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Projective Constant Decompositions of Persistence Modules Over Noetherian Rings

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Nadiya Upegui Keagy

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ABSTRACT OF THE DISSERTATION

Projective Constant Decompositions of
Persistence Modules Over Noetherian Rings

by

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Doctor of Philosophy, Graduate Program in Mathematics
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Dr. Wee Liang Gan, Chairperson

Persistence modules serve as the algebraic foundation for topological data analysis, typically studied as representations of posets over a field. This dissertation extends the structural and decomposition theory of persistence modules to the more general setting of unitary left modules over Noetherian rings. We introduce the notion of a module of projective constant type, a generalization of interval modules that facilitates decomposition results. We characterize the existence of projective constant decompositions for pointwise finitely generated persistence modules indexed by A_n-type quivers, totally ordered sets, and zigzag posets. Our primary results establish that such decompositions exist if and only if specific algebraic criteria are met: the projective colimit conditions (PCC) for quiver and zigzag indexings, and the projectivity of cokernels of internal morphisms for totally ordered indexings. Furthermore, we provide corollaries for the existence of classical interval decompositions over Noetherian rings where finitely generated projective modules are free, such as principal ideal domains (PIDs). Finally, we establish the uniqueness of indecomposable projective constant decompositions within the Krull-Schmidt framework and extend the uniqueness of interval decompositions to persistence modules over integral domains.

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Introduction

Persistence modules arose in topological data analysis as an algebraic framework for encoding how topological features of a space evolve across a filtration. Initially motivated by problems in applied topology and data analysis, a persistence module assigns a vector space to each parameter value and linear maps between them that reflect inclusion-induced structure. Among the most fundamental examples are interval modules, which serve as the basic building blocks in decomposition results and underlie the barcode representation of persistence.

Originally defined as finite sequences of finite-dimensional vector spaces connected by linear maps, persistence modules were later formalized by Zomorodian and Carlsson as quiver representations indexed by the natural numbers and linked to graded modules over polynomial rings [16]. Cohen-Steiner, Edelsbrunner, and Harer extended this framework to real-indexed persistence modules, initiating the study of continuous persistence [5]. Subsequent work by Carlsson and de Silva introduced zigzag modules and related them to finite-dimensional representations of A_n-type quivers, thereby bringing Gabriel's theorem and tools from quiver representation theory into the persistence setting [4].

More recently, Botnan and Crawley-Boevey established structural and interval decomposition theorems for pointwise finite-dimensional persistence modules indexed by totally ordered sets and zigzag paths [3]. Furthermore, Botnan demonstrated the existence of interval decompositions for persistence modules indexed by infinite discrete zigzag posets [2]. We have also contributed alternative constructive proofs for the existence of interval decompositions of persistence modules indexed by both totally ordered sets and zigzag posets [6].

Although persistence modules have predominantly been studied over fields, recent work has extended these investigations to persistence modules over more general coefficient rings. Luo and Henselman-Petrusek established criteria for the existence of interval decompositions of pointwise freely and finitely generated persistence modules over principal ideal domains

(PIDs), initially for modules indexed by finite linear quivers [10], and subsequently for modules indexed by arbitrary totally ordered sets [11].

The purpose of this dissertation is to extend several structural results for persistence modules to the setting of unitary left modules over Noetherian rings R with identity (note that we omit the qualifier "unitary left" as no confusion will arise). To do this, we introduce a generalized notion of an interval module, called a *module of projective constant type* (see Definition 1.6). This module behaves in essentially the same way in that it is isomorphic to some fixed projective R-module on indices lying in the interval and zero elsewhere, with isomorphisms between nonzero components. Some motivation for this definition stems from the fact that an interval module is itself a module of projective constant type.

Fundamentally, we wish to understand when a persistence module can be decomposed as an internal direct sum of modules of projective constant type, which we call a *projective constant decomposition* (see Definition 1.7). In this dissertation, we prove that such a decomposition exists for pointwise finitely generated persistence modules indexed by an A_n-type quiver or a zigzag poset if and only if particular conditions are satisfied, which we call the *projective colimit conditions (PCC)* (see Definition 3.7). We also prove that a pointwise finitely generated persistence module indexed by a totally ordered poset admits a projective constant decomposition if and only if the cokernel of every internal morphism is projective. Our main contributions are stated in the proceeding theorems and corollaries.

Theorem A. Let R be a (not necessarily commutative) Noetherian ring with identity. Let $F : A_n \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated A_n-persistence module. Suppose that F satisfies the projective colimit conditions (PCC). Then F admits a projective constant decomposition.

Theorem B. Let R be a commutative Noetherian ring with identity. Let $F : P \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated and projective totally ordered persistence module.

Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then F admits a projective constant decomposition.

Theorem C. Let R be a commutative Noetherian ring with identity. Let $F : P \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$. Suppose that F satisfies the projective colimit conditions (PCC). Then F admits a projective constant decomposition.

The proofs of Theorem A and of Theorem C for zigzag posets with finite extrema are inspired by Ringel's decomposition of representations of A_n-type quivers into thin representations [14]. The proof of Theorem B generalizes our earlier argument showing the existence of interval decompositions for persistence modules over fields indexed by totally ordered sets [6, §6]. Finally, the proof of Theorem C in the case of zigzag posets with infinite extrema adapts the approach introduced by Botnan for decomposing persistence modules over fields indexed by infinite discrete zigzag posets [2].

Consequently, we also contribute the following corollaries regarding the existence of interval decompositions when, in addition to being a (commutative) Noetherian ring with identity, R has the property that every finitely generated projective R-module is free. Prominent examples of such rings include principal ideal domains, polynomial rings in finite variables over a PID or field, local Noetherian rings with identity, and rings of formal power series over a local Noetherian ring with identity. We discuss these examples further in Corollary 1.11.

Corollary D. Let R be a (not necessarily commutative) Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : A_n \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated A_n-persistence module. Suppose that F satisfies the projective colimit conditions (PCC). Then F admits an interval decomposition.

Corollary E. Let R be a commutative Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : P \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated and projective totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then F admits an interval decomposition.

Corollary F. Let R be a commutative Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : P \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$. Suppose that F satisfies the projective colimit conditions (PCC). Then F admits an interval decomposition.

In the specific case when R is a PID, Corollary E was proven by Luo and Henselman-Petrusek [11].

We contribute the following theorems to confirm that the stated conditions are indeed necessary whenever such projective constant (respectively, interval) decompositions exist.

Theorem G. Let R be a (not necessarily commutative) Noetherian ring with identity. Let $F : A_n \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated A_n-persistence module. Suppose that F admits a projective constant (respectively, interval) decomposition. Then F satisfies the projective colimit conditions (PCC).

Theorem H. Let R be a (not necessarily commutative) Noetherian ring with identity. Let $F : P \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated and projective totally ordered persistence module. Suppose that F admits a projective constant (respectively, interval) decomposition. Then the cokernel of every internal morphism $F_{x,y}$ is projective.

Theorem I. Let R be a (not necessarily commutative) Noetherian ring with identity. Let $F : P \to R\text{-}\mathbf{Mod}$ be a nonzero pointwise finitely generated zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$. Suppose that F admits a projective constant (respectively, interval) decomposition. Then F satisfies the projective colimit conditions (PCC).

In the specific context of interval decompositions over a PID, Theorem H was proven by Luo and Henselman-Petrusek [11].

Finally, we contribute the following uniqueness results for indecomposable projective constant decompositions in the setting of Krull-Schmidt categories and for interval decompositions over integral domains.

Theorem J. Let R be a (not necessarily commutative) ring with identity. Let P be a poset and let $F : P \to R$-**Mod** be a nonzero persistence module. Suppose that R-**Mod** is a Krull-Schmidt category. If F admits a projective constant decomposition, then F admits a unique indecomposable projective constant decomposition up to isomorphism and permutation of the summands.

Theorem K. Let R be an integral domain. Let P be a small poset category and let $F : P \to R$-**Mod** be a nonzero pointwise finitely generated persistence module. If F admits an interval decomposition, then this decomposition is unique up to isomorphism and permutation of the summands.

The proof of Theorem J leverages the renowned Krull-Remak-Schmidt-Azumaya theorem [1], while the proof of Theorem K is inspired by Luo and Henselman-Petrusek's proof of the uniqueness of interval decompositions of pointwise freely and finitely generated persistence modules over PIDs indexed by finite linear quivers [10, §4, Lemma 1].

This dissertation is organized as follows: Chapters 1 and 2 provide background and technical lemmas regarding persistence modules over Noetherian rings that will be useful in proving our main results. In Chapter 3, we define zigzag posets and zigzag persistence modules, and discuss some of their properties. Theorem A and Corollary D are proved in Chapter 4 on A_n-persistence modules; see Theorem 4.3 and Corollary 4.4. Theorem B and Corollary E are proved in Chapter 5 on totally ordered persistence modules; see Theorem 5.19 and Corollary 5.20. Theorem C and Corollary F are proved in Chapter 6 for zigzag persistence modules with finite extrema (see Theorem 6.1 and Corollary 6.2) and in Chapter 7 for zigzag persistence modules with infinite extrema (see Theorem 7.6, Theorem 7.7, and Corollary 7.8). Theorems G, H, and I, which establish the necessity of the given conditions to obtain decomposition results, are all proven in Chapter 8; see Theorems 8.3, 8.4, and 8.5. Theorems J and K regarding uniqueness are proved in Chapter 9; see Theorems 9.5 and 9.8.

This dissertation is essentially self-contained; we use only elementary results in module theory and Zorn's Lemma.

Chapter 1

Definitions and Notation

Fix a Noetherian ring R with identity, and let R-**Mod** denote the category of unitary left R-modules. Throughout this dissertation, we omit the qualifier "unitary left" when referring to such R-modules, as no ambiguity will arise from context. Any poset P defines a category called a poset category. We shall denote the poset category of P also by P.

Definition 1.1. A *P-persistence module* (or just *persistence module*) is a functor $F : P \to R$-**Mod** for some fixed poset category P. A *morphism* of persistence modules indexed by P is, by definition, a natural transformation of functors. The direct sum of persistence modules, persistence submodule, and restriction of a persistence module to a subposet of P are all defined in the natural way.

Notation 1.2. Let $\leq$ denote the partial ordering of the poset P. For $x \leq y$ in P, denote the R-module $F(x)$ by F_x and the *internal morphisms* $F(x \to y)$ by $F_{x,y}$. Since F is a functor, we have $F_{x,x} = \mathrm{id}_{F_x}$ and also $F_{y,z} \circ F_{x,y} = F_{x,z}$ for any $x \leq y \leq z$. Let $\ker[x, y] = \ker(F_{x,y})$, $\mathrm{im}[x, y] = \mathrm{im}(F_{x,y})$ and $\mathrm{coker}[x, y] = \mathrm{coker}(F_{x,y})$ with regard to the internal morphisms.

Definition 1.3. Let P be a poset and let $F : P \to R$-**Mod** be a persistence module. We say F is *pointwise finitely generated (p.f.g.)* if F_x is a finitely generated R-module for every $x \in P$. Likewise, we say F is *pointwise finitely generated and projective (p.f.g.p.)* if F_x is a finitely generated projective R-module for every $x \in P$.

Definition 1.4. Let P be a poset and I a subset of P.

(1) We say that I is *convex* if for all $x, y, z \in P$, we have $y \in I$ whenever $x, z \in I$ and $x \leq y \leq z$.

(2) We say that I is *connected* if for all $x, z \in I$, there exist $y_0, y_1, \ldots, y_n \in I$ such that

$$\begin{cases} y_0 = x, \\ y_n = z, \\ y_{i-1} \leq y_i \text{ or } y_{i-1} \geq y_i \text{ for each } i \in \{1, \dots, n\}. \end{cases}$$

(3) We call I an *interval* in P if I is nonempty, convex, and connected.

Definition 1.5. Let P be a poset and I an interval in P. Given an R-module M, define $F(I, M)$ by:

(1) for all $x \in P$,

$$F(I, M)_x = \begin{cases} M & \text{if } x \in I, \\ 0 & \text{if } x \notin I; \end{cases}$$

(2) for all $x \leq y$ in P,

$$F(I, M)_{x,y} = \begin{cases} \mathrm{id}_M & \text{if } x, y \in I, \\ 0 & \text{if } x \notin I \text{ or } y \notin I. \end{cases}$$

Definition 1.6. Let P be a poset. We say a persistence module indexed by P is *of projective constant type* if it is isomorphic to $F(I, M)$ for some interval I in P and for some projective R-module M.

Definition 1.7. Let P be a poset and let $F : P \to R$-**Mod** be a persistence module. A *projective constant decomposition* of F is an internal direct sum decomposition $F = \bigoplus_{G \in \mathscr{A}} G$ where $\mathscr{A}$ is a set of persistence submodules of F such that every $G \in \mathscr{A}$ is of projective constant type.

Definition 1.8. Let P be a poset. An *interval module* is a persistence module indexed by P which is isomorphic to $F(I, R)$ for some interval I in P.

Definition 1.9. Let P be a poset and let $F : P \to R$-**Mod** be a persistence module. An *interval decomposition* of F is an internal direct sum decomposition $F = \bigoplus_{G \in \mathscr{A}} G$ where $\mathscr{A}$ is a set of persistence submodules of F such that every $G \in \mathscr{A}$ is an interval module. If F admits an interval decomposition, we say F is *interval decomposable*.

Note that an interval decomposition is a special case of a projective constant decomposition since R is free and hence projective. Conversely, if every finitely generated projective R-module is free, we obtain the following useful lemma.

Lemma 1.10. *Let R be a Noetherian ring with identity such that every finitely generated projective R-module is free. Let P be a poset and let $F : P \to R$-**Mod** be a p.f.g. persistence module. Suppose that F admits a projective constant decomposition. Then F is interval decomposable.*

Proof. Let $F = \bigoplus_{G \in \mathscr{A}} G$ be a projective constant decomposition of F. Then each $G \in \mathscr{A}$ is isomorphic to $F(I, M)$ for some interval I in P and for some finitely generated projective R-module M. By assumption, M is free, so G is isomorphic to a direct sum of copies of $F(I, R)$. Thus, each G admits an interval decomposition, which shows that F is interval decomposable. $\square$

The following corollary presents some notable examples of rings that satisfy the conditions of Lemma 1.10.

Corollary 1.11. *Let R be one of the following rings:*

(1) a principal ideal domain;

(2) the polynomial ring in finite variables $k[x_1, \ldots, x_n]$, where k is a principal ideal domain or a field;

(3) a local (commutative) Noetherian ring with identity;

(4) the ring of formal power series $S[[x]]$, where S is a local (commutative) Noetherian ring with identity.

*Let P be a poset and let $F : P \to R$-**Mod** be a p.f.g. persistence module. Suppose that F admits a projective constant decomposition. Then F is interval decomposable.*

Proof. Immediate from Lemma 1.10 by the following observations:

(1) Every projective R-module over a PID is free.

(2) Quillen and Suslin independently proved that when k is a PID or a field, every finitely generated projective module over the polynomial ring $k[x_1, \ldots, x_n]$ is free; this result is now known as the Quillen–Suslin Theorem [13, 15]. Moreover, Hilbert's Basis Theorem ensures that $k[x_1, \ldots, x_n]$ is a Noetherian ring with identity when k is a PID or a field [7].

(3) Kaplansky proved that every projective R-module over a (not necessarily commutative) local ring with identity is free [8].

(4) When S is a local (commutative) Noetherian ring with identity, $S[[x]]$ is also a local (commutative) Noetherian ring with identity. Case (3) implies that every projective $S[[x]]$-module is free.

□

Definition 1.12. Let P be a poset and $x \in P$. We call an index x a *sink* if there are no morphisms $x \to y$ for $x \neq y$. Dually, we call an index x a *source* if there are no morphisms $y \to x$ for $x \neq y$.

Chapter 2

Decomposition Criteria over Noetherian Rings

Recall that a finitely generated module over a Noetherian ring R with identity is itself Noetherian and therefore satisfies the ascending chain condition. This property reflects the primary motivation for working over Noetherian rings, which imposes the necessary finiteness condition to obtain decomposition results. In this chapter, we establish two auxiliary results with regard to Noetherian R-modules. The first shows that certain chains of submodules must be finite, while the second provides a general criterion for decomposing a persistence module as a direct sum of submodules drawn from a specified collection.

Lemma 2.1. *Let R be a Noetherian ring with identity. Let M be a finitely generated projective R-module, and let S be a chain of R-submodules of M. Assume for all $A \in S$ that M/A is projective. Then S is a finite set.*

Proof. Since M is Noetherian, the set S has a maximal element, say A_1. Now M/A_1 projective implies $M = N_1 \oplus A_1$ for some projective R-submodule N_1 of M.

If $S = \{A_1\}$, we are done. If not, the set $S \setminus \{A_1\}$ has a maximal element, say A_2. We have a short exact sequence

$$0 \longrightarrow A_1/A_2 \hookrightarrow M/A_2 \twoheadrightarrow M/A_1 \longrightarrow 0.$$

The sequence splits since M/A_1 is projective, and thus A_1/A_2 is projective as it is a direct summand of M/A_2 which is projective. In particular, there exists a nonzero R-submodule N_2 of A_1 such that $A_1 = N_2 \oplus A_2$.

If $S = \{A_1, A_2\}$, we are done. If not, repeat this process for A_3 and so on. This process must terminate after a finite number of steps, for otherwise

$$N_1 \subset N_1 \oplus N_2 \subset N_1 \oplus N_2 \oplus N_3 \subset \cdots$$

is an infinite ascending chain of R-submodules of M, a contradiction since M satisfies the ascending chain condition. $\square$

The proof of the following lemma is essentially the proof in [6, Lemma 3.6] with the minor addition of Claim 2.2.1.

Lemma 2.2. *Let R be a Noetherian ring with identity. Let P be a poset and let $F : P \to R$-**Mod** be a p.f.g. persistence module. Let $\mathscr{E}$ be a set of nonzero submodules of F. Assume that for each nonzero direct summand H of F, there exists a direct summand G of H such that $G \in \mathscr{E}$. Then there exists a subset $\mathscr{A}$ of $\mathscr{E}$ giving an internal direct sum decomposition $F = \bigoplus_{G\in\mathscr{A}} G$.*

Proof. Let $\mathbf{S}$ be the set of all pairs $(\mathscr{A}, H)$ where:

$\mathscr{A}$ is a subset of $\mathscr{E}$,

H is a submodule of F,

such that we have an internal direct sum decomposition $F = \left(\bigoplus_{G\in\mathscr{A}} G\right) \oplus H$. The set $\mathbf{S}$ is nonempty because $(\emptyset, F) \in \mathbf{S}$.

Define a partial order on $\mathbf{S}$ by $(\mathscr{A}, H) \leq (\mathscr{B}, J)$ if $\mathscr{A} \subseteq \mathscr{B}$ and

$$H = \left(\bigoplus_{G\in\mathscr{B}-\mathscr{A}} G\right) \oplus J.$$

Claim 2.2.1. For any chain $\mathbf{T}$ in $\mathbf{S}$ and fixed $x \in P$, there exists a pair $(\mathscr{B}, J)$ in $\mathbf{T}$ such that for all $(\mathscr{A}, H)$ in $\mathbf{T}$, we have $J_x \subseteq H_x$.

Proof of Claim 2.2.1. Suppose by way of contradiction that no such pair $(\mathscr{B}, J)$ exists. Choose any $(\mathscr{A}^1, H^1)$ in $\mathbf{T}$. Then there exists $(\mathscr{A}^2, H^2)$ in $\mathbf{T}$ such that $H_x^2 \subsetneq H_x^1$. Since $\mathbf{T}$ is a chain, we have either $(\mathscr{A}^1, H^1) \leq (\mathscr{A}^2, H^2)$ or $(\mathscr{A}^2, H^2) \leq (\mathscr{A}^1, H^1)$. The latter case is not possible since this would imply $H^1 \subseteq H^2$. Therefore $(\mathscr{A}^1, H^1) \leq (\mathscr{A}^2, H^2)$.

Repeating this process, we obtain an infinite sequence

$$(\mathscr{A}^1, H^1), (\mathscr{A}^2, H^2), (\mathscr{A}^3, H^3), \ldots$$

in $\mathbf{T}$. But then

$$\bigoplus_{G \in \mathscr{A}^1} G_x \subset \bigoplus_{G \in \mathscr{A}^2} G_x \subset \bigoplus_{G \in \mathscr{A}^3} G_x \subset \cdots$$

is an infinite ascending chain of R-submodules of F_x, a contradiction since F_x is Noetherian and satisfies the ascending chain condition. $\square$

Claim 2.2.2. Every chain in $\mathbf{S}$ has an upper bound. More precisely:

For any chain $\mathbf{T}$ in $\mathbf{S}$, let

$$\mathscr{C} = \bigcup_{(\mathscr{A}, H) \in \mathbf{T}} \mathscr{A}, \qquad N = \bigcap_{(\mathscr{A}, H) \in \mathbf{T}} H.$$

Then $(\mathscr{C}, N)$ lies in $\mathbf{S}$ and is an upper bound of $\mathbf{T}$.

Proof of Claim 2.2.2. There are two things to check:

(1) $F = \left(\bigoplus_{G \in \mathscr{C}} G\right) \oplus N$;

(2) $H = \left(\bigoplus_{G \in \mathscr{C} - \mathscr{A}} G\right) \oplus N$ for all $(\mathscr{A}, H) \in \mathbf{T}$.

Thus for each $x \in P$, we need to check:

(1) $F_x = \left(\bigoplus_{G \in \mathscr{C}} G_x\right) \oplus N_x$;

(2) $H_x = \left(\bigoplus_{G \in \mathscr{C} - \mathscr{A}} G_x\right) \oplus N_x$ for all $(\mathscr{A}, H) \in \mathbf{T}$.

Fix $x \in P$. By Claim 2.2.1, we can choose a pair $(\mathscr{B}, J) \in \mathbf{T}$ such that for all $(\mathscr{A}, H)$ in $\mathbf{T}$, we have $J_x \subseteq H_x$. Now let $(\mathscr{A}, H) \in \mathbf{T}$. We make the following observations:

(a) If $(\mathscr{A}, H) \leq (\mathscr{B}, J)$, then
$$H_x = \left(\bigoplus_{G \in \mathscr{B} - \mathscr{A}} G_x\right) \oplus J_x;$$
hence $J_x \subseteq H_x$.

(b) If $(\mathscr{B}, J) \leq (\mathscr{A}, H)$, then
$$J_x = \left(\bigoplus_{G \in \mathscr{A} - \mathscr{B}} G_x\right) \oplus H_x.$$
The above equality shows $H_x \subseteq J_x$. But also $J_x \subseteq H_x$, hence $J_x = H_x$ and $G_x = 0$ for all $G \in \mathscr{A} - \mathscr{B}$.

(c) By (a) and (b), we have $N_x = J_x$, and $G_x = 0$ for all $G \in \mathscr{C} - \mathscr{B}$.

We deduce that:

$$\begin{aligned} F_x &= \left(\bigoplus_{G \in \mathscr{B}} G_x\right) \oplus J_x \\ &= \left(\bigoplus_{G \in \mathscr{C}} G_x\right) \oplus N_x && \text{by (c).} \end{aligned}$$

If $(\mathscr{A}, H) \leq (\mathscr{B}, J)$, then

$$\begin{aligned} H_x &= \left(\bigoplus_{G \in \mathscr{B} - \mathscr{A}} G_x\right) \oplus J_x \\ &= \left(\bigoplus_{G \in \mathscr{C} - \mathscr{A}} G_x\right) \oplus N_x && \text{by (c).} \end{aligned}$$

If $(\mathscr{B}, J) \leq (\mathscr{A}, H)$, then

$$\begin{aligned} H_x &= J_x && \text{by (b)} \\ &= \left(\bigoplus_{G \in \mathscr{C} - \mathscr{A}} G_x \right) \oplus N_x && \text{by (c).} \end{aligned}$$

□

By Claim 2.2.2, we can apply Zorn's lemma to deduce there exists a pair, say $(\mathscr{A}, H)$, which is maximal in $\mathbf{S}$.

Suppose that $H \neq 0$. Then by assumption there exists an internal direct sum decomposition $H = G \oplus J$ where $G \in \mathscr{E}$. Let $\mathscr{B} = \mathscr{A} \cup \{G\}$. Then we have $(\mathscr{B}, J) \in \mathbf{S}$ and $(\mathscr{A}, H) < (\mathscr{B}, J)$, a contradiction to the maximality of $(\mathscr{A}, H)$.

Therefore $H = 0$. Hence $F = \bigoplus_{G \in \mathscr{A}} G$ is a desired decomposition. □

Chapter 3

Zigzag Posets

In this chapter, we introduce zigzag persistence modules by first providing a precise definition of a zigzag poset. We then present several observations and definitions that will be essential for working with zigzag persistence modules throughout the dissertation, including the projective colimit conditions (PCC), which underpin some of our main results.

Notation 3.1. We write Γ for one of the following sets:

(1) a finite set $\{1, 2, \ldots, n\}$ for $n \geq 2$;

(2) the set of whole numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$; or

(3) the set of integers $\mathbb{Z}$.

The following definition of a zigzag poset is taken from [6, Definition 7.2].

Definition 3.2. Let P be a poset. Let $\{z_i \mid i \in \Gamma\}$ be a sequence of elements of P indexed by Γ (see Notation 3.1). We call P a *zigzag poset with extrema* $\{z_i \mid i \in \Gamma\}$ if there exists a total order $\sqsubseteq$ on the set P satisfying the following conditions:

(1) for all $i, j \in \Gamma$:

$$i < j \quad \Longrightarrow \quad z_i \sqsubset z_j;$$

(2) for all $x, y \in P$:

$$x \leq y \quad \Longrightarrow \quad \text{there exists } i \in \Gamma \text{ such that } \begin{cases} i + 1 \in \Gamma, \\ z_i \sqsubseteq x \sqsubseteq z_{i+1}, \\ z_i \sqsubseteq y \sqsubseteq z_{i+1}; \end{cases}$$

(3) either:

$$\text{for all } i \in \Gamma \text{ such that } i+1 \in \Gamma,$$

$$\text{for all } x, y \in P \text{ such that } z_i \sqsubseteq x \sqsubseteq y \sqsubseteq z_{i+1}\text{:}$$

$$\begin{cases} i \text{ is odd} \implies x \leq y, \\ i \text{ is even} \implies y \leq x, \end{cases}$$

or:

$$\text{for all } i \in \Gamma \text{ such that } i+1 \in \Gamma,$$

$$\text{for all } x, y \in P \text{ such that } z_i \sqsubseteq x \sqsubseteq y \sqsubseteq z_{i+1}\text{:}$$

$$\begin{cases} i \text{ is odd} \implies y \leq x, \\ i \text{ is even} \implies x \leq y. \end{cases}$$

In this case, we call $\sqsubseteq$ the *associated total order.*

Lemma 3.3. *Let P be a zigzag poset with extrema $\{z_i \mid i \in \Gamma\}$. Let $\sqsubseteq$ be the associated total order. Then we have the following.*

(1) For all $x \in P$, there exists $i \in \Gamma$ such that $i+1 \in \Gamma$ and $z_i \sqsubseteq x \sqsubseteq z_{i+1}$.

(2) For all $x, y \in P$ such that $x \leq y$, for all $i \in \Gamma$ such that

$$x \sqsubseteq z_i \sqsubseteq y \quad \text{or} \quad y \sqsubseteq z_i \sqsubseteq x,$$

we have $x = z_i$ or $y = z_i$.

(3) For all $x, y, z \in P$ such that $x \leq y \leq z$, we have

$$x \sqsubseteq y \sqsubseteq z \quad \text{or} \quad z \sqsubseteq y \sqsubseteq x.$$

Proof. (1) Follows from condition (2) in Definition 3.2 when $x = y$.

(2) Suppose that $x \sqsubseteq z_i \sqsubseteq y$. By condition (2) in Definition 3.2, there exists $j \in \Gamma$ such that $j + 1 \in \Gamma$,

$$z_j \sqsubseteq x \sqsubseteq z_{j+1},$$
$$z_j \sqsubseteq y \sqsubseteq z_{j+1}.$$

Hence we have

$$z_j \sqsubseteq x \sqsubseteq z_i \sqsubseteq y \sqsubseteq z_{j+1}.$$

By condition (1) in Definition 3.2, we have $z_i = z_j$ or $z_i = z_{j+1}$. It follows that $z_i = x$ or $z_i = y$. Similarly if $y \sqsubseteq z_i \sqsubseteq x$.

(3) By condition (2) in Definition 3.2, there exists $i \in \Gamma$ such that $i + 1 \in \Gamma$,

$$z_i \sqsubseteq x \sqsubseteq z_{i+1},$$
$$z_i \sqsubseteq z \sqsubseteq z_{i+1}.$$

Then we also have $z_i \sqsubseteq y \sqsubseteq z_{i+1}$ because otherwise, by (2):

$$\begin{aligned} y \sqsubset z_i \quad &\Longrightarrow \quad x = z_i \text{ and } z = z_i \\ &\Longrightarrow \quad y = z_i, \qquad \text{a contradiction;} \end{aligned}$$

similarly,

$$\begin{aligned} z_{i+1} \sqsubset y \quad &\Longrightarrow \quad x = z_{i+1} \text{ and } z = z_{i+1} \\ &\Longrightarrow \quad y = z_{i+1}, \qquad \text{a contradiction.} \end{aligned}$$

It is now easy to deduce the desired result from condition (3) in Definition 3.2. $\square$

Lemma 3.4. *Let P be a zigzag poset with extrema $\{z_i \mid i \in \Gamma\}$. Let $\sqsubseteq$ be the associated total order. Let I be a nonempty subset of P. Then the following statements are equivalent:*

(1) I is an interval in the poset P.

(2) For all $x, y, z \in P$ such that $x \sqsubseteq y \sqsubseteq z$, we have $y \in I$ whenever $x, z \in I$.

Proof. (1)$\Longrightarrow$(2): Let $x, y, z \in P$ such that $x \sqsubseteq y \sqsubseteq z$ and $x, z \in I$.

Since I is connected, there exists $y_0, y_1, \ldots, y_m \in I$ such that $y_0 = x$, $y_m = z$, and

$$y_{i-1} \leq y_i \quad \text{or} \quad y_{i-1} \geq y_i \qquad \text{for all } i \in \{1, \ldots, m\}.$$

Choose the smallest $i \in \{1, \ldots, m\}$ such that $y \sqsubseteq y_i$. Then we have $y_{i-1} \sqsubseteq y \sqsubseteq y_i$.

By condition (2) in Definition 3.2, there exists $j \in \Gamma$ such that $j + 1 \in \Gamma$ and

$$z_j \sqsubseteq y_{i-1} \sqsubseteq z_{j+1},$$
$$z_j \sqsubseteq y_i \sqsubseteq z_{j+1}.$$

Hence we have $z_j \sqsubseteq y_{i-1} \sqsubseteq y \sqsubseteq y_i \sqsubseteq z_{j+1}$.

By condition (3) in Definition 3.2, it follows that

$$y_{i-1} \leq y \leq y_i \qquad \text{or} \qquad y_i \leq y \leq y_{i-1}.$$

Since I is convex, it follows that $y \in I$.

(2)$\Longrightarrow$(1): To prove that I is convex, let $x, y, z \in P$ such that $x, z \in I$ and $x \leq y \leq z$.

By Lemma 3.3(3), we have

$$x \sqsubseteq y \sqsubseteq z \qquad \text{or} \qquad z \sqsubseteq y \sqsubseteq x.$$

Since $x, z \in I$, we have $y \in I$.

To prove that I is connected, let $x, z \in I$.

By Lemma 3.3(1), there exist $i, j \in \Gamma$ such that $i+1, j+1 \in \Gamma$ and

$$z_i \sqsubseteq x \sqsubseteq z_{i+1},$$
$$z_j \sqsubseteq z \sqsubseteq z_{j+1}.$$

If $i = j$, then by condition (3) in Definition 3.2, we have $x \leq z$ or $x \geq z$ (in which case we are done).

Suppose that $i \neq j$. Without loss of generality we may assume $i < j$. By condition (1) in Definition 3.2, we have

$$x \sqsubseteq z_{i+1} \sqsubseteq z_{i+2} \sqsubseteq \cdots \sqsubseteq z_j \sqsubseteq z.$$

It follows that $z_{i+1}, z_{i+2}, \ldots, z_j \in I$. Let

$$\begin{cases} y_0 = x, \\ y_{j-i+1} = z, \\ y_r = z_{i+r} \quad \text{for all } r \in \{1, \ldots, j-i\}. \end{cases}$$

Then by condition (3) in Definition 3.2, we have

$$y_{r-1} \leq y_r \quad \text{or} \quad y_{r-1} \geq y_r \qquad \text{for all } r \in \{1, \ldots, j-i+1\}.$$

□

The above lemmas, in effect, guarantee that our definition of a zigzag poset behaves in the expected way.

Notation 3.5. Let P be a zigzag poset with extrema $\{z_i \mid i \in \Gamma\}$. Let $\sqsubseteq$ be the associated total order. Given $x \sqsubseteq y$ in P, we use the notation below to represent the following intervals in P (when such subsets are nonempty):

(1) $[x, y] = \{z \in P \mid x \sqsubseteq z \sqsubseteq y\}$;

(2) $[x, y) = \{z \in P \mid x \sqsubseteq z \sqsubset y\}$;

(3) $(x, y] = \{z \in P \mid x \sqsubset z \sqsubseteq y\}$;

(4) $(x, y) = \{z \in P \mid x \sqsubset z \sqsubset y\}$.

In particular, when $x = y$ we have $[x, x] = \{x\}$.

Definition 3.6. Let P be a zigzag poset with extrema $\{z_i \mid i \in \Gamma\}$. A functor $F : P \to R$-**Mod** is a *zigzag persistence module with extrema* $\{z_i \mid i \in \Gamma\}$, or more generally a *zigzag persistence module.*

Definition 3.7. Let P be a zigzag poset with extrema $\{z_i \mid i \in \Gamma\}$. Let $\sqsubseteq$ be the associated total order. We say a zigzag persistence module $F : P \to R$-**Mod** satisfies the *projective colimit conditions (PCC)* if the following conditions hold for all indices $x \sqsubseteq y$ in P:

(C1) $\mathrm{colim}(F|_{[x,y]})$ is projective;

(C2) $\mathrm{coker}(F_x \to \mathrm{colim}(F|_{[x,y]}))$ is projective;

(C3) $\mathrm{coker}(F_y \to \mathrm{colim}(F|_{[x,y]}))$ is projective;

(C4) $\mathrm{coker}(F_x \oplus F_y \to \mathrm{colim}(F|_{[x,y]}))$ is projective.

Lemma 3.8. *Let $F : P \to R$-**Mod** be a p.f.g. zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$ satisfying the PCC. Let $\leq$ be the partial order and $\sqsubseteq$ be the associated total order. Then:*

(1) F_x is a finitely generated projective R-module for every $x \in P$;

(2) coker($F_{x,y}$) is projective for every $x \leq y$ in P;

(3) Any direct summand of F also satisfies the PCC.

Proof. (1) Immediate from **(C1)** when $x = y$.

(2) If $x \leq y$ and $x \sqsubseteq y$, then $\operatorname{colim}(F|_{[x,y]}) = F_y$, hence **(C2)** implies $\operatorname{coker}(F_{x,y})$ projective. Similarly, if $x \leq y$ and $y \sqsubseteq x$, then $\operatorname{colim}(F|_{[y,x]}) = F_y$, hence **(C3)** implies $\operatorname{coker}(F_{x,y})$ is projective.

(3) This follows from the fact that colimits commute with coproducts and cokernels in the category R-**Mod** [12, §2.7, Corollary 2]. In particular, it is essential that F is pointwise finitely generated so that if $F = \oplus_{G \in \mathscr{A}} G$ is a decomposition of F, then only finitely many G_x are nontrivial for any fixed $x \in P$. $\square$

To end this chapter, we introduce the notion of a peak of a zigzag persistence module, along with some useful observations regarding peaks that will be used later in this dissertation. Our definition is adapted from Ringel's work on the representation theory of Dynkin quivers [14].

Definition 3.9. Let $F : P \to R$-**Mod** be a zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$. Let $\sqsubseteq$ be the associated total order. We call an index $x \in P$ a *peak* of F provided that for all morphisms $y \to z$:

(1) the map $F_{y,z}$ is injective whenever $y \sqsubseteq z \sqsubseteq x$ or $x \sqsubseteq z \sqsubseteq y$;

(2) the map $F_{y,z}$ is surjective whenever $z \sqsubseteq y \sqsubseteq x$ or $x \sqsubseteq y \sqsubseteq z$.

Lemma 3.10. *Let P be a zigzag poset with extrema $\{z_i \mid i \in \Gamma\}$.*

*(1) If $F : P \to R$-**Mod** is a zigzag persistence module of projective constant type over an interval I in P, then F has a peak at every $x \in I$.*

(2) A direct sum of zigzag persistence modules indexed by P with x as a peak also has x as a peak.

*(3) If $F : P \to R$-**Mod** is a zigzag persistence module with x as a peak, then any direct summand of F also has x as a peak.*

Proof. The above statements are easily verified from Definition 3.9 and the properties of injective and surjective maps with respect to direct sums and direct summands. □

Chapter 4

A_n-Persistence Modules

An A_n-persistence module (when $n \geq 2$) is a special case of a zigzag persistence module with finite extrema (see Definition 3.6). The results on projective constant decompositions of A_n-persistence modules developed in this chapter play a critical role in establishing subsequent results for totally ordered persistence modules in Chapter 5 and zigzag persistence modules in Chapters 6 and 7. For this reason, we devote an entire chapter to this special case. Moreover, A_n-persistence modules bear a strong resemblance to A_n-type quiver representations, albeit with additional structure arising from the persistence setting.

Definition 4.1. Given a quiver of type A_n with the simply laced Dynkin diagram

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & \mathrm{n} \end{tikzcd}$$

we can view A_n as a poset with elements $\{1, 2, \ldots, n\}$ and a partial order denoted by $\leq$. A functor $F : A_n \to R$-**Mod** is an A_n*-persistence module*. It is easy to see that an A_n-persistence module with $n \geq 2$ is in fact a zigzag persistence module with finite extrema and whose associated total order $\sqsubseteq$ is the natural total ordering of the integer indices (see Definitions 3.2 and 3.6).

Definition 4.2. Let $F : A_n \to R$-**Mod** be an A_n-persistence module. We say F satisfies the *projective colimit conditions (PCC)* if:

(1) When $n = 1$: F_1 is a projective R-module;

(2) When $n \geq 2$: F satisfies the PCC when viewed as a zigzag persistence module (see Definition 3.7).

Some care is required in the above definition, since an A_1-persistence module is not, strictly speaking, a zigzag persistence module. However, the conditions align in the obvious way to those provided in Definition 3.7.

We now prove the main result of this chapter, namely Theorem A. The proof is modeled after Ringel's decomposition of representations of A_n-type quivers [14].

Theorem 4.3. *Let $F : A_n \to R$-$\boldsymbol{Mod}$ be a nonzero p.f.g. A_n-persistence module satisfying the PCC. Then F admits a projective constant decomposition.*

Proof. We will proceed by induction on n. The case for $n = 1$ is a single projective R-module F_1, hence trivially of projective constant type.

For $n = 2$, this is the case with either one map $F_{1,2} : F_1 \to F_2$ or $F_{2,1} : F_2 \to F_1$. Without loss of generality, consider the case $F_{1,2} : F_1 \to F_2$. Since $\operatorname{coker}[1, 2]$ is projective by Lemma 3.8, $F_2 = \operatorname{im}[1, 2] \oplus G_2$ for some R-submodule G_2. Moreover, $F_1/\ker[1, 2] \cong \operatorname{im}[1, 2]$ is projective since $\operatorname{im}[1, 2]$ is a direct summand of F_2. Hence $F_1 = \ker[1, 2] \oplus G_1$ for some R-submodule $G_1 \cong \operatorname{im}[1, 2]$. We decompose F in the following way.

$$\begin{array}{ccc}
F_1 & & F_2 \\
\parallel & & \parallel \\
\ker[1,2] & & \\
\oplus & & \\
G_1 & \xrightarrow{\cong} & \operatorname{im}[1,2] \\
 & & \oplus \\
 & & G_2
\end{array}$$

Each row in the diagram represents a submodule of F of projective constant type. Hence F admits a projective constant decomposition.

Now let $n \geq 3$ and assume the inductive hypothesis holds up to $n - 1$. Let $\sqsubseteq$ be the associated total order when viewing F as a zigzag persistence module.

Claim 4.3.1. For any fixed x such that $1 \sqsubset x \sqsubset n$, the persistence module F can be written as $F = G \oplus H \oplus J$, where H has x as a peak (see Definition 3.9), the support of G is in $\{1, \ldots, x-1\}$, and the support of J is in $\{x+1, \ldots, n\}$.

Proof of Claim 4.3.1. Fix $1 \sqsubset x \sqsubset n$. By the inductive hypothesis, $F|_{[1,x]} = G' \oplus H'$, where G' and H' decompose as a direct sum of persistence modules of projective constant type, $G'_x = 0$, and H' has x as a peak (see Lemma 3.10).

Likewise, by the inductive hypothesis, $F|_{[x,n]} = H'' \oplus J''$ where H'' and J'' decompose as a direct sum of persistence modules of projective constant type, $J''_x = 0$, and H'' has x as a peak. Since $G'_x = 0 = J''_x$, we must have $F_x = H'_x = H''_x$. Define the persistence submodules G, H, and J as follows:

$$G_y = \begin{cases} G'_y & \text{if } y \sqsubset x, \\ 0 & \text{if } y \sqsupseteq x; \end{cases} \qquad H_y = \begin{cases} H'_y & \text{if } y \sqsubset x, \\ H''_y & \text{if } y \sqsupseteq x; \end{cases} \qquad J_y = \begin{cases} 0 & \text{if } y \sqsubset x, \\ J''_y & \text{if } y \sqsupseteq x. \end{cases}$$

Then $F = G \oplus H \oplus J$ is the desired decomposition. $\square$

Applying Claim 4.3.1 to F for $x = 2$, we can write $F = G \oplus H \oplus J$, where H has a peak at 2, the support of G is in $\{1\}$, and the support of J is in $\{3, \ldots, n\}$.

Applying Claim 4.3.1 again to H for $x = n-1$, we can write $H = \tilde{G} \oplus \tilde{H} \oplus \tilde{J}$ where $\tilde{H}$ has a peak at $n-1$, the support of $\tilde{G}$ is in $\{1, \ldots, n-2\}$, and the support of $\tilde{J}$ is in $\{n\}$. In particular, note that $\tilde{H}$ also has a peak at 2 as it is a direct summand of H (see Lemma 3.10).

This yields a decomposition $F = G \oplus \tilde{G} \oplus \tilde{H} \oplus \tilde{J} \oplus J$. Now $G, \tilde{G}, J$ and $\tilde{J}$ satisfy the PCC by Lemma 3.8, and hence all admit projective constant decompositions by the inductive hypothesis. It remains to show that $\tilde{H}$ admits a projective constant decomposition, where $\tilde{H}$ has peaks at 2 and $n-1$.

When $n \geq 4$, all the internal morphisms of $\tilde{H}$ between 2 and $n-1$ are, in fact, isomorphisms. Therefore we reduce to the case of showing a projective constant decomposition exists for any A_3-persistence module F with a peak at 2. To do this, we consider three separate cases based on the possible configurations of morphisms in F.

Case 1. Consider the totally ordered A_3-persistence modules with the following diagrams.

$$\begin{tikzcd} F_1 \arrow[r, hook, "F_{1,2}"] \arrow[rr, bend left, "F_{1,3}"] & F_2 \arrow[r, two heads, "F_{2,3}"] & F_3 \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} F_1 & F_2 \arrow[l, two heads, "F_{2,1}"'] & F_3. \arrow[l, hook', "F_{3,2}"'] \arrow[ll, bend right, "F_{3,1}"'] \end{tikzcd}$$

Figure 4.1: A_3-persistence modules in Case 1

Without loss of generality, we will prove the results for the left diagram. Note that $F_{1,2}$ is injective and $F_{2,3}$ is surjective since F has a peak at 2.

Observe from Lemma 3.8 that $\mathrm{coker}[1,3] = F_3/\mathrm{im}[1,3] \cong F_2/(\mathrm{im}[1,2]+\ker[2,3])$ is projective, $F_2/\ker[2,3] \cong F_3$ is projective, and $\mathrm{coker}[1,2] = F_2/\mathrm{im}[1,2]$ is projective. We have the following short exact sequences:

$$0 \longrightarrow \frac{\mathrm{im}[1,2]}{\mathrm{im}[1,2] \cap \ker[2,3]} \hookrightarrow \frac{F_2}{\ker[2,3]} \twoheadrightarrow \frac{F_2}{\mathrm{im}[1,2] + \ker[2,3]} \longrightarrow 0;$$

$$0 \longrightarrow \frac{\ker[2,3]}{\mathrm{im}[1,2] \cap \ker[2,3]} \hookrightarrow \frac{F_2}{\mathrm{im}[1,2]} \twoheadrightarrow \frac{F_2}{\mathrm{im}[1,2] + \ker[2,3]} \longrightarrow 0.$$

Both sequences split, implying $\mathrm{im}[1,2]/(\mathrm{im}[1,2] \cap \ker[2,3])$ and $\ker[2,3]/(\mathrm{im}[1,2] \cap \ker[2,3])$ are projective. Furthermore, we have the following short exact sequences:

$$0 \longrightarrow \mathrm{im}[1,2] \cap \ker[2,3] \hookrightarrow \mathrm{im}[1,2] \twoheadrightarrow \frac{\mathrm{im}[1,2]}{\mathrm{im}[1,2] \cap \ker[2,3]} \longrightarrow 0;$$

$$0 \longrightarrow \mathrm{im}[1,2] \cap \ker[2,3] \hookrightarrow \ker[2,3] \twoheadrightarrow \frac{\ker[2,3]}{\mathrm{im}[1,2] \cap \ker[2,3]} \longrightarrow 0.$$

Both sequences split, yielding the following decompositions and isomorphisms for some R-submodules G, K and H:

$$
\begin{aligned}
&\mathrm{im}[1,2] = (\mathrm{im}[1,2] \cap \ker[2,3]) \oplus G; \\
&\ker[2,3] = (\mathrm{im}[1,2] \cap \ker[2,3]) \oplus K; \\
&\mathrm{im}[1,2] + \ker[2,3] = (\mathrm{im}[1,2] \cap \ker[2,3]) \oplus G \oplus K; \\
&F_2 = (\mathrm{im}[1,2] \cap \ker[2,3]) \oplus G \oplus K \oplus H; \\
&F_3 \cong H \oplus G; \\
&F_1 \cong \mathrm{im}[1,2].
\end{aligned}
$$

Hence, we can decompose F in the following way.

$$
\begin{tikzcd}
F_1 \arrow[r, hook, "F_{1,2}"] & F_2 \arrow[r, two heads, "F_{2,3}"] & F_3 \\
\| & \| & \| \\
 & H \arrow[r, "\cong"] & F_{2,3}(H) \\
 & \oplus & \oplus \\
F_{1,2}^{-1}(G) \arrow[r, "\cong"] & G \arrow[r, "\cong"] & F_{2,3}(G) \\
\oplus & \oplus & \\
F_{1,2}^{-1}(\mathrm{im}[1,2] \cap \ker[2,3]) \arrow[r, "\cong"] & \mathrm{im}[1,2] \cap \ker[2,3] & \\
 & \oplus & \\
 & K &
\end{tikzcd}
$$

Figure 4.2: Decomposition of A_3-persistence module in Case 1

Each row in the diagram represents a submodule of F of projective constant type. Hence F admits a projective constant decomposition.

Case 2. Consider the A_3-persistence module with the following diagram.

$$
\begin{tikzcd}
F_1 \arrow[r, hook, "F_{1,2}"] & F_2 & F_3 \arrow[l, hook', "F_{3,2}"']
\end{tikzcd}
$$

Figure 4.3: A_3-persistence module in Case 2

Note that $F_{1,2}$ and $F_{3,2}$ are both injective since F has a peak at 2. We have $\operatorname{colim}(F|_{[1,3]}) = F_2$, and also $\operatorname{coker}[1,2]$ and $\operatorname{coker}[3,2]$ are projective by Lemma 3.8. Moreover,

$$\operatorname{coker}(F_1 \oplus F_3 \to F_2) = \frac{F_2}{\operatorname{im}[1,2] + \operatorname{im}[3,2]}$$

and is projective by **(C4)**. We have the following short exact sequences:

$$0 \longrightarrow \frac{\operatorname{im}[1,2]}{\operatorname{im}[1,2] \cap \operatorname{im}[3,2]} \hookrightarrow \frac{F_2}{\operatorname{im}[3,2]} \twoheadrightarrow \frac{F_2}{\operatorname{im}[1,2] + \operatorname{im}[3,2]} \longrightarrow 0;$$

$$0 \longrightarrow \frac{\operatorname{im}[3,2]}{\operatorname{im}[1,2] \cap \operatorname{im}[3,2]} \hookrightarrow \frac{F_2}{\operatorname{im}[1,2]} \twoheadrightarrow \frac{F_2}{\operatorname{im}[1,2] + \operatorname{im}[3,2]} \longrightarrow 0.$$

Both sequences split, implying $\operatorname{im}[1,2]/(\operatorname{im}[1,2] \cap \operatorname{im}[3,2])$ and $\operatorname{im}[3,2]/(\operatorname{im}[1,2] \cap \operatorname{im}[3,2])$ are projective. Furthermore, we have the following short exact sequences:

$$0 \longrightarrow \operatorname{im}[1,2] \cap \operatorname{im}[3,2] \hookrightarrow \operatorname{im}[1,2] \twoheadrightarrow \frac{\operatorname{im}[1,2]}{\operatorname{im}[1,2] \cap \operatorname{im}[3,2]} \longrightarrow 0;$$

$$0 \longrightarrow \operatorname{im}[1,2] \cap \operatorname{im}[3,2] \hookrightarrow \operatorname{im}[3,2] \twoheadrightarrow \frac{\operatorname{im}[3,2]}{\operatorname{im}[1,2] \cap \operatorname{im}[3,2]} \longrightarrow 0.$$

Both sequences split, yielding the following decompositions and isomorphisms for some R-submodules G, K and H:

$$\operatorname{im}[1,2] = (\operatorname{im}[1,2] \cap \operatorname{im}[3,2]) \oplus G;$$

$$\operatorname{im}[3,2] = (\operatorname{im}[1,2] \cap \operatorname{im}[3,2]) \oplus K;$$

$$\operatorname{im}[1,2] + \operatorname{im}[3,2] = (\operatorname{im}[1,2] \cap \operatorname{im}[3,2]) \oplus G \oplus K;$$

$$F_2 = (\operatorname{im}[1,2] + \operatorname{im}[3,2]) \oplus H = (\operatorname{im}[1,2] \cap \operatorname{im}[3,2]) \oplus G \oplus K \oplus H;$$

$$F_1 \cong \operatorname{im}[1,2];$$

$$F_3 \cong \operatorname{im}[3,2].$$

Hence, we can decompose F in the following way.

$$
\begin{tikzcd}
F_1 \arrow[r, hook, "F_{1,2}"] \arrow[d, phantom, "\|"] & F_2 \arrow[d, phantom, "\|"] & F_3 \arrow[l, hook', "F_{3,2}"'] \arrow[d, phantom, "\|"] \\
 & H & \\
 & \oplus & \\
F_{1,2}^{-1}(G) \arrow[r, "\cong"] & G & \\
\oplus & \oplus & \\
F_{1,2}^{-1}(\mathrm{im}[1,2] \cap \mathrm{im}[3,2]) \arrow[r, "\cong"] & \mathrm{im}[1,2] \cap \mathrm{im}[3,2] & F_{3,2}^{-1}(\mathrm{im}[1,2] \cap \mathrm{im}[3,2]) \arrow[l, "\cong"'] \\
 & \oplus & \oplus \\
 & K & F_{3,2}^{-1}(K) \arrow[l, "\cong"']
\end{tikzcd}
$$

Figure 4.4: Decomposition of A_3-persistence module in Case 2

Each row in the diagram represents a submodule of F of projective constant type. Hence F admits a projective constant decomposition.

Case 3. Consider the A_3-persistence module with the following diagram.

$$
\begin{tikzcd}
F_1 & F_2 \arrow[l, two heads, "F_{2,1}"'] \arrow[r, two heads, "F_{2,3}"] & F_3
\end{tikzcd}
$$

Figure 4.5: A_3-persistence module in Case 3

Note that $F_{2,1}$ and $F_{2,3}$ are both surjective since F has a peak at 2. We have $\operatorname{colim}(F|_{[1,3]}) = (F_1 \oplus F_3)/E$ where $E = \{(F_{2,1}(a), -F_{2,3}(a)) : a \in F_2\}$. By Lemma 3.8, $F_2/\ker[2,1] \cong \mathrm{im}[2,1] = F_1$ is projective, and similarly $F_2/\ker[2,3] \cong \mathrm{im}[2,3] = F_3$ is projective. Moreover,

$$\frac{F_1 \oplus F_3}{E} \cong \frac{F_2}{\ker[2,1] + \ker[2,3]}$$

and is projective by **(C1)**. We have the following short exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & \dfrac{\ker[2,1]}{\ker[2,1] \cap \ker[2,3]} \arrow[r, hook] & \dfrac{F_2}{\ker[2,3]} \arrow[r, two heads] & \dfrac{F_2}{\ker[2,1] + \ker[2,3]} \arrow[r] & 0;
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \dfrac{\ker[2,3]}{\ker[2,1] \cap \ker[2,3]} \arrow[r, hook] & \dfrac{F_2}{\ker[2,1]} \arrow[r, two heads] & \dfrac{F_2}{\ker[2,1] + \ker[2,3]} \arrow[r] & 0.
\end{tikzcd}
$$

Both sequences split, implying $\ker[2,1]/(\ker[2,1] \cap \ker[2,3])$ and $\ker[2,3]/(\ker[2,1] \cap \ker[2,3])$ are projective. Furthermore, we have the following short exact sequences:

$$0 \longrightarrow \ker[2,1] \cap \ker[2,3] \hookrightarrow \ker[2,1] \twoheadrightarrow \frac{\ker[2,1]}{\ker[2,1] \cap \ker[2,3]} \longrightarrow 0;$$

$$0 \longrightarrow \ker[2,1] \cap \ker[2,3] \hookrightarrow \ker[2,3] \twoheadrightarrow \frac{\ker[2,3]}{\ker[2,1] \cap \ker[2,3]} \longrightarrow 0.$$

Both sequences split, yielding the following decompositions and isomorphisms for some R-submodules G, K and H:

$$\ker[2,1] = (\ker[2,1] \cap \ker[2,3]) \oplus G;$$

$$\ker[2,3] = (\ker[2,1] \cap \ker[2,3]) \oplus K;$$

$$\ker[2,1] + \ker[2,3] = (\ker[2,1] \cap \ker[2,3]) \oplus G \oplus K;$$

$$F_2 = (\ker[2,1] + \ker[2,3]) \oplus H = (\ker[2,1] \cap \ker[2,3]) \oplus G \oplus K \oplus H;$$

$$F_1 \cong F_2/\ker[2,1] \cong K \oplus H;$$

$$F_3 \cong F_2/\ker[2,3] \cong G \oplus H.$$

Hence, we can decompose F in the following way.

$$\begin{tikzcd}
F_1 & & F_2 \arrow[ll, two heads, "F_{2,1}"'] \arrow[rr, two heads, "F_{2,3}"] & & F_3 \\
\| & & \| & & \| \\
 & & G \arrow[rr, "\cong"] & & F_{2,3}(G) \\
 & & \oplus & & \oplus \\
F_{2,1}(H) & & H \arrow[ll, "\cong"'] \arrow[rr, "\cong"] & & F_{2,3}(H) \\
\oplus & & \oplus & & \\
F_{2,1}(K) & & K \arrow[ll, "\cong"] & & \\
 & & \oplus & & \\
 & & \ker[2,1] \cap \ker[2,3] & &
\end{tikzcd}$$

Figure 4.6: Decomposition of A_3-persistence module in Case 3

Each row in the diagram represents a submodule of F of projective constant type. Hence F admits a projective constant decomposition. □

We deduce Corollary D in the following result.

Corollary 4.4. *Let R be a Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : A_n \to R$-**Mod** be a nonzero p.f.g. A_n-persistence module satisfying the PCC. Then F is interval decomposable.*

Proof. Immediate from Theorem 4.3 and Lemma 1.10. □

Chapter 5

Totally Ordered Persistence Modules

Throughout this chapter, we now assume that R is a commutative Noetherian ring with identity, unless otherwise specified. Our goal is to develop decomposition results for totally ordered persistence modules, which will later be necessary in the decomposition of zigzag persistence modules in Chapters 6 and 7. In particular, Lemma 3.8(2) showed that, under the assumption of the projective colimit conditions, the cokernel of every internal morphism of a zigzag persistence module is projective. Motivated by this result, we identify the projectivity of the cokernels of internal morphisms as the key structural condition ensuring the existence of a projective constant decomposition.

We begin by establishing several technical lemmas concerning the existence of compatible chains of images and kernels, along with results on dual spaces and annihilators of projective R-modules. These tools are then used to prove Proposition 5.17, which shows that any totally ordered persistence module admits a decomposition containing a single direct summand of projective constant type. Finally, we prove Theorem 5.19, which guarantees a complete projective constant decomposition of a totally ordered persistence module.

Definition 5.1. Let P be a totally ordered poset. Let $\leq$ denote the total ordering of P. A functor $F : P \to R\text{-}\mathbf{Mod}$ is a *totally ordered persistence module.*

Definition 5.2. Let M be an R-module. We say an internal direct sum decomposition $M = \bigoplus_{B \in \mathscr{B}} B$ is *compatible* with a chain S of R-submodules of M if for every $A \in S$ we have $A = \bigoplus_{B \in \mathscr{A}} B$ for some subset $\mathscr{A} \subseteq \mathscr{B}$.

Definition-Lemma 5.3. Let $F : P \to R\text{-}\mathbf{Mod}$ be a totally ordered persistence module. For any fixed $z \in P$, the set of images $\{\mathrm{im}[x, z] : x \leq z\}$ forms a chain of R-submodules of F_z, called the *chain of images* of F_z. Dually, for any fixed $z \in P$, the set of kernels $\{\ker[z, y] : y \geq z\}$ forms a chain of R-submodules of F_z, called the *chain of kernels* of F_z.

Proof. For a fixed $z \in P$, observe that either $\mathrm{im}[x,z] \subseteq \mathrm{im}[y,z]$ when $x \leq y$, or $\mathrm{im}[y,z] \subseteq \mathrm{im}[x,z]$ when $y \leq x$. Hence $\{\mathrm{im}[x,z] : x \leq z\}$ forms a chain of R-submodules of F_z. The setting for the set of kernels is analogous. $\square$

Lemma 5.4. *Let $F : P \to R$-**Mod** be a p.f.g.p. totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then for any fixed $z \in P$, the chain of images of F_z and the chain of kernels of F_z are both finite sets.*

Proof. Since $F_z/\mathrm{im}[x,z] = \mathrm{coker}[x,z]$ is projective for every $x \leq z$, the chain of images of F_z is finite by Lemma 2.1. Likewise, since $F_z/\ker[z,y] \cong \mathrm{im}[z,y]$ is projective for every $y \geq z$, the chain of kernels of F_z is finite by Lemma 2.1. $\square$

Lemma 5.5. *Let $F : P \to R$-**Mod** be a nonzero p.f.g.p. totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then for any $F_z \neq 0$, there exists an internal direct sum decomposition $F_z = \bigoplus_{G \in \mathscr{A}} G$ of projective R-submodules G that is compatible with the chain of images and the chain of kernels of F_z.*

Proof. Fix $z \in P$ such that $F_z \neq 0$. There is a finite chain of images of F_z and a finite chain of kernels of F_z by Lemma 5.4. Let l and m be the number of unique images and kernels in these chains, respectively. Choose indices $x_i \leq z$ for $1 \leq i \leq l$ so that each $\mathrm{im}[x_i, z]$ is one of the unique images in the chain of images of F_z. Likewise, choose indices $y_j \geq z$ for indices $1 \leq j \leq m$ so that each $\ker[z, y_j]$ is one of the unique kernels in the chain of kernels of F_z. Let $S = \{x_i\}_{i=1}^{l} \cup \{y_j\}_{j=1}^{m} \cup \{z\}$. Then $F|_S$ is a nonzero A_n-persistence module satisfying the PCC, and therefore admits a projective constant decomposition by Theorem 4.3. This decomposition at the index z yields the desired result. $\square$

Definition-Lemma 5.6. Let M be a finitely generated projective R-module. Then there is a natural isomorphism $\varepsilon : M \to M^{**}$ given by $a \mapsto \hat{a}$, where $\hat{a} : M^* \to R$ is the *evaluation map* such that $\hat{a}(f) = f(a)$ for any $f \in M^*$.

Proof. See [9, Corollary 2.10, Remark 2.11]. $\square$

Definition-Lemma 5.7. A finitely generated R-module M is projective if and only if there exists a family of elements $\{a_i : 1 \leq i \leq n\} \subseteq M$ and a family of linear functionals $\{f_i : 1 \leq i \leq n\} \subseteq M^*$ such that, for any $a \in M$,

$$a = \sum_{i=1}^{n} f_i(a) a_i.$$

We call these two families together a *pair of dual bases* of M and denote it by the set of pairs $\{(a_i, f_i) : 1 \leq i \leq n\}$. Moreover, the set $\{a_i : 1 \leq i \leq n\}$ generates M and the set $\{f_i : 1 \leq i \leq n\}$ generates M^*.

Proof. See [9, Lemma 2.9, Remark 2.11]. $\square$

We caution the reader that the term *pair of dual bases* does not refer to bases of free R-modules in the traditional sense. Rather, it refers to a pair of generating sets of a projective R-module and its dual satisfying the conditions of Definition-Lemma 5.7. Although this terminology is nonstandard, it is convenient for our purposes and is consistent with the usage in [9]. Additionally, while the dual of a left R-module is naturally a right R-module, the commutativity of R allows us to treat the dual as a left R-module. Indeed, this identification is the primary motivation for requiring R to be commutative throughout this chapter.

Lemma 5.8. *Let $\{(a_i, f_i) : 1 \leq i \leq n\}$ be a pair of dual bases of a finitely generated projective R-module M. Then $\{(f_i, \hat{a}_i) : 1 \leq i \leq n\}$ is a pair of dual bases of M^*.*

Proof. See [9, Exercise 2.7]. $\square$

Lemma 5.9. *Let $M = A \oplus B$ be a finitely generated projective R-module, and let $\{(a_i, f_i) : 1 \leq i \leq m\}$ and $\{(b_j, g_j) : 1 \leq j \leq n\}$ be pairs of dual bases of A and B, respectively. Define*

$\bar{f}_i : M \to R$ to be the extension of $f_i : A \to R$ with $\bar{f}_i|_A = f_i$ and $\bar{f}_i|_B = 0$. Likewise, define $\bar{g}_j : M \to R$ to be the extension of $g_j : B \to R$ with $\bar{g}_j|_B = g_j$ and $\bar{g}_j|_A = 0$. Then:

(1) The set $\{(a_i, \bar{f}_i) : 1 \leq i \leq m\} \cup \{(b_j, \bar{g}_j) : 1 \leq j \leq n\}$ is a pair of dual bases of M;

(2) We can write $M^* = A' \oplus B'$, where A' is generated by $\{\bar{f}_i : 1 \leq i \leq m\}$ and B' is generated by $\{\bar{g}_j : 1 \leq j \leq n\}$. In particular, $A' \cong A^*$ and $B' \cong B^*$.

Proof. Both claims are easily verified using Definition-Lemma 5.7 and Lemma 5.8. □

Lemma 5.10. *Let $\phi : M_1 \to M_2$ be a homomorphism of finitely generated projective R-modules where $M_1 = A_1 \oplus B_1$ and $M_2 = A_2 \oplus B_2$. Write $M_1^* = A_1' \oplus B_1'$ and $M_2^* = A_2' \oplus B_2'$ as in Lemma 5.9. If $\phi(A_1) \subseteq A_2$ and $\phi(B_1) \subseteq B_2$, then $\phi^*(A_2') \subseteq A_1'$ and $\phi^*(B_2') \subseteq B_1'$. In particular, if ϕ maps A_1 to A_2 bijectively, then ϕ^* maps A_2' to A_1' bijectively.*

Proof. Let $f_2 \in A_2'$. Then $\phi^*(f_2) = f_2 \circ \phi$ is a linear functional in M_1^*. There exist $f_1 \in A_1', g_1 \in B_1'$ such that $f_2 \circ \phi = f_1 + g_1$. Let $b \in B_1$. Then $f_1(b) = 0$. It follows that

$$g_1(b) = f_1(b) + g_1(b) = f_2(\phi(b)) = 0$$

since $\phi(b) \in B_2$ and $f_2|_{B_2} = 0$. Moreover, $g_1(a) = 0$ for any $a \in A_1$. Hence $g_1 = 0$ and $\phi^*(f_2) = f_1$, which shows that $\phi^*(A_2') \subseteq A_1'$. An analogous argument shows $\phi^*(B_2') \subseteq B_1'$.

Now suppose ϕ maps A_1 to A_2 bijectively. We will show that ϕ^* maps A_2' to A_1' bijectively.

Let $f_2 \in A_2'$ and suppose $\phi^*(f_2) = 0$. Then $f_2 \circ \phi = 0$. Let $a_2 \in A_2$. Since ϕ maps A_1 to A_2 surjectively, there exists $a_1 \in A_1$ such that $\phi(a_1) = a_2$. This implies $f_2(a_2) = f_2(\phi(a_1)) = 0$. Moreover, $f_2(b) = 0$ for any $b \in B_2$. Hence $f_2 = 0$, which shows ϕ^* maps A_2' to A_1' injectively.

Now let $f_1 \in A_1'$. Define a map $f_2 \in A_2'$ so that $f_2|_{A_2} = f_1 \circ (\phi|_{A_1})^{-1}$ and $f_2|_{B_2} = 0$.

For any $a \in A_1$,

$$\phi^*(f_2)(a) = f_2(\phi(a)) = f_1 \circ (\phi|_{A_1})^{-1}(\phi(a)) = f_1(a).$$

Moreover, for any $b \in B_1$ we have $\phi^*(f_2)(b) = f_2(\phi(b)) = 0 = f_1(b)$. This shows that $\phi^*(f_2) = f_1$, and hence ϕ^* maps A_2' to A_1' surjectively. $\square$

Definition 5.11. Given a submodule A of an R-module M, the *annihilator* of A is the submodule $A^\perp$ of M^* where $A^\perp = \{f \in M^* : f(a) = 0 \ \forall a \in A\}$.

Lemma 5.12. *Let $\phi : M \to N$ be a homomorphism of R-modules such that coker(ϕ) is projective. Then im(ϕ^*) = ker(ϕ)$^\perp$.*

Proof. First, suppose $f \in \text{im}(\phi^*)$. Then

$$\begin{aligned} f \in \text{im}(\phi^*) &\implies \phi^*(g) = g \circ \phi = f \text{ for some } g \in N^* \\ &\implies (g \circ \phi)(a) = f(a) = 0 \ \forall a \in \ker(\phi) \\ &\implies f \in \ker(\phi)^\perp. \end{aligned}$$

Conversely, suppose $f \in \ker(\phi)^\perp$. Since coker(ϕ) is projective, there exists a submodule $B \subseteq N$ such that $N = \text{im}(\phi) \oplus B$. Likewise, since $M/\ker(\phi)$ is projective, there exists a decomposition $M = \ker(\phi) \oplus A$ where $\phi|_A : A \to \text{im}(\phi)$ is a bijection. Let $(\phi|_A)^{-1} : \text{im}(\phi) \to A$ be the inverse of this bijection and $\pi : N \to \text{im}(\phi)$ be the natural projection map.

Define $g : N \to R$ to be the map with $g = f \circ (\phi|_A)^{-1} \circ \pi$. For any $c \in \ker(\phi)$,

$$\phi^*(g)(c) = g(\phi(c)) = 0 = f(c)$$

since $f \in \ker(\phi)^\perp$.

Moreover, for any $a \in A$ we have

$$\phi^*(g)(a) = (f \circ (\phi|_A)^{-1} \circ \pi)(\phi(a)) = (f \circ (\phi|_A)^{-1})(\phi(a)) = f(a).$$

Hence $\phi^*(g) = f$, which shows $f \in \mathrm{im}(\phi^*)$. □

Lemma 5.13. *Let $M = A \oplus B$ be an R-module. Then $M^* = A^\perp \oplus B^\perp$.*

Proof. Straightforward to verify using Definition 5.11. □

Lemma 5.14. *Let $\phi : M \to N$ be a homomorphism of finitely generated projective R-modules such that coker(ϕ) is projective. Then coker(ϕ^*) is also projective.*

Proof. Clearly $\mathrm{coker}(\phi)$ projective implies $\mathrm{im}(\phi) \cong M/\ker(\phi)$ is projective. Hence there exists a submodule $A \subseteq M$ such that $M = \ker(\phi) \oplus A$. Dualizing yields

$$M^* = \ker(\phi)^\perp \oplus A^\perp = \mathrm{im}(\phi^*) \oplus A^\perp$$

by Lemmas 5.12 and 5.13. Observe $A^\perp$ is projective as a direct summand of M^* which is projective. Therefore $\mathrm{coker}(\phi^*) = M^*/\mathrm{im}(\phi^*) \cong A^\perp$ is projective. □

Lemma 5.15. *Let $\phi : M \to N$ be a surjection of R-modules such that N is projective. If $N = N_1 \oplus N_2$, then we can choose an internal direct sum decomposition $M = M_1 \oplus M_2$ such that:*

(1) $M_1 \cong N_1$;

(2) $\phi(M_1) = N_1$;

(3) $\phi(M_2) = N_2$.

Proof. Since ϕ is surjective and N is projective, there is a homomorphism $\psi : N \to M$ such that $\phi\psi = \mathrm{id}_N$. It follows that $M = \ker(\phi) \oplus \psi(N) = \ker(\phi) \oplus \psi(N_1) \oplus \psi(N_2)$. Set $M_1 = \psi(N_1)$ and $M_2 = \ker(\phi) \oplus \psi(N_2)$ for the desired decomposition. $\square$

Definition 5.16. Let $F : P \to R$-**Mod** be a totally ordered P-persistence module. Let P^{op} be the poset with opposite ordering denoted by $\leq_{\mathrm{op}}$. Define $F^{\vee} : P^{\mathrm{op}} \to R$-**Mod** to be the totally ordered P^{op}-persistence module such that:

(1) $F^{\vee}_x = F^*_x$ for all $x \in P^{\mathrm{op}}$;

(2) $F^{\vee}_{x,y} = F^*_{y,x}$ for all $x \leq_{\mathrm{op}} y$ in P^{op}.

The proof of the following proposition is a generalization of our argument in [6, §6] when proving the existence of interval decompositions of pointwise finite dimensional persistence modules over fields.

Proposition 5.17. *Let $F : P \to R$-**Mod** be a nonzero p.f.g.p. totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then there exists an internal direct sum decomposition $F = \mathbb{I} \oplus G$, where $\mathbb{I}$ is a module of projective constant type.*

Proof. Fix $z \in P$ with $F_z \neq 0$. By Lemma 5.5, there exists an internal direct sum decomposition of F_z that is compatible with the chain of images and the chain of kernels of F_z. In particular, there are only finitely many nonzero summands in this decomposition since F_z is a finitely generated projective R-module. Hence we can write $F_z = \bigoplus_{i=0}^{k} N_i$ for some $k \geq 0$, where each N_i is nonzero. Fix $M = N_0$ and let $N = N_1 \oplus \cdots \oplus N_k$ so that $F_z = M \oplus N$.

The proof proceeds in three main steps. First, we construct a decomposition of F over indices $x \leq z$ that is compatible with the decomposition at F_z; we refer to this as the left-hand decomposition of F. Second, we construct an analogous decomposition over indices

$x \geq z$ using a dualizing argument, which we call the right-hand decomposition. Finally, we combine these two decompositions to obtain the desired result.

Left-hand Decomposition

Let Q be the set of indices $x \leq z$ such that M is a direct summand of $\mathrm{im}[x, z]$. Note that Q is nonempty since $z \in Q$. Our goal is to find a sequence of R-submodules $\{A_x\}_{x \in Q}$ with $A_x \subseteq F_x$ such that:

(1) $A_x \cong M$ for all $x \in Q$;

(2) $F_{x,z}(A_x) = M$ for all $x \in Q$;

(3) $F_{x,y}(A_x) = A_y$ for all $x < y$ in Q.

Let E_x be the preimage of M in F_x for all $x \in Q$. Define Σ to be the set of all families $(A_x)_{x \in Q}$ with the following conditions:

(S1) A_x is a nontrivial submodule with $A_x \subseteq E_x$ for all $x \in Q$;

(S2) $F_{x,y}(A_x) \subseteq A_y$ for all $x < y$ in Q;

(S3) F_x/A_x is projective for all $x \in Q$;

(S4) $F_y/F_{x,y}(A_x)$ is projective for all $x < y$ in Q.

Claim 5.17.1. $(E_x)_{x \in Q}$ is a family in Σ.

Proof of Claim 5.17.1. Conditions **(S1)** and **(S2)** are obvious.

Consider the surjection $F_{x,z} : F_x \to \mathrm{im}[x, z]$. Since $F_{x,z}(E_x) = M$, there is an induced homomorphism $\overline{F_{x,z}} : F_x/E_x \to \mathrm{im}[x, z]/M$. This map is easily verified to be an isomorphism. Hence F_x/E_x is projective since $\mathrm{im}[x, z]/M$ is projective, proving condition **(S3)**.

Consider the surjection $F_{y,z}|_{\mathrm{im}[x,y]} : \mathrm{im}[x,y] \to \mathrm{im}[x,z]$. Since $F_{y,z}(F_{x,y}(E_x)) = M$, there is an induced homomorphism $\overline{F_{y,z}|_{\mathrm{im}[x,y]}} : \mathrm{im}[x,y]/F_{x,y}(E_x) \to \mathrm{im}[x,z]/M$. This map is easily verified to be an isomorphism. Hence $\mathrm{im}[x,y]/F_{x,y}(E_x)$ is projective since $\mathrm{im}[x,z]/M$ is projective. We have a short exact sequence

$$0 \longrightarrow \frac{\mathrm{im}[x,y]}{F_{x,y}(E_x)} \hookrightarrow \frac{F_y}{F_{x,y}(E_x)} \twoheadrightarrow \frac{F_y}{\mathrm{im}[x,y]} \longrightarrow 0.$$

The sequence splits since $\mathrm{coker}[x,y]$ is projective, and thus $F_y/F_{x,y}(E_x)$ is projective as it is isomorphic to a direct sum of projective R-modules. □

Define a partial order on Σ by $(A_x)_{x\in Q} \leq (B_x)_{x\in Q}$ if $B_x \subseteq A_x$ for all $x \in Q$. Let $\{(A_x[\alpha])\}_\alpha$ be a chain of families in Σ indexed by α. Define the family $(B_x)_{x\in Q}$ by

$$B_x = \bigcap_\alpha A_x[\alpha].$$

Claim 5.17.2. $(B_x)_{x\in Q}$ is in Σ.

Proof of Claim 5.17.2. At each fixed index x, $\{A_x[\alpha]\}_\alpha$ is a chain of projective R-submodules of F_x with $F_x/A_x[\alpha]$ projective for every α. Lemma 2.1 guarantees there are only finitely many submodules $A_x[\alpha]$ in this chain. Therefore B_x will be equal to the minimal submodule in the chain $\{A_x[\alpha]\}_\alpha$. This guarantees conditions **(S1)**, **(S3)** and **(S4)** are satisfied.

Finally, since $B_x \subseteq A_x[\alpha]$ for all α, $F_{x,y}(B_x) \subseteq F_{x,y}(A_x[\alpha]) \subseteq A_y[\alpha]$ for all α. Hence $F_{x,y}(B_x) \subseteq \bigcap_\alpha A_y[\alpha] = B_y$, satisfying condition **(S2)**. □

Now $(B_x)_{x\in Q}$ is in Σ and $(A_x[\alpha])_{x\in Q} \leq (B_x)_{x\in Q}$ for every α since $B_x \subseteq A_x[\alpha]$ for every $x \in Q$. This shows $(B_x)_{x\in Q}$ is an upper bound of the chain $\{(A_x[\alpha])\}_\alpha$. Zorn's Lemma implies the existence of a maximal element, say $(A_x)_{x\in Q}$, in Σ.

Define A'_x to be the minimal image of A_r in A_x for all $r < x$ in Q, where the minimal image here is with regard to subset containment.

Claim 5.17.3. The minimal image A'_x exists in A_x.

Proof of Claim 5.17.3. The set of images $\{F_{r,x}(A_r) : r \leq x\}$ form a chain of projective R-submodules of F_x, where $F_x/F_{r,x}(A_r)$ is projective for every $r \leq x$ by condition **(S4)**. Lemma 2.1 guarantees there are only finitely many submodules in this chain. Therefore a minimal image A'_x exists. □

Claim 5.17.4. A'_x maps surjectively to A'_y whenever $x \leq y$.

Proof of Claim 5.17.4. By definition, $A'_x = F_{r,x}(A_r)$ for some $r \leq x$ and $A'_y = F_{w,y}(A_w)$ for some $w \leq y$. We have two cases to check.

Suppose $r \leq w$. Then $F_{r,w}(A_r) \subseteq A_w$ implies $F_{r,y}(A_r) \subseteq F_{w,y}(A_w) = A'_y$. Minimality of A'_y requires $F_{r,y}(A_r) = A'_y$. Hence

$$A'_y = F_{r,y}(A_r) = F_{x,y}F_{r,x}(A_r) = F_{x,y}(A'_x).$$

Now suppose $w < r$. Then $F_{w,r}(A_w) \subseteq A_r$ implies $F_{w,x}(A_w) \subseteq F_{r,x}(A_r) = A'_x$. Minimality of A'_x requires $F_{w,x}(A_w) = A'_x$. Hence

$$F_{x,y}(A'_x) = F_{x,y}F_{w,x}(A_w) = F_{w,y}(A_w) = A'_y.$$

□

Claim 5.17.5. $(A'_x)_{x \in Q}$ is in Σ.

Proof of Claim 5.17.5. By definition, $A'_x = F_{r,x}(A_r)$ for some $r \leq x$. This implies conditions **(S1)** and **(S3)** are immediately satisfied. By Claim 5.17.4., we have $F_{x,y}(A'_x) = A'_y$. Thus condition **(S2)** is satisfied, and moreover $F_y/F_{x,y}(A'_x) = F_y/A'_y$ is projective which satisfies **(S4)**. $\square$

Now $(A'_x)_{x\in Q}$ is in Σ with $(A_x)_{x\in Q} \leq (A'_x)_{x\in Q}$. Therefore maximality of $(A_x)_{x\in Q}$ implies $(A_x)_{x\in Q} = (A'_x)_{x\in Q}$. Since it was shown in Claim 5.17.4 that A'_x maps to A'_y surjectively for all $x \leq y$, we must have A_x maps to A_y surjectively for all $x \leq y$.

Fix $w \in Q$. The map $F_{w,z}|_{A_w} : A_w \to M$ is a surjection. We can apply Lemma 5.15 to decompose $A_w = M_w \oplus K_w$ so that $M_w \cong M, F_{w,z}(M_w) = M$, and $F_{w,z}(K_w) = 0$.

Define the family $(C_x)_{x\in Q}$ by

$$C_x = \begin{cases} F_{x,w}^{-1}(M_w) \cap A_x & \text{if } x < w, \\ M_w & \text{if } x = w, \\ A_x & \text{if } x > w, \end{cases}$$

where $F_{x,w}^{-1}(M_w)$ denotes the preimage of M_w in F_x.

Claim 5.17.6. $(C_x)_{x\in Q}$ is a family in Σ.

Proof of Claim 5.17.6. Condition **(S1)** is obvious for $x \geq w$. When $x < w$, surjectivity of A_x onto A_w guarantees that $F_{x,w}^{-1}(M_w) \cap A_x$ will be a nonzero R-submodule contained in E_x. Condition **(S2)** is obvious for all indices. It remains to carefully justify conditions **(S3)** and **(S4)**.

Condition **(S3)** is immediate when $x > w$. For $x = w$, we have a short exact sequence

$$0 \longrightarrow \frac{A_w}{M_w} \hookrightarrow \frac{F_w}{M_w} \twoheadrightarrow \frac{F_w}{A_w} \longrightarrow 0.$$

The sequence splits, and thus F_w/M_w is projective as it is isomorphic to a direct sum of projective R-modules.

Now suppose $x < w$. Consider the surjection $F_{x,w}|_{A_x} : A_x \to A_w$. There is an induced homomorphism $\overline{F_{x,w}|_{A_x}} : A_x/(F_{x,w}^{-1}(M_w) \cap A_x) \to A_w/M_w$. This map is easily verified to be an isomorphism. Hence $A_x/(F_{x,w}^{-1}(M_w) \cap A_x)$ is projective since A_w/M_w is projective. Moreover, we have a short exact sequence

$$0 \longrightarrow \frac{A_x}{F_{x,w}^{-1}(M_w) \cap A_x} \hookrightarrow \frac{F_x}{F_{x,w}^{-1}(M_w) \cap A_x} \twoheadrightarrow \frac{F_x}{A_x} \longrightarrow 0.$$

The sequence splits, and thus $F_x/(F_{x,w}^{-1}(M_w) \cap A_x)$ is projective as it is isomorphic to a direct sum of projective R-modules. This proves condition **(S3)**.

When $w \leq x < y$ or $x < w \leq y$, condition **(S4)** is obvious. Suppose $x < y < w$. We claim $F_{x,y}(F_{x,w}^{-1}(M_w) \cap A_x) = F_{y,w}^{-1}(M_w) \cap A_y$.

Indeed, the forward inclusion is straightforward. To see the reverse inclusion, suppose $a_y \in F_{y,w}^{-1}(M_w) \cap A_y$. By surjectivity, there exists $a_x \in A_x$ such that $F_{x,y}(a_x) = a_y$. This implies $F_{x,w}(a_x) = F_{y,w}(a_y) \in M_w$. Hence $a_x \in F_{x,w}^{-1}(M_w) \cap A_x$, implying $a_y \in F_{x,y}(F_{x,w}^{-1}(M_w) \cap A_x)$.

The equality above means $F_y/F_{x,y}(F_{x,w}^{-1}(M_w) \cap A_x) = F_y/(F_{y,w}^{-1}(M_w) \cap A_y)$, which was shown to be projective by condition **(S3)**. $\square$

Now $(C_x)_{x\in Q}$ is in Σ with $(A_x)_{x\in Q} \leq (C_x)_{x\in Q}$. Therefore maximality of $(A_x)_{x\in Q}$ implies $(C_x)_{x\in Q} = (A_x)_{x\in Q}$, and in particular that $A_w = M_w$. For every $x \geq w$, surjectivity of A_w onto A_x combined with the fact that $A_w \cong M$ implies A_w maps bijectively to A_x so that $A_x \cong M$ for all $w \leq x \leq z$ in Q. Since w was arbitrary, the maximal element $(A_x)_{x\in Q}$ must be a family satisfying:

(1) $A_x \cong M$ for all $x \in Q$;

(2) $F_{x,z}(A_x) = M$ for all $x \in Q$;

(3) $F_{x,y}(A_x) = A_y$ for all $x < y$ in Q.

Define the persistence submodules $\mathbb{I}^-$ and G^- of $F|_{\{x \leq z\}}$ as follows:

$$\mathbb{I}^-_x = \begin{cases} A_x & \text{if } x \in Q, \\ 0 & \text{if } x \notin Q; \end{cases} \qquad G^-_x = F^{-1}_{x,z}(N) \text{ for all } x \leq z.$$

Then $F|_{\{x \leq z\}} = \mathbb{I}^- \oplus G^-$ where $\mathbb{I}^-$ is of projective constant type.

Right-hand Decomposition

Recall that $F_z = \bigoplus_{i=0}^k N_i$ where $N_0 = M$. Let S be the set of indices $x \geq z$ such that N_0 is not a direct summand of $\ker[z, x]$. Note that S is nonempty since $z \in S$.

We will now focus on the persistence module $F|_S$. We will abuse notation and simply write F instead of $F|_S$ below as it is understood from context, and view F as an S-persistence module.

Consider the S^{op}-persistence module $F^\vee$ where $\leq_{\mathrm{op}}$ denotes the opposite ordering in S^{op} (see Definition 5.16). Since $\operatorname{coker}(F_{x,y})$ is projective for every $x \leq y$ in S, $\operatorname{coker}(F^\vee_{y,x}) = \operatorname{coker}(F^*_{x,y})$ is projective for every $y \leq_{\mathrm{op}} x$ in S^{op} by Lemma 5.14.

By Definition-Lemma 5.7, there exists a pair of dual bases $\{(a_{i,j}, f_{i,j}) : 1 \leq j \leq \iota_i\}$ for each submodule N_i and for some finite number of elements ι_i. Using Lemma 5.9, we can write $F^\vee_z = \bigoplus_{i=0}^k N'_i$ where each N'_i is generated by the set of linear functionals $\{\bar{f}_{i,j} : 1 \leq j \leq \iota_i\}$. Recall here that $\bar{f}_{i,j} : F \to R$ is the extension of the linear functional $f_{i,j} : N_i \to R$ by setting $\bar{f}_{i,j}|_{N_r} = 0$ for all $r \neq i$.

Claim 5.17.7. The decomposition $F^\vee_z = \bigoplus_{i=0}^k N'_i$ is compatible with the chain of images of $F^\vee_z$.

Proof of Claim 5.17.7. To see this, let $x > z$. If $\ker[z, x] = 0$, the claim is immediate. Suppose $\ker[z, x] \neq 0$. After possible reindexing of the submodules N_i (excluding $i = 0$), we can write $\ker[z, x] = N_1 \oplus \cdots \oplus N_d$ for some $d \leq k$.

By Lemma 5.12, $\mathrm{im}(F^{\vee}_{x,z}) = \mathrm{im}(F^{*}_{z,x}) = \ker[z, x]^{\perp}$ for $x \leq_{\mathrm{op}} z$. But it is easy to see that $\ker[z, x]^{\perp} = N'_0 \oplus N'_{d+1} \oplus \cdots \oplus N'_k$, which implies the decomposition is compatible (see Definition 5.2).

Indeed, the right-hand side is obviously in the left-hand side. If $f \in \ker[z, x]^{\perp}$, then we write $f = f_0 + f_1 + \cdots + f_k$ where each $f_i \in N'_i$. If f_i is a nonzero linear functional on N_i for some $1 \leq i \leq d$, then there exists $a \in N_i$ such that $f_i(a)$ is nonzero. But then $f(a) = f_i(a)$ is nonzero, a contradiction. $\square$

Since the decomposition $F^{\vee}_z = \bigoplus_{i=0}^{k} N'_i$ is compatible with the chain of images of $F^{\vee}_z$, we can apply the results of the left-hand decomposition to $F^{\vee}$ so that $F^{\vee} = \mathscr{I} \oplus \mathscr{G}$ with the following properties:

(1) $\mathscr{I}_z = N'_0$ and $\mathscr{G}_z = N'_1 \oplus \cdots \oplus N'_k$;

(2) $F^{\vee}_{x,y}(\mathscr{I}_x) = F^{*}_{y,x}(\mathscr{I}_x) = \mathscr{I}_y$ for all $x \leq_{\mathrm{op}} y$ in S^{op};

(3) $F^{\vee}_{x,y}(\mathscr{G}_x) = F^{*}_{y,x}(\mathscr{G}_x) \subseteq \mathscr{G}_y$ for all $x \leq_{\mathrm{op}} y$ in S^{op};

(4) $\mathscr{I}_x \cong N'_0 \cong N^{*}_0$ for all $x \in S^{\mathrm{op}}$.

Dualizing again, consider the S-persistence module $F^{\vee\vee}$ where $\leq$ denotes the original total ordering on S. Observe by Definition 5.16 that

(1) $F^{\vee\vee}_x = F^{**}_x$ for all $x \in S$;

(2) $F^{\vee\vee}_{x,y} = F^{**}_{x,y}$ for all $x \leq y$ in S.

Since $\operatorname{coker}(F^{\vee}_{y,x})$ is projective for every $y \leq_{\text{op}} x$ in S^{op},

$$\operatorname{coker}(F^{\vee\vee}_{x,y}) = \operatorname{coker}((F^{\vee}_{y,x})^{*}) = \operatorname{coker}(F^{**}_{x,y})$$

is projective for every $x \leq y$ in S by Lemma 5.14. We will now use the decomposition $F^{\vee} = \mathscr{I} \oplus \mathscr{G}$ to construct a right-hand decomposition of $F^{\vee\vee}$.

Observe that $\bigcup_{i=0}^{k}\{(a_{i,j}, \bar{f}_{i,j}) : 1 \leq j \leq \iota_i\}$ is a pair of dual bases of F_z by Lemma 5.9. Thus $\bigcup_{i=0}^{k}\{(\bar{f}_{i,j}, \hat{a}_{i,j}) : 1 \leq j \leq \iota_i\}$ is a pair of dual bases of $F^{\vee}_z$ by Lemma 5.8. In particular, $\mathscr{I}_z$ is generated by the set $\{\bar{f}_{0,j} : 1 \leq j \leq \iota_0\}$ and $\mathscr{G}_z$ is generated by the set $\bigcup_{i=1}^{k}\{\bar{f}_{i,j} : 1 \leq j \leq \iota_i\}$.

Applying Lemma 5.9, we can write $F^{\vee\vee}_z = \mathscr{I}'_z \oplus \mathscr{G}'_z$ where $\mathscr{I}'_z$ is generated by the set $\{\hat{a}_{0,j} : 1 \leq j \leq \iota_0\}$ and $\mathscr{G}'_z$ is generated by the set $\bigcup_{i=1}^{k}\{\hat{a}_{i,j} : 1 \leq j \leq \iota_i\}$. Similarly, for any $x > z$ in S with $F^{\vee}_x = \mathscr{I}_x \oplus \mathscr{G}_x$, we can write $F^{\vee\vee}_x = \mathscr{I}'_x \oplus \mathscr{G}'_x$ as in Lemma 5.9.

Since $F^{\vee}_{y,x}(\mathscr{I}_y) \subseteq \mathscr{I}_x$ and $F^{\vee}_{y,x}(\mathscr{G}_y) \subseteq \mathscr{G}_x$ for any $y \leq_{\text{op}} x$ in S^{op}, Lemma 5.10 implies $F^{\vee\vee}_{x,y}(\mathscr{I}'_x) \subseteq \mathscr{I}'_y$ and $F^{\vee\vee}_{x,y}(\mathscr{G}'_x) \subseteq \mathscr{G}'_y$ for any $x \leq y$ in S. In particular, Lemma 5.10 guarantees that $F^{\vee\vee}_{x,y}$ maps $\mathscr{I}'_x$ to $\mathscr{I}'_y$ bijectively for all $x \leq y$ in S since $F^{\vee}_{y,x}$ maps $\mathscr{I}_y$ to $\mathscr{I}_x$ bijectively for all $y \leq_{\text{op}} x$ in S^{op}.

For all $x \in S$, let $\varepsilon_x : F_x \to F^{\vee\vee}_x$ be the natural isomorphism mapping $a \mapsto \hat{a}$ for any $a \in F_x$ (see Definition-Lemma 5.6). For any $x \leq y$ in S, we have the following commuting diagram of morphisms.

$$\begin{tikzcd}
F_z \arrow[r, "F_{z,x}"] \arrow[rr, bend left, "F_{z,y}"] \arrow[d, "\varepsilon_z"', "\cong"] & F_x \arrow[r, "F_{x,y}"] \arrow[d, "\varepsilon_x"', "\cong"] & F_y \arrow[d, "\varepsilon_y"', "\cong"] \\
F^{\vee\vee}_z = \mathscr{I}'_z \oplus \mathscr{G}'_z \arrow[r, "F^{\vee\vee}_{z,x}"] \arrow[rr, bend right, "F^{\vee\vee}_{z,y}"'] & F^{\vee\vee}_x = \mathscr{I}'_x \oplus \mathscr{G}'_x \arrow[r, "F^{\vee\vee}_{x,y}"] & F^{\vee\vee}_y = \mathscr{I}'_y \oplus \mathscr{G}'_y
\end{tikzcd}$$

Figure 5.1: Commuting Diagram of Morphisms in Right-hand Decomposition

We can use this diagram to pull back the decomposition of $F^{\vee\vee}$ to a decomposition of F for indices $x \in S$. Define the persistence submodules $\mathbb{I}^+$ and G^+ of $F|_{\{x \geq z\}}$ as follows:

$$\mathbb{I}^+_x = \begin{cases} \varepsilon_x^{-1}(\mathscr{I}'_x) & \text{if } x \in S, \\ 0 & \text{if } x \notin S; \end{cases} \qquad G^+_x = \begin{cases} \varepsilon_x^{-1}(\mathscr{G}'_x) & \text{if } x \in S, \\ F_x & \text{if } x \notin S. \end{cases}$$

Then $F|_{\{x\geq z\}} = \mathbb{I}^+ \oplus G^+$ where $\mathbb{I}^+$ is of projective constant type. In particular, we have $\mathbb{I}^+_z = M$ and $G^+_z = N$ since $\varepsilon_z(a_{i,j}) = \hat{a}_{i,j}$ for all such indices i, j.

Overall Decomposition

The left-hand and right-hand decompositions constructed above are compatible at the index z with $\mathbb{I}^-_z = \mathbb{I}^+_z = M$ and $G^-_z = G^+_z = N$. Define the P-persistence submodule $\mathbb{I}$ so that $\mathbb{I}_x = \mathbb{I}^-_x$ if $x \leq z$ and $\mathbb{I}_x = \mathbb{I}^+_x$ if $x > z$. Likewise, define the P-persistence submodule G so that $G_x = G^-_x$ if $x \leq z$ and $G_x = G^+_x$ if $x > z$. Then $F = \mathbb{I} \oplus G$ where $\mathbb{I}$ is a persistence submodule of projective constant type. $\square$

Lemma 5.18. *Let $F : P \to R$-**Mod** be a totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. If $F = G \oplus H$, then the cokernels of every internal morphism $G_{x,y}$ and $H_{x,y}$ are also projective.*

Proof. For any $x \leq y$, it is easy to verify that $\mathrm{im}[x, y] = F_{x,y}(G_x) \oplus F_{x,y}(H_x)$. Observe that

$$\mathrm{coker}[x, y] = \frac{F_y}{\mathrm{im}[x, y]} = \frac{G_y \oplus H_y}{F_{x,y}(G_x) \oplus F_{x,y}(H_x)} \cong \left(\frac{G_y}{F_{x,y}(G_x)}\right) \oplus \left(\frac{H_y}{F_{x,y}(H_x)}\right).$$

Since $\mathrm{coker}[x, y]$ is projective, both $G_y/F_{x,y}(G_x)$ and $H_y/F_{x,y}(H_x)$ are projective. But $G_{x,y}(G_x) = F_{x,y}(G_x)$ and $H_{x,y}(H_x) = F_{x,y}(H_x)$. This implies that $\mathrm{coker}(G_{x,y})$ and $\mathrm{coker}(H_{x,y})$ are both projective for any $x \leq y$. $\square$

The above lemma guarantees that the desired cokernel property descends to direct summands. With this, we are finally ready to state the main theorem of this chapter, namely Theorem B.

Theorem 5.19. *Let $F : P \to R$-**Mod** be a nonzero p.f.g.p. totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then F admits a projective constant decomposition.*

Proof. Immediate from Lemma 2.2, Proposition 5.17 and Lemma 5.18 when letting $\mathscr{E}$ in Lemma 2.2 be the set of submodules of F of projective constant type. $\square$

We deduce Corollary E in the following result.

Corollary 5.20. *Let R be a commutative Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : P \to R$-**Mod** be a nonzero p.f.g.p. totally ordered persistence module. Suppose that the cokernel of every internal morphism $F_{x,y}$ is projective. Then F is interval decomposable.*

Proof. Immediate from Theorem 5.19 and Lemma 1.10. $\square$

The result of Corollary 5.20 when R is a PID was also proved by Luo and Henselman-Petrusek [10, 11].

Chapter 6

Zigzag Persistence Modules: Finite Extrema

Throughout this chapter, we assume that R is a commutative Noetherian ring with identity, unless otherwise specified. Building on the decomposition results for totally ordered persistence modules established in Chapter 5, we now turn to our main decomposition theorem for zigzag persistence modules with finite extrema. The following result is the statement of Theorem C when $\Gamma = \{1, 2, \ldots, n\}$ for $n \geq 2$. The proof follows a similar argument as that of Theorem 4.3.

Theorem 6.1. *Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with extrema $\{z_1, z_2, \ldots, z_n\}$ for $n \geq 2$ satisfying the PCC. Then F admits a projective constant decomposition.*

Proof. We will proceed by induction on n. For the base case of $n = 2$, F is a nonzero p.f.g.p. totally ordered persistence module such that the cokernel of every internal morphism $F_{x,y}$ is projective (see Lemma 3.8). Theorem 5.19 guarantees F admits a projective constant decomposition.

Let $n \geq 3$ and assume the inductive hypothesis holds up to $n - 1$. Let $\sqsubseteq$ be the associated total order. As in the proof of Theorem 4.3, we can reduce to the case of showing that F admits a projective constant decomposition when $n = 3$ and F has a peak at z_2. We will not rewrite all the details of this claim here as they are completely analogous to the proof of Theorem 4.3.

Case 1. Consider the $n = 3$ case with the following diagram.

$$F_{z_1} \longrightarrow F_{z_2} \longrightarrow F_{z_3}$$

Figure 6.1: Zigzag persistence module with $n = 3$ in Case 1

This is a nonzero p.f.g.p. totally ordered persistence module such that the cokernel of every internal morphism $F_{x,y}$ is projective (see Lemma 3.8). Theorem 5.19 guarantees that F admits a projective constant decomposition. When the morphisms are reversed with $z_3 \le z_2 \le z_1$, the same argument holds.

Case 2. Consider the $n = 3$ case with the following diagram.

$$F_{z_1} \longrightarrow F_{z_2} \longleftarrow F_{z_3}$$

Figure 6.2: Zigzag persistence module with $n = 3$ in Case 2

Note all morphisms are injective since z_2 is a peak, and $F_{z_2} \neq 0$.

Observe that $F|_{[z_1,z_2]}$ and $F|_{[z_2,z_3]}$ are both totally ordered persistence modules such that the cokernel of every internal morphism is projective. Lemma 5.4 implies there is a finite chain of images $L = \{\mathrm{im}[x, z_2] : z_1 \sqsubseteq x \sqsubseteq z_2\}$ of F_{z_2} and another finite chain of images $R = \{\mathrm{im}[x, z_2] : z_2 \sqsubseteq x \sqsubseteq z_3\}$ of F_{z_2}.

Let l and r be the number of unique images of L and R, respectively. Choose indices $x_i \sqsubseteq z_2$ for $1 \le i \le l$ so that each $\mathrm{im}[x_i, z_2]$ is one of the unique images in the chain L. Likewise, choose indices $y_j \sqsupseteq z_2$ for indices $1 \le j \le r$ so that each $\mathrm{im}[y_j, z_2]$ is one of the unique images in the chain R.

Let $S = \{x_i\}_{i=1}^{l} \cup \{y_j\}_{j=1}^{r} \cup \{z_2\}$. Then $F|_S$ is a nonzero A_n-persistence module satisfying the PCC, and therefore admits a projective constant decomposition by Theorem 4.3. This decomposition at the index z_2 looks like a finite internal direct sum of projective R-modules $F_{z_2} = \bigoplus_{k=1}^{m} M_k$ that is compatible with both chains of images L and R.

Fix $M = M_1$ and let $N = M_2 \oplus \cdots \oplus M_m$. Then $F_{z_2} = M \oplus N$. Let Q be the set of indices $x \in P$ such that M is a direct summand of $\mathrm{im}[x, z_2]$.

Define $\mathbb{I}_x$ to be the preimage of M in F_x for every $x \in Q$, and $\mathbb{I}_x = 0$ for every $x \notin Q$. For all indices $x \in P$, define G_x to be the preimage of N in F_x. We claim that $F_x = \mathbb{I}_x \oplus G_x$ for every $x \in P$, and also that $\mathbb{I}_x \cong M$ for every $x \in Q$.

Indeed, the claim is obvious for $x \notin Q$ since F_{x,z_2} is injective. For $x \in Q$, consider $\mathrm{im}[x, z_2]$ in F_{z_2}. Then $\mathrm{im}[x, z_2] = \bigoplus_{k \in K} M_k$ for some subset of indices $K \subseteq \{1, \ldots, m\}$. Since F_{x,z_2} is injective, it is easy to see how we pull back this decomposition to yield $F_x = \mathbb{I}_x \oplus G_x$, and moreover that $\mathbb{I}_x \cong M$.

Hence $F = \mathbb{I} \oplus G$ where $\mathbb{I}$ is a persistence submodule of projective constant type. If $G = 0$, we are done. If $G \neq 0$, then G has a peak at z_2 by Lemma 3.10 and satisfies the PCC by Lemma 3.8. We can repeat the above process on G to decompose another direct summand of projective constant type. After m iterations, we obtain a complete projective constant decomposition of F.

Case 3. Consider the $n = 3$ case with the following diagram.

$$F_{z_1} \longleftarrow F_{z_2} \longrightarrow F_{z_3}$$

Figure 6.3: Zigzag persistence module with $n = 3$ in Case 3

Note all morphisms are surjective since z_2 is a peak, and $F_{z_2} \neq 0$.

Observe that $F|_{[z_1,z_2]}$ and $F|_{[z_2,z_3]}$ are both totally ordered persistence modules such that the cokernel of every internal morphism is projective. Lemma 5.4 implies there is a finite chain of kernels $L = \{\ker[z_2, x] : z_1 \sqsubseteq x \sqsubseteq z_2\}$ of F_{z_2} and another finite chain of kernels $R = \{\ker[z_2, x] : z_2 \sqsubseteq x \sqsubseteq z_3\}$.

By an argument analogous to Case 2, F_{z_2} admits an internal direct sum decomposition $F_{z_2} = \bigoplus_{k=1}^{m} M_k$ of projective R-modules that is compatible with both chains of kernels L and R.

Fix $M = M_1$ and $N = M_2 \oplus \cdots \oplus M_m$. Then $F_{z_2} = M \oplus N$. Let Q be the set of indices $x \in P$ such that M is not a direct summand of $\ker[z_2, x]$.

Define $\mathbb{I}_x = F_{z_2,x}(M)$ for every $x \in Q$, and $\mathbb{I}_x = 0$ for every $x \notin Q$. For all indices $x \in P$, define $G_x = F_{z_2,x}(N)$. We claim that $F_x = \mathbb{I}_x \oplus G_x$ for every $x \in P$, and also that $\mathbb{I}_x \cong M$ for every $x \in Q$.

Indeed, the claim is obvious for $x \notin Q$ since $F_{z_2,x}$ is surjective. For $x \in Q$, clearly $\mathbb{I}_x + G_x$ generates all of F_x by surjectivity of $F_{z_2,x}$. To see the sum is direct, suppose $a \in \mathbb{I}_x \cap G_x$. Then there exist $b \in M, c \in N$ such that $F_{z_2,x}(b) = F_{z_2,x}(c) = a$. This implies $b - c \in \ker[z_2, x]$, where $\ker[z_2, x] \subseteq N$ since M is not a direct summand of $\ker[z_2, x]$. Hence $b \in M \cap N \implies b = 0 \implies a = 0$. Moreover, $F_{z_2,x}|_M : M \to \mathbb{I}_x$ is clearly a bijection.

Hence $F = \mathbb{I} \oplus G$ where $\mathbb{I}$ is a persistence submodule of projective constant type. If $G = 0$, we are done. If $G \neq 0$, then G has a peak at z_2 by Lemma 3.10 and satisfies the PCC by Lemma 3.8. We can repeat the above process on G to decompose another direct summand of projective constant type. After m iterations, we obtain a complete projective constant decomposition of F. □

We deduce Corollary F when $\Gamma = \{1, 2, \ldots, n\}$ for $n \geq 2$ in the following result.

Corollary 6.2. *Let R be a commutative Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with extrema $\{z_1, z_2, \ldots, z_n\}$ for $n \geq 2$ satisfying the PCC. Then F is interval decomposable.*

Proof. Immediate from Theorem 6.1 and Lemma 1.10. □

Chapter 7

Zigzag Persistence Modules: Infinite Extrema

Throughout this chapter, we assume that R is a commutative Noetherian ring with identity, unless otherwise specified. Several of the arguments in this chapter are inspired by Botnan's approach to decomposing zigzag persistence modules over fields indexed by infinite discrete zigzag posets [2]. As in Chapter 5 on totally ordered persistence modules, we first prove in Proposition 7.5 that a pointwise finitely generated zigzag persistence module with infinite extrema satisfying the PCC admits a decomposition containing a single direct summand of projective constant type. We then use this result to prove Theorems 7.6 and 7.7.

Notation 7.1. Let P be a zigzag poset. Recall the four interval notations $[x, y]$, $[x, y)$, $(x, y]$ and (x, y) outlined in Notation 3.5. We use bracket notation to indicate the following:

(1) $\langle x, y]$ is an interval of the form $(x, y]$ or $[x, y]$;

(2) $[x, y\rangle$ is an interval of the form $[x, y)$ or $[x, y]$;

(3) $\langle x, y)$ is an interval of the form (x, y) or $[x, y)$;

(4) $(x, y\rangle$ is an interval of the form (x, y) or $(x, y]$;

(5) $\langle x, y\rangle$ is any one of the four intervals $[x, y]$, $[x, y)$, $(x, y]$ or (x, y).

Additionally, we will denote a persistence module of projective constant type over the interval $\langle x, y\rangle$ by $\mathbb{I}^{\langle x,y\rangle}$, with notation for other intervals defined analogously.

Lemma 7.2. *Let $F : P \to R$-**Mod** be a p.f.g. zigzag persistence module with extrema $\{z_i \mid i \in \mathbb{Z}\}$ satisfying the PCC. Let $\sqsubseteq$ be the associated total order. If there exists an interval $[z_s, z_t]$ such that the projective constant decomposition of $F|_{[z_s,z_t]}$ includes a submodule supported on $\langle r, w\rangle$ with $z_s \sqsubset r \sqsubset w \sqsubset z_t$, then F admits a decomposition $F = \mathbb{I}^{\langle r,w\rangle} \oplus G$ for some submodule $\mathbb{I}^{\langle r,w\rangle}$ of projective constant type.*

Proof. Note that $F|_{[z_s,z_t]}$ admits a projective constant decomposition by Theorem 6.1. Suppose J is a direct summand in the projective constant decomposition of $F|_{[z_s,z_t]}$ that is supported on $\langle r,w\rangle$ with $z_s \sqsubset r \sqsubset w \sqsubset z_t$. Then we can write $F|_{[z_s,z_t]} = J \oplus H$ where H is the direct sum of the remaining direct summands in the decomposition.

Define the persistence submodules $\mathbb{I}^{\langle r,w\rangle}$ and G as follows:

$$\mathbb{I}_x^{\langle r,w\rangle} = \begin{cases} J_x & \text{if } z_s \sqsubseteq x \sqsubseteq z_t, \\ 0 & \text{otherwise;} \end{cases} \qquad G_x = \begin{cases} H_x & \text{if } z_s \sqsubseteq x \sqsubseteq z_t, \\ F_x & \text{otherwise.} \end{cases}$$

This results in a decomposition $F = \mathbb{I}^{\langle r,w\rangle} \oplus G$ as desired, where $\mathbb{I}^{\langle r,w\rangle}$ is a persistence submodule of projective constant type supported on $\langle r,w\rangle$. □

For a totally ordered poset with distinct minimum and maximum elements, a totally ordered persistence module indexed by this poset can be treated as a zigzag persistence module with two finite extrema consisting of the minimum and maximum, and whose associated total order coincides with the total ordering of P. The following lemmas characterize the conditions under which such a persistence module decomposes into an internal direct sum of a single module of projective constant type and a complementary submodule.

Lemma 7.3. *Let P be a totally ordered poset with distinct minimum and maximum elements; that is, there exist $r,w \in P$ with $r \neq w$ such that $r \leq x$ for all $x \in P$ and $w \geq x$ for all $x \in P$. Suppose that $F : P \to R$-**Mod** is a totally ordered persistence module such that:*

(a) coker$(F_{r,w})$ is projective;

(b) every morphism $F_{x,y}$ is injective;

(c) $F_r = M \oplus N$ for some nonzero projective R-submodule M.

Then there exists a decomposition $F = \mathbb{I}^{[r,w]} \oplus G$, where:

(1) $\mathbb{I}^{[r,w]}$ *is a persistence module of projective constant type supported on* $[r, w]$*;*

(2) $\mathbb{I}_x^{[r,w]} \cong M$ *for all* $r \leq x \leq w$*;*

(3) $\mathbb{I}_r^{[r,w]} = M$ *and* $G_r = N$.

Proof. Define $\mathbb{I}_x^{[r,w]} = F_{r,x}(M)$ for all $r \leq x \leq w$. Observe that $\mathbb{I}_r^{[r,w]} = M$. Additionally, define $N_w = F_{r,w}(N)$.

By injectivity, we have $\mathrm{im}[r, w] = \mathbb{I}_w^{[r,w]} \oplus N_w$. Since $\mathrm{coker}[r, w]$ is projective by assumption, there exists a submodule H such that $F_w = \mathrm{im}[r, w] \oplus H$.

Define $G_w = N_w \oplus H$. Then $F_w = \mathbb{I}_w^{[r,w]} \oplus G_w$. Additionally, define $G_x = F_{x,w}^{-1}(G_w)$ for all $r \leq x < w$. Observe that $G_r = N$. We claim $F_x = \mathbb{I}_x^{[r,w]} \oplus G_x$ for all $x \in P$. Indeed, let $a \in F_x$. Then $F_{x,w}(a) = b + c$ for some $b \in \mathbb{I}_w^{[r,w]}, c \in G_w$. Moreover, there exists $d \in \mathbb{I}_r^{[r,w]}$ such that $F_{r,w}(d) = b$. We have the following implications:

$$\begin{aligned}
&F_{x,w}(a) = b + c = F_{x,w}F_{r,x}(d) + c \\
\implies &F_{x,w}(a - F_{r,x}(d)) = c \\
\implies &a - F_{r,x}(d) \in G_x \\
\implies &a \in \mathbb{I}_x^{[r,w]} + G_x.
\end{aligned}$$

To show the sum is direct, let $a \in \mathbb{I}_x^{[r,w]} \cap G_x$. Then $F_{x,w}(a) \in \mathbb{I}_w^{[r,w]} \cap G_w$. Hence $F_{x,w}(a) = 0$ and injectivity implies $a = 0$. This proves the claim, and moreover by construction we have $F_{x,y}(\mathbb{I}_x^{[r,w]}) = \mathbb{I}_y^{[r,w]}$ and $F_{x,y}(G_x) \subseteq G_y$ for all $x \leq y$. Hence $F = \mathbb{I}^{[r,w]} \oplus G$ is the desired decomposition. $\square$

Lemma 7.4. *Let* P *be a totally ordered poset with distinct minimum and maximum elements; that is, there exist* $r, w \in P$ *with* $r \neq w$ *such that* $r \leq x$ *for all* $x \in P$ *and* $w \geq x$ *for all* $x \in P$*. Suppose that* $F : P \to R$-***Mod*** *is a totally ordered persistence module such that:*

(a) every morphism $F_{x,y}$ is surjective;

(b) F_w is a projective R-module;

(c) $F_w = M \oplus N$ for some nonzero projective R-submodule M.

Then there exists a decomposition $F = \mathbb{I}^{[r,w]} \oplus G$, where:

(1) $\mathbb{I}^{[r,w]}$ is a persistence module of projective constant type supported on $[r,w]$;

(2) $\mathbb{I}_x^{[r,w]} \cong M$ for all $r \leq x \leq w$;

(3) $\mathbb{I}_w^{[r,w]} = M$ and $G_w = N$.

Proof. Since $F_{r,w}$ is surjective with $F_w = M \oplus N$ projective, by Lemma 5.15 we can choose a decomposition $F_r = \mathbb{I}_r^{[r,w]} \oplus G_r$ such that:

(i) $\mathbb{I}_r^{[r,w]} \cong M$;

(ii) $F_{r,w}(\mathbb{I}_r^{[r,w]}) = M$;

(iii) $F_{r,w}(G_r) = N$.

Define $\mathbb{I}_x^{[r,w]} = F_{r,x}(\mathbb{I}_r^{[r,w]})$ and $G_x = F_{r,x}(G_r)$ for all $r < x \leq w$. Observe that $\mathbb{I}_w^{[r,w]} = M$ and $G_w = N$. We claim $F_x = \mathbb{I}_x^{[r,w]} \oplus G_x$ for all $x \in P$.

Indeed, clearly $\mathbb{I}_x^{[r,w]} + G_x$ generates F_x by surjectivity of $F_{r,x}$. To see the sum is direct, suppose $a \in \mathbb{I}_x^{[r,w]} \cap G_x$. Then there exists $b \in \mathbb{I}_r^{[r,w]}$ with $F_{r,x}(b) = a$. In particular, we have

$$F_{r,w}(b) = F_{x,w}F_{r,x}(b) = F_{x,w}(a) = 0$$

since $F_{x,w}(a) \in M \cap N$. But $F_{r,w}$ maps $\mathbb{I}_r^{[r,w]}$ to $\mathbb{I}_w^{[r,w]}$ bijectively, hence $F_{r,w}(b) = 0 \implies b = 0 \implies a = 0$.

This proves the claim, and moreover by construction we have $F_{x,y}(\mathbb{I}_x^{[r,w]}) = \mathbb{I}_y^{[r,w]}$ and $F_{x,y}(G_x) \subseteq G_y$ for all $x \leq y$. Therefore $F = \mathbb{I}^{[r,w]} \oplus G$ is the desired decomposition. $\square$

Proposition 7.5. *Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with extrema $\{z_i \mid i \in \mathbb{Z}\}$ satisfying the PCC. Then there exists an internal direct sum decomposition $F = \mathbb{I} \oplus G$, where $\mathbb{I}$ is a module of projective constant type.*

Proof. Let $\sqsubseteq$ denote the associated total order of the zigzag poset P. Without loss of generality, assume that for all $x, y \in P$ such that $z_i \sqsubseteq x \sqsubseteq y \sqsubseteq z_{i+1}$:

$$\begin{cases} i \text{ is odd} \quad \implies \quad x \leq y; \\ i \text{ is even} \quad \implies \quad y \leq x. \end{cases}$$

See Definition 3.2 for a reminder on the above statement.

Recall by Theorem 6.1 that $F|_{[z_{-n}, z_n]}$ admits a projective constant decomposition for any positive integer n. We split this proof into two cases.

Case 1. Suppose there exists a positive integer n such that the projective constant decomposition of $F|_{[z_{-n}, z_n]}$ contains a submodule supported on $\langle r, w \rangle$ with $-n \sqsubset r \sqsubset w \sqsubset n$. Then we are done by Lemma 7.2.

Case 2. Suppose no such n exists. Fix an index z_i such that $F_{z_i} \neq 0$. Note that such an index exists since if all $F_{z_i} = 0$, then we would have Case 1.

Claim 7.5.1. There exists $z_t \sqsupset z_i$ such that $F_{x,y}$ is injective and $F_{y,x}$ is surjective for all $z_t \sqsubseteq x \sqsubseteq y$, when such maps are defined. Dually, there exists $z_s \sqsubset z_i$ such that $F_{y,x}$ is injective and $F_{x,y}$ is surjective for all $x \sqsubseteq y \sqsubseteq z_s$, when such maps are defined.

Proof of Claim 7.5.1. To see this, we will show there are finitely many morphisms $F_{x,y}$ that are non-injective for $z_i \sqsubset x \sqsubseteq y$. Suppose by way of contradiction there is no $z_t \sqsupset z_i$ such that all maps $F_{x,y}$ are injective for $z_t \sqsubseteq x \sqsubseteq y$. Then there exist indices $x_1, y_1 \sqsupset z_i$ with $x_1 \leq y_1$ such that $\ker[x_1, y_1] \neq 0$.

Recall from the Definition 3.2(2) there exists $k_1 \in \mathbb{Z}$ such that $z_{k_1-1} \sqsubseteq x_1 \sqsubseteq y_1 \sqsubseteq z_{k_1}$. In particular, we have $x_1 \leq y_1 \leq z_{k_1}$ with respect to the partial order. Since $\ker[x_1, y_1] \neq 0$, it follows that $\ker[x_1, z_{k_1}] \neq 0$.

Looking at the projective constant decomposition of $F|_{[z_i, z_{k_1}]}$ which exists by Theorem 6.1, there exists a submodule $\mathbb{I}^{\langle r_1, w_1 \rangle}$ of projective constant type in the decomposition with $z_i \sqsubseteq r_1 \sqsubset w_1 \sqsubset z_{k_1}$. Now if $z_i \sqsubset r_1 \sqsubset w_1 \sqsubset z_{k_1}$, then we would have Case 1. Therefore $r_1 = z_i$ and the submodule of projective constant type $\mathbb{I}^{[z_i, w_1\rangle}$ is supported on the interval $[z_i, w_1\rangle$.

Continuing in this manner, there exists an $x_2 \sqsupset z_{k_1}$ and $z_{k_2} \geq x_2$ such that $\ker[x_2, z_{k_2}] \neq 0$. The projective constant decomposition of $F|_{[z_i, z_{k_2}]}$ will contain a submodule $\mathbb{I}^{[z_i, w_2\rangle}$ of projective constant type supported on the interval $[z_i, w_2\rangle$ where $w_1 \sqsubset z_{k_1} \sqsubset w_2$. Repeating this process indefinitely results in an infinite ascending chain of R-submodules

$$\mathbb{I}_{z_i}^{[z_i, w_1\rangle} \subseteq \mathbb{I}_{z_i}^{[z_i, w_1\rangle} \oplus \mathbb{I}_{z_i}^{[z_i, w_2\rangle} \subseteq \mathbb{I}_{z_i}^{[z_i, w_1\rangle} \oplus \mathbb{I}_{z_i}^{[z_i, w_2\rangle} \oplus \mathbb{I}_{z_i}^{[z_i, w_3\rangle} \subseteq \cdots$$

of F_{z_i}, a contradiction since F_{z_i} is Noetherian. Therefore such a sufficiently large index z_t must exist. The settings for surjections and for $z_s \sqsubset z_i$ are analogous. □

Let z_s and z_t be the indices as described in Claim 7.5.1. Without loss of generality, we can choose s and t to be odd integers.

By Theorem 6.1, $F|_{[z_s, z_t]}$ admits a projective constant decomposition. Choose a submodule $\mathbb{I}^{\langle r, w \rangle}$ in this decomposition supported on $\langle r, w \rangle$ so that we write $F|_{[z_s, z_t]} = \mathbb{I}^{\langle r, w \rangle} \oplus H$ where

H is the direct sum of the remaining submodules in the decomposition. Our goal is to extend this decomposition to all of F so that we can write $F = \mathbb{I} \oplus G$.

Define $G_x = H_x$ and $\mathbb{I}_x = \mathbb{I}_x^{\langle r,w \rangle}$ for all $z_s \sqsubseteq x \sqsubseteq z_t$. It remains to carefully define what happens for indices $x \sqsupset z_t$ and $x \sqsubset z_s$.

First we will focus on indices $x \sqsupset z_t$. If $\mathbb{I}_{z_t}^{\langle r,w \rangle} = 0$ and $H_{z_t} = F_{z_t}$, then the decomposition is easy. Define $G_x = F_x$ and $\mathbb{I}_x = 0$ for all $x \sqsupset z_t$.

If $\mathbb{I}_{z_t}^{\langle r,w \rangle} \neq 0$, define $\mathbb{I}_{z_t} = \mathbb{I}_{z_t}^{\langle r,w \rangle}$ and $G_{z_t} = H_{z_t}$. Since t is odd, we have $z_t < z_{t+1}$ with respect to the partial order. In particular, $F|_{[z_t, z_{t+1}]}$ is a totally ordered persistence module indexed by the poset $[z_t, z_{t+1}]$ with a minimum z_t and maximum z_{t+1} such that:

(a) $\operatorname{coker}(F_{z_t, z_{t+1}})$ is projective;

(b) every morphism $F_{x,y}$ is injective for $z_t \leq x \leq y \leq z_{t+1}$;

(c) $F_{z_t} = \mathbb{I}_{z_t} \oplus G_{z_t}$, where $\mathbb{I}_{z_t}$ is a nonzero projective R-module.

By Lemma 7.3, there exists a decomposition $F|_{[z_t, z_{t+1}]} = \mathbb{I}^{[z_t, z_{t+1}]} \oplus H^{(t)}$ such that:

(1) $\mathbb{I}^{[z_t, z_{t+1}]}$ is a persistence submodule of projective constant type supported on $[z_t, z_{t+1}]$;

(2) $\mathbb{I}_x^{[z_t, z_{t+1}]} \cong \mathbb{I}_{z_t}$ for all $z_t \sqsubseteq x \sqsubseteq z_{t+1}$;

(3) $\mathbb{I}_{z_t}^{[z_t, z_{t+1}]} = \mathbb{I}_{z_t}$ and $H_{z_t}^{(t)} = G_{z_t}$.

Define $\mathbb{I}_x = \mathbb{I}_x^{[z_t, z_{t+1}]}$ and $G_x = H_x^{(t)}$ for all $z_t \sqsubset x \sqsubseteq z_{t+1}$. This, in effect, has extended the decomposition over $[z_s, z_t]$ to $[z_s, z_{t+1}]$.

Similarly, since $t+1$ is even, we have $z_{t+2} < z_{t+1}$ with respect to the partial order. In particular, $F|_{[z_{t+1}, z_{t+2}]}$ is a totally ordered persistence module indexed by the poset $[z_{t+1}, z_{t+2}]$ with a minimum z_{t+2} and maximum z_{t+1} such that:

(a) every morphism $F_{y,x}$ is surjective for $z_{t+1} \sqsubseteq x \sqsubseteq y \sqsubseteq z_{t+2}$;

(b) $F_{z_{t+1}}$ is a projective R-module;

(c) $F_{z_{t+1}} = \mathbb{I}_{z_{t+1}} \oplus G_{z_{t+1}}$, where $\mathbb{I}_{z_{t+1}}$ is a nonzero projective R-module.

By Lemma 7.4, there exists a decomposition $F|_{[z_{t+1},z_{t+2}]} = \mathbb{I}^{[z_{t+1},z_{t+2}]} \oplus H^{(t+1)}$ such that:

(1) $\mathbb{I}^{[z_{t+1},z_{t+2}]}$ is a persistence submodule of projective constant type supported on $[z_{t+1}, z_{t+2}]$;

(2) $\mathbb{I}_x^{[z_{t+1},z_{t+2}]} \cong \mathbb{I}_{z_{t+1}}$ for all $z_{t+1} \sqsubseteq x \sqsubseteq z_{t+2}$;

(3) $\mathbb{I}_{z_{t+1}}^{[z_{t+1},z_{t+2}]} = \mathbb{I}_{z_{t+1}}$ and $H_{z_{t+1}}^{(t+1)} = G_{z_{t+1}}$.

Define $\mathbb{I}_x = \mathbb{I}_x^{[z_{t+1},z_{t+2}]}$ and $G_x = H_x^{(t+1)}$ for all $z_{t+1} \sqsubset x \sqsubseteq z_{t+2}$. This, in effect, has extended the decomposition over $[z_s, z_{t+1}]$ to $[z_s, z_{t+2}]$.

We can repeat this process indefinitely over the intervals $[z_{t+2}, z_{t+3}], [z_{t+3}, z_{t+4}]$, and so on to obtain a decomposition $F_x = \mathbb{I}_x \oplus G_x$ for all $x \sqsupset z_t$.

Dually, we can apply the above argument for indices $x \sqsubset z_s$. This results in a decomposition $F = \mathbb{I} \oplus G$ as desired. $\square$

We now proceed to the main results of this chapter. Namely, the following two theorems are the statement of Theorem C when $\Gamma = \mathbb{Z}$ and $\Gamma = \mathbb{N}$.

Theorem 7.6. *Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with extrema $\{z_i \mid i \in \mathbb{Z}\}$ satisfying the PCC. Then F admits a projective constant decomposition.*

Proof. Immediate from Lemma 2.2, Proposition 7.5 and Lemma 3.8 when letting $\mathscr{E}$ in Lemma 2.2 be the set of submodules of F of projective constant type. $\square$

Theorem 7.7. *Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with extrema $\{z_i \mid i \in \mathbb{N}\}$ satisfying the PCC. Then F admits a projective constant decomposition.*

Proof. There is an obvious way to embed a zigzag persistence module F with extrema $\{z_i \mid i \in \mathbb{N}\}$ into a zigzag persistence module $\bar{F}$ with extrema $\{z_i \mid i \in \mathbb{Z}\}$ so that $\bar{F}|_{\{x \sqsupseteq z_0\}} = F$ and $\bar{F}|_{\{x \sqsubset z_0\}} = 0$. The projective constant decomposition of $\bar{F}$ guaranteed by Theorem 7.6 yields a projective constant decomposition of F. $\square$

We deduce Corollary F when $\Gamma = \mathbb{Z}$ and $\Gamma = \mathbb{N}$ in the following result.

Corollary 7.8. *Let R be a commutative Noetherian ring with identity such that every finitely generated projective R-module is free. Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with either extrema $\{z_i \mid i \in \mathbb{Z}\}$ or extrema $\{z_i \mid i \in \mathbb{N}\}$ satisfying the PCC. Then F is interval decomposable.*

Proof. Immediate from Theorem 7.6, Theorem 7.7 and Lemma 1.10. $\square$

Chapter 8

Necessity

Throughout this chapter, we assume that R is a (not necessarily commutative) Noetherian ring with identity, unless otherwise specified. In the preceding chapters, we showed that the projective colimit conditions are sufficient to guarantee the existence of projective constant decompositions of A_n-persistence modules and zigzag persistence modules. We further established that requiring the cokernels of all internal morphisms to be projective is sufficient to guarantee the existence of a projective constant decompositions of totally ordered persistence modules. These conditions also implied the existence of interval decompositions when R had the added property that all finitely generated projective R-modules are free. We now show that these conditions are in fact necessary whenever a projective constant (respectively, interval) decomposition exists.

Lemma 8.1. *Let $F : P \to R$-**Mod** be a nonzero zigzag persistence module with finite extrema $\{z_1, z_2, \cdots, z_n\}$ for $n \geq 2$ of projective constant type. Then:*

(N1) *$colim(F)$ is projective;*

(N2) *$coker(F_{z_1} \to colim(F))$ is projective;*

(N3) *$coker(F_{z_n} \to colim(F))$ is projective;*

(N4) *$coker(F_{z_1} \oplus F_{z_n} \to colim(F))$ is projective.*

Proof. Let $\sqsubseteq$ denote the associated total order. By Definition 1.6, F is isomorphic to $F(I, M)$ for some interval I in P and for some projective R-module M. It suffices to prove the conditions hold for $F = F(I, M)$. Recall this means that $F_x = M$ for all $x \in I$ and $F_x = 0$ otherwise. Additionally, $F_{x,y} = \mathrm{id}_{x,y}$ for all $x \leq y$ in I and $F_{x,y} = 0$ otherwise.

When $n = 2$, either $\mathrm{colim}(F) = F_{z_2}$ if $z_1 < z_2$ or $\mathrm{colim}(F) = F_{z_1}$ if $z_2 < z_1$. This observation makes the four conditions immediate.

Let $n \geq 3$. We divide the remainder of this proof into three cases based on whether z_1 and z_n are sinks or sources (see Definitions 1.12 and 3.2). In each case, we show that the four conditions hold.

Case 1. Suppose that z_1 and z_n are both sinks. This implies that n is odd, and that for all $x, y \in P$ such that $z_i \sqsubseteq x \sqsubseteq y \sqsubseteq z_{i+1}$:

$$\begin{cases} i \text{ is odd} & \Longrightarrow \quad y \leq x; \\ i \text{ is even} & \Longrightarrow \quad x \leq y. \end{cases}$$

Below is an example diagram of F; the thick lines indicate the interval I where F takes the value M, and the arrows indicate the direction of the morphisms.

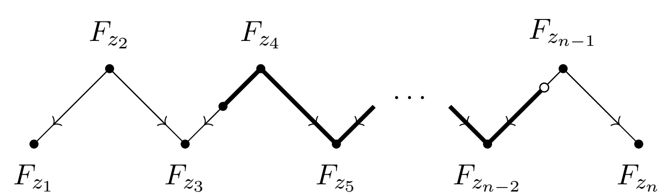

Figure 8.1: Example of Zigzag Persistence Module $F(I, M)$

Let $K = \{1, 3, \ldots, n-2, n\}$ be the odd integers of the sinks z_i of F, and let $C = \{2, 4, \ldots, n-3, n-1\}$ be the even integers of the sources z_i of F. It follows that

$$\mathrm{colim}(F) = \frac{\bigoplus_{i \in K} F_{z_i}}{E} = \frac{F_{z_1} \oplus F_{z_3} \oplus \cdots \oplus F_{z_{n-2}} \oplus F_{z_n}}{E}.$$

The submodule E above is generated by the set of elements

$$\{\iota_{z_{i-1}}(F_{z_i, z_{i-1}}(a)) - \iota_{z_{i+1}}(F_{z_i, z_{i+1}}(a)) \mid a \in F_{z_i}, i \in C\},$$

where $\iota_{z_j} : F_{z_j} \to \bigoplus_{i \in K} F_{z_i}$ is the natural inclusion map for all sinks z_j with $j \in K$. This defines an equivalence relation $\sim$ on $\bigoplus_{i \in K} F_{z_i}$ that identifies

$$\iota_{z_{i-1}}(F_{z_i, z_{i-1}}(a)) \sim \iota_{z_{i+1}}(F_{z_i, z_{i+1}}(a))$$

for all sources z_i with $i \in C$ and for all $a \in F_{z_i}$.

Consider the interval I. If no extrema z_i lie in I, then $\operatorname{colim}(F) = 0$ since $F_{z_i} = 0$ for all sinks z_i with $i \in K$. The four conditions are immediate.

Now suppose that at least one extremum z_i lies in I. Let z_s and z_t be the minimum and maximum extrema in I, respectively, with respect to the associated total order. Note that it is possible for $z_s = z_t$ when exactly one extremum lies in I.

Claim 8.1.1. If z_s or z_t are sources, then $\operatorname{colim}(F) = 0$.

Proof of Claim 8.1.1. Without loss of generality, assume that z_s is a source. Hence $F_{z_s} = M$. Since z_s is the minimum extrema in I, this implies $F_{z_{s-1}} = 0$. Based on the equivalence relation $\sim$, we identify

$$0 = \iota_{z_{s-1}}(F_{z_s, z_{s-1}}(a)) \sim \iota_{z_{s+1}}(F_{z_s, z_{s+1}}(a)) \text{ for all } a \in F_{z_s}.$$

If z_s is the maximum source in P, then $z_{s+1} = z_n$. Hence all elements of $\bigoplus_{i \in K} F_{z_i}$ get identified with 0 under the equivalence relation $\sim$, and thus $\operatorname{colim}(F) = 0$.

If z_s is not the maximum source in P, then z_{s+2} is a source in P. Based on the equivalence relation $\sim$, we identify

$$\iota_{z_{s+1}}(F_{z_{s+2}, z_{s+1}}(a)) \sim \iota_{z_{s+3}}(F_{z_{s+2}, z_{s+3}}(a)) \text{ for all } a \in F_{z_{s+2}}.$$

But this implies

$$0 \sim \iota_{z_{s+1}}(F_{z_s,z_{s+1}}(a)) \sim \iota_{z_{s+1}}(F_{z_{s+2},z_{s+1}}(a)) \text{ for all } a \in F_{z_{s+2}} \subseteq F_{z_s}.$$

Continuing this process, it is easy to see that all elements of $\bigoplus_{i\in K} F_{z_i}$ get identified with 0 under the equivalence relation $\sim$, and hence $\mathrm{colim}(F) = 0$. $\square$

Based on the setting of Claim 8.1.1 when z_s or z_t are sources, the four conditions are immediate. Finally, assume that z_s and z_t are both sinks. Then we can equivalently describe the submodule E as

$$E = \{(a_{z_1}, a_{z_3}, \ldots, a_{z_{n-2}}, a_{z_n}) : \textstyle\sum_{i\in K} a_{z_i} = 0\}.$$

We omit the detailed calculations of this as it is easy to verify using the key observation that if z_i is a source in I, then z_{i-1} and z_{i+1} are also in I.

Let $\theta : \bigoplus_{i\in K} F_{z_i} \to M$ be the map given by

$$(a_{z_1}, a_{z_3}, \ldots, a_{z_{n-2}}, a_{z_n}) \mapsto \textstyle\sum_{i\in K} a_{z_i}.$$

Then $\ker(\theta) = E$, and moreover θ is surjective. Thus

$$\mathrm{colim}(F) = \frac{\bigoplus_{i\in K} F_{z_i}}{E} = \frac{\bigoplus_{i\in K} F_{z_i}}{\ker(\theta)} \cong \mathrm{im}(\theta) = M$$

by the First Isomorphism Theorem, which proves condition **(N1)** that $\mathrm{colim}(F)$ is projective.

If $F_{z_1} = 0$ then **(N2)** is immediate. If $F_{z_1} = M$, then $z_1 = z_s$. The map $F_{z_1} \to \mathrm{colim}(F)$ is easily seen to be surjective. Hence $\mathrm{coker}(F_{z_1} \to \mathrm{colim}(F)) = 0$ is trivially projective, which shows **(N2)**. Analogous arguments show that conditions **(N3)** and **(N4)** also hold.

Case 2. Suppose that z_1 and z_n are both sources. This implies that n is odd, and that for all $x, y \in P$ such that $z_i \sqsubseteq x \sqsubseteq y \sqsubseteq z_{i+1}$:

$$\begin{cases} i \text{ is odd} \quad \implies \quad x \leq y; \\ i \text{ is even} \quad \implies \quad y \leq x. \end{cases}$$

If $n = 3$, then $\mathrm{colim}(F) = F_{z_2}$ where $F_{z_2} = 0$ or $F_{z_2} = M$. Hence $\mathrm{colim}(F)$ is projective satisfying **(N1)**, and the other three conditions follow trivially.

If $n > 3$, then $\mathrm{colim}(F) = \mathrm{colim}(F|_{[z_2, z_{n-1}]})$ where z_2 and z_{n-1} are both sinks. Hence we apply Case 1 to show that $\mathrm{colim}(F)$ is projective for condition **(N1)**.

If $\mathrm{colim}(F|_{[z_2, z_{n-1}]}) = 0$, then the other three conditions are immediate. Suppose that $\mathrm{colim}(F|_{[z_2, z_{n-1}]}) \cong M$. Then Claim 8.1.1 implies that the minimum and maximum extrema z_s and z_t, respectively, in $I \cap [z_2, z_{n-1}]$ are both sinks.

If $F_{z_1} = 0$, then **(N2)** is immediate. If $F_{z_1} = M$, then $z_1 \in I$. Since $z_1 \sqsubseteq z_2 \sqsubseteq z_s$ with $z_1, z_s \in I$, Lemma 3.4 implies $z_2 \in I$, and hence $z_2 = z_s$ by minimality of z_s. It is easy to verify from this observation that $F_{z_1} \to \mathrm{colim}(F)$ is a surjective map. Hence $\mathrm{coker}(F_{z_1} \to \mathrm{colim}(F)) = 0$ is trivially projective, which shows **(N2)** is satisfied. Conditions **(N3)** and **(N4)** follow similarly.

Case 3. Suppose that z_1 is a sink and z_n is a source. This implies that n is even, and that for all $x, y \in P$ such that $z_i \sqsubseteq x \sqsubseteq y \sqsubseteq z_{i+1}$:

$$\begin{cases} i \text{ is odd} \quad \implies \quad y \leq x; \\ i \text{ is even} \quad \implies \quad x \leq y. \end{cases}$$

Observe that $\mathrm{colim}(F) = \mathrm{colim}(F|_{[z_1, z_{n-1}]})$ where z_1 and z_{n-1} are both sinks. Hence we apply Case 1 to show that $\mathrm{colim}(F)$ is projective for condition **(N1)**.

If $\mathrm{colim}(F|_{[z_1,z_{n-1}]}) = 0$, then the other three conditions are immediate. Suppose that $\mathrm{colim}(F|_{[z_1,z_{n-1}]}) \cong M$. Then Claim 8.1.1 implies that the minimum and maximum extrema z_s and z_t, respectively, in $I \cap [z_1, z_{n-1}]$ are both sinks.

If $F_{z_1} = 0$, then **(N2)** is immediate. If $F_{z_1} = M$, then $z_1 \in I$. Hence $z_1 = z_s$ by minimality of z_s. It is easy to verify that $F_{z_1} \to \mathrm{colim}(F)$ is a surjective map. Hence $\mathrm{coker}(F_{z_1} \to \mathrm{colim}(F)) = 0$ is trivially projective, which shows **(N2)**.

If $F_{z_n} = 0$, then **(N3)** is immediate. If $F_{z_n} = M$, then $z_n \in I$. Since $z_t \sqsubseteq z_{n-1} \sqsubseteq z_n$ with $z_t, z_n \in I$, Lemma 3.4 implies $z_{n-1} \in I$, and hence $z_{n-1} = z_t$ by maximality of z_t. It is easy to verify with this observation that $F_{z_n} \to \mathrm{colim}(F)$ is a surjective map. Hence $\mathrm{coker}(F_{z_n} \to \mathrm{colim}(F)) = 0$ is trivially projective, which shows **(N3)**.

Condition **(N4)** follows similarly as **(N2)** and **(N3)**. The case for when z_1 is a source and z_n is a sink is analogous. $\square$

Lemma 8.2. *Let $F : P \to R$-**Mod** be a nonzero zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$ of projective constant type. Let $\sqsubseteq$ be the associated total order. Then F satisfies the projective colimit conditions; that is, for any $x \sqsubseteq y$ of P:*

(C1) *$\mathrm{colim}(F|_{[x,y]})$ is projective;*

(C2) *$\mathrm{coker}(F_x \to \mathrm{colim}(F|_{[x,y]}))$ is projective;*

(C3) *$\mathrm{coker}(F_y \to \mathrm{colim}(F|_{[x,y]}))$ is projective;*

(C4) *$\mathrm{coker}(F_x \oplus F_y \to \mathrm{colim}(F|_{[x,y]}))$ is projective.*

Proof. By Definition 1.6, F is isomorphic to $F(I, M)$ for some interval I in P and for some projective R-module M. It suffices to prove the conditions hold for $F = F(I, M)$.

Let $x \sqsubseteq y$ be indices in P. When $x = y$, the four conditions are immediate. Similarly, when $[x, y] \cap I = \emptyset$, the four conditions are immediate. Let $x \sqsubset y$ such that $[x, y] \cap I \neq \emptyset$.

There exist $s, t \in \Gamma$ such that $z_s \sqsubseteq x \sqsubseteq z_{s+1}$ and $z_t \sqsubseteq y \sqsubseteq z_{t+1}$ by Lemma 3.3. Observe that $F|_{[x,y]}$ is a nonzero zigzag persistence module of projective constant type with either:

(1) finite extrema $\{x, y\}$ if $s = t$;

(2) finite extrema $\{x, z_{s+1}, z_{s+2}, \ldots, z_{t-1}, z_t, y\}$ if $s < t$.

In either case, Lemma 8.1 guarantees the four conditions are satisfied. $\square$

The following result is the statement of Theorem I regarding zigzag persistence modules.

Theorem 8.3. *Let $F : P \to R$-**Mod** be a nonzero p.f.g. zigzag persistence module with extrema $\{z_i \mid i \in \Gamma\}$. If F admits a projective constant (respectively, interval) decomposition, then F satisfies the projective colimit conditions.*

Proof. Let $F = \bigoplus_{G \in \mathscr{A}} G$ be a projective constant decomposition of F. Then each G satisfies the PCC by Lemma 8.2. It is clear that F will also satisfy the PCC based on the fact that colimits commute with coproducts and cokernels in the category R-**Mod** [12, §2, Corollary 2]. In particular, it is essential that F is pointwise finitely generated so that only finitely many G_x are nontrivial for any fixed $x \in P$. An interval decomposition of F is a special case of a projective constant decomposition, which makes the respective claim immediate. $\square$

Consequently, we also establish Theorem G regarding A_n-persistence modules.

Theorem 8.4. *Let $F : A_n \to R$-**Mod** be a nonzero p.f.g. A_n-persistence module. If F admits a projective constant (respectively, interval) decomposition, then F satisfies the projective colimit conditions.*

Proof. Suppose that F admits a projective constant decomposition. When $n = 1$, the result is immediate. When $n \geq 2$, F can be viewed as a nonzero p.f.g. zigzag persistence module with finite extrema which satisfies the PCC by Theorem 8.3. An interval decomposition of F is a special case of a projective constant decomposition, which makes the respective claim immediate. $\square$

Finally, we state Theorem H regarding totally ordered persistence modules.

Theorem 8.5. *Let $F : P \to R$-**Mod** be a nonzero p.f.g.p. totally ordered persistence module. If F admits a projective constant (respectively, interval) decomposition, then the cokernel of every internal morphism $F_{x,y}$ is projective.*

Proof. When $x = y$, the claim is immediate. When $x < y$, $F|_{[x,y]}$ can be viewed as a nonzero p.f.g. zigzag persistence module with finite extrema $\{x, y\}$. Since F admits a projective constant decomposition, $F|_{[x,y]}$ also admits a projective constant decomposition and therefore $F|_{[x,y]}$ satisfies the PCC by Theorem 8.3. In particular, $\mathrm{coker}(F_{x,y})$ is projective by Lemma 3.8. An interval decomposition of F is a special case of a projective constant decomposition, which makes the respective claim immediate. $\square$

Note in Theorems 8.3, 8.4, and 8.5, the necessity of the conditions hold regardless of whether R is commutative or noncommutative. This is in contrast to the sufficiency results, which in some cases required R to be commutative.

Chapter 9

Uniqueness

We conclude this dissertation by investigating the uniqueness of projective constant decompositions. Given that modules over Noetherian rings need not admit a unique decomposition into indecomposable summands, uniqueness of projective constant decompositions cannot be expected in full generality.

Nevertheless, by imposing additional conditions on R-**Mod**, we obtain meaningful uniqueness results. We first demonstrate that if R-**Mod** is a Krull-Schmidt category—an additive category where objects decompose uniquely into finite direct sums of indecomposables with local endomorphism rings—then every projective constant decomposition of a persistence module admits a refinement to an indecomposable version (see Definition 9.3). This decomposition is unique up to isomorphism and permutation of summands. Furthermore, we show that when R is an integral domain, an interval decomposition of any pointwise finitely generated persistence module indexed by a small poset category likewise satisfies this uniqueness property. Notably, we relax the Noetherian requirement on R throughout this chapter, assuming only that R is a unital ring and that R-**Mod** denotes the category of all unitary left R-modules. Additionally, the results are established for persistence modules indexed by arbitrary posets, extending beyond the cases of A_n-type quivers, totally ordered sets, or zigzag posets presented in this dissertation.

Definition 9.1. Let P be a poset. We say a persistence module $F : P \to R$-**Mod** is *indecomposable* if F is nonzero and $F \cong G \oplus H$ implies either $G = 0$ or $H = 0$. Likewise, an R-module M is *indecomposable* if M is nonzero and $M \cong N \oplus L$ implies either $N = 0$ or $L = 0$.

Lemma 9.2. *Let P be a poset and let $F : P \to R$-**Mod** be a module of projective constant type with $F \cong F(I, M)$ for some interval I in P and for some indecomposable projective R-module M. Then F is indecomposable.*

Proof. It suffices to prove the result for $F(I, M)$. Suppose that $F(I, M)$ admits a decomposition $F(I, M) \cong G \oplus H$. Then for each $x \in I$, we have $F(I, M)_x \cong G_x \oplus H_x$. Since $F(I, M)_x = M$ is an indecomposable projective R-module, it follows that either $G_x = 0$ for all $x \in I$ or $H_x = 0$ for all $x \in I$, implying that $F(I, M)$ is indecomposable. □

Definition 9.3. Let P be a poset and let $F : P \to R$-**Mod** be a persistence module. An *indecomposable projective constant decomposition* of F is a projective constant decomposition $F = \bigoplus_{G \in \mathscr{A}} G$ such that for every $G \in \mathscr{A}$, we have $G \cong F(I, M)$ for some interval I in P and for some indecomposable projective R-module M.

Lemma 9.4. *Let P be a poset and let $F : P \to R$-**Mod** be a module of projective constant type with $F \cong F(I, M)$ for some interval I in P and for some indecomposable projective R-module M. If R-**Mod** is a Krull-Schmidt category, then F has a local endomorphism ring.*

Proof. It suffices to prove the result for $F(I, M)$. Let $f : F(I, M) \to F(I, M)$ be a morphism. Observe that for all $r, w \in I$ such that $r \leq w$, we have

$$F(I, M)_{r,w} \circ f_r = f_w \circ F(I, M)_{r,w}.$$

Hence $f_r = f_w$ since $F(I, M)_{r,w} = \mathrm{id}_M$.

Now consider any $x, y \in I$. Since I is connected, there exist $y_0, y_1, \ldots, y_n \in I$ such that Definition 1.4(2) holds. By the observation above, we will have $f_x = f_{y_0} = \cdots = f_{y_n} = f_y$.

This yields an isomorphism

$$\mathrm{End}(F(I, M)) \cong \mathrm{End}_R(M),$$

where $\mathrm{End}(F(I, M))$ and $\mathrm{End}_R(M)$ are the endomorphism rings of $F(I, M)$ and M, respectively. As M is indecomposable, $\mathrm{End}_R(M)$ is local, which in turn ensures that $\mathrm{End}(F(I, M))$ is local. $\square$

We now state and prove Theorem J.

Theorem 9.5. *Let P be a poset and let $F : P \to R$-**Mod** be a nonzero persistence module. Suppose that R-**Mod** is a Krull-Schmidt category. If F admits a projective constant decomposition, then F admits a unique indecomposable projective constant decomposition up to isomorphism and permutation of the summands.*

Proof. Let $F = \bigoplus_{G \in \mathscr{A}} G$ be a projective constant decomposition of F. For each $G \in \mathscr{A}$, we have $G \cong F(I, M)$ for some interval I in P and for some projective R-module M. Since R-**Mod** is a Krull-Schmidt category, we can decompose M into a finite direct sum of indecomposable projective R-modules. This induces a finite decomposition $G = \bigoplus_{i=1}^{n} G^{(i)}$ where each $G^{(i)} \cong F(I, N_i)$ for some indecomposable projective R-module N_i. Thus each $G \in \mathscr{A}$ admits an indecomposable projective constant decomposition, and hence F admits an indecomposable projective constant decomposition.

To see uniqueness, let $F = \bigoplus_{H \in \mathscr{B}} H$ be an indecomposable projective constant decomposition of F. By Lemmas 9.2 and 9.4, each $H \in \mathscr{B}$ is indecomposable with local endomorphism ring. The Krull-Remak-Schmidt-Azumaya theorem [1] guarantees this decomposition is unique up to isomorphism and permutation of the summands. $\square$

Corollary 9.6. *Let P be a poset and let $F : P \to R$-**Mod** be a nonzero persistence module. Suppose that R-**Mod** is a Krull-Schmidt category and that R is an indecomposable R-module.*

If F admits an interval decomposition, then the interval decomposition is unique up to isomorphism and permutation of the summands.

Proof. Immediate from Theorem 9.5, Lemma 9.2 and Lemma 9.4. $\square$

The assumption in Corollary 9.6 that R is indecomposable as an R-module is essential. This condition ensures that an interval decomposition is indeed an indecomposable projective constant decomposition.

Recall that a category is *small* if both its collection of objects and its collection of morphisms form sets. The poset categories of A_n-type quivers, totally ordered sets, and zigzag posets are all small. For persistence modules indexed by small poset categories, we establish the uniqueness of interval decompositions over integral domains without assuming the Krull-Schmidt property. This is achieved via scalar extension to the field of fractions, which reduces the problem to the known uniqueness results for persistence modules over fields.

Definition 9.7. Let R be an integral domain, $\mathfrak{F}$ the field of fractions of R, and $\mathrm{Vec}(\mathfrak{F})$ the category of $\mathfrak{F}$-vector spaces. Let P be a poset and let $F : P \to R$-**Mod** be a persistence module. Define $\mathfrak{F} \otimes F : P \to \mathrm{Vec}(\mathfrak{F})$ to be the persistence module such that:

- $(\mathfrak{F} \otimes F)_x = \mathfrak{F} \otimes_R F_x$ for all $x \in P$;
- $(\mathfrak{F} \otimes F)_{x,y} = \mathrm{id}_{\mathfrak{F}} \otimes F_{x,y}$ for all $x \leq y$ in P.

We now state and prove Theorem K. This result is obtained by generalizing the proof of [10, §4, Lemma 1], which establishes the uniqueness of interval decompositions for pointwise freely and finitely generated persistence modules over PIDs indexed by finite linear quivers.

Theorem 9.8. *Let R be an integral domain. Let P be a small poset category and let F : $P \to R$-**Mod** be a nonzero p.f.g. persistence module. If F admits an interval decomposition, then this decomposition is unique up to isomorphism and permutation of the summands.*

Proof. Let $\mathfrak{F}$ be the field of fractions of R and $\mathrm{Vec}(\mathfrak{F})$ the category of $\mathfrak{F}$-vector spaces. Let $F = \bigoplus_{G\in\mathscr{A}} G$ be an interval decomposition of F. For each $x \in P$, we have

$$(\mathfrak{F} \otimes F)_x = \mathfrak{F} \otimes_R F_x = \mathfrak{F} \otimes_R \left(\bigoplus_{G\in\mathscr{A}} G_x \right) \cong \bigoplus_{G\in\mathscr{A}} (\mathfrak{F} \otimes_R G_x) \cong \bigoplus_{G\in\mathscr{A}} (\mathfrak{F} \otimes G)_x,$$

where the first isomorphism follows from the fact that only finitely many G_x are nontrivial. Hence there is an obvious way to define a persistence module isomorphism such that

$$\mathfrak{F} \otimes F \cong \bigoplus_{G\in\mathscr{A}} (\mathfrak{F} \otimes G).$$

In particular, if G is supported on the interval I in P, then $\mathfrak{F} \otimes G$ is an interval module supported on I. Indeed, for all $x \in I$ we have

$$(\mathfrak{F} \otimes G)_x = \mathfrak{F} \otimes_R G_x \cong \mathfrak{F} \otimes_R R \cong \mathfrak{F},$$

and moreover for all $x, y \in I$ such that $x \leq y$, we see that the morphism

$$(\mathfrak{F} \otimes G)_{x,y} = \mathrm{id}_{\mathfrak{F}} \otimes G_{x,y}$$

is a surjection between one-dimensional $\mathfrak{F}$-vector spaces, and hence an isomorphism. Thus $\mathfrak{F} \otimes F$ is interval decomposable, and this decomposition is unique up to isomorphism and permutation of the summands whenever P is a small poset category [3, §2]. The obvious bijection between the interval summands of F and those of $\mathfrak{F} \otimes F$ guarantees the interval decomposition of F is likewise unique up to isomorphism and permutation of the summands. □

Corollary 9.9. *Let R be one of the following rings:*

(1) a principal ideal domain;

(2) the polynomial ring in finite variables $k[x_1, \ldots, x_n]$, where k is a principal ideal domain or a field;

(3) a local Noetherian integral domain;

(4) the ring of formal power series $S[[x]]$, where is S is a local Noetherian integral domain.

*Let P be a poset and let $F : P \to R$-**Mod** be a nonzero p.f.g. persistence module. If F admits an interval decomposition, then this decomposition is unique up to isomorphism and permutation of the summands.*

Proof. Immediate from Theorem 9.8 noting every ring listed is an integral domain. □

Recall that the rings listed in Corollary 9.9 have the property that every finitely generated projective module is free (see Corollary 1.11). Consequently, Corollaries D, E, F, and 9.9 collectively establish the existence and essential uniqueness of interval decompositions for persistence modules over such rings indexed by A_n-type quivers, totally ordered sets, and zigzag posets, provided the relevant necessary and sufficient conditions are met.

References

[1] Gorô Azumaya, *Corrections and supplementaries to my paper concerning Krull–Remak–Schmidt's theorem*, Nagoya Mathematical Journal **1** (1950), 117–124.

[2] Magnus Bakke Botnan, *Interval decomposition of infinite zigzag persistence modules*, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3571–3577.

[3] Magnus Bakke Botnan and William Crawley-Boevey, *Decomposition of persistence modules*, Proceedings of the American Mathematical Society (2018).

[4] Gunnar Carlsson and Vin de Silva, *Zigzag Persistence*, Foundations of Computational Mathematics **10** (2010), no. 4, 367–405.

[5] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer, *Stability of Persistence Diagrams*, Discrete & Computational Geometry **37** (2007), no. 1, 103–120.

[6] Wee Liang Gan and Nadiya Upegui Keagy, *On interval decomposition of persistence modules* (2025), arXiv 2507.01303, primary class math.AT

[7] David Hilbert, *Über die Theorie der algebraischen Formen*, Mathematische Annalens, vol. 36, 1890.

[8] Irving Kaplansky, *Projective modules*, Ann. of Math (2) **68** (1958), 372-377.

[9] T.Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, Springer New York, NY, 2012.

[10] Jiajie Luo and Gregory Henselman-Petrusek, *Interval Decomposition of Persistence Modules over a Principal Ideal Domain*, Foundations of Computational Mathematics, posted on 2025, DOI 10.1007/s10208-025-09732-8.

[11] Jiajie Luo and Gregory Henselman-Petrusek, *Interval Decomposition of Infinite Persistence Modules over a Principal Ideal Domain* (2025), arXiv 2511.07614, primary class math.AT

[12] Bodo Pareigis, *Categories and functors*, Pure and Applied Mathematics, vol. 39, Academic Press, New York-London, 1970.

[13] Daniel Quillen, *Projective modules over polynomial rings*, Inventiones Mathematicae, vol. 36, 1976.

[14] Claus Michael Ringel, *Representation theory of Dynkin quivers. Three contributions*, Front. Math. China **11** (2016), no. 4, 765–814.

[15] Andrei Suslin, *Projective modules over polynomial rings are free*, Soviet Mathematics, vol. 17, 1976.

[16] Afra Zomorodian and Gunnar Carlsson, *Computing Persistent Homology*, Discrete and Computational Geometry **33** (2005), no. 2, 249–274.