,, \sin(\theta 3(t)) \qquad (2.1)$$
$$Rod4 = (\cos(\theta 4(t)\,, \sin(\theta 4(t))$$

Next, as shown in Figure 1, Five bar linkage mechanism, there are five nodes in the linkage mechanism, the coordinate of each node is:

$$O = (0\,, 0)$$
$$nA = \mathrm{L} \times \ \mathrm{Rod1}$$
$$nB = \mathrm{nA} + (\mathrm{L} \times \ \mathrm{Rod2}) \qquad (2.2)$$
$$nC = (2\mathrm{L}\,, 0) + (\mathrm{L} \times \ \mathrm{Rod4})$$
$$nD = 2L$$

Rod1 through Rod4, describe each linkage coordinate, however to use Lagrangian method, to derive the equation of motion, one needs to describe the center of mass of each link; therefore, by knowing the length of each rod, the coordinate of the center of mass of each link can be written as:

$$R1 = \frac{\mathrm{L}}{2}\,Rod1$$

$$R2 = \mathrm{nA} + \frac{\mathrm{L}}{2}Rod2$$

$$R3 = nC + \frac{L}{2}Rod3 \qquad (2.3)$$

$$R4 = (2\mathrm{L}\,,0\,) + \left(\frac{\mathrm{L}}{2}\,Rod4\right)$$

As shown in the equation of Rod1 thought Rod4, the angle of each link, θ is a function of time, so by taking the time derivative of each equation (Rod1 through Rod4) we can write the velocity of each rod as:

$$Rod_n{}' = \frac{\mathrm{dRod_n}}{\mathrm{dt}} = \left(-\sin\left(\theta n(t)\right)\theta n'(t), cos\left(\theta n(t)\right)\theta n'(t)\right) \qquad (2.4)$$

$$where\; n = 1{,}2{,}3{,}4$$

We can do the same for the velocity of the center of mass of each rod

$$R_n{}' = \frac{\mathrm{dR_n}}{\mathrm{dt}} \qquad (2.5)$$

Now the we have the kinematics for the linkage mechanism, we need to write the kinematics for the inverted pendulum as well, the figure to the right shows the three-unit

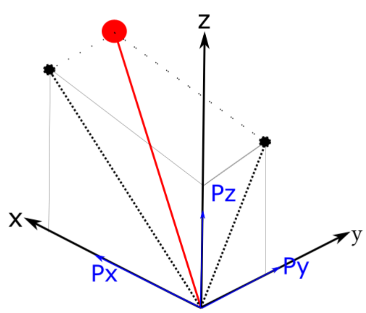

Figure 2, Inverted Pendulum

vector of Px, Py, and Pz. Using these unit vectors we can easily define the position of the pendulum as:

$$Rod_{Pen} = (\mathrm{Px}(t), \mathrm{Py}(\mathrm{t}), \mathrm{Pz}(t)) \tag{2.6}$$

However, just as we did for the linkage mechanism, we need to define the position of the center of mass of the pendulum with respect to the origin of our system. The inverted pendulum is mounted at node B of the linkage, and assuming that it has length ℓ, we can write its position as:

$$\begin{gathered} R_{pen} = (L\cos(\theta 1(\mathrm{t})) + L\cos(\theta 2(\mathrm{t})) + \frac{1}{2}\ell \mathrm{px}(t), \frac{1}{2}\ell \mathrm{py}(t) + \\ L\sin(\theta 1(\mathrm{t})) + L\sin(\theta 2(\mathrm{t})), \frac{1}{2}\ell \mathrm{pz}(t)) \end{gathered} \tag{2.7}$$

R_{pen} represents the position of the center of mass of the inverted pendulum which is mounted on node B of the linkage, with respect to the origin of the system which is at point O. Now that we have the position of both the inverted pendulum and its center of mass, we can write their velocities by taking time derivative of equations Rod_{pen} and R_{pen}.

$$\begin{gathered} Rod_{Pen}' = \frac{\mathrm{dRod_{Pen}}}{\mathrm{dt}} \\ R_{Pen}' = \frac{\mathrm{dR_{Pen}}}{\mathrm{dt}} \end{gathered} \tag{2.8}$$

All the results and detail calculations are shown in the appendix. Now, that we have the kinematics of the system, the next step is to look at the dynamics of the system in order to write the Lagrangian equation.

2.2. Dynamics

The dynamics of the system involves its kinetic and potential energy. To use the Lagrangian method, first we need to find the kinetic and potential energy of the system. When finding the kinetic energy of the system, we need to look the effect of both mass and the inertia of the system. Keeping this in mind, we need to write down 8 different kinetic energy equations. Four of these equations are for the four links of the system, one is for the base of the inverted pendulum which includes the sensors and other parts, and one is for the inverted pendulum itself.

$$T_n = \frac{1}{2}(mR_n'^2 + JRod_n'^2) \quad Where\ n = 1,2,3,4\ and\ J = \frac{L^2 m}{12}$$

$$T_5 = \frac{1}{2}MnB'^2 \tag{2.9}$$

$$T_6 = (\frac{1}{2}m_{pen}R_{pen}'^2 + J_{pen}Rod_{pen}'^2 \quad Where\ \ J_{pen} = \frac{\ell^2 m_{pen}}{12}$$

We also need to write down the kinetic energy of each the shaft of DC motors, used in the system. There are two motors mounted at ends the first and the last link. The equations blow represent the kinetic energy of each shaft.

$$\mathrm{T_{Motor1}} = \left(\frac{1}{2}\right)\mathrm{Im} * \left(\frac{\mathrm{d}\theta_1[t]}{t}\right)^2$$

$$\mathrm{T_{Motor4}} = \left(\frac{1}{2}\right)\mathrm{Im} * \left(\frac{\mathrm{d}\theta_4[t]}{t}\right)^2 \tag{2.10}$$

Combining the total of eight kinetic energy equations, we can write the total kinetic energy of the system as:

$$T = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_{Motor1} + T_{Motor4} \tag{2.11}$$

Where T represent the total kinetic energy of the system.

In terms of potential energy of the system, the only component that contribute to this, is the inverted pendulum. Since the linkage mechanism is on a x-y plane, the links do not have any contribution to the potential energy. Knowing this we can write the potential energy of the inverted pendulum as weight times the position of the pendulum which gives us:

$$V = m_{pen}g \times R_{pen} = \frac{1}{2} m_{pen}\, g\, \ell\, P_z(t) \tag{2.12}$$

2.2.1 Constraints

The linkage mechanism consists of 5 links and 5 joints. Using equation 3, this results in a cylindrical mechanism. To control a 2 DOF linkage, 2 inputs are required;

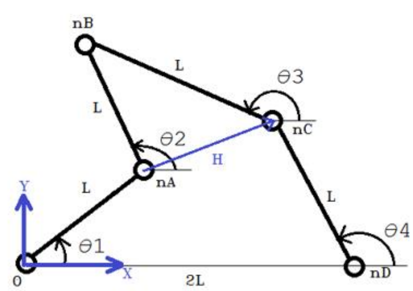

Figure 3, linkage mechanism for constraints therefore, we have 2 DC motors mounted on the first and last link. As shown in figure 3, the linkage mechanism consists of four angles, which two of these angles are controlled by the inputs. Thinking down the line about our states for our system model, we cannot have $\theta 2(t)\ and\ \theta 3(t)$ as states, since there are no sensors at node A, B, nor C; therefore, a relationship between the two controlled angles $(\theta 1[t], \theta 4[t])$ and the unknown angles $(\theta 2[t], \theta 3[t])$ may be derived.

As shown in the figure to the right, let's draw a vector from node A to node C and call this vector H.

$$H^2 = (Cx - Ax)^2 + (Cy - Ay)^2 \qquad (2.13)$$

Where Ax and Cx represent the x components of node A and C, and Ay and Cy represent the y components of those nodes. By using the triangle formed with ABC and law of cosine, we can write the following:

$$H = L^2 + L^2 - 2L\cos[\theta 3(t) - \theta 2(t)] \qquad (2.14)$$

By solving equation (2.14) we can write $\theta 3(t)$ as function of $\theta 1(t), \theta 2(t),\ and\ \theta 4(t)$. In order to solve for $\theta 2(t)$ we can use the fact that the length of all the links are equal to each other and write:

$$(Bx - Ax)^2 + (By - Ay)^2 = (Bx - Cx)^2 + (By - Cy)^2 \quad (2.15)$$

By solving equation (2.15) we can write $\theta 3(t)$ as function of $\theta 1(t)\ and\ \theta 4(t)$. As a result, $\theta 2(t)\ and\ \theta 3(t)$ can be written as a function of the inputs.

$$\theta 2 = \sec^{-1}\left(\frac{f_{cos(\theta 1,\theta 4)}}{f_{cos(\theta 1,\theta 4)} + f_{sin(\theta 1,\theta 4)} + f_{sin^2(\theta 1,\theta 4)}}\right)$$

$$\theta 3 = \theta 2 + \cos^{-1}(f_{\cos(\theta 1,\theta 2)}) \quad (2.16)$$

As shown in equation (2.16) the result of solving for $\theta 2(t)\ and\ \theta 3(t)$is costly. Both $\theta 2(t)\ and\ \theta 3(t)$ are highly non-linear and if they were to be using in our model, they would be costly in terms of simulation and computation time. Thus, we need to drive a new set of constraints so we can use in Lagrangian Multipliers.

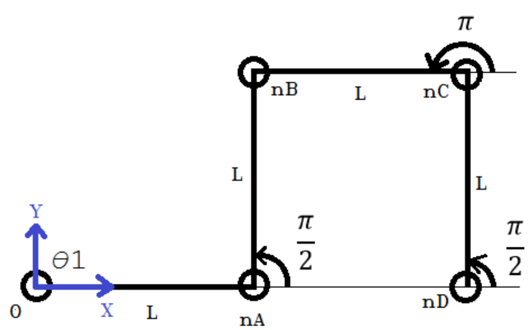

Figure 4, Five bar linkage constraints

In equation (2.2), nC was defined in a counter clock wise direction of the links. Looking and node C, we can write it in two ways. First going in counter clock wise direction as was done in (2.2), and second in the clock wise direction. Since in both ways we end up at a same point we can write a set of x and y coordinate as constraints of the linkage.

$$nB = nC - LRod3 \rightarrow\ nC + LRod3 - nB = 0 \tag{2.17}$$

Which results in 2 sets of constraints:

$$\begin{aligned} &Constraint1{:}\ 2L - L\cos(\theta 1(t)) - L\cos(\theta 2(t)) + L\cos(\theta 3(t)) \\ &\quad + L\cos(\theta 4(t)) = 0 \\ &Constraint2{:}\ -L\sin(\theta 1(t)) - L\sin(\theta 2(t)) + L\,s\text{in}(\theta 3(t)) \\ &\quad + L\sin(\theta 4(t)) = 0 \end{aligned} \tag{2.18}$$

For the inverted pendulum, since $p_x, p_y, and\ p_z$are defined as unit vectors, the norm of Rod_{pen} should equal to 1. The constraint equation for the inverted pendulum can be written as

$$Constraint3{:}\ Rod_{pen}^2 = 1 \rightarrow p_x(t)^2 + p_y(t)^2 + p_z(t)^2 = 1 \tag{2.19}$$

2.2.2 DC Motor

There are two Brush DC Motors in the system. They are used to control the input of the system. The fundamental input of the system can be recognized as the voltage. The suppled voltage into the motors, cause the motor to change position with a certain velocity and acceleration. To develop equation of motions for the entire system, we need to understand and have a model for the motors as well.

The circuit in figure 5 represent the electrical and torque characteristics of a DC

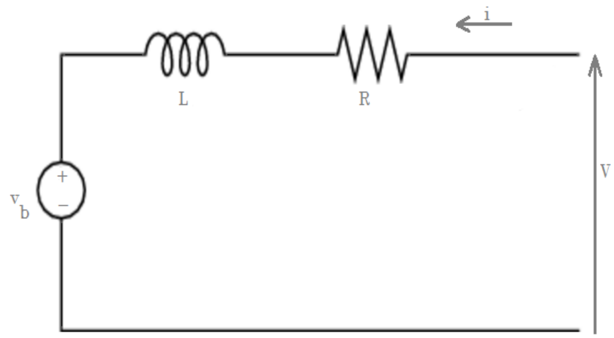

Figure 5, DC motor circuit

motor. v_b represent the back emf that is induce by the permanent magnets on to the motor armature. Back emf is a function of the angular velocity of the motor armature, and it is directly proportional to the angular velocity ω.

$$v_b = k_v\omega \tag{2.20}$$

The torque of the DC motor is directly proportional to the current, and it can be written as

$$\tau = k_t i \tag{2.21}$$

$k_v, and\ k_t$ are the back emf constant and torque constant of the motor. Assuming that the mechanical and electrical power are equal, and making sure that all the parameters have units that are in SI, we can say that $k_v = k_t$.

Using Kirchhoff's voltage law around the circuit loop shown in figure 5, we can write the following ordinary differential equation:

$$V(t) = R\, i(t) + L\frac{di}{dt} + v_b(t) \tag{2.22}$$

By substituting equations 2.20 and 2.21 into 2.22 we get

$$V(t) = R\frac{\tau(t)}{k_t} + L\frac{d\tau}{dt}\frac{1}{k_t} + k_v\omega(t)$$

$$\frac{d\tau}{dt} = -\frac{R\,\tau(t)}{L} - \frac{k_v k_t}{L}\omega(t) + \frac{k_t V(t)}{L} \tag{2.23}$$

Solving the ordinary differential equation (2.23), and assuming that the transient $e^{-\frac{R}{L}t}$ is very fast, we can approximate the torque of the DC motor as:

$$\tau(t) = -\frac{k_t k_V}{R}\omega(t) + \frac{k_t}{R}V(t) \tag{2.24}$$

Using a gearbox with a DC motor is essential since the DC motors which run on 12Volt, usually do not have enough torque to carry out the load of the system. They are also high speed which makes them hard to control. Adding a gearbox to the system, increase the torque and decrease the angular velocity of the DC motor.

$$\tau(t)_{Gearbox} = Ratio \times \tau(t)_{motor}$$

$$\tau(t)_{Gearbox} = Ratio \times \left[-\frac{k_t k_V}{R}\omega(t) + \frac{k_t}{R}V(t)\right] \tag{2.25}$$

Since angular velocity is the time derivative of the angular position, we can write the torque of the motors as a function of $\theta 1(t)\ and\ \theta 4(t)$ which are the input angles of the linkage.

$$\tau_1(t) = Ratio \times \left[-\frac{k_t k_V}{R}\theta_1{}'(t) \ + \frac{k_t}{R}V_1(t)\right]$$

$$\tau_4(t) = Ratio \times \left[-\frac{k_t k_V}{R}\theta_4{}'(t) \ + \frac{k_t}{R}V_4(t)\right] \quad (2.26)$$

In equation above (2.26), the torque provided by the DC motor at each link, is written in terms of the corresponding angular velocity and voltage.

2.3. Lagrangian Dynamics and Equation of Motion

In the previous sections the kinetic and potential energy of the system were written in equations (2.11) and (2.12). Using Lagrangian method the equation of motion can be developed. Using equations (1.1) and (1.2), we can write the equation of motion as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau \quad (2.27)$$

Where

$$L = T - V - \lambda(t) \times Constraints \quad (2.28)$$

$$L = T - V - \lambda 1(t)Constraint1 - \ \lambda 2(t)Constraint2 - \lambda 3(t)Constraint3$$

q is a vector with the system parameters

$$q = [\theta 1(t) \quad \theta 2(t) \quad \theta 3(t) \quad \theta 4(t) \quad p_x(t) \quad p_y(t) \quad p_z(t)\,]$$

$$\dot{q} = \tfrac{dq}{dt} = [\theta 1'(t) \quad \theta 2'(t) \quad \theta 3'(t) \quad \theta 4'(t) \quad p_x{}'(t) \quad p_y{}'(t) \quad p_z{}'(t)\,]$$

$$\ddot{q} = \frac{d\dot{q}}{dt} = [\theta 1''(t) \quad \theta 2''(t) \quad \theta 3''(t) \quad \theta 4''(t) \quad p_x''(t) \quad p_y''(t) \quad p_z{}''(t)\,]$$

τ uses the torques form equation (2.26) which we can write as a vector for each element of q.

$$\tau = \begin{bmatrix} ratio \times \tau_1(t) \\ 0 \\ 0 \\ ratio \times \tau_4(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.29}$$

Also, using the constraints of the system 2.18 and 2.19, we can construct a constraint matrix for the system by taking the time derivative of our 3 equations with respect to q:

$$A = \begin{bmatrix} L\sin[\theta 1(t)] & -L\cos[\theta 1(t)] & 0 \\ L\sin[\theta 2(t)] & -L\cos[\theta 2(t)] & 0 \\ -L\sin[\theta 3(t)] & L\cos[\theta 3(t)] & 0 \\ -L\sin[\theta 4(t)] & L\cos[4(t)] & 0 \\ 0 & 0 & 2p_x(t) \\ 0 & 0 & 2p_y(t) \\ 0 & 0 & 2p_z(t) \end{bmatrix} \tag{2.30}$$

Using equation (2.27) we get a set of 10 equations of motion which are functions of $q,\ \dot{q},\ \ and\ \ddot{q}$. 7 equations for q, and 3 equations corresponds to the constraints of the system λ. Writing the equation of motion in a matrix form results in:

$$\begin{gathered} M(q)\,\ddot{q} + A(q)\,\lambda = g(q,\dot{q}) \\ A^T(q)\,\ddot{q} = -\dot{A}^T(q)\,\dot{q} = h(q,\dot{q}) \end{gathered} \tag{2.31}$$

Equation (2.13) is the equation of motion for our system in a matrix form which was developed using Lagrangian method from 2.27 where M, g, and h matrix are:

$M=$

$$\begin{bmatrix} I_m+\frac{1}{3}L^2\left(4m+3(M+m_{pen})\right) & \frac{1}{2}L^2\left(m+2(M+m_{pen})\right)\cos[\theta1(t)]-\theta2(t)] & 0 & 0 & -\frac{1}{2}L\,m_{pen}\ell\sin[\theta1(t)] & \frac{1}{2}L\,m_{pen}\ell cos[\theta1(t)] & 0 \\ \frac{1}{2}L^2\left(m+2(M+m_{pen})\right)\cos[\theta1(t)]-\theta2(t)] & \frac{1}{3}L^2\left(m+3(M+m_{pen})\right) & 0 & 0 & -\frac{1}{2}L\,m_{pen}\ell\sin[\theta2(t)] & \frac{1}{2}L\,m_{pen}\ell cos[\theta2(t)] & 0 \\ 0 & 0 & \frac{L^2m}{3} & \frac{1}{2}L^2m\cos[\theta3(t)-\theta4(t)] & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}L^2m\cos[\theta3(t)-\theta4(t)] & I_m+\frac{4L^2m}{3} & 0 & 0 & 0 \\ -\frac{1}{2}L\,m_{pen}\ell\sin[\theta1(t)] & -\frac{1}{2}L\,m_{pen}\ell\sin[\theta2(t)] & 0 & 0 & \frac{m_{pen}\ell^2}{3} & 0 & 0 \\ \frac{1}{2}L\,m_{pen}\ell\cos[\theta1(t)] & \frac{1}{2}L\,m_{pen}\ell cos[\theta2(t)] & 0 & 0 & 0 & \frac{m_{pen}\ell^2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{m_{pen}\ell^2}{3} \end{bmatrix}$$

$g=$

$$\begin{bmatrix} \frac{-2\,\mathrm{k_t}\,\mathrm{ratio}\,\mathrm{V1(t)}+2\,\mathrm{k}_t\,\mathrm{k_v}\,\mathrm{ratio}^2\,\theta1'(t)+L^2\left(m+2(M+\mathrm{m_{Pen}})\right)R\sin[\theta1(t)-\theta2(t)]\,\theta2'(t)^2}{2R} \\ \frac{1}{2}L^2\left(m+2(M+\mathrm{m_{Pen}})\right)\sin[\theta1(t)-\theta2(t)]\,\theta1'(t)^2 \\ -\frac{1}{2}L^2m\sin[\theta3(t)-\theta4(t)]\,\theta4'(t)^2 \\ \frac{2\mathrm{k_t}\,\mathrm{ratio}\,\mathrm{V4}(t)+L^2m\,R\,\sin[\theta3(t)-\theta4(t)]\theta3'(t)^2-2\,\mathrm{k_t}\mathrm{k_v}\,\mathrm{ratio}^2\,\theta4'(t)}{2R} \\ \frac{1}{2}L\,\mathrm{m_{Pen}}\ell\,\left(\cos[\theta1(t)]\,\theta1'(t)^2+\cos[\theta2(t)]\,\theta2'(t)^2\right) \\ \frac{1}{2}L\,\mathrm{m_{Pen}}\ell\,\left(\sin[\theta1(t)]\,\theta1'(t)^2+\sin[\theta2(t)]\,\theta2'(t)^2\right) \\ -\frac{1}{2}g\,m_{pen}\ell \end{bmatrix}$$

$h=$

$$\begin{bmatrix} L\,(-\cos[\theta1(t)]\theta1'(t)^2-\cos[\theta2(t)]\theta2'(t)^2+\cos[\theta3(t)]\theta3'(t)^2+\cos[\theta4(t)]\theta4'(t)^2) \\ L\,(-\sin[\theta1(t)]\theta1'(t)^2-\sin[\theta2(t)]\theta2'(t)^2+\sin[\theta3(t)]\theta3'(t)^2+\sin[\theta4(t)]\theta4'(t)^2) \\ -2\left(p_x'(t)^2+p_y'(t)^2+p_z'(t)^2\right) \end{bmatrix}$$

2.4. Reduced Model

In the previous section, we derived the equation of motion for our system. This equation is a non-linear differential equation. To develop a state space model for the system, we need to linearize this system around it equilibrium point. The goal in this and next section is to develop a state space model for the system, and reduce the state space model as much as we can.

To reduce the state space, we can use our equation of motion and the constraints of the system. We have the following

$$M(q)\,\ddot{q} + A(q)\,\lambda = g(q,\dot{q})$$

$$A^T(q)\,\dot{q} = 0 \qquad (2.32)$$

Now we can define a state like variable x where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Now we can rewrite equation (2.32) as

$$M(x_1)\,\dot{x_2} + A(x_1)\,\lambda = g(x_1, x_2)$$

$$A^T(x_1)\,x_2 = 0 \qquad (2.33)$$

$$\dot{x_1} = x_2$$

Now $A^T(x_1)\,x_2 = 0$ has a solution $S(x_1)$ which is defined as

$$S(x_1) = \left[Null[A^T(x_1)]\right]^T \qquad (2.34)$$

Next, we can define a new matrix SA as

$$SA = \begin{bmatrix} S^T(x_1) \\ A^T(x_1) \end{bmatrix}$$

Using $A^T(x_1)\,x_2 = 0$, and the new matrix SA

$$SA \cdot x_2 = \begin{bmatrix} S^T(x_1) \\ A^T(x_1) \end{bmatrix} x_2 = \begin{bmatrix} P \\ 0 \end{bmatrix} \tag{2.35}$$

Where $P = \begin{bmatrix} p_1[t] & p_2[t] & p_3[t] & p_4[t] \end{bmatrix}$ is a new vector state.

Now, we can solve for x_2 and by taking the time derivative for $\dot{x_2}$

$$x_2 = T \cdot P$$

$$Where\ T = (SA)^{-1} \tag{2.36}$$

$$\dot{x_2} = \dot{T}(x_1)P + T(x_1)\dot{p}$$

Next, we can rewrite our reduced equation of motion and constraints in terms of x

$$T^T(x_1)M(x_1)T(x_1)\dot{P} = T^T(x_1)g(x_1, TP) - T^T(x_1)M(x_1)\dot{T}(x_1)P$$

$$\dot{x_1} = T \cdot P \tag{2.37}$$

2.5. Linearization and Space State Model

To linearize the system, we need to identify the equilibrium of the system, and then we can linearize the reduced non-linear model around its equilibrium. The vector q was defined as $q = [\theta 1(t) \quad \theta 2(t) \quad \theta 3(t) \quad \theta 4(t) \quad p_x(t) \quad p_y(t) \quad p_z(t)\,]$. The inverted pendulum has two equilibriums. When $p_z = -1\ or\ 1\ for\ p_x = p_y = 0$. Using figure 4 the equilibrium of the linkage mechanism can be identified. To summarize, the equilibrium of the model is:

$$q = \begin{bmatrix} 0 & \frac{\pi}{2} & \pi & \frac{\pi}{2} & 0 & 0 & 1 \end{bmatrix}, \dot{q} = 0,$$

$$p = 0, \dot{p} = 0 \tag{2.38}$$

$$V_1[t] = V_4[t] = 0$$

Now that we have the equilibrium of the system, we can start linearization to develop the state space model. To do so, first we need to define the output, the states, and the input of the system. The output is defined in equation (2.39), where $arctan\left(\frac{p_z(t)}{p_y(t)}\right)$ is the angle of the inverted pendulum in the y-z place (α) and $Arctan\left(\frac{p_z(t)}{p_x(t)}\right)$ is in x-z place (β)

$$Output = \begin{bmatrix} \theta_1(t) \\ \theta_4(t) \\ Arctan\left(\frac{p_z(t)}{p_y(t)}\right) \\ Arctan\left(\frac{p_z(t)}{p_x(t)}\right) \end{bmatrix} \tag{2.39}$$

States and input of the system can be written as

$$X = \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \\ p_x(t) \\ p_y(t) \\ p_z(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \\ x_7(t) \\ x_8(t) \\ x_9(t) \\ x_{10}(t) \\ x_{11}(t) \end{bmatrix}, \quad u = \begin{bmatrix} V_1(t) \\ V_2(2) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{2.40}$$

The reduced equation of motion (Equation 2.37) can be written as

$$\begin{gathered} M_p(x_1)\dot{p} = g_p(x_1, x_2, u) \\ \dot{x}_1 = x_2 = T(x_1) \cdot P \end{gathered} \tag{2.41}$$

Equation (2.41) can be written as

$$\begin{gathered} \dot{p} = M_p(x_1)^{-1} g_{xp}(x_1, T(x_1) \cdot P, u) \\ \dot{x}_1 = T(x_1) \cdot P \end{gathered} \tag{2.42}$$

Linearizing equations (2.42) around the system's equilibrium and taking the partial differentiation we get

$$\begin{gathered} \partial\dot{p} = A_{11}\partial p + A_{12}\partial x_1 + B_{11}\partial u \\ \partial\dot{x}_1 = A_{21}\partial p + A_{22}\partial x_1 + B_{21}\partial u \end{gathered} \tag{2.43}$$

Where $A_{1n} = M(\bar{x}_1)^{-1} \cdot \frac{\partial g_{xp}}{\partial x}$, $A_{2n} = \frac{\partial[T(x_1 \cdot P)}{\partial x}$, $B_{1n} = M(\bar{x}_1)^{-1} \cdot \frac{\partial g_{xp}}{\partial u}$, $B_{2n} = 0$, solved at the equilibrium of the system.

Form the constraints equation (2.33) we get $A^T \partial x_1 = 0$ using a change of variable we can set $\partial x_1 = \bar{S}\,\partial z$ and $\partial \dot{x}_1 = \bar{T}^T \bar{S}\,\partial \dot{z}$ where $\bar{S}$ and $\bar{T}$ are S and T Matrices evaluated at the equilibrium of the system. By doing this change of variable we end up with

$$\begin{aligned} \partial \dot{p} &= A_{11}\partial p + A_{12}\bar{S}\,\partial z + B_{11}\partial u \\ \partial \dot{z} &= \bar{T}^T A_{21}\partial p + \bar{T}^T A_{22}\bar{S}\,\partial z + \bar{T}^T B_{21}\partial u \end{aligned} \tag{2.44}$$

Next, we can write the space state model of the system in the form of equation (1.4)

Where the A, B, C, and D matrices are defined as

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12}\bar{S} \\ \bar{T}^T A_{21} & \bar{T}^T A_{22}\bar{S} \end{bmatrix}, B = \begin{bmatrix} B11 \\ 0 \end{bmatrix} \\ C &= \frac{\partial(output)}{\partial x}, D = \frac{\partial(output)}{\partial u} = 0 \end{aligned} \tag{2.45}$$

Now that we have A, B, C, and D matrices defined, we can write the space state model and use it to design the controller for the system. In the appendix, Mathematica file, the detail calculation and numerical results can be found.

Chapter 3

Controller Design and Simulation Result

Now that we have a state space model we can start to design the controller for the system. The first step is to define the parameters of the system. The table blow shows the numerical values of the system.

Table 1, System parameters numerical value

Parameter	Symbol	Value	Unit
Gravity	g	9.81	m/s^2
Length of each link	L	0.3	m
Mass of each link	m	0.1131	kg
Mass of Pendulum Box	M	0.5	kg
Motor shaft Inertia	Im	6.033×10^{-7}	$kg\ m^2$
Motor internal resistance	R	5.714	Ω
Motor back emf constant	k_v	0.00907	$V\ s/rad$
Motor torque constant	k_t	0.00907	$V\ s/rad$
Gearbox, gear ratio	ratio	227	
Length of the inverted pendulum	ℓ	0.2	m
Mass of the inverted pendulum	mPen	0.25	kg

Table 2, Space state model of the system (Numerical)

0	0	12.3706	0	91.848	0	0	0	0	-12.0047
0	0	0	12.3706	0	91.848	0	0	12.0047	0
0	0	-10.9961	0	-16.3094	0	0	0	0	10.6708
0	0	0	-10.9961	0	-16.3094	0	0	-10.6708	0
1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	$\frac{1}{2}$	0	0	0	0	0	0	0
0	0	0	$\frac{1}{2}$	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	-1	0	0	0	0
0	0	0	0	-1	0	0	0	0	0

Table 2, shows the numerical result of the space state model of the system using system parameter data table.

3.1. Open Loop Simulation

As mentioned one of the objective of this project was to simulate the system in the non-linear form to have more accurate result. This simulation was done using Wolfarm

Mathematica software. The simulation chapter in the Mathematica notebook shown in appendix covers all details of the simulation and the code.

To understand the system better, first let's look at the poles and zeros of the system. One way of finding the open loop poles of the system is to look at the eigenvalues of the A matrix. Using the values shown in table 2, we can find the eigenvalues of A.

$$\begin{gathered} Poles = -13.3395, -13.3395, 9.04323, 9.04323, -6.6998, -6.6998, 0, 0 \\ Zeros = -8.57321\,, 8.57321\,, -8.57321, 8.57321, 0, 0 \end{gathered} \tag{3.1}$$

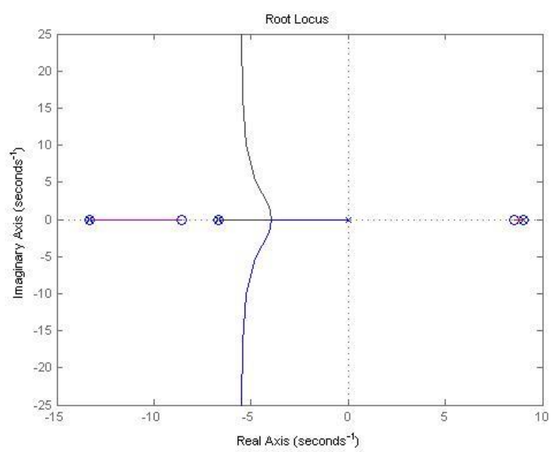

Figure 6, Root Locus Plot from input one

As expected the system is unstable with two poles on the left-hand side of the complex plane and two poles as zero. Also, we see that the poles are coupled, which is also expected since α *and* β act in the same way on the system. In figure 6, the root locus plot of the system from the first input, confirms that the system is unstable. We can also look at the system transfer function and see the system has been decoupled and essentially look at the angles of the inverted pendulum separately.

Table 3, Open loop Transfer Function

$\frac{-392.152+5.3354\,s^2}{-808.212\,s-91.848\,s^2+10.9961\,s^3+s^4}$	$0.$
$0.$	$\frac{-392.152+5.3354\,s^2}{-808.212\,s-91.848\,s^2+10.9961\,s^3+s^4}$
$-\frac{12.0047\,s}{-808.212-91.848\,s+10.9961\,s^2+s^3}$	$0.$
$-\frac{1.16415\times10^{-10}}{653206.+148465.\,s-9338.25\,s^2-3636.36\,s^3-62.7824\,s^4+21.9922\,s^5+s^6}$	$\frac{12.0047\,s}{-808.212-91.848\,s+10.9961\,s^2+s^3}$

Now that we have more understanding of the system, we can solve the non-linear equation of motion shown in 2.31, using computer software (Mathematica) and look at the response of the system. By solving the non-linear equation of motion, we can see the more accurate response of the system.

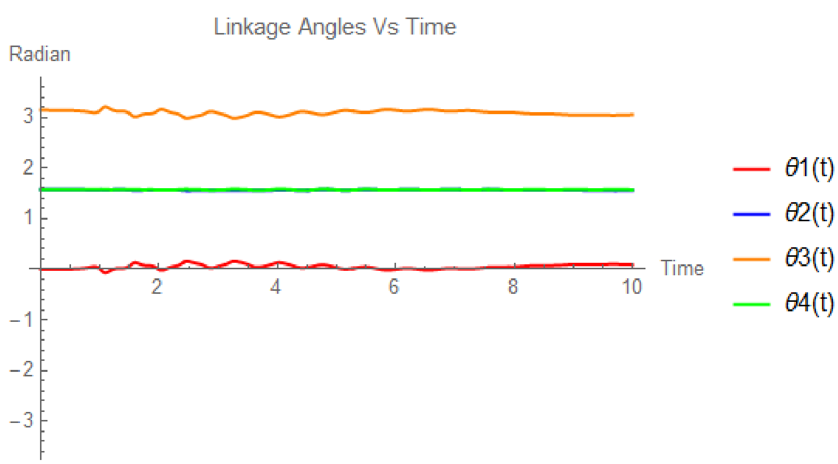

Figure 7, Open loop simulation of linkage angles

Figure 7, shows the open loop simulation of the linkage angles. Since there is no input to the system, there is no significant motion in the linkages. As we see in the figure

7, the angles of the linkage only oscillate slightly due and they are within ±0.2 radians (12 degrees) of their equilibrium position. This small oscillation is due to the drop of the inverted pendulum.

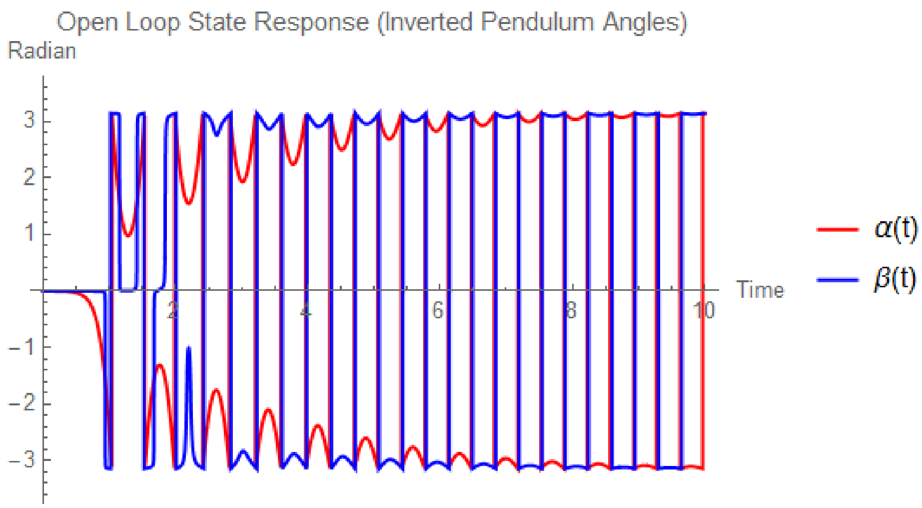

Figure 8, Open loop state response of the inverted pendulum

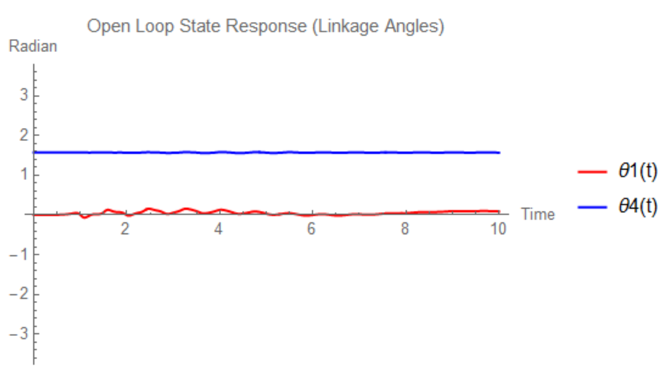

Figure 9, Open loop state response of the linkage angles

As expected without any control input to the system, the inverted pendulum will fall over. As shown in figure 8, the pendulum drops down and oscillates around its bottom equilibrium position at π radian or 180 degree.

In figure 9, we see that the inputs angles are relatively at their equilibrium since there is no control input to the system. The slight oscillation of the states is due to the motion of the inverted pendulum.

To make sure that our solver did hold the constraints of the system we plot the constraints equation verses time, and as shown in figure 10, we see that throughout solving the equations of motion we did not violate the constraints equations.

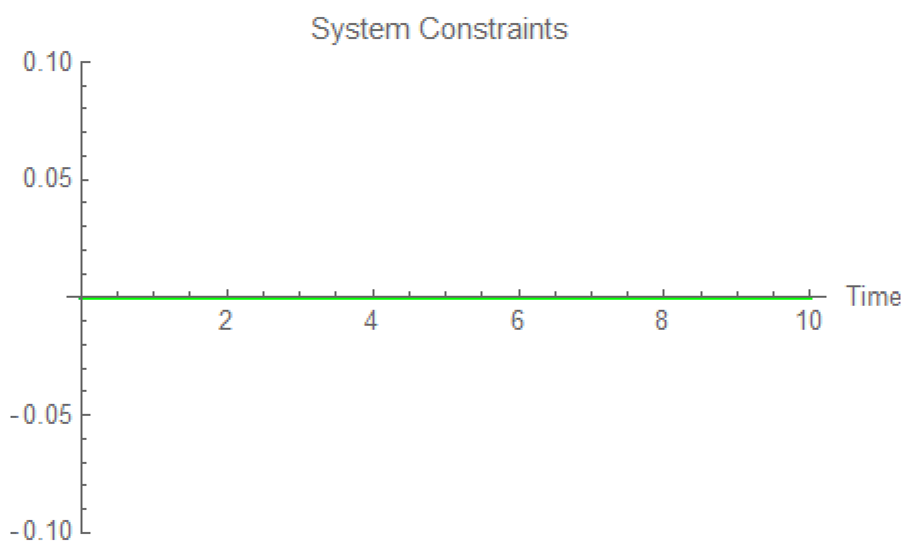

Figure 10, System Constraints

3.2. Close Loop Simulation

LQ problem is a case where the system dynamics are described by a set of differential equations and the cost is described by a quadratic function [8]. Minimizing the cost function can result in an optimal control for a dynamic system. LQR is a state feedback back controller with the following equations:

Using space state model (1.) system, with a quadratic cost function

$$J = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \tag{3.2}$$

Where x and u are the state and input of the LTI system (1.). The feedback control law which minimizes the cost function (3.2) is given as:

$$u = Kx \tag{3.3}$$

Where the gain K is given as:

$$K = -R^{-1}B^T X \tag{3.4}$$

In (3.4) X is the solution of an Algebraic Riccati Equation (ARE):

$$A^T X + XA - XBR^{-1}B^T X + Q = 0 \tag{3.5}$$

Using this method, we can obtain an optimal control gain K which is independent of the initial condition of the system.

When designing the controller, we set Q as the following matrix:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix} \qquad \text{Where } \rho = 50$$

And for R matrix we set it equal to:

$$R = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \qquad \text{Where } \gamma = 1.25$$

Now that we have a Q and R matrix we can solve the Algebraic Riccati Equation (3.5) and find the solution for X. To do so we can use Mathematica, since we also have A and B matrix from the state space model in table 2. The solution to the ARE for X is a 8 by 8 matrix. Once we find X, we can then solve for the gain K using equation (3.4).

$$K = \begin{bmatrix} 0 & -4.464 & 0 & -3.074 & 0 & -39.84 & 0 & -6.324 \\ 4.464 & 0 & 3.074 & 0 & 39.84 & 0 & 6.324 & 0 \end{bmatrix} \tag{3.6}$$

Linear Quadratic Estimation is an algorithm that estimates the unknown variables utilizing a series of measurements observed overtime.

Consider the LTI system:

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_w w \\ y &= C_y x + D_{yw} w \\ z &= C_z x \end{aligned} \tag{3.7}$$

With the cost function, of

$$J = \lim_{n \to \infty} E\left[\left(z(t) - \hat{z}(t)\right)^T \left(z(t) - \hat{z}(t)\right)\right] \tag{3.8}$$

And the observer

$$\begin{aligned} \dot{x} &= A\hat{x} + F(\hat{y} - y) \\ \hat{y} &= C_y \hat{x} \\ \hat{z} &= C_z \hat{x} \end{aligned} \tag{3.9}$$

Where the estimation gain F is given as:

$$F = -YC_y^T\left(D_{yw}WD_{yw}^T\right)^{-1} \quad (3.10)$$

Where Y is the solution of the ARE:

$$AY + YA^T - YC_y^T\left(D_{yw}WD_{yw}^T\right)^{-1}C_yY + B_wWB_w^T = 0 \quad (3.11)$$

Like LQR it turns out that we can use the same Q and R matrix for state estimation. By using equation (3.11) we can solve the ARE for Y and use it to find F. Since $B_w = 0$ in our system, and by setting $C_y = C$ we can solve the ARE to get Y which is an 8 by 8 matrix. By substituting Y in equation 3.11, we get a matrix for the state estimation F.

$$F = \begin{bmatrix} 0 & 4.331 & 0 & 163.446 \\ -4.331 & 0 & 163.446 & 0 \\ 0 & -0.389 & 0 & -14.711 \\ 0.389 & 0 & -14.711 & 0 \\ 0 & 0.478 & 0 & 18.096 \\ -0.478 & 0 & 18.096 & 0 \\ 0 & -6.337 & 0 & -0.478 \\ 6.337 & 0 & -0.478 & 0 \end{bmatrix} \quad (3.12)$$

Now that we have both the LQR and estimation gain we can use LQG to put our controller together.

Linear Quadratic Gaussian (LQG) control is one of the fundamental controller in optimal control. It is a combination of Linear Quadratic Estimator (LQE) with Linear Quadratic Regulator (LQR). LQG is an observer based controller with the following equations:

Consider the LTI system:

$$\dot{x} = Ax + B_uu + B_ww \quad (3.13)$$

$$y = C_y x + D_{yw} w$$

$$z = C_z x + D_{zu} u$$

And the observer based controller

$$\dot{x} = A\hat{x} + B_u u + F(\hat{y} - y)$$

$$\hat{y} = C_y \hat{x} \qquad (3.14)$$

$$u = K\hat{x}$$

With the cost function of:

$$J = \lim_{n \to \infty} E[z(t)^T z(t)] \qquad (3.15)$$

Where (K, F) stabilize the closed loop system are:

$$K^* = -(D_{zu}^T D_{zu})^{-1} B_u^T X^* \qquad (3.16)$$

$$F^* = -Y^* C_y^T \left(D_{yw} W D_{yw}^T\right)^{-1} \qquad (3.17)$$

$X^* and\ F^*$are the solution of the ARE

$$A^T X^* + X^* A - X^* B_u (D_{zu}^T D_{zu})^{-1} B_u^T X^* + C_z^T C_z = 0 \qquad (3.18)$$

$$AY^* + Y^* A^T - Y^* C_y^T \left(D_{yw} W D_{yw}^T\right)^{-1} C_y Y^* + B_w W B_w^T = 0 \qquad (3.19)$$

In equation (3.14) we can substitute $\hat{y}\ and\ u$ in the $\dot{x}$ equation. This results in

$$\dot{x} = A\hat{x} + BK\hat{x} + F(C\hat{x} - y)$$

$$\dot{x} = A\hat{x} + BK\hat{x} + FC\hat{x} - Fy$$

$$\dot{x} = (A + BK + FC)\hat{x} - Fy \qquad (3.20)$$

Equation (3.20) is the controller for the system. By inserting the parameters values in there we can get a state space model for the controller:

Table 4, Controller Space State Model

-53.5929	0.	-24.5384	0.	-549.901	0.	-71.5933	0.	0	-4.33084	0	-163.446
0.	-53.5929	0.	-24.5384	0.	-549.901	0.	-71.5933	4.33084	0	-163.446	0
47.6381	0.	21.8119	0.	423.559	0.	67.0987	0.	0	0.389458	0	14.7107
0.	47.6381	0.	21.8119	0.	423.559	0.	67.0987	-0.389458	0	14.7107	0
1.	0.	0.	0.	-18.096	0.	0.478291	0.	0	-0.478291	0	-18.096
0.	1.	0.	0.	0.	-18.096	0.	0.478291	0.478291	0	-18.096	0
0.	0.	0.5	0.	0.478291	0.	-6.33725	0.	0	6.33725	0	0.478291
0.	0.	0.	0.5	0.	0.478291	0.	-6.33725	-6.33725	0	0.478291	0
0	-4.46434	0	-3.07455	0	-39.8431	0	-6.32456	0	0	0	0
4.46434	0	3.07455	0	39.8431	0	6.32456	0	0	0	0	0

If we look at the eigenvalue of the A matrix of the controller:

$$\begin{gathered} Controller\ Poles = \\ -25.84 + 22.74i, -25.84 - 22.74i, -25.84 + 22.74i, -25.84 \\ + 22.74i, -6.41, -6.41, 1.873, 1.873 \end{gathered} \tag{3.21}$$

We see that the controller itself is unstable. There are poles in the right-hand side of complex plane; therefore, we are using an unstable controller to stabilize an unstable system. However, if we look at the poles of the closed loop system, in an other words the eigenvalues of A matrix for the closed loop system, we can see if the closed loop system is stable.

The closed loop system is given by:

$$\begin{gathered} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & B_u K \\ -FC_y & A + B_u K + FC_y \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_w \\ -FB_w \end{bmatrix} w \\ z = \begin{bmatrix} C_z & D_{zu} K \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \end{gathered} \tag{3.22}$$

$$Closed\ Loop\ Poles =$$

$$-13.36 + 1.11 \times 10^{-10}i, -13.36 - 1.11 \times 10^{-10}i, -13.05 + 1.61 \times 10^{-10}i, -13.05 - 1.61 \times 10^{-10}i, -9.21 + 4.82 \times 10^{-9}i, -9.21 - 4.829 \times 10^{-9}i, -9.08 + 5.06 \times 10^{-9}i, -9.08 - 5.06 \times 10^{-9}i, -6.66 + 6.57 \times 10^{-10}i, -6.66 - 6.57 \times 10^{-10}i, -6.31 + 4.72 \times 10^{-10}i, -6.31 - 4.72 \times 10^{-10}i, -6.16 + 8.48 \times 10^{-10}i, -6.16 - 8.48 \times 10^{-10}i, -3.34, -3.34$$

Now that we have the closed loop system model, we can simulate the non-linear system from chapter 2, with the controller added. The figure below shows the response of the system where the initial angle of the inverted pendulum is at 20 degrees.

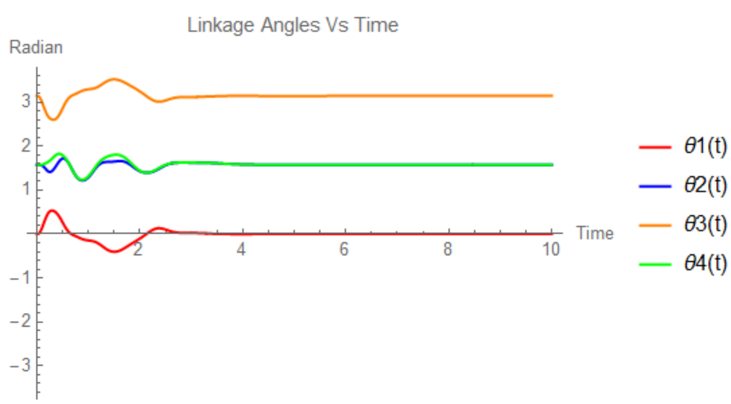

Figure 11, Linkage angels behavior with 20 degree initial condition of the inverted pendulum

Figure 11, shows the plot of the angles of the five-bar linkage with the controller added to the system.

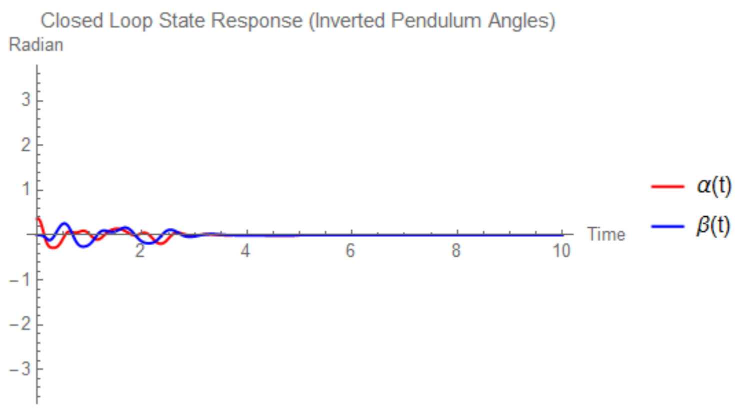

Figure 12, closed loop state response of inverted pendulum

Figure 12, shows the closed loop state response for the inverted pendulum. As shown in the plot the settling time is about 4 seconds.

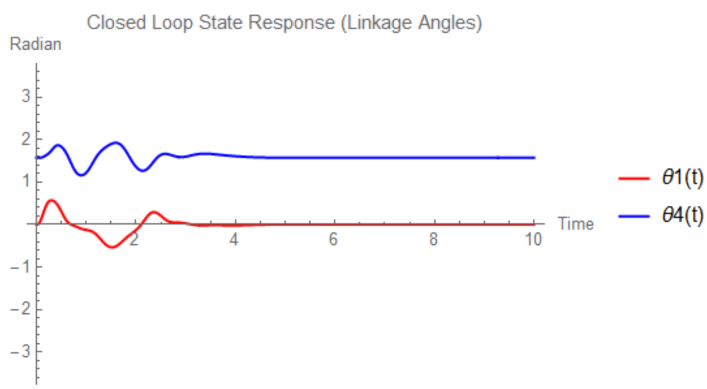

Figure 13, Closed loop state response of the linkage angles

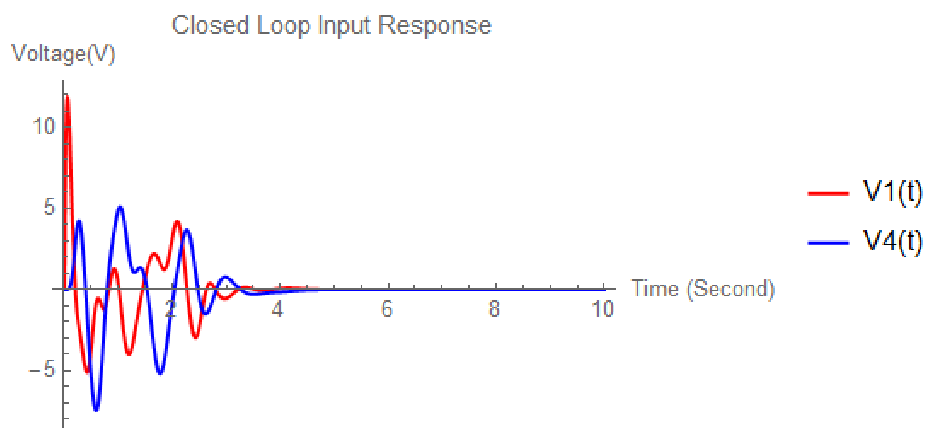

Figure 14, Closed loop input response

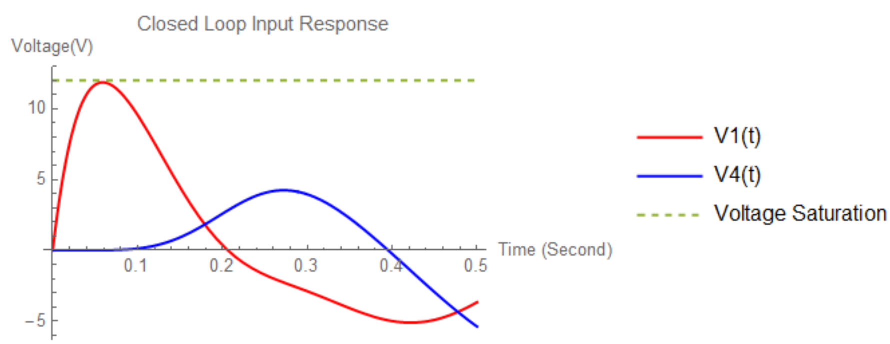

Figure 15, Closed loop input response (first half second)

Figure 14 and 15 shows the input voltage of the system at each motor. In figure 15, we can see the voltage saturation line in which, we can make sure that the input voltage does not exceed 12 volts.

Chapter 4

Experimental Setup and Conclusion

The experimental setup of the system has not yet been complete. Future work will

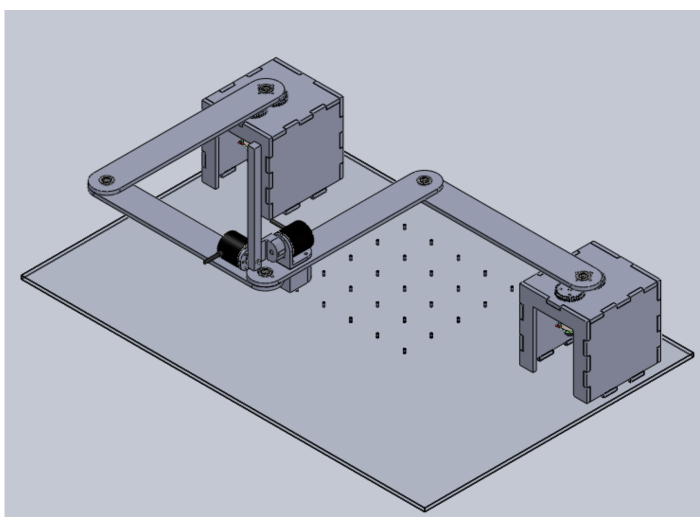

Figure 16, CAD model of the system hardware

be done in order to complete the experiment. So far, we have competed the hardware part of the experiment, and more work on implementing the control algorithm needs to be done.

Figure 16, shows the CAD model of the system. It consists of 2 motor box, 4 links, and the inverted pendulum part.

Each motor box consists of a DC motor and an encoder, with a 1:1 gear contacting them together. The encoder is directly attached to the links, which minimizes the effect of the backlash of the gear box of the motor. Two encoders are used to measure the angles of the inverted pendulum. There are more work needs to be done on the experimental part of this project.

By simulating the non-linear model, and using it to adjust the controller we were able to achieve stabilization in a case where the inverted pendulum is theoretically 20 degrees off from vertical axis. In future work this can be check against the simulation of the linearized model to see the impact of nonlinear modeling.

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