Submitted to Nuclear Physics B

LBL-3891 c.1
Preprint

SLOPE RENORMALIZATION OF THE ONE LOOP PLANAR STRING DIAGRAM

Robert Roth

June 18, 1975

Prepared for the U.S. Energy Research and
Development Administration under Contract W-7405-ENG-48

DISCLAIMER

SLOPE RENORMALIZATION OF THE ONE LOOP PLANAR STRING DIAGRAM*

Robert Roth

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

June 18, 1975

Abstract: It has been previously shown that for the number of dimensions $d < 25$, the one loop planar string diagram is simply a multiplicative (coupling constant) renormalization of the Born term. It is shown here that for $d = 25$ or 26 the extra divergent term gives, in addition to a further multiplicative renormalization, a renormalization of the slope of the Regge trajectories.

I. Introduction

The N-point Veneziano amplitude is a tree diagram approximation to the strong interaction scattering amplitude. Thus one is led to examine loop diagrams in the hope of obtaining a more realistic amplitude. As in quantum electrodynamics, these diagrams are found to be divergent. A renormalization procedure must then be found in which all infinities are absorbed into a redefinition of the physical parameters of the theory. This was done for the one loop planar diagram for $d < 25$ by Neveu and Scherk [1], and the result was found to be a simple multiplicative renormalization of the Born term. In this paper we do this for the one loop planar diagram in the critical number of dimensions. We show that the divergent part of the amplitude

* This work was supported by the Energy Research and Development Agency (ERDA).

can at the one loop level be written as

$$c_1 \left(B_N(m,\ s_{ij}) + c_2 \frac{\partial B(m,\ s_{ij})}{\partial m} \right) = c_1\, B_N(m + c_2,\ s_{ij})$$

where c_1, c_2 are constants, s_{ij} are the planar subenergies, m is the slope of the Regge trajectories, and B_N is the Veneziano amplitude. Thus at this level, multiplicative and slope renormalizations are all that are needed to render the integral finite.

II. Method

In the interacting string picture, the single loop planar amplitude for N scalars is given by the following expression, after a Jacobi transformation has been made on the usual variables of integration [2]

$$\int_0^\infty dq \int_0^{2\pi} d\phi_1 \int_0^{\phi_1} d\phi_2 \cdots \int_0^{\phi_{N-2}} d\phi_{N-1} \left[q^{-(d+11)/12} \right] f(\ell n\ q)$$

$$\times \prod_{1 \leq r < s \leq N} \exp\left(-p_r \cdot p_s\ N(\rho_r,\ \rho_s)\right)$$

where

$$\exp\left(-p_r \cdot p_s\ N(\rho_r,\ \rho_s)\right) = \left(-\frac{4\pi}{\ell n\ q} \sin \frac{1}{2} (\phi_r - \phi_s) \right.$$

$$\times \left\{ \prod_{n=1}^{\infty} \left(1 - q^{2n} e^{i(\phi_r - \phi_s)}\right) \right.$$

$$\left. \left. \times \left(1 - q^{2n} e^{-i(\phi_r - \phi_s)}\right) \left(1 - q^{2n}\right)^{-2} \right\} \right)^{-2p_r \cdot p_s} \qquad (1)$$

and where $f(\ln q)$ is a function of $\ln q$ (no powers), $\phi_N = 0$, and the square bracket should read instead q^{-3} for the special case $d = 26$. The integral diverges near $q = 0$; that is, the region where the loop shrinks to a point. We note for later use that this point (the loop at $q = 0$) is located at $i\infty$ in the ϕ-plane. To examine the integral in the $q = 0$ region, we expand the curly bracket in a power series in q^2 and find that for $d < 25$, only the constant term leads to a divergent integral. This term was shown by Neveu and Scherk[1] to be simply a multiplicative renormalization of the Veneziano amplitude. For $d \geqslant 25$, the linear term in q^2 in the power series expansion also leads to a divergent integral. It is this integral that we examine here. We change to the more convenient variables of integration $u_i = \tan \frac{\phi_i}{2}$. In the limit $q \to 0$, these u variables are related to the string diagram variables ρ by the usual tree diagram transformation

$$\rho = \sum_{r=1}^{N} \alpha_r \ln (u - u_r) \tag{2}$$

with the cut now at $u = i$. We then obtain as the coefficient of the infinite q integration

$$\sum_{1 \leqslant a < b \leqslant N} 2\, p_a \cdot p_b \int_0^0 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{N-2}} du_{N-1} \frac{(u_a - u_b)^2}{(u_a^2 + 1)(u_b^2 + 1)} \times \prod_{1 \leqslant i < j \leqslant N} (u_i - u_j)^{-2p_i \cdot p_j} \tag{3}$$

where the range of integration is restricted only by $u_N \equiv 0 < u_{N-1} < \cdots < u_1 < 0 = u_N$ (which is just a convenient notation for $0 < u_{N-1} < \cdots < u_r < \infty$ and $-\infty < u_{r-1} < \cdots < u_1 < 0$ for some r).

Even after factoring out the infinite q integral, we find that the remaining integral (3) diverges. This remaining divergence cannot still be due to the loop shrinking to a point since the integral (3) diverges only for particular configurations of the u's. In fact the remaining divergence is due to configurations of the one loop diagram that correspond to external line self-energy insertions. Suppose we factor out the self-energy part in one of these configurations. Then we are left with a tree level diagram with exactly the same incoming states and momenta, or else the contribution is not divergent. Thus we expect our divergences to be simply an infinite constant multiple of B_N.

In order to evaluate the contribution of (3) to the amplitude, we must first choose some cutoff procedure rendering the integral finite. To this end, we temporarily suspend momentum conservation by introducing a new incoming momentum k (see Fig. 1). We tentatively choose it to enter the string diagram at the position of the loop (remember that we have taken the limit $q \to 0$ which corresponds to the loop shrinking to a point), but we shall see that we will have to modify this slightly. We expect this procedure to eliminate our infinities, since now all self-energy insertions have incoming and outgoing momenta which differ by k. In the limit $k \to 0$, we should then recover a constant multiple of B_N as the divergent part. Thus instead of the normal energy-momentum conservation equation, we have

$$\sum_{i=1}^{N} p_i + k = 0$$

where k is the new momentum introduced.

This new momentum introduces an extra term $i\sum_{i=1}^{d-2} k_i X^i(\rho_c)$ to the exponential of the functional integral for the S matrix. This leads to the extra term in (1)

$$\exp\left\{\sum_{i=1}^{d-2}\sum_{r=1}^{N} k_i p_r^i N(\rho_r,\rho_c) + \frac{1}{2} N(\rho_c,\rho_c)\sum_{i=1}^{d-2} k_i^2\right\} .$$

The second term, which is infinite, is similar to an infinite term obtained in the conventional path integral interacting string formalism[2]. As in the latter case, it can be absorbed into the volume element since it has no dependence on the integration variables. The first term in the exponent changes (3) to the expression

$$\sum_{1\leq a<b\leq N} 2\, p_a\cdot p_b \int_0^0 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{N-2}} du_{N-1} \frac{(u_a-u_b)^2}{(u_a^2+1)(u_b^2+1)} \times \prod_{1\leq i<j\leq N} (u_i-u_j)^{-2p_i\cdot p_j} \prod_r \left[\frac{(u_r-u_c)^2}{u_r^2+1}\right]^{-p_r\cdot k}$$

$$+ \sum_a 2p_a\cdot k \int_0^0 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{N-2}} du_{N-1} \frac{(u_a-u_c)^2}{(u_a^2+1)(u_c^2+1)} \times \prod_{1\leq i<j\leq N} (u_i-u_j)^{-2p_i\cdot p_j} \prod_r \left[\frac{(u_r-u_c)^2}{u_r^2+1}\right]^{-p_r\cdot k} \qquad (4)$$

where we have defined $k_{\pm} = 0$ to change the $d - 2$ product to a covariant d product, $u_c \equiv i$ is the point to which the loop has been mapped, and we have neglected terms in k^2 since they are second order in a small quantity. The fact that $(u_c^2 + 1)^{-1} = (i^2 + 1)^{-1}$ appearing in the second term is undefined is a point we shall deal with later. We can write the last factor of the first term in (4) as

$$\prod_r \left[\frac{(u_r-u_c)^2}{u_r^2+1}\right]^{-p_r\cdot k} = \exp\left\{-\sum_r p_r\cdot k\ \ln\left[\frac{(u_r-u_c)^2}{u_r^2+1}\right]\right\}$$

$$= 1 - \sum_r p_r\cdot k\ \ln\left[\frac{(u_r-u_c)^2}{u_r^2+1}\right] + O(k^2) . \qquad (5)$$

This expansion is valid in the range of integration since the u_r's are real and $u_c = i$, so that the argument of the logarithm never blows up. Also since the logarithm is always well-behaved, the convergence properties of all the terms in the series are the same. Thus since we will see that the first term in the series behaves as k^{-1} as $k \to 0$ (this behavior is expected of an external line self-energy insertion), we can neglect the terms of order k^2 and higher in (5). Thus we have left the first two terms in the expansion (5) in addition to the second term in (4). We refer to these throughout the rest of this paper as terms I, II, and III. We point out that although term I has exactly the same form as the original divergent expression (3), it is now well-behaved due to the new energy-momentum conservation equation.

III. Evaluation of the N-Point Function

Consider term I for the N-point function for a particular choice of $a, b \neq N$ and $u_a < 0$ (just as an illustration). Then we have

$$2p_a\cdot p_b \int_{-\infty}^{0} du_a \int_{0}^{u_a} du_b \int_{u_a}^{0} du_1 \int_{u_a}^{u_1} du_2 \cdots \int_{u_a}^{u_{a-2}} du_{a-1} \int_{u_{a+2}}^{u_a} du_{a+1} \cdots$$

$$\times \int_{u_b}^{u_{b-2}} du_{b-1} \int_{0}^{u_b} du_{b+1} \cdots \int_{0}^{u_{N-2}} du_{N-1}$$

$$\times \prod_{i>j} (u_i - u_j)^{-2p_i\cdot p_j} \; \frac{(u_a - u_b)^2}{(u_a^2 + 1)(u_b^2 + 1)}$$

$$= 2p_a\cdot p_b \int_{-\infty}^{0} \frac{du_a}{u_a^2 + 1} \int_{0}^{-1} \frac{d\alpha_b}{(\alpha_b^2 u_a^2 + 1)} (1 - \alpha_j)^2 \int_{-1}^{0} d\alpha_1 \int_{-1}^{\alpha_1} d\alpha_2 \cdots$$

$$\times \int_{-1}^{\alpha_{a-2}} d\alpha_{a-1} \int_{\alpha_b}^{-1} d\alpha_{a+1} \cdots \int_{\alpha_b}^{\alpha_{b-2}} d\alpha_{b-1} \int_{0}^{\alpha_b} d\alpha_{b+1} \cdots \int_{0}^{\alpha_{N-2}} d\alpha_{N-1}$$

$$\times \prod_{i>j} (\alpha_i - \alpha_j)^{-2p_i\cdot p_j} = \qquad (6a)$$

(Equation continued)

$$\pi p_a\cdot p_b \left[\int_{0}^{\eta} + \int_{\eta}^{\eta^{-1}} + \int_{\eta^{-1}}^{\infty} + \int_{-\infty}^{-\eta^{-1}} + \int_{-\eta^{-1}}^{-1} \right] d\alpha_b \; \frac{(1 + \alpha_b)^2}{1 + |\alpha_b|}$$

$$\times \left\{ \int_{-1}^{0} d\alpha_1 \int_{-1}^{\alpha_1} d\alpha_2 \cdots \int_{-1}^{\alpha_{a-2}} d\alpha_{a-1} \int_{\alpha_b}^{-1} d\alpha_{a+1} \cdots \right.$$

$$\left. \times \int_{\alpha_b}^{\alpha_{b-2}} d\alpha_{b-1} \int_{0}^{\alpha_b} d\alpha_{b+1} \cdots \int_{0}^{\alpha_{N-2}} d\alpha_{N-1} \prod_{i>j} (\alpha_i - \alpha_j)^{-2p_i\cdot p_j} \right\} . \qquad (6b)$$

In (6a), we have made the substitution $\alpha_i = u_i/u_a$. In (6b), we have done the u_a integration and broken up the α_b integral as shown. If all the ϵ_i were zero, then the quantity in the curly brackets would be $\frac{1}{\alpha_b(1 + \alpha_b)} B_N$ where B_N is the N-point Veneziano (Koba-Nielsen) formula. In the limit $\epsilon_i \to 0$, the second and the fifth α_b integrals are still finite. Thus we are permitted to take the limit before doing the α_b integral, and these two α_b integrals contribute just a constant multiple of B_N. Notice that this result is independent of the value of $\eta > 0$, and we can choose it to be as small as we like. In particular we can let $\eta \to 0$, as long as this limit is taken _after_ the $\epsilon_i \to 0$, and we choose to do so for convenience.

Now let us examine the first α_b integration. Since the range of the α_b integration is infinitesimal, the only possible contribution to this term can arise when the integrand blows up for $\epsilon_i = 0$. This occurs only when all but one of the u's are equal. A detailed calculation for several N confirms this, but we know this must be the case in general since this region corresponds to the configuration where the loop is in one of the strings and far from the interaction region. Since we already have $\alpha_b = u_b/u_a \approx 0$, the only possibility for the u's in which all but one are equal is $u_i \approx 0$, $i \neq a$. Thus we can restrict the other α's to be less than some number ξ, where we can clearly choose $\eta \ll \xi \ll 1$. In fact, after a little thought, it is clear that, in addition to the above inequality, we can take ξ as small as we like, by simultaneously making η smaller if necessary. Doing this, we obtain

$$\pi p_a \cdot p_b \int_0^{\eta} d\alpha_b \int_{-\xi}^{0} d\alpha_1 \int_{-\xi}^{\alpha_1} d\alpha_2 \cdots \int_{-\xi}^{\alpha_{a-2}} d\alpha_{a-1} \int_{\alpha_b}^{\xi} d\alpha_{a+1} \cdots$$

$$\times \int_{\alpha_b}^{\alpha_{b-2}} d\alpha_{b-1} \int_0^{\alpha_b} d\alpha_{b+1} \cdots \int_0^{\alpha_{N-2}} d\alpha_{N-1} \prod_{\substack{i>j \\ i,j \neq a}} (\alpha_i - \alpha_j)^{-2p_i \cdot p_j} \tag{7a}$$

(Equation continued)

$$= \pi p_a \cdot p_b \int_0^{\eta} \frac{d\alpha_b}{\alpha_b^{1+\epsilon_a}} \int_{-\xi/\alpha_b}^{0} d\beta_1 \int_{-\xi/\alpha_b}^{\beta_1} d\beta_2 \cdots$$

$$\times \int_{-\xi/\alpha_b}^{\beta_{a-2}} d\beta_{a-1} \int_1^{\xi/\alpha_b} d\beta_{a+1} \cdots \int_1^{\beta_{b-2}} d\beta_{b-1} \int_0^1 d\beta_{b+1} \int_0^{\beta_{b+1}} d\beta_{b+2} \cdots$$

$$\times \int_0^{\beta_{N-2}} d\beta_{N-1} \prod_{\substack{i>j \\ i,j \neq a}} (\beta_i - \beta_j)^{-2p_i \cdot p_j} \tag{7b}$$

$$= \pi p_a \cdot p_b \int_0^{\eta} \frac{d\alpha_b}{\alpha_b^{1+\epsilon_a}} \int_{-\infty}^{-\alpha_b/\xi} d\gamma_1 \int_{\gamma_1}^{-\alpha_b/\xi} d\gamma_2 \cdots$$

$$\times \int_{\gamma_{a-2}}^{-\alpha_b/\xi} d\gamma_{a-1} \int_{\alpha_b/\xi}^{1} d\gamma_{a+1} \cdots \int_{\gamma_{b-2}}^{1} d\gamma_{b-1} \int_1^{\infty} d\gamma_{b+1} \int_{\gamma_{b+1}}^{\infty} d\gamma_{b+2} \cdots$$

$$\times \int_{\gamma_{N-2}}^{\infty} d\gamma_{N-1} \prod_{\substack{i>j \\ i,j \neq a,N}} (\gamma_j - \gamma_i)^{-2p_i \cdot p_j} \prod_{\ell \neq a,b,N} \gamma_\ell^{-(2p_\ell \cdot p_a + \epsilon_\ell)} \tag{7c}$$

(Equation continued)

$$= \pi p_a \cdot p_b \, \frac{\eta^{-\epsilon_a}}{-\epsilon_a} \int_{-\infty}^{0} d\gamma_1 \int_{\gamma_1}^{0} d\gamma_2 \cdots \int_{\gamma_{a-2}}^{0} d\gamma_{a-1} \int_{0}^{1} d\gamma_{a+1} \cdots$$

$$\times \int_{\gamma_{b-2}}^{1} d\gamma_{b-1} \int_{1}^{\infty} d\gamma_{b+1} \int_{\gamma_{b+1}}^{\infty} d\gamma_{b+2} \cdots \int_{\gamma_{N-2}}^{\infty} d\gamma_{N-1}$$

$$\times \widetilde{\prod_{\substack{i > j \\ i,\gamma \neq a,N}}} (\gamma_j - \gamma_i)^{-2p_i \cdot p_j} \widetilde{\prod_{\ell \neq a,b,N}} (\gamma_\ell - \gamma_a)^{-(2p_\ell \cdot p_a + \epsilon_\ell)} \; . \tag{7d}$$

In (7a) we have excluded those factors with i or j equal to a since $\alpha_a = 1$ and all other $\alpha_i \approx 0$. Equation (7b) is obtained by the substitution $\beta_i = \alpha_i/\alpha_b$ and (7c) by $\gamma_i = \beta_i^{-1}$, and where we have used the altered energy-momentum condition extensively. Equation (7d) follows only if the extra pieces added (by changing the limits of integration) contribute nothing to the integral. This will occur only if $s_{ja} < -2$, $j = 1, \cdots, a-1$ and $s_{aj} < -2$, $j = a+1, \cdots, b-1$ where $s_{\alpha\beta} = (p_\alpha + p_{\alpha+1} + \cdots + p_\beta)^2$.

It is quite tempting to identify (7d) immediately as a linear combination of derivatives of B_N with respect to $p_a \cdot p_\ell$, $\ell \neq a,b,N$. However, this is quite misleading, since the $p_i \cdot p_j$ are not all independent due to the N relations

$$p_i \cdot \left[\sum_{j=1}^{N} p_j + k \right] = 0 \; . \tag{8}$$

If we were to use these relations to eliminate N of the $p_i \cdot p_j$, then the products $p_a \cdot p_\ell$, $\ell \neq a,b,N$ would appear elsewhere, and our simple argument would break down.

In order to see that (7d) does involve the derivative of B_N, it will be convenient to change to the variables $s_{\alpha\beta}$ defined above. These have the advantage that they are all independent (we count $s_{1,\alpha-1} = s_{\alpha N}$ as one, etc.), unlike the $p_i \cdot p_j$ which are restricted by (8). We must however decide where we will put k in the definition of the $s_{\alpha\beta}$. That is, we could choose

$$s_{\alpha\beta} = (p_\alpha + p_{\alpha+1} + \cdots + p_\beta + k)^2 = (p_{\beta+1} + p_{\beta+2} + \cdots + p_{\alpha-1})^2 \; ,$$

or we could put the k in the last expression. We note that using the wrong $s_{\alpha\beta}$ in the divergent term leads to extra finite terms in the final result. Since the divergent part of the term we are dealing with here is proportional to $\frac{1}{\epsilon_a} B_N$, it arises from the configuration where the loop is in string a. Thus the arguments of the B_N should be the kinematic variables with p_a replaced by $p_a + k$. We therefore use the $s^a_{\alpha\beta}$ defined so that the k appears in that sum of momenta that contains p_a. Then it is not hard to show (working backwards) that

$$\widetilde{\prod_{i > j}} (\gamma_j - \gamma_i)^{-2p_i \cdot p_j} = \widetilde{\prod_{\ell}} (\gamma_\ell - \gamma_a)^{\epsilon_\ell} \left[\frac{(\gamma_{a+1} - \gamma_a)(\gamma_a - \gamma_{a-1})}{\gamma_{a+1} - \gamma_{a-1}} \right]^{\epsilon_a}$$

$$\times \widetilde{\prod_{\text{all } s^a_{ij}}} \left\{ \frac{(\gamma_j - \gamma_i)(\gamma_{j+1} - \gamma_{i+1})}{(\gamma_j - \gamma_{i-1})(\gamma_{j+1} - \gamma_i)} \right\}^{-s^a_{ij} - 2} \; . \tag{9}$$

If one of the γ's is infinite (here γ_N), then this formula still holds and all factors with that γ cancel. Also we can write the square bracket in (9) as

$$\left[\frac{(\gamma_{a+1}-\gamma_a)(\gamma_a-\gamma_{a-1})}{\gamma_{a+1}-\gamma_{a-1}}\right]^{\epsilon_a} = \prod_{\ell\leqslant a\leqslant m<b}\left[\frac{(\gamma_m-\gamma_\ell)(\gamma_{m+1}-\gamma_{\ell-1})}{(\gamma_m-\gamma_{\ell-1})(\gamma_{m+1}-\gamma_\ell)}\right]^{\epsilon_a} \tag{10}$$

where the above result depends crucially on the relations $\gamma_a = 0$, $\gamma_b = 1$, $\gamma_c = \infty$. Then using (9) and (10), (7d) becomes

$$\pi p_a\cdot p_b\,\frac{\eta^{-\epsilon_a}}{\epsilon_a}\int_{-\infty}^{0} d\gamma_1\int_{\gamma_1}^{0} d\gamma_2\cdots\int_{\gamma_{a-2}}^{0} d\gamma_{a-1}\int_{0}^{1} d\gamma_{a+1}\cdots$$

$$\times\int_{\gamma_{b-2}}^{1} d\gamma_{b-1}\int_{1}^{\infty} d\gamma_{b+1}\cdots\int_{\gamma_{N-2}}^{\infty} d\gamma_{N-1}$$

$$\times\prod_{\text{all } s^a_{ij}}\left[\frac{(\gamma_j-\gamma_i)(\gamma_{j+1}-\gamma_{i-1})}{(\gamma_j-\gamma_{i-1})(\gamma_{j+1}-\gamma_i)}\right]^{-s^a_{ij}-2}$$

$$\times\prod_{\ell\leqslant a\leqslant m<b}\left[\frac{(\gamma_m-\gamma_\ell)(\gamma_{m+1}-\gamma_{\ell-1})}{(\gamma_m-\gamma_{\ell-1})(\gamma_{m+1}-\gamma_\ell)}\right]^{\epsilon_a}$$

$$= \pi p_a\cdot p_b\left[\frac{1}{\epsilon_a}\,B_N(s^a_{ij}) + \sum_{\substack{\ell\leqslant a\leqslant m<b\\ \ell\neq m}}\frac{\partial B(s_{ij})}{\partial s_{\ell m}}\right] \tag{11}$$

in the limit $\epsilon_a \to 0$. Terms proportional to $\ell n\,\eta$ have been dropped,

since it is known that they cancel with terms in other α_b integrals (this is due to the fact that η is an arbitrary division point of an integral.). Notice that it is unimportant which s_{ij}'s we use as the argument of the derivatives of B_N in the limit $\epsilon_i \to 0$.

By a similar argument, the third and fourth α_b integrals each lead to an identical expression to (7d) except that ϵ_b replaces ϵ_a in the coefficient, and analogous steps lead to an expression similar to (11). If $u_a > 0$ we get, in addition to (11), one more term identical to (11). Although the calculation is slightly different, the result holds over if one of a or b equals N . However, in order to obtain (7d) for all a,b, we must have $s_{\alpha\beta} < -2$ for all planar channels. This poses no problem as the $s_{\alpha\beta}$ are all independent. Later the proof holds also in the physical region by analytic continuation.

Adding up all the contributions, we obtain for term I

$$2\pi\left[\sum_{a<b}\left(\frac{1}{\epsilon_a}+\frac{1}{\epsilon_b}\right)p_a\cdot p_b\,B_N(s^a_{ij}) + \sum_{\substack{a,b,\ell,m\\ \text{with } \ell\leqslant a\leqslant m<b\\ \text{or } a<\ell\leqslant b\leqslant m<N}} p_a\cdot p_b\,\frac{\partial B(s_{ij})}{\partial s_{\ell m}}\right]$$

$$\xrightarrow[\epsilon_a\to 0]{} -2\pi\left[-\left(N-\sum_a\frac{1}{2\epsilon_a}\right)B_N(s_{ij}) + \sum_{1\leqslant \ell<m<N} s_{\ell m}\,\frac{\partial B_N(s_{ij})}{\partial s_{\ell m}}\right]$$

$$= -2\pi\left[\kappa\,B_N(s_{ij}) + \frac{\partial}{\partial m}\,B_N(s_{ij})\right]$$

where κ is a constant, m is the slope, and we have used the relations

$$2p_a \cdot p_b = s_{ab} - s_{a,b-1} - s_{a+1,b} + s_{a+1,b-1}$$

(in which we must use the definitions $s_{aa} = -1$, $s_{a+1,a} = 0$). This is the desired result, and thus we have a universal renormalization of the slope of the Regge trajectories.

To complete the proof, we must show that terms II and III do not affect our result. Term II is actually a sum of terms with the term I integrand and the extra pieces

$$-\sum_r \frac{\epsilon_r}{2} \ln\left[\frac{(u_r - i)^2}{u_r^2 + 1}\right] .$$

Since the extra term is well-behaved throughout the range of integration and is first order in ϵ_i, we neglect all but the divergent part of the integral. If $r \neq a$ or b, then this always occurs for $u_r \approx 0$ and we can write

$$\frac{\epsilon_r}{2} \ln\left[\frac{(u_r - i)^2}{u_r^2 + 1}\right] = -\frac{i\pi\, \epsilon_r}{2}$$

and the result is a multiple of the divergent part for $d < 25$. For $r = a$ or b, we write

$$\ln \frac{(u_r - i)^2}{u_r^2 + 1} = -2i \tan^{-1} \frac{1}{u_r} .$$

Then the u_a integration is modified using

$$\int du_i \frac{\tan^{-1} \frac{1}{u_i}}{(u_i^2 + 1)(u_i^2\alpha_j^2 + 1)} = \begin{cases} \frac{1}{2}\left(\frac{\pi}{2}\right)^2 & \text{for } \alpha_j \to 0 \\ \frac{1}{\alpha_j}\left(\frac{\pi}{2}\right)^2 & \text{for } \alpha_j \to \infty \end{cases}$$

$$\int du_i \frac{\tan^{-1} \frac{1}{u_i\alpha_j}}{(u_i^2 + 1)(u_i^2\alpha_j^2 + 1)} = \begin{cases} \left(\frac{\pi}{2}\right)^2 & \text{for } \alpha_j \to 0 \\ \frac{1}{2\alpha_j}\left(\frac{\pi}{2}\right)^2 & \text{for } \alpha_j \to \infty . \end{cases}$$

Adding all the terms up, we find that term II is proportional to

$$\sum_{a>b} \frac{\pi}{2} p_a \cdot p_b \left\{ -i\pi\left(\frac{1}{\epsilon_a} + \frac{1}{\epsilon_b}\right) \sum_{r \neq a,b} \epsilon_r - 2i\left[\epsilon_a\left(\frac{\pi}{4}\frac{1}{\epsilon_a} + \frac{\pi}{2}\frac{1}{\epsilon_b}\right) + \epsilon_b\left(\frac{\pi}{2}\frac{1}{\epsilon_a} + \frac{\pi}{4}\frac{1}{\epsilon_b}\right)\right]\right\} B_N$$

$$= \frac{N\pi^2 i}{4} B_N .$$

As we remarked earlier, term III contains the explicit factor $(i^2 + 1)^{-1}$ which must be removed. This can be done by displacing the point of entry of the new momentum k to a fixed point in the string diagram infinitesimally close to the loop. Since we are dealing with the case where the loop has shrunk to a point, we can use the tree diagram transformation (2) to find the displacement of the point of entry of the loop in the u plane. This gives

$$\Delta u_c = \Delta \rho_c \left(\sum_r \frac{\alpha_r}{i - u_r} \right)^{-1}$$

which changes term III to read

$$\frac{1}{2i(\Delta \rho_c)} \sum_a \epsilon_a \int_0^0 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{N-2}} du_{N-1} \frac{(u_a - i)^2}{u_a^2 + 1}$$

$$\times \left(\sum_r \frac{\alpha_r}{i - u_r} \right) \prod_{1 \leqslant i < j \leqslant N} (u_i - u_j)^{-2p_i \cdot p_j}$$

where we have left out the terms with ϵ_r in the exponent, since we already have a factor ϵ_a.

Unfortunately this expression is still divergent. This divergence, already seen in the old renormalization calculation, occurs when the loop approaches the boundary of the string diagram, or in the region of integration where all the u's are equal. To remedy the situation we introduce another cutoff to eliminate this region of integration, and later take the limit as the cutoff goes away. We do this in the following way. Since the whole term has a coefficient linear in ϵ, the only contributions will come from the region where the integral diverges, i.e., the region where all but one (at least) of the u's, say u_k, are equal to some value u_A. Then we have

$$\sum_r \frac{\alpha_r}{i - u_r} = \frac{\alpha_k}{i - u_k} + \frac{1}{i - u_A} \sum_{r \neq k} \alpha_r = \alpha_k \left[\frac{1}{i - u_k} - \frac{1}{i - u_A} \right] .$$

So for the case where $a \neq N$ and $k \neq N$ or a, we can write term III in the above region as

$$\frac{\alpha_k}{2i(\Delta \rho_c)} \sum_{a \neq N} \epsilon_a \int_{\xi}^{-\xi} du_k \int_{-\eta}^{0} du_a \left[\frac{1}{i - u_k} - \frac{1}{i - u_a} \right] \frac{(u_a - i)^2}{u_a^2 + 1}$$

$$\times \int_{u_a}^{0} du_1 \int_{u_a}^{u_1} du_2 \cdots \int_{u_a}^{u_{a-2}} du_{a-1} \int_{u_k}^{u_a} du_{a+1} \cdots$$

$$\times \int_{u_k}^{u_{k-2}} du_{k-1} \int_0^{u_k} du_{k+1} \cdots \int_0^{u_{N-2}} du_{N-1} \prod_{1 \leqslant i < j \leqslant N} (u_i - u_j)^{-2p_i \cdot p_j} \tag{12}$$

where $\eta \ll \epsilon$. The u_k integral has been restricted so that $|u_k| > \xi$. Since all the other u's are near zero $(= u_N)$, this has the effect of eliminating the region where all the u's are equal. The u_a integral is restricted by $u_a > \eta$ (this should read $u_a < \eta$ if $a < k$) so that we exclude the region $u_k \sim u_a$. The remaining u's actually have been left unrestricted since there will be no contribution anyway unless they are all near zero. Finally, u_A has been set equal to u_a, which is permissible since all u_i, $i \neq k$ are equal. It is clear then from (12) that our result is just a constant multiple of B_N. The terms $a = N$ and $k = N$ or a, although somewhat different, are similar and give the same result. Also we can easily convince ourselves that changing u_c by an infinitesimal amount cannot change our result for term II (since it is finite). Thus we conclude that both terms II and III simply add to the multiplicative renormalization and do not affect the slope renormalization.

We should point out that we have been using the fact that the s_{ij} are all independent. If the number of particles is greater than 26, the number of dimensions, then this is not strictly true. However, we note that throughout the derivation of the interacting string amplitude, no use was made of the number of dimensions. Thus we would have written down exactly the same expression no matter how many dimensions we were working in. We therefore calculate always in more dimensions than the number of particles we are dealing with, and are confident that the result will be valid in fewer dimensions.

We have now shown that the single loop amplitude for N scalar particles is a slope renormalization. By factorization, we trivially obtain the same result for N excited particles.

Acknowledgments

With great pleasure I acknowledge the patience, assistance, and guidance of Professor S. Mandelstam throughout the preparation of this work. I would also like to thank him and Hessamaddin Arfaei for carefully reading over the manuscript and suggesting useful additions and corrections. Finally, I would like to thank Warren Siegel for help with a tricky summation.

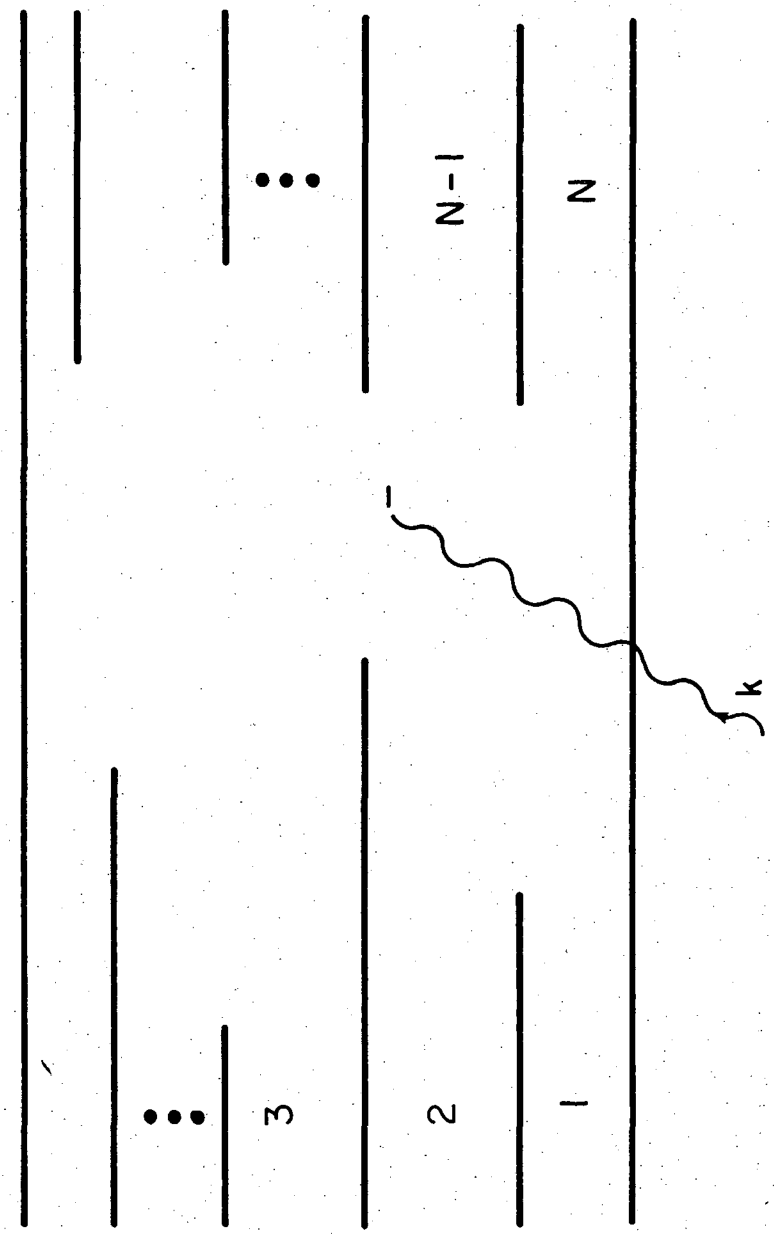
Footnotes and References

* This work was supported by the Energy Research and Development Agency (ERDA).

1. A. Neveu and J. Scherk, Phys. Rev. D1, 2355 (1970).
2. For a more complete discussion of this, see S. Mandelstam, Physics Reports 13C, 259 (1974).

Figure Captions

Fig. 1. N-point function with new momentum k entering.



XBL757-3344

Fig. 1

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720