

UCLA

UCLA Electronic Theses and Dissertations

Title

Stimulated Raman Scattering of Lasers with Orbital Angular Momentum in the Kinetic Regime

Permalink

<https://escholarship.org/uc/item/8cn96472>

ISBN

9798252423968

Author

Chase, Sarah Elizabeth

Publication Date

2026-06-12

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Stimulated Raman Scattering
of Lasers with Orbital Angular Momentum
in the Kinetic Regime

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Physics

by

Sarah Chase

2026

© Copyright by

Sarah Chase

2026

ABSTRACT OF THE DISSERTATION

Stimulated Raman Scattering of Lasers with Orbital Angular Momentum in the Kinetic Regime

by

Sarah Chase

Doctor of Philosophy in Physics

University of California, Los Angeles, 2026

Professor Warren B. Mori, Chair

The nonlinear optics of plasmas (NLOP) is a fundamental topic within plasma physics. A sub-area of NLOP is laser-plasma interactions (LPI) that involve three-wave coupling. Three-wave LPI processes are of interest for inertial fusion energy (IFE) and are a fundamental nonlinear plasma physics research area. Stimulated Raman scattering (SRS) is one LPI of interest where an incident pump laser decays into a scattered electromagnetic wave (EM) and an electron plasma wave (EPW). There is much interest in SRS backscatter where the direction of the scattered EM wave is opposite that of the pump. Much previous research on SRS has involved the study of how plane waves propagate in plasmas. However, in IFE the lasers are broken up into multiple finite-width and length speckles which are related to Gaussian-limited laser pulses, which can be decomposed into Laguerre-Gaussian modes. A Laguerre-Gaussian mode can have orbital angular momentum (OAM).

The interaction of finite-width Gaussian laser pulses with plasmas is investigated through theory, fluid simulations, and fully kinetic particle-in-cell simulations. As a starting point,

the reasoning for why a finite-width laser can have orbital angular momentum in vacuum is carefully reviewed, including the conclusion that within a Laguerre-Gaussian mode defined by a spot size w_0 with an angular dependence $e^{i\ell\varphi}$ each photon has an angular momentum $\ell\hbar$. This is referred to as the photon's orbital angular momentum (OAM). The properties of Laguerre-Gaussian modes are examined in terms of the laser's power, peak intensity, and spot size. The regimes of growth for one-dimensional SRS backscatter are also outlined and reviewed. A regime of interest for IFE is the strongly damped regime where damping of the EPW is sufficiently large that its spatio-temporal growth is altered.

Starting from the coupled envelope equations for SRS in a uniform plasma within the paraxial ray approximation including a phenomenological damping term for the EPW, multi-dimensional conservation laws for wave action, energy, axial and transverse momentum, and angular momentum are derived. Sink terms for the otherwise conserved quantities from the EPW damping are included. The expression for OAM in SRS is shown to be the sum over the OAM expression for each of the three waves. The expression for the OAM of each of the EM waves in plasma is identical to the known expressions for an EM wave in vacuum. In normalized units the OAM of the EPW has the same form as that for the EM wave. However, for the EPW the OAM is shown to be the result of the angular momentum of the currents within the finite-width wave and not from the Poynting flux as is the case for the EM waves.

It is shown that conservation of energy and momentum are equivalent to ω and k matching of the three waves because ω and k are constant properties of the carrier and not the envelope. A general wave is not a pure azimuthal harmonic so an effective or average ℓ is defined. The average ℓ number is a property of the envelope and not the carrier of the wave. It is thus argued that ℓ matching is not an absolute concept and is only true when the ℓ of each wave is constant.

Simulations of SRS of a single Gaussian speckle are studied using the fluid code pF3D which solves the three coupled envelope equations for SRS and the fully kinetic particle-in-

cell code OSIRIS. Results from an ensemble of pF3D simulations are presented that explore the effects of OAM, intensity, power, and spot size of the laser, as well as the temperature of the plasma which controls the dimensionless parameter, $k\lambda_D$, which is a key determiner of the collisionless damping of the plasma wave where k is the wavenumber of the plasma wave and λ_D is the Debye length. The pF3D simulations show that for both SRS growing from noise or well-defined seeds, OAM can be transferred from the pump to the decay EM and EPW waves. It is found that the degree of SRS growth is mostly determined by the peak intensity of the Laguerre-Gaussian mode indicating that for constant power and spot size, SRS is reduced for pumps with larger ℓ numbers. Conservation of the effective ℓ number arises for pumps with the highest intensities and the decay waves are not pure Laguerre-Gaussian modes even when the effective ℓ number is near an integer. When SRS grows from a well-defined seed and the reflectivity is low the EM mode often has no OAM.

Fully kinetic particle-in-cell (PIC) simulations of SRS from a single Gaussian speckle reveal a different behavior. For pumps without OAM, SRS grows more rapidly with higher peak and average reflectivity for otherwise similar parameters in the PIC simulations, and it comes in narrower and more intense bursts. However, the trend that SRS reflectivity is reduced for pumps with OAM and fixed laser power and spot size is also seen in PIC simulations. OAM is seen to be transferred from the pump to the scattered light wave and angular momentum is transferred to the particles. In cases where the pump has OAM, no obvious additional kinetic effects of the EPW are observed.

The dissertation of Sarah Chase is approved.

E. Paulo Da Costa Alves

George J. Morales

Chandra J. Joshi

Warren B. Mori, Committee Chair

University of California, Los Angeles

2026

To my mother

TABLE OF CONTENTS

1	Introduction	1
2	Theory	10
2.1	Introduction	10
2.2	Angular momentum of light	12
2.2.1	Laguerre-Gaussian functions	13
2.2.2	OAM of an EM wave	17
2.2.3	SAM of an EM wave	20
2.2.4	Quanta of angular momentum per photon	22
2.3	SRS	22
2.3.1	Single particle motion in a laser field	23
2.3.2	Laser plasma coupling	26
2.3.3	SRS growth	37
2.3.4	Reflectivity	43
2.3.5	Wave-particle interactions	45
2.3.6	Conservation laws	47
2.4	Summary	58
3	Fluid simulations of Stimulated Raman Scattering	60
3.1	Introduction	60
3.2	Overview of pF3D	61
3.3	Spontaneous SRS	74

3.3.1	Simulations without OAM at constant temperature	77
3.3.2	Simulations with different temperatures	79
3.3.3	Simulations with constant temperatures	82
3.4	Growth of a well-defined seed	100
3.4.1	Spatial shape and OAM indices	103
3.5	Summary	109
4	Particle-in-Cell Simulations of Stimulated Raman Scattering with Orbital Angular Momentum	111
4.1	Introduction	111
4.2	Simulation setup	112
4.3	Diagnostic Methods	116
4.3.1	Measuring Reflectivity	116
4.3.2	Measuring Orbital Angular Momentum	119
4.4	Results and Discussion	122
4.4.1	Comparison between pF3D and OSIRIS	123
4.4.2	Field Dynamics and Wave Structures	128
4.4.3	Kinetic Effects and Particle Dynamics	131
4.4.4	OAM in the waves	134
4.5	Summary	138
5	Conclusion and directions for future work	140
A	Conservation laws for fields and particles	146
A.1	Conservation of Energy	148

A.2	Conservation of Momentum	149
A.3	Conservation of Angular Momentum	152
A.4	Rewriting particle contributions as fluid plasma quantities	154
A.4.1	Conservation of energy with fluid quantities	154
A.4.2	Conservation of momentum with plasma fluid quantities	157
A.4.3	Conservation of angular momentum with plasma fluid quantities . . .	158
References		161

LIST OF FIGURES

- 3.1 (a-c) Real and imaginary parts of the EPW frequency normalized to ω_p and EPW wavenumber normalized to λ_D^{-1} as a function of density and temperature computed from a non-relativistic kinetic solver. We are interested in $-\text{Im}(\omega_2)$ because strong damping occurs for larger negative values. (d) Real part of ω_2 as a function of k_2 compared to the Bohm-Gross dispersion relation $\frac{\omega_{BG}}{\omega_p} = \sqrt{1 + 3(k_2\lambda_D)^2}$. (e) The negative imaginary part of ω_2 as a function of k_2 compared to the asymptotic Landau damping rate $-\frac{\gamma_L}{\omega_p} = \frac{\sqrt{\pi}}{k_2\lambda_D(1+3(k_2\lambda_D)^2)^{3/2}} e^{-\frac{1}{2(k_2\lambda_D)^2} - \frac{3}{2}}$. The curves in (d-e) correspond to a density of $0.14n_c$ and a range of temperatures. 65
- 3.2 Three conditions for a transition from the strongly to weakly damped regime shown on linear (a) and logarithmic (b) scales. Each line represents the intensity where the transition condition is satisfied as a function of $k_2\lambda_D$. Both $k_2\lambda_D$ and the damping rate, ν , depend on the plasma density and temperature. The blue line shows the intensity for which the temporal growth rate is greater than the damping rate (the strongly-damped temporal transition) as described in Chapter 2. The highest transition intensity (orange) corresponds to the transition from the weakly-damped to the strongly-damped spatial regime. The green line shows the intensity where the spatial gain exponent of the field (power) exceeds 10 (20) so that the backscatter grows by a factor of e^{10} . The gain exponent is $\beta_s L$ where L is the simulation box length which is $200\lambda_0$. The space between the blue and orange lines represents parameters where SRS is simultaneously in the weakly-damped temporal regime and the strongly-damped spatial regime. 67

3.3	Radial laser profiles for $\ell = 0, 1, 2, 3$. To compare lasers with different amounts of OAM, we match two of the following three parameters: power, peak intensity, and w_0 . In (a), power and w_0 are the same for each laser; in panel (b), peak intensity and w_0 are the same; and, in panel (c), power and peak intensity are matched. In (c), w_0 changes for each ℓ while the effective width, w_ℓ , is constant. For (a-b), w_0 is $4\lambda_0$ and for (c), w_ℓ is $4\lambda_0$	72
3.4	The ratio of the integral of $e^{2\beta_s L}$ to the power at best focus for varying intensity and power. Panels (a)-(c) use laser profiles that correspond to figure 3.3 (a)-(c) respectively.	74
3.5	At the start of the simulation, random thermal noise exists in the EPW as shown in panels (d) and (f). Panels (a) and (c) show the pump laser intensity ($ b_0 ^2$) and (b) and (e) show the reflected light intensity ($ b_1 ^2$). In the top row, the pump intensity is zero, i.e., no laser exists. In the second row, a small pump is present and therefore a corresponding signal appears in panel (e) due to the interaction of the pump and the random thermal plasma noise. The units in panels (d) and (e) are W/cm^2 , and panels (c) and (f) are in units of the critical density.	76
3.6	Panel (a) shows $\hat{R}(t)$, reflectivity variations over time for 10 ps. Each line is a time series from a simulation with a different peak laser intensity and the intensities are listed in the legend in (b). Panel (b) shows average reflectivity for many simulations with different intensities. The markers on the curve in (b) show $\langle \hat{R} \rangle$, the average reflectivity, for the eight cases where $\hat{R}(t)$ is plotted in (a). The time average is taken over the entire simulation (10 ps). The black dashed line in (b) shows the threshold for transition from the strongly-damped to weakly-damped spatial regime which occurs at $2.52 \times 10^{16} \text{ W}/\text{cm}^2$. The dashed curve labeled “R (theory)” shows the numerical solution to the equation 2.187. Every simulation uses a Gaussian pump laser with no OAM.	78

- 3.7 Average reflectivity vs $k_2\lambda_D$ for simulations with different OAM. In (a) and (d), the lasers have the same power and same w_0 , while the peak intensities differ. The peak intensity for the case without OAM is 2×10^{16} W/cm². This is the same in each plot so the blue line is the same in each plot. In (b) and (e), the peak intensity is the same (2×10^{16} W/cm²) for all OAM, while the power differs, and in (c) and (f) w_0 changes so that power and peak intensity are the same for all OAM (constant w_ℓ). Panels (a)-(c) use a spot size of $4\lambda_0$ for $\ell = 0$ and panels (d)-(f) use a spot size of $8\lambda_0$ for $\ell = 0$. The threshold line shows the value of $k_2\lambda_D$ where SRS transitions from the weakly damped to strongly damped spatial regime. In (b-c) and (e-f), the threshold is $k_2\lambda_D = 0.31$ because all simulations have the same I_{peak} . In (a) and (d), the peak intensity depends on ℓ so the threshold value of $k_2\lambda_D$ changes. The dashed lines labeled R (theory) represent the solution to equation 2.187 where the best fit parameters are determined separately for $\ell = 0$ and $\ell = 1$. The fit parameters are given in the text. 80
- 3.8 Average reflectivity, $\langle \hat{R} \rangle$, vs power of the incident laser for different amounts of OAM. The transition intensity from the strongly to weakly damped spatial regime is 1.87×10^9 W for the $\ell = 0$ beam but increases for other ℓ due to shape. Transitions for higher OAM exceed the bounds of the plot. The dashed curves labeled $\hat{R}$ (theory) represent solutions to equation 2.187 for fitted parameters which are separate for $\ell = 0$ and $\ell = 1$. The fitting parameters are shown in the text. 83
- 3.9 Average reflectivity $\langle \hat{R} \rangle$ vs I_{peak} for lasers with different OAM. The transition intensity is the same for each ℓ and is equal to 6.04×10^{16} W/cm². The spot size is $4\lambda_0$ in each case and the OAM beams have more power at the same intensity due to their different shape. The dashed lines labeled R (theory) show solutions to equation 2.187 with fitting parameters in the text. 85

3.10	Reflectivity $\langle \hat{R} \rangle$ vs. intensity and power when the different OAM modes have the same effective spot size, w_ℓ , of $4\lambda_0$ and therefore w_0 is $(4.0, 2.43, 2.08, 1.89)\lambda_0$ for $\ell_0 = 0, 1, 2, 3$. Adjusting w_0 with ℓ allows for the different OAM cases to be compared at the same power and intensity simultaneously. The transition, which depends only on peak intensity and plasma conditions, is the same for all ℓ . The dashed lines labeled $\hat{R}$ (theory) represent solutions to equation 2.187 for fitting parameters given in the text.	87
3.11	Effective OAM for all simulations as a function of z at $t = 5.4$ ps. Panels (a), (b), (c), and (d) show simulations where the pump laser has OAM equal to $\ell_0 = 0, 1, 2$ and 3 . The color of each line depends on the incident peak intensity of the pump light where a darker color represents a higher intensity laser. Higher intensities also correspond to greater reflectivity as shown in previous figures. . .	89
3.12	Effective OAM at constant peak intensity for four simulations with different OAM in the pump light at 5.4 ps. The intensity is 2.5×10^{16} W/cm ² in each case for which the reflectivity is low. The blue lines show the OAM of the pump light which is constant in every case because pump depletion is low. The orange line shows the OAM in the backscattered light and the green line shows the OAM in the EPW. The red line shows how well OAM is conserved through the variation in $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$	92
3.13	Intensities of the three waves from simulations with different pump OAM and SRS growth from noise. Each column represents one simulation. The top row shows the intensity of the pump light, the middle row shows the intensity of the scattered light, and the bottom row shows the intensity of the plasma wave. Although the pump is tightly focused at $z = 0$, the scattered waves are strongest near z_{min} . The vertical dashed white line shows the location of the transverse slices in figure 3.14. The peak intensity for the case without OAM is 2.4×10^{16} W/cm ² and for the three cases with OAM it is 2.0×10^{16} W/cm ²	94

3.14	Transverse slices of intensity for the pump and daughter waves for different pump OAM. The slices are taken near z_{min} as shown by the dashed white lines in figure 3.13. The top row shows the pump light intensity, the middle row shows the scattered light intensity, and the bottom row shows the EPW.	96
3.15	Cross sections for different pump intensities from simulations which use pumps with $\ell = 0$ and $\ell = 1$. The top half shows the results for $\ell = 0$ and the bottom half shows results for $\ell = 1$. The top row of each half shows the real part of the pump wave, the second row shows the real part of the scattered light wave, and the third row shows the real part of the plasma wave. Each column corresponds to a different pump intensity. The intensities are listed at the top and are equal to 1×10^{16} , 2×10^{16} , 3×10^{16} , 4×10^{16} , and 5×10^{16} W/cm ² . Each plot is on a different color scale in order to maximize visible detail. Colorbars are excluded for brevity.	98
3.16	Cross sections for the three waves for different pump intensities from simulations which use pumps with $\ell = 2$ and $\ell = 3$. The top half shows the results for $\ell = 2$ and the bottom half shows results for $\ell = 3$. The top row of each half shows the real part of the pump wave, the second row shows the real part of the scattered light wave, and the third row shows the real part of the plasma wave. Each column corresponds to a different pump intensity. The intensities are listed at the top and are equal to 1×10^{16} , 2×10^{16} , 3×10^{16} , 4×10^{16} , and 5×10^{16} W/cm ² . Each plot is on a different color scale in order to maximize visible detail. Colorbars are excluded for brevity.	99

3.17	The solid lines show the gain from simulations with pumps and seeds with different OAM. Panel (a) shows cases where the pump laser has no OAM. The blue lines are for a seed with no OAM, the orange lines for a seed with $\ell_1 = 1$ and the green lines for a seed with $\ell_1 = 2$. Panels (b) and (c) show the same for pumps with $\ell_0 = 1$ and $\ell_0 = 2$. The dashed lines are the calculated predictions from equations 3.35 and 3.34. The fitting parameters are in the text.	102
3.18	Two dimensional slices in (x, y) and (z, y) of the pump, seed, and EPW. The pump and seed both have $\ell = 0$ (No OAM). The (z, y) slice is shown in the right column, which is a slice at $x = 0$. The left column is an (x, y) slice at the location shown by the dashed white line on the right plot, near $z = -108\lambda_0$	104
3.19	Two dimensional slices in (x, y) and (z, y) of the pump, seed, and EPW. The pump and seed both have $\ell = 1$. The (z, y) slice is shown in the right column, which is a slice at $x = 0$. The left column is an (x, y) slice at the location shown by the dashed white line on the right plot, near $z = -108\lambda_0$	105
3.20	Same plots as in figures 3.18 and 3.19 except the pump laser does not have OAM but the counter-propagating seed has $\ell_1 = 1$. The plasma wave develops OAM. The amplified seed and plasma wave profiles in both (x, y) and (z, y) are not radially symmetric, indicating a change in OAM due to SRS.	106
3.21	Same plots as in figures 3.18 - 3.20 except a seed without OAM is amplified by an $\ell_0 = 1$ pump. The plasma wave develops OAM, and the shape of the seed is modified.	108
3.22	The effective OAM of the seed and plasma wave changes when the pump has no OAM. As the seed grows, it loses OAM. The plasma wave initially has OAM opposite the seed and then it also evolves toward zero. The intensity for the pump is $4.9 \times 10^{16} \text{ W/cm}^2$	108

3.23	The effective OAM when the pump has $\ell_0 = 1$ and the seed has either $\ell_1 = 0$ (a) or $\ell_1 = 2$ (b). The intensity in both cases is $4.9 \times 10^{16} \text{ W/cm}^2$. The OAM of the seed does not change into the OAM of the pump like in the $\ell_0 = 0$ case.	109
4.1	Electric field (in the x_3 direction) of the incident pump lasers early in time. The plots are slices in (x_2, x_3) at x_1 shown by the white dashed line and in (x_1, x_2) at $x_3 = 0$. A Gaussian laser with no OAM (top) and a Laguerre-Gaussian mode with $\ell_0 = 1$ (bottom). The right column shows the wave propagation and the left column shows a transverse cut taken at best focus which is shown as the dashed white line on the right side plots.	115
4.2	OAM of the pump laser early in time. $\bar{\ell}$ is measured for a simulation with an antenna that launches a pump with $\ell_0 = 0$ and an antenna that launches a pump with $\ell_0 = 1$. Additionally, $\bar{\ell}_1$ is measured for the background noise before any SRS signals have appeared.	121
4.3	Particle phase space distribution at the start of the simulation. The horizontal axis is momentum in the x_1 direction and the vertical axis is angular momentum in the x_1 direction. The color represents the count of particles at each point in the (p_1, m_1) plane where the brightest spot is the highest concentration of particles.	122
4.4	Comparison of reflectivity for pF3D simulations for each simulation in table 4.1. The top row shows results with $\ell_0 = 0$ and the bottom row corresponds to simulations where $\ell_0 = 1$. The results in left panels use a box length of $100\lambda_0$ and the right panels use a box length of $200\lambda_0$	124
4.5	Reflectivity in OSIRIS (blue) and pF3D (orange) for simulations B-D in table 4.1. pF3D simulates longer times than OSIRIS. Reflectivity in OSIRIS comes in sharp bursts and average reflectivity is greater.	124
4.6	Comparison of reflectivity for OSIRIS (blue) and pF3D (orange) for simulations E-I in table 4.1. OSIRIS simulations have higher peak and average reflectivities.	125

4.7	Reflectivity over time for OSIRIS simulations with and without OAM. At later times and for the highest intensity (red curve), the use of periodic boundary conditions begins to alter the results.	126
4.8	We simulated two cases with a pump laser at lower intensity for a longer time in order to measure low reflectivities. For the case without OAM, the peak value of a_0 is 0.036, and for the $\ell_0 = 1$ case the peak value is $a_0 = 0.030$, corresponding to simulations A and E in table 4.1. Both simulations show very little SRS until $\omega_0 t > 6000$	127
4.9	Average reflectivity versus a_0 (left) and power (right) for OSIRIS and pF3D simulations. The blue line represents PIC simulations where the pump laser has $\ell = 0$ and the green line shows PIC simulations where the pump laser has $\ell_0 = 1$. The corresponding pF3D simulations are also shown in orange (for $\ell_0 = 0$) and red (for $\ell_0 = 1$). The trend for OSIRIS and pF3D simulations is similar between $\ell_0 = 0$ and $\ell_0 = 1$, but the OAM simulations show higher reflectivity for the same a_0 and lower reflectivity for the same power.	128
4.10	Fourier transform of E_1 and E_3 at a single (x_2, x_3) position. For the simulation without OAM (top row), the lineout is along the axis at $x_2 = 0, x_3 = 0$ and for the $\ell_0 = 1$ case (bottom row), the lineout is at $x_2 = 0, x_3 = w_0/\sqrt{2} \approx 18c/\omega_0$ which is near the r of maximum amplitude for an LG mode. The left column shows E_1 , the middle column shows E_3 , and the right column shows a lineout at $\omega_0 t = 2210$. The three dashed vertical lines in the plots in the right column represent the k of each wave.	129
4.11	EM fields at the beginning of a burst of SRS in a simulation where the incident light does not have OAM. The top panel shows the field of the Gaussian pump laser, the middle panel shows the backscattered light wave, and the bottom panel shows the field of the EPW.	131

4.12	EM fields during a burst of SRS for a simulation where the pump is a Laguerre-Gaussian mode with $\ell = 1$. The top panel shows the incident light, the middle panel shows the scattered light, and the bottom panel shows the plasma wave.	132
4.13	Nonlinear wavenumber (frequency) shifts are shown in k space over time at a single radial position for a simulation with $\ell_0 = 0$ (left) and $\ell_0 = 1$ (second to the right). The white line shows the reflectivity at the same times. The second and fourth panel show the change in the slope of the distribution function where brighter spots correspond to a flatter distribution function. For the case with $\ell_0 = 1$ (right), the change in f is significant at later times due to the development of a current.	132
4.14	The change in the slope of the distribution function over time for simulations C, D, G, and H in table 4.1. Brighter spots correspond to flattening of f . For the two cases with $\ell_0 = 1$, a current develops at later times which affects the distribution function even far from v_{ph} .	134
4.15	Light wave OAM at $\omega_0 t = 3400$ when reflectivity is high. The pump in this simulation (G) has $\ell_0 = 1$ and $a_0 = 0.053$. The OAM of the pump (blue) is approximately the same as its initial value of $\ell_0 = 1$. The OAM of the scattered light (orange) varies across the box with a maximum $\bar{\ell}_1 \approx 2$.	135
4.16	Average linear and angular momentum of the particles. The left column shows data for simulations C (blue) and D (orange) in which the pump does not have OAM. The right column shows simulations G (blue), H (orange), and I (green) which all include a pump with $\ell_0 = 1$. In every case with $\ell_0 = 1$, the plasma develops an average p_1 that is negative, and an average m_1 that varies but is nonzero. The top row shows reflectivity measured from the Poynting flux in order to demonstrate the correlation of the timing of the SRS bursts and changes to the distribution function.	137

4.17 Phase space distributions for simulations B (top) and G (bottom). The left column shows the distribution of particles in p_1 and x_1 , the middle column shows p_1 and m_1 , and the right column shows the average angular momentum along x_1 . For both simulations, the data is shown at a time when reflectivity is at its peak. 138

LIST OF TABLES

2.1	Growth rates for purely temporal or spatial growth with weak or strong EPW damping. In a real scenario, both temporal and spatial gain can occur and damping may be neither negligible nor dominant.	42
3.1	The relationship between power and peak intensity depends on ℓ for lasers with OAM. For a Gaussian laser ($\ell = 0$), the ratio of the power to peak intensity is $\frac{\pi}{2}w_0^2$, where w_0 is the spot size. For nonzero ℓ , we define an effective width w_ℓ so that $\frac{P}{I_{peak}} = \frac{\pi}{2}w_\ell^2$. The table shows how w_ℓ/w_0 varies with ℓ for $\ell = 0, 1, 2, 3, 4, 5$	71
4.1	Summary of parameters for SRS simulations with and without OAM. Individual simulations are labeled A through I. The antenna in simulations with OAM uses $\ell_0 = 1$ and those without OAM use $\ell_0 = 0$	112
4.2	Parameters used for all simulations in both normalized and physical units. The box length is $100\lambda_0$, the spot size is $4\lambda_0$, and the box width is $8w_0$ or $32\lambda_0$	113
A.1	Scalars and tensors derived from conservation laws for fields and particles including plasma fluid moments of the distribution function	160

ACKNOWLEDGMENTS

I would like to extend my sincere gratitude to the many people and institutions that supported the research presented in this dissertation.

First and foremost, I thank my advisor, Professor Warren Mori, for his scientific guidance and insight. I would also like to express my gratitude to the members of my thesis committee: Professor Chandrashekhar Joshi, Professor George Morales, and Professor E. Paulo Alves. Their feedback and consultation have been instrumental throughout this process. Although Dr. Denise Hinkel was not a formal member of my committee, I want to acknowledge that she played a role similar to that of a committee member. Her mentorship was greatly appreciated.

Additionally, I am grateful to my fellow members of the research group and colleagues at UCLA, whose collaboration and insightful discussions were a constant source of motivation. I also wish to thank the scientists who mentored me at UCLA and LLNL, including Dr. Frank Tsung and Dr. Mike Belyaev. I am grateful to the broader plasma physics community. The opportunity to present my work and learn from others at numerous conferences has been essential to my development as a scientist.

This work would not have been possible without generous institutional and financial support. I gratefully acknowledge the support of Lawrence Livermore National Laboratory through the multi-year ACT-UP award. I also thank the Meyers family for their generous support of the UCLA Simulation of Plasmas Group. Their gift allowed me to continue my research when other funding sources ended.

Finally, I wish to deeply thank my family and friends for their love and encouragement throughout this journey.

VITA

- 2016 B.A. (Physics) and B.A. (Statistics), University of California Berkeley.
- 2019 M.S. (Physics), University of California Los Angeles.

PUBLICATIONS

V. Patel, S. Chase, F. S. Tsung, J. P. Palastro, D. E. Hinkel, and W. B. Mori. Generalized multi-dimensional conservation laws for stimulated Raman and Brillouin scattering in a density gradient. March 2026

CHAPTER 1

Introduction

Fusion has long been promised as a clean, abundant, and sustainable source of energy for humanity's future. For over half a century, researchers have aimed to replicate the mechanism of energy production in the Sun: thermonuclear fusion of hydrogen, other isotopes of hydrogen, or other elements. With success of experiments at the National Ignition Facility [BHin, Linin] culminating with the demonstration of ignition [Abu24], in recent years interest in fusion has shifted from the realm of science fiction to one with well-defined roadmaps.

In the Sun, gravitational forces create the high densities and temperatures required for hydrogen nuclei to overcome electrostatic repulsion, fuse into helium, and release energy. On Earth such conditions can be met by either confining a moderate density plasma for long periods of time (even steady state) via magnetic fields or a high plasma density for short periods of time via inertia. A large device called ITER [SCM07] is being built in France with the hope of demonstrating a burning plasma where more energy is produced than is input into the device. Another approach for controlled fusion is to recreate the extreme conditions at the core of the Sun on Earth through laser-driven inertial confinement fusion (ICF), in which powerful lasers are used to either directly [CABin] or indirectly [Linin] drive a rapid compression of a small fuel target containing deuterium and tritium, two heavy isotopes of hydrogen. With respect to ICF, ignition refers to the threshold where the fusion reaction produces more energy than the fuel absorbs, and, in December 2022, the National Ignition Facility (NIF) reached this milestone for the first time [Abu24], marking a significant breakthrough for ICF and fusion research in general. Extending the success of

ICF experiments towards energy on the grid constitutes the subject of inertial fusion energy (IFE).

In direct-drive ICF [CABin], multiple laser beams are spherically uniformly aimed directly at a spherical capsule of D-T fuel. The temporal profile of the laser energy is designed so that the pressure it provides leads to an energy-efficient compression of the fuel causing it to ignite. In indirect-drive ICF [Linin], the capsule is positioned at the center of a hohlraum, a cylindrical container coated with a high-Z material. Laser beams enter through holes on either end of the hohlraum and are aimed as uniformly as possible at the inner walls. As the interior walls heat, they in turn re-radiate the energy in the form of X-rays creating a uniform blackbody spectrum within the hohlraum. This creates a spherically uniform pressure bath of X-rays based on the Stefan-Boltzmann law. The radiation pressure causes the outer layers of the capsule to ablate, driving a symmetric compression of the core, which initiates fusion of the D-T fuel. As for direct drive, the temporal intensity profile is also shaped to provide an efficient implosion. Compared to indirect drive where laser energy is first converted into X-rays, direct-drive ICF can be more efficient. However, at present only ignition from indirect drive has been demonstrated.

In indirect drive, as the walls are heated they can ablate and fill the hohlraum. As this occurs, they can block the path of the incoming lasers thereby limiting the duration of the laser and laser energy that can be used. This can limit the achievable energy gain. If the hohlraum is filled with a low-density gas it can become ionized and heated to provide a back pressure on the walls thereby preventing blowoff and permitting higher gain [Linin]. A gas fill also helps to stabilize the plasma environment and ensure X-ray uniformity.

In the presence of the gas fill the incoming lasers must propagate through long distances of tenuous plasma. Even at moderate plasma densities, the lasers are susceptible to laser-plasma instabilities (LPI) [Kru88, Mic23] which can thus become a significant challenge in gas-filled hohlraums. These instabilities reduce laser efficiency by deflecting and absorbing energy, which can compromise the symmetry and performance of the implosion. In stimulated

Brillouin scattering (SBS) [FKLinb, DKL74], a portion of the laser beam decays into a low-frequency ion acoustic wave (IAW) in the plasma. Related to SBS is a process called cross-beam energy transfer (CBET) [KWA96, MLG96, MDW09]. CBET leads to the exchange of energy between laser beams that cross at small angles. This can be used to redistribute energy between crossing laser beams to tune the distribution of X-rays [MDW09]. At higher intensities it can also lead to an unpredictable redistribution of laser energy. Additionally, stimulated Raman scattering (SRS) [FKLinb, DKL74] can generate an electron plasma wave (EPW), which not only scatters the laser but may also preheat the fuel [EKL80, FKLina, LLK79] by trapping and accelerating background plasma electrons. Mitigating these LPIs - particularly SRS - is essential for improving laser-target coupling and optimizing performance in gas-filled hohlraums.

Understanding and controlling SRS is therefore a central focus of efforts to optimize future ICF experiments. Furthermore, the physical mechanisms behind SRS and related instabilities are not limited to fusion applications alone. They are examples of a broader class of wave interactions known as the nonlinear optics of plasmas (NLOP) [Kru88, Mic23]. This field has been investigated for over half a century, initially driven by applications in both laser-driven ICF and ionospheric profile modification. At its core, NLOP encompasses a set of parametric wave-wave interactions such as SRS, SBS, and the two-plasmon decay instability ($2\omega_p$) [FKLinb, DKL74, LR76, SSW83]. In these interactions, a laser propagating through plasma couples to new modes - scattering into an electromagnetic (EM) wave and an EPW in the case of SRS, or into an EM wave and an IAW in the case of SBS. In the $2\omega_p$ process, the laser decays into two EPWs. In certain regimes, SRS and $2\omega_p$ can become coupled where one of the decay waves is neither purely an EM or an EPW, forming what is known as the high-frequency hybrid instability (HFHI) [AW95].

These interactions are governed by conservation laws [MR07, FKLInb, Kru88, Mic23, BK95]: conservation of energy (frequency matching), conservation of momentum (wavevector matching), and conservation of action (photon number). Because the wavevector is a vector

quantity, the scattered waves can emerge at angles relative to the incoming beam, an effect referred to as side scatter [LRW73].

Research on NLOP has steadily expanded over the decades. Early studies modeled idealized plane waves in uniform plasmas, but later work incorporated more realistic features such as density gradients [Ros72, LRW73], plasma flows, pump depletion [BSW98], dissipation, and kinetic effects [MO72a, WFM07, BBC13, VDB02]. However, these models often still assumed plane-wave or monochromatic lasers, which differ markedly from the actual conditions in ICF experiments.

In practice, ICF lasers are far from what is assumed in the basic theory. Instead, large beams are broken into patterns of near-diffraction-limited beamlets, called speckles, through the use of phase plates [SSK89]. In some designs, bandwidth is introduced to make the speckle pattern evolve in time, with the lifetime of each speckle inversely proportional to the bandwidth [LSB87]. As a result, the fundamental unit of an ICF laser beam is best modeled as a single, localized Gaussian pulse (speckle). The plasma interaction dynamics of such finite-width pulses have recently gained renewed attention, particularly in the context of Raman and Brillouin amplification [VTA16a, ATH13], where a short seed pulse counter-propagates against a longer pump. Despite their importance, single-speckle interactions remain underexplored, especially in terms of the richer physics and additional conservation laws they reveal within the framework of nonlinear optics.

Building on this foundation, recent work has begun to examine how beams with nontrivial spatial phase profiles influence laser-plasma interactions. One especially promising avenue is the use of lasers carrying orbital angular momentum (OAM) [ABS92, PCA04], which offer both new physical insights and potential strategies for mitigating SRS.

It is well-understood that light carries spin angular momentum (SAM) when it is circularly polarized [Poy09, Bet36]. In fact, the mechanical torque exerted by circularly polarized light was measured as early as 1936 [Bet36], demonstrating that angular momentum can be transferred from light to matter. More recently, it was shown that light can also carry orbital

angular momentum when it has a spatially structured phase front [ABS92]. In contrast to spin angular momentum (SAM), which is associated with polarization, OAM arises from the helical structure of the wavefront itself, giving rise to a characteristic ring-shaped intensity profile. For a laser with OAM, the direction of the polarization remains constant while the phase varies across the spot. The interaction of OAM-carrying laser modes with plasma is now an active area of research [cTT09, VTA16a, VTA16b, BNK19, BNK20, SVT18, HZN20, TKD20], with implications for both fundamental physics and applications like instability control in ICF.

Any finite-width laser can be described with various basis functions for the transverse coordinates. A common choice is the Hermite- Gaussian (HG) [ABS92, PCA04] functions in Cartesian coordinates (x, y) . Another common choice is to use the Laguerre-Gaussian (LG) [ABS92] expansion for the laser envelope in the variables (r, φ) . The LG functions have an angular index ℓ and radial index p which are both integers. The radial index quantifies the number $(p + 1)$ of intensity rings in r . The angular index appears in the phase factor $e^{i\ell\varphi}$. LG modes ($|\ell| > 0$) have helical wavefront surfaces and a phase singularity along the axis of propagation.

In Ref. [ABS92], it was shown that the ℓ number of an LG wave describes the amount of OAM. We call this the OAM number. As the LG solutions to the paraxial Helmholtz equation constitute a complete eigenbasis, any light wave can be described as a composition of pure OAM modes. Thus, in general, the OAM number of a light wave can be viewed as a weighted sum over each LG mode, and is not necessarily an integer.

A variety of experiments have demonstrated the utility of light with OAM. LG beams are able to manipulate single particles in the form of optical tweezers [OMA02] and to trap multiple particles in an LG trap array [PMF04], which has widespread applications for biophysics and biotechnology. OAM light has been used for resilient information encoding [GCP04], and for quantum optics, with implications for quantum cryptography [MVW01].

In general, generating LG light can be achieved starting from non-OAM light. A specific

LG mode can be generated by propagating a lowest order LG mode through a spiral phase plate [BVS90]. Furthermore, LG modes have been generated from HG laser modes through the use of two cylindrical lenses [BAv93]. A plasma can also be used to generate or amplify OAM modes. Because it has been shown that angular momentum is conserved between SAM and OAM in a plasma [HZN20], LG light can be generated from circularly polarized light [Wan19].

While experiments with OAM lasers can still be challenging, LPI involving OAM can be studied in simulations. Simulations have shown that OAM light can produce plasma waves with helical shapes [MAT09], and that LG light waves in a plasma can induce large static axial magnetic fields [SVT18, LF21]. Additionally, high-intensity OAM light can be generated through Raman amplification [VTA16a] and Brillouin amplification [ATH13]. High OAM harmonic generation can be tuned according to an OAM selection rule [VTA16b]. High-intensity OAM pulses may also be achieved through reflection of an OAM pulse off of an overdense plasma whose surface was modified by an intense non-OAM pulse [Wan19]. In all of this work, pure LG modes were considered for both the incident and any nonlinearly-generated light.

In this research, we examine SRS backscatter of single-speckle pulses with and without OAM both as a basic science problem and due to its relevance to ICF. The issue of SRS in ICF was addressed by lowering the density of the fill gas in the hohlraum, but this solution leads to a lower fusion gain for a fixed laser energy.

As noted above, Vieira et al. [VTA16a] examined SRS in the context of Raman amplification. This work was limited to non-kinetic and strongly coupled regimes where wave-particle interactions are not important. In addition, as a seed (down-shifted in frequency from the pump by the EPW frequency) and pump pulses with integer OAM numbers were used, they were able to argue that, in addition to frequency and wavenumber matching, an ℓ matching condition (OAM conservation) must also exist. The plasma wave is created by the ponderomotive force from the beating of the pump and seed, so it can also be created with helical

fronts. As a result, it is possible to create plasma waves with OAM. This is in contrast to SRS of lasers with SAM, where the incoming laser is circularly polarized. In the case of SAM, the resulting plasma wave cannot have angular momentum due to the form of the ponderomotive force [cTT09].

Wave-particle interactions within the EPW complicate the growth of SRS. It is well-known that linear (small amplitude) plasma waves can damp without collisions through Landau damping [Lan46]. This damping depends on the slope of the distribution function at the phase velocity of the EPW. For typical parameters on the NIF, SRS operates in the strongly damped regime where $k\lambda_D \gtrsim 0.3$, and the growth rate predicted by linear theory is smaller than the Landau damping rate. However, nonlinear effects associated with trapped particles (those moving near the phase velocity of the wave) can lead to waves evolving towards undamped or BGK waves that also have small nonlinear frequency shifts and no damping [MO72a, ON65]. This occurs within several bounce periods for trapped particles. Plasma wave experiments have, in fact, found greater backscatter growth than expected based on linear theory in the kinetic regime [FCM00]. If the plasma wave grows to a sufficiently large amplitude where the bounce time is smaller than the Landau damping time, then the wave will become undamped and SRS transitions from the strongly damped to the weakly damped regime. This transition was identified in kinetic simulations and is referred to as inflation [VDB02, ESW12].

More recently, collisionless damping of waves with OAM is being explored [Men12, BNK19]. Simulations show that a similar damping mechanism exists for plasma waves with OAM and that the particle distribution function is modified in the presence of the wave [BNK20]. However, it is still not clear how the process of inflation is modified when the plasma wave has OAM. Electron trapping and its role in EPW damping in a helical plasma wave is not yet well understood. If such a wave were to reduce particle trapping compared to a plasma wave without OAM, the SRS growth rate should be modified and the transition to inflation delayed.

Simulating SRS in nonlinear kinetic regimes is very challenging. Modeling only a single speckle in three dimensions using particle-in-cell simulations [BL18] is state-of-the-art [YAR09] and it is thus not possible to carry out large parameter scans. Therefore, when modeling a full multi-speckle NIF beam, software that simulates SRS based on the envelopes of each wave using a reduced fluid description is used. Developing phenomenological descriptions of nonlinear kinetic effects of the plasma waves is ongoing. While the goal is to include accurate nonlinear kinetic damping models into fluid descriptions, this has yet to be achieved.

In Chapter 3, we carry out parameter scans of SRS of a single speckle in a uniform plasma with and without OAM using the fluid code pF3D [BSW98]. While pF3D can include ion physics and non-uniform plasmas, we operate it in a mode where the ion density does not evolve and is uniform. For the simulations presented later, pF3D solves the envelope equations for the three relevant modes (vector potential for the incident laser and reflected light wave, and density fluctuation of the EPW) of SRS backscatter and it includes Landau damping for the EPW. A pump which can have OAM is incident on a plasma whose noise source is determined from thermal fluctuations of the density in equilibrium. In some cases, we send a seed pulse with well-defined integer OAM value backwards towards the pump.

This effort is a precursor to investigating nonlinear kinetic effects of SRS of lasers with OAM. We explore a range of $k_2\lambda_D$ in pF3D simulations with a particular focus on the kinetic regime, $0.3 \leq k_2\lambda_D \leq 0.4$, where linear theory predicts that the damping rate is large and SRS can be in the strongly damped regime. As pF3D does not include nonlinearities of the plasma wave, inflation is precluded. However, as we show in Chapter 3, pF3D provides an excellent test bed for studying geometric effects associated with SRS of OAM pulses.

As we show in Chapter 2, the envelope equations have 3D conservation laws for action, energy, momentum, and angular momentum. The conservation laws thus provide expressions for the angular momentum of each wave. These expressions apply for when the envelopes consist of mixtures of LG modes. Thus, the conservation of angular momentum is more general than just ℓ number conservation. Diagnostics for angular momentum can thus be

included within pF3D.

The simulations show that when comparing SRS of lasers with and without OAM that there are three important variables, the spot size (which defines the diffraction length), the power, and the peak intensity. For LG modes, the spot size defines the diffraction length, regardless of the ℓ value. This spot size is different from that obtained by the weighted average of r^2 over the intensity. However, for a given spot size and fixed total power, the peak intensity decreases with the ℓ number. The simulations show a reduction in laser reflectivity when pumps with OAM are involved when the power and peak intensity of the incident light are kept constant. Additionally, results with a well-defined seed are compared to an ensemble of 1D numeric calculations of seed gain. Conservation of OAM is confirmed through measurements of angular momentum for the various waves. Simulations show that the reflected light and the EPW do not necessarily have pure OAM values throughout the simulation box.

In Chapter 4, we present three-dimensional PIC simulations using OSIRIS [FST02] of SRS of lasers with and without OAM. The simulations are computationally expensive so only a limited set of cases were carried out over a more limited time domain. The simulations also indicate that SRS reflectivity is reduced for lasers with OAM if the power and spot size are kept fixed. The behavior of SRS is also shown to be qualitatively and quantitatively different in the kinetic simulations. The reflectivity occurs in bursts and no steady-state spatial growth is seen. The peak amplitude of the reflectivity is higher and the duration of the peaks is narrower. Nonlinear frequency shifts of the EPW are seen for pumps with and without OAM.

In Chapter 5, the results are summarized and some final thoughts on directions for future work are offered. Finally, in Appendix A a review of general conservation laws for energy, momentum, and angular momentum including fields and discrete or continuum particle descriptions is provided.

CHAPTER 2

Theory

2.1 Introduction

As described in Chapter 1, the aim of this dissertation is to address new aspects of stimulated Raman scattering (SRS) that arise when the pump and/or decay waves have intrinsic angular momentum. In this chapter, we lay the groundwork for Chapters 3 and 4 by reviewing previous work and providing new analysis of SRS involving pumps and/or scattered light with orbital angular momentum (OAM).

It has been known for more than a century that electromagnetic waves can have angular momentum and this can be transferred to particles [Poy09]. In Appendix A, we review the general conservation laws including that for angular momentum between fields and particles. The fact that electromagnetic (EM) waves described by Laguerre-Gaussian (LG) solutions to the paraxial wave equation in vacuum can have both quantized spin angular momentum (SAM) and orbital angular momentum (OAM) appears to have been first described by Allen et al. in their seminal 1992 paper [ABS92].

This work shows that the concept of OAM for an EM wave is inherently a 3D effect and can easily be understood by considering LG modes, which are characterized by an expansion in azimuthal mode numbers, $e^{i\ell\varphi}$ [ABS92, PCA04]. In their seminal work, Allen et al. argued that for a pure LG (a single ℓ value) mode, the average OAM per photon is $\ell\hbar$, while for a circularly polarized beam, the average SAM per photon is $\pm\hbar$. [ABS92]

OAM is not a type of polarization, but rather a property of the beam's phase [PCA04].

It arises when the phase of a wave depends on the azimuthal angle, φ , producing a helical phase front structure. To study SRS involving OAM, one must consider a full 3D description of finite-width modes (where ∂_φ does not vanish) and cannot rely on the normal plane wave analysis. The combination of angular momentum due to polarization (SAM) and phase (OAM) can become complicated [APB99].

Polarization and OAM are independent concepts that can be combined such as the case of a circularly-polarized LG mode. The electric field of a laser defines the polarization vector. In linearly and circularly polarized beams, the polarization vector does not vary within a wavefront (i.e. across the laser spot). In a circularly polarized beam, the polarization vector varies along the propagation direction. Vector beams, on the other hand, have spatially-varying polarization across the phase front. One example of a vector beam, is a wave that is radially polarized where the polarization vector, $\hat{\mathbf{e}}$, points outward from the propagation axis like spokes of a wheel, $\hat{\mathbf{e}} = \hat{\mathbf{r}}$. By rewriting $\hat{\mathbf{r}} = \frac{x}{r}\hat{\mathbf{x}} + \frac{y}{r}\hat{\mathbf{y}}$ and recognizing the field amplitude is a Gaussian times r , it can be seen that such a mode is a superposition of low order Hermite- Gaussian (HG) modes with cross-polarization [APB99]. In general, vector beams are defined by polarization vectors that depend on r and φ , or x and y . A vector beam may also have OAM, that is, both the phase and polarization vary across the wavefront [APB99]. It is even possible for the polarization pattern to be entangled with the helical phase structure due to OAM [PCA04].

In this chapter, we set the stage for the rest of this dissertation by first providing a review of angular momentum for EM waves in vacuum. This includes describing the family of LG modes in vacuum or linear media in both real and Fourier space. We then review how a single or pure LG mode carries quanta of OAM and how circular polarization adds quanta of SAM. Next, we review the theory of SRS for plane waves and derive the 3D paraxial ray equations and envelope equations. SRS of plane waves has been studied extensively [FKLinb, Kru88, Mic23]. The effect of SAM in the form of circular polarization on SRS for plane waves has also been addressed [VTA16b]. However, there have been limited simulations and theoretical

studies of SRS of 3D Gaussian lasers, including those with OAM [MTT09, VTA16a]. In this chapter, we turn our attention to SRS of lasers with OAM by deriving the coupled equations for the fluid regimes so that we may later investigate kinetic and finite-width effects through pF3D and particle-in-cell (PIC) simulations using OSIRIS [BDG01, LSA16]. We consider the 1D limit of these equations to review various regimes (weakly damped, strongly damped, temporal growth, and spatial growth), followed by a discussion of how nonlinear kinetic effects related to electrons trapped in electron plasma waves (EPW) can transition the relevant regime for SRS from a kinetic to a fluid regime [MO72b, VDB02, YAR09].

We finish the chapter by using the 3D envelope equations to obtain conservation laws for action, energy, momentum, and angular momentum in terms of the wave envelopes and discuss the relationship between the local energy and momentum conservation laws and ω and k matching for plane waves. We also offer a physical interpretation for the resulting expression for the OAM of an electron plasma wave and we conclude by showing how the conservation law for OAM pertinent to SRS is related to what others have referred to as ℓ matching [VTA16a].

2.2 Angular momentum of light

In Appendix A, we reviewed the derivation of general conservation laws for fields and particles for energy, momentum, and angular momentum. These are based on the full set of Maxwell's equations and the Vlasov equation [BL18]. However, to investigate SRS it is useful to obtain formulae for angular momentum of the fields based on the paraxial ray approximation (in terms of wave envelopes).

Consider a light wave propagating in a plasma. The vector potential, $\mathbf{A}$, obeys the wave equation

$$(c^2 \nabla^2 - \partial_t^2 - \omega_p^2) \mathbf{A} = 0 \quad (2.1)$$

where ω_p is the plasma frequency. We assume the vector potential can be written as,

$$\mathbf{A} = \frac{\mathbf{u}}{2} e^{ikz - i\omega t} + \text{c.c.} \quad (2.2)$$

where k is the wavenumber, ω is the frequency, and $\omega^2 = \omega_p^2 + c^2 k^2$ where c is the speed of light in vacuum. We define the envelope, $\mathbf{u}(z, \mathbf{x}_\perp, t) = u\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is the polarization vector and we assume that u has a slowly varying amplitude and phase, i.e.,

$$|\partial_t u| \ll \omega |u| \quad (2.3)$$

$$|\partial_z u| \ll k |u| \quad (2.4)$$

Under these assumptions and assuming $\nabla \hat{\mathbf{e}} = 0$, the envelope satisfies the paraxial wave-like equation (PWE) [DM00, DMT96],

$$\omega \left(\partial_t + v_g \partial_z - \frac{ic^2}{2\omega} \nabla_\perp^2 \right) u = 0 \quad (2.5)$$

where v_g is the group velocity,

$$v_g = \frac{c^2 k}{\omega} \quad (2.6)$$

In vacuum, $\omega = ck$ and $v_g = c$. For some situations of interest, the laser pulse length is much longer than a Rayleigh or diffraction length. In that case, ∂_t can be dropped leading to the common form for the PWE,

$$\omega \left(v_g \partial_z - \frac{ic^2}{2\omega} \nabla_\perp^2 \right) u = 0 \quad (2.7)$$

Equation 2.7 has general solutions that form a complete set for a given frequency based on either Hermite- Gaussian (HG) or Laguerre-Gaussian (LG) modes [APB99]. With respect to investigating the angular momentum of light, the LG modes are most useful.

2.2.1 Laguerre-Gaussian functions

It can be shown that there exist solutions to the PWE in cylindrical coordinates that form a complete set. These so-called Laguerre-Gaussian modes form an eigenbasis indexed by

two integers, p and ℓ . The radial index, p , is associated with the number of rings in the transverse profile of the envelope. The azimuthal index, ℓ , describes the helicity and, as we will show, the orbital angular momentum. LG solutions to the PWE are given by envelopes $u_{p\ell}$,

$$u_{p\ell}(r, \varphi, z) = \frac{C}{\sqrt{1 + \frac{z^2}{z_R^2}}} \left(\frac{r\sqrt{2}}{w(z)} \right)^{|\ell|} L_p^{|\ell|} \left(\frac{2r^2}{w^2(z)} \right) e^{-\frac{r^2}{w^2(z)} + \frac{ikr^2z}{2(z^2 + z_R^2)} + i\ell\varphi + i(2p + \ell + 1) \tan^{-1} \frac{z}{z_R}} \quad (2.8)$$

where z_R is the Rayleigh length, $w(z)$ is the beam waist, and C is a constant. L_p^ℓ is the associated Laguerre polynomial, also called the generalized Laguerre polynomial, which is defined as

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) \quad (2.9)$$

The first two associated Laguerre polynomials are

$$L_0^\alpha(x) = 1 \quad (2.10)$$

$$L_1^\alpha(x) = 1 + \alpha - x \quad (2.11)$$

The beam is focused at $z = 0$ and the width of the beam varies as

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}} \quad (2.12)$$

where w_0 is the size at best focus. The expressions for $u_{p\ell}(r, \varphi, z)$ are based on the PWE, and thus they are approximate solutions to Maxwell's equations. However, at a given z and for a given frequency, the $u_{p\ell}$ form a complete set of functions.

2.2.1.1 LG modes in Fourier space

We can look to the Fourier transform to understand the propagation and k -space structure of a specific mode. For simplicity, we consider the envelope at $z = 0$ for modes with $p = 0$,

$$u_{0\ell}(r, \varphi) = C \left(\frac{r\sqrt{2}}{w_0} \right)^{|\ell|} e^{-\frac{r^2}{w_0^2} + i\ell\varphi} \quad (2.13)$$

The Fourier transform of a function $F(x, y)$ is

$$\tilde{F}(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-i\mathbf{k} \cdot \mathbf{x}_{\perp}} F(x, y) \quad (2.14)$$

where $\mathbf{k} = k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}}$ and $\mathbf{x}_{\perp} = x \hat{\mathbf{x}} + y \hat{\mathbf{y}}$. We can rewrite equation 2.14 as

$$\tilde{F}(k_{\perp}, \theta) = \frac{1}{2\pi} \int_0^{\infty} dr r \int_0^{2\pi} d\varphi e^{-ik_{\perp} r \cos(\theta - \varphi)} F(r, \varphi) \quad (2.15)$$

where $k_{\perp} = \sqrt{k_x^2 + k_y^2}$ and $\theta = \arctan(k_y/k_x)$. Therefore, the Fourier transform of $u_{0\ell}(r, \varphi)$ is

$$\tilde{u}_{0\ell}(k_{\perp}, \theta) = \frac{C}{2\pi} \left(\frac{\sqrt{2}}{w_0} \right)^{|\ell|} \int_0^{\infty} dr r^{|\ell|+1} e^{-\frac{r^2}{w_0^2}} \int_0^{2\pi} d\varphi e^{i\ell\varphi - ik_{\perp} r \cos(\theta - \varphi)} \quad (2.16)$$

The integral in φ can be simplified using the Bessel function identity,

$$\int_0^{2\pi} d\psi e^{i\ell\psi - iz \cos \psi} = 2\pi i^{\ell} J_{\ell}(z) \quad (2.17)$$

where J_{ℓ} is the Bessel function of the first kind. Substituting equation 2.17 using $\psi = \varphi - \theta$ and $z = k_{\perp} r$ leaves

$$\tilde{u}_{0\ell}(k_{\perp}, \theta) = i^{\ell} e^{i\ell\theta} C \left(\frac{\sqrt{2}}{w_0} \right)^{|\ell|} \int_0^{\infty} dr r^{|\ell|+1} e^{-\frac{r^2}{w_0^2}} J_{\ell}(k_{\perp} r) \quad (2.18)$$

The term $e^{i\ell\varphi}$ in $u_{p\ell}$, which produces a helical wavefront, is replicated in k -space. For $\ell = 0$, equation 2.18 reduces to the well-known radially-symmetric result for a Gaussian beam,

$$\tilde{u}_{00}(k_{\perp}, \theta) = C \frac{w_0^2}{2} e^{-\frac{k_{\perp}^2 w_0^2}{4}} \quad (2.19)$$

or

$$\tilde{u}_{00}(k_x, k_y) = C \frac{w_0^2}{2} e^{-\frac{(k_x^2 + k_y^2) w_0^2}{4}} \quad (2.20)$$

where we have used the identity,

$$\int_0^{\infty} dr r e^{-(ar)^2} J_0(br) = \frac{1}{2a^2} e^{-\frac{b^2}{4a^2}} \quad (2.21)$$

with $a = \frac{1}{w_0}$ and $b = k_{\perp}$. For $\ell = 1$, equation 2.18 becomes

$$\tilde{u}_{01}(k_{\perp}, \theta) = i e^{i\theta} C \frac{\sqrt{2}}{w_0} \int_0^{\infty} dr r^2 e^{-\frac{r^2}{w_0^2}} J_1(k_{\perp} r) \quad (2.22)$$

which is also a standard integral and has the solution

$$\tilde{u}_{01}(k_{\perp}, \theta) = ik_{\perp} e^{-i\theta} C \frac{w_0^2}{2^{3/2}} e^{-\frac{k_{\perp}^2 w_0^2}{4}} \quad (2.23)$$

or

$$\tilde{u}_{01}(k_x, k_y) = i(k_x + ik_y) C \frac{w_0^2}{2^{3/2}} e^{-\frac{(k_x^2 + k_y^2) w_0^2}{4}} \quad (2.24)$$

In general, understanding the $k_{\perp}$ dependence requires calculation of the Gaussian-weighted ℓ -order Hankel transform of $r^{|\ell|} e^{-\frac{r^2}{w_0^2}}$,

$$\int_0^{\infty} dr r^{\mu+1} e^{-a^2 r^2} J_{\mu}(br) = \frac{b^{\mu}}{(2a^2)^{\mu+1}} \Gamma(\mu+1) e^{-\frac{b^2}{4a^2}} \quad (2.25)$$

Therefore, the solution to equation 2.18 for all ℓ is

$$\tilde{u}_{0\ell}(k_{\perp}, \theta) = i^{\ell} k_{\perp}^{|\ell|} e^{i\ell\theta} C \frac{w_0^{|\ell|+2}}{2^{|\ell|/2+1}} e^{-\frac{k_{\perp}^2 w_0^2}{4}} \Gamma(|\ell|+1) \quad (2.26)$$

or

$$\tilde{u}_{0\ell}(k_x, k_y) = i^{\ell} C (k_x + \text{sign}(\ell) ik_y)^{|\ell|} \frac{w_0^{|\ell|+2}}{2^{|\ell|/2+1}} e^{-\frac{(k_x^2 + k_y^2) w_0^2}{4}} \Gamma(|\ell|+1) \quad (2.27)$$

where Γ is the Gamma function, which is simply the factorial function for integer arguments,

$$\Gamma(n+1) = n! \quad \text{for } n \in \mathbb{Z}_{\geq 0} \quad (2.28)$$

Because $|\ell|$ is a positive integer for $u_{p\ell}$, $\Gamma(|\ell|+1)$ in $\tilde{u}_{0\ell}$ can be replaced with $|\ell|!$. Rearranging terms leads to the general result for the Fourier transform of a Laguerre-Gaussian mode with $p=0$ at $z=0$,

$$\tilde{u}_{0\ell}(k_{\perp}, \theta) = C \frac{w_0^2}{2} |\ell|! \left(\frac{ik_{\perp} w_0}{\sqrt{2}} \right)^{|\ell|} e^{-\frac{k_{\perp}^2 w_0^2}{4} + i\ell\theta} \quad (2.29)$$

or

$$\tilde{u}_{0\ell}(k_x, k_y) = C \frac{w_0^2}{2} |\ell|! \left[\frac{iw_0}{\sqrt{2}} (k_x + \text{sign}(\ell) ik_y) \right]^{|\ell|} e^{-\frac{(k_x^2 + k_y^2) w_0^2}{4}} \quad (2.30)$$

We have shown that, in addition to the dominant k in the $\hat{z}$ -direction, the LG modes also have $k_{\perp}$ that rotates about the z -axis. If $\mathbf{k}$ is interpreted as representing the magnitude and direction of the momentum, then this shows that the LG mode carries field angular momentum.

2.2.2 OAM of an EM wave

In the previous subsection, we found that the Fourier transform of the wave envelope depends on the azimuthal coordinate in k -space, indicating that the fields have angular momentum. To make this clearer, we next follow the derivation in Ref. [ABS92] to show that a wave whose envelope is described by the LG mode, $u_{p\ell}$, has orbital angular momentum proportional to ℓ and the average angular momentum per photon is quantized.

It is well known [APB99] that the fields of an EM wave have momentum density given as

$$\mathbf{G} \equiv \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \quad (2.31)$$

and, therefore, the angular momentum density in the fields of a light wave, $\mathbf{M}$, can be defined as,

$$\mathbf{M} \equiv \mathbf{r} \times \mathbf{G} = \frac{1}{4\pi c} \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \quad (2.32)$$

where it is the time average, $\langle \mathbf{M} \rangle$, that is of interest. Equation 2.32 is valid when $\mathbf{E}$ and $\mathbf{B}$ of an EM wave are described exactly. However, much can be learned if they are approximated using the LG modes which are based on the PWE. The time-averaged linear momentum density is half the real part of $\mathbf{G}$,

$$\langle \mathbf{G} \rangle = \frac{1}{8\pi c} \Re(\mathbf{E} \times \mathbf{B}) = \frac{1}{16\pi c} (\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*) \quad (2.33)$$

To get an approximate expression for $\langle \mathbf{G} \rangle$ based on the envelope, we start from

$$\mathbf{A}(z, \mathbf{x}_\perp, t) = \frac{1}{2} \mathbf{u}(z, \mathbf{x}_\perp, t) e^{ikz - i\omega t} + \text{c.c.} \approx \frac{1}{2} \mathbf{u}(\mathbf{x}_\perp) e^{ikz - i\omega t} + \text{c.c.} \quad (2.34)$$

Although we assume a mostly transverse wave, $\mathbf{u}$ must have a z component to satisfy the gauge condition $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge or Lorenz gauge if $\varphi = 0$),

$$\nabla \cdot \mathbf{A} = \nabla \cdot \frac{1}{2} (\mathbf{u}_\perp + u_z \hat{\mathbf{z}}) e^{ikz - i\omega t} + \text{c.c.} = \frac{1}{2} (\nabla_\perp \cdot \mathbf{u}_\perp + ik u_z) e^{ikz - i\omega t} + \text{c.c.} = 0 \quad (2.35)$$

Therefore, we can rewrite the envelope as

$$\mathbf{u} = \mathbf{u}_\perp + \frac{i\hat{\mathbf{z}}}{k} (\nabla_\perp \cdot \mathbf{u}_\perp) \quad (2.36)$$

Substituting equation 2.34 and equation 2.36, we can write the electric field as

$$\mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A} = \frac{i\omega}{2c} \mathbf{u} e^{ikz-i\omega t} + \text{c.c.} = \frac{i\omega}{2c} \left[\mathbf{u}_\perp + \frac{i\hat{\mathbf{z}}}{k}(\nabla_\perp \cdot \mathbf{u}_\perp) \right] e^{ikz-i\omega t} + \text{c.c.} \quad (2.37)$$

Furthermore, the magnetic field can be expressed in terms of the envelope as,

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} &= \nabla \times \left(\frac{1}{2} \mathbf{u} e^{ikz-i\omega t} \right) + \text{c.c.} = \\ &= \frac{1}{2} e^{ikz-i\omega t} (\nabla_\perp + ik\hat{\mathbf{z}}) \times \left[\mathbf{u}_\perp + \frac{i\hat{\mathbf{z}}}{k}(\nabla_\perp \cdot \mathbf{u}_\perp) \right] + \text{c.c.} \end{aligned} \quad (2.38)$$

Substituting $\mathbf{E}$ and $\mathbf{B}$ into equation 2.33 to calculate the linear momentum density yields

$$\begin{aligned} \langle \mathbf{G} \rangle &= -\frac{i\omega}{16\pi c^2} \mathbf{u}^* \times (\nabla_\perp + ik\hat{\mathbf{z}}) \times \mathbf{u} + \text{c.c.} \\ &= -\frac{i\omega}{16\pi c^2} \left[\mathbf{u}_\perp^* - \frac{i\hat{\mathbf{z}}}{k}(\nabla_\perp \cdot \mathbf{u}_\perp^*) \right] \times (\nabla_\perp + ik\hat{\mathbf{z}}) \times \left[\mathbf{u}_\perp + \frac{i\hat{\mathbf{z}}}{k}(\nabla_\perp \cdot \mathbf{u}_\perp) \right] + \text{c.c.} \end{aligned} \quad (2.39)$$

Using the following vector identity,

$$\mathbf{a} \times (\nabla \times \mathbf{b}) - \mathbf{b} \times (\nabla \times \mathbf{a}) = \mathcal{A}(\mathbf{a}, \mathbf{b}) + \nabla \times (\mathbf{a} \times \mathbf{b}) - \mathbf{a}(\nabla \cdot \mathbf{b}) + \mathbf{b}(\nabla \cdot \mathbf{a}) \quad (2.40)$$

where we have defined an operator $\mathcal{A}$ whose i th component is

$$\mathcal{A}_i(\mathbf{a}, \mathbf{b}) = a_j \partial_i b_j - b_j \partial_i a_j \quad (2.41)$$

and noting that $\nabla \approx \nabla_\perp$ because we have assumed the envelopes are slowly varying in z , we can substitute $\mathbf{a} = \mathbf{u}^*$ and $\mathbf{b} = \mathbf{u}$ and rearrange the vector identity to get

$$\mathbf{u}^* \times (\nabla_\perp \times \mathbf{u}) - \mathbf{u} \times (\nabla_\perp \times \mathbf{u}^*) = \mathcal{A}(\mathbf{u}^*, \mathbf{u}) + \nabla \times (\mathbf{u}^* \times \mathbf{u}) - \mathbf{u}^*(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{u}^*) \quad (2.42)$$

We can also show that, using the properties of vector triple products,

$$\mathbf{u}^* \times (\hat{\mathbf{z}} \times \mathbf{u}) + \mathbf{u} \times (\hat{\mathbf{z}} \times \mathbf{u}^*) = 2|\mathbf{u}|^2 \hat{\mathbf{z}} - \mathbf{u}^*(\hat{\mathbf{z}} \cdot \mathbf{u}) - \mathbf{u}(\hat{\mathbf{z}} \cdot \mathbf{u}^*) \quad (2.43)$$

By utilizing equations 2.42 and 2.43, we can rewrite equation 2.39,

$$\begin{aligned} \langle \mathbf{G} \rangle &= -\frac{i\omega}{16\pi c^2} [\mathcal{A}(\mathbf{u}^*, \mathbf{u}) + \nabla_\perp \times (\mathbf{u}^* \times \mathbf{u}) - \mathbf{u}^*(\nabla_\perp \cdot \mathbf{u}_\perp) + \mathbf{u}(\nabla_\perp \cdot \mathbf{u}_\perp^*) \\ &\quad + 2ik|\mathbf{u}|^2 \hat{\mathbf{z}} - ik\mathbf{u}^*(\hat{\mathbf{z}} \cdot \mathbf{u}) - ik\mathbf{u}(\hat{\mathbf{z}} \cdot \mathbf{u}^*)] \end{aligned} \quad (2.44)$$

where we have also replaced $\nabla_{\perp} \cdot \mathbf{u}$ with $\nabla_{\perp} \cdot \mathbf{u}_{\perp}$. Next, we substitute the z component of the envelope,

$$\hat{\mathbf{z}} \cdot \mathbf{u} = \frac{i}{k} \nabla_{\perp} \cdot \mathbf{u}_{\perp} \quad (2.45)$$

and note that several terms will cancel because

$$\mathbf{u}^* (\nabla_{\perp} \cdot \mathbf{u}_{\perp}) + ik \mathbf{u}^* (\hat{\mathbf{z}} \cdot \mathbf{u}) = 0 \quad (2.46)$$

and, similarly,

$$\mathbf{u} (\nabla_{\perp} \cdot \mathbf{u}_{\perp}^*) - ik \mathbf{u} (\hat{\mathbf{z}} \cdot \mathbf{u}^*) = 0. \quad (2.47)$$

Therefore, we can write $\langle \mathbf{G} \rangle$ as

$$\langle \mathbf{G} \rangle = -\frac{i\omega}{16\pi c^2} [\mathcal{A}(\mathbf{u}^*, \mathbf{u}) + \nabla_{\perp} \times (\mathbf{u}^* \times \mathbf{u})] + \frac{k\omega}{8\pi c^2} |\mathbf{u}|^2 \hat{\mathbf{z}} \quad (2.48)$$

If we now assume linear polarization, $\mathbf{u}_{\perp} = u \hat{\mathbf{x}}$, equation 2.48 becomes

$$\langle \mathbf{G} \rangle = -\frac{i\omega}{16\pi c^2} (u^* \partial_y u - u \partial_y u^*) \hat{\mathbf{y}} + \frac{k\omega}{8\pi c^2} |u|^2 \hat{\mathbf{z}} \quad (2.49)$$

where terms smaller by factors of $\frac{1}{k}$ or more have been dropped so that the gauge condition implies $\partial_x u$ is negligible. We can write a general form for the momentum density of a linearly polarized wave as

$$\langle \mathbf{G} \rangle_{lin} = -\frac{i\omega}{16\pi c^2} (u^* \nabla_{\perp} u - u \nabla_{\perp} u^*) + \frac{k\omega}{8\pi c^2} |u|^2 \hat{\mathbf{z}} \quad (2.50)$$

When considering angular momentum, we are typically interested in M_z , so the term with $u^* \nabla_{\perp} u - u \nabla_{\perp} u^*$ is most relevant. We will show later on that the same $u^* \nabla_{\perp} u - u \nabla_{\perp} u^*$ expression appears in conservation laws for the wave envelopes in SRS.

Substituting u_{pl} for a Laguerre-Gaussian wave, we find

$$u^* \nabla_{\perp} u - u \nabla_{\perp} u^* = 2i|u|^2 \left(\frac{krz}{z^2 + z_R^2} \hat{\mathbf{r}} + \frac{\ell}{r} \hat{\boldsymbol{\varphi}} \right) \quad (2.51)$$

which can be used to write equation 2.50 as,

$$\langle \mathbf{G} \rangle = \frac{\omega}{8\pi c^2} |u|^2 \left(\frac{krz}{z^2 + z_R^2} \hat{\mathbf{r}} + \frac{\ell}{r} \hat{\boldsymbol{\varphi}} + k \hat{\mathbf{z}} \right) \quad (2.52)$$

Equation 2.52 implies that the Poynting vector, which is equal to $c^2\langle\mathbf{G}\rangle$ and measures energy flux, points along a twisting spiral [AP00] for a Laguerre-Gaussian beam due to the $\hat{\varphi}$ component. The $\hat{\mathbf{r}}$ component contributes to diffraction, while the $\hat{\mathbf{z}}$ component represents forward propagation.

Thus, the time averaged angular momentum density can be obtained by substituting equation 2.52 into equation 2.32, and calculating the cross product assuming a linearly-polarized Laguerre-Gaussian wave,

$$\langle\mathbf{M}\rangle = \frac{\omega}{8\pi c^2}|u|^2 \left[-\frac{z\ell}{r}\hat{\mathbf{r}} + kr \left(\frac{z^2}{z^2 + z_R^2} - 1 \right) \hat{\varphi} + \ell\hat{\mathbf{z}} \right] \quad (2.53)$$

Dividing equation 2.53 by power and integrating over the beam profile, i.e., $\int_0^\infty dr r \int_0^{2\pi} d\varphi$, leaves only the $\hat{\mathbf{z}}$ -component because the $\hat{\mathbf{r}}$ and $\hat{\varphi}$ components are cylindrically symmetric. Thus, we find the relationship between ℓ and M_z for an LG wave,

$$\frac{8\pi c^2}{\omega^2}\langle M_z \rangle = \frac{\ell}{\omega}|u|^2 \quad (2.54)$$

The OAM, which points in the direction of propagation, is directly proportional to ℓ , the azimuthal index.

2.2.3 SAM of an EM wave

When the wave is circularly polarized, spin angular momentum also contributes to M_z . To derive the SAM contribution, we assume $\mathbf{u} = u\hat{\mathbf{e}}$ where $\hat{\mathbf{e}}$ is a circular polarization vector which can be written as

$$\hat{\mathbf{e}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\sigma\hat{\mathbf{y}}) \quad (2.55)$$

where σ depends on polarization,

$$\sigma = \begin{cases} -1 & \text{right-hand circular polarization} \\ +1 & \text{left-hand circular polarization} \\ 0 & \text{linear polarization} \end{cases} \quad (2.56)$$

In equation 2.48, the term with $\mathbf{u}^* \times \mathbf{u}$ cancels for linear polarization where $\hat{\mathbf{e}}^* = \hat{\mathbf{e}}$. However, substituting equation 2.55 and calculating the cross product explicitly yields

$$\nabla_{\perp} \times (\mathbf{u}^* \times \mathbf{u}) = \nabla_{\perp} \times (i\sigma|u|^2\hat{\mathbf{z}}) = i\sigma\nabla_{\perp}|u|^2 \times \hat{\mathbf{z}} \quad (2.57)$$

which we substitute into equation 2.48 to arrive at the circular polarization form of equation 2.48,

$$\begin{aligned} \langle \mathbf{G} \rangle_{\text{circ}} = \langle \mathbf{G} \rangle_{\text{lin}} + \frac{\omega\sigma}{16\pi c^2} \nabla_{\perp}|u|^2 \times \hat{\mathbf{z}} = \\ \frac{\omega}{8\pi c^2} \left[\frac{u^* \nabla_{\perp} u - u \nabla_{\perp} u^*}{2i} + \frac{\sigma}{2} \nabla_{\perp}|u|^2 \times \hat{\mathbf{z}} + k|u|^2 \hat{\mathbf{z}} \right] \end{aligned} \quad (2.58)$$

We can write the cross product of the gradient of the intensity in cylindrical coordinates as,

$$\nabla_{\perp}|u|^2 \times \hat{\mathbf{z}} = \frac{1}{r} \partial_{\varphi}|u|^2 \hat{\mathbf{r}} - \partial_r|u|^2 \hat{\boldsymbol{\varphi}} \quad (2.59)$$

For a Laguerre-Gaussian wave, the intensity $|u_{pl}|^2$ has no azimuthal dependence, i.e., $\partial_{\varphi}|u_{pl}|^2 = 0$. Therefore, for a circularly polarized LG wave, the linear momentum density is

$$\langle \mathbf{G} \rangle = \frac{\omega}{8\pi c^2} \left[\frac{krz}{z^2 + z_R^2} |u|^2 \hat{\mathbf{r}} + \left(\frac{\ell}{r} |u|^2 - \frac{\sigma}{2} \partial_r |u|^2 \right) \hat{\boldsymbol{\varphi}} + k|u|^2 \hat{\mathbf{z}} \right] \quad (2.60)$$

and the angular momentum density is

$$\begin{aligned} \langle \mathbf{M} \rangle = \frac{\omega}{8\pi c^2} \left[-\frac{z}{r} (\ell|u|^2 - \sigma r \partial_r |u|^2) \hat{\mathbf{r}} + kr \left(\frac{z^2}{z^2 + z_R^2} - 1 \right) |u|^2 \hat{\boldsymbol{\varphi}} + \right. \\ \left. \left(\ell|u|^2 - \frac{\sigma r}{2} \partial_r |u|^2 \right) \hat{\mathbf{z}} \right] \end{aligned} \quad (2.61)$$

The angular momentum in the propagation direction, with both OAM and SAM, can now be written as

$$\frac{8\pi c^2}{\omega^2} \langle M_z \rangle = \frac{\ell}{\omega} |u|^2 - \frac{\sigma r}{2\omega} \partial_r |u|^2 \quad (2.62)$$

where the SAM term depends on σ , and equation 2.62 reduces to the linear polarization case (equation 2.54) when $\sigma = 0$. Note the SAM depends on the gradient of $|u|^2$, and we comment on the implication of this in the next section.

2.2.4 Quanta of angular momentum per photon

Although the calculations are purely classical, it is possible to define quanta of OAM and SAM per photon. Integrating equation 2.62 over the transverse directions,

$$\frac{8\pi c^2}{\omega^2} \int d\mathbf{x}_\perp \langle M_z \rangle = \frac{\ell}{\omega} \int d\mathbf{x}_\perp |u|^2 - \frac{\sigma}{2\omega} \int dr d\varphi r^2 \partial_r |u|^2 = \frac{\ell + \sigma}{\omega} \int d\mathbf{x}_\perp |u|^2 \quad (2.63)$$

and dividing by the number of photons, $\int |u|^2 d\mathbf{x}_\perp / (\hbar\omega)$, we determine that the average OAM per photon is $\ell\hbar$ and the average SAM per photon is $\sigma\hbar$. The total angular momentum per photon is a combination of SAM and OAM, $(\ell + \sigma)\hbar$. Note that for SAM the M_z contribution depends on $\partial_r |u|^2$ and thus, to go to the plane wave limit, one must first integrate over r rather than first assuming $\partial_r |u|^2$ vanishes. Otherwise the SAM of a plane wave would vanish.

We have shown that light waves described by LG envelopes have a well-defined OAM proportional to ℓ . Next, we will describe SRS of a laser in plasma in terms of the wave envelopes in order to arrive at conservation laws and an expression for the OAM of a plasma wave.

2.3 SRS

In this section, we review the physics behind stimulated Raman (SRS) backscatter. SRS is one process encompassed by the subject of the nonlinear optics of plasmas (NLOP). The focus of this dissertation is new aspects of SRS when pumps with OAM are considered.

Within the NLOP are 3-wave parametric processes in which an incident light wave decays into a scattered light wave and a forward going longitudinal mode such as an electron plasma wave (EPW) and an ion acoustic wave (IAW). For both processes, it is important to understand how an individual electron behaves in the fields of single or multiple lasers.

2.3.1 Single particle motion in a laser field

Before moving to the plasma fluid model, we derive the force on a single particle in terms of the laser envelopes. The equation of motion for an electron is given by

$$\frac{d\mathbf{p}}{dt} = -e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \quad (2.64)$$

where $\mathbf{p}$ is the momentum of the electron, $\mathbf{v}$ is the velocity, and e is the elementary charge. Assuming a transverse light wave, the scalar potential can be neglected, i.e., $\phi = 0$. Under this assumption, the force on an electron, in terms of the vector potential, is

$$-e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) = -e \left(-\frac{1}{c} \partial_t \mathbf{A} + \frac{\mathbf{v}}{c} \times \nabla \times \mathbf{A} \right) \quad (2.65)$$

Expanding the cross product, equation 2.64 can be rewritten as

$$\frac{d\mathbf{p}}{dt} = -e \left(-\frac{1}{c} \partial_t \mathbf{A} - \frac{\mathbf{v}}{c} \cdot \nabla \mathbf{A} + \nabla \mathbf{A} \cdot \frac{\mathbf{v}}{c} \right) \quad (2.66)$$

Upon recognizing that the time rate of change of $\mathbf{A}$ as seen by the electron is

$$\frac{d\mathbf{A}}{dt} = \partial_t \mathbf{A} + \mathbf{v} \cdot \nabla \mathbf{A} \quad (2.67)$$

then equation 2.66 may be rearranged as

$$\frac{d}{dt} \left(\mathbf{p} - \frac{e\mathbf{A}}{c} \right) = -e \nabla \mathbf{A} \cdot \frac{\mathbf{v}}{c} \quad (2.68)$$

which shows that the canonical momentum is conserved in directions for which there is translational invariance. For an electron in a laser, $\mathbf{p}$ and $e\mathbf{A}/c$ are preferentially in the perpendicular direction and thus,

$$\frac{d}{dt} \left(\mathbf{p}_\perp - \frac{e\mathbf{A}_\perp}{c} \right) = -e \nabla_\perp \mathbf{A}_\perp \cdot \frac{\mathbf{v}_\perp}{c} \quad (2.69)$$

Furthermore, for a finite-width wave for which there is not translational invariance, the right-hand side scales as $\frac{1}{k_0 w} \frac{eA_\perp}{mc^2}$ times smaller than the terms on the left-hand side where w is the

beam waist. Therefore, for $k_0 w \gg 1$ (which is assumed when using the PWE), we find, to a high degree of accuracy,

$$\mathbf{p}_\perp \approx \frac{e\mathbf{A}_\perp}{c} \quad (2.70)$$

where it is also assumed that the electron starts at rest in front of the laser. In inertial fusion energy (IFE), the laser intensities are not large enough for the electrons to acquire relativistic speeds. In this case, the normalized form of equation 2.70 can be written as

$$\frac{\mathbf{p}_\perp}{mc} = \frac{\mathbf{v}_\perp}{c} = \frac{e\mathbf{A}_\perp}{mc^2} \equiv \mathbf{a} \quad (2.71)$$

where $\mathbf{a}$ is called the normalized vector potential (or quiver velocity),

$$\mathbf{a} = \frac{e\mathbf{A}_\perp}{mc^2} = 0.853 \times 10^{-9} \sqrt{I(\text{W/cm}^2)} \lambda_{\text{nm}} \quad (2.72)$$

We are interested in the time average of equation 2.69 which leads to slow changes to the particle's motion. Substituting $\mathbf{a}$ on the right-hand side and averaging yields

$$\frac{d}{dt} \langle \mathbf{p} \rangle = -mc^2 \langle \nabla \mathbf{a} \cdot \mathbf{a} \rangle = -\frac{mc^2}{2} \nabla |\mathbf{a}|^2 \quad (2.73)$$

The time averaged force is therefore

$$\langle \mathbf{F} \rangle = \mathbf{F}_p \equiv -\frac{mc^2}{2} \nabla |\mathbf{a}|^2 \quad (2.74)$$

and this is referred to as the ponderomotive force of a laser [Kru88, Daw61]. The ponderomotive force points in the direction of lower intensity, $-\nabla |\mathbf{a}|^2$. The force is independent of the sign of the particle because the phenomenon is second order, and the behavior depends on the kinetic energy, $|\mathbf{v}|^2$. However, $\mathbf{F}_p$ is inversely proportional to mass ($|\mathbf{a}| \propto 1/m$), so the ponderomotive force is much smaller on ions.

Motivated by investigating three-wave instabilities, consider a vector potential made of two light waves at different frequencies with the same polarization, $\hat{\mathbf{e}}$. For SRS or SBS, the waves are the incident and scattered light. In this case, the normalized vector potential is

$$\mathbf{a} = \hat{\mathbf{e}} (b_0 e^{ik_0 z - i\omega_0 t} + b_1 e^{ik_1 z - i\omega_1 t}) + \text{c.c.} \quad (2.75)$$

Substituting equation 2.75 into the ponderomotive force, we find that an oscillating force exists at the beat frequency of the two waves,

$$\mathbf{F}_p = -\frac{mc^2}{2}\nabla (|b_0|^2 + |b_1|^2 + b_0b_1^*e^{i\Delta kz-i\Delta\omega t} + b_0^*b_1e^{-i\Delta kz+i\Delta\omega t}) \quad (2.76)$$

where $\Delta k = k_0 - k_1$ and $\Delta\omega = \omega_0 - \omega_1$. This force will drive a resonant response in a plasma when Δk and $\Delta\omega$ satisfy the dispersion relation for an EPW [FKLinb, Kru88]. Similarly, SBS is the case where Δk and $\Delta\omega$ excite an ion acoustic wave instead of an EPW. For SBS, $\mathbf{F}_p$ still acts on the electrons and the resulting space charge force acts on the ions. So $\Delta k = k_2$ and $\Delta\omega = \omega_2$ are still at the wavevector and frequency of the third wave, which is, in this case, the IAW.

The ponderomotive force changes the momentum of the particles. Assuming the envelope is slowly varying compared to the phase, the equation of motion in the direction of propagation is

$$\frac{d}{dt} \left(\frac{p_z}{mc} \right) = \frac{F_{pz}}{mc} = -i\frac{ck_2}{2}b_0b_1^*e^{ik_2z-i\omega_2t} + \text{c.c.} \quad (2.77)$$

This is the dominant motion in 1D SRS or in SRS when the light waves obey the PWE. However, for light waves with OAM, the plasma wave may also have helicity.

2.3.1.1 Helical plasma wave generation

When either or both of the incident and scattered waves have OAM, then $\mathbf{F}_p$ transfers the net OAM to the plasma. Consider two lasers with different helicities, i.e., different ℓ and different OAM. For a pump signal with $b_0 \sim e^{i\ell_0\varphi}$, and a backscattered signal with $b_1 \sim e^{i\ell_1\varphi}$, the ponderomotive force will now have a cross term with $\ell_2 = \Delta\ell = \ell_0 - \ell_1$. A plasma wave driven by this force develops a helical structure due to the $e^{i\ell_2\varphi}$ dependence. The phase mismatch can be used to understand the spatial distribution of the force,

$$\mathbf{F}_p \sim \nabla [f_p(r)e^{ik_2z-i\omega_2t+i\ell_2\varphi}] = \left[\left(i\frac{\ell_2}{r}\hat{\boldsymbol{\varphi}} + ik_2\hat{\mathbf{z}} \right) f_p(r) + \nabla f_p(r) \right] e^{ik_2z-i\omega_2t+i\ell_2\varphi} \quad (2.78)$$

where f_p is proportional to the product of the laser amplitudes (b_0b_1) and $\nabla f_p(r)$ can be dropped to lowest order. The singularity at $r = 0$ in the $\hat{\boldsymbol{\varphi}}$ direction is resolved by the fact

that at least one of the lasers must have zero intensity on axis when ℓ_2 is nonzero. The strength of the azimuthal force is proportional to the helical index of the plasma wave, ℓ_2 . When the two lasers have equal helicity ($\ell_0 = \ell_1$), the resulting plasma wave can have zero azimuthal dependence, i.e., $\ell_2 = 0$, even if the transverse profile has a central dark spot due to the overlap of the laser profiles. In reality, the physics is more complicated, as the spot size, w , can be different for the two waves, and they may be made of a superposition of ℓ numbers (combinations of LG modes).

This view of the ponderomotive force as a driver of an EPW does not paint a fully self-consistent picture, but it is a useful tool to isolate the evolution of a plasma wave. For a deeper understanding of SRS, we must also consider the way particle motion affects light waves in the many-body setting that is a plasma. There has been some recent work in this area [VDB02, YAR09, ESW12].

2.3.2 Laser plasma coupling

In the previous section, we used single particle motion to describe how lasers with OAM could couple to a plasma through the ponderomotive force, $\mathbf{F}_p$. In SRS, the $\mathbf{F}_p$ of a pump EM wave and a backscattered EM wave drives an EPW. The pump then scatters off the EPW, reinforcing the backscattered wave. Thus, a full description of SRS requires equations for how an EPW is driven and how the presence of an EPW modifies the propagation of EM waves.

To make analytical progress, we will resort to the two fluid description of a plasma. We quickly review where this fits within the hierarchy of plasma descriptions.

The most exact description is through the Klimontovich distribution function (phase space density),

$$F_s(\mathbf{x}, \mathbf{v}, t) = \sum_p^N \delta(\mathbf{x} - \mathbf{x}_p(t)) \delta(\mathbf{v} - \mathbf{v}_p(t)) \quad (2.79)$$

where s stands for species, p for particle, N for the total number of particles, and $\mathbf{x}_p(t)$ and

$\mathbf{v}_p(t)$ for the position and velocity of particle p . We assume all particles have non-relativistic speeds. The Klimontovich, F_s satisfies

$$\partial_t F_s + \mathbf{v} \cdot \nabla F_s + \frac{\mathbf{F}}{m_s} \cdot \nabla_v F_s = 0 \quad (2.80)$$

where $\mathbf{F}/m_s$ is the acceleration of a particle at position $\mathbf{x}$ and velocity $\mathbf{v}$, and ∇_v has components $[\nabla_v]_i = \partial_{v_i}$. Equation 2.80 says that phase space density is constant following particles (it is either zero or a delta function). The phase-space probability distribution function, f_s is obtained by averaging F_s over an ensemble of equivalent (same macroscopic properties) Klimontovich distributions. The smoothed F_s is called the single particle distribution function. For a large number of particles per Debye sphere, which has volume $\frac{4}{3}\pi n\lambda_D^3$, where $\lambda_D = v_e/\omega_p$ is the Debye length, f_s obeys the Vlasov equation,

$$\partial_t f_s + \mathbf{v} \cdot \nabla f_s + \frac{\mathbf{F}_s}{m_s} \cdot \nabla_v f_s = 0 \quad (2.81)$$

where the force $\mathbf{F}_s$ is also smoother than the $\mathbf{F}$ in equation 2.80 as it is based on the fields generated by a smoother distribution function.

When details of how the distribution function, f_s , depends on $\mathbf{v}$ are not important, a fluid description can be appropriate. The fluid equations are obtained from moments of the Vlasov equation [BL18]. Taking the zeroth moment of equation 2.81, yields

$$\int d\mathbf{v} \mathbf{v}^0 \left[\partial_t f_s + \mathbf{v} \cdot \nabla f_s + \frac{\mathbf{F}}{m_s} \cdot \nabla_v f_s \right] = 0 \quad (2.82)$$

which becomes

$$\partial_t n_s + \nabla \cdot n_s \mathbf{u}_s = 0 \quad (2.83)$$

where $n_s \equiv \int d\mathbf{v} f_s$ is the density of species s and the fluid velocity, $\mathbf{u}_s$, is

$$\mathbf{u}_s \equiv \frac{\int d\mathbf{v} \mathbf{v} f_s}{n_s} \quad (2.84)$$

Taking the first moment gives

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{e}{m} \left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right) - \frac{1}{mn} \mathbf{P} \quad (2.85)$$

where we drop the subscript s and assume electrons, and define the pressure tensor as

$$\mathbf{P} \equiv m \int (d\mathbf{v})(\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u})f_s \quad (2.86)$$

Rather than taking the next moment to obtain an equation for $\mathbf{P}$, we close the system of equations by assuming an equation of state. The simplest choice for SRS is an adiabatic equation of state,

$$P = P_0 \left(\frac{n}{n_0} \right)^{\gamma_s} = k_B T n_0 \left(\frac{n}{n_0} \right)^{\gamma_s} \quad (2.87)$$

where k_B is the Boltzmann constant and $\mathbf{P} = P\mathbf{I}$, i.e., the pressure is isotropic. This is justified by carrying out a fully kinetic description using the Vlasov equation and arguing that the divergence of the heat flux is small in an EPW. Thus, the full set of Maxwell, two-fluid equations are

$$-\nabla \times \mathbf{E} = \frac{1}{c} \partial_t \mathbf{B} \quad (2.88)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \partial_t \mathbf{E} \quad (2.89)$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (2.90)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.91)$$

where the current density is $\mathbf{J} = \sum_s q_s n_s \mathbf{u}_s$ and the charge density is $\rho = \sum_s q_s n_s$; and

$$\partial_t n_s + \nabla \cdot n_s \mathbf{u}_s = 0 \quad (2.92)$$

$$\partial_t \mathbf{u}_s + \mathbf{u}_s \cdot \nabla \mathbf{u}_s = \frac{q_s}{m_s} \left(\mathbf{E} + \frac{\mathbf{u}_s}{c} \times \mathbf{B} \right) - \frac{P_{0s}}{m_s n_s} \nabla \left(\frac{n_s}{n_{0s}} \right)^{\gamma_s} \quad (2.93)$$

where $\gamma_e = 3$ (adiabatic) and $\gamma_i = 1$ (isothermal). We can also rewrite the pressure term,

$$\frac{P_0}{mn} \nabla \left(\frac{n}{n_0} \right)^{\gamma} = \frac{\gamma k_B T}{m} \left(\frac{n}{n_0} \right)^{\gamma-2} \nabla \frac{n}{n_0} \quad (2.94)$$

which, for $\gamma = 3$, becomes $\frac{3}{2} v_e^2 \nabla \left(\frac{n}{n_0} \right)^2$ where the electron thermal velocity is $v_e = \sqrt{k_B T / m}$.

Next, we use these equations to describe SRS, motivated by our interest in OAM. Thus, we retain the three-dimensional effects. The analysis follows unpublished notes from a graduate class at UCLA. We also assume the ions are immobile and drop the s in the equations

as these are only equations for the electron density and fluid velocity. We can substitute $\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ into equation 2.93 to obtain

$$(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{e}{m} \left(-\nabla\phi - \frac{1}{c}\partial_t\mathbf{A} + \frac{\mathbf{u}}{c} \times \nabla \times \mathbf{A} \right) - \frac{3}{2}v_e^2 \nabla \left(\frac{n}{n_0} \right)^2 \quad (2.95)$$

which can be rewritten as

$$\partial_t \left(\mathbf{u} - \frac{e\mathbf{A}}{mc} \right) = \frac{e}{m} \nabla\phi + \mathbf{u} \times \nabla \times \left(\mathbf{u} - \frac{e\mathbf{A}}{mc} \right) - \nabla \frac{|\mathbf{u}|^2}{2} - \frac{3}{2}v_e^2 \nabla \left(\frac{n}{n_0} \right)^2 \quad (2.96)$$

where we have used the vector identity,

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times \nabla \times \mathbf{u} \quad (2.97)$$

Upon taking the curl of equation 2.96 we obtain

$$\partial_t \nabla \times \left(\mathbf{u} - \frac{e\mathbf{A}}{mc} \right) = \nabla \times \mathbf{u} \times \nabla \times \left(\mathbf{u} - \frac{e\mathbf{A}}{mc} \right) \quad (2.98)$$

which is an equation for $\mathcal{P} = m\mathbf{u} - \frac{e}{c}\mathbf{A}$, the canonical momentum. We define $\nabla \times \mathcal{P} = \mathbf{V}$ as the general or canonical vorticity so that equation 2.98 becomes

$$\partial_t \mathbf{V} - \nabla \times \mathbf{u} \times \mathbf{V} = 0 \quad (2.99)$$

Suppose $\mathbf{V} = 0$ everywhere at $t = 0$. Then the vorticity equation says that $\mathbf{V}$ will remain zero everywhere for all t . If there is no applied magnetic field, and the plasma starts at rest with $\mathbf{A} = 0$ (laser is far from focus at $z \rightarrow -\infty$), then, as the laser propagates from $z \rightarrow -\infty$, $\mathbf{u}$ will adjust so that $\mathbf{V} = 0$, or equivalently

$$\nabla \times \mathbf{u} = \nabla \times \frac{e\mathbf{A}}{mc} = \frac{e\mathbf{B}}{mc} \quad (2.100)$$

Therefore, we may neglect the term in equation 2.96 containing $\nabla \times (\mathbf{u} - e\mathbf{A}/mc)$. Substituting $\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}$ and multiplying through by m , equation 2.96 now reduces to

$$\partial_t(m\mathbf{u}) = -e\mathbf{E} - \nabla \frac{m|\mathbf{u}|^2}{2} - \frac{3}{2}mv_e^2 \nabla \left(\frac{n}{n_0} \right)^2 \quad (2.101)$$

where $-\nabla m|\mathbf{u}|^2/2$ is the fluid version of the ponderomotive force (equation 2.74).

The next step is to obtain paraxial ray approximation equations for SRS that encompass lowest order finite spot size effects and that describe SRS for lasers with OAM. In SRS a laser (pump EM wave) enters a plasma and decays into an EPW and a backscattered EM wave (daughter waves). Starting from Ampere's Law and rewriting $\mathbf{E}$ and $\mathbf{B}$ in terms of the vector potential gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \partial_t^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} \quad (2.102)$$

where the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, has been used. The current due to the electron motion is $-en\mathbf{u}$. In general, from the canonical vorticity we have

$$\nabla \times \left(\mathbf{u} - \frac{e\mathbf{A}}{mc} \right) = 0 \quad (2.103)$$

As mentioned above, to lowest order, a laser is described by its components perpendicular to $\mathbf{k} = \hat{\mathbf{z}}k_0$, or by $\mathbf{A}_\perp$. Consider a laser linearly polarized along the $\hat{\mathbf{x}}$ direction. Then, from equation 2.103, we can write

$$\partial_z \left(u_x - \frac{eA_x}{mc} \right) = \partial_x \left(u_z - \frac{eA_z}{mc} \right) \quad (2.104)$$

where all quantities oscillate at the laser frequency, and thus, the components are related as

$$u_x - \frac{eA_x}{mc} \approx \frac{1}{ik_0 w_0} \left(u_z - \frac{eA_z}{mc} \right) \quad (2.105)$$

Furthermore, based on the Coulomb gauge, $|A_z| \sim |\frac{1}{k_0 w_0} A_\perp|$ and $u_z \sim |A_\perp|^2$. If A_y is not zero, an equation similar to 2.105 can be written for $u_y - eA_y/mc$ and we reach the same conclusion. Thus,

$$\mathbf{u}_\perp - \frac{e\mathbf{A}_\perp}{mc} \approx O \left(\frac{1}{(k_0 w_0)^2}, \frac{A_\perp^2}{k_0 w_0} \right) \quad (2.106)$$

or $\mathbf{u}_\perp = e\mathbf{A}_\perp/mc$ to a high degree of accuracy even in multi-dimensions. Therefore, the current may be written

$$\mathbf{J}_\perp = -en\mathbf{u}_\perp = -\frac{e^2 n}{m} \mathbf{A}_\perp \quad (2.107)$$

If we define the plasma frequency,

$$\omega_p = \sqrt{\frac{4\pi e^2 n_0}{m}} \quad (2.108)$$

then replacing $\mathbf{J}_\perp$ in equation 2.102 yields

$$\nabla^2 \mathbf{A}_\perp - \frac{1}{c^2} \partial_t^2 \mathbf{A}_\perp - \frac{\omega_p^2}{c^2} \frac{n}{n_0} \mathbf{A}_\perp = 0 \quad (2.109)$$

Next, we separate the electron density into $n = n_0 + n_1$, where n_1 is a density perturbation about n_0 , and multiply through by c^2 to obtain

$$(\partial_t^2 + \omega_p^2 - c^2 \nabla^2) \frac{e \mathbf{A}_\perp}{mc^2} = -\omega_p^2 \frac{n_1}{n_0} \frac{e \mathbf{A}_\perp}{mc^2} \quad (2.110)$$

The density perturbation is normalized to n_0 and $\mathbf{A}_\perp$ to mc^2/e and, substituting the normalized vector potential, we have

$$(\partial_t^2 + \omega_p^2 - c^2 \nabla^2) \mathbf{a} = -\omega_p^2 \frac{n_1}{n_0} \mathbf{a} \quad (2.111)$$

In SRS, recall that an incoming laser decays into a backscattered EM wave and a forward going EPW. The instability arises when the RHS of equation 2.110 resonantly drives the LHS. This requires ω and k matching. We also need an equation for a density perturbation driven by the two light waves. To obtain this equation we start from the continuity equation for electrons,

$$\partial_t n + \nabla \cdot n \mathbf{u} = 0 \quad (2.112)$$

or

$$\partial_t n_1 + n_0 \nabla \cdot \mathbf{u} + \nabla \cdot n_1 \mathbf{u} = 0 \quad (2.113)$$

The third term drives harmonics, but if n_1 is $O(\epsilon)$ then this term is $O(\epsilon^2)$ and can be dropped.

Taking the time derivative of the continuity equation gives

$$\partial_t^2 n_1 + n_0 \nabla \cdot \partial_t \mathbf{u} = 0 \quad (2.114)$$

which, upon normalizing to n_0 and substituting equation 2.101 divided by mass, leads to

$$\partial_t^2 \frac{n_1}{n_0} + \nabla \cdot \left[-\frac{e}{m} \mathbf{E} - \nabla \frac{|\mathbf{u}|^2}{2} - \frac{3}{2} v_e^2 \nabla \left(\frac{n}{n_0} \right)^2 \right] = 0 \quad (2.115)$$

The last term can be rewritten by separating n into $n_0 + n_1$ and assuming n_1^2 is $O(\epsilon^2)$,

$$\nabla \cdot \frac{3}{2} v_e^2 \nabla \left(\frac{n}{n_0} \right)^2 = \frac{3}{2} v_e^2 \nabla^2 \left(\frac{n_0 + n_1}{n_0} \right)^2 \approx 3 v_e^2 \nabla^2 \frac{n_1}{n_0} \quad (2.116)$$

Additionally, by using Gauss' Law, $\nabla \cdot \mathbf{E} = -4\pi e n_1$, equation 2.115 reduces to

$$\partial_t^2 \frac{n_1}{n_0} + \frac{4\pi e^2}{m} n_1 - 3 v_e^2 \nabla^2 \frac{n_1}{n_0} = \frac{1}{2} \nabla^2 |\mathbf{u}|^2 \quad (2.117)$$

and, rearranging, we reach

$$(\partial_t^2 + \omega_p^2 - 3 v_e^2 \nabla^2) \frac{n_1}{n_0} = \frac{1}{2} \nabla^2 |\mathbf{u}|^2 \quad (2.118)$$

We can separate out the contributions of $\mathbf{u}$ to those from the laser and those from the EPW,

$$\mathbf{u} = \mathbf{u}_L + \mathbf{u}_1 \quad (2.119)$$

and recall for the laser the $\perp$ components dominate while for the EPW the z -component dominates. Thus,

$$u^2 = u_{\perp L}^2 + u_{z1}^2 \quad (2.120)$$

and the u_{z1}^2 only generates harmonics of the EPW. Furthermore, recall that the laser contribution can be written in terms of the normalized vector potential,

$$\frac{\mathbf{u}_{\perp L}}{c} = \frac{e \mathbf{A}_{\perp}}{m c^2} = \mathbf{a} \quad (2.121)$$

and so, using this definition, equation 2.118 becomes

$$(\partial_t^2 + \omega_p^2 - 3 v_e^2 \nabla^2) \frac{n_1}{n_0} = \frac{c^2}{2} \nabla^2 |\mathbf{a}|^2 \quad (2.122)$$

Finally, the two coupled differential equations describing both the EM and EPW waves in stimulated Raman scattering (in unnormalized form) are

$$(\partial_t^2 + \omega_p^2 - c^2 \nabla^2) \frac{e \mathbf{A}}{m c^2} = -\omega_p^2 \frac{n_1}{n_0} \frac{e \mathbf{A}}{m c^2} \quad (2.123)$$

$$(\partial_t^2 + \nu \partial_t + \omega_p^2 - 3 v_e^2 \nabla^2) \frac{n_1}{n_0} = \frac{c^2}{2} \nabla^2 \left| \frac{e \mathbf{A}}{m c^2} \right|^2 \quad (2.124)$$

where we drop the $\perp$ subscript on $\mathbf{A}$. To be consistent with the literature, we have introduced a phenomenological plasma wave damping factor, ν , and assumed the light wave is undamped. However, if ν represents Landau damping, then it is improper to include it in this way as it is not a local operator in space and time. We drop it for now but add it back when dealing with the envelope equations where it is properly defined.

In order to understand the creation of a backscattered light wave and a plasma wave in the presence of an incident laser, we separate the vector potential into a larger, pump component, $\mathbf{A}_0$, and the responsive scattered light, $\mathbf{A}_1$. To first order, the pump light is not modified by the interaction. However, with large amounts of SRS, the incident laser can become depleted as the backscatter grows. Separating $\mathbf{A}$ and keeping only the first order terms, we arrive at the following wave equations,

$$(\partial_t^2 + \omega_p^2 - c^2 \nabla^2) \mathbf{A}_1 = -\omega_p^2 \frac{n_1}{n_0} \mathbf{A}_0 \quad (2.125)$$

$$(\partial_t^2 + \omega_p^2 - 3v_e^2 \nabla^2) \frac{n_1}{n_0} = \frac{e^2}{m^2 c^2} \nabla^2 (\mathbf{A}_0 \cdot \mathbf{A}_1) \quad (2.126)$$

The equation for pump depletion must include the second order term, $n_1 \mathbf{A}_1$,

$$(\partial_t^2 + \omega_p^2 - c^2 \nabla^2) \mathbf{A}_0 = -\omega_p^2 \frac{n_1}{n_0} \mathbf{A}_1 \quad (2.127)$$

A dispersion relation can be obtained by assuming

$$\frac{e \mathbf{A}_0}{mc^2} = \frac{\mathbf{a}_0}{2} e^{i(k_0 z - \omega_0 t)} + \text{c.c.} \quad (2.128)$$

$$\frac{e \mathbf{A}_1}{mc^2} = \frac{\mathbf{a}_1}{2} e^{i(k_1 z - \omega_1 t)} + \text{c.c.} \quad (2.129)$$

$$\frac{n_1}{n_0} = \frac{\bar{n}_1}{2} e^{i(k_2 z - \omega_2 t)} + \text{c.c.} \quad (2.130)$$

where $\mathbf{a}_0$, $\mathbf{a}_1$, and $\bar{n}_1$ are constant, and ω and k matching is assumed,

$$\omega_0 = \omega_1 + \omega_2 \quad (2.131)$$

$$k_0 = k_1 + k_2 \quad (2.132)$$

and ω_j and k_j are complex. However, if pump depletion is to be included then the use of the envelope equations is required. The envelope equations are also useful for identifying different regimes of growth.

2.3.2.1 Envelope equations

When studying three-wave interactions it is often useful to use the envelope equations. In this approach, rather than letting ω_j and k_j be complex, we separate each wave into a normalized, complex envelope and a phase factor,

$$\frac{e\mathbf{A}_0}{mc^2} = \hat{\mathbf{e}}b_0e^{ik_0z-i\omega_0t} + \text{c.c.} \quad (2.133)$$

$$\frac{e\mathbf{A}_1}{mc^2} = \hat{\mathbf{e}}b_1e^{ik_1z-i\omega_1t} + \text{c.c.} \quad (2.134)$$

$$\frac{n_1}{n_0} = b_2e^{ik_2z-i\omega_2t} + \text{c.c.} \quad (2.135)$$

where now ω 's and k 's are real. The amplitudes, b_j , are now complex functions of position and time. The polarization vector is $\hat{\mathbf{e}}$ and is constant. In the case of stimulated Raman backscatter, we define ω_1 and k_1 so that v_{g1} is negative and the scattered light wave propagates backwards. Substituting these forms for the envelopes and dropping second derivatives in t and z by assuming $\partial_t^2 b \ll \omega^2 b$ and $\partial_z^2 b \ll k^2 b$, i.e., the envelopes are slowly varying compared to the fast phase factor, leads to

$$\partial_t^2 (be^{ikz-i\omega t}) \approx [-2i\omega (\partial_t b) - \omega^2 b] e^{ikz-i\omega t} \quad (2.136)$$

$$\nabla^2 (be^{ikz-i\omega t}) \approx [2ik (\partial_z b) - k^2 b + \nabla_\perp^2 b] e^{ikz-i\omega t} \quad (2.137)$$

Equation 2.125 in terms of b_1 then becomes

$$[-2i\omega_1 (\partial_t b_1) - 2ik_1 c^2 (\partial_z b_1) + (-\omega_1^2 + \omega_p^2 + c^2 k_1^2) b_1 - c^2 \nabla_\perp^2 b_1] e^{ik_1 z - i\omega_1 t} + \text{c.c.} = -\omega_p^2 \frac{n_1}{n_0} \frac{e\mathbf{A}_0}{mc^2} \quad (2.138)$$

We assume ω_j and k_j for each wave satisfies its own fluid dispersion relation. Thus, $-\omega_1^2 + \omega_p^2 + c^2 k_1^2$ sums to zero, so this term cancels out. The left-hand side has a term involving

$b_1 e^{ik_1 z - i\omega_1 t}$ and a separate term for the complex conjugate, $b_1^* e^{-ik_1 z + i\omega_1 t}$. The right-hand side thus has 4 terms after being converted into the envelope format,

$$\frac{n_1}{n_0} \frac{eA_0}{mc^2} = b_0 b_2 e^{i(k_0+k_2)z - i(\omega_0+\omega_2)t} + b_0 b_2^* e^{i(k_0-k_2)z - i(\omega_0-\omega_2)t} + \text{c.c.} \quad (2.139)$$

For SRS backscatter to grow, the ω and k matching conditions (equations 2.131 and 2.132) must be satisfied for resonant interactions. These conditions are equivalent to a statement of conservation of energy and momentum, which will be discussed in detail in the section on conservation laws later in this chapter. Thus, only the term with $b_0 b_2^*$ should be included on the right-hand side of the envelope equation when b_1 is on the left. (The complex conjugate, $b_0^* b_2$, drives b_1^* .)

Therefore, the equation for the evolution of the envelope of the backscattered light is

$$\omega_1 \left(\partial_t + \frac{c^2 k_1}{\omega_1} \partial_z - i \frac{c^2}{2\omega_1} \nabla_\perp^2 \right) b_1 = -i \frac{\omega_p^2}{2} b_0 b_2^* \quad (2.140)$$

Following the same method, the plasma wave equation becomes

$$\begin{aligned} & [-2i\omega_2 (\partial_t b_2) - 6ik_2 v_e^2 (\partial_z b_2) + (-\omega_2^2 + \omega_p^2 + 3v_e^2 k_2^2) b_2 - \\ & - 3v_e^2 \nabla_\perp^2 b_2] e^{ik_2 z - i\omega_2 t} + \text{c.c.} = \frac{e^2}{m^2 c^2} \nabla^2 (\mathbf{A}_0 \cdot \mathbf{A}_1) \end{aligned} \quad (2.141)$$

which can be rewritten as

$$-2i\omega_2 e^{ik_2 z - i\omega_2 t} \left(\partial_t + \frac{3v_e^2 k_2}{\omega_2} \partial_z - i \frac{3v_e^2}{2\omega_2} \nabla_\perp^2 \right) b_2 + \text{c.c.} = \frac{e^2}{m^2 c^2} \nabla^2 (\mathbf{A}_0 \cdot \mathbf{A}_1) \quad (2.142)$$

when we assume $-\omega_2^2 + \omega_p^2 + 3v_e^2 k_2^2 = 0$. The right-hand side corresponds to the ponderomotive force and can be written as

$$\begin{aligned} \frac{e^2}{m^2 c^2} \nabla^2 (\mathbf{A}_0 \cdot \mathbf{A}_1) & \approx -c^2 (k_0 - k_1)^2 b_0 b_1^* e^{i(k_0-k_1)z - i(\omega_0-\omega_1)t} \\ & - c^2 (k_0 + k_1)^2 b_0 b_1 e^{i(k_0+k_1)z - i(\omega_0+\omega_1)t} + \text{c.c.} \end{aligned} \quad (2.143)$$

where we drop the $\nabla_\perp^2$ contribution. Neglecting the term with $b_0 b_1$ and including only the resonant mode which satisfies the matching condition yields the envelope equation for the evolution of b_2 ,

$$\omega_2 \left(\partial_t + \nu + \frac{3v_e^2 k_2}{\omega_2} \partial_z - i \frac{3v_e^2}{2\omega_2} \nabla_\perp^2 \right) b_2 = -i \frac{k_2^2 c^2}{2} b_0 b_1^* \quad (2.144)$$

where we have included a phenomenological damping term $\nu(\omega_2, k_2)$ which can correspond to Landau damping if needed. Lastly, the envelope equation for b_0 resembles that of b_1 , and we choose $b_1 b_2$ on the right in order to satisfy the matching conditions for resonant coupling,

$$\omega_0 \left(\partial_t + \frac{c^2 k_0}{\omega_0} \partial_z - i \frac{c^2}{2\omega_0} \nabla_\perp^2 \right) b_0 = -i \frac{\omega_p^2}{2} b_1 b_2 \quad (2.145)$$

Upon defining a normalized thermal velocity,

$$u_e^2 \equiv 3 \frac{v_e^2}{c^2} \quad (2.146)$$

we can rewrite the full set of coupled envelope equations,

$$\omega_0 \left(\partial_t + v_{g0} \partial_z - i \frac{c^2}{2\omega_0} \nabla_\perp^2 \right) b_0 = -i c_0 b_1 b_2 \quad (2.147)$$

$$\omega_1 \left(\partial_t + v_{g1} \partial_z - i \frac{c^2}{2\omega_1} \nabla_\perp^2 \right) b_1 = -i c_1 b_0 b_2^* \quad (2.148)$$

$$\omega_2 \left(\partial_t + \nu + v_{g2} \partial_z - i \frac{u_e^2 c^2}{2\omega_2} \nabla_\perp^2 \right) b_2 = -i c_2 b_0 b_1^* \quad (2.149)$$

The envelopes, b_0 , b_1 , and b_2 , are complex functions and c_0 , c_1 and c_2 , are real constants. The coupling factors are $c_0 = c_1 = \frac{\omega_p^2}{2}$ and $c_2 = \frac{c^2 k_2^2}{2}$, and the group velocities of each wave are $v_{g0} = k_0 c^2 / \omega_0$, $v_{g1} = k_1 c^2 / \omega_1$, and $v_{g2} = u_e^2 k_2 c^2 / \omega_2$. We have also assumed that $|\nabla_\perp| \ll |k|$ which is the basic assumption behind the paraxial ray approximation.

Note that we have assumed the plasma density is uniform ($\nabla n_0 = 0$) and neglected damping of the EM waves. It is straightforward to include both effects in these equations. However, for simplicity throughout the remainder of the dissertation, we assume a uniform plasma and negligible damping of EM waves.

2.3.3 SRS growth

Next, we review the growth rates (and, similarly, the characteristic gain length) in four regimes relevant to the simulations to be presented in Chapters 3 and 4. These regimes are delineated by whether the damping of the EPW is important and by whether we are interested in temporal and spatial growth. We will assume that $\nabla_{\perp}^2$ can be neglected to simplify the derivations. We also briefly address convective and absolute growth. The material in this section has benefited greatly from ref. [Mic23] and unpublished class notes from a graduate course at UCLA.

2.3.3.1 Temporal growth

In order to derive the temporal growth rate, we neglect ∂_z in equations 2.148 and 2.149. This is equivalent to examining the growth of a noise source with the k of the fastest growing mode present at $t = 0$. We are then left with ∂_t and ν in the plasma wave equation (equation 2.149). In the case of weak damping, ν is negligible. On the other hand, if damping is strong, ν is much greater than ∂_t and damping is the dominant effect on b_2 . Marginal damping, where neither ∂_t nor ν can be neglected, can also be studied.

Weak damping For weakly-damped temporal growth of SRS, we drop ∂_z and ν in equations 2.148 and 2.149. We are therefore left with

$$\omega_1 \partial_t b_1 = -i c_1 b_0 b_2^* \quad (2.150)$$

$$\omega_2 \partial_t b_2 = -i c_2 b_0 b_1^* \quad (2.151)$$

The coupled equations can be combined to form a second order differential equation for b_1 or b_2 . Taking ∂_t of equation 2.150 and substituting the complex conjugate of equation 2.151 gives,

$$\partial_t^2 b_1 = -i \frac{c_1}{\omega_1} b_0 (\partial_t b_2^*) = -i \frac{c_1}{\omega_1} b_0 \left(i \frac{c_2}{\omega_2} b_0^* b_1 \right) = \frac{c_1 c_2}{\omega_1 \omega_2} |b_0|^2 b_1 = \gamma_0^2 b_1 \quad (2.152)$$

where we have defined the temporal growth rate,

$$\gamma_0 \equiv \sqrt{\frac{c_1 c_2}{\omega_1 \omega_2}} |b_0| = \frac{c \omega_p k_2}{2 \sqrt{\omega_1 \omega_2}} |b_0| \quad (2.153)$$

We are assuming that pump depletion is negligible and the pump is non-evolving so that $\partial_t b_0 = 0$. The solutions to equation 2.152 are exponential in b_1 with growth rate γ_0 i.e., $e^{\gamma_0 t}$. The growth rate, γ_0 , is proportional to the amplitude of the pump wave, b_0 . In a low density and cold plasma $\omega_1 \approx \omega_0$, $\omega_2 \approx \omega_p$, and $k_2 \approx 2\omega_0/c$, in which case

$$\gamma_0 \approx \omega_p \sqrt{\frac{\omega_p}{\omega_0}} |b_0| \quad (2.154)$$

Strong damping The SRS growth rate is modified when plasma wave damping is significant i.e., $\nu \gg \partial_t$. Consider equation 2.148 and equation 2.149 when the dominant term on the left-hand side of equation 2.149 is νb_2 . The two coupled equations become

$$\omega_1 \partial_t b_1 = -i c_1 b_0 b_2^* \quad (2.155)$$

$$\omega_2 \nu b_2 = -i c_2 b_0 b_1^* \quad (2.156)$$

where the equation for b_1 is the same as the weakly damped case (equation 2.152), but the EPW equation has changed. Substituting the complex conjugate of equation 2.156 into equation 2.155 gives

$$\partial_t b_1 = -i \frac{c_1}{\omega_1} b_0 \left(i \frac{c_2}{\omega_2 \nu} b_0^* b_1 \right) = \frac{c_1 c_2}{\nu \omega_1 \omega_2} |b_0|^2 b_1 = \frac{\gamma_0^2}{\nu} b_1 \quad (2.157)$$

where by assumption the damping rate, ν is real. We define the strongly damped growth rate as

$$\gamma_s \equiv \frac{\gamma_0^2}{\nu} \quad (2.158)$$

Solutions to equation 2.157 take the form $e^{\gamma_s t}$. In this case, the growth rate is proportional to the square of the pump amplitude, $|b_0|^2$, which is the pump intensity. We have assumed ∂_t/ν was small, which can be checked a posteriori,

$$\frac{\partial_t}{\nu} = \frac{\gamma_0^2}{\nu^2} \quad (2.159)$$

so strong damping arises when $\gamma_0 \ll \nu$ and the ratio of the strongly damped growth rate to the weakly damped growth rates is

$$\frac{\gamma_s}{\gamma_0} = \frac{\gamma_0}{\nu} \quad (2.160)$$

which, as expected, is also small in this case.

2.3.3.2 Spatial gain

When the backscatter grows spatially, as opposed to purely temporally, we calculate the growth rate by neglecting ∂_t in equations 2.148 and 2.149. This is equivalent to assuming the noise source is an EM antenna oscillating at a fixed ω_1 at a given location. Just as in the previous case, we can look at both the weakly damped scenario where $\nu \ll v_{g2}\partial_z$ and the strong damping scenario where ν is sufficiently large so that $\nu \gg v_{g2}\partial_z$.

Weak damping For weakly-damped spatial growth we drop ∂_t and ν everywhere in the coupled equations to obtain

$$\omega_1 v_{g1} \partial_z b_1 = -i c_1 b_0 b_2^* \quad (2.161)$$

$$\omega_2 v_{g2} \partial_z b_2 = -i c_2 b_0 b_1^* \quad (2.162)$$

We can combine these equations by substituting the complex conjugate of 2.162 into the spatial derivative of 2.161 to obtain

$$\partial_z^2 b_1 = \frac{c_1 c_2 |b_0|^2}{\omega_1 \omega_2 v_{g1} v_{g2}} b_1 = \frac{\gamma_0^2}{v_{g1} v_{g2}} b_1 = \beta_0^2 b_1 \quad (2.163)$$

where we have maintained the assumption of a constant pump, b_0 . Solutions to equation 2.163 take the form $e^{\beta_0 z}$, where the growth rate is

$$\beta_0 \equiv \frac{\gamma_0}{\sqrt{v_{g1} v_{g2}}} \quad (2.164)$$

Note that, for spatial growth, one must determine the direction for growth. We can argue on physical grounds that, for SRS, it grows backwards, because the group velocity of the light

wave is much greater than that of the forward moving EPW. To determine the direction mathematically requires a complicated analysis of inverting the Laplace Fourier Transform of the Green's function for different types of noise sources that naturally introduces the idea of convective and absolute instabilities [Bri64, Ros72].

Strong damping If the damping rate, ν , is sufficiently large, we can neglect ∂_z in equation 2.149, which yields

$$\omega_1 v_{g1} \partial_z b_1 = -ic_1 b_0 b_2^* \quad (2.165)$$

$$\omega_2 \nu b_2 = -ic_2 b_0 b_1^* \quad (2.166)$$

where equation 2.165 is the same as in the weakly-damped spatial case. The group velocity of the plasma wave, v_{g2} , no longer plays a role in the SRS gain. Substituting the complex conjugate of equation 2.166 into equation 2.165 gives

$$\partial_z b_1 = \frac{\gamma_0^2}{v_{g1} \nu} b_1 = \beta_s b_1 \quad (2.167)$$

The solutions have a gain rate of β_s and grow as $e^{\beta_s z}$ where

$$\beta_s \equiv \frac{\gamma_0^2}{v_{g1} \nu} = \beta_0^2 \frac{v_{g2}}{\nu} = \frac{\gamma_s}{v_{g1}} \quad (2.168)$$

which is the temporal growth rate for strong damping divided by the group velocity of the backscattered light wave. The condition for being in the strongly damped spatial regime is $v_{g2} \partial_z \ll \nu$ or

$$\frac{v_{g2}}{\nu} \beta_s = \frac{v_{g2}}{v_{g1}} \frac{\gamma_0^2}{\nu^2} \ll 1 \quad (2.169)$$

Note that one can simultaneously be in the strongly damped spatial regime but in the weakly damped temporal regime because $v_{g1} \gg v_{g2}$. In that case, both of the following conditions are satisfied,

$$\frac{\nu}{\gamma_0} \ll 1 \quad (2.170)$$

$$\sqrt{\frac{v_{g1}}{v_{g2}}} \frac{\nu}{\gamma_0} \gg 1 \quad (2.171)$$

In the above, we concentrated on purely temporal or spatial growth. These are strictly relevant when the noise source is either a sinusoidal perturbation at the wave number with the maximum growth rate or an antenna with the frequency of maximum growth at one point in space. In reality the noise sources are more complicated. If the noise source is a delta function in both space and time, then the instability can be classified as absolute or convective. Briggs [Bri64] provided precise methods for understanding the behavior of instabilities based on the properties of the dispersion relation or equivalently the Green's function. The analysis is beyond the scope of this dissertation, but we do provide a high-level summary of such an analysis for SRS.

In a convective instability, the asymptotic behavior is described by a signal that has a finite width that convects such that the growth at one point in space initially grows and then decays, while an absolute instability spreads out and grows indefinitely at one point in space. Thus, a requirement for an absolute instability is that the group velocity of the two daughter waves are in opposite directions.

For the three wave equations, as the pulse convects and grows, the location of the maximum growth convects at a velocity $\approx (v_{g1} + v_{g2})/2$ and the peak grows with the temporal growth rate. Thus, for SRS where $v_{g2} \ll v_{g1}$, the maximum amount of growth is $e^{2\gamma_0 L/v_{g1}}$ where L is the distance the packet has convected. This corresponds to a spatial growth rate of $2\gamma_0/v_{g1}$. As convective growth occurs in response to many noise sources, eventually the growth can reach a steady state with the spatial growth $\beta_0 \equiv \frac{\gamma_0}{\sqrt{v_{g1}v_{g2}}}$ which is $\sqrt{v_{g1}/v_{g2}}$ times larger than the convective gain rate. The convective growth can be extended to include damping [Mic23]. In this case the growth of the peak and the spatial growth rate are the same.

SRS can also grow as an absolute instability with a growth rate,

$$\gamma_{abs} = \frac{2\gamma_0 \sqrt{|v_{g1}||v_{g2}|} - \nu|v_{g1}|}{|v_{g1}| + |v_{g2}|} \approx 2\gamma_0 \sqrt{\frac{|v_{g2}|}{|v_{g1}|}} - \nu \quad (2.172)$$

In the absence of EPW damping the absolute growth rate is γ_{abs} and the threshold condition

	Weak Damping	Strong Damping
Temporal	$\gamma_0 = \frac{c\omega_p k_2}{2\sqrt{\omega_1 \omega_2}} b_0 $	$\gamma_s = \frac{\gamma_0^2}{\nu}$
Spatial	$\beta_0 = \frac{\gamma_0}{\sqrt{v_{g1} v_{g2}}}$	$\beta_s = \frac{\gamma_0^2}{v_{g1} \nu}$

Table 2.1: Growth rates for purely temporal or spatial growth with weak or strong EPW damping. In a real scenario, both temporal and spatial gain can occur and damping may be neither negligible nor dominant.

for absolute growth is,

$$\gamma_0 > \frac{\nu}{2} \sqrt{\frac{|v_{g1}|}{|v_{g2}|}}. \quad (2.173)$$

Note that this is also the approximate condition for transitioning from weakly to strongly damped spatial growth. For parameters of relevance to NIF, the threshold for absolute growth is not satisfied and furthermore SRS occurs in a density gradient which can also prevent absolute growth.

2.3.3.3 Growth rates summarized

The temporal and spatial growth rates are summarized in table 2.1. For weak damping, the growth rate is always linear in $|b_0|$, the amplitude of the pump laser. Whereas, for strong damping, the growth rates are proportional to the intensity of the pump, $|b_0|^2$. The growth rates derived here are a useful tool to explore the limits of extremely weak or strong damping, and purely temporal or spatial growth.

At the National Ignition Facility (NIF), the lasers enter the laser entrance hole and then propagate in a density gradient that begins at less than 5% of the critical density and rises towards above the critical density as the laser approaches the wall. The resulting SRS EPW has a $k_2 \lambda_D$ that decreases from larger than 1 at the lower densities to around 0.3 near 10% critical density. At this $k_2 \lambda_D$ SRS can begin to be significant. The laser intensity is 10^{15} W/cm² and the laser wavelength is 351 nm, which corresponds to a normalized vector

potential of $a_0 = 0.01$. For $n_0/n_c \approx 0.1$, the parameter $k_2\lambda_D$ lies between 0.3 and 0.35, and thus for this value of a_0 , $\gamma_0/\nu \approx 0.3$ and $(\gamma_0/\nu)\sqrt{v_{g2}/v_{g1}} \approx 0.1$. Thus SRS generally begins in the strongly damped regime. As we discuss below, it can transition into the weakly damped regime if the EPW nonlinearly evolves into an undamped BGK-like mode.

2.3.4 Reflectivity

In the previous subsections, we quantified the growth by assuming b_0 remained constant. In reality, the pump will deplete thereby lowering and saturating the reflectivity. The regime of interest in this dissertation is the strongly damped limit. In this subsection, we derive well known expressions for the reflectivity in this regime that account for pump depletion [BSW98, FKLInb]. The derivation also introduces the idea of conservation of wave action that is discussed in more detail later. We consider steady state reflectivity and start from equations 2.165 and 2.166 augmented with the equation for the pump,

$$\omega_0 v_{g0} \partial_z b_0 = -i c_0 b_1 b_2 \quad (2.174)$$

and substitute equation 2.166 into equations 2.165 and 2.174 to obtain

$$\omega_0 v_{g0} \partial_z b_0 = -\frac{c_0 c_2}{\nu \omega_2} |b_1|^2 b_0 \quad (2.175)$$

$$\omega_1 v_{g1} \partial_z b_1 = \frac{c_1 c_2}{\nu \omega_2} |b_0|^2 b_1 \quad (2.176)$$

where $v_{g0} > 0$ and $v_{g1} < 0$. Note that if $|b_0|^2$ is constant (no pump depletion), then equation 2.176 recovers equation 2.167 where $b_1 \propto e^{\beta_s z}$ and $\beta_s \equiv \frac{c_1 c_2}{\nu \omega_1 \omega_2 |v_{g1}|} |b_0|^2$ is the spatial gain coefficient in the strongly damped regime.

To include pump depletion, we take linear combinations of equations 2.175 and 2.176 to obtain

$$\omega_0 v_{g0} (b_0^* \partial_z b_0 + b_0 \partial_z b_0^*) = \partial_z (\omega_0 v_{g0} |b_0|^2) = -2 \frac{c_0 c_2}{\nu \omega_2} |b_1|^2 |b_0|^2 \quad (2.177)$$

$$\omega_1 v_{g1} (b_1^* \partial_z b_1 + b_1 \partial_z b_1^*) = \partial_z (\omega_1 v_{g1} |b_1|^2) = 2 \frac{c_1 c_2}{\nu \omega_2} |b_0|^2 |b_1|^2 \quad (2.178)$$

Adding equations 2.177 and 2.178 and noting that $c_0 = c_1$ leads to

$$\partial_z(\omega_0 v_{g0}|b_0|^2 + \omega_1 v_{g1}|b_1|^2) = 0 \quad (2.179)$$

or

$$\partial_z(\omega_0 v_{g0}|b_0|^2 - \omega_1 |v_{g1}||b_1|^2) = 0 \quad (2.180)$$

and we define $N_0 = \omega_0 v_{g0}|b_0|^2$ and $N_1 = \omega_1 |v_{g1}||b_1|^2$ where N_0 is related to the wave action or number of photons as described later. Using these definitions, equations 2.177 and 2.178 can be rewritten as

$$\partial_z N_0 = -\Gamma_0 N_0 N_1 \quad (2.181)$$

$$\partial_z N_1 = -\Gamma_0 N_0 N_1 \quad (2.182)$$

where

$$\Gamma_0 \equiv \frac{2c_0 c_2}{\nu \omega_0 \omega_1 \omega_2 |v_{g1}| |v_{g0}|} \quad (2.183)$$

We consider a plasma slab of length L where the pump enters at $z = 0$ and the backscattered light originates at $z = L$. Thus, we have $\partial_z(N_0 - N_1) = 0$ which leads to,

$$N_0(z) - N_1(z) = N_0(0) - N_1(0) = N_0(L) - N_1(L) \quad (2.184)$$

Using equation 2.184, equation 2.182 can be rewritten as,

$$\partial_z N_1 = -\Gamma_0 N_1 [N_1 + N_0(0) - N_1(0)] \quad (2.185)$$

Defining $\frac{N_1}{N_0(0)} \equiv \tilde{N}_1$, $\frac{N_1(0)}{N_0(0)} \equiv R$, and $\frac{N_1(L)}{N_0(0)} \equiv b$, and integrating equation 2.185 leads to

$$\frac{\tilde{N}_1(z)}{\tilde{N}_1(z) + (1 - R)} = C_0 e^{-2\beta_s(1-R)z} \quad (2.186)$$

where C_0 is an integration constant.

Evaluating equation 2.186 at $z = 0$ gives $R = C_0$ and then evaluating equation 2.186 at $z = L$ gives [BSW98, FKLInb],

$$R = \frac{b}{1 - R + b} e^{2\beta_s(1-R)L} \quad (2.187)$$

which shows that

$$R = \frac{\omega_1 v_{g1} b_1^2}{\omega_0 v_{g0} b_0^2} = b e^{2\beta_s L} \quad (2.188)$$

when $R \ll 1$ and $b \ll 1$. As R increases, the scaling with β_s becomes less strong due to pump depletion.

2.3.5 Wave-particle interactions

The above analysis is based on the fluid approximation where wave-particle interactions are neglected. While the envelope approximation is advantageous in understanding the evolution of SRS, we have only included lowest order wave-particle interactions as a damping term in equation 2.149 which can be used to include Landau damping. In the previous section, we only allowed for a damping rate of the form $\nu \partial_t b_2$ in the equation for the EPW where ν is a constant. However, in regimes of interest, we must study the origin of the damping rate, ν , and estimate the ways in which ν can evolve due to changes to the plasma distribution function as SRS proceeds.

We assume the distribution function of a thermal plasma to be Maxwellian with respect to velocity in all directions. An electron plasma wave with phase velocity, v_{ph} , near the thermal velocity, v_e , can interact with particles whose velocity is near v_{ph} . For SRS and $k_2 \lambda_D \lesssim 0.3$, the phase velocity of the wave is resonant with the tail of the distribution function and does not affect the bulk of the particles. However, particles with velocities near v_{ph} can oscillate in the potential of the wave, which is moving very slowly in their reference frame. The EPW has periodic potential wells in which particles can become trapped and bounce back and forth. The wave accelerates particles with velocities below v_{ph} and decelerates particles with velocities above v_{ph} which leads to a flattening of the distribution function.

2.3.5.1 Bounce frequency

We can determine the bounce frequency by assuming that the electric field of a plasma wave is $E_0 \sin kx$ in the frame of a resonant particle ($v \approx v_{ph}$). For a particle trapped in the wave executing small oscillations, the electric force is nearly linear because $\sin \theta \approx \theta$ for $\theta \ll 1$. Therefore, the particle behavior may be approximated as

$$\frac{d^2x}{dt^2} = -\frac{e}{m} E_0 \sin kx \approx -\frac{eE_0 k}{m} x \quad (2.189)$$

and we define the bounce frequency, ω_B , from the second order differential equation as

$$\omega_B \equiv \sqrt{\frac{eE_0 k}{m}} \quad (2.190)$$

Trapped particles, in turn, affect the growth of the EPW. Landau damping occurs when there are more particles with velocities slower than v_{ph} than with velocities faster than v_{ph} . If the slope of the distribution function near v_{ph} is large and negative, the Landau damping rate is significant. Although Landau damping is a linear process, the effect of trapped particles can be complicated [Lan46, ON65, MO72b].

2.3.5.2 Inflation

In the linear phase of growth for SRS, the EPW can be linearly Landau damped. However, after a bounce period, $T_B = 2\pi/\omega_B$, the distribution function, f , flattens near v_{ph} . While this is a nonlinear process, it can also be viewed as a renormalization of the distribution function [MO72b]. The flattened distribution can then be used to determine a modified Landau damping rate,

$$\nu \propto -\left. \frac{df}{dv} \right|_{v_{ph}} \quad (2.191)$$

The damping rate may no longer be significant because the magnitude of the slope of f near v_{ph} has decreased. The modified ν can be much lower, transitioning SRS into the weakly damped regime where growth is stronger. This process is referred to as inflationary SRS [VDB02, VDB07].

Inflationary SRS results in high levels of reflectivity for conditions where fluid theory predicts a strongly-damped regime and little growth. We detect inflation in simulations where we can analyze the particle distribution function by measuring the slope near v_{ph} . There has been much interest in mitigating inflationary SRS by disrupting the wave-particle interactions that lead to distribution function flattening. An applied magnetic field is one idea to do this [VDB02, VDB07, SBM17, Mic23, RFT26].

The motion of trapped particles in helical plasma waves is therefore of interest in this effort. SRS of a laser with OAM can produce a plasma wave with OAM. The azimuthal variation of the wave will affect particle resonance and delay the transition to inflation. Thus, we are motivated to study SRS of light with OAM in the regime for which linear theory predicts strong damping. In Chapter 4, we perform PIC simulations of SRS with OAM pumps to investigate if EPWs with OAM delay inflation.

2.3.6 Conservation laws

We are motivated to understand the exchange of angular momentum in SRS. In this subsection, we derive conservation laws for the multi-dimensional SRS envelope equations (equations 2.147, 2.148, 2.149) based on the paraxial ray and fluid approximations. To our knowledge these have yet to appear in the literature. We aim to derive conservation laws for action, energy, momentum, and angular momentum in terms of the wave envelopes, b_0 , b_1 , and b_2 . At the end of this research it was recognized that the 3 envelope equations, equations 2.147 through 2.149, can also be obtained from a Lagrangian density. A Lagrangian can be used to obtain these conservation laws through Noether's theorem [PCT26]. We note that some of the conservation laws obtained from the Lagrangian are identical to what is described below for uniform plasmas.

For brevity, we define 3 operators,

$$\mathcal{L}_0 = \omega_0 \left(\partial_t + v_{g0} \partial_z - i \frac{c^2}{2\omega_0} \nabla_{\perp}^2 \right) \quad (2.192)$$

$$\mathcal{L}_1 = \omega_1 \left(\partial_t + v_{g1} \partial_z - i \frac{c^2}{2\omega_1} \nabla_{\perp}^2 \right) \quad (2.193)$$

$$\mathcal{L}_2 = \omega_2 \left(\partial_t + \nu + v_{g2} \partial_z - i \frac{u_e^2 c^2}{2\omega_2} \nabla_{\perp}^2 \right) \quad (2.194)$$

so that we can rewrite equations 2.147, 2.148, and 2.149 as

$$\mathcal{L}_0 b_0 = -ic_0 b_1 b_2 \quad (2.195)$$

$$\mathcal{L}_1 b_1 = -ic_1 b_0 b_2^* \quad (2.196)$$

$$\mathcal{L}_2 b_2 = -ic_2 b_0 b_1^* \quad (2.197)$$

and we define a real constant,

$$\Gamma_2 \equiv \frac{c_2}{c_0} = \frac{c^2 k_2^2}{\omega_p^2} \quad (2.198)$$

Noting that $c_0 = c_1$, we identify the following property,

$$b_0^* \mathcal{L}_0 b_0 = -b_1 (\mathcal{L}_1 b_1)^* = -\frac{1}{\Gamma_2} b_2 (\mathcal{L}_2 b_2)^* = -ic_0 b_0^* b_1 b_2 \quad (2.199)$$

which we will use to form linear combinations of equations 2.195, 2.196, and 2.197 in order to obtain conservation laws for action, energy, momentum, and angular momentum. The conservation laws will be in the form

$$\partial_t Q + \nabla \cdot \Gamma_Q = 0 \quad (2.200)$$

where Q is a scalar or a vector and Γ_Q is the flux of Q .

2.3.6.1 Conservation of action

We begin by examining the action of each wave. This leads to what is commonly referred to as the Manley-Rowe relations [MR07]. To derive the conservation of action, we begin with the linear combination,

$$b_0^* \mathcal{L}_0 b_0 + b_1 (\mathcal{L}_1 b_1)^* = 0 \quad (2.201)$$

Substituting the operators $\mathcal{L}_0$ and $\mathcal{L}_1$, equation 2.201 becomes

$$\omega_0 b_0^* \partial_t b_0 + \omega_1 b_1 \partial_t b_1^* + \omega_0 v_{g0} b_0^* \partial_z b_0 + \omega_1 v_{g1} b_1 \partial_z b_1^* - \frac{ic^2}{2} (b_0^* \nabla_\perp^2 b_0 - b_1 \nabla_\perp^2 b_1^*) = 0 \quad (2.202)$$

Next, in order to form total derivatives and find a conservation law resembling 2.200, we add the complex conjugate to equation 2.202 and rearrange to find

$$\begin{aligned} \partial_t (\omega_0 |b_0|^2 + \omega_1 |b_1|^2) + \partial_z (\omega_0 v_{g0} |b_0|^2 + \omega_1 v_{g1} |b_1|^2) \\ + \nabla_\perp \cdot -\frac{ic^2}{2} [(b_0^* \nabla_\perp b_0 - b_0 \nabla_\perp b_0^*) + (b_1^* \nabla_\perp b_1 - b_1 \nabla_\perp b_1^*)] = 0 \end{aligned} \quad (2.203)$$

where we have used the fact that $b^* \partial b + b \partial b^* = \partial |b|^2$ and

$$b^* \partial^2 b - b \partial^2 b^* = \partial (b^* \partial b - b \partial b^*) \quad (2.204)$$

The right-hand side of equation 2.203 would be nonzero if we included light wave damping in equations 2.147 and 2.148. We identify equation 2.203 as a conservation law for the action $\omega_0 |b_0|^2 + \omega_1 |b_1|^2$. For the light waves, the classical action is related to the number of photons which is conserved [MR07, Kru88] as we discuss below.

We can show that $\omega_0 |b_0|^2 + \frac{\omega_2}{\Gamma_2} |b_2|^2$ is conserved through the same approach, except now a sink is present due to EPW damping. We take the following linear combination,

$$b_0^* \mathcal{L}_0 b_0 + \frac{1}{\Gamma_2} b_2 (\mathcal{L}_2^* b_2)^* + \text{c.c.} = 0 \quad (2.205)$$

which becomes, by the same simplifications,

$$\begin{aligned} \partial_t \left(\omega_0 |b_0|^2 + \frac{\omega_2}{\Gamma_2} |b_2|^2 \right) + \partial_z \left(\omega_0 v_{g0} |b_0|^2 + \frac{\omega_2 v_{g2}}{\Gamma_2} |b_2|^2 \right) \\ + \nabla_\perp \cdot -\frac{ic^2}{2} \left[(b_0^* \nabla_\perp b_0 - b_0 \nabla_\perp b_0^*) + \frac{u_e^2}{\Gamma_2} (b_2^* \nabla_\perp b_2 - b_2 \nabla_\perp b_2^*) \right] = -2\nu \frac{\omega_2}{\Gamma_2} |b_2|^2 \end{aligned} \quad (2.206)$$

We note that it is also possible to obtain an equation for the conservation of $\omega_1 |b_1|^2 - \frac{\omega_2}{\Gamma_2} |b_2|^2$ by the same approach.

As noted above, $\frac{\omega_j}{\Gamma_j} |b_j|^2$ is the action and can be identified as the number of quanta per unit volume for the j th mode. Consider the EM waves, $j = 0$ and 1 with $\Gamma_j = 1$. For these

waves, b_j is the normalized vector potential and

$$|\mathbf{E}| \propto |\partial_t \mathbf{A}| \propto \omega b \quad (2.207)$$

Thus, $\omega b^2 \propto |\mathbf{E}|^2/\omega$ and recall that the photon density is $|\mathbf{E}|^2/\hbar\omega$. Thus, aside from the normalizations, ωb^2 is the number of photons divided by volume. It is also referred to as the action of the wave, $a = \omega b^2$. If equation 2.203 is integrated over all space then it says that the time derivative of the number of photons is zero. Therefore, equation 2.203 implies that as the pump loses a photon, it shows up as a photon in the backscattered wave.

2.3.6.2 Conservation of energy

In the case of conservation of energy, we begin the derivation by constructing a single equation which is a linear combination with all 3 waves included,

$$\omega_0 b_0^* \mathcal{L}_0 b_0 + \omega_1 b_1 (\mathcal{L}_1 b_1)^* + \frac{\omega_2}{\Gamma_2} b_2 (\mathcal{L}_2 b_2)^* = 0 \quad (2.208)$$

where we have used the frequency matching condition, $\omega_0 - \omega_1 - \omega_2 = 0$ to eliminate the coupling terms. Adding the complex conjugate and rearranging yields,

$$\begin{aligned} \partial_t \left(\omega_0^2 |b_0|^2 + \omega_1^2 |b_1|^2 + \frac{\omega_2^2}{\Gamma_2} |b_2|^2 \right) + \partial_z \left(\omega_0^2 v_{g0} |b_0|^2 + \omega_1^2 v_{g1} |b_1|^2 + \frac{\omega_2^2 v_{g2}}{\Gamma_2} |b_2|^2 \right) \\ + \nabla_\perp \cdot -\frac{ic^2}{2} \left[\omega_0 (b_0^* \nabla_\perp b_0 - b_0 \nabla_\perp b_0^*) + \omega_1 (b_1^* \nabla_\perp b_1 - b_1 \nabla_\perp b_1^*) \right. \\ \left. + \frac{\omega_2 u_e^2}{\Gamma_2} (b_2^* \nabla_\perp b_2 - b_2 \nabla_\perp b_2^*) \right] = -2\nu \frac{\omega_2^2}{\Gamma_2} |b_2|^2 \quad (2.209) \end{aligned}$$

We identify this as the energy density because $\omega^2 b^2 \propto |\mathbf{E}|^2$. The energy, $\omega_0^2 |b_0|^2 + \omega_1^2 |b_1|^2 + \frac{\omega_2^2}{\Gamma_2} |b_2|^2$, is thus a conserved quantity in the interaction, and the plasma wave damping creates an energy sink proportional to the energy of the plasma wave. As noted in reference [Mic23], for the 1D case where $\nabla_\perp$ is neglected, this equation is not linearly independent from the set of conservation of action equations.

2.3.6.3 Conservation of momentum

Because momentum is a vector, we need conservation laws for each component. For the z component, we take the linear combination,

$$k_0 b_0^* \mathcal{L}_0 b_0 + k_1 b_1 (\mathcal{L}_1 b_1)^* + \frac{k_2}{\Gamma_2} b_2 (\mathcal{L}_2 b_2)^* = 0 \quad (2.210)$$

where we use wave number matching which implies $k_0 - k_1 - k_2 = 0$ to eliminate the coupling terms. Adding the complex conjugate and making some algebraic manipulations leads to

$$\begin{aligned} \partial_t \left(k_0 \omega_0 |b_0|^2 + k_1 \omega_1 |b_1|^2 + \frac{k_2 \omega_2}{\Gamma_2} |b_2|^2 \right) + \partial_z \left(k_0 \omega_0 v_{g0} |b_0|^2 + k_1 \omega_1 v_{g1} |b_1|^2 + \frac{k_2 \omega_2 v_{g2}}{\Gamma_2} |b_2|^2 \right) \\ + \nabla_\perp \cdot -\frac{ic^2}{2} \left[k_0 (b_0^* \nabla_\perp b_0 - b_0 \nabla_\perp b_0^*) + k_1 (b_1^* \nabla_\perp b_1 - b_1 \nabla_\perp b_1^*) \right. \\ \left. + \frac{k_2 u_e^2}{\Gamma_2} (b_2^* \nabla_\perp b_2 - b_2 \nabla_\perp b_2^*) \right] = -2\nu \frac{k_2 \omega_2}{\Gamma_2} |b_2|^2 \quad (2.211) \end{aligned}$$

where, again, the plasma wave damping term leads to a sink.

To examine the transverse momentum, $\mathbf{p}_\perp$, we note that Eq. 2.209 relates the momentum and energy flux (up to a factor of c^2). Therefore, we expect an equation for the transverse momentum to take the following form,

$$\begin{aligned} \mathbf{p}_\perp = -\frac{i}{2} \left[\omega_0 (b_0^* \nabla_\perp b_0 - b_0 \nabla_\perp b_0^*) + \omega_1 (b_1^* \nabla_\perp b_1 - b_1 \nabla_\perp b_1^*) + \right. \\ \left. \frac{\omega_2}{\Gamma_2} (b_2^* \nabla_\perp b_2 - b_2 \nabla_\perp b_2^*) \right] \quad (2.212) \end{aligned}$$

A conservation law for the transverse momentum must include $\partial_t \mathbf{p}_\perp$. We note that the following combination will be included,

$$\begin{aligned} \partial_t \left[-\frac{i\omega_0}{2} (b_0^* \nabla_\perp b_0 - b_0 \nabla_\perp b_0^*) \right] = -\frac{i}{2} [(\nabla_\perp b_0) - b_0 \nabla_\perp] \omega_0 \partial_t b_0^* \\ + \frac{i}{2} [(\nabla_\perp b_0^*) - b_0^* \nabla_\perp] \omega_0 \partial_t b_0 \quad (2.213) \end{aligned}$$

Thus, we construct a linear combination of the envelope equations which, in this case, does not sum to zero, but instead can be written as the total derivative of a coupling term,

$$i [(\nabla_{\perp} b_0) - b_0 \nabla_{\perp}] (\mathcal{L}_0 b_0)^* + i [(\nabla_{\perp} b_1) - b_1 \nabla_{\perp}] (\mathcal{L}_1 b_1)^* - \frac{i}{\Gamma_2} [(\nabla_{\perp} b_2^*) - b_2^* \nabla_{\perp}] \mathcal{L}_2 b_2 = c_0 \nabla_{\perp} (b_0 b_1 b_2^*) \quad (2.214)$$

Adding the complex conjugate to equation 2.214 and regrouping yields a conservation law for the expected form of $\mathbf{p}_{\perp}$. To show this we expand the terms with $\mathcal{L}_0 b_0$ and $(\mathcal{L}_0 b_0)^*$ to obtain

$$i [(\nabla_{\perp} b_0) - b_0 \nabla_{\perp}] (\mathcal{L}_0 b_0)^* - i [(\nabla_{\perp} b_0^*) - b_0^* \nabla_{\perp}] \mathcal{L}_0 b_0 = \partial_t [i\omega_0 (b_0^* \nabla_{\perp} b_0 - b_0 \nabla_{\perp} b_0^*)] + \partial_z [i\omega_0 v_{g0} (b_0^* \nabla_{\perp} b_0 - b_0 \nabla_{\perp} b_0^*)] + \nabla_{\perp} \cdot c^2 \left[\frac{b_0^* \nabla_{\perp}^2 b_0 + b_0 \nabla_{\perp}^2 b_0^*}{2} - \nabla_{\perp} b \nabla_{\perp} b^* \right] \quad (2.215)$$

The same can be done for the $\mathcal{L}_1 b_1$ and $(\mathcal{L}_1 b_1)^*$, and $\mathcal{L}_2 b_2$ and $(\mathcal{L}_2 b_2)^*$ terms. Defining a symmetric rank-2 tensor, $\mathbf{T}$, as

$$\mathbf{T} \equiv c^2 \left[\frac{b_0^* \nabla_{\perp}^2 b_0 + b_0 \nabla_{\perp}^2 b_0^*}{2} \mathbf{I} - \nabla_{\perp} b_0 \nabla_{\perp} b_0^* + \frac{b_1^* \nabla_{\perp}^2 b_1 + b_1 \nabla_{\perp}^2 b_1^*}{2} \mathbf{I} - \nabla_{\perp} b_1 \nabla_{\perp} b_1^* - \frac{u_e^2}{\Gamma_2} \left(\frac{b_2^* \nabla_{\perp}^2 b_2 + b_2 \nabla_{\perp}^2 b_2^*}{2} \mathbf{I} - \nabla_{\perp} b_2 \nabla_{\perp} b_2^* \right) + c_0 (b_0 b_1 b_2^* + b_0^* b_1^* b_2) \mathbf{I} \right] \quad (2.216)$$

and grouping terms from equation 2.215 and the analogous equations for b_1 and b_2 , we arrive at the conservation law for $\mathbf{p}_{\perp}$,

$$\partial_t \left[i\omega_0 (b_0^* \nabla_{\perp} b_0 - b_0 \nabla_{\perp} b_0^*) + i\omega_1 (b_1^* \nabla_{\perp} b_1 - b_1 \nabla_{\perp} b_1^*) + \frac{i\omega_2}{\Gamma_2} (b_2^* \nabla_{\perp} b_2 - b_2 \nabla_{\perp} b_2^*) \right] + \partial_z \left[i\omega_0 v_{g0} (b_0^* \nabla_{\perp} b_0 - b_0 \nabla_{\perp} b_0^*) + i\omega_1 v_{g1} (b_1^* \nabla_{\perp} b_1 - b_1 \nabla_{\perp} b_1^*) + \frac{i\omega_2 v_{g2}}{\Gamma_2} (b_2^* \nabla_{\perp} b_2 - b_2 \nabla_{\perp} b_2^*) \right] + \nabla_{\perp} \cdot \mathbf{T} = -2\nu \frac{i\omega_2}{\Gamma_2} (b_2^* \nabla_{\perp} b_2 - b_2 \nabla_{\perp} b_2^*) \quad (2.217)$$

$\mathbf{T}$ represents the flux of momentum in the transverse direction, and $\mathbf{I}$ is the identity matrix. The symmetry of $\mathbf{T}$ has physical implications. Namely, the plasma has no external rotation and no net torque from internal forces.

2.3.6.4 Conservation of angular momentum

In this dissertation, we are primarily interested in the angular momentum. More specifically, we are most interested in the component of angular momentum in the propagation direction, M_z . The angular momentum in $\hat{\mathbf{z}}$, M_z , follows from the conservation law for $\mathbf{p}_\perp$. We start from the definition of M_z ,

$$M_z = xp_y - yp_x = (\mathbf{x}_\perp \times \mathbf{p}_\perp) \cdot \hat{\mathbf{z}} = rp_\varphi \quad (2.218)$$

where $r = \sqrt{x^2 + y^2}$, and note that

$$(\mathbf{x}_\perp \times \nabla_\perp) \cdot \hat{\mathbf{z}} = \partial_\varphi \quad (2.219)$$

in cylindrical coordinates. We can rewrite equation 2.217 as a conservation law for M_z by taking the cross product with $\mathbf{x}_\perp$ and noting that $\partial_t \mathbf{x}_\perp$ and $\partial_z \mathbf{x}_\perp$ are zero. Substituting equation 2.219 yields

$$\begin{aligned} \partial_t \left[i\omega_0 (b_0^* \partial_\varphi b_0 - b_0 \partial_\varphi b_0^*) + i\omega_1 (b_1^* \partial_\varphi b_1 - b_1 \partial_\varphi b_1^*) + \frac{i\omega_2}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \right] \\ + \partial_z \left[i\omega_0 v_{g0} (b_0^* \partial_\varphi b_0 - b_0 \partial_\varphi b_0^*) + i\omega_1 v_{g1} (b_1^* \partial_\varphi b_1 - b_1 \partial_\varphi b_1^*) + \right. \\ \left. \frac{i\omega_2 v_{g2}}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \right] + \mathbf{x}_\perp \times (\nabla_\perp \cdot \mathbf{T}) = -2\nu \frac{i\omega_2}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \quad (2.220) \end{aligned}$$

Equation 2.220 is not quite a conservation law because the term $\mathbf{x}_\perp \times (\nabla_\perp \cdot \mathbf{T})$ is not yet a total derivative. Examining the components, we find

$$[\mathbf{x}_\perp \times (\nabla_\perp \cdot \mathbf{T})]_i = \epsilon_{ijk} x_j \partial_l T_{lk} = \partial_l (\epsilon_{ijk} x_j T_{lk}) - \epsilon_{ijk} T_{lk} \partial_l (x_j) \quad (2.221)$$

and, upon recognizing that $\partial_l x_j = \delta_{jl}$, we can rewrite the last term as

$$\epsilon_{ijk} T_{lk} \delta_{jl} = \epsilon_{ijk} T_{jk} \quad (2.222)$$

However, $\mathbf{T}$ is a symmetric tensor, $T_{jk} = T_{kj}$, and ϵ_{ijk} is anti-symmetric. The product of symmetric and anti-symmetric tensors vanishes. Therefore, we can rewrite this term as a total derivative,

$$[\mathbf{x}_\perp \times (\nabla_\perp \cdot \mathbf{T})]_i = \epsilon_{ijk} x_j \partial_l T_{lk} = \partial_l (\epsilon_{ijk} x_j T_{lk}) = [\nabla_\perp \cdot (\mathbf{x}_\perp \times \mathbf{T})]_i \quad (2.223)$$

Substituting equation 2.223 into equation 2.220 yields a conservation law for M_z ,

$$\begin{aligned} \partial_t \left[i\omega_0 (b_0^* \partial_\varphi b_0 - b_0 \partial_\varphi b_0^*) + i\omega_1 (b_1^* \partial_\varphi b_1 - b_1 \partial_\varphi b_1^*) + \frac{i\omega_2}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \right] \\ + \partial_z \left[i\omega_0 v_{g0} (b_0^* \partial_\varphi b_0 - b_0 \partial_\varphi b_0^*) + i\omega_1 v_{g1} (b_1^* \partial_\varphi b_1 - b_1 \partial_\varphi b_1^*) + \right. \\ \left. \frac{i\omega_2 v_{g2}}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \right] + \nabla_\perp \cdot (\mathbf{x}_\perp \times \mathbf{T}) = -2\nu \frac{i\omega_2}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \quad (2.224) \end{aligned}$$

Just as for action, energy, and momentum, the plasma wave damping can act as an OAM sink proportional to ν . The conserved angular momentum in the $\hat{\mathbf{z}}$ -direction for the three waves is therefore

$$M_z = i\omega_0 (b_0^* \partial_\varphi b_0 - b_0 \partial_\varphi b_0^*) + i\omega_1 (b_1^* \partial_\varphi b_1 - b_1 \partial_\varphi b_1^*) + \frac{i\omega_2}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \quad (2.225)$$

which we have derived directly from the envelope equations. The angular momentum of a single EM wave with envelope b and frequency ω therefore takes the form

$$M_z = i\omega (b^* \partial_\varphi b - b \partial_\varphi b^*) \quad (2.226)$$

while the angular momentum of the EPW is

$$M_z = \frac{i\omega_2}{\Gamma_2} (b_2^* \partial_\varphi b_2 - b_2 \partial_\varphi b_2^*) \quad (2.227)$$

Note that the expression in equation 2.226 is identical to that in Allen et al., up to normalization factors, where now the waves are propagating in a plasma.

2.3.6.5 Physical interpretation of plasma wave OAM

The expressions for transverse momentum, $\mathbf{p}_\perp$, and the angular momentum, M_z , for the light waves are the same as those from the work of [ABS92]. However, it is worth noting that the expressions derived here are valid in a plasma (medium) where k_i is not simply ω_i . In addition, the analogous expression for the transverse and angular momentum for a plasma wave has not appeared before. It is thus important to attach a physical interpretation for this expression.

To lowest order, a plasma wave is longitudinal (electrostatic) and can thus be described by a scalar potential. Therefore, there is no field momentum associated with it as $\mathbf{B}$ vanishes; the momentum density must therefore come from the fluid motion, $\mathbf{p}_\perp = nm\mathbf{u}_\perp$. As for the EM waves, we are interested in the time average of this quantity.

We assume that ϕ , n , and $\mathbf{u}_\perp$ are the complex amplitudes of the envelopes which all have the same frequency and wavenumber. We can write Gauss' Law as

$$-\nabla^2\phi = (k_\perp^2 - \nabla_\perp^2)\phi = -4\pi en \quad (2.228)$$

where we have assumed $\partial_z\phi$ is negligible. Using a Taylor expansion in $\nabla_\perp^2$ to approximately invert equation 2.228, we obtain

$$\phi = -\frac{4\pi e}{k_\perp^2} \frac{n}{(1 - \frac{\nabla_\perp^2}{k_\perp^2})} = -\frac{4\pi e}{k_\perp^2} \left[1 + \frac{1}{k_\perp^2} \nabla_\perp^2 + O\left(\frac{\nabla_\perp^4}{k_\perp^4}\right) \right] n \quad (2.229)$$

Next, we rearrange the momentum equation (Euler's Law) assuming $\partial_t\mathbf{u}_\perp = -i\omega_2\mathbf{u}_\perp$

$$\mathbf{u}_\perp = \frac{ie}{m\omega_2} \nabla_\perp \phi - 3i \frac{v_e^2}{\omega_2 n_0} \nabla_\perp n \quad (2.230)$$

Upon substituting equation 2.229, equation 2.230 can be rewritten as

$$\begin{aligned} \mathbf{u}_\perp &\approx -\frac{i}{\omega_2} \nabla_\perp \left[\frac{4\pi e^2}{mk_\perp^2} \left(1 + \frac{1}{k_\perp^2} \nabla_\perp^2 \right) + 3 \frac{v_e^2}{n_0} \right] n \\ &= -\frac{i\omega_p^2}{\omega_2 k_\perp^2} \nabla_\perp \left(1 + 3k_\perp^2 \lambda_D^2 + \frac{1}{k_\perp^2} \nabla_\perp^2 \right) \frac{n}{n_0} \end{aligned} \quad (2.231)$$

where we have used the definition of the Debye length

$$\lambda_D = \frac{v_e}{\omega_p} \quad (2.232)$$

From the fluid dispersion relation for an EPW, we can substitute ω_2^2/ω_p^2 for $1 + 3k_\perp^2 \lambda_D^2$ in equation 2.231. Therefore, replacing n/n_0 with b_2 , we can write equation 2.231 as

$$\mathbf{u}_\perp = -\frac{i\omega_p^2}{\omega_2 k_\perp^2} \nabla_\perp \left[\frac{\omega_2^2}{\omega_p^2} + \frac{1}{k_\perp^2} \nabla_\perp^2 \right] b_2 \approx -i \frac{\omega_2}{k_\perp^2} \nabla_\perp b_2 \quad (2.233)$$

where we can obtain higher order terms through the Taylor expansion approach. Thus, the time average of $\mathbf{p}_\perp$ can be expressed in terms of b_2 ,

$$\left\langle \frac{\mathbf{p}_\perp}{n_0 m c} \right\rangle = \frac{1}{4 n_0 c} (n^* \mathbf{u}_\perp + n \mathbf{u}_\perp^*) = \frac{1}{4} \left(b_2^* \frac{\mathbf{u}_\perp}{c} + b_2 \frac{\mathbf{u}_\perp^*}{c} \right) = \frac{i \omega_2}{4 c k_2^2} (-b_2^* \nabla_\perp b_2 + b_2 \nabla_\perp b_2^*) \quad (2.234)$$

Recalling that $\Gamma_2 = \frac{c^2 k_2^2}{\omega_p^2}$, equation 2.234 becomes

$$\left\langle \frac{\mathbf{p}_\perp}{n_0 m c} \right\rangle = -\frac{i}{4 \Gamma_2} \frac{\omega_2}{\omega_p} \left(b_2^* \frac{c}{\omega_p} \nabla_\perp b_2 - b_2 \frac{c}{\omega_p} \nabla_\perp b_2^* \right) \quad (2.235)$$

which is the same expression as that in equation 2.212 up to normalization constants. Thus, we have shown that the expression for $\mathbf{p}_\perp$ obtained from the conservation laws which were based on the envelope equations can also be given a physical interpretation obtained by a Taylor approximation approach.

2.3.6.6 OAM index of an arbitrary wave

In this subsection, we consider under what circumstances the conservation of M_z can be described as an ℓ matching condition, in analogy with how the conservation of energy and momentum are described by ω and k matching conditions. Recalling the connection to the derivation of the OAM per photon of a pure LG mode in vacuum as $\ell \hbar$, we can use the conservation law to define an effective OAM index for any envelope. The effective OAM index for wave j is the ratio of the angular momentum density integrated over $d\mathbf{x}_\perp$ to the wave action integrated over $d\mathbf{x}_\perp$,

$$\bar{\ell}_j = \frac{-\frac{i \omega_j}{2} \int (b_j^* \partial_\varphi b_j - b_j \partial_\varphi b_j^*) d\mathbf{x}_\perp}{\omega_j \int |b_j|^2 d\mathbf{x}_\perp}. \quad (2.236)$$

In general, we can expand an arbitrary envelope, b , as a superposition of OAM states each with integer ℓ ,

$$b(\mathbf{x}_\perp) = \sum_{\ell=-\infty}^{\infty} b_\ell(r) e^{i \ell \varphi} \quad (2.237)$$

The effective OAM index, $\bar{\ell}$, (equation 2.236) is the weighted average,

$$\bar{\ell} = \frac{\sum_\ell \ell |b_\ell|^2}{\sum_{\ell'} |b_{\ell'}|^2} \quad (2.238)$$

For a pure Laguerre-Gaussian envelope in vacuum with envelope $e^{i\ell_0\varphi}$, the index ℓ_0 is recovered. Note that the action of the wave with frequency ω_j is defined as $a_j \equiv \omega_j \sum_{\ell'} |b_{\ell'}|^2$ so the OAM of the wave is $\bar{\ell}_j a_j$.

In the literature, the concept of ℓ matching has been invoked with respect to SRS [cTT09]. This can be easily argued if each wave is a pure OAM Laguerre-Gaussian mode. Consider $b_0 \propto e^{i\ell_0\varphi}$, $b_1 \propto e^{i\ell_1\varphi}$, and $b_2 \propto e^{i\ell_2\varphi}$. In order for each wave to be resonantly driven, the RHS of each wave equation (equations. 2.147 to 2.149) must have the same ℓ as that on the LHS, which leads directly to $\ell_0 = \ell_1 + \ell_2$.

However, this implies that the ℓ_j number of each wave cannot change. The same assumption is true for ω and k matching, where it is implicitly assumed that the ω_j and k_j of each wave does not change. However, these quantities are not part of the envelope, while OAM is an intrinsic property of the envelope itself. Thus, the OAM selection rule or ℓ matching is more subtle than for ω and k .

To clarify the above discussion, it is illustrative to see how ω matching follows from the conservation of energy equation. The same arguments apply to k matching and the conservation of momentum equation. Consider the conservation of energy equation after integrating over $d\mathbf{x}_\perp$,

$$\partial_t (\omega_0 \bar{a}_0 + \omega_1 \bar{a}_1 + \omega_2 \bar{a}_2) + \partial_z (v_{g0} \omega_0 \bar{a}_0 + v_{g1} \omega_1 \bar{a}_1 + v_{g2} \omega_2 \bar{a}_2) = 0 \quad (2.239)$$

where $\bar{a} \equiv \int d\mathbf{x}_\perp a$ and a_2 is defined as $\frac{1}{\Gamma_2} |b_2|^2$. This equation can be expanded as,

$$\begin{aligned} \bar{a}_0 (\partial_t + v_{g0} \partial_z) \omega_0 + \bar{a}_1 (\partial_t + v_{g1} \partial_z) \omega_1 + \bar{a}_2 (\partial_t + v_{g2} \partial_z) \omega_2 \\ + \omega_0 (\partial_t \bar{a}_0 + \partial_z v_{g0} \bar{a}_0) + \omega_1 (\partial_t \bar{a}_1 + \partial_z v_{g1} \bar{a}_1) + \omega_2 (\partial_t \bar{a}_2 + \partial_z v_{g2} \bar{a}_2) = 0 \end{aligned} \quad (2.240)$$

which can then be rewritten as

$$\begin{aligned} \bar{a}_0 (\partial_t + v_{g0} \partial_z) \omega_0 + \bar{a}_1 (\partial_t + v_{g1} \partial_z) \omega_1 + \bar{a}_2 (\partial_t + v_{g2} \partial_z) \omega_2 \\ + (\omega_0 - \omega_1 - \omega_2) (\partial_t \bar{a}_0 + \partial_z v_{g0} \bar{a}_0) = 0 \end{aligned} \quad (2.241)$$

if the conservation of action equations (equations 2.203 and 2.206) are used. Equation 2.241 is satisfied if $\omega_0 - \omega_1 - \omega_2 = 0$, and each frequency is constant. Thus, ω matching follows from energy conservation only if each ω_j does not change. Analogous conclusions can be made for k matching by combining the momentum and conservation of action equations.

Following the same arguments, we can combine conservation of OAM with the conservation of action equations,

$$\begin{aligned} \bar{a}_0(\partial_t + v_{g0}\partial_z)\bar{\ell}_0 + \bar{a}_1(\partial_t + v_{g1}\partial_z)\bar{\ell}_1 + \bar{a}_2(\partial_t + v_{g2}\partial_z)\bar{\ell}_2 \\ + (\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2)(\partial_t\bar{a}_0 + \partial_z v_{g0}\bar{a}_0) = 0 \end{aligned} \quad (2.242)$$

which makes clear that $\bar{\ell}$ matching, $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2 = 0$ is only true if each $\bar{\ell}$ remains constant. This is generally not true, so $\bar{\ell}$ matching is not an absolute concept in SRS while the conservation equation for M_z is absolute.

In the next chapter, we explore OAM resonance and under what conditions ℓ matching is satisfied through simulations of SRS growth from noise and with various combinations of pump and seed OAM. We analyze the evolution and exchange of OAM with equation 2.236 and study conservation of angular momentum in pF3D and PIC simulations.

2.4 Summary

In this chapter, we presented a theoretical framework for the study of stimulated Raman scattering with orbital angular momentum. First, we reviewed the derivation for the orbital angular momentum of light and the modes whose envelopes can be described by Laguerre-Gaussian modes which are expansions in azimuthal mode numbers, $e^{i\ell\varphi}$. The OAM per photon in a single mode is $\ell\hbar$. Additionally, we reviewed the derivation of the contribution to angular momentum from circular polarization. These reviews were done with significant detail. Next, we discussed the motion of a single particle in an EM field and reviewed the derivation of the ponderomotive force. We expanded the single particle picture to a plasma

by deriving the fluid plasma equations from the Vlasov equation. From the continuity and momentum equation we obtained two coupled differential equations describing both the EM and plasma waves in SRS including the lowest order effects of finite-width lasers. We then split the light wave into a large pump wave and a smaller seed or backscattered wave and assumed all waves could be described by a slowly varying envelope. This led to the full set of coupled envelope equations.

By examining the temporal and spatial limits of the envelope equations, we defined growth rates for strong and weak EPW damping. These growth rates will be used in the next chapter. We briefly discussed the relationship to the growth rate of convective and absolute instabilities; and discussed the reflectivity in the strongly damped regime. Next, we discussed inflation and the behavior of a single particle trapped in a plasma wave which can also affect SRS growth. Finally, we derived conservation laws for action, energy, momentum, and angular momentum from the envelope equations which led to a physical interpretation of the OAM of a plasma wave. To our knowledge, these conservation laws for SRS backscatter within the validity of the paraxial wave equation have heretofore not appeared in the literature. We used the conservation law for angular momentum to show under what conditions there is an effective ℓ matching condition.

This chapter outlined the necessary theory and equations to compare simulations of stimulated Raman scattering with and without orbital angular momentum. The next chapter (Chapter 3) discusses fluid simulations using pF3D and compares the gain of a seed to the growth rates derived here. The following chapter (Chapter 4) covers particle-in-cell simulations using OSIRIS and the changes to SRS which arise when single particle or kinetic behavior is considered.

CHAPTER 3

Fluid simulations of Stimulated Raman Scattering

3.1 Introduction

As described in the previous chapters, this dissertation is focused on studying Stimulated Raman Scattering (SRS) of finite-width lasers (a single speckle), with an emphasis on studying how SRS differs when the incident speckle does or does not have orbital angular momentum (OAM). In SRS, a pump laser incident on an underdense plasma can spontaneously decay into a backscattered light wave and a plasma wave. SRS can also involve the amplification of a well-defined backscattered seed from a pump. In Chapter 2, key physics concepts involved in multidimensional SRS were reviewed, and new results regarding conservation laws (for action, energy, linear momentum, and OAM) were presented that are appropriate within the fluid and paraxial ray approximations. The role of kinetic effects in plasma waves with and without OAM was also discussed.

To understand how the presence of OAM affects SRS, it is thus important to isolate the roles of fluid effects and kinetic effects. Therefore, in general, although the physics of SRS can depend on wave-particle interactions, we can still learn about fundamental trends and important behavior by using a fluid code. In this chapter, we present results from fluid simulations of SRS of speckles with and without OAM. The results will be compared against theory and will be interpreted using concepts from Chapter 2. The results will also be used as a starting point for Chapter 4 where results from fully kinetic simulations will be described.

The fluid code to be used in this chapter is pF3D [BSW98, BDG01, LSA16, WHL12]

which is a general purpose code that can model SRS as well as Stimulated Brillouin Scattering (SBS), self-focusing, and filamentation in plasmas including plasma flows and heat flux. It can simulate non-uniform plasmas and model multi-speckle lasers generated from phase plates, including smoothing from spectral dispersion (SSD). pF3D is based on the envelope approximations and the paraxial ray approximation that were used to generate the three coupled wave equations in Chapter 2. Landau damping for the EPW and collisional damping of the EM waves can also be included. For the purposes of this dissertation, the plasmas are uniform, plasma flow and ion dynamics are disabled, and damping of only the EPW from Landau damping is considered. For SRS the only nonlinearity is pump depletion. For the purposes of this work, the antenna in pF3D was modified to launch a single speckle with OAM.

In this chapter, we first examine SRS growth from noise, in contrast to Raman amplification of a well-defined seed pulse, which is explored subsequently. Small background fluctuations can be amplified spontaneously, and a spectrum of modes within some resonant region can grow into backscatter, even in the absence of a well-defined seed. We study how the presence of OAM in the pump laser affects the reflectivity and compare to a Gaussian laser with no OAM based on various comparison metrics. We also explore the effect of Landau damping by changing the temperature to scan the behavior of SRS between the strongly-damped and weakly-damped regimes. We compare the gain of a seed for different OAM combinations of the pump and seed. Because each simulation is not computationally intensive, we are able to generate a large amount of data to study different laser and plasma configurations.

3.2 Overview of pF3D

In this section, we provide a brief overview of the relevant equations in pF3D that are used to simulate SRS. The code solves three envelope equations based on the fluid dispersion

relations which we derived in the previous chapter (equations 2.147, 2.148, and 2.149) to advance the three waves (pump, backscattered wave, and forward propagating EPW). For convenience, we repeat equations 2.133, 2.134, and 2.135, which show how the complex envelopes, b_0 , b_1 , and b_2 , are defined,

$$\frac{e\mathbf{A}_0}{mc^2} = \hat{\mathbf{e}}_0 b_0 e^{ik_0 z - i\omega_0 t} + \text{c.c.} \quad (3.1)$$

$$\frac{e\mathbf{A}_1}{mc^2} = \hat{\mathbf{e}}_0 b_1 e^{ik_1 z - i\omega_1 t} + \text{c.c.} \quad (3.2)$$

$$\frac{n_1}{n_0} = b_2 e^{ik_2 z - i\omega_2 t} + \text{c.c.} \quad (3.3)$$

where $\mathbf{A}_0$ and $\mathbf{A}_1$ are the vector potentials of the incident and backscattered light, and n_1 is the plasma density perturbation. We simulate linearly polarized light ($\hat{\mathbf{e}}_0 = \hat{\mathbf{x}}$) which precludes spin angular momentum (the equations can be modified to include circularly polarized light). As shown in Chapter 2 the envelope equations for SRS in a uniform plasma are

$$\omega_0 \left(\partial_t + v_{g0} \partial_z - i \frac{c^2}{2\omega_0} \nabla_\perp^2 \right) b_0 = -ic_0 b_1 b_2 \quad (3.4)$$

$$\omega_1 \left(\partial_t + v_{g1} \partial_z - i \frac{c^2}{2\omega_1} \nabla_\perp^2 \right) b_1 = -ic_1 b_0 b_2^* \quad (3.5)$$

$$\omega_2 \left(\partial_t + \nu + v_{g2} \partial_z - i \frac{u_e^2 c^2}{2\omega_2} \nabla_\perp^2 \right) b_2 = -ic_2 b_0 b_1^* \quad (3.6)$$

where the ω 's and k 's satisfy the fluid dispersion relations,

$$\omega_0^2 = \omega_p^2 + c^2 k_0^2 \quad (3.7)$$

$$\omega_1^2 = \omega_p^2 + c^2 k_1^2 \quad (3.8)$$

$$\omega_2^2 = \omega_p^2 + 3k_2^2 v_e^2 \quad (3.9)$$

as well as the three-wave matching conditions,

$$\omega_0 = \omega_1 + \omega_2 \quad (3.10)$$

$$k_0 = k_1 + k_2. \quad (3.11)$$

As noted previously, for this convention $\omega_1 > 0$ and $k_1 < 0$. The OAM number for each wave, $\bar{\ell}_0$, $\bar{\ell}_1$, and $\bar{\ell}_2$, can be calculated from the envelopes using equation 2.236,

$$\bar{\ell}_j = \frac{-\frac{i\omega_j}{2} \int (b_j^* \partial_\varphi b_j - b_j \partial_\varphi b_j^*) d\mathbf{x}_\perp}{\omega_j \int |b_j|^2 d\mathbf{x}_\perp}. \quad (3.12)$$

which we also use to confirm our implementation of pure OAM (Laguerre-Gaussian) laser modes created by the antenna.

In pF3D, SRS can grow from noise or from a well-defined seed where a small amplitude Laguerre-Gaussian mode is launched from the right boundary (the pump is launched from the left boundary). The initial values for b_2 are determined based on assuming a thermal plasma and the initial values for b_1 come from the interaction of the pump and the plasma.

The noise source is random so there should be negligible azimuthal asymmetry (or OAM). To confirm this we measured the angular momentum in both the background plasma and electromagnetic noise by calculating $\bar{\ell}_2$ and $\bar{\ell}_1$ from equation 2.236 from the wave envelopes at the start of the simulation, and found them to be zero. Therefore, the initial state of the plasma has no OAM.

The plasma density is $0.14n_c$ where n_c is the critical density which is defined as

$$n_c \equiv \frac{m\omega_0^2}{4\pi e^2} = \frac{\omega_0^2}{\omega_p^2} n_0. \quad (3.13)$$

The density is uniform except for a ramp at the entrance and exit planes of the pump laser. The laser is focused at the center of the simulation box and has a wavelength, λ_0 , which corresponds to $0.3512\mu\text{m}$. The size of the simulation region is $48\lambda_0 \times 48\lambda_0 \times 200\lambda_0$. The cell size is $0.5\lambda_0$ in the transverse directions and $4\lambda_0$ in the propagation (z) direction. When the beam waist is $w_0 = 4\lambda_0$, the box length is about $4z_R$ where $z_R \equiv \frac{\pi}{\lambda_0} w_0^2 = 16\pi\lambda_0$. Recall that this expression for z_R holds for all Laguerre-Gaussian modes with the same w_0 in the exponential. We also performed simulations with $w_0 = 8\lambda_0$, and for these cases the box length was either z_R (same length in laser wavelengths as before) or $4z_R$. We explored a range of cell sizes and found that the results were well converged for the chosen resolution.

3.2.0.1 Obtaining the normal EPW modes

In the envelope equations $\nu = \nu_2$ comes from Landau damping. Collisional damping for b_1 and b_2 is negligible for the parameters considered. For plasma conditions of interest, determining the real and imaginary parts for ω_2 requires solving the fully kinetic dispersion relation where the wavenumbers, frequencies, and EPW damping rate depend on the plasma density and temperature. We use an analytical solver to find all parameters for various plasma densities and electron temperatures. We assume the plasma is in local equilibrium so the electron distribution function is a Maxwellian. The necessary equations start with the kinetic dispersion relation for an EPW,

$$1 - \frac{1}{2k_2^2\lambda_D^2}Z'\left(\frac{\omega_2}{\sqrt{2}k_2v_e}\right) = 0 \quad (3.14)$$

where Z' is the derivative of the plasma dispersion function [FC61],

$$Z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - \zeta} dt \quad (3.15)$$

and the Debye length, λ_D , is the ratio of electron thermal velocity to the plasma frequency,

$$\lambda_D \equiv \frac{v_e}{\omega_p} = \sqrt{\frac{k_B T_e}{4\pi n_0 e^2}} \quad (3.16)$$

where k_B is the Boltzmann constant. The Debye length is a fundamental length scale in a plasma which characterizes the collective shielding behavior for electric fields. The two matching conditions for the real parts of ω and k (equations 3.10 and 3.11), and the three dispersion relations form a set of equations for the frequencies and wavenumbers of the three waves. We assume a fluid dispersion relation for each of the undamped EM waves,

$$\omega_i^2 = \omega_p^2 + c^2 k_i^2 \quad (3.17)$$

where i is 0 for the pump and 1 for the backscatter or seed, and the wavenumbers and frequencies are real. For frequency and wave number matching, we allow complex ω_2 , where the imaginary part determines the EPW damping rate, ν . The kinetic solver calculates frequencies and wavenumbers given n_0/n_c and v_e/c .

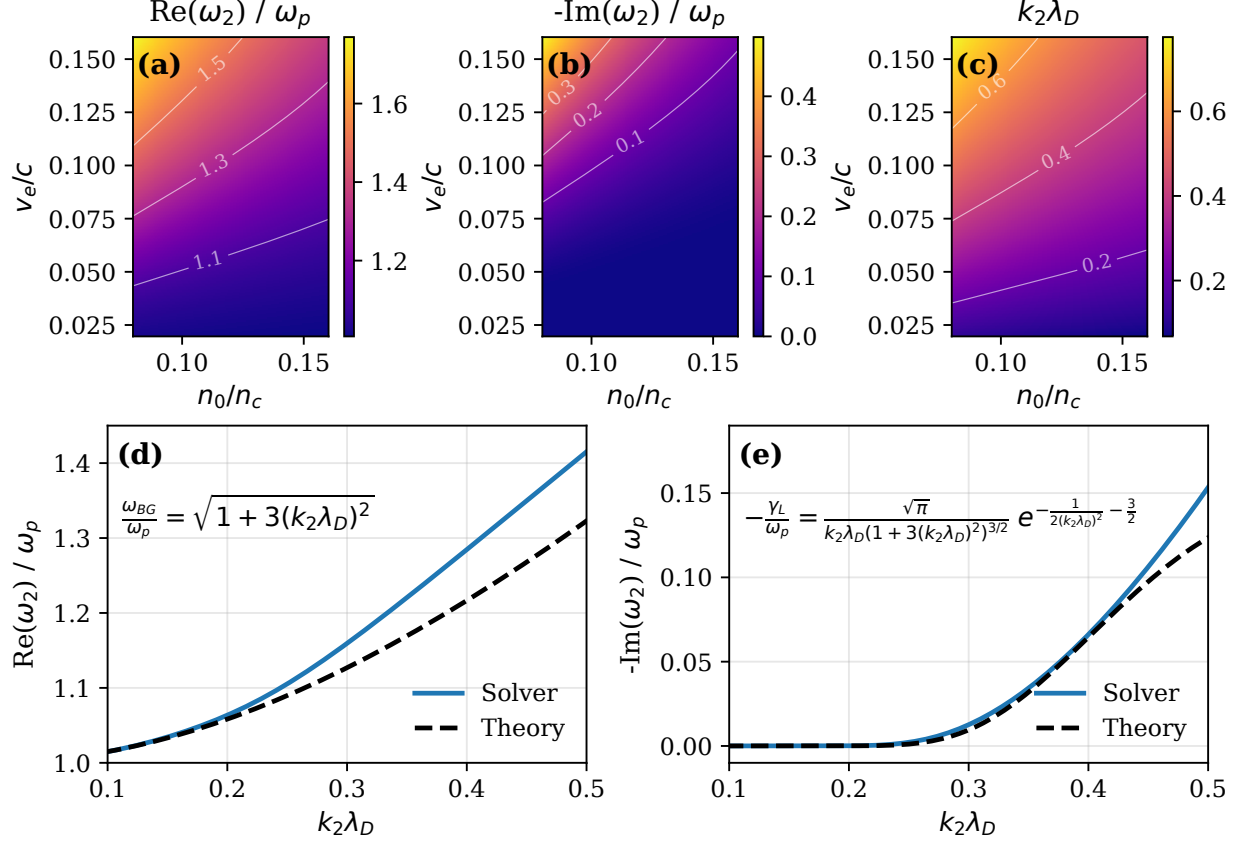


Figure 3.1: (a-c) Real and imaginary parts of the EPW frequency normalized to ω_p and EPW wavenumber normalized to λ_D^{-1} as a function of density and temperature computed from a non-relativistic kinetic solver. We are interested in $-\text{Im}(\omega_2)$ because strong damping occurs for larger negative values. (d) Real part of ω_2 as a function of k_2 compared to the Bohm-Gross dispersion relation $\frac{\omega_{BG}}{\omega_p} = \sqrt{1 + 3(k_2 \lambda_D)^2}$. (e) The negative imaginary part of ω_2 as a function of k_2 compared to the asymptotic Landau damping rate $-\frac{\gamma_L}{\omega_p} = \frac{\sqrt{\pi}}{k_2 \lambda_D (1 + 3(k_2 \lambda_D)^2)^{3/2}} e^{-\frac{1}{2(k_2 \lambda_D)^2} - \frac{3}{2}}$. The curves in (d-e) correspond to a density of $0.14n_c$ and a range of temperatures.

In figure 3.1 we show the EPW parameters for a range of densities and temperatures. In (a-b), we plot the normalized real and imaginary parts of ω_2 for densities from $0.08n_c$ to $0.16n_c$ and temperatures from 1.27 keV to 13.1 keV (v_e from $0.05c$ to $0.16c$). At the highest temperatures relativistic corrections can be important, but we have not included them here. The damping rate, ν , is $-\text{Im}(\omega_2)$. For low density and high temperature, $\text{Re}(\omega_2)/\omega_p$ is 1.7. Conversely, for high density and low temperature, $\text{Re}(\omega_2)/\omega_p$ is near 1. In panel (b), $-\text{Im}(\omega_2)/\omega_p$ ranges from 0 to 0.5. Panel (c) shows the normalized wavenumber, $k_2\lambda_D$ which varies from 0.1 to 0.7 for the parameters scanned. The white lines in (a-c) show contours of constant values. In panel (d), we plot the real part of the frequency vs $k_2\lambda_D$ and compare to the Bohm-Gross dispersion relation for a Langmuir electron plasma wave,

$$\frac{\omega_{BG}}{\omega_p} = \sqrt{1 + 3(k_2\lambda_D)^2}. \quad (3.18)$$

Additionally, in (e), we compare the solver results for $-\text{Im}(\omega_2)$ vs $k_2\lambda_D$ to the analytical and asymptotic Landau damping rate formula [Lan46, Jac99],

$$-\frac{\gamma_L}{\omega_p} = \frac{\sqrt{\pi}}{k_2\lambda_D(1 + 3(k_2\lambda_D)^2)^{3/2}} e^{-\frac{1}{2(k_2\lambda_D)^2} - \frac{3}{2}}. \quad (3.19)$$

For panels (d-e), the density is constant at $0.14n_c$ and the temperature varies such that $k_2\lambda_D$ scans from 0.1 to 0.5. As expected, the Bohm-Gross theory aligns with the solver results for low $k_2\lambda_D$ (below 0.2) and diverges as $k_2\lambda_D$ increases. The Landau formula matches the solver best for $k_2\lambda_D$ from about 0.35 to 0.4 but matches well for all values below 0.4. For this research, we focus on the strongly damped regime where $k_2\lambda_D$ is greater than 0.3.

These parameters relate to those in plasmas where SRS occurs on NIF in gas-filled hohlraums. However, we are only studying the basic physics of a single speckle in a uniform plasma and not the behavior of multi-speckle NIF beams under complex plasma and laser conditions.

Figure 3.2 shows curves in $k_2\lambda_D$ vs. intensity space in order to examine various damping regimes as described in Chapter 2. It is assumed that $n_0 = 0.14n_c$. In (a), intensity is

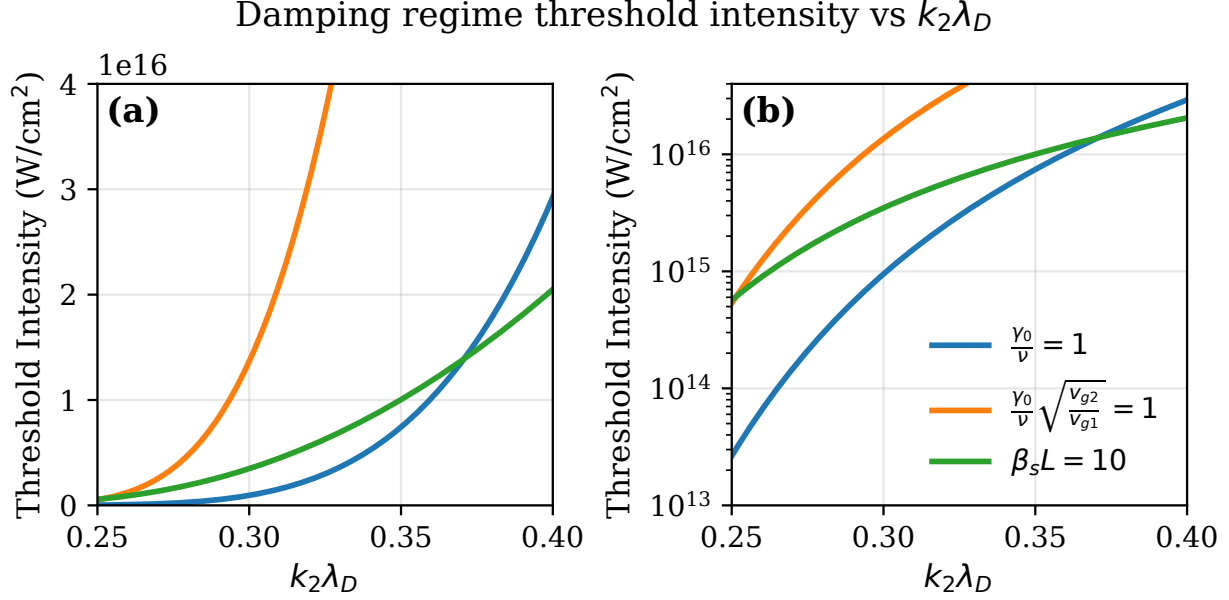


Figure 3.2: Three conditions for a transition from the strongly to weakly damped regime shown on linear (a) and logarithmic (b) scales. Each line represents the intensity where the transition condition is satisfied as a function of $k_2\lambda_D$. Both $k_2\lambda_D$ and the damping rate, ν , depend on the plasma density and temperature. The blue line shows the intensity for which the temporal growth rate is greater than the damping rate (the strongly-damped temporal transition) as described in Chapter 2. The highest transition intensity (orange) corresponds to the transition from the weakly-damped to the strongly-damped spatial regime. The green line shows the intensity where the spatial gain exponent of the field (power) exceeds 10 (20) so that the backscatter grows by a factor of e^{10} . The gain exponent is $\beta_s L$ where L is the simulation box length which is $200\lambda_0$. The space between the blue and orange lines represents parameters where SRS is simultaneously in the weakly-damped temporal regime and the strongly-damped spatial regime.

on a linear scale, while in (b) it is on a logarithmic scale. In both plots three curves are shown, which correspond to three regime transitions or boundaries. The temporal behavior transitions at the boundary,

$$\frac{\gamma_0}{\nu} = 1 \quad (3.20)$$

where γ_0 is the temporal growth rate and ν is the EPW damping rate which comes from the kinetic solver. This corresponds to the transition from the weakly damped ($\gamma_0/\nu \gg 1$) to the strongly damped ($\gamma_0/\nu \ll 1$) temporal regimes. The orange curve shows the transition boundary from the weakly damped to strongly damped spatial regimes,

$$\frac{\gamma_0}{\nu} \sqrt{\frac{v_{g2}}{|v_{g1}|}} = 1 \quad (3.21)$$

where v_{g2} is the group velocity of the EPW and v_{g1} is the group velocity of the backscattered light. The green curve corresponds to the parameters where the strongly damped spatial gain exponent exceeds 10 so that the noise source for the field amplitude grows by 10 e-foldings, e^{10} amplification,

$$\beta_s L = 10 \quad (3.22)$$

where the box length L is $200\lambda_0$ which is about $4z_R$ when the beam waist w_0 is $4\lambda_0$. We often show plots of the reflectivity of the intensity rather than amplitude so this boundary corresponds to 20 e-foldings of growth of that quantity. We will indicate the spatial regime threshold in the reflectivity data plots presented later in this chapter, in order to compare pF3D simulations to 1D theory, as well as plots of the predicted reflectivity from equation 2.187 in Chapter 2.

In the simulations presented here, the simulation box length is significantly larger than the Rayleigh length, z_R , and therefore the intensity of the Gaussian beam is 20% of the peak value at the entrance and exit of the plasma. For lasers with finite-width, the peak value for a_0 is used to examine the regimes of growth and thus the peak intensity is used to calculate β_s in equation 2.168. If $\int \beta_s dz$ is used instead of $\beta_s L$, the estimated spatial gain exponent decreases by a factor of two.

As mentioned in Chapter 2, one can simultaneously be in the strongly damped spatial regime but in the weakly damped temporal regime. That is demonstrated in figure 3.2 where the temporal threshold (blue) is far below the spatial threshold (orange) for high $k_2\lambda_D$ and high intensity. For most cases of interest ($k_2\lambda_D \approx 0.34$), SRS is in the strongly damped spatial regime.

3.2.0.2 OAM laser shape

Beyond understanding the effect of damping, we are interested in how SRS proceeds in the presence of OAM. In order to properly examine the effect of OAM, we first compare the transverse shapes of lasers with $\ell = 0, 1, 2, 3$. The intensity profiles are radially symmetric even when the envelope includes $e^{i\ell\varphi}$. When $\ell \neq 0$, there is zero intensity on axis, and the profile forms a ring (or multiple rings when $p \neq 0$ which is not considered here). For the rest of this chapter, we assume ℓ is always positive, so that we can neglect the absolute value whenever we give relevant formulae. We compare simulations across ℓ by matching two out of the three parameters that characterize the laser envelopes, peak intensity, power, and w_0 . The relationship between these three is explained in detail below.

The intensity of light with OAM is not in general peaked on the $r = 0$ axis. Specifically, I_ℓ , goes as $|u_{0\ell}|^2$ which is $r^{2\ell}|u_0|^2$ where u_0 is equal to $u_{p\ell}$ in equation 2.8 with p and ℓ equal to 0, i.e., $u_0 = u_{00}$ [ABS92]. The peak intensity and power of the beam depend on ℓ . In what follows we only consider the $p = 0$ situation.

We calculate the location of the peak of the intensity profile by taking the derivative,

$$\partial_r I_\ell = \partial_r |u_{0\ell}|^2 = \partial_r \left[C^2 \left(\frac{\sqrt{2}r}{w_0} \right)^{2\ell} e^{-\frac{2r^2}{w_0^2}} \right] = \left(\frac{2\ell}{r} - \frac{4r}{w_0^2} \right) |u_{0\ell}|^2 \quad (3.23)$$

which is zero at a radius $w_0\sqrt{\ell/2}$. The peak value of the intensity is thus

$$I_{peak} = I_\ell \left(r = w_0\sqrt{\frac{\ell}{2}} \right) = C^2 \ell^\ell e^{-\ell} \quad (3.24)$$

Recall that for any OAM, this definition of w_0 defines a diffraction length $z_R = \pi w_0^2/\lambda$,

$$u_{p\ell} \propto \frac{1}{\sqrt{1 + \frac{z^2}{z_R^2}}}. \quad (3.25)$$

As ℓ increases, the intensity is distributed at larger r for a fixed w_0 . Additionally, the power of a laser is the integral of the intensity over the transverse plane,

$$P = \int d\mathbf{x}_\perp I = \int_0^{2\pi} d\varphi \int_0^\infty dr r I \quad (3.26)$$

For the envelope $u_{0\ell}$, the power is

$$P = \int d\mathbf{x}_\perp I_\ell = 2\pi \int_0^\infty dr |u_{0\ell}|^2 = 2\pi C^2 \left(\frac{\sqrt{2}}{w_0}\right)^{2\ell} \int_0^\infty dr r^{2\ell+1} e^{-\frac{2r^2}{w_0^2}} \quad (3.27)$$

Using a variable substitution to simplify the integral leaves

$$P = \pi C^2 \frac{w_0^2}{2} \int_0^\infty dx x^\ell e^{-x} = \frac{\pi C^2 w_0^2}{2} \Gamma(\ell + 1) = \frac{\pi C^2 w_0^2}{2} \ell! \quad (3.28)$$

where Γ is the Gamma function and, because ℓ is a positive integer, $\Gamma(\ell + 1)$ can be replaced with ℓ factorial. Therefore, the ratio of power to peak intensity is

$$\frac{P}{I_{peak}} = \frac{\pi w_0^2}{2} \frac{\ell!}{\ell^\ell} e^\ell \quad (3.29)$$

For a Gaussian laser ($\ell = 0$), the ratio of power to peak intensity is $\pi w_0^2/2$. We can thus define an effective width for a laser with OAM,

$$w_\ell \equiv w_0 \sqrt{\frac{\ell! e^\ell}{\ell^\ell}} \quad (3.30)$$

such that the ratio of power to peak intensity may be written in the same form as for a Gaussian beam,

$$\frac{P}{I_{peak}} = \frac{\pi w_\ell^2}{2} \quad (3.31)$$

The parameter w_ℓ is therefore the width of an equivalent Gaussian beam with the same total power and peak intensity as the LG mode with OAM ℓ . We note that w_ℓ is neither a geometric beam size like the radius of peak intensity, $r_0 = w_0 \sqrt{\ell/2}$, the root-mean-square

OAM index, ℓ	Effective beam width, $\frac{w_\ell}{w_0} = \sqrt{\frac{\ell!e^\ell}{\ell^\ell}}$	Approximate numerical value
0	1	1
1	$\sqrt{e}$	1.65
2	$\frac{e}{\sqrt{2}}$	1.92
3	$\frac{\sqrt{2}}{3}e^{3/2}$	2.11
4	$\frac{\sqrt{6}}{8}e^2$	2.26
5	$\frac{2\sqrt{6}}{25}e^{5/2}$	2.39

Table 3.1: The relationship between power and peak intensity depends on ℓ for lasers with OAM. For a Gaussian laser ($\ell = 0$), the ratio of the power to peak intensity is $\frac{\pi}{2}w_0^2$, where w_0 is the spot size. For nonzero ℓ , we define an effective width w_ℓ so that $\frac{P}{I_{peak}} = \frac{\pi}{2}w_\ell^2$. The table shows how w_ℓ/w_0 varies with ℓ for $\ell = 0, 1, 2, 3, 4, 5$.

width of the intensity distribution, $\sqrt{\langle r^2 \rangle} = w_0\sqrt{(\ell+1)/2}$, nor a spot size, w_0 , that defines a diffraction length $z_R = \pi w_0^2/\lambda$. Rather, the effective width provides a convenient measure of how the transverse distribution of intensity associated with OAM reduces the peak intensity for a fixed power. As ℓ increases, w_ℓ increases, representing the decreasing concentration of power near the radius of maximum intensity. Table 3.1 shows the numerical values of w_ℓ/w_0 when $\ell = 0, 1, 2, 3, 4, 5$.

In practice, we only define ℓ , I_{peak} , and w_0 for a laser in pF3D simulations. It is not possible to compare lasers with different OAM that have the same power, peak intensity, and w_0 simultaneously. Instead, we adjust the values of I_{peak} and w_0 depending on ℓ to match two out of three (power, peak intensity, and w_0).

Figure 3.3 shows the intensity profiles for the three approaches for matching lasers with OAM used in this chapter. The laser without OAM ($\ell = 0$) is the same in every case, but the lasers with OAM vary in either I_{peak} or w_0 . In (a), the power and w_0 are kept the same but I_{peak} is varied, in (b) the peak intensity and w_0 are kept the same but power is varied, and, in (c), the power and peak intensity are kept the same and w_0 is varied. The laser

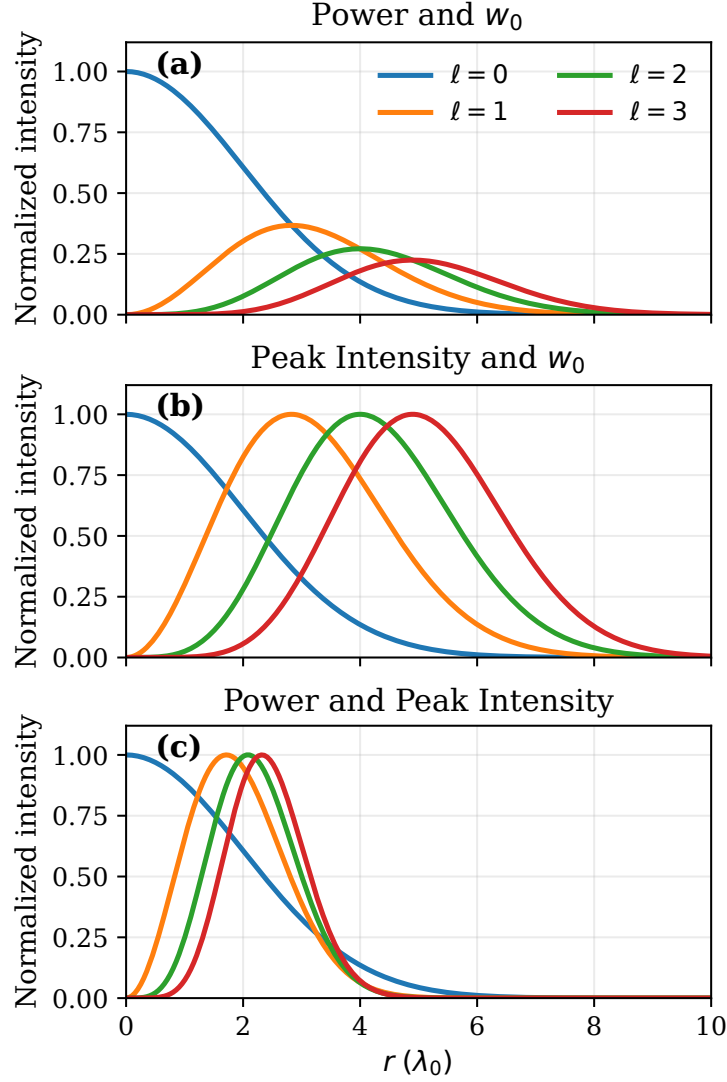


Figure 3.3: Radial laser profiles for $\ell = 0, 1, 2, 3$. To compare lasers with different amounts of OAM, we match two of the following three parameters: power, peak intensity, and w_0 . In (a), power and w_0 are the same for each laser; in panel (b), peak intensity and w_0 are the same; and, in panel (c), power and peak intensity are matched. In (c), w_0 changes for each ℓ while the effective width, w_ℓ , is constant. For (a-b), w_0 is $4\lambda_0$ and for (c), w_ℓ is $4\lambda_0$.

profile for each ℓ is plotted at the plane of best focus, which is at the center of the box.

In (a) and (b), $w_0 = 4\lambda_0$ in each case. In (c), to choose w_0 so that $w_\ell = 4\lambda_0$ and P/I_{peak} is the same for all ℓ , we set w_0 to $(4.0, 2.4, 2.1, 1.9)\lambda_0$ for $\ell = 0, 1, 2, 3$ respectively. These are tightly focused lasers. Figure 3.3 shows that power, peak intensity, and the beam waist must all be considered.

With respect to inertial fusion energy (IFE), the quantity that matters is the reflectivity. With this in mind, we comment on some observations from Figure 3.3. As can be shown from equations 3.26-3.31 and seen in figure 3.3, the distribution of the intensity with radius varies with respect to the ℓ number depending on which pair of the three quantities, power, peak intensity, and spot size, are kept fixed. In (a) (and from equation 3.29) it is seen that if the power and spot size (w_0) are kept fixed then the peak intensity decreases and the effective spot size (w_ℓ) increases with increasing ℓ . In this case, the peak laser intensity is lower but intensity is spread out over a wider range of radii. In (b) it can be seen that for fixed peak intensity and spot size, the width of the intensity distribution with r is nearly the same, but the radius at which the intensity is peaked is larger for higher ℓ . Finally, in (c) for fixed power and peak intensity it can be seen that both the location of the peak and the distribution of the intensity remain fairly similar for all values of ℓ shown. As ℓ increases w_0 must decrease to keep P/I_{peak} constant which also decreases the Rayleigh length.

Keeping in mind that the reflectivity is the gain of SRS across the spot divided by the total power and the gain is proportional to the intensity in the strongly damped regime, we can make some qualitative inferences. For (c) it would be expected that the reflectivity would be similar for nonzero ℓ values since the power is fixed and the intensity has a similar distribution (so long as diffraction is not an issue). In the case of (b), it would appear that the total gain would be larger for nonzero ℓ values since the cross-sectional area of large intensity increases with ℓ . However, the power also increases, so the normalized reflectivity would remain similar. On the other hand for (a), the power is fixed while the peak intensity decreases with ℓ indicating that the reflectivity would decrease with ℓ for this case.

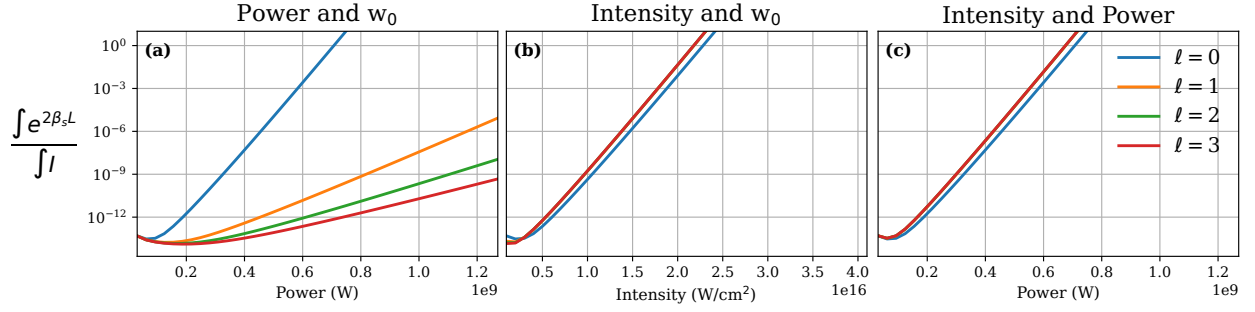


Figure 3.4: The ratio of the integral of $e^{2\beta_s L}$ to the power at best focus for varying intensity and power. Panels (a)-(c) use laser profiles that correspond to figure 3.3 (a)-(c) respectively.

In the next section, we present results of pF3D simulations of SRS. We measure the average reflectivity to quantify the amount of SRS and compare to the threshold for the strongly damped spatial regime. However, to illustrate some of what was just discussed in the preceding paragraph, we plot $\frac{\int dr 2\pi r \exp[2\beta_s(r)L]}{\int dr 2\pi r I(r)}$. The results are shown in figure 3.4 for the three cases which were explained in figure 3.3. It can be seen, as just discussed, that for fixed power and spot size there is progressively smaller gain as ℓ is increased due to I_{peak} gradually decreasing while being spread out to larger r . However, for the other two cases there is very little variation with ℓ . As we show next, for the pF3D simulations there can be variations with ℓ for the latter two cases because of pump depletion, diffraction, and spatially varying noise effects.

3.3 Spontaneous SRS

We aim to compare SRS growth from noise for different amounts of OAM in the pump laser, different laser intensities, and different plasma temperatures. For a broad comparison, we measure the average reflectivity of each simulation. In this chapter, reflectivity is the ratio of backscattered “power” ($|b_1|^2$) to incident “power” ($|b_0|^2$) at the left boundary,

$$\hat{R}(t) = \frac{\int |b_1(\mathbf{x}_\perp, z_{min}, t)|^2 d\mathbf{x}_\perp}{\int |b_0(\mathbf{x}_\perp, z_{min}, t)|^2 d\mathbf{x}_\perp} \quad (3.32)$$

where z_{min} is the entrance plane of the pump laser. We note that the b_j 's are the vector potentials so the ratio of

$$\frac{E_1^2}{E_0^2} = \frac{\omega_1^2}{\omega_0^2} \hat{R}(t) \quad (3.33)$$

and in Chapter 2 we had defined $R = \frac{k_1 b_1^2}{k_0 b_0^2}$.

In the first few picoseconds of a simulation, reflectivity grows quickly when SRS is present. Afterwards, the reflectivity fluctuates due to the finite transit time of the light waves and pump depletion. We determine a single average reflectivity value, $\langle \hat{R} \rangle$, for each simulation by averaging $\hat{R}(t)$ over a time duration much longer than the transit time. Instantaneous and average reflectivity are used to assess the amount of SRS in a simulation.

The random noise in pF3D can be created in either the b_1 (EM) or b_2 (EPW) fields. While details of the full options for setting the noise levels in pF3D can be found in [BSW98], for simulations carried out for this dissertation there was only an initial noise for the EPW which was set assuming a local thermodynamic equilibrium for the chosen absolute density and temperature. For the analysis of the reflectivity to be presented later, we note that we have found empirically that this noise source for b_2 has an average amplitude near [BSW98] 10^{-11} in units of the critical density when the electron temperature is 4.0 keV. The plasma noise is randomly distributed in space. There is no external noise source for the backscattered light (b_1). Instead, the backscattered light wave only grows in the presence of the pump and the EPW. Spontaneous SRS creates the backscattered wave when the pump light interacts with the EPW thermal noise. Therefore, the spatial distribution of the amplitude of b_1 depends on the transverse shape of b_0 even when the pump intensity is very low and reflectivity is small. Figure 3.5 shows the noise signals at the right side of the simulation box, opposite the laser entrance plane. The top row shows the intensity of the pump, backscatter, and EPW when the incident light amplitude is zero. In the second row, a low amplitude pump excites a small signal in b_1 . In both cases, random thermal noise appears in b_2 , the EPW. We have run with other pump strengths and found that the initial b_1 level scales with b_0 .

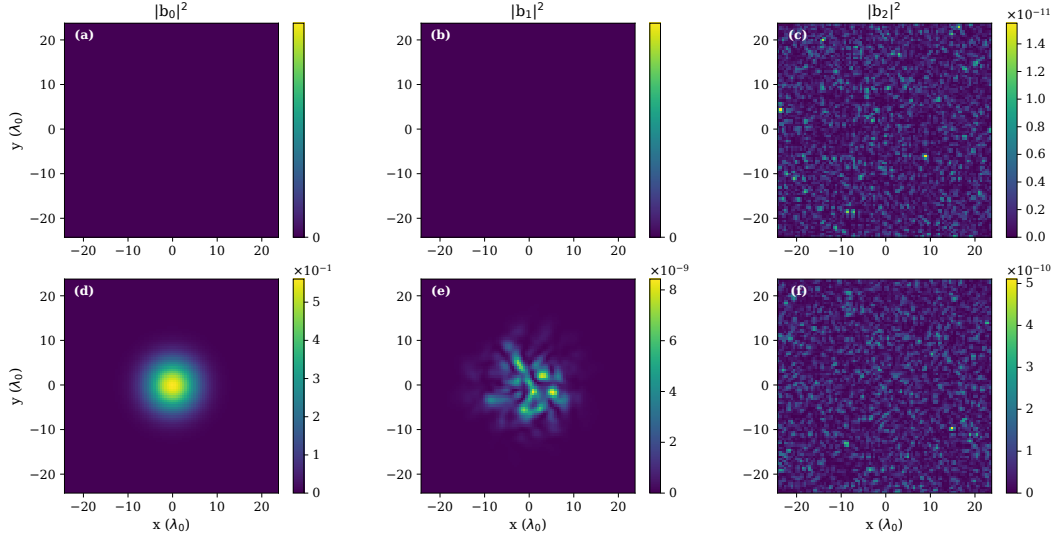


Figure 3.5: At the start of the simulation, random thermal noise exists in the EPW as shown in panels (d) and (f). Panels (a) and (c) show the pump laser intensity ($|b_0|^2$) and (b) and (e) show the reflected light intensity ($|b_1|^2$). In the top row, the pump intensity is zero, i.e., no laser exists. In the second row, a small pump is present and therefore a corresponding signal appears in panel (e) due to the interaction of the pump and the random thermal plasma noise. The units in panels (d) and (e) are W/cm^2 , and panels (c) and (f) are in units of the critical density.

3.3.1 Simulations without OAM at constant temperature

As a reference guide, we carried out a sequence of pF3D simulations for $\ell_0 = 0$ lasers with different peak intensities under the same plasma conditions. Figure 3.6 shows the variation in reflectivity where the pump laser has no OAM, $w_0 = 4\lambda_0$, and the peak intensity varies from $1 \times 10^{15} \text{ W/cm}^2$ to $4 \times 10^{16} \text{ W/cm}^2$. Panel (a) in figure 3.6 shows instantaneous reflectivity, $\hat{R}(t)$, over 10 ps for 8 intensities. The density is $0.14n_c$ and the electron temperature is 4.09 keV. According to the numerical solver, $k_2\lambda_D$ is about 0.32 and ν/ω_p is about 0.02.

Instantaneous reflectivity rises sharply at the start of the simulation, and then oscillates around an average level in every case. The reflectivity peaks are seen to occur at the same time for all simulations because the box size, EPW noise level, and group velocity of the EM waves are constant. As expected both the average reflectivity and peak instantaneous reflectivity increase as the laser intensity increases. Figure 3.6 (b) shows the relationship between average reflectivity, $\langle \hat{R} \rangle$, and intensity, where the average is taken over the entire 10 ps simulation. Average reflectivity grows steeply for intensities between $5 \times 10^{15} \text{ W/cm}^2$ and $1.8 \times 10^{16} \text{ W/cm}^2$, and the curve is flatter at the highest and lowest intensities. The approximate threshold intensity for transition from the strongly to the weakly damped spatial regime, from equation 3.21, is $2.52 \times 10^{16} \text{ W/cm}^2$. Therefore, the strongly damped regime is relevant for all intensities where the reflectivity rapidly increases and initially saturates.

In figure 3.6 (b) we also show the predicted reflectivity from equation 2.187 in Chapter 2. The prediction depends on the noise level and the spatial growth rate, β_s , and their dependence on r . According to the best fit of the data shown here, the appropriate noise level, $b \equiv |k_1||b_1(L)|^2$, is about 10^{-12} where the definition of b is explained in the previous chapter. The exponent in equation 2.187 is $2\beta_s(1 - R)L$ where $R \equiv |k_1||b_1(0)|^2$, which was derived from a 1D model. In practice, the amplitude of the pump wave and noise source depend not only on the radial position but also on the axial position, z , because we are modeling a tightly-focused Gaussian laser. Therefore, $\beta_s L$ should be replaced with

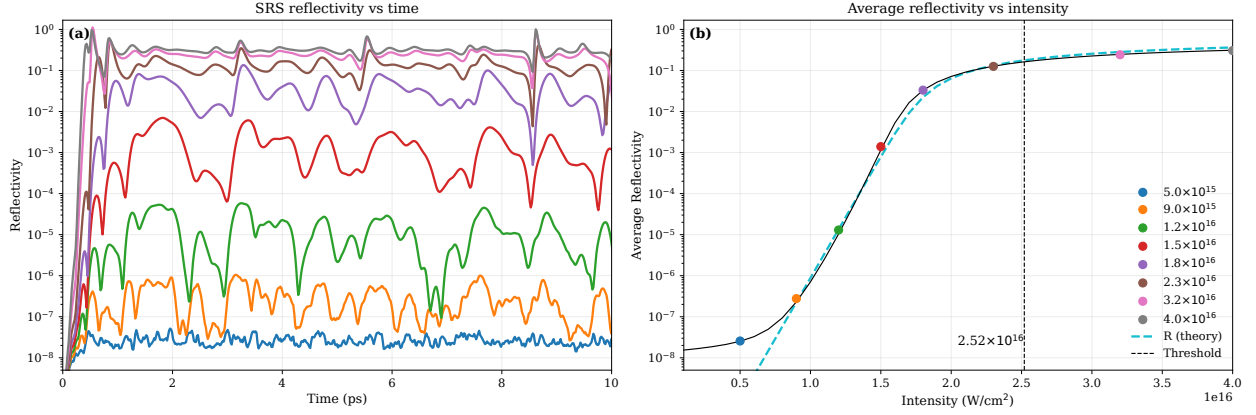


Figure 3.6: Panel (a) shows $\hat{R}(t)$, reflectivity variations over time for 10 ps. Each line is a time series from a simulation with a different peak laser intensity and the intensities are listed in the legend in (b). Panel (b) shows average reflectivity for many simulations with different intensities. The markers on the curve in (b) show $\langle \hat{R} \rangle$, the average reflectivity, for the eight cases where $\hat{R}(t)$ is plotted in (a). The time average is taken over the entire simulation (10 ps). The black dashed line in (b) shows the threshold for transition from the strongly-damped to weakly-damped spatial regime which occurs at $2.52 \times 10^{16} \text{ W/cm}^2$. The dashed curve labeled “R (theory)” shows the numerical solution to the equation 2.187. Every simulation uses a Gaussian pump laser with no OAM.

an effective average $\alpha \int dz \beta_s$ where α is a numerical factor. We note that there is not a straightforward way to include this into the theory on reflectivity given in Chapter 2. Through experimentation with methods of adjusting $\beta_s L$, we found that the best fit comes with a factor of about $\alpha = 0.62$ to incorporate the reduction in gain due to diffraction. For a plasma length of $200\lambda_0$, a reduction to 62% produces an effective length of $124\lambda_0$. In the next section we discuss how to best fit the effective length for simulations with OAM.

3.3.2 Simulations with different temperatures

We analyzed the relationship between average reflectivity, $\langle \hat{R} \rangle$, and $k_2 \lambda_D$ for different amounts of OAM in the pump laser. The density is constant at $0.14n_c$ and the electron temperature varies. In these cases β_s varies due to changes in damping for each scan with ℓ .

In figure 3.7, we plot $\langle \hat{R} \rangle$ as $k_2 \lambda_D$ is varied from 0.25 to 0.4 for different values of ℓ of the pump compared with three techniques. When $\ell = 0$, the pump intensity is $2 \times 10^{16} \text{ W/cm}^2$. Panels (a-c) use $w_0 = 4\lambda_0$ for the case with no OAM and (d-f) use $8\lambda_0$. The box size is constant. In (a) and (d), the different OAM simulations have the same power and w_0 . Therefore, the peak intensities for the cases with OAM are lower and more diffuse in r . In (b) and (e), w_0 and peak intensity are both held fixed, while the power of the pump varies. In (c) and (f), the power and peak intensity are both held fixed but w_0 is varied to maintain constant w_ℓ . For the simulations in (c) and (f) the peak intensity is $2 \times 10^{16} \text{ W/cm}^2$, and the effective width, w_ℓ , is constant. Therefore, for these panels, the spot size w_0 decreases as ℓ increases and it is approximately $2\lambda_0[4\lambda_0]$ for $\ell = 3$ in (c) [(f)].

The threshold for the transition to the strongly-damped spatial regime depends on $k_2 \lambda_D$ and I_{peak} . Therefore, the threshold is the same for all OAM in panels (b-c) and (e-f). On the other hand, in (a) and (d), the threshold varies as I_{peak} varies. When the peak intensity is $2 \times 10^{16} \text{ W/cm}^2$, the transition occurs at $k_2 \lambda_D = 0.31$. The transitions are shown as the vertical dashed lines. All simulations follow the trend that $\langle \hat{R} \rangle$ decreases as $k_2 \lambda_D$ increases. The damping rate, ν , increases with $k_2 \lambda_D$ and increased EPW damping decreases

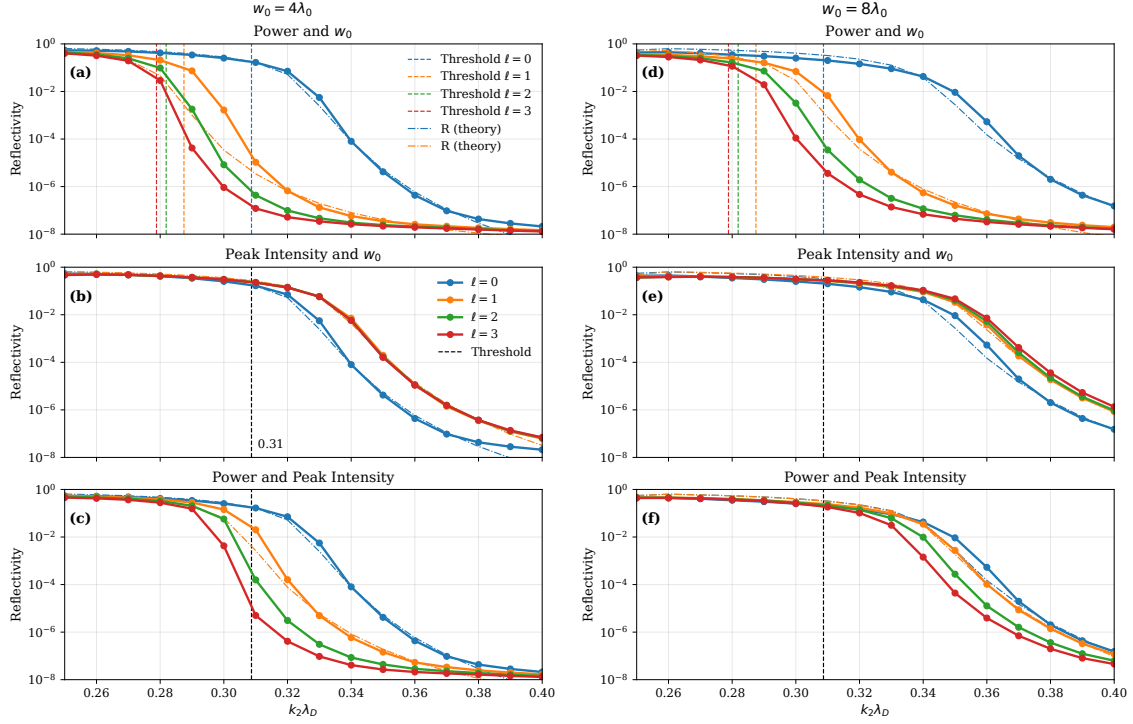


Figure 3.7: Average reflectivity vs $k_2\lambda_D$ for simulations with different OAM. In (a) and (d), the lasers have the same power and same w_0 , while the peak intensities differ. The peak intensity for the case without OAM is $2 \times 10^{16} \text{ W/cm}^2$. This is the same in each plot so the blue line is the same in each plot. In (b) and (e), the peak intensity is the same ($2 \times 10^{16} \text{ W/cm}^2$) for all OAM, while the power differs, and in (c) and (f) w_0 changes so that power and peak intensity are the same for all OAM (constant w_ℓ). Panels (a)-(c) use a spot size of $4\lambda_0$ for $\ell = 0$ and panels (d)-(f) use a spot size of $8\lambda_0$ for $\ell = 0$. The threshold line shows the value of $k_2\lambda_D$ where SRS transitions from the weakly damped to strongly damped spatial regime. In (b-c) and (e-f), the threshold is $k_2\lambda_D = 0.31$ because all simulations have the same I_{peak} . In (a) and (d), the peak intensity depends on ℓ so the threshold value of $k_2\lambda_D$ changes. The dashed lines labeled R (theory) represent the solution to equation 2.187 where the best fit parameters are determined separately for $\ell = 0$ and $\ell = 1$. The fit parameters are given in the text.

the reflectivity.

The trends seen in figure 3.7 can be qualitatively understood based on the discussion of figures 3.3 and 3.4 and the analytic formula for reflectivity, equation 2.187. We are plotting the reflectivity vs. $k_2\lambda_D$ and not intensity as was the case in Figure 3.3. However, the spatial gain coefficient β_s is proportional to intensity and inversely proportional to ν . And in figure 3.1 (e) it can be seen that ν varies nearly linearly in $k_2\lambda_D$ when $k_2\lambda_D$ falls in the range 0.3 to 0.4. Thus, arguments for how the reflectivity depends on intensity can be applied for how it depends on $k_2\lambda_D$.

As discussed earlier, there are significant differences in the reflectivity as a function of ℓ when the power and spot size are held fixed. In this case, the peak intensity decreases and the distribution of intensity spreads out to larger r as ℓ increases. The decrease in I_{peak} leads to a noticeable decrease in the reflectivity. On the other hand, when the peak intensity and spot size remain fixed, the simple geometrical arguments given above in section 3.2.0.2 indicate that the reflectivity would be insensitive to the ℓ number, and this is confirmed in the pF3D simulation results shown in panels (b) and (e). This is because although the value of r for which the location of highest intensity, and hence the cross-sectional area for which that corresponds, increase with ℓ , the reflectivity is normalized to power, which also increases with ℓ in proportion. For constant power and peak intensity, panels (a) and (d), the simple arguments indicate no difference in reflectivity with ℓ . However, as can be seen in panels (c) and (f), there are differences although these are less than in panels (a) and (d). In this case diffraction becomes progressively more important for higher ℓ because in order to keep power and peak intensity constant the spot size needs to be reduced. The fact that diffraction is important is confirmed when comparing panels (c) and (f) as there is less difference for the $8\lambda_0$ size where the box length is kept fixed in laser wavelengths and not Rayleigh lengths.

We have also included plots of the reflectivity by using equation 2.187 for $\ell = 0, 1$ in each panel. We experimented with the coefficients and show examples that roughly agree

with the plots. For $w_0 = 4\lambda_0$ (panels (a)-(c)) we used $b = 5.9 \times 10^{-12}$ and $\alpha = 0.377$ for $\ell = 0$. For $\ell = 1$ we used $b = 1.3 \times 10^{-9}$ and $\alpha = 0.095$ in panel (a), $b = 1.34 \times 10^{-11}$ and $\alpha = 0.45$ in panel (b), and $b = 1.11 \times 10^{-10}$ and $\alpha = 0.205$ in panel (c). For the $w_0 = 8\lambda_0$ simulations (panels (d)-(f)), we used $b = 1.81 \times 10^{-11}$ and $\alpha = 0.52$ for $\ell = 0$. For $\ell = 1$ we used $b = 3.73 \times 10^{-10}$ and $\alpha = 0.6$ in panel (d), $b = 2.59 \times 10^{-11}$ and $\alpha = 0.6$ in panel (e), and $b = 1.25 \times 10^{-11}$ and $\alpha = 0.52$ in panel (f).

We do not have precise arguments for how to choose the noise level, b , and the β_s reduction factor α . However, we can comment that as expected in each case the reduction factor is smaller for the $w_0 = 4\lambda_0$ case because of diffraction effects and that the reduction factor is smaller for $\ell = 1$ when power and spot size are kept constant.

3.3.3 Simulations with constant temperatures

Due to the nonlinear dependence of the EPW damping $k_2\lambda_D$, it is not trivial to separate the $\langle\hat{R}\rangle$ dependence on ℓ and $k_2\lambda_D$ over the full parameter space. Thus, in this section we study the dependence of $\langle\hat{R}\rangle$ solely on the laser parameters. We keep the density and temperature constant at $n/n_c = 0.14$ and 4.69 keV which yields $k_2\lambda_D$ of 0.34 according to the kinetic solver. At this temperature and density, the damping rate ν is significant as indicated in figure 3.1. The spot size, w_0 , is set to $4\lambda_0$ except when we simultaneously keep the power and peak intensity (and by definition w_ℓ) fixed as OAM number changes, i.e., w_0 must decrease as ℓ is increased.

We measured average reflectivity for a range of intensities and powers with $\ell_0 = 0, 1, 2, 3$. Figure 3.8 shows the change in average reflectivity as the power increases from 10^8 to 2.1×10^9 W for different amounts of OAM in the incident laser. We note that for $w_0 = 4\lambda_0$ and $\ell = 0$ this corresponds to the peak intensity ranging from 3.2×10^{15} W/cm² to 6.7×10^{16} W/cm². As was the case when varying the electron temperature, a laser without OAM ($\ell_0 = 0$) shows significantly more SRS than a laser with OAM for fixed power. Furthermore, the average reflectivity decreases as the OAM of the pump increases for the same power. For $\ell_0 = 0$, $\langle\hat{R}\rangle$

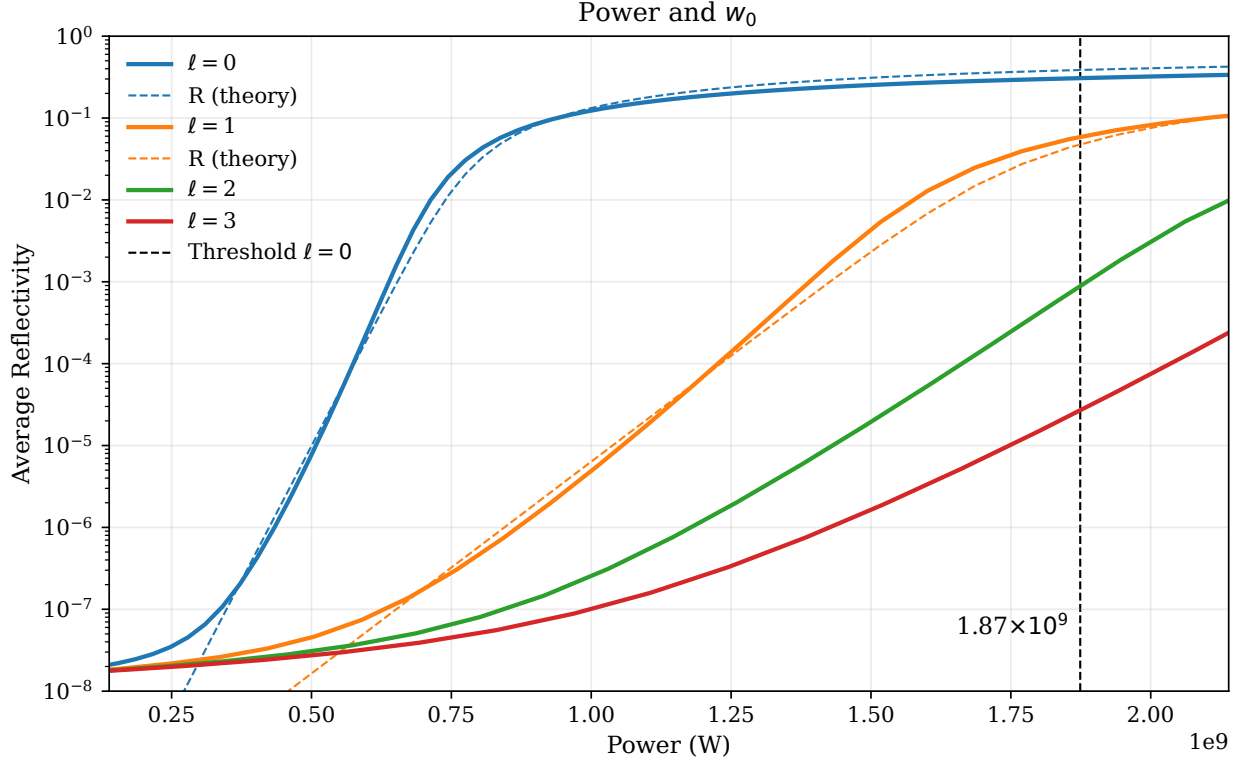


Figure 3.8: Average reflectivity, $\langle \hat{R} \rangle$, vs power of the incident laser for different amounts of OAM. The transition intensity from the strongly to weakly damped spatial regime is 1.87×10^9 W for the $\ell = 0$ beam but increases for other ℓ due to shape. Transitions for higher OAM exceed the bounds of the plot. The dashed curves labeled $\hat{R}$ (theory) represent solutions to equation 2.187 for fitted parameters which are separate for $\ell = 0$ and $\ell = 1$. The fitting parameters are shown in the text.

increases steeply as power increases until reaching a saturated state where further increases in power have little effect. This indicates that pump depletion has occurred. On the other hand for nonzero OAM number $\langle \hat{R} \rangle$ increases more slowly with power and the increase is less steep as ℓ increases from 1 to 3. For these nonzero OAM numbers $\langle \hat{R} \rangle$ does not saturate within the range of powers shown.

This comparison clearly illustrates that for fixed power and spot size there is a drastic difference in reflectivity for pumps with different OAM for a given power. As described earlier, when power and w_0 are constant for different ℓ , the lasers with OAM have a much lower peak intensity. (The top panel of figure 3.3 shows the profiles for this case.) The power threshold is 1.87×10^9 W for the $\ell_0 = 0$ mode and increases with OAM so much that it is not within the bounds of the plot.

Curves generated from equation 2.187 are also shown in figures 3.8, 3.9, and 3.10. Good agreement was found for $\ell = 0, 1$ if $b = 2.67 \times 10^{-12}$ [$b = 4.25 \times 10^{-11}$] and $\alpha = 0.517$ [$\alpha = 0.554$] for $\ell = 0[1]$ for figures 3.8 and 3.9. In Figure 3.10, the parameters for $\ell = 0$ are the same, but for $\ell = 1$, $b = 1.32 \times 10^{-11}$ and $\alpha = 0.303$.

We note that this situation is the most relevant for the IFE application where the laser power is the metric that determines the strength of the pressure on the capsule. Thus, if it were possible to make each speckle have some OAM it would be possible to suppress SRS. This suppression is a purely geometrical effect that indicates it is useful to limit the peak intensity of individual speckles, while keeping the power fixed.

We also compared lasers with different OAM at the same peak intensity and w_0 . Figure 3.9 shows the relationship between intensity and reflectivity for a range of intensities from 10^{15} to 7×10^{16} W/cm². A laser with OAM has a moderately higher average reflectivity than a laser with $\ell_0 = 0$. The reflectivity is nearly insensitive to the OAM number for $\ell_0 \neq 0$.

All lasers with OAM have zero intensity on axis and, as ℓ increases, the radial position of maximum intensity moves further from the axis. The $\ell_0 = 0$ case (no OAM) is unique in that

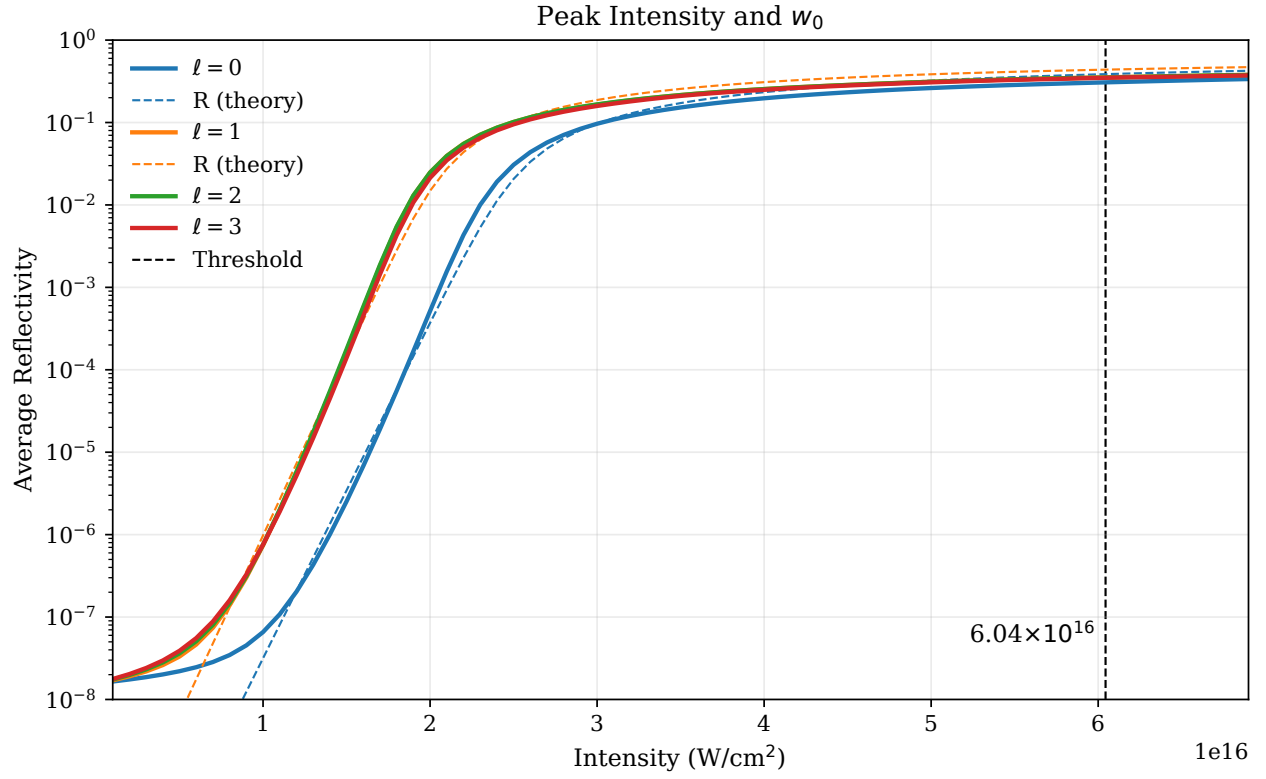


Figure 3.9: Average reflectivity $\langle \hat{R} \rangle$ vs I_{peak} for lasers with different OAM. The transition intensity is the same for each ℓ and is equal to 6.04×10^{16} W/cm². The spot size is $4\lambda_0$ in each case and the OAM beams have more power at the same intensity due to their different shape. The dashed lines labeled R (theory) show solutions to equation 2.187 with fitting parameters in the text.

there is a signal at $r = 0$. Therefore, a clear distinction exists between $\ell_0 = 0$ and $\ell_0 \neq 0$, but more OAM seems not to significantly change $\langle \hat{R} \rangle$. This was also seen and explained in figures 3.4 and 3.7.

Figures 3.8 and 3.9 suggest that the reduction in average reflectivity for OAM light at the same power is due to the reduction in peak intensity. When compared at the same peak intensity, the relative difference between the average reflectivity for pumps with and without OAM flips because in this case the higher OAM lasers have more power. We have also shown that the shape of the $\langle \hat{R} \rangle$ versus intensity curve for SRS growth is the same regardless of OAM, suggesting that the same saturation mechanism applies in all cases, but the threshold power shifts when the pump has OAM.

To further understand the effect of peak intensity and OAM, we adjust the shapes so that we can compare SRS of lasers with different OAM at the same power and the same intensity simultaneously by matching w_ℓ . Figure 3.10 shows the average reflectivity in this case, plotted against peak intensity (bottom) and power (top). In general, reflectivity decreases as ℓ_0 increases even when compared at the same power and the same peak intensity. Thus, the change in reflectivity due to OAM cannot simply be explained by power or intensity. Instead, the shape of the laser profile also plays a role in determining the amount of SRS.

There is a clear reduction in reflectivity for OAM light when the power is held constant. We have shown this for both constant w_0 and constant peak intensity. pF3D simulations show that the spatial distribution of intensity due to OAM affects the average reflectivity, $\langle \hat{R} \rangle$, and, thereby, the amount of SRS. A laser with more OAM shows less SRS due to this geometric effect in two out of three cases.

There are no individual particles in pF3D, and the Landau damping only depends on $k_2 \lambda_D$ calculated at $t = 0$. Therefore, the only culprit for the change due to OAM is the distribution of intensity across the transverse plane. Particle-in-cell (PIC) simulations are required to study any other kinetic effects. We address this with OSIRIS simulations in the next chapter.

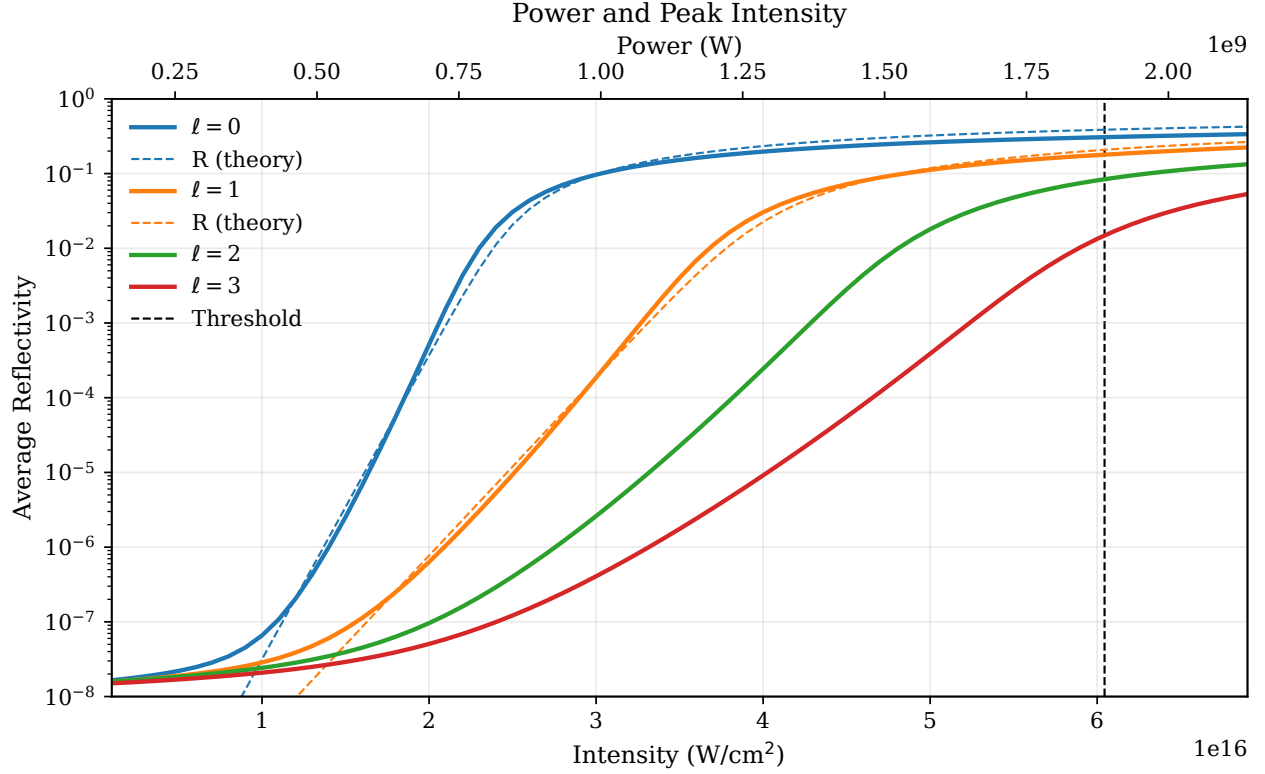


Figure 3.10: Reflectivity $\langle \hat{R} \rangle$ vs. intensity and power when the different OAM modes have the same effective spot size, w_ℓ , of $4\lambda_0$ and therefore w_0 is $(4.0, 2.43, 2.08, 1.89)\lambda_0$ for $\ell_0 = 0, 1, 2, 3$. Adjusting w_0 with ℓ allows for the different OAM cases to be compared at the same power and intensity simultaneously. The transition, which depends only on peak intensity and plasma conditions, is the same for all ℓ . The dashed lines labeled $\hat{R}$ (theory) represent solutions to equation 2.187 for fitting parameters given in the text.

3.3.3.1 OAM and $\bar{\ell}$ matching

In Chapter 2 we discussed how the 3-wave equations naturally lead to an equation for conservation of OAM and how this equation leads to a conservation equation for the effective ℓ number which we label $\bar{\ell}$. This equation shows that $\bar{\ell}$ matching, $\bar{\ell}_0 = \bar{\ell}_1 + \bar{\ell}_2$, only arises in regions of time and space where a steady state is reached. In this section we examine how the effective ℓ numbers for each wave evolve and under what conditions $\bar{\ell}$ matching arises in the pF3D simulations.

The OAM in each wave changes throughout the simulation when SRS backscatter is significant. We use equation 2.236 to examine the effective OAM index of b_0 , b_1 , and b_2 . At the start of a simulation, ℓ_0 is set to 0, 1, 2, or 3 (pure LG modes are launched from the left boundary) , and b_1 and b_2 have no initial OAM (in the absence of a seed laser). As the amplitudes of b_1 and b_2 increase, $\bar{\ell}_1$ and $\bar{\ell}_2$ become detectable. For low values of wave amplitude the effective OAM indices fluctuate significantly. For larger pump intensities and hence larger reflectivities for $\ell_0 \neq 0$ both $\bar{\ell}_1$ and $\bar{\ell}_2$ can acquire larger (and non-integer) shifts from 0 and evolve slowly in space (and time) on the entrance side of the simulation box. For $\ell_0 = 0$ both $\bar{\ell}_1$ and $\bar{\ell}_2$ are also 0 for high intensities and reflectivities. In all values of ℓ_0 at high intensities, once the $\bar{\ell}_j$ vary smoothly, $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ is approximately 0.

The definition of the OAM index based on equation 2.236 results in non-integer values of $\bar{\ell}$ that are function of time and space. We will show that on the right side of the box, away from the laser entrance plane, the backscatter amplitude is near the noise level, leading to large fluctuations of $\bar{\ell}$ even when reflectivity is large. Even for large intensities and near steady state the effective OAM, $\bar{\ell}_j$, for the daughter waves is not necessarily an integer, even when ℓ_0 is a fixed integer for the pump. We calculate the OAM after the simulation reaches a near steady state.

These results are summarized in figure 3.11 , which shows $\bar{\ell}_0$, $\bar{\ell}_1$, and $\bar{\ell}_2$ for many simulations at 5.4 picoseconds. The color of each line is determined by the initial peak intensity

Effective OAM for simulations with different OAM

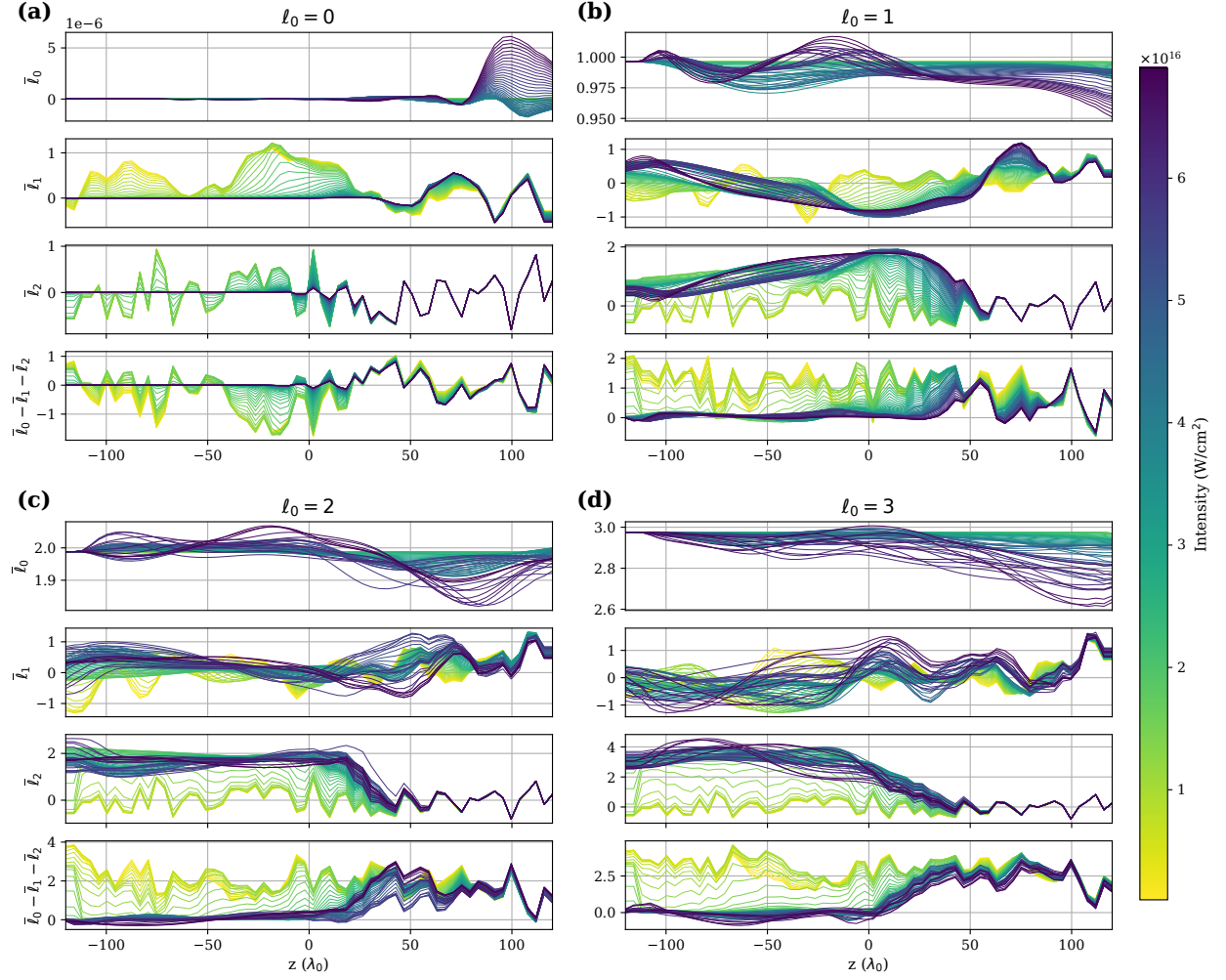


Figure 3.11: Effective OAM for all simulations as a function of z at $t = 5.4$ ps. Panels (a), (b), (c), and (d) show simulations where the pump laser has OAM equal to $\ell_0 = 0, 1, 2$ and 3. The color of each line depends on the incident peak intensity of the pump light where a darker color represents a higher intensity laser. Higher intensities also correspond to greater reflectivity as shown in previous figures.

of the pump. The lines in panel (a) correspond to results from simulations with $\ell_0 = 0$ (no OAM) while in panels (b), (c), and (d), $\ell_0 = 1, 2$, and 3 respectively. The yellower lines correspond to cases where the intensity (and therefore reflectivity) is low. The simulations with and without OAM are compared at the same w_0 and peak intensity.

Several observations can be made. Note that the diagnostic for calculating $\bar{\ell}$ does not give an exact integer for $\bar{\ell}_0$ due to the use of finite difference $\nabla_\perp$ operators. For the pump, the effective OAM of the pump tends to be constant near the initial value of ℓ_0 for the lower intensities. However, at the highest intensities, where there is significant pump depletion on the right-hand side of the box, there can be modifications to ℓ_0 leading to variations up to 10%. In addition, there can also be detectable changes to $\bar{\ell}_0$ on the left-hand side if the reflection is large enough. For high reflectivity, the beating of the reflected light and EPW can alter the OAM of the pump even in regions of space towards the entrance plane where its amplitude has not changed much.

It can also be clearly seen in each panel that as the intensity and hence reflectivity increase, the effective OAM of each wave stabilizes (there are only gradual changes in z) near the entrance plane of the laser; however next to the exit plane they continue to show fluctuations in regions where the scattered intensity and EPW amplitude are low. In general, low amplitude waves seem to show more fluctuations in their effective OAM based on its definition having the power in the denominator.

It can also be seen that the OAM of the EPW is independent of the pump intensity for $z > 50\lambda_0$. This is because there has been little amplification of the EPW within this region and $\bar{\ell}_2$ in this region is essentially that of the noise. Although the noise levels of the OAM of the EPW are very small, the OAM index can be large since it is the ratio of the OAM to the power which are both small numbers.

In the $\ell_0 = 0$ case (panel (a) of figure 3.11), the scattered light and EPW also have an effective OAM index of zero. When $\ell_0 = 1, 2$, and 3 (panels (b), (c), and (d)), the effective OAM of the scattered light and the EPW increase with increasing reflectivity. The effective

OAM can also be negative. It can also be seen that $\bar{\ell}_1 \approx 0$ for $I \leq 2.5 \times 10^{16}$ W/cm², i.e., when the reflectivity is low.

Despite all of the above complications, it can be seen in the bottom row of each panel that OAM matching roughly holds on the left side of the simulation box for sufficiently large pump intensities. In these panels, $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ is plotted as a function of z . For large pump intensities, the values of $\bar{\ell}_1$ and $\bar{\ell}_2$ vary slowly in z on the entrance half of the simulation box where the amplitudes of these waves are also large.

It is important to note that although $\bar{\ell}$ matching roughly holds in this region, neither $\bar{\ell}_1$ nor $\bar{\ell}_2$ are integers. Thus, these waves are not pure LG modes. In order to study the way OAM is distributed between b_1 and b_2 and the change in $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ in z as one moves closer to the entrance plane, we examine a single intensity for pumps with different OAM.

In figure 3.12 we show the effective OAM index $\bar{\ell}$ for each of the three waves, as well as the sum, $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ from four simulations with ℓ_0 varying from 0,1,2, and 3 while the pump intensity was kept constant at 2.5×10^{16} W/cm² and all lasers use a spot size of $4\lambda_0$. This is 35.7% of the highest intensity used for figure 3.11 and the data corresponds to the same time, 5.4 ps.

As can be seen in the top left panel of figure 3.12 when $\ell_0 = 0$, there are large fluctuations in $\bar{\ell}_j$ (it is noisy) for $j = 1, 2$ at values of z in the exit half of the box, and the fluctuations range from -1 to 1 approximately. For values of z in the entrance half of the box, however, all waves have $\bar{\ell}_j$ equal to zero and the matching condition is well satisfied.

For the $\ell_0 = 1$ case (top right), the $\bar{\ell}_j$ of the EPW and backscattered light waves in the left half of the box are not constant and they do not have integer values. At the entrance plane where backscattered amplitude is greatest, $\bar{\ell}_1$ is about 0.25, $\bar{\ell}_2$ is about 0.75, and $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2 \approx 0$. On the right side of the box the OAM indices fluctuate for all waves except the pump. Thus, $\bar{\ell}$ matching is roughly satisfied in regions where the $\bar{\ell}_j$'s have stabilized.

In the cases with $\ell_0 = 2$ and $\ell_0 = 3$ the backscatter wave has very little OAM (small

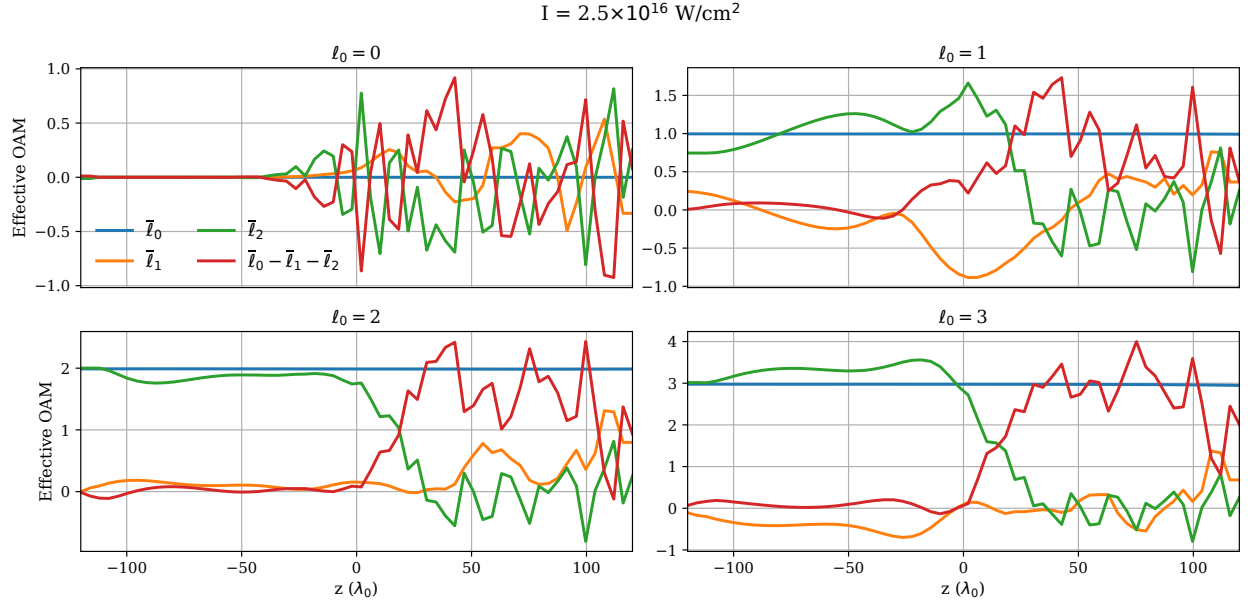


Figure 3.12: Effective OAM at constant peak intensity for four simulations with different OAM in the pump light at 5.4 ps. The intensity is $2.5 \times 10^{16} \text{ W/cm}^2$ in each case for which the reflectivity is low. The blue lines show the OAM of the pump light which is constant in every case because pump depletion is low. The orange line shows the OAM in the backscattered light and the green line shows the OAM in the EPW. The red line shows how well OAM is conserved through the variation in $\bar{l}_0 - \bar{l}_1 - \bar{l}_2$.

values of $\bar{\ell}_1$) relative to the plasma wave ($\bar{\ell}_2$). The plasma wave eventually acquires an OAM index approximately equal to that of the pump while that of the scattered light roughly remains close to zero. In all cases shown, the pump OAM, $\bar{\ell}_0$ stays at its original value, ℓ_0 , because there is little pump depletion.

Conservation of $\bar{\ell}$ is represented by the red line which plots $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$. As noted above, when the pump has OAM, the summed value is small but not exactly zero. Although the simulations have reached a steady state, the sum of the effective $\bar{\ell}$ numbers is approximately but not perfectly conserved. We comment that for higher intensities (as seen in figure 3.11) the value of $\bar{\ell}_1$ can deviate significantly from 0 for all values of ℓ_0 except 0.

3.3.3.2 Spatial shape

As shown above, the OAM indices of the waves are not integers indicating that the waves are not pure LG modes. Thus, it is of interest to investigate the structure of the modes more closely. This is best done by examining the modes in real space, using 2D cuts in both the (x, z) plane at $y = 0$ and the (x, y) plane at a z near the entrance plane of the pump laser.

Reflectivity $\langle \hat{R} \rangle$ measures the power of the reflected signal which is integrated over $\mathbf{x}_\perp$ and the OAM index is a measure of a quantity obtained by averaging over $\mathbf{x}_\perp$. Therefore, in this section we investigate the shape in the 2D cuts of the wave intensities for cases with different OAM. In all cases the pump has a spot size of w_0 equal to $4\lambda_0$.

In figure 3.13, we show the spatial distribution of the intensity in cuts in (x, z) of each wave to demonstrate that $|b_1|^2$ and $|b_2|^2$ do not have the same distribution of intensity as $|b_0|^2$ when the pump has OAM. The peak intensity for the case without OAM is $2.4 \times 10^{16} \text{ W/cm}^2$ and the peak intensity for the three cases with OAM is $2.0 \times 10^{16} \text{ W/cm}^2$. It can be seen that the incident light is tightly focused at $z = 0$ and the two daughter waves are largest near the laser entrance, at z_{min} . Each column shows a simulation with a different amount of OAM. The top row is the intensity of the incident pump light in W/cm^2 , the middle row is

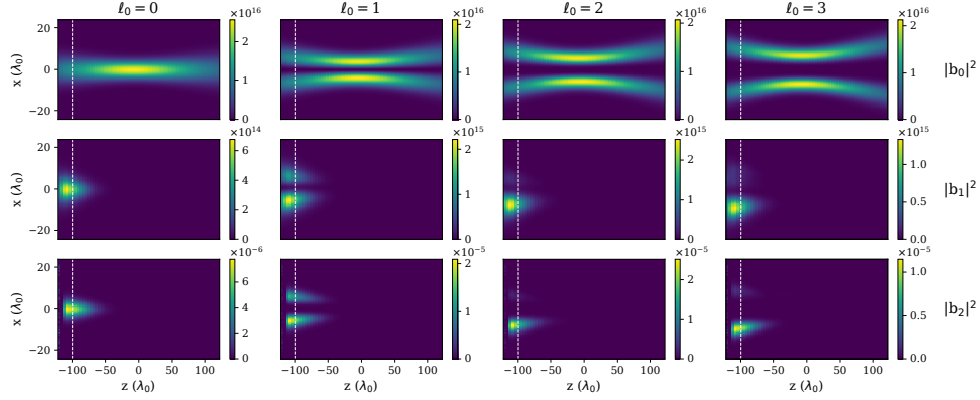


Figure 3.13: Intensities of the three waves from simulations with different pump OAM and SRS growth from noise. Each column represents one simulation. The top row shows the intensity of the pump light, the middle row shows the intensity of the scattered light, and the bottom row shows the intensity of the plasma wave. Although the pump is tightly focused at $z = 0$, the scattered waves are strongest near z_{min} . The vertical dashed white line shows the location of the transverse slices in figure 3.14. The peak intensity for the case without OAM is $2.4 \times 10^{16} \text{ W/cm}^2$ and for the three cases with OAM it is $2.0 \times 10^{16} \text{ W/cm}^2$.

the intensity of the backscattered light in W/cm^2 , and the bottom row is the EPW density in units of n_c . All cases are in a highly-damped, low reflectivity regime, where the pump signal is not perturbed by SRS (if pump depletion occurs the pump is depleted near the exit plane).

The backscattered light and EPW show similar spatial intensity patterns which is consistent with the growth of SRS. When the pump has no OAM both daughter waves are intense near the axis, $x = 0$; and they appear symmetric about x . On the other hand, for cases with OAM, the signals are strongest off axis as is the case for the pump. As ℓ_0 increases, the regions of largest intensity for the daughter waves moves further from the axis. Additionally, the pump light is more spread out because of the OAM.

The main point is that the scattered waves with OAM in figure 3.13 appear asymmetric with respect to the propagation axis. Recalling that the intensity distribution for a pure LG mode is symmetric about the z axis, it can be concluded that neither daughter wave is a pure LG mode. This is consistent for situations where the effective OAM number is not an integer. However, this indicates that even when the index is nearly an integer, the mode is still a superposition of pure LG modes.

This is more easily seen by examining cuts of both the intensity and the fields in the (x, y) plane for values of z where the daughter waves are large. By inspecting equation 2.8, it is clear that the intensity of a pure LG mode does not vary in φ . In Figure 3.14 we show the intensity cross-section for each wave near z_{min} where the scattered signals are largest. As was the case for figure 3.13 the top row is for $|b_0|^2$, the middle row is for $|b_1|^2$ and the bottom row is for $|b_2|^2$; and the columns are simulations with different OAM in the pump. For $\ell_0 = 0$ the two daughter waves both have intensity patterns that match the pump and which have no φ dependence. On the other hand, when $\ell_0 \neq 0$, the scattered light and electron plasma wave both show a distinct spiral pattern, where the number of lobes is 2 (for $\ell_0 = 1$) or 3 (for both $\ell_0 = 2, 3$). This indicates that the mode is a superposition of pure modes.

To understand this structure in more detail, in figures 3.15 and 3.16 the real part of

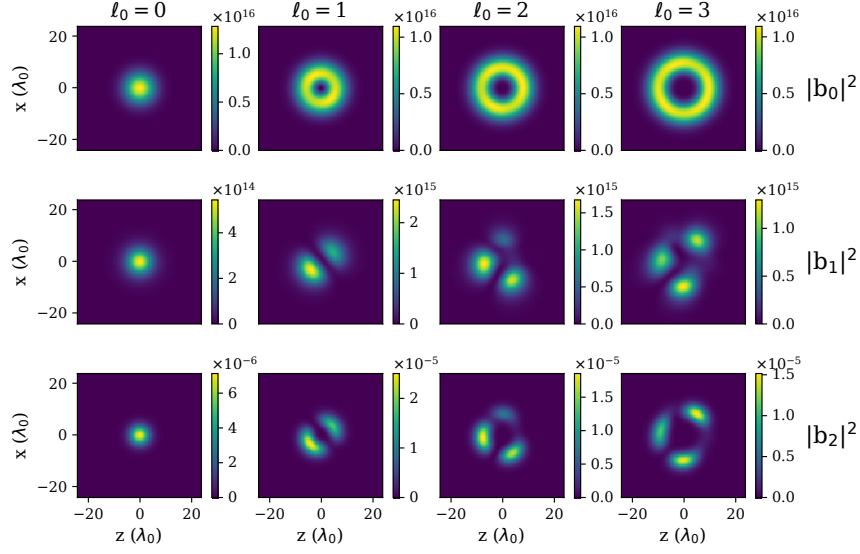


Figure 3.14: Transverse slices of intensity for the pump and daughter waves for different pump OAM. The slices are taken near z_{min} as shown by the dashed white lines in figure 3.13. The top row shows the pump light intensity, the middle row shows the scattered light intensity, and the bottom row shows the EPW.

each wave for five different intensities and therefore different reflectivities is displayed for $\ell_0 = 0, 1, 2, 3$.

Figure 3.15 shows the real parts of the wave envelopes for $\ell_0 = 0, 1$. The top half of the figure is for a pump with no OAM. Each column is a simulation with a different pump intensity. The intensities are listed at the top of the columns and are equal to 1×10^{16} , 2×10^{16} , 3×10^{16} , 4×10^{16} , and $5 \times 10^{16} \text{ W/cm}^2$. The reflectivity increases with pump intensity so the amplitude of the daughter waves increases from left to right in the figure. The scales change between columns so that the signal can be clearly seen in each case. This sequence of plots confirms that for $\ell_0 = 0$ the two daughter waves appear symmetric about the axis, i.e., they have no φ dependence. Their spot sizes are both narrower than that of the pump. They also appear to have a nonzero p dependence based on equation 2.8. It can also be seen for the highest intensities (and highest reflectivities) that the pump is locally depleted near the axis.

The lower half of the figure shows data for a pump with $\ell_0 = 1$. Unlike the case without OAM, the signals are asymmetric about the axis. The pump has a spiraling shape where the amplitude of the wave clearly oscillates once when rotating in φ . Dark blue and bright yellow correspond to different signs for the fields. Just as for the $\ell_0 = 0$ case, the scattered light wave and EPW do not necessarily have the same width as the pump. Furthermore, it can be seen that the EPW has a similar dependence in φ as the pump while the scattered light wave has a more complicated profile. It appears to be made of a superposition of ℓ and p values.

Figure 3.16 shows the same organization of plots for pumps with $\ell_0 = 2$ and $\ell_0 = 3$. The fact that the EPW has a profile that more closely resembles that of the pump is more clear. It is also clear that the profile is less uniform, indicating the EPW is made up of a superposition of pure LG modes. The scattered light profile is even more complicated. It is interesting that as seen in figures 3.11 and 3.12 the EPW structure and its $\bar{\ell}_2$ tracks more closely that of the pump while the light wave's $\bar{\ell}_1$ number is closer to 0. However, the phase

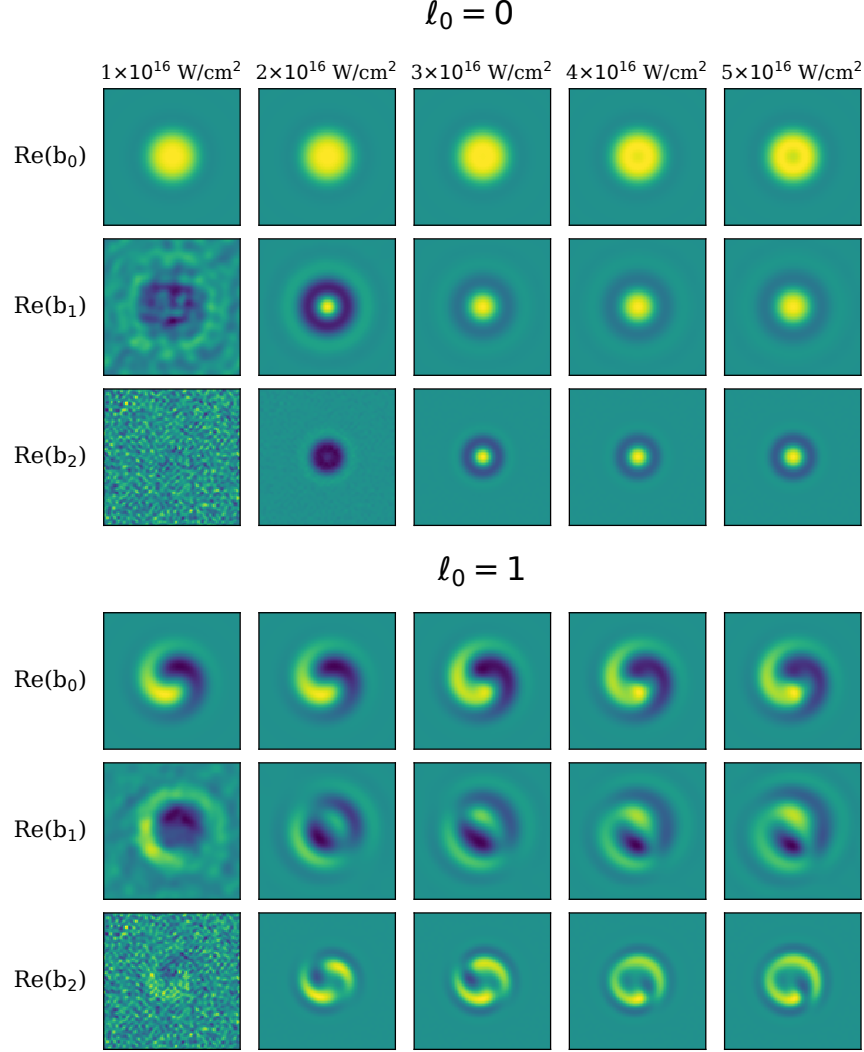


Figure 3.15: Cross sections for different pump intensities from simulations which use pumps with $\ell = 0$ and $\ell = 1$. The top half shows the results for $\ell = 0$ and the bottom half shows results for $\ell = 1$. The top row of each half shows the real part of the pump wave, the second row shows the real part of the scattered light wave, and the third row shows the real part of the plasma wave. Each column corresponds to a different pump intensity. The intensities are listed at the top and are equal to 1×10^{16} , 2×10^{16} , 3×10^{16} , 4×10^{16} , and $5 \times 10^{16} \text{ W/cm}^2$. Each plot is on a different color scale in order to maximize visible detail. Colorbars are excluded for brevity.

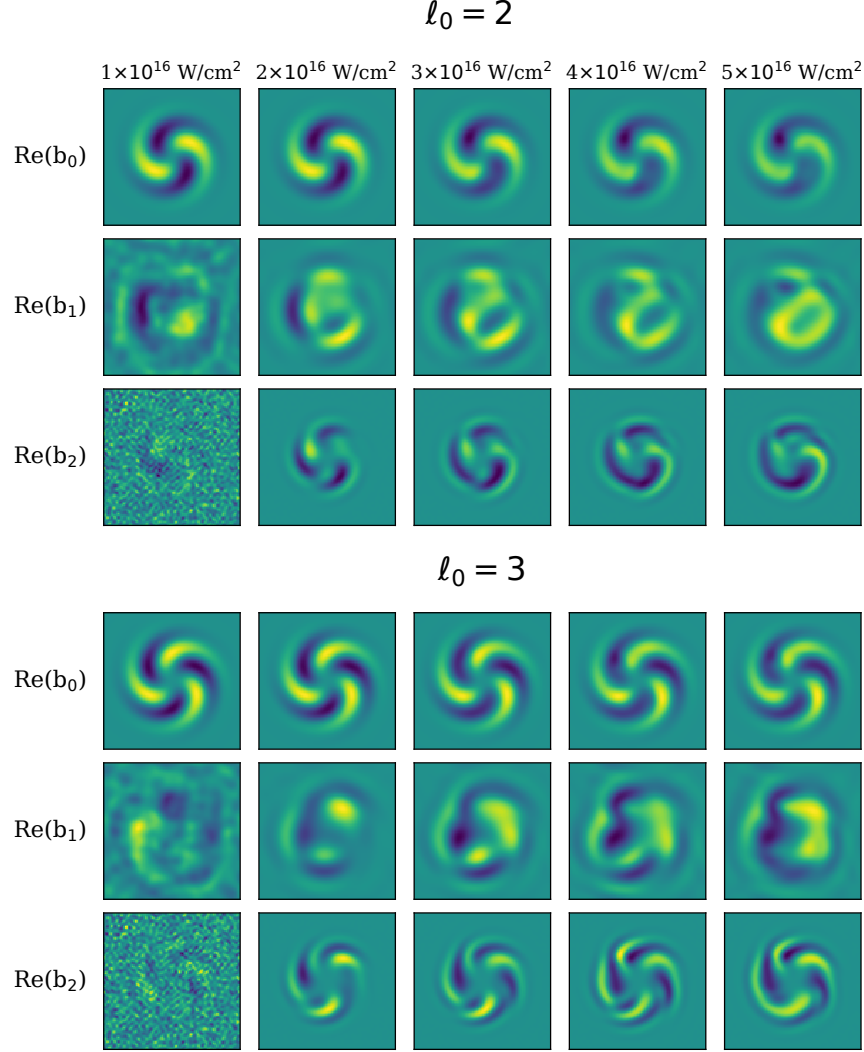


Figure 3.16: Cross sections for the three waves for different pump intensities from simulations which use pumps with $\ell = 2$ and $\ell = 3$. The top half shows the results for $\ell = 2$ and the bottom half shows results for $\ell = 3$. The top row of each half shows the real part of the pump wave, the second row shows the real part of the scattered light wave, and the third row shows the real part of the plasma wave. Each column corresponds to a different pump intensity. The intensities are listed at the top and are equal to 1×10^{16} , 2×10^{16} , 3×10^{16} , 4×10^{16} , and $5 \times 10^{16} \text{ W/cm}^2$. Each plot is on a different color scale in order to maximize visible detail. Colorbars are excluded for brevity.

and radial structure of b_1 are still complicated. As the reflectivity increases, the spiraling shapes of the backscatter wave and EPW evolve. The waves also diffract as they propagate whereas the slice shown in the figure is at a particular value of z .

We have demonstrated that the daughter waves generated from spontaneous SRS are not pure Laguerre-Gaussian modes. In the next section, in order to isolate the coupling between specific OAM modes of the pump and backscattered light, we examine the amplification of a well-defined seed laser.

3.4 Growth of a well-defined seed

So far we have explored spontaneous SRS from a well-defined pump and from a broad noise source for the EPW. In this section, we examine the amplification of a smaller seed laser with a well-defined OAM (a pure LG mode) to understand the resonance of individual modes in the backscattered light. We modify the initialization of the simulations to include a counter-propagating seed laser launched from the right-hand boundary. This also permits the exploration of the coupling between different OAM pumps and seeds in a precise manner and allows examining the local conservation of OAM, and the evolution of each wave's OAM index in a more controlled way. For simplicity the spot size of both the pump and seed are $w_{0,1} = 4\lambda_0$ and $k_1 = 2\pi/\lambda_1$ is determined through ω and k matching based on the density ($n_0 = 0.14n_c$) and temperature ($T_e = 4.72$ keV). The pump and seed are both focused at the center of the box at $z = 0$. The intensity of the seed laser is 10^{-6} times the intensity of the pump laser in all cases. We explored a range of pump intensities from 10^{15} W/cm² to 2.9×10^{16} W/cm².

We compare the growth of a seed to linear gain calculations using the growth rates derived in Chapter 2. We examine the gain of 1D SRS in terms of $R = k_1|b_1(z_{min})|^2$ and $b = k_1|b_1(z_{max})|^2$ where $R/b = |b_1(z_{min})|^2/|b_1(z_{max})|^2$ is the ratio of the intensity of the seed at z_{min} (post-amplification) and the intensity of the seed at z_{max} (pre-amplification). The

reflectivity, R , is calculated from a modified version of equation 2.187,

$$R = \frac{b}{1 - R + b} e^{(1-R)\bar{\alpha} \int dz 2\beta_s} \quad (3.34)$$

where β_s is the 1D spatial growth rate which we now heuristically allow to depend on the local pump amplitude, and $\bar{\alpha}$ is a fitting parameter (which differs from α we used earlier because in equation 2.187 we let the argument in the exponential be $\alpha 2\beta_s L$ where β_s was determined with the peak intensity). The shape of the pump laser determines $\beta_s = \beta_s(\mathbf{x}_\perp, z)$ as a function of position and the noise level b comes from the shape of the seed at z_{max} . Therefore, we can calculate $R = R(\mathbf{x}_\perp)$ for every point in the transverse plane.

We assume every point in the plane (rings for azimuthal symmetry) grows independently of its neighbors in order to estimate an average gain, G_{eff} . For a seed with an initial transverse normalized intensity profile $b(\mathbf{x}_\perp)$, we construct a weighted average of the amplified seed intensity. The 3D gain, G_{eff} , is a combination of growth rate calculations at each location in the transverse plane, which is defined as

$$G_{eff} \equiv \ln \left(\frac{R_{3d}}{b_{3d}} \right) = \ln \left[\frac{\int d\mathbf{x}_\perp R(\mathbf{x}_\perp)}{\int d\mathbf{x}_\perp b(\mathbf{x}_\perp)} \right] \quad (3.35)$$

where $R(\mathbf{x}_\perp)$ comes from solving equation 3.34. The integral becomes a sum over discrete cells in the simulation. Thus, we can estimate a gain coefficient when the pump and seed have different transverse profiles, which correspond to different OAM. When diffraction arises, each ring does not behave independently, which is why we need the fitting parameter $\bar{\alpha}$.

If R is small, then $R \approx b e^{\bar{\alpha} \int dz 2\beta_s} = b e^G$ where $G \equiv \bar{\alpha} \int dz 2\beta_s$, and equation 3.35 becomes

$$G_{R \ll 1} \equiv \ln \left(\frac{R_{3d}}{b_{3d}} \right) \approx \ln \left[\frac{\int d\mathbf{x}_\perp b(\mathbf{x}_\perp) e^{G(\mathbf{x}_\perp)}}{\int d\mathbf{x}_\perp b(\mathbf{x}_\perp)} \right] \quad (3.36)$$

From the simulation data, we measured the gain of the seed as $G = \ln[|b_1(z_{min})|/|b_1(z_{max})|]$, using the seed amplitude at the left and right sides of the simulation. We applied this equation to the data to find $\bar{\alpha}$ and then compared the theory and simulation results for G and G_{eff} .

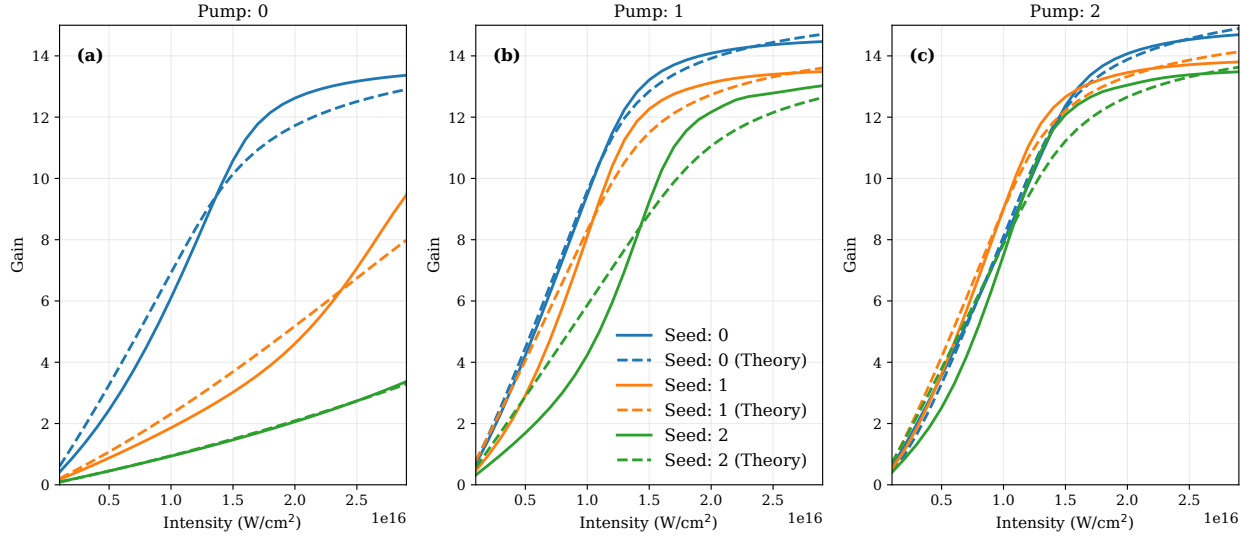


Figure 3.17: The solid lines show the gain from simulations with pumps and seeds with different OAM. Panel (a) shows cases where the pump laser has no OAM. The blue lines are for a seed with no OAM, the orange lines for a seed with $\ell_1 = 1$ and the green lines for a seed with $\ell_1 = 2$. Panels (b) and (c) show the same for pumps with $\ell_0 = 1$ and $\ell_0 = 2$. The dashed lines are the calculated predictions from equations 3.35 and 3.34. The fitting parameters are in the text.

Figure 3.17 summarizes the results for pumps with ℓ_0 equal to 0, 1, or 2 and seeds with ℓ_1 equal to 0, 1, or 2. The dashed lines show the calculation from equations 3.34 and 3.35. The solid lines show the simulation data for the combinations of pumps and seeds. Panel (a) shows the results for a pump with no OAM, panel (b) shows the results for a pump with $\ell_0 = 1$ and panel (c) shows the results for a pump with $\ell_0 = 2$.

The value of the fitting parameter $\bar{\alpha}$ varies for each pump and seed combination. For the $\ell = 0$ pump, $\bar{\alpha}$ is 0.69 when the seed has $\ell_1 = 0$, 0.38 when the seed has $\ell_1 = 1$, and 0.32 when the seed has $\ell_1 = 2$. When the pump has $\ell_0 = 1$, the value of $\bar{\alpha}$ is 0.82 for a seed with $\ell_1 = 0$, 0.76 for a seed with $\ell_1 = 1$, and 0.60 for a seed with $\ell_1 = 2$. Finally, for a pump with $\ell_0 = 2$, $\bar{\alpha}$ is 0.74, 0.81, and 0.75 for seeds with $\ell_1 = 0, 1$, and 2 respectively. Determining

how to choose $\bar{\alpha}$ before a simulation is performed is left for future work.

3.4.1 Spatial shape and OAM indices

In this section, we present more results from simulations with pump and seed combinations with $\ell_0 = 0, 1, 2, 3$ where all lasers have w_0 of $4\lambda_0$ and the box length is $240\lambda_0$. We also show details on the spatial profiles of each wave and how OAM is transferred between the three waves.

Figure 3.18 shows the intensity of the pump and seed, and the density of the plasma wave for a simulation where neither the pump laser nor seed laser has OAM. The right column shows a longitudinal slice along $x = 0$. In the left column, we show a transverse slice of each wave. This slice corresponds to the dashed white line in the plots in the right column, at approximately $z = -108\lambda_0$. The seed reaches its maximum intensity due to SRS amplification as it exits the plasma at the left boundary around $z = -112\lambda_0$. It is seen that as expected all three waves have a similar profile with an intensity peak on axis.

Next, in figure 3.19, the pump and seed both have OAM and $\ell_0 = \ell_1 = 1$. All three waves have no intensity on axis as is the case for pure waves with OAM. However, the plasma wave does not necessarily have OAM, even though the density forms a ring in the cross-sectional slice. Recall, in order to have OAM the phase of the wave (not intensity) must vary in φ .

We also simulate Raman amplification of a seed which has a different OAM than the pump. Figure 3.20 shows the wave envelopes for a simulation where the pump does not have OAM ($\ell_0 = 0$) and the seed has $\ell_1 = 1$ initially. Due to SRS, the shape of the seed evolves to a transverse profile which is no longer radially symmetric. The resulting shape suggests that the seed is no longer an OAM mode with $\bar{\ell}_1 = 1$. The plasma wave develops a similar transverse shape and the effective OAM index, $\bar{\ell}_2$, grows in order to conserve OAM of the three waves and compensate for changes to $\bar{\ell}_1$.

When the OAM combination of the pump and seed is reversed, i.e., $\ell_0 = 1$ and $\ell_1 = 0$,

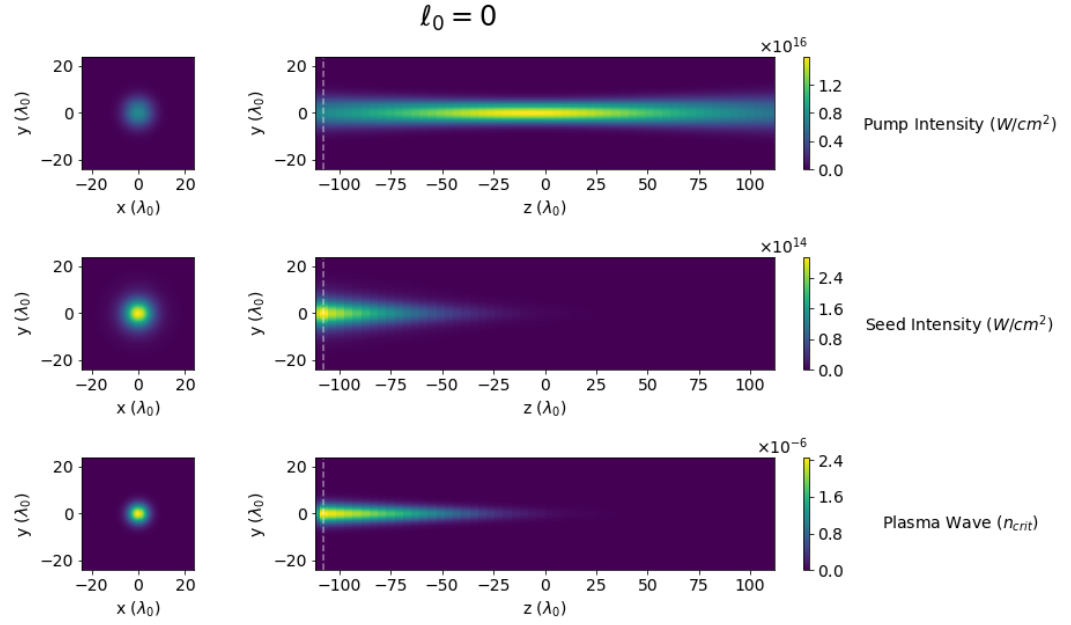


Figure 3.18: Two dimensional slices in (x, y) and (z, y) of the pump, seed, and EPW. The pump and seed both have $\ell = 0$ (No OAM). The (z, y) slice is shown in the right column, which is a slice at $x = 0$. The left column is an (x, y) slice at the location shown by the dashed white line on the right plot, near $z = -108\lambda_0$.

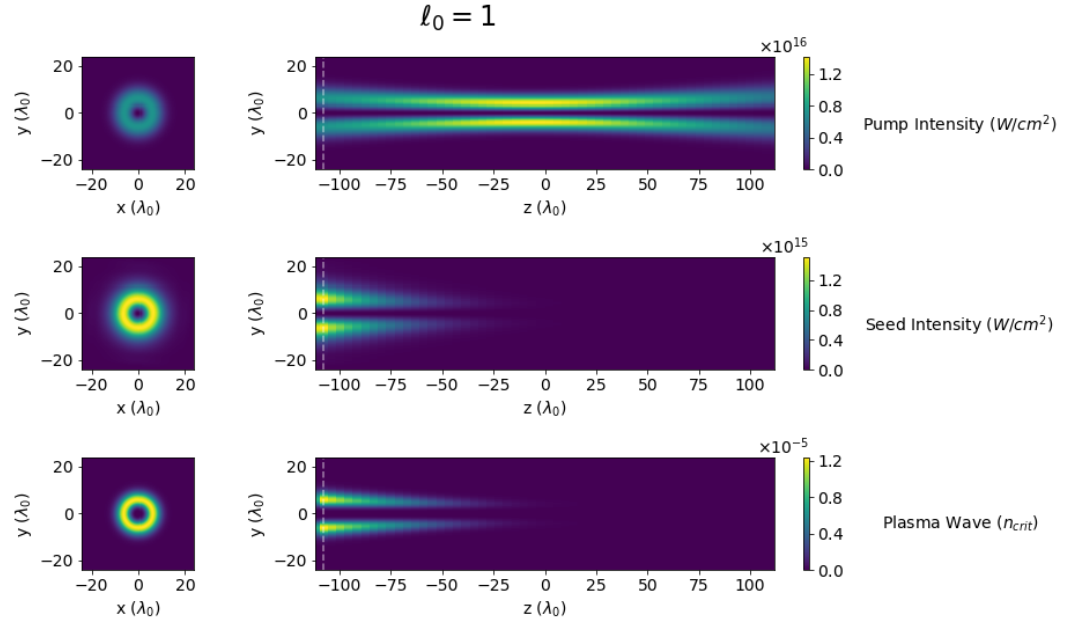


Figure 3.19: Two dimensional slices in (x, y) and (z, y) of the pump, seed, and EPW. The pump and seed both have $\ell = 1$. The (z, y) slice is shown in the right column, which is a slice at $x = 0$. The left column is an (x, y) slice at the location shown by the dashed white line on the right plot, near $z = -108\lambda_0$

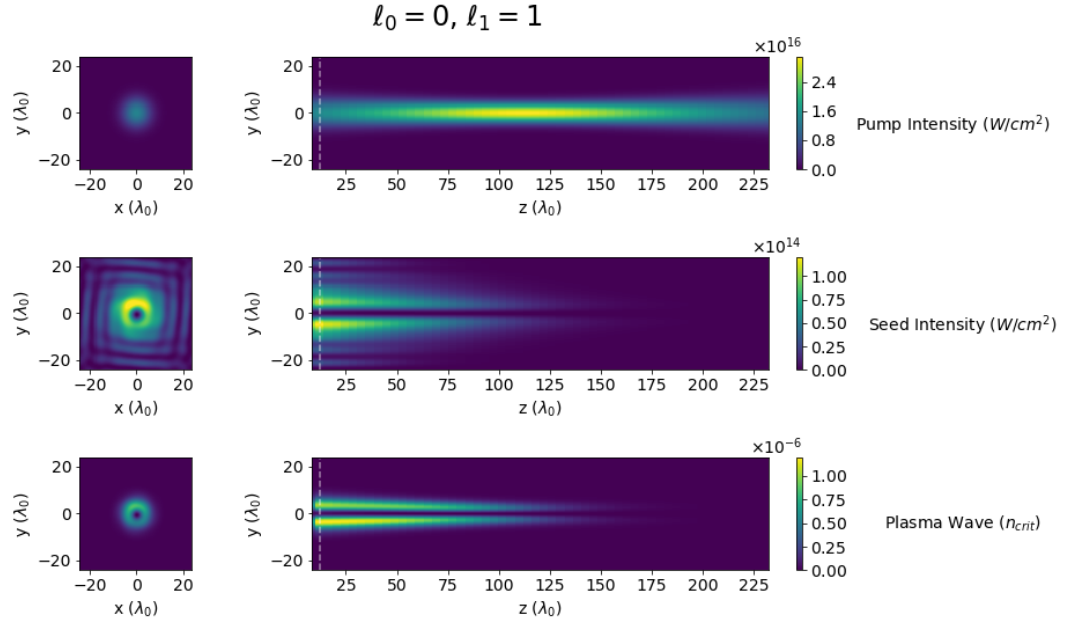


Figure 3.20: Same plots as in figures 3.18 and 3.19 except the pump laser does not have OAM but the counter-propagating seed has $\ell_1 = 1$. The plasma wave develops OAM. The amplified seed and plasma wave profiles in both (x, y) and (z, y) are not radially symmetric, indicating a change in OAM due to SRS.

the seed OAM does not change as much as in the previous case. However, the transverse profile of the seed is amplified in an off-axis ring as shown in Figure 3.21. The intensity on axis is not as strong but is not necessarily 0 due to the transverse group velocity of the waves.

In order to illustrate how OAM is transferred between the three waves, in figure 3.22 we plot $\bar{\ell}_j$ for each wave. As the seed grows it does not remain a pure OAM mode. Figure 3.22 shows the change in OAM when the pump has no OAM and the seed has $\ell_1 = 1$ or $\ell_1 = 2$. When the pump is small, the plasma wave has OAM opposite that of the seed in order to conserve OAM. As the seed grows, it loses OAM due to the non-OAM pump and it becomes an $\ell = 0$ mode. The plasma wave also evolves to a mode with no OAM. However, this transition happens at a slightly different axial position. Therefore the total OAM, $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ is nonzero for a small region (where the $\bar{\ell}_j$'s are not slowly evolving) as shown in the figure.

In figure 3.23 we plot $\bar{\ell}_j$ of each wave for simulation where the pump has $\ell_0 = 1$ and the seed has $\ell_1 = 0$ or $\ell_1 = 2$. In the left panel, the seed maintains its lack of OAM and $\bar{\ell}_1 \approx 0$. The OAM of the plasma wave is approximately equal to the OAM of the pump, $\bar{\ell}_2 \approx \ell_0$, and the pump OAM appears unchanged. Thus, OAM conservation holds across the entire simulation region. When the seed has $\ell_1 = 2$ (panel (b)), the plasma wave develops $\bar{\ell}_2 = -1$ on the right-hand side of the box where the OAM of the seed is also the same as its original value, $\bar{\ell}_1 = \ell_1 = 2$. Additionally, the pump loses some OAM in this region and $\bar{\ell}_0$ decreases by about 20%. On the left side of the box where the seed amplitude is larger, the OAM of the pump is constant at its original value, the OAM of the seed tends towards the OAM of the pump, and the OAM of the EPW increases toward 0. OAM is not as well conserved on the left side of the simulation region, as well as the far right side which is shown by the red line. Panel (b) shows a more complicated distribution of OAM compared to panel (a) where the seed does not have OAM, and figure 3.22 where the pump does not have OAM.

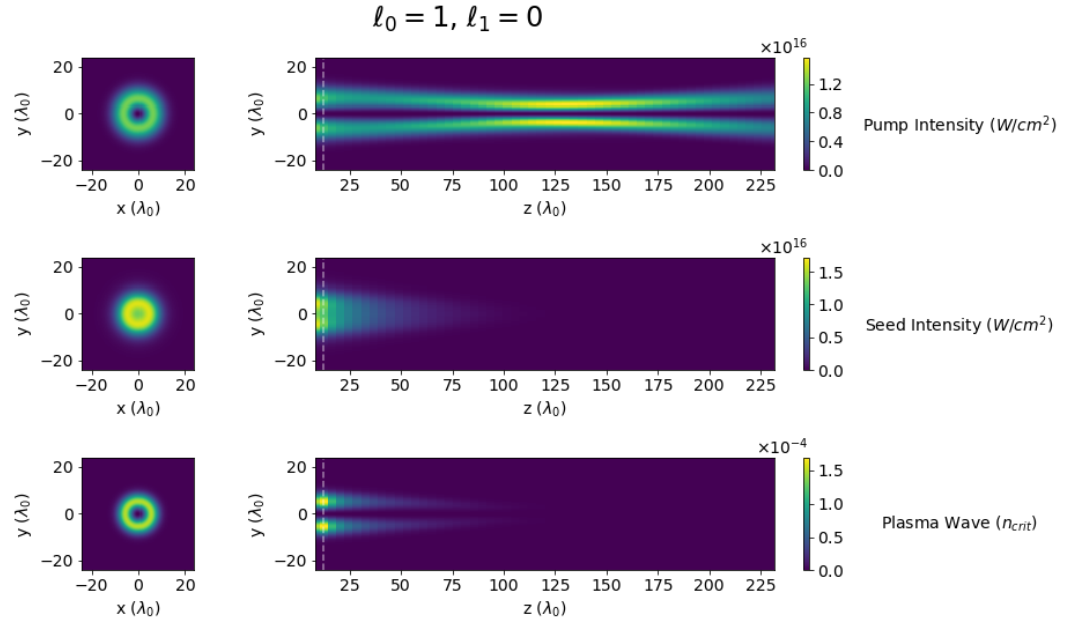


Figure 3.21: Same plots as in figures 3.18 - 3.20 except a seed without OAM is amplified by an $\ell_0 = 1$ pump. The plasma wave develops OAM, and the shape of the seed is modified.

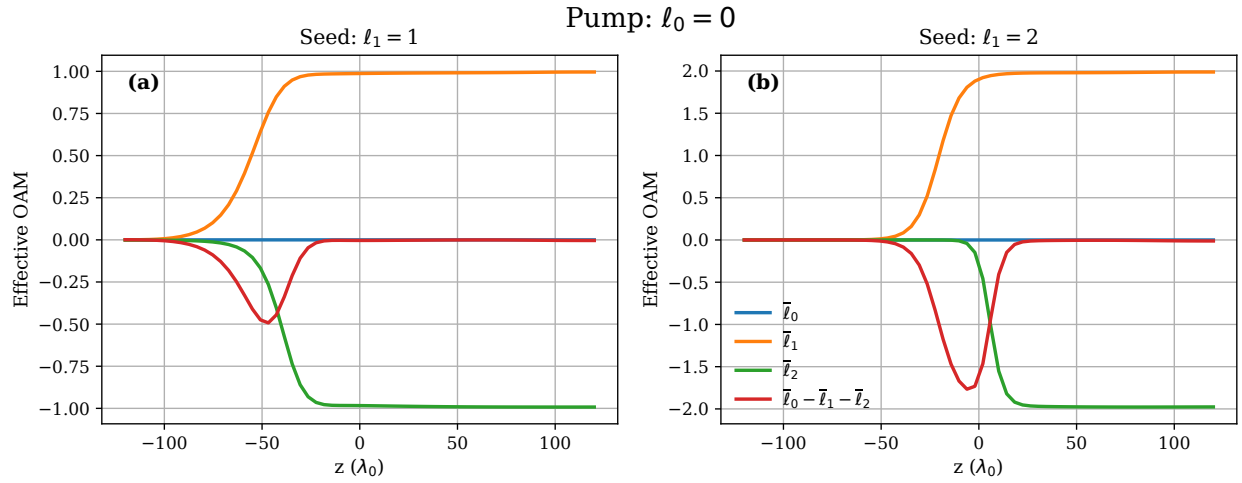


Figure 3.22: The effective OAM of the seed and plasma wave changes when the pump has no OAM. As the seed grows, it loses OAM. The plasma wave initially has OAM opposite the seed and then it also evolves toward zero. The intensity for the pump is $4.9 \times 10^{16} \text{ W/cm}^2$.

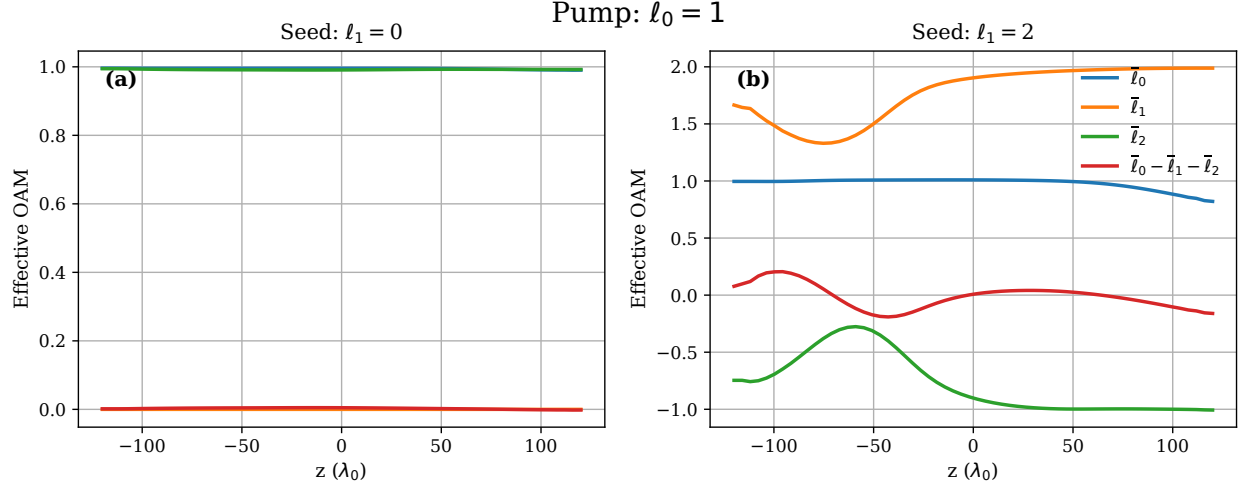


Figure 3.23: The effective OAM when the pump has $\ell_0 = 1$ and the seed has either $\ell_1 = 0$ (a) or $\ell_1 = 2$ (b). The intensity in both cases is $4.9 \times 10^{16} \text{ W/cm}^2$. The OAM of the seed does not change into the OAM of the pump like in the $\ell_0 = 0$ case.

3.5 Summary

This chapter presented the results of pF3D simulations of SRS of pumps with and without OAM where the instability grows from noise and from a seed. We discussed the use of an analytical solver to obtain the normal EPW modes where the frequency ω_2 is complex in the kinetic dispersion relation and all other frequencies and wavenumbers are real. The negative imaginary part of ω_2 is the Landau damping rate which we use along with the peak intensity of the pump to determine the damping regimes discussed in Chapter 2 for both temporal and spatial growth. We compared results for reflectivity to the thresholds for transition from the weakly-damped to strongly-damped regime.

It was also shown that lasers with OAM can be compared at the same power and w_0 , same peak intensity and w_0 , or same power and peak intensity where we introduced the effective width w_ℓ . The spatial growth rate β_s is a function of intensity and can therefore be considered a local quantity. We used $\int \beta_s dz$ to predict the amount of SRS for every

(x, y) and combined the results to determine an average theoretical value. We introduced the fitting parameter α to modify the growth rate and, in general, reduce it.

We analyzed many pF3D simulations to scan a range of intensities as well as temperatures while keeping the density constant but varying $k_2\lambda_D$. We found that lasers with more OAM produce lower reflectivity when compared at the same power. However, when compared at the same peak intensity, it is the non-OAM laser which has the lowest reflectivity. This trend holds over variation in $k_2\lambda_D$ for a single intensity, and when intensity is varied for a single $k_2\lambda_D$. We showed that the difference between reflectivity with and without OAM is reduced when the spot size is $8\lambda_0$ instead of $4\lambda_0$.

We examined the OAM of each wave and found the $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ is best conserved where the intensity of each wave is large. Next, we considered Raman amplification of a seed where the seed can also have OAM which may or may not match that of the pump. We showed that seeds without OAM tend to have the greatest gain. Finally, we showed that the OAM of the seed and plasma wave can change along z as reflectivity grows.

The large ensemble of pF3D simulations we have presented lays the groundwork for comparison to a small number of OSIRIS simulations in the following chapter.

CHAPTER 4

Particle-in-Cell Simulations of Stimulated Raman Scattering with Orbital Angular Momentum

4.1 Introduction

In Chapter 3, fluid simulations were conducted to investigate how the use of pump lasers with OAM modified the behavior of SRS in the regime of high $k_2\lambda_D$ where linear Landau damping is important. The only nonlinearity included within pF3D is pump depletion. In Chapter 2, the concept of kinetic inflation of SRS was also reviewed, where nonlinear modifications to the damping of the EPW can enhance SRS; these modifications can also modify the real part of the frequency and detune SRS. In this chapter, results from fully kinetic particle-in-cell (PIC) OSIRIS simulations of SRS of lasers with OAM are presented. The central goal of the PIC simulations is to confirm whether, when power is held fixed, OAM reduces SRS, if OAM is transferred from the pump to the daughter waves when kinetic effects are included, and if there are noticeable differences between the wave-particle interactions within the EPW arising from SRS driven by pumps with and without OAM. In PIC simulations, the motions of individual particles are followed self-consistently. Unlike the large parameter scans conducted with pF3D, the extreme computational cost of 3D particle-in-cell (PIC) simulations limited our study to a few detailed cases. Each simulation provides great detail at high spatial and temporal resolutions, allowing us to study the small-scale kinetic effects that change SRS of waves with OAM.

Case	a_0	Intensity (W/cm ²)	Power (GW)	$\frac{\gamma_0}{\nu}$	$\frac{\gamma_0}{\nu} \sqrt{\frac{v_{g2}}{v_{g1}}}$
$\ell_0 = 0$					
A	0.036	1.4×10^{16}	0.45	2.52	0.78
B	0.043	2.1×10^{16}	0.64	3.01	0.94
C	0.054	3.2×10^{16}	1.00	3.78	1.18
D	0.065	4.7×10^{16}	1.46	4.55	1.42
$\ell_0 = 1$					
E	0.030	1.0×10^{16}	0.84	2.10	0.65
F	0.040	1.8×10^{16}	1.50	2.80	0.87
G	0.053	3.1×10^{16}	2.63	3.71	1.15
H	0.060	4.0×10^{16}	3.37	4.20	1.31
I	0.073	5.9×10^{16}	4.99	5.11	1.59

Table 4.1: Summary of parameters for SRS simulations with and without OAM. Individual simulations are labeled A through I. The antenna in simulations with OAM uses $\ell_0 = 1$ and those without OAM use $\ell_0 = 0$.

4.2 Simulation setup

The simulations were performed using the OSIRIS framework [FST02], a fully relativistic, three-dimensional, parallelized PIC code. OSIRIS models the plasma response to an incident light wave from first principles by solving Maxwell’s equations for the electromagnetic fields and the Lorentz force equation for the charged particles. SRS emerges from the evolution of these fundamental equations as was described theoretically in Chapter 2.

Table 4.1 explains the nine PIC simulations carried out to generate the data described in this chapter. Table 4.2 lists parameters that were unchanged for all simulations. We used a laser spot size of $w_0 = 4\lambda_0$ and we modeled SRS growth from noise using pump beams with

Parameter	Normalized	Physical
Density	$0.14 n_c$	$1.26 \times 10^{21}/\text{cm}^3$
Thermal Velocity	$0.0888 c$	4.03 keV
Box Length (x_1)	$625 c/\omega_0 \approx 100 \lambda_0$	35.0 μm
Box Width (x_2 and x_3)	$200 c/\omega_0$	11.2 μm
Wavelength	$6.28 c/\omega_0$	0.3512 μm
Spot Size	$4 \lambda_0$	1.40 μm
Max Time	$5100 \omega_0^{-1}$	0.95 ps
Debye Length	$0.237 c/\omega_0$	0.0133 μm
Time step	$0.17 \omega_0^{-1}$	0.0317 fs
Cell size	$0.295 c/\omega_0$	0.0166 μm
Particles per cell	64	

Table 4.2: Parameters used for all simulations in both normalized and physical units. The box length is $100\lambda_0$, the spot size is $4\lambda_0$, and the box width is $8w_0$ or $32\lambda_0$.

either a Gaussian ($\ell_0 = 0$) or a Laguerre-Gaussian ($\ell_0 = 1$) profile. The plasma density is $n_e = 0.14n_c$, and the temperature is $T_e = 4.03$ keV which corresponds to an electron thermal velocity of $v_e = 0.0888c$. For this density and temperature, a numerical solver predicts $k_2\lambda_D = 0.32$. Even with numerical dispersion corrections, this is the approximate value seen in the PIC simulations.

In all cases, the box length is $625 c/\omega_0$ which is about $100 \lambda_0$ or $2z_R$. There are 64 particles per cell and $678 \times 678 \times 2121$ cells. Thus, the total number of particles is 6.24×10^{10} and the cell size is $0.295 c/\omega_0$ in the propagation direction, corresponding to ≈ 24 cells per laser wavelength in the plasma. The transverse size of the simulation box is $200 c/\omega_0$ ($\approx 8w_0$ and $32\lambda_0$); the cell size is also $0.295 c/\omega_0$ in the transverse direction. We use quadratic particle shapes and the simulation boundaries are periodic in the two transverse directions, open for light waves, and a thermal bath in the axial direction. The Debye length is about $0.237 c/\omega_0$ so the cell size is about $1.25 \lambda_D$ (≈ 15 cells per EPW wavelength) in each direction. The time step is $0.17 \omega_0^{-1}$ which is 0.0317 fs. Most simulations have a maximum time of 0.95 ps which is $5100 \omega_0^{-1}$ and 3×10^4 iterations. The two lowest amplitude simulations were run for twice as long, i.e., 1.90 ps, $10200 \omega_0^{-1}$, and 6×10^4 iterations. For comparison, recall that pF3D simulation times are greater than 10 ps and the box length is up to $240\lambda_0$ with a cell size of $4\lambda_0$. In this chapter, we also present pF3D results for simulations with the same domain size as for the OSIRIS simulations.

The laser is launched from an antenna which stays on during the entire simulation. The nine different cases are summarized in table 4.1. For the cases without OAM ($\ell_0 = 0$), we used a laser with a peak value of a_0 of 0.036 , 0.043 , 0.054 , and 0.065 , which correspond to peak intensities of 1.4×10^{16} , 2.1×10^{16} , 3.2×10^{16} , and 4.7×10^{16} W/cm² and powers of 0.45 , 0.64 , 1.0 , and 1.5 GW for the laser spot size. The ratios of the temporal growth rate to the Landau damping rate (predicted by a numerical solver) are 2.52 , 3.01 , 3.78 , and 4.55 . The value of $\frac{\gamma_0}{\nu} \sqrt{\frac{v_{g2}}{v_{g1}}}$ is 0.78 , 0.94 , 1.18 , and 1.42 . For the cases with OAM, the laser has $\ell_0 = 1$ and the values of peak a_0 we explored are 0.030 , 0.040 , 0.053 , 0.060 , and 0.073 . These peak

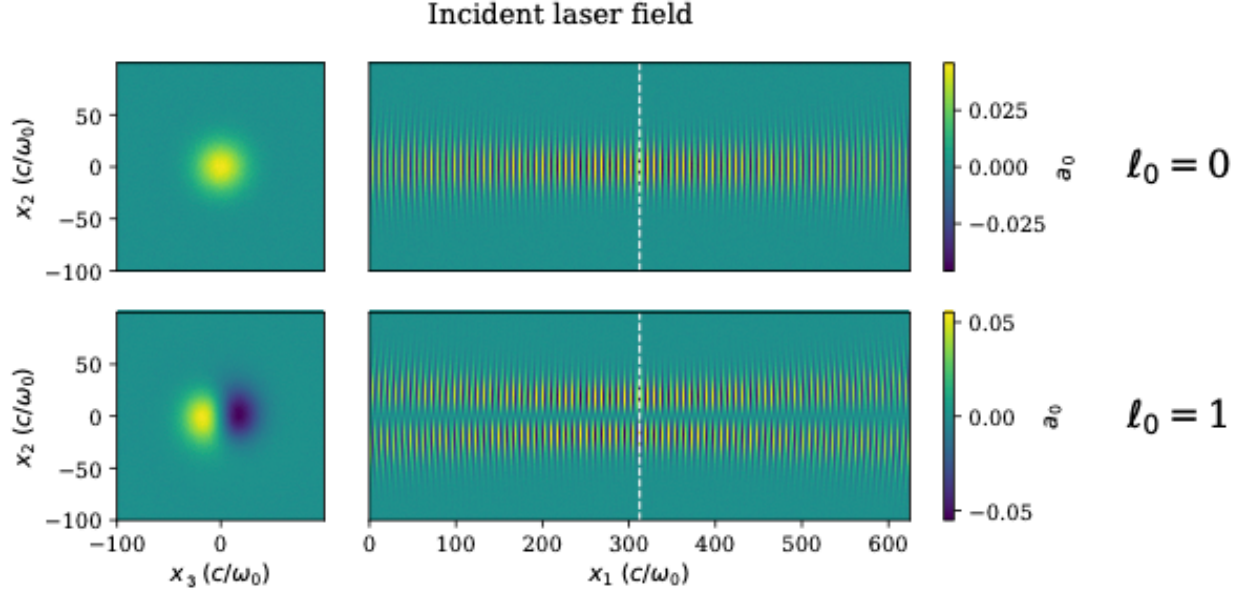


Figure 4.1: Electric field (in the x_3 direction) of the incident pump lasers early in time. The plots are slices in (x_2, x_3) at x_1 shown by the white dashed line and in (x_1, x_2) at $x_3 = 0$. A Gaussian laser with no OAM (top) and a Laguerre-Gaussian mode with $\ell_0 = 1$ (bottom). The right column shows the wave propagation and the left column shows a transverse cut taken at best focus which is shown as the dashed white line on the right side plots.

amplitudes correspond to an intensity of 1.0×10^{16} , 1.8×10^{16} , 3.1×10^{16} , 4.0×10^{16} , and 5.9×10^{16} W/cm² and a power of 0.84, 1.50, 2.63, 3.37, and 4.99 GW respectively. Recall that the ratio of power to peak intensity also depends on the OAM index of the laser, ℓ_0 , even when w_0 is constant. For the cases with OAM, γ_0/ν is 2.10, 2.80, 3.71, 4.20, and 5.11, and $\frac{\gamma_0}{\nu} \sqrt{\frac{v_{g2}}{v_{g1}}}$ is 0.65, 0.87, 1.15, 1.31, and 1.59 respectively. Thus, the simulations explore the strongly and weakly damped spatial regime as described in Chapter 2.

We confirmed that the antenna launches a Laguerre-Gaussian beam. Figure 4.1 shows the electric field of the incident pump waves at the start of the simulations. The incident light propagates through the entire simulation region and then maintains its maximum amplitude well before reflectivity occurs. The top row of figure 4.1 corresponds to a case without OAM.

The upper left plot shows a transverse slice of the field at best focus and the upper right plot shows a slice in the (x_1, x_2) plane along $x_3 = 0$ after the wave has propagated across the entire box. The dashed white line represents the location of the transverse slice (in the (x_2, x_3) plane). The lasers are focused at the center of the box. The bottom row shows the same slices for a case where the antenna launches a Laguerre-Gaussian OAM laser mode with $\ell_0 = 1$. The pattern of an LG $\ell = 1$ mode in which there is a null in the field on axis is seen in both slices. Although not shown, we verified that the fields have the expected Rayleigh length. The plots show the value of a_0 or equivalently v_{osc}/c . The maximum value for the case without OAM is 0.041 and the maximum value for the case with OAM (bottom row) is 0.052.

4.3 Diagnostic Methods

To extract physical quantities from the raw field and particle data generated by OSIRIS, we employed several diagnostic techniques. Some of these techniques or similar methods have been broadly utilized in two-dimensional OSIRIS simulations of laser-plasma instabilities including SRS. Three-dimensional OSIRIS simulations of laser-plasma instabilities are not as common, so we discuss the diagnostics in more detail, including those developed for this research on angular momentum and OAM of light waves.

4.3.1 Measuring Reflectivity

In order to measure reflectivity in OSIRIS, we start from the cold plasma limit of equation A.80,

$$\frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} + \frac{1}{2} m n v^2 \right) + \nabla \cdot \left(\frac{c(\mathbf{E} \times \mathbf{B})}{4\pi} + \mathbf{v} \frac{m n v^2}{2} \right) = 0. \quad (4.1)$$

To order of the field amplitudes squared, there is no plasma kinetic energy flux, so the total wave energy flux comes entirely from the Poynting flux for $a_0 \lesssim 1$,

$$\mathbf{S} \equiv \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}). \quad (4.2)$$

The Poynting flux measures the flow of energy and electromagnetic momentum, $\mathbf{G}$ (see Appendix A, $\mathbf{G} = \mathbf{S}/c^2$). The incident laser energy flows predominantly in the $+x_1$ direction and, when integrated across x_2 and x_3 , is constant when pump depletion is negligible. Energy in the scattered wave flows in the opposite direction. We measure the time-averaged change in S_1 to analyze reflectivity,

$$\langle S_1 \rangle = \frac{c}{4\pi} \langle E_2 B_3 - E_3 B_2 \rangle \quad (4.3)$$

The time averaging over several periods of both counter-propagating light waves which have different frequencies, ω_0 and ω_1 produces a smooth signal. Next, we integrate over the transverse plane to reduce the data to the $\mathbf{x}_1$ dimension and then select a particular slice in x_1 in order to identify a single time-series. The pump laser creates positive $\langle S_1 \rangle$, and the bursts of SRS have negative $\langle S_1 \rangle$. The net time averaged Poynting flux is the sum,

$$\langle S_1 \rangle = \langle S_1 \rangle_{pump} + \langle S_1 \rangle_{SRS} \quad (4.4)$$

The reflectivity is normalized to the incident pump. When determining the reflected light we subtract the Poynting flux the pump would have had in the plasma if there had not been SRS, and call this the unperturbed pump which is constant in time. Thus to isolate the effects of SRS we calculate the difference between the net time-averaged Poynting flux and that caused by the pump alone. We calculate the constant pump flux level directly from the simulations by measuring it at early times when there is no SRS,

$$\langle S_1 \rangle_{pump}(\mathbf{x}, t) \approx \langle S_1 \rangle(\mathbf{x}, t = 0) \quad (4.5)$$

Thus, the flux due to SRS is $\langle S_1 \rangle_{pump}$ subtracted from the total flux,

$$\langle S_1 \rangle_{SRS}(\mathbf{x}, t) = \langle S_1 \rangle(\mathbf{x}, t) - \langle S_1 \rangle(\mathbf{x}, t = 0) \quad (4.6)$$

and $\langle S_1 \rangle_{SRS}$ is negative because light is scattered in the $(-\hat{\mathbf{x}}_1)$ direction. We examine a single value of x_1 near the entrance plane of the laser and integrate over x_2 and x_3 to find the flux at a particular plane over time. Additionally, we normalize $\langle S_1 \rangle_{SRS}$ to $\langle S_1 \rangle_{pump}$ integrated over the same plane to determine the reflectivity,

$$\tilde{R}(t) = \frac{-\int d\mathbf{x}_\perp \langle S_1 \rangle_{SRS}}{\int d\mathbf{x}_\perp \langle S_1 \rangle_{pump}} = 1 - \frac{\int d\mathbf{x}_\perp \langle S_1 \rangle(x_1 = 0, \mathbf{x}_\perp, t)}{\int d\mathbf{x}_\perp \langle S_1 \rangle(x_1 = 0, \mathbf{x}_\perp, t = 0)} \quad (4.7)$$

where we note that $\tilde{R} = (E_1 B_1)/(E_0 B_0) = (k_1 \omega_1 b_1^2)/(k_0 \omega_0 b_0^2) = (\omega_1/\omega_0)R$ where the subscripts refer to the wave and not the component of the field and R was defined in Chapter 2. Equation 4.7 allows us to compare simulations with $\ell_0 = 0$ and $\ell_0 = 1$ because of the integration over $\mathbf{x}_\perp$. We calculate the average reflectivity, $\langle \tilde{R} \rangle$, by averaging over the entire simulation time which can capture multiple reflectivity spikes representing bursts of SRS. The average value is sensitive to the maximum time of the simulation. We note that when an EM wave exits a plasma that $\mathbf{E}$ does not change but $\mathbf{B} \rightarrow \mathbf{E}$ and thus $\mathbf{E} \times \mathbf{B} \rightarrow |\mathbf{E}|^2$. Thus, in vacuum the reflectivity will be $\frac{E_1^2}{E_0^2} = \frac{\omega_1 k_0}{\omega_0 k_1} \tilde{R} = \frac{\omega_1^2 b_1^2}{\omega_0^2 b_0^2}$. In what follows in this chapter, we plot $\tilde{R}$ for both the OSIRIS and pF3D data.

The finite-width reflected waves vary across $\mathbf{x}_\perp$ depending on the OAM of the pump, producing a different shape in k -space for $\ell_0 = 0$ and $\ell_0 = 1$. In combination with equation 4.7, the Fourier transform of E_1 and E_3 over time on axis for $\ell_0 = 0$ and near $r = w_0/\sqrt{2}$ for $\ell_0 = 1$ are used to measure reflectivity by filtering the field near the k of the backscattered light.

For a pump laser polarized in x_3 , the backscattered light is also in the E_3 component, and the (longitudinal) EPW appears in the E_1 component. The Fourier transforms of E_1 and E_3 show distinct signals for each wave: the incident light, the backscattered light, and the EPW. From the k -space diagnostic, we can also see nonlinear frequency shifts in the waves which we will discuss later.

4.3.2 Measuring Orbital Angular Momentum

We are interested in OAM in the propagation direction, M_1 , which depends on S_2 and S_3 and therefore on every component of the electric and magnetic fields. For an EM wave, the dominant field is $\mathbf{E}_\perp$ (the $\hat{x}_2$ and $\hat{x}_3$ directions for our geometry), but there is a small E_1 component to satisfy Gauss's law or the Coulomb gauge condition ($\nabla \cdot \mathbf{A} = 0$ when using $\mathbf{A}$),

$$E_1 = \frac{i}{k} \nabla_\perp \cdot \mathbf{E}_\perp \quad (4.8)$$

Additionally, the magnetic field is

$$\mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E} \quad (4.9)$$

where to a high degree of accuracy $\mathbf{k} \approx k\hat{x}_1$. The Poynting flux, which is the cross product of $\mathbf{E}$ and $\mathbf{B}$,

$$\mathbf{E} \times \mathbf{B} = \mathbf{E} \times \left(\frac{1}{\omega} \mathbf{k} \times \mathbf{E} \right) = \frac{1}{\omega} |\mathbf{E}|^2 \mathbf{k} - \frac{i}{\omega} (\nabla_\perp \cdot \mathbf{E}_\perp) \mathbf{E} \quad (4.10)$$

can be split into a term which points in the propagation direction, $\mathbf{k}$, which is proportional to $|\mathbf{E}|^2$, and a term perpendicular to $\mathbf{k}$ that represents transverse energy flux. When integrated across the transverse coordinates the net transverse flux vanishes.

As is well known [SDM98], the momentum flux from the fields, $\mathbf{G} = \mathbf{S}/c^2$ so to get the momentum flux we can also use the above expression for $\mathbf{E} \times \mathbf{B}$. The field angular momentum can thus be written as,

$$\mathbf{M} = \mathbf{x} \times \frac{1}{4\pi c} (\mathbf{E} \times \mathbf{B}) \propto \frac{|\mathbf{E}|^2}{\omega} (\mathbf{x} \times \mathbf{k}) - \frac{\mathbf{E} \cdot \mathbf{k}}{\omega} (\mathbf{x} \times \mathbf{E}) \quad (4.11)$$

and in the propagation direction which is along k_1 , the term with $\mathbf{x} \times \mathbf{k}$ vanishes leaving only the term with $\mathbf{E} \cdot \mathbf{k}$. Therefore, the angular momentum is related to the product $(\mathbf{x} \times \mathbf{E}_\perp)(\nabla_\perp \cdot \mathbf{E}_\perp)$ which, in the x_1 direction, strongly resembles the expression based on the envelopes $b\partial_\phi b$.

In the previous chapter, we used equation 2.236 to calculate $\bar{\ell}_j$ in terms of the wave

envelopes,

$$\bar{\ell}_j = \frac{-\frac{i\omega_j}{2} \int (b_j^* \partial_\varphi b_j - b_j \partial_\varphi b_j^*) d\mathbf{x}_\perp}{\omega_j \int |b_j|^2 d\mathbf{x}_\perp}. \quad (4.12)$$

Identifying that $|\mathbf{E}^{(j)}|^2 \propto \omega_j^2 b_j^2$ and considering a laser polarized in x_3 , the analogous expression for the effective OAM index of wave j in terms of the fields is

$$\bar{\ell}_j = 2k_j \frac{\int d\mathbf{x}_\perp x_2 E_1^{(j)} E_3^{(j)}}{\int d\mathbf{x}_\perp |\mathbf{E}^{(j)}|^2} \quad (4.13)$$

where $E_a^{(j)}$ is the a th component of the electric field filtered for the signal of the pump ($j = 0$) and scattered ($j = 1$) light wave. The effective OAM varies in both time and along x_1 , i.e., $\bar{\ell}_j = \bar{\ell}_j(x_1, t)$. We have averaged it over several wavelengths in the x_1 direction to reduce the noise.

We estimate the OAM of the EM waves by filtering the E_1 and E_3 fields for the corresponding k and analyzing equation 4.13. For a particular k_j , the filtered version of E_a is

$$E_a^{(j)}(x_1, \mathbf{x}_\perp, t) = \mathcal{F}_{x_1}^{-1} \left[\tilde{E}_a(k, \mathbf{x}_\perp, t) \text{ for } |k - k_j| < \Delta k \right] \quad (4.14)$$

where the width of the filter, Δk , is determined through experimentation. The effective OAM indices of each wave are not necessarily integers. When launched from the antenna, the incident light has either $\ell_0 = 0$ or $\ell_0 = 1$, but as the simulation proceeds the $\bar{\ell}$ of each wave can vary to any value.

Figure 4.2 demonstrates that the OAM diagnostic gives the correct $\bar{\ell}$ when a laser is launched using the OSIRIS antenna configuration with either $\ell_0 = 0$ or 1, simulations B and G (which have $a_0 = 0.043$ and 0.053) in table 4.1. Although the signal is noisy, the OAM of the pump light according to equation 4.13 with $j = 0$ is clearly near its original value of either zero (orange) or one (green). The OAM of the background noise is also measured and found to be zero (blue). The line for the pump without OAM is less noisy than the background noise data because the background amplitude is small so the denominator in equation 4.13 is smaller than in the case where a large signal is present and measured. The

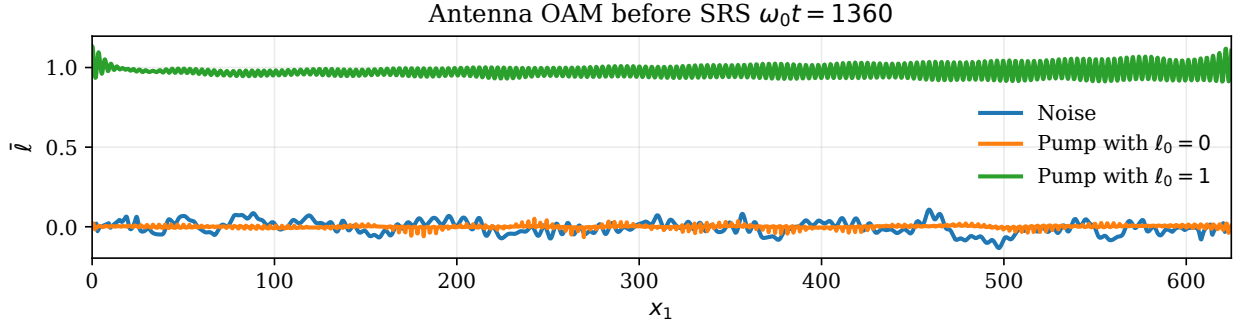


Figure 4.2: OAM of the pump laser early in time. $\bar{\ell}$ is measured for a simulation with an antenna that launches a pump with $\ell_0 = 0$ and an antenna that launches a pump with $\ell_0 = 1$. Additionally, $\bar{\ell}_1$ is measured for the background noise before any SRS signals have appeared.

OAM was measured at a time $\omega_0 t = 1360$ by which time the laser has fully propagated across the box, but for which no reflectivity has appeared.

The plasma wave OAM, on the other hand, is stored in the particles and not the fields because the wave is electrostatic (the momentum flux of the fields vanishes). At the writing of this dissertation, we do not have a satisfactory diagnostic for the OAM of the EPW. For future work, there are several options to explore. One idea would be to filter out the density perturbation at the k of the EPW and then obtain the envelope through various techniques. The OAM fluid operator could then be applied. Another option is to determine the average angular momentum per particle and divide this by the average momentum/ k . This would give the OAM number. However, the momentum of a plasma wave is proportional to the current. While a wave generates a current, this induces an electric field that tries to cancel it. As a result it would be necessary to separate out the average current from the wave and that from induced electric field. There is also a drift induced by the EM waves so this also needs to be separated out. Another option used by Blackman et al. [BNK19, BNK20, BNK22] would be to concentrate on the electrons near the phase velocity of the wave. Determining the OAM of the plasma wave from SRS simulations is an area for future work.

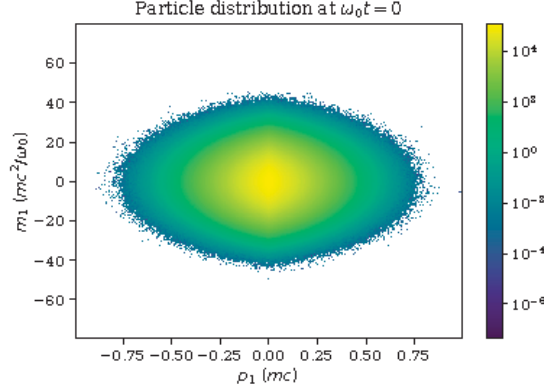


Figure 4.3: Particle phase space distribution at the start of the simulation. The horizontal axis is momentum in the x_1 direction and the vertical axis is angular momentum in the x_1 direction. The color represents the count of particles at each point in the (p_1, m_1) plane where the brightest spot is the highest concentration of particles.

We note that for data plotted in a later section, we did calculate the angular momentum in the x_1 direction for the electrons. We summed

$$m_1 = x_2 p_3 - x_3 p_2 \quad (4.15)$$

over all particles in the transverse direction to obtain the total particle OAM in the x_1 direction. As a reference, figure 4.3 shows the distribution of particles at the start of the simulation in terms of momentum, p_1 and angular momentum, m_1 . The average value of both p_1 and m_1 is zero at $t = 0$.

4.4 Results and Discussion

We now present the results of SRS from the PIC simulations, organized to first show the differences in reflectivity from pF3D and OSIRIS and the high-level impact of OAM on SRS reflectivity, followed by a detailed analysis to illustrate some of the underlying field and particle dynamics that explain these effects.

4.4.1 Comparison between pF3D and OSIRIS

For reference, we show results from pF3D for identical physical parameters as for the PIC simulations. In Chapter 3 the axial box length was $200\lambda_0$; however due to computational constraints the PIC simulations had a box length of $100\lambda_0$. We thus start by showing in figure 4.4 the reflectivity over time for pF3D simulations with the same ℓ_0 and a_0 that are simulated in OSIRIS for the box lengths of $100\lambda_0$ and $200\lambda_0$. Reflectivity curves are from pF3D simulations for the parameters of each simulation in table 4.1 for axial box lengths of both 100 and 200 λ_0 . In figure 4.4, we define the reflectivity as $\frac{\omega_1 k_1 |b_1|^2}{\omega_0 k_0 |b_0|^2}$ as discussed in section 4.3.1.

In the top row, results from pF3D simulations corresponding to simulations A-D in table 4.1 are shown. On the left, the results correspond to $L_z = 100\lambda_0$ while on the right, the box length is $L_z = 200\lambda_0$. As expected, the longer box length provides more gain and higher peak and average reflectivity. Furthermore, there is negligible reflectivity for the smallest intensities, $a_0 = 0.036$ and 0.043 on the left, and for just $a_0 = 0.036$ on the right. In the bottom row, results for pF3D simulations corresponding to simulations E-I in table 4.1 are shown where $\ell_0 = 1$ in each case. For all $L_z = 100\lambda_0$ cases, observable reflectivity is only seen for $a_0 \gtrsim 0.05$. On the other hand, observable reflectivity is seen for $a_0 \gtrsim 0.04$ for $\ell_0 = 0$ or $\ell_0 = 1$ for $L_z = 200\lambda_0$.

With these results in mind, we now compare pF3D results with those from OSIRIS. In OSIRIS the simulations were run up to $5000\omega_0^{-1}$ or about 1 ps, while in pF3D they were run up to $20\,000\omega_0^{-1}$ or about 4 ps. In each PIC simulation, the incident laser starts from $\omega_0 x_1/c = 0$ and propagates to the right ($+\hat{x}_1$). Once the laser reaches its maximum amplitude, the Poynting flux is constant. When bursts of SRS occur, the Poynting flux decreases because the backscattered light is traveling in the opposite direction of the incident laser ($-\hat{x}_1$). Averaging over several periods of both the incident and backscattered light and integrating across the transverse plane generates a smooth reflectivity curve.

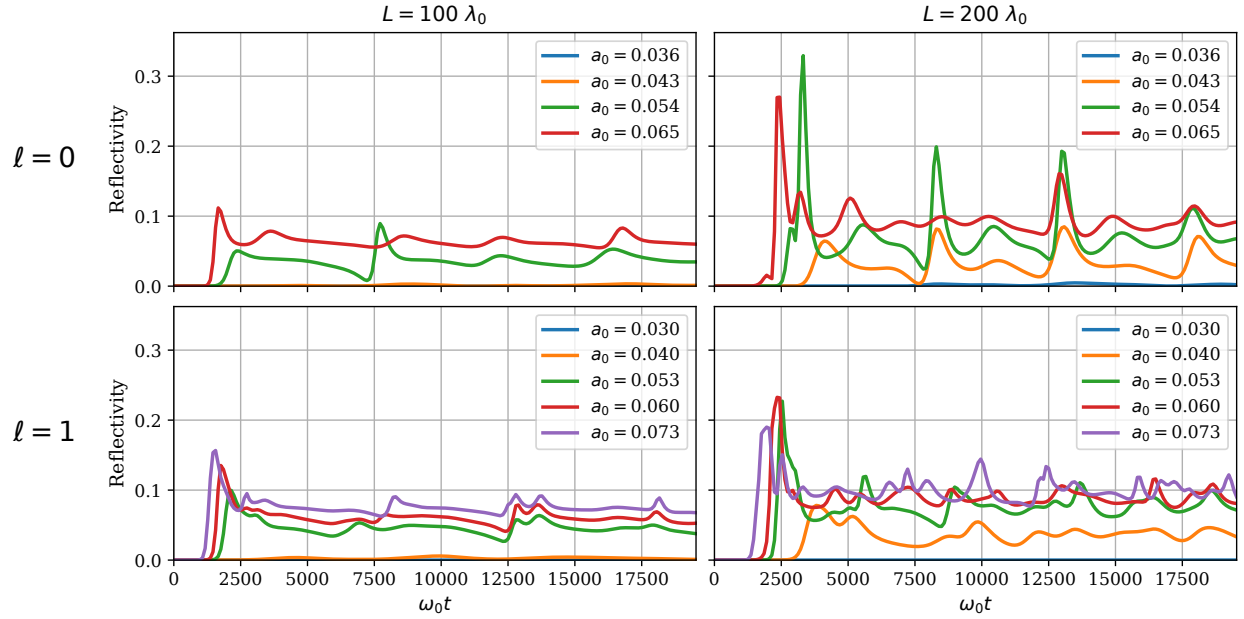


Figure 4.4: Comparison of reflectivity for pF3D simulations for each simulation in table 4.1. The top row shows results with $\ell_0 = 0$ and the bottom row corresponds to simulations where $\ell_0 = 1$. The results in left panels use a box length of $100\lambda_0$ and the right panels use a box length of $200\lambda_0$.

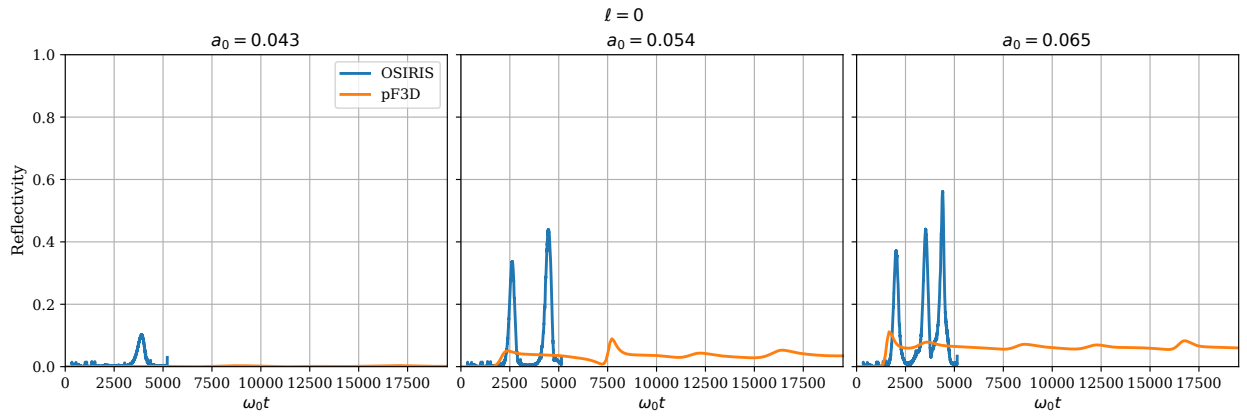


Figure 4.5: Reflectivity in OSIRIS (blue) and pF3D (orange) for simulations B-D in table 4.1. pF3D simulates longer times than OSIRIS. Reflectivity in OSIRIS comes in sharp bursts and average reflectivity is greater.

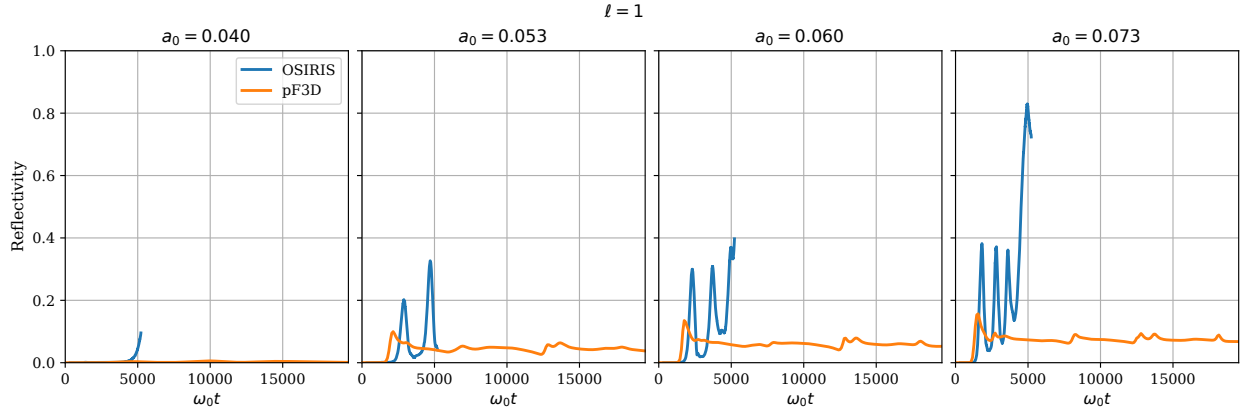


Figure 4.6: Comparison of reflectivity for OSIRIS (blue) and pF3D (orange) for simulations E-I in table 4.1. OSIRIS simulations have higher peak and average reflectivities.

A comparison of the reflectivity between OSIRIS and pF3D is shown in figures 4.5 and 4.6. It can clearly be seen that for pumps with either $\ell_0 = 0$ or 1 that the reflectivity in PIC is described by bursts which are narrower and the peak reflectivity is higher than in pF3D. The reasons for this are discussed later. In the left panel of figure 4.5 ($a_0 = 0.043$) there is no reflectivity for pF3D while there is a very small burst between 2500 and $5000 \omega_0^{-1}$ in the OSIRIS simulation. For progressively higher peak intensities the bursts in OSIRIS show more reflectivity, and there are more of them. In all cases, the reflectivity is nearly zero for the first $\approx 1000 \omega_0^{-1}$ and the first spike appears at gradually earlier times as the intensity is increased. The average reflectivity also increases in both OSIRIS and pF3D which will be quantified later. In figure 4.6, these trends are also seen for the $\ell_0 = 1$ cases. Furthermore, to make the behavior of the reflectivity clearer in the different PIC simulations, we replot the OSIRIS data in figure 4.7 where the time axis only goes out to $5000 \omega_0^{-1}$. It can be seen that the peak reflectivity of each burst is the same, but as the intensity is increased the number of bursts increases.

We also ran the two low amplitude cases (simulations A and E) for twice as long in order to study the reflectivity near threshold. Figure 4.8 shows the two cases where the time of the onset of SRS is now greater than $6000 \omega_0^{-1}$ for both simulations. The incident lasers in

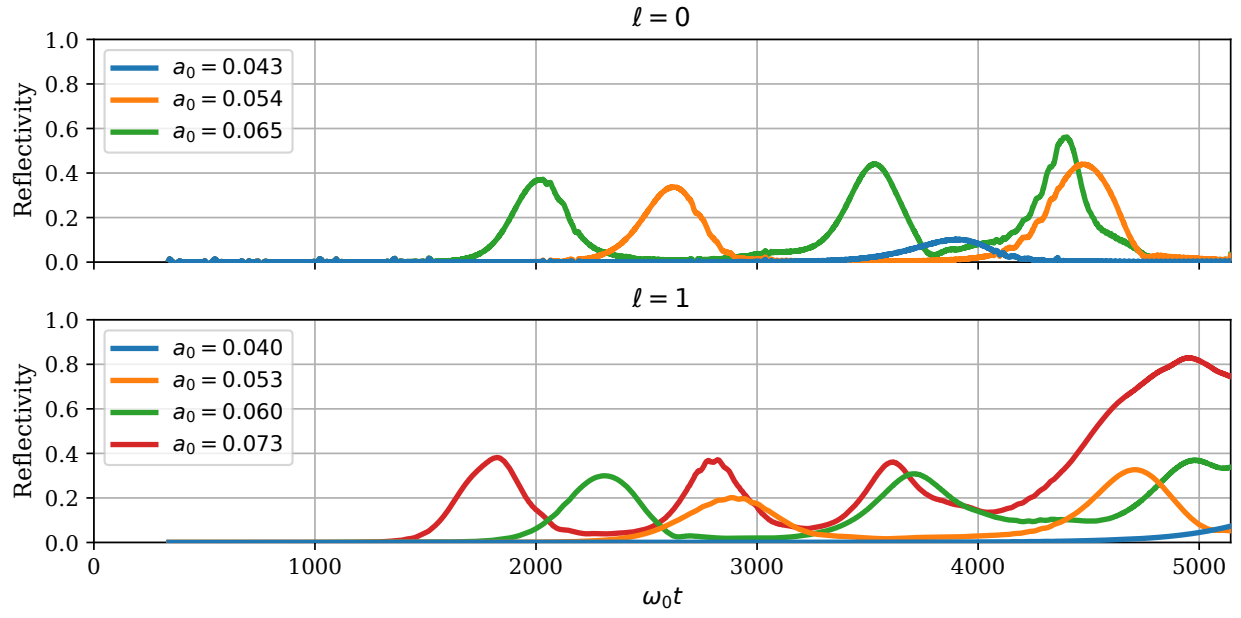


Figure 4.7: Reflectivity over time for OSIRIS simulations with and without OAM. At later times and for the highest intensity (red curve), the use of periodic boundary conditions begins to alter the results.

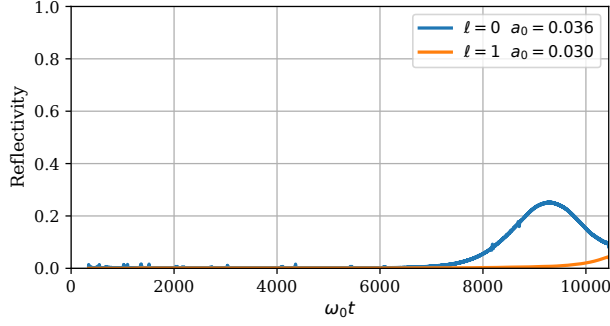


Figure 4.8: We simulated two cases with a pump laser at lower intensity for a longer time in order to measure low reflectivities. For the case without OAM, the peak value of a_0 is 0.036, and for the $\ell_0 = 1$ case the peak value is $a_0 = 0.030$, corresponding to simulations A and E in table 4.1. Both simulations show very little SRS until $\omega_0 t > 6000$.

these two simulations are: one without OAM and $a_0 = 0.036$ and another with $\ell_0 = 1$ and $a_0 = 0.030$. For both, the first burst of SRS is just beginning or is still in progress. In contrast, for the highest intensity cases shown in figure 4.7 the first SRS burst grows rapidly near $1500\omega_0^{-1}$ and there are about three reflectivity bursts .

We also quantified the average reflectivity. In figure 4.9 the average reflectivity is calculated from $1600\omega_0^{-1} - 5100\omega_0^{-1}$ for each simulation in table 4.1. For reference, the average reflectivity is also shown for the corresponding pF3D cases. The left panel shows the relationship between average reflectivity and peak amplitude, a_0 , for lasers with and without OAM (and with the same spot size). For the same amplitude (peak intensity), the OAM pumps ($\ell_0 = 1$) provide slightly greater amounts of SRS. This trend is also seen in the pF3D simulations although they have smaller reflectivity than the PIC simulations. In the kinetic simulations, it appears that average reflectivity starts to deviate significantly between cases with and without OAM at the highest intensities. More simulations are needed in this regime to make a definitive statement. On the other hand, in the right panel, the dependence of reflectivity on laser power (and with the same spot size) is shown. In this case the lasers with OAM produce less SRS than the lasers without OAM for the same power which is also

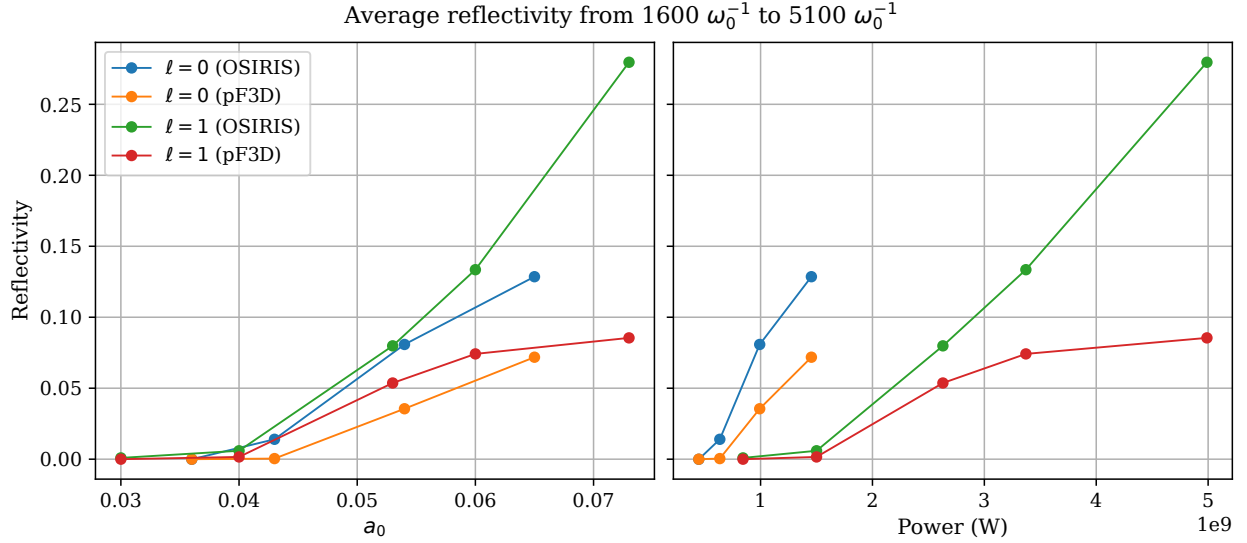


Figure 4.9: Average reflectivity versus a_0 (left) and power (right) for OSIRIS and pF3D simulations. The blue line represents PIC simulations where the pump laser has $\ell = 0$ and the green line shows PIC simulations where the pump laser has $\ell_0 = 1$. The corresponding pF3D simulations are also shown in orange (for $\ell_0 = 0$) and red (for $\ell_0 = 1$). The trend for OSIRIS and pF3D simulations is similar between $\ell_0 = 0$ and $\ell_0 = 1$, but the OAM simulations show higher reflectivity for the same a_0 and lower reflectivity for the same power.

seen in the pF3D simulations. As explained in Chapter 3, this is due to the fact that for a fixed power the OAM laser has a lower peak intensity. In this chapter, all simulations use the same spot size. Therefore, it is not possible to match amplitude and power of the laser simultaneously between the $\ell_0 = 0$ and $\ell_0 = 1$ cases.

4.4.2 Field Dynamics and Wave Structures

OSIRIS simulations provide detailed and well-resolved field data. We analyzed the Fourier transform (FFT) along x_1 of a single (x_2, x_3) position over time in figure 4.10. The top row corresponds to simulation C where the pump does not have OAM ($\ell_0 = 0$) and a_0 is 0.054, and the bottom row is from simulation G where the pump has $\ell_0 = 1$ and a_0 is

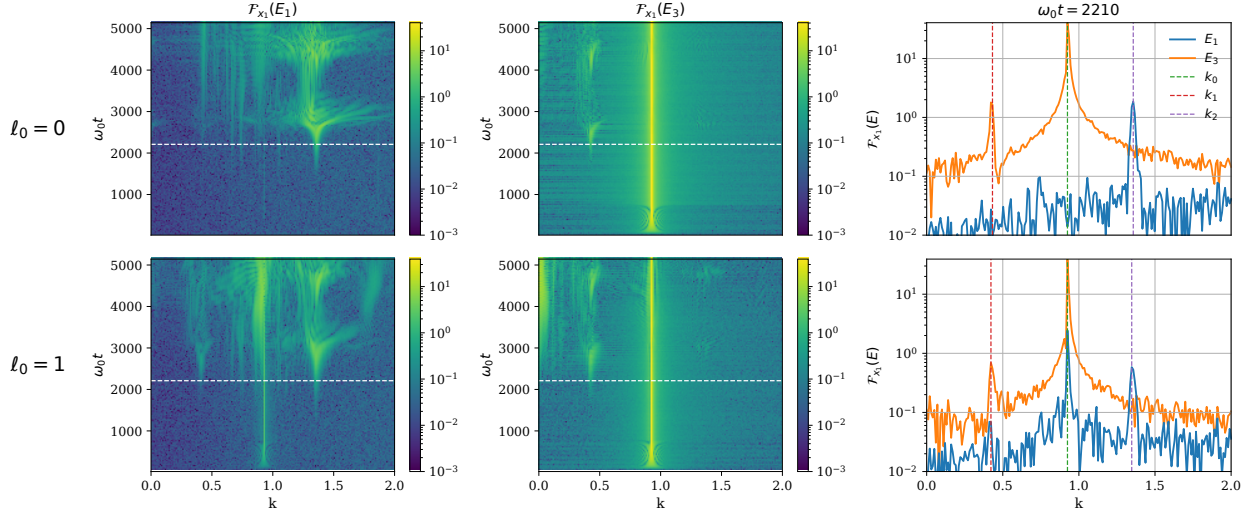


Figure 4.10: Fourier transform of E_1 and E_3 at a single (x_2, x_3) position. For the simulation without OAM (top row), the lineout is along the axis at $x_2 = 0$, $x_3 = 0$ and for the $\ell_0 = 1$ case (bottom row), the lineout is at $x_2 = 0$, $x_3 = w_0/\sqrt{2} \approx 18c/\omega_0$ which is near the r of maximum amplitude for an LG mode. The left column shows E_1 , the middle column shows E_3 , and the right column shows a lineout at $\omega_0 t = 2210$. The three dashed vertical lines in the plots in the right column represent the k of each wave.

0.053. For the pump without OAM, we take a lineout on axis at $x_2 = 0$, $x_3 = 0$ and for the pump with OAM, we use a position near $x_2 = 0$, $x_3 = w_0/\sqrt{2} \approx 18c/\omega_0$ where the field amplitude is large. For a Laguerre-Gaussian laser with $\ell_0 = 1$, there is zero intensity on axis. The left column of figure 4.10 shows the Fourier transform (FFT) of E_1 along a line in x_1 at every tenth timestep. The middle column shows the same for E_3 , the field component which contains the dominant pump signal. The rightmost column is a lineout at $\omega_0 t = 2210$ showing the Fourier transform (FFT) of each field at this time. Additionally, the three dashed vertical lines in the plots on the right show the values of k_0 , k_1 , and k_2 . It is clear that the signal peaks align with the k of each wave. The time is chosen when SRS is just beginning.

At later times, nonlinear frequency (wavenumber) shifts (due to kinetic effects) appear as

shown by the widening signal in the k vs t plots. The scattered wave and EPW are no longer at a single k_1 or k_2 mode, but instead the structure becomes broader and more complicated. For the case with $\ell_0 = 0$ shown here, the peak value of a_0 is about 0.054 (C in table 4.1) and for the $\ell_0 = 1$ case, it is 0.053 (G in table 4.1).

We can isolate the field structure of each wave individually in real space by filtering the wave number. In order to determine filtering ranges, we use the theoretical wavenumbers. The wavenumber for the pump k_0 is $0.93\omega_0/c$, for the seed k_1 is $0.41\omega_0/c$, and for the EPW k_2 is $1.34\omega_0/c$. We isolate $k \pm 0.16\omega_0/c$ for each mode. Figure 4.11 shows each wave at $\omega_0 t = 2600$ during a burst of reflectivity for a simulation without OAM. In the top panel, the pump laser is shown. The middle panel shows the backscattered light which is maximized on the left side of the box. The wavefront appears to have significant curvature.

This is partly due to the structure of the EPW. The bottom panel of figure 4.11 shows the plasma wave oscillation where there is wavefront bowing due to nonlinear frequency shifts of the EPW (larger shifts on axis where the field amplitude is largest) from kinetic effects. Such structures were shown previously in 2008 by Yin et al. [YAB08]. Although the plasma wave is growing to the left, the waves themselves travel to the right. The EPW is narrower than the scattered light wave. The backscattered light diffracts as it propagates backwards while diffraction of the EPW is much slower.

Figure 4.12 shows a burst of SRS in the fields of a simulation where the pump laser has OAM ($\ell_0 = 1$) at $\omega_0 t = 2992$. The top panel shows a slice of the OAM incident light where it has propagated across the box. For this slice, the single laser looks like two lasers moving side by side (one on the “top” and one on the “bottom”) propagating near each other. The middle panel shows the backscattered light. Although the pump has no amplitude on axis, the backscattered light is nonzero there due to diffraction. This is different than in the pF3D simulations (figure 3.13) where the scattered light wave also has no signal on axis.

The wavefronts of the EPW inside both the “top” and “bottom” transverse slices show wavefront bowing due to nonlinear frequency shifts. This indicates that to lowest order SRS

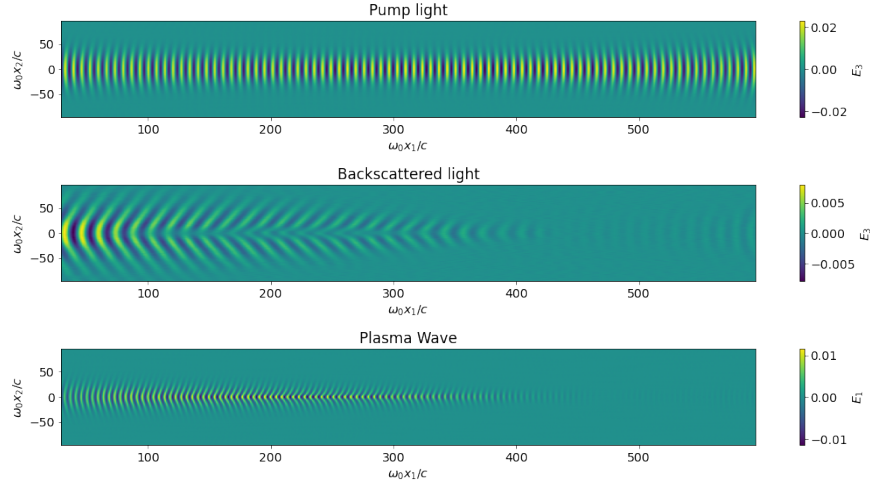


Figure 4.11: EM fields at the beginning of a burst of SRS in a simulation where the incident light does not have OAM. The top panel shows the field of the Gaussian pump laser, the middle panel shows the backscattered light wave, and the bottom panel shows the field of the EPW.

for a laser with OAM occurs independently for each transverse slice. However, the width of the EPW is narrow (only a few laser wavelengths) leading to significant diffraction of the laser so it no longer has a null on axis.

4.4.3 Kinetic Effects and Particle Dynamics

As SRS proceeds, the particle distribution function is modified. The slope of the distribution function near the phase velocity of the EPW becomes flatter as trapped electrons execute bounce motion in the wave. The flattening can reduce the damping rate ($\propto \partial f / \partial v$) and decrease the frequency of the EPW ($\propto \partial^2 f / \partial v^2$) [MO72b]. By measuring the slope over time we can correlate the timing of the changes to the slope with the timing of SRS bursts (determined via other diagnostic techniques). Figure 4.13 shows the correlation between SRS reflectivity, the shift in wave number, and slope of the distribution function near the phase velocity vs time. The data is taken from simulations C and G in table 4.1.

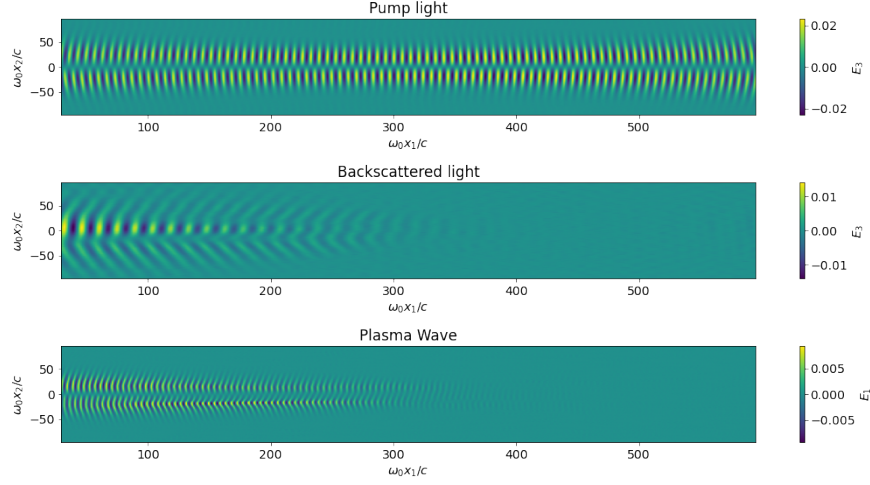


Figure 4.12: EM fields during a burst of SRS for a simulation where the pump is a Laguerre-Gaussian mode with $\ell = 1$. The top panel shows the incident light, the middle panel shows the scattered light, and the bottom panel shows the plasma wave.

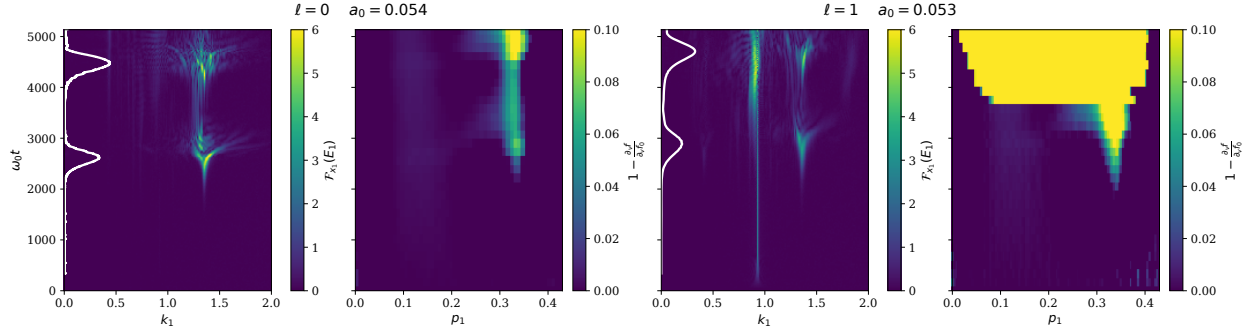


Figure 4.13: Nonlinear wavenumber (frequency) shifts are shown in k space over time at a single radial position for a simulation with $\ell_0 = 0$ (left) and $\ell_0 = 1$ (second to the right). The white line shows the reflectivity at the same times. The second and fourth panel show the change in the slope of the distribution function where brighter spots correspond to a flatter distribution function. For the case with $\ell_0 = 1$ (right), the change in f is significant at later times due to the development of a current.

The left two panels correspond to a simulation where the pump does not have OAM and the peak a_0 is 0.054. The first panel shows the Fourier transform of the E_1 field on axis. The field is maximum on axis when the pump is an $\ell_0 = 0$ mode. The white line shows the reflectivity as measured by the Poynting flux diagnostic. The reflectivity peaks occur at the same time as the bright bursts near the k_2 of the plasma wave. In the second panel, we show the change in the slope of the distribution function by plotting $1 - \frac{\partial_v f}{\partial_v f_0}$ to examine how the slope deviates from the initial distribution function at $t = 0$. The bright spots indicate where the distribution function is flatter than it was initially (where $1 - \frac{\partial_v f}{\partial_v f_0} > 0$). This data is taken over the entire simulation domain while the flattening will be strongest in regions where the EPW is largest. So the actual change in the slope of f is likely to be much larger than these measurements show. We can infer that changes to $\partial_v^2 f$ are largest where changes to $\partial_v f$ are largest. In which case nonlinear frequency shifts would also be large in the bright regions.

The right two panels come from a simulation with $\ell_0 = 1$ and $a_0 = 0.053$. In the rightmost panel the change in the slope of the distribution function is shown. After the first burst of SRS has ended, the plasma develops a current which will be discussed in more detail later. Therefore, the average p_1 is shifted so much that the slope is significantly changed in a very wide region. Thus, wave-particle interactions appear to be different for SRS driven by pumps with OAM. Quantifying this better is an area for future work.

The Fourier transform (FFT) of the E_1 field for the $\ell_0 = 1$ case is calculated at a single slice at a radial distance of $\approx \frac{w_0}{\sqrt{2}}$ where the pump field is maximum. The timing of the bursts of SRS correlate with the bright spots in the Fourier transform. The line near $k_1 = 1$ that appears in the $\ell_0 = 1$ plot and not the $\ell_0 = 0$ plot comes from the E_1 component of the pump laser. The time when the frequency spreads out in k space is the same as the time the burst ends. Thus, the nonlinear wavenumber (frequency) shift of the EPW due to changes in $\partial_v^2 f$ is also correlated to changes in $\partial f / \partial v$ and the detuning affects the reflectivity.

In order to understand the evolution of the distribution function in more detail, we plot

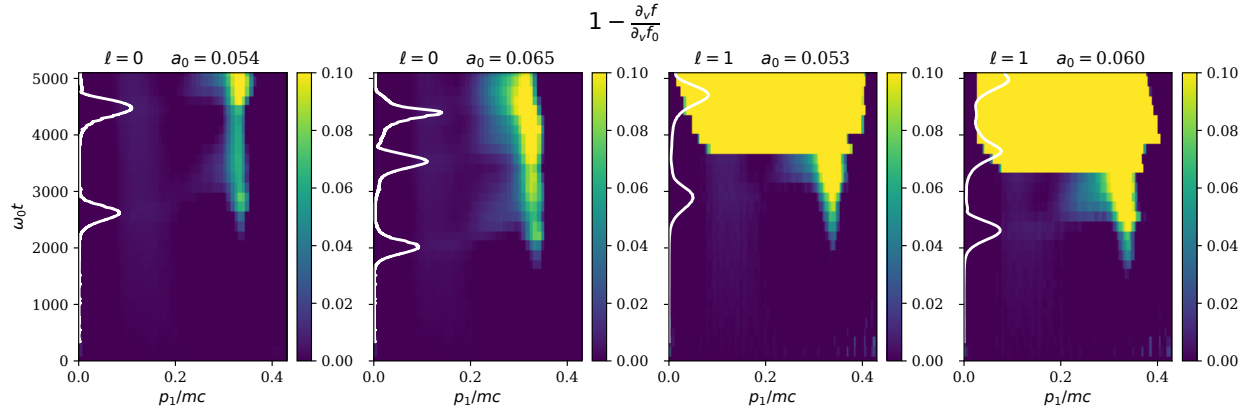


Figure 4.14: The change in the slope of the distribution function over time for simulations C, D, G, and H in table 4.1. Brighter spots correspond to flattening of f . For the two cases with $\ell_0 = 1$, a current develops at later times which affects the distribution function even far from v_{ph} .

the change in the slope for different values of a_0 and ℓ in figure 4.14. The four panels correspond to simulations C, D, G, and H in table 4.1 where the antenna in C and D does not have OAM and the antenna in G and H has $\ell_0 = 1$. It is clear that changes to $\partial f / \partial v$ begin around the same time as the first burst of reflectivity. It is also clear that when the pump has OAM, the distribution function is more drastically changed.

Next, we examine the average momentum and angular momentum of the particles as well as the change in $\bar{\ell}_0$ and $\bar{\ell}_1$, to determine how the presence of OAM in the pump affects the wave-particle interactions.

4.4.4 OAM in the waves

We measure the OAM in the pump and backscattered EM waves using equation 4.13. Figure 4.15 shows the OAM of each wave at a time when reflectivity is high, $\omega_0 t = 3400$. The simulation (G) has a pump with $\ell_0 = 1$ and $a_0 = 0.053$. It is clear that $\ell_0 = 1$ for the pump across the entire simulation box. On the other hand the effective ℓ_1 of the scattered light

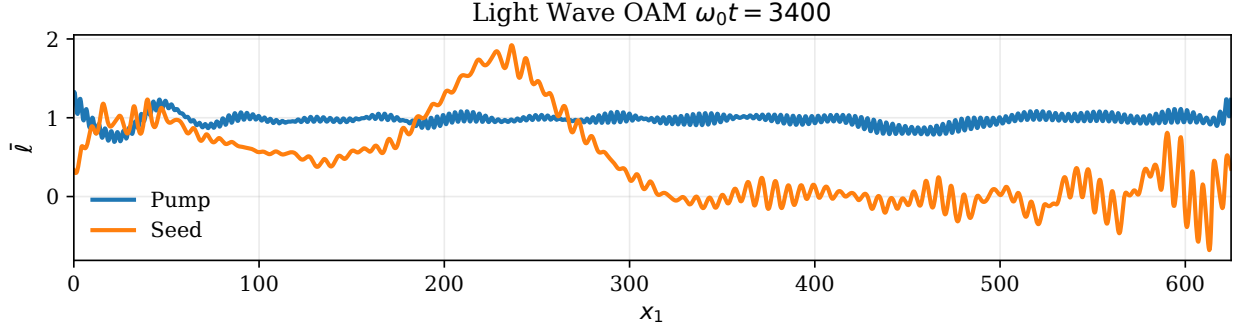


Figure 4.15: Light wave OAM at $\omega_0 t = 3400$ when reflectivity is high. The pump in this simulation (G) has $\ell_0 = 1$ and $a_0 = 0.053$. The OAM of the pump (blue) is approximately the same as its initial value of $\ell_0 = 1$. The OAM of the scattered light (orange) varies across the box with a maximum $\bar{\ell}_1 \approx 2$.

wave starts at 0 at the right edge and then grows to ≈ 1 at the left edge. This shows that OAM can be transferred to the scattered light wave growing from noise. As noted earlier, this occurs to lowest order if SRS grows independently within each transverse slice. At the moment, we do not have an accurate diagnostic of the OAM for the EPW. However, if OAM matching is approximately satisfied then the OAM of the EPW would be ≈ 0 .

Although we do not yet have a clear diagnostic of $\bar{\ell}_2$, we can measure the average momentum, p_1 , and angular momentum, m_1 in the particles from the distribution function. In figure 4.16, we show how the average momentum and angular momentum change over time and how that change aligns temporally with reflectivity. The left column shows two cases where the pump does not have OAM. The two simulations use pumps with $a_0 = 0.054$ (blue) and 0.065 (orange). When the pump laser does not have OAM, the average p_1 and average m_1 are zero. However, when the laser has $\ell_0 = 1$, a current develops as shown in the right column of figure 4.16 where three cases are compared. Each case has a pump with $\ell_0 = 1$ and $a_0 = 0.053$ (blue), 0.060 (orange), and 0.073 (green). This corresponds to simulations G, H, and I in table 4.1. The average momentum deviates from zero at approximately $\omega_0 t = 3000$. In all cases, the significant change in average p_1 occurs after the first burst of reflectivity has

ended. The average momentum is larger for cases with higher a_0 .

The bottom right panel of figure 4.16 shows the average angular momentum in the particles for the three simulations where the pump has $\ell_0 = 1$. The particles develop an average m_1 early in the simulation. The timing aligns with when the pump laser has traveled across most of the simulation region. At later times, the average m_1 returns to zero and then, in all cases, m_1 is briefly positive. When $a_0 = 0.060$, the average m_1 is positive, then slightly negative, and then constant and positive again at $\omega_0 t = 5000$. In the other two cases with $\ell_0 = 1$, the average m_1 is positive at first and then negative and does not rise again before the end of the simulation.

We also show the particle distribution in phase space for two simulations in figure 4.17. In the top row, the simulation uses a pump laser with $\ell = 0$, $a_0 = 0.043$ (B) and the data is shown at $\omega_0 t = 3910$. The bottom row shows data from a simulation with a pump that has $\ell = 1$, $a_0 = 0.053$ (G) and the time is $\omega_0 t = 2890$. For both cases the time chosen aligns with the peak of the first burst of reflectivity.

The left column of figure 4.17 shows the particle distribution in terms of x_1 and p_1 limited to the region where $p_1 > 0$. In both rows, the distribution is not uniform in x_1 , meaning that there is some spatial variation. In the bottom row, there is an increase in average momentum at higher values of x_1 , on the right side of the box. The second column shows the distribution of p_1 and m_1 , the angular momentum in the x_1 direction. In the bottom row, it is clear that a wider spread develops in m_1 at higher values of p_1 . The third column shows the average value of m_1 vs x_1 . For the pump without OAM, the average value is nearly zero. At lower values of x_1 , the average m_1 is slightly greater than at higher values of x_1 , but the average is zero. However, for the case with the $\ell_0 = 1$ pump (bottom), the difference in m_1 is significant between the left side of the box and the right side, and the average m_1 is nonzero.

Figure 4.17 demonstrates that SRS involving a pump with $\ell_0 = 1$ affects the angular momentum of the particles. Although we have not calculated $\bar{\ell}_2$, we have shown that the

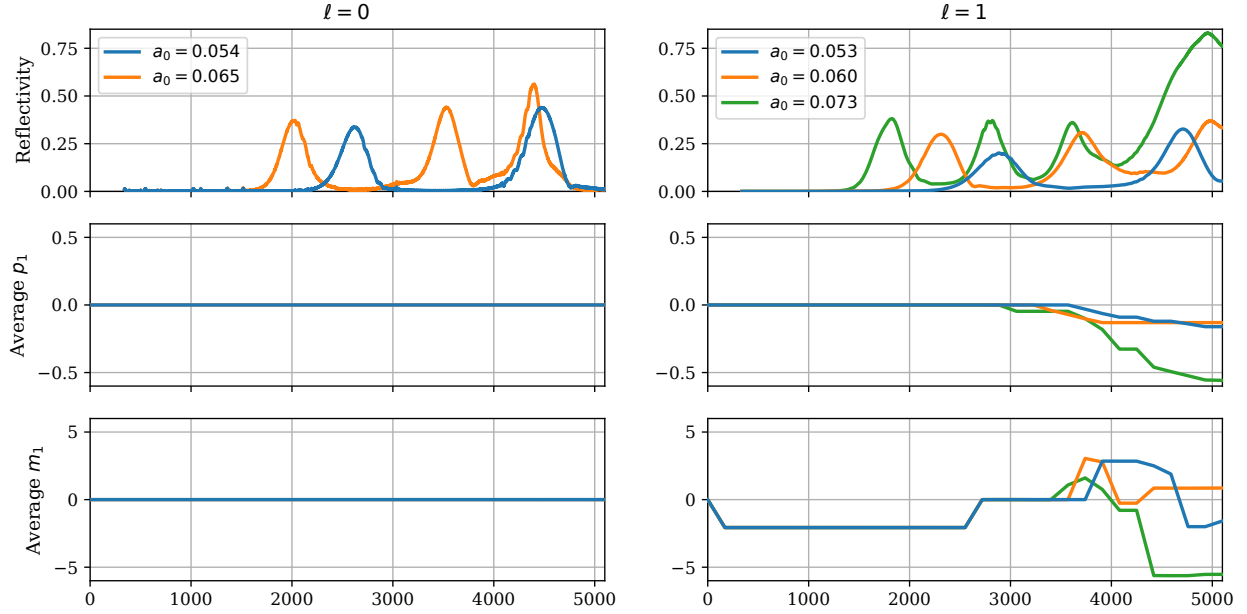


Figure 4.16: Average linear and angular momentum of the particles. The left column shows data for simulations C (blue) and D (orange) in which the pump does not have OAM. The right column shows simulations G (blue), H (orange), and I (green) which all include a pump with $\ell_0 = 1$. In every case with $\ell_0 = 1$, the plasma develops an average p_1 that is negative, and an average m_1 that varies but is nonzero. The top row shows reflectivity measured from the Poynting flux in order to demonstrate the correlation of the timing of the SRS bursts and changes to the distribution function.

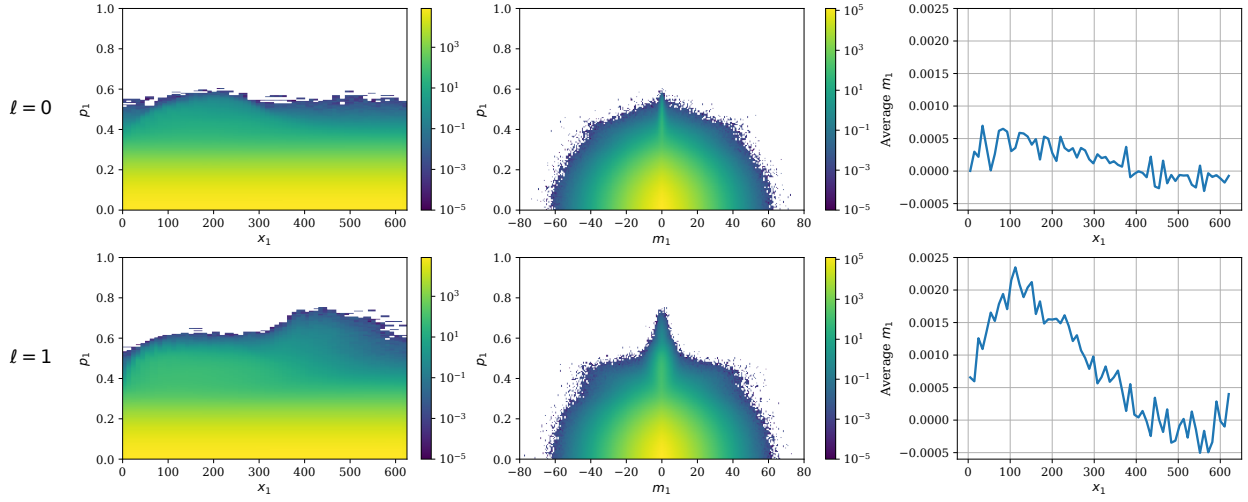


Figure 4.17: Phase space distributions for simulations B (top) and G (bottom). The left column shows the distribution of particles in p_1 and x_1 , the middle column shows p_1 and m_1 , and the right column shows the average angular momentum along x_1 . For both simulations, the data is shown at a time when reflectivity is at its peak.

average angular momentum increases in the whole simulation region. Additionally, the average is higher in the left half of the box where reflectivity is high. Thus, future work is needed to examine the average m_1 in a smaller region where the EPW is large, and the lower right panel of figure 4.17 suggests that it will be even higher because m_1 is not uniform in x_1 .

4.5 Summary

We have presented simulations from OSIRIS of SRS of lasers with and without OAM. As OAM is intrinsically a three-dimensional process, we carried out 3D OSIRIS simulations which are computationally intensive. These simulations provide copious amounts of data. We compared the reflectivity from OSIRIS to that from pF3D simulations with otherwise identical parameters. We simulated a few intensities for light with $\ell_0 = 0$ and $\ell_0 = 1$ and

used $w_0 = 4\lambda_0$ for all cases. We notice the same trend as in Chapter 3 where the pump with OAM shows lower reflectivity when compared at the same power but higher reflectivity when compared at the same intensity (a_0). We found that SRS reflectivity comes in intense bursts in the case of PIC with higher average reflectivity as well when compared to pF3D. Exploring the importance of the noise level on these differences is an area for future work. It was also confirmed that OAM is transferred to the backscattered light wave when the pump has OAM and that changes to the distribution function are correlated with the behavior of SRS. These changes can modify the damping rate and frequency of the EPW. Quantifying the effect of these changes is an area for future work.

CHAPTER 5

Conclusion and directions for future work

In this dissertation, results from an investigation of new aspects of stimulated Raman backscatter (SRBS) that arise when the pump wave has intrinsic or orbital angular momentum (OAM) were presented. In SRBS an incoming electromagnetic (EM) wave decays into a backscattered EM wave and a forward-propagating electron plasma wave (EPW). This concluding chapter summarizes the overarching findings and offers some suggestions for future work. An electromagnetic wave that carries OAM is three-dimensional by nature in the sense that the laser has a finite extent in directions transverse to its propagation direction.

In Chapter 2, the dissertation includes a review of key aspects of SRBS in plane waves, and to set the stage for studying SRBS of lasers with OAM, a review of Laguerre-Gaussian modes is offered. Additionally, a derivation is given of the multi-dimensional envelope equations for SRBS valid within the paraxial ray approximation in a uniform plasma including a phenomenological damping term for the EPW. Starting from these envelope equations, we identified several conserved quantities defined in terms of the envelopes b_0 , b_1 , and b_2 of the pump, scattered EM wave, and EPW, and their spatial derivatives. These quantities were action, energy, axial momentum, transverse momentum, and angular momentum. Conservation of momentum is derived separately for the axial and transverse components. While the conservation laws for action, energy, and axial momentum are well known within the one-dimensional limit, the multi-dimensional versions are yet to appear in the literature. The resulting expressions also included sink terms from the EPW damping. The angular momentum in the axial direction can be defined and it depends on the transverse momentum

(and transverse derivatives). It is now commonly referred to as the OAM of a wave. We derived an angular momentum conservation law where the total OAM is simply a sum over the OAM of each light wave and the EPW. For each light wave, the angular momentum is

$$M_z = i\omega(b^*\partial_\varphi b - b\partial_\varphi b^*) \quad (5.1)$$

and the angular momentum of the EPW is

$$M_z = \frac{i\omega_2}{\Gamma_2}(b_2^*\partial_\varphi b_2 - b_2\partial_\varphi b_2^*) \quad (5.2)$$

The OAM of the light waves in a plasma is identical to that previously obtained in vacuum through the momentum associated with the Poynting flux. Thus, it was found that the OAM of an EM wave in a tenuous plasma is the same as that in vacuum. Additionally, we found that the OAM of the EPW has the same form but with different normalizations as that of an EM wave although the EPW has no Poynting flux. We then connected the expression for the OAM of the plasma wave to a physical interpretation based on Gauss' Law and Euler's equation where the OAM is carried by the current within the EPW.

We investigated the OAM of an arbitrary wave envelope by measuring the effective OAM index, $\bar{\ell}_j$ which can be a function of z . An arbitrary wave is a superposition of Laguerre-Gaussian modes with different ℓ numbers. The effective ℓ number is a weighted average and is thus not necessarily an integer. Finally, we addressed the question of ℓ -matching and explained that a matching condition for the ℓ number of each wave only holds when the OAM of each wave is constant. This is a new result. Thus, while ω and k matching are fundamental concepts associated with three-wave couplings such as SRBS, ℓ matching is not fundamental. This is because ω and k are properties of the carrier which are constant while ℓ is a property of the envelope which evolves. We analyzed $\bar{\ell}_0 - \bar{\ell}_1 - \bar{\ell}_2$ in simulations to show that ℓ -matching is indeed not necessarily satisfied everywhere.

In Chapter 3, we used the fluid simulation code pF3D to study SRS of finite-width lasers with different amounts of OAM. In pF3D the only nonlinearity is pump depletion. We

simulated SRS for a range of intensities and plasma temperatures to scan different growth regimes where the theory presented in Chapter 2 would predict that SRS can occur in either strongly or weakly damped temporal or either strongly or weakly damped spatial growth. We focused on the strongly damped spatial regime as it is relevant to laser-plasma interactions in inertial fusion energy (IFE) and we compared our data to a theoretical model of SRS reflectivity even in the presence of pump depletion.

We found that to compare lasers with different OAM, there are three parameters that must be considered, the peak intensity, the power, and the spot size, w_0 . Only two of these can be simultaneously held constant. We showed that in order to compare beams with the same intensity and power simultaneously for different ℓ , the effective beam width, w_ℓ (defined as $\pi w_\ell^2/2 \equiv \text{Power}/I_{\text{peak}}$) should be matched instead of the spot size, w_0 (the parameter that defines the diffraction or Rayleigh length for each Laguerre-Gaussian mode), and we derived the formula for w_ℓ/w_0 . For the same w_0 , a beam with OAM has more power at the same peak intensity than a beam without OAM. Thus, when comparing SRBS of beams with the same power and w_0 , it needs to be taken into account that the OAM light has a lower peak intensity and its energy is spread out over a larger transverse extent. Within IFE, the laser power needs to be at a fixed value. Thus, the use of lasers with OAM might be a path to reduce SRBS reflectivity.

We measured reflectivity when SRS grows from noise in pF3D using pump lasers with $\ell_0 = 0, 1, 2$, and 3 . We scanned different temperatures to change the damping rate of the plasma wave which depends on $k_2\lambda_D$ which increases as temperature increases.

When SRS grows from noise, lasers with OAM show lower reflectivity if compared at the same power. However, when compared at the same peak intensity, a laser with OAM shows higher reflectivity than an $\ell_0 = 0$ beam. When we adjusted the spot size, lasers with OAM show lower reflectivity. However, when we used an effective spot size of $8\lambda_0$ instead of $4\lambda_0$, but kept the plasma length the same, the difference decreases. This is because diffraction is less important for the larger spot size. We fit the reflectivity data to theoretical predictions

involving the spatial gain rate calculated at different radial positions and integrated over the transverse coordinates to account for the different shapes of the laser. Through this technique, we were able to utilize the 1D theory to make predictions for differently shaped 3D beams.

We measured the effective OAM for each wave and found that the variation in $\bar{\ell}$ changed as intensity and reflectivity changed. As noted above, we showed that the effective OAM is not necessarily an integer. We found that OAM was best conserved when SRS reached a steady state. When we included a seed laser, the plasma wave OAM changes in order to maintain conservation if possible. We showed that in regions where reflectivity was low, OAM is not conserved in our simulations, even when the laser does not have any OAM.

We examined the 3D shape of daughter waves from pumps with OAM and showed that the scattered light develops a similar spatial structure with a central dark spot and a spiral structure which depends on the ℓ_0 of the pump. The scattered light and plasma wave create doughnut shapes even if they do not all have large amounts of angular momentum as long as the pump has OAM. Every wave is a superposition of LG modes and not a pure OAM beam.

We included a seed with OAM to examine the resonance for different combinations of ℓ_0 and ℓ_1 . When an $\ell_0 = 0$ pump amplifies an $\ell_1 = 1$ seed, the effective OAM of the seed decreases and tends toward the OAM of the pump. However, when a pump with OAM amplifies a non-OAM seed, it is the plasma wave which develops OAM equal to that of the pump in order to maintain OAM conservation.

In Chapter 4 we presented results from particle-in-cell OSIRIS simulations, which include kinetic effects and nonlinear frequency shifts that are not modeled by pF3D. The combination of different simulation techniques allows for a diverse set of data that probes different SRS regimes. We simulated pumps with $\ell_0 = 0$ and $\ell_0 = 1$ in a plasma without a seed so SRS grows from noise. We measured reflectivity through changes in the Poynting flux. We showed that reflectivity is higher for pumps with OAM when compared at the same peak

intensity. We ran five simulations with OAM pumps and four simulations using pumps that do not have OAM. We simulated about one picosecond for seven of the cases and about two picoseconds for the other two cases. When reflectivity is compared at the same power, the lasers without OAM show more SRS. We filtered the data in k -space to separate out the pump and scattered EM wave. We were then able to determine the OAM of each wave and showed that OAM is transferred to the scattered light wave.

We showed nonlinear frequency shifts in the EPW and hence the scattered light waves. Additionally, we separated the waves by frequency to explore the spatial behavior and displayed wavefront bowing of the scattered light. We also showed that the plasma distribution function changes during bursts of SRS. In particular, the slope becomes flatter near the phase velocity of the plasma wave. We showed that the change in slope coincides temporally with bursts of reflectivity in the Poynting flux as well as frequency-resolved signals.

We close by noting that the results from this dissertation open up new ideas and directions for future research. For example, towards the end of this research it was recognized that a Lagrangian exists whose Euler-Lagrange equations directly lead to the three envelope equations. Furthermore, this Lagrangian can also account for non-uniform plasmas in which the density depends on the axial coordinate. This Lagrangian can then be used to derive conservation laws using Noether's theorem. Such a Lagrangian can also be used to examine convective growth through the use of trial functions. Another area for future research is to examine the OAM of multi-speckle beams. Even if the OAM of the plasma and pump are essentially 0 at the beginning, there can be local regions of space and time in which OAM arises from SRBS. Investigating if local regions develop OAM even if there is no global OAM is another area for further investigation. It would also be of interest to see if methods can be developed such that a multi-speckle beam has global OAM and/or individual speckles have OAM.

There is also much work left for studying OAM in kinetic simulations. This includes studying wave-particle interactions within EPWs with OAM and running simulations longer

and with more particles per cell. The PIC data is vast so it is important to strategically dump particle data in relevant regions of space and time. It is also important to develop diagnostics of the OAM of the EPW. This is complicated by the fact that it is hard to filter out the contributions of a single particle's linear and angular momentum from each wave and that there is return current generated that needs to be filtered out when calculating the wave momentum. So developing better data analysis is important. Last, we note that the simulations indicate that lasers with OAM can reduce SRBS reflectivity when the power is held fixed as this leads to broader transverse laser profiles so that the peak intensity is lower. If such a geometrical effect and not an OAM effect is the reason for the reduction of SRBS then 2D and not 3D simulations can be used to simulate how higher order laser modes can reduce backscatter. In these 2D simulations the laser would be expanded into Hermite-Gaussian rather than Laguerre-Gaussian modes. This would permit PIC simulations of multi-speckle beams.

APPENDIX A

Conservation laws for fields and particles

Conservation laws are useful and powerful tools to understand fundamental constraints on field and particle dynamics in plasma. In this appendix, we review how to connect the microscopic and macroscopic descriptions of the interaction of charged particles and electromagnetic fields. We follow discussions in [SDM98, Jac99, Cha60]. We derive conservation laws for energy, momentum, and angular momentum systematically, and then transform from particle-based to fluid-based representations. This derivation lays the groundwork for inclusion of arbitrary laser fields, such as those with orbital angular momentum, by explicitly relating the fields to the moments of the distribution function. In Chapter 2, the conservation laws for energy, momentum, and angular momentum are obtained for the paraxial ray and envelope equations pertinent to SRS.

Conservation laws take the general form

$$\partial_t \phi + \nabla \cdot \Phi = 0 \tag{A.1}$$

where $\phi(\mathbf{x}, t)$ is the density of the conserved quantity and $\Phi(\mathbf{x}, t)$ is its associated flux. While equation A.1 is for a scalar, it can also be generalized to situations where ϕ is a vector or a tensor, as is the case for momentum. The aim of the following derivations is to write conservation laws for energy, momentum, and angular momentum in terms of $\mathbf{E}$, $\mathbf{B}$, and moments of the plasma distribution function, f : the density, n , bulk flow velocity, $\mathbf{V}$, pressure tensor, $\mathbf{P}$, and heat flux tensor, $\mathbf{Q}$.

In order to treat $\mathbf{E}$ and $\mathbf{B}$ symmetrically, we temporarily allow for the existence of magnetic monopoles in the form of a magnetic charge on a particle, q_m . To derive conservation

laws, we write Maxwell's equations and the equation of motion as if there were a magnetic charge density, ρ_m , and current density, $\mathbf{J}_m$. Once we have derived the conservation laws in terms of charge and current densities and we are ready to move on to the plasma fluid model, we will set q_m to zero for all species, as is physically accurate.

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (\text{A.2})$$

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m \quad (\text{A.3})$$

$$-\nabla \times \mathbf{E} = \frac{4\pi}{c}\mathbf{J}_m + \frac{1}{c}\partial_t\mathbf{B} \quad (\text{A.4})$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t\mathbf{E} \quad (\text{A.5})$$

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q_m\mathbf{B} + \frac{q\mathbf{v}}{c} \times \mathbf{B} - \frac{q_m\mathbf{v}}{c} \times \mathbf{E} \quad (\text{A.6})$$

The charge and current densities can be written as a sum over particles with electric charge q_s , magnetic charge $q_{m,s}$, velocity $\mathbf{v}_s$, and position $\mathbf{x}_s$,

$$\rho(\mathbf{x}, t) = \sum_s q_s \delta(\mathbf{x} - \mathbf{x}_s) \quad (\text{A.7})$$

$$\rho_m(\mathbf{x}, t) = \sum_s q_{m,s} \delta(\mathbf{x} - \mathbf{x}_s) \quad (\text{A.8})$$

$$\mathbf{J}(\mathbf{x}, t) = \sum_s q_s \mathbf{v}_s \delta(\mathbf{x} - \mathbf{x}_s) \quad (\text{A.9})$$

$$\mathbf{J}_m(\mathbf{x}, t) = \sum_s q_{m,s} \mathbf{v}_s \delta(\mathbf{x} - \mathbf{x}_s) \quad (\text{A.10})$$

In Chapter 2, we reviewed the concept of phase space and the Klimontovich distribution function, F . In terms of F ,

$$\rho = \int d\mathbf{p}_s q_s F_s \quad (\text{A.11})$$

$$\mathbf{J} = \int d\mathbf{p}_s q_s \mathbf{v}_s F_s \quad (\text{A.12})$$

When moving to a continuum distribution function, f_s , the same definitions hold.

A.1 Conservation of Energy

We begin our search for a conservation of energy law by evaluating the time derivative of the total energy in the particles which is the sum of the energy of each particle. The energy of a particle with mass m_s and momentum $\mathbf{p}_s$ is

$$\gamma_s m_s c^2 = \sqrt{1 + \frac{p_s^2}{m_s^2 c^2}} m_s c^2 \quad (\text{A.13})$$

where $p_s^2 = \mathbf{p}_s \cdot \mathbf{p}_s$. Examining the momentum of a particle will lead to a conservation law because we know how to relate the time derivative of the momentum to the EM fields. Differentiating the energy then isolates $d\mathbf{p}_s/dt$,

$$\frac{d}{dt} \sum_s \gamma_s m_s c^2 = \sum_s \frac{1}{\gamma_s} \frac{\mathbf{p}_s}{m_s} \cdot \frac{d\mathbf{p}_s}{dt} = \sum_s \mathbf{v}_s \cdot \frac{d\mathbf{p}_s}{dt} \quad (\text{A.14})$$

We can substitute $\frac{d\mathbf{p}_s}{dt}$ for each particle using equation A.6 to get

$$\sum_s \mathbf{v}_s \cdot \frac{d\mathbf{p}_s}{dt} = \sum_s \mathbf{v}_s \cdot \left(q_s \mathbf{E} + q_{m,s} \mathbf{B} + \frac{q_s \mathbf{v}_s}{c} \times \mathbf{B} - \frac{q_{m,s} \mathbf{v}_s}{c} \times \mathbf{E} \right) \quad (\text{A.15})$$

The terms with $\mathbf{v}_s \cdot (\mathbf{v}_s \times \mathbf{B})$ and $\mathbf{v}_s \cdot (\mathbf{v}_s \times \mathbf{E})$ are identically zero because the cross product of two vectors is orthogonal to both vectors. Noting that we can substitute $\mathbf{J}$ and $\mathbf{J}_m$ into equation A.15, the remaining terms become

$$\sum_s \mathbf{v}_s \cdot (q_s \mathbf{E} + q_{m,s} \mathbf{B}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{B} \quad (\text{A.16})$$

Next, we rearrange equations A.4 and A.5 to isolate the currents,

$$\mathbf{J} = \frac{c}{4\pi} \left(\nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E} \right) \quad (\text{A.17})$$

$$\mathbf{J}_m = -\frac{c}{4\pi} \left(\nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} \right) \quad (\text{A.18})$$

which can be substituted into A.16,

$$\mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{B} = \frac{c}{4\pi} \left(\nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E} \right) \cdot \mathbf{E} - \frac{c}{4\pi} \left(\nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} \right) \cdot \mathbf{B} \quad (\text{A.19})$$

The left-hand side will later be replaced by plasma fluid quantities. The right-hand side will be simplified into the form of a conservation law. The goal is to only be left with time and spatial derivatives. Upon noting that for both $\mathbf{E}$ and $\mathbf{B}$,

$$\frac{1}{4\pi}(\partial_t \mathbf{E}) \cdot \mathbf{E} = \partial_t \left(\frac{E^2}{8\pi} \right) \quad (\text{A.20})$$

and using a vector identity for the divergence of a cross product,

$$(\nabla \times \mathbf{E}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{E} = \nabla \cdot (\mathbf{E} \times \mathbf{B}) \quad (\text{A.21})$$

we arrive at

$$\mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{B} = -\partial_t \left(\frac{E^2 + B^2}{8\pi} \right) - \nabla \cdot \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \quad (\text{A.22})$$

At this point, we can set $\mathbf{J}_m = 0$ and write everything in the conventional form of a conservation law:

$$\partial_t \varepsilon + \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} = 0 \quad (\text{A.23})$$

We have labeled the field energy as

$$\varepsilon \equiv \frac{E^2 + B^2}{8\pi} \quad (\text{A.24})$$

and defined the Poynting vector,

$$\mathbf{S} \equiv \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \quad (\text{A.25})$$

The Poynting vector represents the flow of energy in the EM fields. We will show that it is proportional to the field momentum when we derive the conservation law for momentum. The term $\mathbf{J} \cdot \mathbf{E}$ will be addressed in the section on plasma fluid quantities. For now, it represents the energy and energy flux outside the EM fields.

A.2 Conservation of Momentum

We will follow a similar method as before, combining Maxwell's equations and the equation of motion. Starting from the time derivative of the total momentum summed over particles,

we plug in equation A.6 to get

$$\frac{d}{dt} \sum_s \mathbf{p}_s = \sum_s \frac{d\mathbf{p}_s}{dt} = \sum_s \left(q_s \mathbf{E} + q_{m,s} \mathbf{B} + \frac{q_s \mathbf{v}_s}{c} \times \mathbf{B} - \frac{q_{m,s} \mathbf{v}_s}{c} \times \mathbf{E} \right) \quad (\text{A.26})$$

where we can rewrite the right-hand side in terms of the charge and current densities,

$$\sum_s \frac{d\mathbf{p}_s}{dt} = \rho \mathbf{E} + \rho_m \mathbf{B} + \frac{1}{c} (\mathbf{J} \times \mathbf{B} - \mathbf{J}_m \times \mathbf{E}) \quad (\text{A.27})$$

Using Maxwell's equations to replace charge and current densities with $\mathbf{E}$ and $\mathbf{B}$ yields

$$\begin{aligned} \sum_s \frac{d\mathbf{p}_s}{dt} = \\ \frac{1}{4\pi} \left[(\nabla \cdot \mathbf{E}) \mathbf{E} + (\nabla \cdot \mathbf{B}) \mathbf{B} + \left(\nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E} \right) \times \mathbf{B} - \left(-\nabla \times \mathbf{E} - \frac{1}{c} \partial_t \mathbf{B} \right) \times \mathbf{E} \right] \end{aligned} \quad (\text{A.28})$$

For both $\mathbf{E}$ and $\mathbf{B}$, we use the following vector identity,

$$(\nabla \times \mathbf{E}) \times \mathbf{E} = (\mathbf{E} \cdot \nabla) \mathbf{E} - \frac{1}{2} \nabla E^2 \quad (\text{A.29})$$

which, when added to $(\nabla \cdot \mathbf{E}) \mathbf{E}$, forms a sum that can be written as the divergence of a dyadic product, which is the rank-2 tensor formed by the outer product of the vector with itself. Noting that,

$$(\nabla \cdot \mathbf{E}) \mathbf{E} + (\mathbf{E} \cdot \nabla) \mathbf{E} = \nabla \cdot \mathbf{E} \mathbf{E} \quad (\text{A.30})$$

we identify the tensor $\mathbf{E} \mathbf{E} = \mathbf{E} \otimes \mathbf{E}$, whose components are

$$(\mathbf{E} \mathbf{E})_{ij} = (\mathbf{E} \otimes \mathbf{E})_{ij} = E_i E_j \quad (\text{A.31})$$

Thus, we can simplify the sum appearing in equation A.28 to the divergence of a rank-2 tensor,

$$(\nabla \cdot \mathbf{E}) \mathbf{E} + (\nabla \times \mathbf{E}) \times \mathbf{E} = (\nabla \cdot \mathbf{E}) \mathbf{E} + (\mathbf{E} \cdot \nabla) \mathbf{E} - \frac{1}{2} \nabla E^2 = -\nabla \cdot \left(\frac{1}{2} E^2 \mathbf{I} - \mathbf{E} \mathbf{E} \right) \quad (\text{A.32})$$

where $\mathbf{I}$ is the identity matrix, whose components satisfy $\mathbf{I}_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker-delta,

$$\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (\text{A.33})$$

Next, the terms in equation A.28 with partial time derivatives can be combined because

$$(\partial_t \mathbf{E}) \times \mathbf{B} - (\partial_t \mathbf{B}) \times \mathbf{E} = (\partial_t \mathbf{E}) \times \mathbf{B} + \mathbf{E} \times (\partial_t \mathbf{B}) = \partial_t (\mathbf{E} \times \mathbf{B}) \quad (\text{A.34})$$

Therefore, equation A.28 becomes

$$\sum_s \frac{d\mathbf{p}_s}{dt} = -\partial_t \left(\frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \right) - \nabla \cdot \left(\frac{E^2 + B^2}{8\pi} \mathbf{I} - \frac{\mathbf{E}\mathbf{E} + \mathbf{B}\mathbf{B}}{4\pi} \right) \quad (\text{A.35})$$

Defining the field momentum vector,

$$\mathbf{G} \equiv \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{\mathbf{S}}{c^2} \quad (\text{A.36})$$

and plugging in equation A.27 with $\rho_m = 0$ and $\mathbf{J}_m = 0$, we arrive at the conservation law for momentum,

$$\partial_t \mathbf{G} + \nabla \cdot \mathbf{T} + \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = 0 \quad (\text{A.37})$$

where we have defined the electromagnetic stress tensor,

$$\mathbf{T} \equiv \varepsilon \mathbf{I} - \frac{\mathbf{E}\mathbf{E} + \mathbf{B}\mathbf{B}}{4\pi} \quad (\text{A.38})$$

whose components are

$$T_{ij} = \varepsilon \delta_{ij} - \frac{E_i E_j + B_i B_j}{4\pi} \quad (\text{A.39})$$

The divergence of a rank-2 tensor such as $\mathbf{T}$ is a vector, and the conservation law for momentum is a vector equation. The conservation law for the i th component of momentum is

$$\partial_t G_i + \partial_j T_{ij} + \rho E_i + \frac{1}{c} \epsilon_{ijk} J_j B_k = 0 \quad (\text{A.40})$$

where ϵ_{ijk} is the Levi-Civita symbol,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{if any 2 of } (i, j, k) \text{ are equal} \end{cases} \quad (\text{A.41})$$

and the i th component of the field momentum vector is

$$G_i = \frac{1}{4\pi c} \epsilon_{ijk} E_j B_k \quad (\text{A.42})$$

In equation A.40, the repeated indices (j and k) are summed over.

A valid conservation law must only contain terms which are time or spatial derivatives. Thus, we will later rewrite $\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}$ as the derivative of fluid quantities.

A.3 Conservation of Angular Momentum

As before, we begin with the time derivative of the angular momentum of the particles and isolate $\frac{d\mathbf{p}_s}{dt}$. Using the fact that $\frac{d\mathbf{x}_s}{dt} = \frac{\mathbf{p}_s}{m_s}$ is parallel to $\mathbf{p}_s$ to eliminate $\frac{d\mathbf{x}_s}{dt} \times \mathbf{p}_s = 0$, we have

$$\frac{d}{dt} \sum_s \mathbf{x}_s \times \mathbf{p}_s = \sum_s \left(\frac{d\mathbf{x}_s}{dt} \times \mathbf{p}_s + \mathbf{x}_s \times \frac{d\mathbf{p}_s}{dt} \right) = \sum_s \mathbf{x}_s \times \frac{d\mathbf{p}_s}{dt} \quad (\text{A.43})$$

Substituting equation A.6 and replacing the sum over particles with current and charge densities gives

$$\begin{aligned} \sum_s \mathbf{x}_s \times \left(q_s \mathbf{E} + q_{m,s} \mathbf{B} + \frac{q_s \mathbf{v}_s}{c} \times \mathbf{B} - \frac{q_{m,s} \mathbf{v}_s}{c} \times \mathbf{E} \right) = \\ \mathbf{x} \times \left(\rho \mathbf{E} + \rho_m \mathbf{B} + \frac{\mathbf{J}}{c} \times \mathbf{B} - \frac{\mathbf{J}_m}{c} \times \mathbf{E} \right) \end{aligned} \quad (\text{A.44})$$

which indicates that we should calculate the cross product of the position vector and equation A.37. We will go through each term in this equation and rewrite it as a total derivative to show that angular momentum is conserved.

$$\mathbf{x} \times \left[\partial_t \mathbf{G} + \nabla \cdot \mathbf{T} + \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right] = 0 \quad (\text{A.45})$$

The position vector is time independent so we can simplify,

$$\mathbf{x} \times \partial_t \mathbf{G} = \partial_t (\mathbf{x} \times \mathbf{G}) \quad (\text{A.46})$$

Examining the cross product of $\mathbf{x}$ and the divergence of the EM stress tensor, we have

$$[\mathbf{x} \times \nabla \cdot \mathbf{T}]_i = \epsilon_{ijk} x_j \partial_l T_{kl} \quad (\text{A.47})$$

Given that $\partial_l x_j = \delta_{lj}$, we can rewrite this as

$$\epsilon_{ijk} x_j \partial_l T_{kl} = \epsilon_{ijk} [\partial_l (x_j T_{kl}) - \delta_{jl} T_{kl}] = \partial_l (\epsilon_{ijk} x_j T_{kl}) - \epsilon_{ijk} T_{kj} \quad (\text{A.48})$$

The outer product of a vector with itself, like $\mathbf{E}\mathbf{E}$ or $\mathbf{B}\mathbf{B}$, is a symmetric tensor, as is $\mathbf{I}$. Therefore, the EM stress tensor, $\mathbf{T}$ is symmetric. The Levi-Civita symbol, on the other hand, is antisymmetric. The contraction of a symmetric and antisymmetric tensor, like $\epsilon_{ijk} \delta_{jk}$, vanishes.

$$\epsilon_{ijk} A_{jk} = 0 \quad \text{if} \quad A_{jk} = A_{kj} \quad (\text{A.49})$$

Therefore, we have shown that the cross product of $\mathbf{x}$ and the divergence of a symmetric tensor can be written as a derivative of a single expression,

$$\mathbf{x} \times (\nabla \cdot \mathbf{A}) = \nabla : (\mathbf{x} \times \mathbf{A}) \quad (\text{A.50})$$

whose components are

$$[\nabla : (\mathbf{x} \times \mathbf{A})]_i = \partial_j \epsilon_{jkl} x_k A_{li} \quad (\text{A.51})$$

The resulting derivative is suitable for inclusion in a conservation law, and thus, the conservation law for angular momentum is

$$\partial_t (\mathbf{x} \times \mathbf{G}) + \nabla : (\mathbf{x} \times \mathbf{T}) + \mathbf{x} \times \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) = 0 \quad (\text{A.52})$$

The full set of conservation laws for energy, momentum, and angular momentum are

$$\partial_t \varepsilon + \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} = 0 \quad (\text{A.53})$$

$$\partial_t \mathbf{G} + \nabla \cdot \mathbf{T} + \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = 0 \quad (\text{A.54})$$

$$\partial_t (\mathbf{x} \times \mathbf{G}) + \nabla : (\mathbf{x} \times \mathbf{T}) + \mathbf{x} \times \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) = 0 \quad (\text{A.55})$$

A.4 Rewriting particle contributions as fluid plasma quantities

To connect the conservation laws to plasma properties and include only derivative terms, we need to rewrite the expressions containing ρ and $\mathbf{J}$ in terms of fluid quantities. In terms of the plasma distribution function, f , the current and charge densities are

$$\mathbf{J} = \int d\mathbf{v} f(\mathbf{v}, \mathbf{x}, t) q \mathbf{v} \quad (\text{A.56})$$

$$\rho = \int d\mathbf{v} f(\mathbf{v}, \mathbf{x}, t) q \quad (\text{A.57})$$

We have already eliminated magnetic monopoles in the previous section, so we do not need to consider them here.

A.4.1 Conservation of energy with fluid quantities

Defining the force on a particle as $\mathbf{F}$,

$$\mathbf{F} \equiv q\mathbf{E} + \frac{q\mathbf{v}}{c} \times \mathbf{B} \quad (\text{A.58})$$

we note that the particle contribution term in the conservation law for energy, $\mathbf{J} \cdot \mathbf{E}$, can be written as

$$\mathbf{J} \cdot \mathbf{E} = \int d\mathbf{v} f q \mathbf{v} \cdot \mathbf{E} = \int d\mathbf{v} f \mathbf{v} \cdot \left(\mathbf{F} - \frac{q\mathbf{v}}{c} \times \mathbf{B} \right) = \int d\mathbf{v} f \mathbf{v} \cdot \mathbf{F} \quad (\text{A.59})$$

where $\mathbf{v} \cdot \mathbf{v} \times \mathbf{B}$ vanishes because $\mathbf{v}$ is orthogonal to any cross product with $\mathbf{v}$. When we rewrite $\mathbf{v}$ as a gradient,

$$\mathbf{v} = \nabla_v \frac{v^2}{2} \quad (\text{A.60})$$

where

$$(\nabla_v)_i = \partial_{v_i} \quad (\text{A.61})$$

we note that $\nabla_v \cdot \mathbf{F} = 0$. Assuming that f tends to 0 for particles with high energy so that $\frac{v^2}{2} \mathbf{F} f \Big|_{-\infty}^{\infty}$ is negligible, integrating equation A.59 by parts yields

$$\mathbf{J} \cdot \mathbf{E} = \int d\mathbf{v} f \left(\nabla_v \frac{v^2}{2} \right) \cdot \mathbf{F} = - \int d\mathbf{v} \frac{v^2}{2} \nabla_v \cdot (f \mathbf{F}) = - \int d\mathbf{v} \frac{v^2}{2} \mathbf{F} \cdot \nabla_v f \quad (\text{A.62})$$

In a Maxwellian plasma, for example, the distribution function depends on velocity as e^{-v^2/v_e^2} where v_e is the thermal velocity, equal to $\sqrt{T/m}$ where T is the temperature, and we typically have that $v_e \ll c$, so f tends to zero sufficiently fast.

Next, we use the Vlasov equation,

$$\partial_t f + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \nabla_v f = \left. \frac{df}{dt} \right|_c \quad (\text{A.63})$$

which can be rearranged as,

$$\frac{\mathbf{F}}{m} \cdot \nabla_v f = -\partial_t f - \mathbf{v} \cdot \nabla f + \left. \frac{df}{dt} \right|_c \quad (\text{A.64})$$

where $\left. \frac{df}{dt} \right|_c$ represents changes in f due to collisions. Substituting equation A.64, equation A.62 becomes

$$\mathbf{J} \cdot \mathbf{E} = \int d\mathbf{v} \frac{mv^2}{2} \left(\partial_t f + \mathbf{v} \cdot \nabla f - \left. \frac{df}{dt} \right|_c \right) \quad (\text{A.65})$$

which, upon noting the independence of $\mathbf{x}$, $\mathbf{v}$, and t in order to move ∂_t and ∇ outside the integrals over $\mathbf{v}$, can be rearranged as a sum of 3 integrals,

$$\mathbf{J} \cdot \mathbf{E} = \partial_t \int d\mathbf{v} \frac{mv^2}{2} f + \nabla \cdot \int d\mathbf{v} \frac{mv^2}{2} \mathbf{v} f - \int d\mathbf{v} \frac{mv^2}{2} \left. \frac{df}{dt} \right|_c \quad (\text{A.66})$$

To make progress in simplifying equation A.66, we separate velocity into bulk flow and fluctuations,

$$\mathbf{v} = \mathbf{V} + \mathbf{v}_r \quad (\text{A.67})$$

where

$$\int d\mathbf{v} \mathbf{v} f = n\mathbf{V} \quad (\text{A.68})$$

and

$$\int d\mathbf{v} \mathbf{v}_r f = 0 \quad (\text{A.69})$$

By definition, the fluctuations are symmetric about $\mathbf{V}$ in terms of velocity dependence, and, while $\mathbf{v}_r = \mathbf{v}_r(\mathbf{v}, \mathbf{x}, t)$, $\mathbf{V}$ is only a function of $\mathbf{x}$ and t . Substituting equation A.67 into the first integral in equation A.66 yields

$$\int d\mathbf{v} \frac{mv^2}{2} f = \frac{m}{2} V^2 \int d\mathbf{v} f + m\mathbf{V} \cdot \int d\mathbf{v} \mathbf{v}_r f + \frac{m}{2} \int d\mathbf{v} v_r^2 f = \frac{1}{2} mnV^2 + \frac{1}{2} P_{ii} \quad (\text{A.70})$$

where we have defined the pressure tensor

$$\mathbf{P} \equiv \int d\mathbf{v} m \mathbf{v}_r \mathbf{v}_r f = \int d\mathbf{v} m (\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f \quad (\text{A.71})$$

and P_{ii} is the trace of $\mathbf{P}$. The pressure tensor depends only on the fluctuations and not the bulk flow. Substituting equation A.67 into the next integral in equation A.66, the integral becomes

$$\int d\mathbf{v} \frac{mv^2}{2} \mathbf{v} f = \int d\mathbf{v} \frac{mv^2}{2} (\mathbf{V} + \mathbf{v}_r) f = \frac{1}{2} (mnV^2 + P_{ii}) \mathbf{V} + m\mathbf{V} \cdot \int d\mathbf{v} \mathbf{v}_r \mathbf{v}_r f + \frac{m}{2} \int d\mathbf{v} v_r^2 \mathbf{v}_r f \quad (\text{A.72})$$

and, noting that

$$m\mathbf{V} \cdot \int d\mathbf{v} \mathbf{v}_r \mathbf{v}_r f = \mathbf{V} \cdot \mathbf{P} \quad (\text{A.73})$$

equation A.72 simplifies to

$$\int d\mathbf{v} \frac{mv^2}{2} \mathbf{v} f = \frac{1}{2} (mnV^2 + P_{ii}) \mathbf{V} + \mathbf{V} \cdot \mathbf{P} + \int d\mathbf{v} \frac{mv_r^2}{2} \mathbf{v}_r f \quad (\text{A.74})$$

The dot product of a vector, $\mathbf{V}$, and a rank-2 tensor, $\mathbf{P}$, is a vector. Next, we must define a higher-order quantity, $\mathbf{Q}$,

$$\mathbf{Q} \equiv \int d\mathbf{v} m \mathbf{v}_r \mathbf{v}_r \mathbf{v}_r f = \int d\mathbf{v} m (\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f \quad (\text{A.75})$$

so that

$$\int d\mathbf{v} \frac{mv_r^2}{2} v_{rj} f = Q_{jii} \quad (\text{A.76})$$

$\mathbf{Q}$ is the rank-3 heat flux tensor, which is the third moment of the distribution function, whereas the pressure tensor $\mathbf{P}$ is the second moment. Plugging in equation A.76, we have

$$\nabla \cdot \int d\mathbf{v} \frac{mv^2}{2} \mathbf{v} f = \nabla \cdot \left[\frac{1}{2} (mnV^2 + P_{ii}) \mathbf{V} + \mathbf{V} \cdot \mathbf{P} + \mathbf{Q}_{ii} \right] \quad (\text{A.77})$$

which can also be written as

$$\nabla \cdot \int d\mathbf{v} \frac{mv^2}{2} \mathbf{v} f = \partial_j \left[\frac{1}{2} (mnV^2 + P_{ii}) V_j + V_i P_{ij} + Q_{jii} \right] \quad (\text{A.78})$$

Substituting equation A.70 and equation A.78, equation A.66 simplifies to

$$\mathbf{J} \cdot \mathbf{E} = J_i E_i = \partial_t \left(\frac{1}{2} mn V^2 + \frac{1}{2} P_{ii} \right) + \partial_j \left[\frac{1}{2} (mn V^2 + P_{ii}) V_j + V_i P_{ij} + Q_{jii} \right] - \int d\mathbf{v} \frac{mv^2}{2} \frac{df}{dt} \Big|_c \quad (\text{A.79})$$

The last term cannot be modified without assuming a model for collisions. We neglect that discussion in this dissertation and instead refer to [Cha60].

Plugging equation A.79 into equation A.53 produces a valid conservation law for energy in terms of EM fields and plasma fluid quantities. The full form is

$$\begin{aligned} \partial_t \left[\frac{E^2 + B^2}{8\pi} + \frac{1}{2} mn V^2 + \frac{1}{2} P_{ii} \right] + \\ \partial_j \left[\frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})_j + \frac{1}{2} (mn V^2 + P_{ii}) V_j + V_i P_{ij} + Q_{jii} \right] = \int d\mathbf{v} \frac{mv^2}{2} \frac{df}{dt} \Big|_c \end{aligned} \quad (\text{A.80})$$

or, equivalently,

$$\begin{aligned} \partial_t \left[\varepsilon + \frac{1}{2} mn V^2 + \frac{1}{2} P_{ii} \right] + \\ \nabla \cdot \left[\mathbf{S} + \frac{1}{2} (mn V^2 + P_{ii}) \mathbf{V} + \mathbf{V} \cdot \mathbf{P} + \mathbf{Q}_{ii} \right] = \int d\mathbf{v} \frac{mv^2}{2} \frac{df}{dt} \Big|_c \end{aligned} \quad (\text{A.81})$$

This equation relates the energy and energy flux in the EM fields to the plasma fluid quantities and the right-hand side allows for a sink or source of energy due to collisions.

A.4.2 Conservation of momentum with plasma fluid quantities

The plasma contribution in the conservation law for momentum is $\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}$ which, upon plugging in ρ , $\mathbf{J}$, and $\mathbf{F}$, can be rewritten

$$\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = \int d\mathbf{v} \left(q \mathbf{E} + \frac{q\mathbf{v}}{c} \times \mathbf{B} \right) f = \int d\mathbf{v} \mathbf{F} f \quad (\text{A.82})$$

With the intention of isolating $\mathbf{F} \cdot \nabla_v f$ so that we may substitute the Vlasov equation as in the previous case, we explicitly insert a factor of $\nabla_v \cdot \mathbf{v}$ in order to integrate by parts:

$$\int d\mathbf{v} \mathbf{F} f = \int d\mathbf{v} (\nabla_v \cdot \mathbf{v}) \mathbf{F} f = - \int d\mathbf{v} \mathbf{v} \nabla_v \cdot (\mathbf{F} f) = - \int d\mathbf{v} \mathbf{v} \mathbf{F} \cdot \nabla_v f \quad (\text{A.83})$$

Substituting this result into equation A.82, we have

$$\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = - \int d\mathbf{v} \mathbf{v} (\mathbf{F} \cdot \nabla_v f) \quad (\text{A.84})$$

Plugging in equation A.64, this becomes

$$\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = \int d\mathbf{v} m \mathbf{v} \left(\partial_t f + \mathbf{v} \cdot \nabla f - \frac{df}{dt} \Big|_c \right) \quad (\text{A.85})$$

which simplifies to

$$\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = \partial_t m n \mathbf{V} + \nabla \cdot \int d\mathbf{v} m \mathbf{v} \mathbf{v} f - \int d\mathbf{v} m \mathbf{v} \frac{df}{dt} \Big|_c \quad (\text{A.86})$$

and substituting

$$\int d\mathbf{v} m \mathbf{v} \mathbf{v} f = \int d\mathbf{v} m (\mathbf{V} + \mathbf{v}_r)(\mathbf{V} + \mathbf{v}_r) f = m n \mathbf{V} \mathbf{V} + \mathbf{P} \quad (\text{A.87})$$

where $\mathbf{V} \mathbf{V}$ is the dyadic product, into equation A.86 yields

$$\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} = \partial_t m n \mathbf{V} + \nabla \cdot (m n \mathbf{V} \mathbf{V} + \mathbf{P}) - \int d\mathbf{v} m \mathbf{v} \frac{df}{dt} \Big|_c \quad (\text{A.88})$$

Revisiting equation A.37, we arrive at the conservation law for momentum in terms of fluid quantities by plugging in equation A.88,

$$\partial_t (\mathbf{G} + m n \mathbf{V}) + \nabla \cdot (\mathbf{T} + m n \mathbf{V} \mathbf{V} + \mathbf{P}) = \int d\mathbf{v} m \mathbf{v} \frac{df}{dt} \Big|_c \quad (\text{A.89})$$

A.4.3 Conservation of angular momentum with plasma fluid quantities

The last plasma term remaining in the set of conservation laws is $\mathbf{x} \times (\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B})$, which encompasses the angular momentum in the particles. Following the same procedure of substituting ρ and $\mathbf{J}$ in terms of integrals, integrating by parts, and substituting the Vlasov equation, we have

$$\mathbf{x} \times \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) = -\mathbf{x} \times \int d\mathbf{v} \mathbf{v} \mathbf{F} \cdot \nabla_v f = \mathbf{x} \times \int d\mathbf{v} m \mathbf{v} \left(\partial_t f + \mathbf{v} \cdot \nabla f - \frac{df}{dt} \Big|_c \right) \quad (\text{A.90})$$

Applying the result for $\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}$ (equation A.88) yields

$$\mathbf{x} \times \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) = \mathbf{x} \times \left[\partial_t mn \mathbf{V} + \nabla \cdot (mn \mathbf{V} \mathbf{V} + \mathbf{P}) - \int d\mathbf{v} m \mathbf{v} \frac{df}{dt} \Big|_c \right] \quad (\text{A.91})$$

We note that

$$\mathbf{x} \times \partial_t mn \mathbf{V} = \partial_t (\mathbf{x} \times mn \mathbf{V}) \quad (\text{A.92})$$

because $\partial_t \mathbf{x} = 0$. The divergence term can be rewritten,

$$[\mathbf{x} \times \nabla \cdot (mn \mathbf{V} \mathbf{V} + \mathbf{P})] = \nabla : [\mathbf{x} \times (mn \mathbf{V} \mathbf{V} + \mathbf{P})] \quad (\text{A.93})$$

using the identity in equation A.50 because $\mathbf{V} \mathbf{V}$ and $\mathbf{P}$ are symmetric tensors. Thus, the complete conservation law for angular momentum is

$$\partial_t [\mathbf{x} \times (\mathbf{G} + mn \mathbf{V})] + \nabla : [\mathbf{x} \times (\mathbf{T} + mn \mathbf{V} \mathbf{V} + \mathbf{P})] = \int d\mathbf{v} (\mathbf{x} \times m \mathbf{v}) \frac{df}{dt} \Big|_c \quad (\text{A.94})$$

or, equivalently, the i th component is

$$\partial_t [\epsilon_{ijk} x_j (G_k + mn V_k)] + \partial_j [\epsilon_{jkl} x_k (T_{li} + mn V_l V_i + P_{li})] = \int d\mathbf{v} m (\epsilon_{ijk} x_j v_k) \frac{df}{dt} \Big|_c \quad (\text{A.95})$$

The term on the right-hand side represents a sink or source of angular momentum due to collisions. We define the EM angular momentum as

$$\mathbf{M} \equiv \mathbf{x} \times \mathbf{G} = \mathbf{x} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \quad (\text{A.96})$$

We have derived conservation laws for fields and fluid quantities from Maxwell's equations and the equation of motion. Table A.1 summarizes the important quantities we have defined for both the fields and the particles.

Variable	Definition	Meaning	Tensor rank
ε	$\frac{E^2+B^2}{8\pi}$	Field energy	0
$\mathbf{S}$	$\frac{c}{4\pi}(\mathbf{E} \times \mathbf{B})$	Poynting flux, flow of energy	1
$\mathbf{G}$	$\frac{\mathbf{E} \times \mathbf{B}}{4\pi c}$	Field momentum	1
$\mathbf{T}$	$\varepsilon \mathbf{I} - \frac{\mathbf{E}\mathbf{E}+\mathbf{B}\mathbf{B}}{4\pi}$	Flow of momentum	2
$\mathbf{M}$	$\mathbf{x} \times \mathbf{G}$	Field angular momentum	1
n	$\int d\mathbf{v} f$	Plasma density	0
$\mathbf{V}$	$\frac{1}{n} \int d\mathbf{v} \mathbf{v} f$	Bulk flow velocity	1
$\mathbf{P}$	$\int d\mathbf{v} m(\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f$	Plasma pressure	2
$\mathbf{Q}$	$\int d\mathbf{v} m(\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f$	Plasma heat flux	3

Table A.1: Scalars and tensors derived from conservation laws for fields and particles including plasma fluid moments of the distribution function

REFERENCES

- [ABS92] Les Allen, Marco W Beijersbergen, RJC Spreeuw, and JP Woerdman. “Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes.” *Physical review A*, **45**(11):8185, 1992.
- [Abu24] H. Abu-Shawareb et al. “Achievement of Target Gain Larger than Unity in an Inertial Fusion Experiment.” *Physical Review Letters*, **132**:065102, 2024.
- [AP00] L Allen and M J Padgett. “The Poynting vector in Laguerre±Gaussian beams and the interpretation of their angular momentum density.” *Optics Communications*, 2000.
- [APB99] L. Allen, M. J. Padgett, and M. Babiker. “IV The Orbital Angular Momentum of Light.” In E. Wolf, editor, *Progress in Optics*, volume 39, pp. 291–372. Elsevier, January 1999.
- [ATH13] Paulo Alves, Raoul Trines, Kathryn Humphrey, Robert Bingham, Alan Cairns, Frederico Fiuza, Ricardo Fonseca, Luis Silva, and Peter Norreys. “A robust plasma-based laser amplifier via stimulated Brillouin scattering.” *arXiv e-prints*, p. arXiv:1311.2034, November 2013.
- [AW95] Bedros B Afeyan and Edward A Williams. “Unified theory of stimulated Raman scattering and two-plasmon decay in inhomogeneous plasmas: High frequency hybrid instability.” *Physical review letters*, **75**(23):4218, 1995.
- [BAv93] M.W. Beijersbergen, L. Allen, H.E.L.O. van der Veen, and J.P. Woerdman. “Astigmatic laser mode converters and transfer of orbital angular momentum.” *Optics Communications*, **96**(1):123–132, 1993.
- [BBC13] R. L. Berger, S. Bruner, T. Chapman, L. Divol, Still C.H., and E.J. Valeo. “Electron and ion kinetic effects on non-linearly driven electron plasma and ion acoustic waves.” *Phys. Plasmas*, **20**:032107, March 2013.
- [BDG01] R. L. Berger, L. M. Divol, S. H. Glenzer, D. E. Hinkel, R. K. Kirkwood, A. B. Langdon, James Dwayne Moody, C. H. Still, Larry J. Suter, E. A. Williams, and Peter E. Young. “Modeling with backscatter and transmitted light of high-power smoothed beams with pF3D: a massively parallel laser plasma interaction code.” In *ECLIM 2000: 26th European Conference on Laser Interaction with Matter*, volume 4424, pp. 206–211. SPIE, April 2001.
- [Bet36] Richard A. Beth. “Mechanical Detection and Measurement of the Angular Momentum of Light.” *Phys. Rev.*, **50**:115–125, Jul 1936.

- [BHin] R Betti and OA Hurricane. “Inertial-confinement fusion with lasers.” *Nature Physics*, **12**(5):435–448, 2016; and references therein.
- [BK95] AJ Brizard and AN Kaufman. “Local Manley-Rowe relations for noneikonal wave fields.” *Physical review letters*, **74**(23):4567, 1995.
- [BL18] C.K. Birdsall and A.B Langdon. *Plasma Physics via Computer Simulation*. CRC Press, 0 edition, October 2018.
- [BNK19] D. R. Blackman, R. Nuter, Ph. Korneev, and V. T. Tikhonchuk. “Kinetic plasma waves carrying orbital angular momentum.” *Physical Review E*, **100**(1):013204, July 2019.
- [BNK20] D. R. Blackman, R. Nuter, Ph. Korneev, and V. T. Tikhonchuk. “Nonlinear Landau damping of plasma waves with orbital angular momentum.” *Physical Review E*, **102**(3):033208, September 2020.
- [BNK22] D. R. Blackman, R. Nuter, Ph. Korneev, A. Arefiev, and V. T. Tikhonchuk. “Kinetic phenomena of helical plasma waves with orbital angular momentum.” *Physics of Plasmas*, **29**(7):072105, July 2022.
- [Bri64] Richard J. Briggs. *Electron-stream interactions with plasmas*. The MIT Press, 1964.
- [BSW98] RL Berger, CH Still, EA Williams, and AB Langdon. “On the dominant and subdominant behavior of stimulated Raman and Brillouin scattering driven by nonuniform laser beams.” *Physics of Plasmas*, **5**(12):4337–4356, 1998.
- [BVS90] V. Bazhenov, Mikhail Vasnetsov, and Marat Soskin. “Laser beams with screw dislocations in their wavefronts.” *Optical and Quantum Electronics*, **24**:951–962, 01 1990.
- [CABin] RS Craxton, KS Anderson, TR Boehly, VN Goncharov, DR Harding, JP Knauer, RL McCrory, PW McKenty, DD Meyerhofer, JF Myatt, et al. “Direct-drive inertial confinement fusion: A review.” *Physics of Plasmas*, **22**(11), 2015; and references therein.
- [Cha60] Subrahmanyan Chandrasekhar. *Plasma Physics: A Course Given at the University of Chicago*. University of Chicago Press, 1960. Compiled by S.K. Trehan.
- [cTT09] J. T. Mendonça, B. Thidé, and H. Then. “Stimulated Raman and Brillouin Backscattering of Collimated Beams Carrying Orbital Angular Momentum.” *Phys. Rev. Lett.*, **102**:185005, May 2009.
- [Daw61] John Dawson. “On Landau Damping.” *Physics of Fluids*, **4**(7):869, 1961.

- [DKL74] James F Drake, Predhiman K Kaw, Yee-Chun Lee, G Schmid, Chuan S Liu, and Marshall N Rosenbluth. “Parametric instabilities of electromagnetic waves in plasmas.” *Physics of Fluids*, **17**(4):778–785, 1974.
- [DM00] Brian J Duda and Warren B Mori. “Variational principle approach to short-pulse laser-plasma interactions in three dimensions.” *Physical Review E*, **61**(2):1925, 2000.
- [DMT96] CD Decker, WB Mori, K-C Tzeng, and T Katsouleas. “The evolution of ultra-intense, short-pulse lasers in underdense plasmas.” *Physics of Plasmas*, **3**(5):2047–2056, 1996.
- [EKL80] Kent Estabrook, William L. Kruer, and Barbara F. Lasinski. “Heating by Raman backscatter and forward scatter.” *Physical Review Letters*, **45**(17):1399, 1980.
- [ESW12] Ian N. Ellis, David J. Strozzi, Benjamin J. Winjum, Frank S. Tsung, Thomas Grismayer, Warren B. Mori, Jay E. Fahlen, and Edward A. Williams. “Convective Raman Amplification of Light Pulses Causing Kinetic Inflation in Inertial Fusion Plasmas.” *Physics of Plasmas*, **19**(11):112704, November 2012. arXiv:1208.1576 [physics].
- [FC61] Burton D. Fried and Samuel D. Conte. *The Plasma Dispersion Function*. Academic Press, 1961.
- [FCM00] Juan C. Fernández, J. A. Cobble, D. S. Montgomery, M. D. Wilke, and B. B. Afeyan. “Observed insensitivity of stimulated Raman scattering on electron density.” *Physics of Plasmas*, **7**(9):3743–3750, 09 2000.
- [FKLina] DW Forslund, JM Kindel, and EL Lindman. “Plasma simulation studies of stimulated scattering processes in laser-irradiated plasmas.” *The Physics of Fluids*, **18**(8):1017–1030, 1975; and references therein.
- [FKLinb] DW Forslund, JM Kindel, and EL Lindman. “Theory of stimulated scattering processes in laser-irradiated plasmas.” *The Physics of Fluids*, **18**(8):1002–1016, 1975; and references therein.
- [FST02] R. A. Fonseca, L. O. Silva, F. S. Tsung, V. K. Decyk, W. Lu, C. Ren, W. B. Mori, S. Deng, S. Lee, T. Katsouleas, and J. C. Adam. “OSIRIS: A three-dimensional, fully relativistic particle in cell code for modeling plasma based accelerators.” In *International Conference on Computational Science*, pp. 342–351. Springer, 2002.
- [GCP04] Graham Gibson, Johannes Courtial, Miles J. Padgett, Mikhail Vasnetsov, Valeriy Pas’ko, Stephen M. Barnett, and Sonja Franke-Arnold. “Free-space information transfer using light beams carrying orbital angular momentum.” *Opt. Express*, **12**(22):5448–5456, Nov 2004.

- [HZN20] Chen-Kang Huang, Chaojie Zhang, Zan Nie, Kenneth A. Marsh, Chris E. Clayton, and Chandrashekhar Joshi. “Conservation of angular momentum in second harmonic generation from under-dense plasmas.” *Communications Physics*, **3**(1):1–9, November 2020. Number: 1 Publisher: Nature Publishing Group.
- [Jac99] John David Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1999.
- [Kru88] W. Kruer. *The Physics Of Laser Plasma Interactions*. Avalon Publishing, 1988.
- [KWA96] William L. Kruer, Scott C. Wilks, Bedros B. Afeyan, and Robert K. Kirkwood. “Energy transfer between crossing laser beams.” *Physics of Plasmas*, **3**(1):382–385, 1996.
- [Lan46] L. Landau. “On the vibrations of the electronic plasma.” *Journal of Physics*, **10**(1):25–34, 1946.
- [LF21] Andrew Longman and Robert Fedosejevs. “Kilo-Tesla axial magnetic field generation with high intensity spin and orbital angular momentum beams.” *Phys. Rev. Res.*, **3**:043180, Dec 2021.
- [Linin] John Lindl. “Development of the indirect-drive approach to inertial confinement fusion and the target physics basis for ignition and gain.” *Physics of Plasmas*, **2**(11):3933–4024, 11 1995; and references therein.
- [LLK79] A. Bruce Langdon, BF Lasinski, and WL Kruer. “Nonlinear saturation and recurrence of the Two-Plasmon Decay Instability.” *Physical Review Letters*, **43**(2):133, 1979.
- [LR76] CS Liu and Marshall N Rosenbluth. “Parametric decay of electromagnetic waves into two plasmons and its consequences.” *The Physics of Fluids*, **19**(7):967–971, 1976.
- [LRW73] C. S. Liu, M. N. Rosenbluth, and R. B. White. “Parametric Scattering Instabilities in Inhomogeneous Plasmas.” *Phys. Rev. Lett.*, **31**:697–700, Sep 1973.
- [LSA16] Steven Langer, David Strozzi, Peter Amendt, Thomas Chapman, Laura Hopkins, Andrea Kritcher, and Scott Sepke. “pF3D Simulations of SBS and SRS in NIF Hohlraum Experiments.” *Bulletin of the American Physical Society*, **2016**:UO9.001, October 2016. Conference Name: APS Division of Plasma Physics Meeting Abstracts ADS Bibcode: 2016APS..DPPUO9001L.
- [LSB87] Robert Lehmberg, Andrew Schmitt, and Stephen Bodner. “Theory of induced spatial incoherence.” *Journal of Applied Physics*, **62**:2680 – 2701, 11 1987.

- [MAT09] J. T. Mendonca, S. Ali, and B. Thidé. “Plasmons with orbital angular momentum.” *Physics of Plasmas*, **16**(11):112103, November 2009.
- [MDW09] P Michel, L Divol, EA Williams, CA Thomas, DA Callahan, S Weber, SW Haan, JD Salmonson, NB Meezan, OL Landen, et al. “Energy transfer between laser beams crossing in ignition hohlraums.” *Physics of Plasmas*, **16**(4), 2009.
- [Men12] J. T. Mendonça. “Kinetic description of electron plasma waves with orbital angular momentum.” *Physics of Plasmas*, **19**(11):112113, November 2012.
- [Mic23] Pierre Michel. *Introduction to Laser-Plasma Interactions*. Graduate Texts in Physics. Springer International Publishing, 2023.
- [MLG96] CJ McKinstrie, JS Li, RE Giacone, and HX Vu. “Two-dimensional analysis of the power transfer between crossed laser beams.” *Physics of Plasmas*, **3**(7):2686–2692, 1996.
- [MO72a] G. J. Morales and T. M. O’Neil. “Nonlinear Frequency Shift of an Electron Plasma Wave.” *Phys. Rev. Lett.*, **28**:417–420, Feb 1972.
- [MO72b] G. J. Morales and T. M. O’Neil. “Nonlinear Frequency Shift of an Electron Plasma Wave.” *Physical Review Letters*, **28**(7):417–420, February 1972.
- [MR07] JM Manley and HE Rowe. “Some general properties of nonlinear elements-Part I. General energy relations.” *Proceedings of the IRE*, **44**(7):904–913, 2007.
- [MTT09] J. T. Mendonça, B. Thidé, and H. Then. “Stimulated Raman and Brillouin Backscattering of Collimated Beams Carrying Orbital Angular Momentum.” *Physical Review Letters*, **102**(18):185005, May 2009.
- [MVW01] Alois Mair, Alipasha Vaziri, Gregor Weihs, and Anton Zeilinger. “Entanglement of the orbital angular momentum states of photons.” *Nature*, **412**(6844):313–316, July 2001.
- [OMA02] A. T. O’Neil, I. MacVicar, L. Allen, and M. J. Padgett. “Intrinsic and Extrinsic Nature of the Orbital Angular Momentum of a Light Beam.” *Physical Review Letters*, **88**(5):053601, January 2002. Publisher: American Physical Society.
- [ON65] Thomas O’Neil. “Collisionless Damping of Nonlinear Plasma Oscillations.” *Physics of Fluids*, **8**(12):2255, 1965.
- [PCA04] Miles Padgett, Johannes Courtial, and Les Allen. “Light’s Orbital Angular Momentum.” *Physics Today*, **57**(5):35–40, May 2004.

- [PCT26] Vijay Patel, Sarah Chase, Frank S. Tsung, John P. Palastro, Denise E. Hinkel, and Warren B. Mori. “Generalized multi-dimensional conservation laws for stimulated Raman and Brillouin scattering in a density gradient.” Manuscript in preparation, 2026.
- [PMF04] P. A. Prentice, M. P. MacDonald, T. G. Frank, A. Cuschieri, G. C. Spalding, W. Sibbett, P. A. Campbell, and K. Dholakia. “Manipulation and filtration of low index particles with holographic Laguerre-Gaussian optical trap arrays.” *Opt. Express*, **12**(4):593–600, Feb 2004.
- [Poy09] John Henry Poynting. “The wave motion of a revolving shaft, and a suggestion as to the angular momentum in a beam of circularly polarised light.” *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, **82**(557):560–567, 07 1909.
- [RFT26] E. Rovere, R. K. Follett, F. S. Tsung, B. J. Winjum, J. J. Santos, G. Pérez-Callejo, R. Florido, A. Bordón-Sanchez, M. Caetano de Sousa, M. A. Gigosos, F. N. Beg, and M. Bailly-Grandvaux. “Effects of a magnetic field on hot electron generation from laser-plasma instabilities.” *Physics of Plasmas*, **33**(2):022503, 02 2026.
- [Ros72] Marshall N. Rosenbluth. “Parametric Instabilities in Inhomogeneous Media.” *Phys. Rev. Lett.*, **29**:565–567, Aug 1972.
- [SBM17] D. J. Strozzi, D. S. Bailey, P. Michel, L. Divol, S. M. Sepke, G. D. Kerbel, C. A. Thomas, J. E. Ralph, J. D. Moody, and M. B. Schneider. “Interplay of Laser-Plasma Interactions and Inertial Fusion Hydrodynamics.” *Physical Review Letters*, **118**(2):025002, January 2017. Publisher: American Physical Society.
- [SCM07] M. Shimada, D.J. Campbell, V. Mukhovatov, M. Fujiwara, N. Kirneva, K. Lackner, M. Nagami, V.D. Pustovitov, N. Uckan, J. Wesley, N. Asakura, A.E. Costley, A.J.H. Donné, E.J. Doyle, A. Fasoli, C. Gormezano, Y. Gribov, O. Gruber, T.C. Hender, W. Houlberg, S. Ide, Y. Kamada, A. Leonard, B. Lipschultz, A. Loarte, K. Miyamoto, V. Mukhovatov, T.H. Osborne, A. Polevoi, and A.C.C. Sips. “Progress in the ITER Physics Basis.” *Nuclear Fusion*, **47**(6):S1–S17, 2007.
- [SDM98] Julian Schwinger, Lester L. DeRaad, Kimball A. Milton, and Wu-yang Tsai. *Classical Electrodynamics*. Perseus Books, 1998.
- [SSK89] S Skupsky, RW Short, T Kessler, RS Craxton, S Letzring, and JM Soures. “Improved laser-beam uniformity using the angular dispersion of frequency-modulated light.” *Journal of Applied Physics*, **66**(8):3456–3462, 1989.
- [SSW83] A Simon, RW Short, EA Williams, and T Dewandre. “On the inhomogeneous two-plasmon instability.” *The Physics of fluids*, **26**(10):3107–3118, 1983.

- [SVT18] Y. Shi, J. Vieira, R. M. G. M. Trines, R. Bingham, B. F. Shen, and R. J. Kingham. “Magnetic Field Generation in Plasma Waves Driven by Copropagating Intense Twisted Lasers.” *Phys. Rev. Lett.*, **121**:145002, Oct 2018.
- [TKD20] V.T. Tikhonchuk, Ph. Korneev, E. Dmitriev, and R. Nuter. “Numerical study of momentum and energy transfer in the interaction of a laser pulse carrying orbital angular momentum with electrons.” *High Energy Density Physics*, **37**:100863, 2020.
- [VDB02] H. X. Vu, D. F. DuBois, and B. Bezzerides. “Kinetic inflation of stimulated Raman backscatter in regimes of high linear Landau damping.” *Physics of Plasmas*, **9**(5):1745–1763, 05 2002.
- [VDB07] H. X. Vu, D. F. DuBois, and B. Bezzerides. “Inflation threshold: A nonlinear trapping-induced threshold for the rapid onset of stimulated Raman scattering from a single laser speckle.” *Physics of Plasmas*, **14**(1):012702, January 2007.
- [VTA16a] J. Vieira, R. M. G. M. Trines, E. P. Alves, R. A. Fonseca, J. T. Mendonça, R. Bingham, P. Norreys, and L. O. Silva. “Amplification and generation of ultra-intense twisted laser pulses via stimulated Raman scattering.” *Nature Communications*, **7**(1):10371, January 2016. Number: 1 Publisher: Nature Publishing Group.
- [VTA16b] J. Vieira, R. M. G. M. Trines, E. P. Alves, R. A. Fonseca, J. T. Mendonça, R. Bingham, P. Norreys, and L. O. Silva. “High Orbital Angular Momentum Harmonic Generation.” *Physical Review Letters*, **117**(26):265001, December 2016.
- [Wan19] J. W. Wang. “Intense attosecond pulses carrying orbital angular momentum using laser plasma interactions | Nature Communications.”, 2019.
- [WFM07] B.J. Winjum, J. Fahlan, and W.B. Mori. “The relative importance of fluid and kinetic frequency shifts of an electron plasma wave.” *Phys. Plasmas*, **14**:102104, October 2007.
- [WHL12] E. A. Williams, D. E. Hinkel, A. B. Langdon, S. H. Langer, R. L. Berger, and C. H. Still. “pF3D modeling of NIF Ignition experiments.” *Bulletin of the American Physical Society*, **54**:CP8.085, October 2012. Conference Name: APS Division of Plasma Physics Meeting Abstracts ADS Bibcode: 2012APS..DPPCP8085W.
- [YAB08] L. Yin, B. J. Albright, K. J. Bowers, W. Daughton, and H. A. Rose. “Saturation of backward stimulated scattering of laser in kinetic regime: Wavefront bowing, trapped particle modulational instability, and trapped particle self-focusing of plasma waves.” *Physics of Plasmas*, **15**(1):013109, 01 2008.

- [YAR09] L. Yin, B. J. Albright, H. A. Rose, K. J. Bowers, B. Bergen, D. S. Montgomery, J. L. Kline, and J. C. Fernández. “Onset and saturation of backward stimulated Raman scattering of laser in trapping regime in three spatial dimensions.” *Physics of Plasmas*, **16**(11):113101, November 2009.