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Immersed Virtual Element Methods for Elliptic Interface Equations

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Virtual Element Method for Elliptic Interface Problem

DISSERTATION

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by

Frank M. Lin

Dissertation Committee:
Professor Long Chen, Chair
Professor Song-Ying Li
Professor Jack Xin

2021

DEDICATION

I dedicate this thesis to my parents for always encouraging me to pursue knowledge and supporting me in every aspect of life.

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VITA

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ABSTRACT OF THE DISSERTATION

Virtual Element Method for Elliptic Interface Problem

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2021

Professor Long Chen, Chair

We developed a simplified formulation of the existing immersed finite element (IFE) methods for solving the second order elliptic interface problems. A virtual body-fitted mesh is generated using the intersection points of the interface and the underlying shape regular triangulation. We define an immersed virtual element local space. By deriving an error equation and performing convergence analysis base on it, we not only create a more concise formulation and convergence proof of partially penalized IFE method, but also brings a connection between various methods such as body-fitted FEM, IFEM, VEM, etc. In addition, our approach has the advantage that it is easier to generalize into three dimension case.

Chapter 1

Finite Element Methods Review

This will be a short chapter, on reviewing the basics of conforming linear finite element method on Poisson's equation. We aim to set the foundation here, as all the methods in later chapters can be viewed as generalization or extension of it.

Finite element methods are based on the variational/weak formulation of the partial differential equation. We break the given domain into union of small triangles(the mesh), define the finite dimensional discrete function space on it. On the discrete space we then solve the approximated (finite dimensional linear algebra) problem, and proceed to prove the error of the approximated solution converges in desired way.

1.1 Equation and Discrete Problem

The model problem is the Poisson equation with Dirchlet boundary condition.

$$-\Delta u = f \text{ on } \Omega$$

and

$$u = 0 \text{ on } \partial\Omega$$

Through integration by parts, the variational formulation is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$ and $(f, v) = \int_\Omega f v dx$.

Under the assumption that $f \in L^2(\Omega)$, there is a well known regularity result, that the solution $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{1.1}$$

The domain Ω is cut into union of simplex (Triangle or Tetrahedron, for mesh generating algorithm, see for example [16]), called mesh. Then we define the discrete, finite dimensional function space using the mesh, and find an approximated solution. We say a triangulation $\mathcal{T}_h$ satisfies the minimum/maximum angle condition if minimum/maximum angle of each triangle in $\mathcal{T}_h$ is uniformly bounded below/above. We say a triangulation is shape-regular if it satisfies both minimum/maximum angle conditions.

Given a shape regular triangulation $\mathcal{T}_h$, the discrete linear finite element space is defined to be $V_h := \{v | v \in C(\overline{\Omega}), v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}_h\}$. Then we solve the discrete problem, find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \; \forall v_h \in V_h$$

.

For each triangle τ, the local stiffness matrix A_τ (matrix form by $\int_\tau \nabla\phi_i \cdot \nabla\phi_j dx$ where ϕ_i) can be computed exactly, and is given below,

Theorem 1.1.1 (Local stiffness matrix). *Let θ_i denotes the interior angles of τ,*

$$A_\tau = \frac{1}{2}\begin{bmatrix} \cot(\theta_2)+\cot(\theta_3) & -\cot(\theta_3) & -\cot(\theta_2) \\ -\cot(\theta_3) & \cot(\theta_1)+\cot(\theta_3) & -\cot(\theta_1) \\ -\cot(\theta_2) & -\cot(\theta_1) & \cot(\theta_1)+\cot(\theta_2) \end{bmatrix}$$

1.2 FEM error estimate

To estimate u_h, we first establish the projection and interpolation estimates, then we use an orthogonality argument to show that u_h converges in the same way.

Theorem 1.2.1 (Bramble Hilbert Lemma (Special Case)[11]). *For $u \in H^2(K)$, where $K \subset \mathbb{R}^2$ is a bounded star-shaped domain with diameter h_K, there exists a linear polynomial p, such that,*

$$\|u-p\|_{0,\Omega} + h_K|u-p|_{1,\Omega} \lesssim h_K^2|u|_{2,\Omega}$$

where the constant is independent of h_K but depends on the star-shaped constant.

For $u \in H^1$, we define the nodal interpolation u_I by $u_I \in V_h$, u_I and u match values on verticies of all triangles $\tau \in \mathcal{T}_h$

For the nodal interpolation u_I we have the following estimate,

Theorem 1.2.2 (Interpolation Estimate[11]). *For $u \in H^2(\Omega)$, $\Omega \subset \mathbb{R}^2$, V_h linear finite element space on shape-regular quasi-uniform triangulations $\mathcal{T}_h$, we have*

$$|u-u_I|_{1,\Omega} \lesssim h|u|_{2,\Omega}$$

Then we have the following H^1 estimate of the discrete solution

Theorem 1.2.3 (Interpolation Estimate)**.** *For u and u_h being the solutions of the continuous and discrete equations, when $u \in H^2(\Omega) \cap H_0^1(\Omega)$, we have the following estimate.*

$$|u - u_h|_{1,\Omega} \lesssim h\|u\|_{2,\Omega}$$

Proof. From the continuous and discrete equation, we have

$$a(u, v) = (f, v), \forall v \in H_0^1(\Omega)$$

and

$$a(u_h, v) = (f, v), \forall v \in V_h$$

Therefore for all $v \in V_h$ we have the orthogonality $a(u - u_h, v_h) = 0$, which implies

$$\|\nabla(u - u_h)\| = \inf_{v_h \in V_h} \|\nabla(u - v_h)\|$$

The equation above combines with Theorem 1.2.3 gives the desire result. □

Chapter 2

Virtual Element Methods Review

In this chapter we review the frameworks of the virtual element methods, and some lemmas needed for the convergence analysis.

2.1 Introduction and model problem

Virtual element method (VEM) [6] can be viewed as an extension of the finite element method (FEM) on triangle and tetrahedral elements to general polygonal and polyhedral elements. Since some local basis functions have no closed form formulas, not every quantities can be computed by degrees of freedom (see more detail in section 3.1). The degrees of freedom in the VEM space are defined in such a way that many integral quantities or matrices (e.g. stiffness matrix) can be approximated from them without computing non-polynomial basis functions.

The aim of this and next chapter is to present the optimal order of error estimates of VEM with relaxed geometric assumptions on the three dimensional mesh. Consider the following weak formulation for a model Poisson equation with zero Dirichlet boundary condition in a

3-dimensional Lipschitz domain Ω: given an $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

and $(\cdot, \cdot)$ is the inner product on $L^2(\Omega)$.

We will define the VEM spaces in section 3.1. Roughly speaking, the two dimensional local VEM space of order $k \in \mathbb{N}$ is defined by the space of functions that are continuous piecewise polynomial of degree at most k on the boundary, and inside the domain the Laplacian of the function is a degree at most $k-2$ polynomial. For three dimensional local VEM space, inside the domain the Laplacian of the function is still a degree at most $k-2$ polynomial, while on each face on the boundary the function is in a modified VEM space so that the L^2 projection is computable from degrees of freedom. The basis functions of the local VEM space are well-defined, but their values or gradient are not explicitly computed. Each local bilinear form contains an orthogonal H^1-projection term that can be computed exactly from the degrees of freedom. The global conforming VEM space of degree k, denoted as V_h (See definition 3.1.9), glues the local spaces together using continuity condition across inter-element faces. i.e.,

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \Big[\big(\nabla \Pi_K u, \nabla \Pi_K v\big)_K + S_K(u, v) \Big], \tag{2.1}$$

where Π_K is the H^1-projection operator (see Definition 3.1.3) to the space of degree k polynomials on K, and $S_K(\cdot, \cdot)$ is a stabilization term to ensure the coercivity.

In the traditional norm equivalence VEM approaches, the stabilization term is defined to satisfy k-consistency,

$$a(u,p) = a_h(u,p) \text{ for } p \in \mathbb{P}_k(K) \ \forall K \in \mathcal{T}_h$$

and the following norm equivalence between the exact form on H_0^1 and the approximated form on V_h(See definition 3.1.9),

$$a(u,u) \lesssim a_h(u,u) \lesssim a(u,u), \quad \text{for} \ \ u \in V_h \tag{2.2}$$

in which both constants in the inequalities are independent of u. With this property the finite dimensional approximation to the weak formulation using the VEM discretization (3.10)

is well-posed, and the H^1 seminorm error estimate is optimal. One possible choice of the local stabilization on K is given in [6]: for u, v in the VEM space

$$S_K^{\text{orig}}(u,v) = \sum_{r=1}^{N_K} \chi_r(u - \Pi_K u)\chi_r(v - \Pi_K v) \tag{2.3}$$

where N_K is the number of degrees of freedom (see Definition 3.1.2) on K, χ_r is each individual degree of freedom.

Under certain geometric assumptions of the polytopal mesh, the aforementioned norm equivalence (2.2) is established with a proper choice of the stabilization, and the optimal order error estimates can be achieved (see [6]). Typical geometric assumptions include that (1) the mesh is star-shaped with chunkiness parameter [12] uniformly bounded below, and (2) the distance between neighboring vertices are comparable to the diameter of the element.

However, it has been observed in numerical experiments that the optimal convergence rates for the virtual element methods can be achieved with relatively little geometric assumptions

[16]. In [8], different choices of stabilization terms are analyzed in detail, and further relaxed the geometric assumptions for two dimensional mesh by including short edges. Recently, in [12], it is shown that one can allow small faces on a three dimensional mesh and still achieve the optimal order with only uniform star shape assumption. However, several error estimates still require faces to be uniform star shape and the error estimate depends on the logarithm of the longest to the shortest edge ratio of the faces.

We shall use a different approach, which was first proposed in [14] to handle the 2D cut mesh, to relax the geometric assumptions further on three dimensional meshes and still achieve the optimal order.

Instead of working on a stronger H^1-seminorm, the error analysis is performed toward a weaker "energy norm" $|\!|\!|\cdot|\!|\!| := a_h^{1/2}(\cdot,\cdot)$. Similar to that of the Discontinuous Galerkin (DG)-type methods, an error equation for $|\!|\!|u_h - u_I|\!|\!|$, is derived. This error equation breaks down the $|\!|\!|u_h - u_I|\!|\!|$ into several standard projection and interpolation error estimates. Our method does not rely on the norm equivalence property of the stabilization term.

Instead, different from the above so-called identity matrix stabilization (2.3) above, the stabilization term is concocted from the "boundary term" emerged from the integration by parts (see section 3.2 for detail), while equipped with correct weights to remain the optimal order for the error estimates.

The following new stabilization term is proposed in this chapter, which partly agrees with the conjecture in [12] and is "singularly conforming" in the sense that the term which keeps the conformity of the method may have a small constant in front it.

$$S_K(u,v) = h_K^{-1} \sum_{F\subset\partial K} \Big[\big(Q_K u - Q_F u, Q_K v - Q_F v\big)_F + \epsilon_F h_F \sum_{e\subset\partial F} \big(u - Q_F u, v - Q_F v\big)_e\Big], \tag{2.4}$$

where Q_K, Q_F are L^2-projection operators (see Definition 3.1.4) to the local space of degree k polynomials on K respectively. The ϵ_F is related to the chunkiness parameter of ρ_F (see Definition 2.2.1) of each polygonal face F on the boundary of a polyhedral element K.

This new approach, comparing to the existing approaches, allows us to deal with the mesh that has less constraints on the shape regularity. For example, the chunkiness parameter ρ_F of each face F on an element K may no longer be uniformly bounded below. In addition, the constants in the new estimates do not depend on logarithm of the longest and the shortest edge of each face. As a result, we obtain the optimal order error estimate on a weaker energy norm (3.3.5) with a set of relaxed geometric assumptions 4.2.1 that are introduced in section 3.3.

A final remark before we go to the next section, for each inequality with constant we will put extra emphasis on whether the hidden constant depends on the chunkiness parameter of the domain or not. In the next chapter, our main result, the error equation will be derived, and then based on this error equation, the optimal order of the a priori error estimate under appropriate geometric assumptions can be achieved.

2.2 Approximation Theory on Star-Shaped or Convex Domain

In this section, we shall review some existing results on VEM projection (2.2.6) and interpolation error estimates (2.2.8).

Definition 2.2.1 (Star-shaped polytope)**.** *Let D be a simple polygon or polyhedron. We said D is star-shaped with respect to a disc/ball B if for every point $y \in D$, the convex hull of $\{y\} \cup B$ is contained in D. If D is star-shaped with respect to a disc/ball with radius ρh_D. We define the supremum of ρ to be chunkiness parameter ρ_D.*

Lemma 2.2.2 (Bramble-Hilbert estimates on star-shaped domain [10])**.** *Let D be a star-shaped domain, then we have the following estimates,*

$$\inf_{q \in \mathbb{P}_l} |u - q|_{H^m(D)} \leq C(\rho_D) h_D^{l+1-m} |u|_{H^{l+1}(D)}, \ \forall u \in H^{l+1}(D), l = 0, 1, ..., k, m \leq l \tag{2.5}$$

where $C(\rho_D)$ is inverse proportional to ρ_D.

As $\rho_D \to 0$, the above estimate will be less suitable to apply (e.g. to a face with small ρ_D). In the following version of Bramble-Hilbert estimates the constant is independent of ρ_D.

Lemma 2.2.3 (Bramble-Hilbert estimates on convex domains [20])**.** *Let D be a convex domain, there exists a constant C independent of ρ_D,*

$$\inf_{q \in \mathbb{P}_l} |u - q|_{H^m(D)} \leq C h_D^{l+1-m} |u|_{H^{l+1}(D)}, \ \forall u \in H^{l+1}(D), l = 0, 1, ..., k, m \leq l \tag{2.6}$$

The following scaled trace inequalities are often used when we need to bound norm on boundary faces by norm on elements.

Lemma 2.2.4 (Trace inequalities on star-shaped domain [12]). *Let D be a star-shaped domain, then*

$$\|u\|^2_{L^2(\partial D)} \lesssim h_D^{-1}\|u\|^2_{L^2(D)} + h_D|u|^2_{H^1(D)}, \forall u \in H^1(D) \tag{2.7}$$

Let F be a face of $D \subset \mathbb{R}^3$, we have

$$h_D|u|^2_{H^1(F)} \lesssim |u|^2_{H^1(D)} + h_D^2|u|^2_{H^2(D)}, \forall u \in H^2(D) \tag{2.8}$$

where in both cases, the constant in $\lesssim$ is inverse proportional to ρ_D.

In the following Poincare inequality, the constant in estimation can be written explicitly in term of only the diameter, if the domain is convex.

Theorem 2.2.5 (Poincaré inequality on convex domain [4]). *Let $D \subset \mathbb{R}^n$ be a convex domain with diameter h_D. Then*

$$\|u\|_{L^2(D)} \leq \frac{h_D}{\pi}\|\nabla u\|_{L^2(D)}$$

for all $u \in H^1(D)$ satisfying

$$\int_D u(x)dx = 0$$

By combining the Bramble-Hilbert Estimates, and the stability of projection operators (Q_D and Π_D) [12] in L^2, H^1, and H^2 norm, we can obtain the following projection error estimates.

Theorem 2.2.6 (Projection Error Estimate). *Let D be a star-shaped domain. Let Π be Π_D or Q_D then for $m, l, k \in \mathbb{N}$, $0 \leq m \leq 2$, $\min(1, m) \leq l \leq k$, $u \in H^{l+1}(D)$ we have*

$$\|u - \Pi u\|_{m,D} \leq [C(\rho_D)h_D]^{l+1-m}|u|_{l+1,D} \tag{2.9}$$

where $C(\rho_D)$ is inverse proportional to ρ_D.

Theorem 2.2.7 (Projection Error Estimate on Convex Domains). *Let D be a convex domain, then there exists a constant C independent of ρ_D such that*

$$\|u - Q_D u\|_{0,D} \leq C h_D^{l+1} |u|_{l+1,D}, \tag{2.10}$$

and

$$|u - \Pi_D u|_{1,D} \leq C h_D^{l} |u|_{l+1,D}. \tag{2.11}$$

The optimal order of interpolation operators are much harder to prove. Brenner and Sung [12] construct an auxiliary semi-norm to prove the following interpolation estimates. We will list the result here and refer the reference for the detail (Although the estimate of $|u - Q_D u_I|_{1,D}$ is not explicitly given in [12], the derivation follows from H^1 stability of Q_D and almost identical procedure of deriving the estimate of $|u - \Pi_D u_I|_{1,D}$). The interpolation estimates for three dimensional element require the uniform star-shaped condition.

Theorem 2.2.8 (Interpolation Error Estimate). *Let D be a star-shaped domain. Let u_I be the nodal interpolation of the function on the local VEM space defined in 3.1.10. We have, for $1 \leq l \leq k$, $\forall u \in H^{l+1}(D)$*

$$|u - u_I|_{1,D} + |u - \Pi_D u_I|_{1,D} + |u - Q_D u_I|_{1,D} \lesssim [C(\rho_D) h_D]^l |u|_{l+1,D} \tag{2.12}$$

$$|u - \Pi_D u_I|_{2,D} \lesssim [C(\rho_D) h_D]^{l-1} |u|_{l+1,D} \tag{2.13}$$

$$\|u - u_I\|_{0,D} + \|u - Q_D u_I\|_{0,D} + \|u - \Pi_D u_I\|_{0,D} \lesssim [C(\rho_D) h_D]^{l+1} |u|_{l+1,D} \qquad (2.14)$$

The constants $C(\rho_D)$ *is inverse proportional to* ρ_D*.*

Chapter 3

A Conforming VEM for Poisson Equation in Three Dimensions

In this chapter, we go into detail of the virtual element space apply virtual element method to solve 3D Poisson's equation.

3.1 VEM Preliminaries

In this section, we introduce the definition of the VEM space, modified spaces and the corresponding degrees of freedom. The motivation behind the modified space is to make L^2 projection of the VEM function computable from degrees of freedom.

In this section, we will use standard notations for differential operators, function spaces and norms that can be found, for examples in [1].

The domain Ω is partitioned into a three dimensional mesh $\mathcal{T}_h$, and for simplicity Ω is assumed to have a polygonal boundary so that there is no geometric error of $\mathcal{T}_h$ on $\partial\Omega$. Let

K be a simple polyhedral element in $\mathcal{T}_h$. F denotes a face of the element, and e denotes an edge of a face. D denotes a general domain in two or three dimensions, and h_D is the diameter of D. $e \subset \partial F$ or $F \subset \partial K$ are used to denote the edge or face is one of the edges or faces on the boundary of F or K.

3.1.1 VEM spaces

To define the three dimensional VEM space, first we need to define the two dimensional local VEM space V_k ([5]) and the modified space W_k ([2]). Notice when defining the local VEM space on a face, the surface Laplacian operator Δ_F on a face F shall be used. Let $k \in \mathbb{N}$ and let $\mathbb{P}_k(D)$ be the space of polynomial functions with degree up to k (where $\mathbb{P}_{-1}$ contains only zero polynomial) on D.

Definition 3.1.1 (Local two dimensional VEM space on a face F)**.**

$$V_k(F) := \big\{v \in H^1(F) : \Delta_F v \in \mathbb{P}_{k-2}(F), v|_{\partial F} \in B_k(\partial F)\big\}, \tag{3.1}$$

where

$$B_k(\partial F) := \{v \in C^0(\partial F) : v|_e \in \mathbb{P}_k(e) \textit{ for all } e \subset \partial F\}. \tag{3.2}$$

The degrees of freedom of the space in Definition 3.1.1 can be defined using the scaled monomials.

Let D be a two dimensional simple polygon or three dimensional simple polygonal domain, and (x_c, y_c, z_c) be the center of mass of D. Then the scaled monomials are polynomials of the form $m_\alpha = (\frac{x-x_c}{h_D})^{\alpha_1}(\frac{y-y_c}{h_D})^{\alpha_2}(\frac{z-z_c}{h_D})^{\alpha_3}$ where α_1, α_2, α_3 are non-negative integers. We define the degree of the polynomial to be $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ (or without z and α_3 if in two dimension).

Definition 3.1.2 (Degrees of freedom)**.** *The degrees of freedom of v_h in $V_k(F)$ are defined as follows:*

1. *The value of v_h at the vertices of F.*
2. *The moments up to order $k-2$ of v_h in each edge e. That is, $\frac{1}{|e|}\int_e v_h m_\alpha$ where m_α is a scaled monomial for $\alpha \leq k-2$.*
3. *the moments up to order $k-2$ of v_h in F. That is, $\frac{1}{|F|}\int_F v_h m_\alpha$ where m_α is a scaled monomial for $\alpha \leq k-2$.*

With the degrees of freedom above, the following projection operator Π_K in the gradient inner product can be defined.

Definition 3.1.3 (Gradient orthogonal projection operator)**.** $\Pi_D^k : H^1(D) \to \mathbb{P}_k(D)$*,* $v \mapsto \Pi_D^k v$ *satisfies*

$$\left(\nabla(\Pi_D^k v - v), \nabla p\right)_D = 0, \quad \forall p \in \mathbb{P}_k(D).$$

where the constant kernel is determined by the following constraint:

$$\int_D (\Pi_D^k v - v) = 0, \quad k \geq 2, \tag{3.3}$$

or

$$\int_{\partial D} (\Pi_D^k v - v) = 0, \quad k = 1. \tag{3.4}$$

On a polygonal domain D, to compute the gradient projection of $v_h \in V_k(D)$ to $\mathbb{P}_k(D)$, it is sufficient to compute $(\nabla v_h, \nabla q)_D$ for all $q \in \mathbb{P}_k(D)$. Integration by parts

$$(\nabla v_h, \nabla q)_D = -(v_h, \Delta q)_D + (v_h, \nabla q \cdot n)_{\partial D}, \tag{3.5}$$

then the first term of the right hand side can be computed via internal moments of v_h in D (See Definition 3.1.2), and the second term can be computed because it's a polynomial integral (See Definition 3.1.1).

However, for a three dimensional polyhedron D, a naive generalization of the degrees of freedom for the local space $V_k(D)$ mimicking what of the polygonal version in Definition 3.1.2 is not sensible. In the three dimensional case, part of the second term $(v_h, \nabla q \cdot n)_F$ in equation (3.5) is a surface moment integral on F that is not computable if F is not triangular. The reason is that only the moments of v_h on a face $F \subset \partial D$ up to degree $k-2$ are given as degrees of freedom (See definition 3.1.2), yet for $q \in \mathbb{P}_k(D)$, $\nabla q \cdot n|_F \in \mathbb{P}_{k-1}(F)$. To compute this, we need to be able to compute the L^2-projection onto $\mathbb{P}_{k-1}(F)$ for a VEM function v_h. To this end, modified face spaces such as $W_k(F)$ or $\tilde{W}_k(F)$ are to be introduced.

Definition 3.1.4 (L^2 orthogonal projection operator)**.** $Q_D^k : L^2(D) \to \mathbb{P}_k(D)$, $v \mapsto Q_D^k v$ *satisfies*

$$\left(Q_D^k v - v, q\right)_D = 0, \quad \forall q \in \mathbb{P}_k(D).$$

When D is a polygonal face on the boundary of a polyhedron K, the above L^2 projection is not computable through the internal moment degrees of freedom for $V_k(F)$ in Definition 3.1.2, in that the moments $(v_h, q)_D$ for polynomial q being degree k or $k-1$ are unknown.

However the space $V_k(F)$ defined above can be enriched in a certain way ([2, 12], see definition 3.1.5 and 3.1.8) such that the L^2-projection is computable from the same degrees of freedom. These are the motivations behind defining the modified space such as $W_k(F)$ and $\tilde{W}_k(F)$, instead of using a direct generalization from $V_k(F)$ to $V_k(D)$ for a polyhedron D.

When the order of the projection operators are omitted, we assume it is k, the same as the order of the VEM space.

Definition 3.1.5 (Local modified VEM space)**.** *Let $\widetilde{\mathbb{P}}_k(e)$ be the space of degree exactly k monomials, then the local modified VEM space can be defined as:*

$$W_k(F) := \big\{v \in H^1(F) : \ \Delta_F v \in \mathbb{P}_k(F), v|_{\partial F} \in B_k(\partial F), \\ (v, q)_F = (\Pi_F^k v, q)_F, \ \forall q \in \widetilde{\mathbb{P}}_k(F) \cup \widetilde{\mathbb{P}}_{k-1}(F)\big\}. \tag{3.6}$$

Note that W_k and V_k share the same degrees of freedom, but the L^2 projection of a function in W_k is now computable. In W_k we can replace $(v_h, q)_K$ by $(\Pi_K^k v_h, q)_K$ for q being degree k or $k-1$ and the later integral is computable (just polynomial integral).

The three dimensional local VEM space can be defined as follows:

Definition 3.1.6 (Local three dimensional VEM space on an element K)**.**

$$V_k(K) := \big\{v \in H^1(K) : \Delta v \in \mathbb{P}_{k-2}(K), v|_{\partial K} \in B_k(\partial K)\big\}, \tag{3.7}$$

where $B_k(\partial K) := \big\{v \in C^0(\partial K) : v|_F \in W_k(F), v|_e \in \mathbb{P}_k(e)\big\}$.

Any function in $V_k(K)$ can be uniquely determined by its degrees of freedom ([5]) defined in the following paragraph.

Definition 3.1.7 (Degrees of freedom of three dimensional VEM space)**.** *We can take the following degrees of freedom of v_h in $V_k(K)$, where K is a three dimensional element.*

1. *The value of v_h at the vertices of K*

2. *The moments on each edge e up to degree $k-2$. That is, $\frac{1}{|e|}\int_e v_h m_\alpha$ where m_α is the scaled monomials with $\alpha \leq k-2$.*

3. *The moments on each face F up to degree $k-2$. That is, $\frac{1}{|F|}\int_F v_h m_\alpha$ where m_α is the scaled monomials with $\alpha \leq k-2$.*

4. *The moments on the element K up to degree $k-2$. That is, $\frac{1}{|K|}\int_K v_h m_\alpha$ where m_α is the scaled monomials with $\alpha \leq k-2$.*

An alternative definition of the modified VEM space [12], that allows us to compute both H^1 and L^2 projection from degrees of freedom is the following. We denote such a space $\tilde{W}_k(D)$, where D can be a polyhedron domain in any dimension. For convenience we shall define $\tilde{W}_k(e) = \mathbb{P}_k(e)$ for e being 1 dimensional edge and higher dimension spaces are defined recursively.

Definition 3.1.8 (The modified local $\tilde{W}_k$ space)**.** *Let D be a two or three dimensional polygon or polygonal domain, define the space $\tilde{W}_k(D)$ by $v_h \in \tilde{W}_k(D)$ satisfies,*

1. *v_h is continuous on ∂D.*

2. *v_h restrict to each $F \subset \partial D$ (face of polyhedron, edge of polygon), is a function in $\tilde{W}_k(F)$*

3. *Δv_h is a polynomial of degree k in K.*

4. *$\Pi^k_D v_h - Q^k_D v_h$ is a polynomial of degree at most $k-2$.*

When computing L^2 projection in $\tilde{W}_k$, we first write $Q^k_K v_h = \Pi^k_K v_h + w$, $w \in \mathbb{P}_{k-2}$, and the corresponding integrals can be computed using internal degree up to $k-2$.

We shall use the following $\tilde{W}_h$ for the global VEM space for the rest of the chapter.

Definition 3.1.9 (The global spaces). *For a given mesh size h. We define V_h, W_h, $\tilde{W}_h$ the global VEM space. That is, u is in the corresponding local VEM space V_k, W_k, $\tilde{W}_k$ for each element and is continuous on the boundary between elements.*

We shall only use the $\tilde{W}_h$ for the global VEM space for the rest of the chapter, so that L^2 projection is computable for any three dimensional element.

We then have the following natural definition of the nodal interpolation.

Definition 3.1.10 (The local nodal interpolation of the function). *Let K be an element and $u \in H^1(K)$, then we define u_I as a function in $\tilde{W}_k(K)$ that has the same degrees of freedom as u.*

Definition 3.1.11 (The global nodal interpolation of the function). *Let $u \in H^1(\Omega)$, then we define u_I as a function in $\tilde{W}_h$ that has the same degrees of freedom as u.*

We use the same notation u_I for both local and global interpolation, but under the proper context it should not be confused.

The following choice of stabilization term is motivated by the error equation that will be derived in the next chapter.

Definition 3.1.12 (Stabilization term and discrete bilinear form). *On an element K, the stabilization term is defined as follows:*

$$\begin{aligned} S_K(u,v) = h_K^{-1} \sum_{F \subset \partial K} \Bigg[&(Q_K u - Q_F u, Q_K v - Q_F v)_F \\ &+ \epsilon_F h_F \sum_{e \subset \partial F} ((u - Q_F u), (v - Q_F v))_e \Bigg], \end{aligned} \tag{3.8}$$

where $\epsilon_F \propto \rho_F^{-1}$ is a mesh-dependent parameter (where ρ_F is the chunkiness parameter of

F), and the discrete bilinear form is given by

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \Big[\big(\nabla\Pi_K u, \nabla\Pi_K v\big)_K + S_K(u,v) \Big]. \tag{3.9}$$

Then the VEM approximation problem is: to seek $u_h \in \tilde{W}_h$, where $\tilde{W}_h$ is the virtual element space (see definition 3.1.9)

$$a_h(u_h,v_h) = \sum_{K\in\mathcal{T}_h} (f, Q_K v_h) \quad \forall v_h \in \tilde{W}_h. \tag{3.10}$$

3.2 A priori error estimate

In this section, we first derive identities on the discrete bilinear form. A priori error estimate (3.2.4) then can be derived from the error equation (3.12). After that the optimal order of error estimate can be derived from estimating each term in the error equation.

Recall that on an element K, the bilinear form and the stabilization term are defined in equations (3.9) and (3.8) and the VEM approximation problem is (3.10).

The bilinear form (3.9) can be used to induce a seminorm $|||u||| = a_h^{1/2}(u,u)$, and the following lemma verifies that it is a norm on the VEM space with the boundary condition imposed.

Lemma 3.2.1. $|||\cdot|||$ *is a norm on* $V_h \cap H_0^1(\Omega)$*.*

Proof. It suffices to verify that if $|||v||| = 0$, then $v \equiv 0$. By definition, when $a_h(v,v) = 0$, we have $\Pi_K v_h = 0$ on each K, and $Q_K v_h = Q_F v_h$ on each $F \subset \partial K$.

By the boundary condition of the space, $Q_F v_h = 0$ for F on $\partial\Omega$. Because $Q_K v_h = Q_F v_h$ on each $F \subset \partial K$, that makes $Q_K v_h = 0$ for K contains at least a boundary face. For the same reasons, $Q_K v_h = 0$ for K that shares a face with K', an element that contains at least a boundary face. Repeat this argument we have $Q_K v_h = Q_F v_h = 0$ for each K.

In addition, on each face $v_h = Q_F v_h = 0$. That makes the degrees of freedom of v_h on each K equal 0, that implies $v_h = 0$ by unisolvence of the VEM space [5]

, which completes the proof. □

Lemma 3.2.2 (The approximated bilinear form)**.** *For u that is the solution to* (4.3)*, u_h that is the solution to* (3.10)*, and any $v \in V_h$, the following identity holds,*

$$a_h(u_h, v) = \sum_{K\in\mathcal{T}_h} (\nabla\Pi_K u, \nabla\Pi_K v)_K + \sum_{K\in\mathcal{T}_h} \langle \nabla(u - \Pi_K u)\cdot n, Q_K v - Q_F v\rangle_{\partial K} \tag{3.11}$$

Proof. First we apply the integration by parts to $u, v, \Pi_K u, Q_K v$, and use the definitions of H^1-projection Π_K and L^2-projection Q_K to get

$$\begin{aligned} -(\Delta u, Q_K v)_K &= (\nabla\Pi_K u, \nabla Q_K v)_K + \langle \nabla u\cdot n, Q_K v\rangle_{\partial K}, \\ (\Delta\Pi_K u, Q_K v)_K &= -(\nabla\Pi_K u, \nabla Q_K v)_K - \langle \nabla\Pi_K u\cdot n, Q_K v\rangle_{\partial K}, \\ \text{and} \quad -(\Delta\Pi_K u, v)_K &= (\nabla\Pi_K u, \nabla v)_K + \langle \nabla\Pi_K u\cdot n, Q_F v\rangle_{\partial K}. \end{aligned}$$

Adding above equations together and notice that $(\Delta\Pi_K u, Q_K v)_K = (\Delta\Pi_K u, v)_K$. By the definition of Q_K, we get

$$\begin{aligned} -(\Delta u, Q_K v)_K =& (\nabla\Pi_K u, \nabla v)_K \\ &+ \langle (\nabla u - \nabla\Pi_K u)\cdot n, Q_K v\rangle_{\partial K} \\ &+ \langle \nabla\Pi_K u\cdot n, Q_F v\rangle_{\partial K}. \end{aligned}$$

By definition, the first term can be rewritten as $(\nabla\Pi u, \nabla\Pi v)_K$. By the continuity of $\nabla u\cdot n$ across the interelement faces, and the fact that $Q_F v$ is single value at the face F, $\sum_{K\in\mathcal{T}_h}\langle\nabla u\cdot$

$n, Q_F v\rangle_{\partial K} = 0$. As a result, recalling the VEM approximation problem in equation (3.10), we arrive at the following identity

$$
\begin{aligned}
a_h(u_h, v) &= -\sum_{K\in\mathcal{T}_h} (\Delta u, Q_K v)_K \\
&= \sum_{K\in\mathcal{T}_h} (\nabla\Pi_K u, \nabla\Pi_K v)_K + \sum_{K\in\mathcal{T}_h} \langle\nabla(u-\Pi_K u)\cdot n, Q_K v - Q_F v\rangle_{\partial K}.
\end{aligned}
$$

□

Theorem 3.2.3 (Error equation)**.** *Under the same setting with Lemma 3.2.2, let u_I be the VEM interpolation in* (3.1.11)*, the following identity holds,*

$$
\begin{aligned}
a_h\big(u_h - u_I, v_h\big) = \sum_{K\in\mathcal{T}_h} \Bigg[&(\nabla\Pi_K(u-u_I), \nabla\Pi_K v_h)_K \\
&+ \sum_{F\subset\partial K} \Big((\nabla(\Pi_K u - u)\cdot n, Q_K v_h - Q_F v_h)_F \\
&- h_K^{-1}(Q_K u_I - Q_F u_I, Q_K v_h - Q_F v_h)_F \\
&- \epsilon_F h_F \sum_{e\subset\partial F} (u_I - Q_F u_I, v_h - Q_F v_h)_e \Big) \Bigg]
\end{aligned}
\tag{3.12}
$$

Proof. It follows directly from Lemma 3.2.2 and stabilization term definition in 3.8. □

Corollary 3.2.4 (*A priori* error bound)**.** *The following a priori error estimate holds for u_h and u_I (defined in 3.1.11) with a constant independent of the chunkiness parameter for ρ_F of each face in the underlying mesh, and $\epsilon_F \propto \rho_F$*

$$
\begin{aligned}
|||u_h - u_I|||^2 \lesssim \sum_{K\in\mathcal{T}_h} \Bigg[&\|\nabla\Pi_K(u-u_I)\|_{0,K}^2 + \sum_{F\subset\partial K} \Big(h_K \|\nabla(\Pi_K u - u)\cdot n\|_{0,F}^2 \\
&+ h_K^{-1}\|Q_K u_I - Q_F u_I\|_{0,F}^2 + \epsilon_F \sum_{e\subset\partial F} \|u_I - Q_F u_I\|_{0,e}^2 \Big) \Bigg]
\end{aligned}
\tag{3.13}
$$

Proof. From the error equation, plug in $v_h = u_h - u_I$ and apply Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|||u_h - u_I|||^2 \lesssim \sum_{K\in\mathcal{T}_h} \Big[&\|(\nabla\Pi_K(u-u_I)\|_{0,K}\|\nabla\Pi_K v_h\|_{0,K} \\
&+ \sum_{F\subset\partial K} \Big(h_K^{1/2}\|\nabla(\Pi_K u - u)\cdot n\|_{0,F} h_K^{-1/2}\|Q_K v_h - Q_F v_h\|_{0,F} \\
&+ h_K^{-1/2}\|Q_K u_I - Q_F u_I\|_{0,F} h_K^{-1/2}\|Q_K v_h - Q_F v_h\|_{0,F} + \\
&\sum_{e\subset\partial F} \epsilon_F\|(u_I - \Pi_F u_I)\|_{0,e}\epsilon_F\|(v_h - \Pi_F v_h)\|_{0,e}\Big)\Big]
\end{aligned} \tag{3.14}$$

The second part of each term is clearly parts of $|||u_h - u_I|||$ and therefore can be bounded by $|||u_h - u_I|||$. After cancelling $|||u_h - u_I|||$ we get the estimate. □

3.3 Geometric conditions and error estimations

In this section, based on the a priori error estimate in Corollary 3.2.4, the energy norm estimate follows from estimating each term in (3.13). The necessary geometry conditions motivated by (3.13) to have optimal order of convergence are proposed as follows.

Assumption 3.3.1 (Geometric conditions)**.** *For each element $K \in \mathcal{T}_h$, the following three geometric conditions are met:*

1. *Number of faces in K is uniformly bounded.*
2. *K is star-shaped with the chunkiness parameter ρ_K defined in 2.2.1 bounded below.*
3. *For each $F \subset \partial K$, F is star-shaped, but the star-shape constant may not be uniformly bounded.*

Lemma 3.3.2 (Optimal order error estimate of the stabilization term on face)**.** *Let* $u \in H^{k+1}(K)$*, and* u_I *be the VEM space interpolation defined in 3.1.10. Suppose the geometric assumptions 3.3.1 hold, then*

$$h_K^{-1/2}\|Q_K u_I - Q_F u_I\|_{0,F} \lesssim h_K^k |u|_{k+1,K}, k \geq 1 \tag{3.15}$$

Proof. By triangle inequality,

$$\begin{aligned} h_K^{-1/2}\|Q_K u_I - Q_F u_I\|_{0,F} &= h_K^{-1/2}\|Q_F(Q_K u_I - u_I)\|_{0,F} \\ &\lesssim h_K^{-1/2}\|Q_K u_I - u_I\|_{0,F} \\ &\lesssim h_K^{-1/2}(\|Q_K u_I - u\|_{0,F} + \|u - u_I\|_{0,F}) \\ &\lesssim h_K^{-1}(\|Q_K u_I - u\|_{0,K} + \|u - u_I\|_{0,K}) \\ &+ (|Q_K u_I - u|_{1,K} + |u - u_I|_{1,K}) \\ &\lesssim h_K^k |u|_{k+1,K} \end{aligned} \tag{3.16}$$

where Theorem 2.2.6, 2.2.8, 2.2.4 are applied. □

Lemma 3.3.3 (Optimal order error estimate of stabilization term on edge)**.** *Let* $u \in H^{k+1}(K)$*, and* u_I *be the VEM space interpolation defined in 3.1.10. Suppose the geometric assumption 3.3.1 hold, then for a mesh dependent constant* $\epsilon_F \propto \rho_F^k$

$$\epsilon_F\|u_I - Q_F u_I\|_{0,e} \lesssim h_F^k |u|_{k+1,K}, k \geq 1 \tag{3.17}$$

Proof. By the Theorem 2.2.4 and triangle inequality, under the star-shaped condition 2.2.1,

$$\begin{aligned}\|u_I - Q_F u_I\|_{0,e} &\lesssim \epsilon_F^{-1}(h_F^{-1/2}\|u_I - Q_F u_I\|_{0,F} + h_F^{1/2}|u_I - Q_F u_I|_{1,F}) \\ &\lesssim \epsilon_F^{-1}(h_F^{-1/2}(\|u_I - u\|_{0,F} + \|u - Q_F u_I\|_{0,F}) + h_F^{1/2}(|u_I - u|_{1,F} \\ &+ |u - Q_F u_I|_{1,F}))\end{aligned} \tag{3.18}$$

where each except the last term has optimal error order by Theorem 2.2.8. In order to use Theorem 2.2.8 on the face, ρ_F need to be included because we do not assume it is uniformly bounded below, therefore the constant $\epsilon_F \propto \rho_F^k$ is introduced. For the last term, we apply the polynomial norm equivalence and the triangle inequality.

$$\begin{aligned}|u - Q_F u_I|_{1,F} &\lesssim |u - \Pi_F u_I|_{1,F} + |\Pi_F u_I - Q_F u_I|_{1,F} \\ &\lesssim |u - \Pi_F u_I|_{1,F} + \epsilon_F^{-1} h_F^{-1}\|\Pi_F u_I - Q_F u_I\|_{0,F} \\ &\lesssim |u - \Pi_F u_I|_{1,F} + \epsilon_F^{-1} h_F^{-1}\|\Pi_F u_I - u_I\|_{0,F} + \epsilon_F^{-1} h_F^{-1}\|u_I - Q_F u_I\|_{0,F}\end{aligned} \tag{3.19}$$

where each term has optimal error order by Theorem 2.2.8. Similarly the inverse inequality(polynomial norm equivalence) depends on ρ_F [12] (and we do not assume ρ_F is uniformly bounded below), a mesh dependent constant $\epsilon_F \propto \rho_F^k$ is introduced to compensate. □

Now we derived the estimates of the other terms in the a priori

error bound (3.2.4).

Lemma 3.3.4 (The projection type error estimates)**.** *Let u_I be the interpolation defined in*

3.1.10, then

$$\|\nabla\Pi_K(u-u_I)\|_{0,K} \lesssim h_K^{k-1}|u|_{k,K}, k \geq 2, \tag{3.20}$$

$$h_K^{1/2}\|\nabla(\Pi_K u-u)\|_{0,F} \lesssim h_K^{k-1}|u|_{k,K}, k \geq 2. \tag{3.21}$$

Proof. By Theorem 2.2.6 and Π_K is the projection under $|\cdot|_{1,K}$,

$$\|\nabla\Pi_K(u-u_I)\|_{0,K} \lesssim |u-u_I|_{1,K} \lesssim h_K^{k-1}|u|_{k,K}$$

In addition, by Theorems 2.2.4 and 2.2.6

$$h_K^{1/2}\|\nabla(\Pi_K u-u)\|_{0,F} \lesssim |\Pi_K u-u|_{1,K} + h_K|\Pi_K u-u|_{2,K} \lesssim h_K^{k-1}|u|_{k,K}$$

□

Theorem 3.3.5 (Energy norm error estimate)**.** *Let u_I be the interpolation defined in 3.1.11, then Suppose the geometric assumptions 4.2.1 hold, then for $k \geq 1$ the followings hold,*

$$|||u_h-u_I||| \lesssim h^k|u|_{k+1} \tag{3.22}$$

Proof. (3.22) follows immediately from the bound of each term in the a priori error estimate (3.13) by Lemmas 3.3.4, 3.3.2, and 3.3.3. □

Chapter 4

Elliptic Interface Problem

4.1 Introduction

In this chapter we go over the equations of the elliptic interface problem, and give an overview of various numerical methods ([30, 31, 26]) to solve it. We shall go over the detail of each method in the next few chapters.

We consider a two dimensional domain Ω, which is formed by two different materials separated by a closed smooth curve $\Gamma \in C^{1,1}$, i.e., Γ separates Ω into sub-domains Ω^+ and Ω^- such that $\overline{\Omega} = \overline{\Omega^+ \cup \Omega^- \cup \Gamma}$, cf. Fig. 4.1. The coefficient β is assumed to be a piecewise positive constant function on Ω:

$$\beta(x, y) = \begin{cases} \beta^+, & (x, y) \in \Omega^+, \\ \beta^-, & (x, y) \in \Omega^-. \end{cases}$$

We consider the approximationm problem to the following elliptic interface problem,

$$\begin{aligned} -\nabla \cdot (\beta \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

We further assume that the data $f \in L^2(\Omega)$ and the true solution satisfies the following jump conditions

$$u^+ = u^-, \quad \beta^+ \nabla u^+ \cdot \mathbf{n}^+ + \beta^- \nabla u^- \cdot \mathbf{n}^- = 0 \qquad \text{on } \Gamma \tag{4.2}$$

where $\mathbf{n}^\pm$ denotes the unit outward normal vector to the corresponding domains Ω^+ and Ω^- on Γ, respectively. With slight abuse of the point-wise values of u, for a point $x \in \Gamma$, u^+ and u^- are defined as $u^+(x) := \lim_{\epsilon \to 0^+} u(x - \epsilon \cdot \mathbf{n}^+)$ and $u^-(x) := \lim_{\epsilon \to 0^+} u(x - \epsilon \cdot \mathbf{n}^-)$

The weak formulation of (4.1) is: to seek the solution $u \in H_0^1(\Omega)$ such that

$$a(u, v) := (\beta \nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{4.3}$$

where $(\cdot, \cdot)$ is the L^2-inner product on Ω. The existence and uniqueness of the solution is well-known by Lax-Milgram lemma due to the positivity of β and the Poincaré inequality.

Under the assumption that $f \in L^2(\Omega)$ and $\Gamma \in C^{1,1}$, it can be shown that (see e.g. [28, 29, 19]) the solution $u \in H^2(\Omega^+ \cup \Omega^-)$ and

$$\|u\|_{H^2(\Omega^+ \cup \Omega^-)} \le C_{\beta^\pm} \|f\|_{L^2(\Omega)}, \tag{4.4}$$

where, for $k > 1$,

$$H^k(\Omega^+ \cup \Omega^-) = \left\{ u \in H^1(\Omega) \text{ and } u^\pm \in H^k(\Omega^\pm) \right\}$$

The main challenge of using standard finite element methods is that the solution of equation

(4.3) is not in $H^2(\Omega)$, which makes the well-known optimal approximation results for linear finite element methods (FEM), $\|u-u_h\|_{H^1(\Omega)} \lesssim h|u|_{H^2(\Omega)}$ not achievable if the mesh does not fit the interface. In order to achieve the optimal convergence order, two major approaches in the literature are proposed: (i) modify the mesh to fit the interface and then apply either continuous or discontinuous Galerkin formulation, (ii) modify the finite element spaces to encode the jump conditions into the discretization.

The first approach, also known as the body-fitted finite element methods, generates a shape regular mesh in a way that the interface cannot intersect element interior and can be well-approximated by edges of elements [16, 29]. The approximated interface separates the domain to $\Omega_h^{\pm}$. Defining $\beta_h = \beta^{\pm}$ according to the subdomains $\Omega_h^{\pm}$ and modifying the bilinear from in (4.3) to be $a_h(u, v) = (\beta_h \nabla u, \nabla v)$. The error estimates then follow the standard Céa's Lemma by taking advantage of the fact that the error caused by the mismatch between the interface and the approximated interface is of higher order of the desired rate of convergence [34, 9, 18, 29].

The latter approach aims to circumvent the burden of generating interface-fitted mesh as this procedure could be non-trivial and expensive for geometrically complicated or moving interface. It includes, for example, the CutFEM [13], the mutiscale FEM (MsFEM) [19] and the immersed finite element (IFE) to be discussed in this paper and many others. The methods modify finite element spaces on interface elements, i.e., those elements are by interface, and thus can be used on unfitted meshes and still obtain the optimal convergence order where the hidden constant is independent of interface location relative to the mesh. In particular, for the IFE method, a set of local basis functions on the interface elements are devised as piecewise polynomials that include jump conditions (4.2) in their connection in a pointwise or averaging sense. The convergence of IFE methods have been established in [30, 31, 26] and improved recently in [21, 24].

We shall present a new formulation and a convergence analysis of the immersed finite element method using the methodology of virtual element methods (VEM) [5, 7]. In this new approach, there is a underling linear virtual element space that is defined on an interface-fitted polygon mesh. As an analogy to VEM, we define $\Pi_K u_h$ to be the projection to IFE space on non-fitted mesh. A stabilization term is then added to the discrete bilinear form to ensure the coercivity. To summarize, the proposed discrete bilinear form follows the standard VEM projection–stabilization split as follows,

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \Big[(\beta_h \nabla \Pi_K u_h, \nabla \Pi_K v_h)_K + S_K(u_h - \Pi_K u_h, v_h - \Pi_K v_h) \Big], \tag{4.5}$$

where Π_K is an orthogonal projection to a local IFE space defined in definition 5.9, and $S_K(\cdot, \cdot)$ is a stabilization term that preserves the approximation property to the correct order. The choice of stabilization for VEM is flexible, and in this paper, we opt for a tangential derivative-type stabilization in [33, 8].

This new formulation may inherit the advantages of both VEM and IFE in the following sense. First, it is still able to solve the interface problems on unfitted meshes with optimal convergence order

$$|\!|\!| u_h - u |\!|\!| \lesssim h \|u\|_{H^2(\Omega^+ \cup \Omega^-)}$$

where $|\!|\!|\cdot|\!|\!|$ is defined using $a_h(\cdot, \cdot)$. Second, compared with other penalty-type methods in the literature [24, 31], the proposed new formulation based on VEM needs only on edge term and is more local as the stabilization term does not need interaction of neighbor elements. Third, compared with the anisotropic analysis for VEM [12, 14], as the virtual space is projected onto the IFE spaces, the anisotropic subelement shape of interface elements can be easily handled by the properties of IFE spaces including their trace inequalities and approximation capabilities which are all independent of interface location, i.e., the cut points.

4.2 Body-Fitted FEM and VEM

In this section, we shall review the body-fitted FEM and VEM method of solving the elliptic interface problem.

Let $\mathcal{T}_h = \{K\}$ be a shape regular triangulation of the domain Ω. A triangle K is defined as an interface triangle if $\text{meas}(K \cap \Omega^+) > 0$ and $\text{meas}(K \cap \Omega^-) > 0$. Throughout the paper we make the following assumptions on the interface Γ and the triangulation $\mathcal{T}_h$.

Assumption 4.2.1 (Assumptions on meshes and the interface)**.** *For each interface triangle K:*

1. *$\Gamma \cap \partial K$ consists of exactly 2 points, called cut points;*
2. *these two cut points cannot be on the same edge of K.*

These assumptions can be satisfied if the triangulation $\mathcal{T}_h$ is fine enough [19].

Due to the discontinuity of the coefficient β, the solution to (4.3) is not in $H^2(\Omega)$ globally. Under the assumption that $f \in L^2(\Omega)$ and Γ is of class C^2 (e.g. [28, 29, 19]), it can be shown that the solution $u \in H^2(\Omega^+ \cup \Omega^-)$ and

$$\|u\|_{H^2(\Omega^+ \cup \Omega^-)} \leq C_{\beta^\pm} \|f\|_{L^2(\Omega)}, \tag{4.6}$$

where, for $k > 1$,

$$H^k(\Omega^+ \cup \Omega^-) = \left\{u \in H^1(\Omega) \text{ and } u^\pm \in H^k(\Omega^\pm)\right\}$$

and the piecewise H^k norm is defined by $\|u\|^2_{H^k(\Omega^+ \cup \Omega^-)} = \|u\|^2_{H^k(\Omega^+)} + \|u\|^2_{H^k(\Omega^-)}$ for any $u \in H^k(\Omega^+ \cup \Omega^-)$.

4.2.1 The body-fitted Finite Element Method

Given a shape regular triangulation $\mathcal{T}_h$ which may not resolve the interface, we can use the Delaunay triangulation of the interface points, that include vertices of all interface element and cut points (algorithm details can be found in [16]), to generate an auxiliary body-fitted triangulation $\overline{\mathcal{T}}_h$. It has been proved in [16] that the so-called maximum angle condition 4.2.2 is satisfied.

Definition 4.2.2 (Maximum Angle Condition)**.** *We said that a family of meshes $\mathcal{T}_h$ satisfied the maximum angle condition if the maximum angle in $\mathcal{T}_h$ is bounded uniformly away from π as $h \to 0$.*

Using the procedure in [16], we can generate a discrete interface Γ_h, a subset of edges of triangles in $\overline{\mathcal{T}}_h$, such that $\Gamma_h \subset S_\delta(\Gamma)$ and $\delta = \mathcal{O}(h^2)$, where $S_\delta(\Gamma)$ is the tubular neighborhoods of the interface Γ: $S_\delta(\Gamma) := \{x \in \Omega : \text{dist}(x, \Gamma) < \delta\}$. The domain Ω is split into $\Omega_h^\pm$ by Γ_h. In addition, define $\beta_h = \beta^\pm$ on $\Omega_h^\pm$. Note that $\beta \neq \beta_h$ since in general $\Omega_h^\pm \cap \Omega^\mp \neq \varnothing$, but the mismatch of β and β_h occurs in an area with small measurement, e.g., K_δ in Figure 4.2.

As $\Gamma_h \subset S_\delta$, we have the following trivial conclusion.

Lemma 4.2.3 (Γ_h approximation property)**.** *Suppose the curvature of Γ is bounded above and the mesh is fine enough so assumptions 4.2.1 holds. Let $\Omega_\delta := \{x \in \Omega : \beta \neq \beta_h\}$. We have $|\Omega_\delta| = \mathcal{O}(h^2)$.*

Proof. Under the assumption on Γ and mesh, we have $\delta = \mathcal{O}(h^2)$, that makes $|K_\delta| = \mathcal{O}(h^3)$. Because the number of interface triangle is of $\mathcal{O}(h^{-1})$, we reach the conclusion $|\Omega_\delta| = \mathcal{O}(h^2)$. □

It can be shown that, under the maximum angle condition, the body-fitted linear finite element method can achieve the optimal order of convergence [17]. We first present an inequality [17, 29], that could be derived from Gagliardo–Nirenberg interpolation inequality.

Lemma 4.2.4. *[17, 29] Let D be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ_D. Let $S_\sigma(\Gamma_D) = \{x \in D : dist(x, \Gamma_D) < \sigma\}$ be the $\sigma-$neighborhood of Γ_D. Then we have*

$$\|u\|_{L^2(S_\sigma(\Gamma_D))} \lesssim \sqrt{\sigma}\|u\|_{H^1(D)}$$

for all $u \in H^1(D)$. Furthermore, the constant is independent of σ.

We then define the local and global discrete spaces of the body-fitted linear FEM.

Definition 4.2.5 (Local linear FEM Space)**.** *Given a triangle K, the local standard FEM space is simply $\mathbb{P}_1(K)$, the linear polynomial space on K. The degrees of freedom are values of the function at vertices of K. We emphasize that these include nodal values at cut points.*

Definition 4.2.6 (Global body-fitted linear FEM Space)**.** *For any triangulation $\mathcal{T}_h$ of Ω, define $\bar{V}_h$ as the continuous and piecewise linear finite element space on the auxiliary body-fitted mesh $\bar{\mathcal{T}}_h$ satisfying the boundary condition of $H_0^1(\Omega)$.*

The standard conforming finite element approximation in the body-fitted finite element space is: find $u_h \in \bar{V}_h$ such that

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} (\beta_h \nabla u_h, \nabla v_h)_K = (f, v_h) \quad \forall v_h \in \bar{V}_h. \tag{4.7}$$

The following lemma [34, 18, 29] on the interpolation error will be useful for the error analysis of the solution on the body-fitted meshes. For the completeness and later usage, we present the results below.

Lemma 4.2.7 (Approximation of the nodal interpolation)**.** *Let $u \in H^2(\Omega^+ \cup \Omega^-) \cap H_0^1(\Omega)$ and let $\bar{\mathcal{T}}_h$ be the body-fitted mesh generated by Algorithm 1 in [16] that satisfies maximum* *angle condition. Let $I_h u$ be the nodal interpolation to the body-fitted linear finite element space $\bar{V}_h$. Then we have the following estimate with the constant independent of β, β_h:*

$$\|u - I_h u\|_{0,\Omega} + h\|\nabla(u - I_h u)\|_{0,\Omega} \lesssim h^2 \|u\|_{H^2(\Omega^+ \cup \Omega^-)}$$

Proof. Apply Sobolev extension theorems [32] to $u^{\pm}$ to get $Eu^{\pm} \in H^2(\Omega)$ and $\|Eu^{\pm}\|_{H^2(\Omega)} \lesssim \|u^{\pm}\|_{H^2(\Omega^{\pm})}$ where the constant depends on Ω only. For $K \subset \Omega_h^{\pm}$, write $u - I_h u = (u - Eu^{\pm}) + (Eu^{\pm} - I_h Eu^{\pm})$ and by triangle inequality it is sufficient to estimate each term. Note that $I_h Eu^{\pm} = I_h u^{\pm}$. Then, apply the standard interpolation error estimate on triangles with the maximal angle condition [3], we obtain

$$\|Eu^+ - I_h Eu^+\|_{0,K} + h\|\nabla(Eu^+ - I_h Eu^+)\|_{0,K} \lesssim h^2 \|Eu\|_{H^2(K)} \lesssim h^2 \|u\|_{H^2(K)}$$

for $K \subset \Omega_h^+$, and we write the similar inequality on u^- for $K \subset \Omega_h^-$.

On the other hand, for $u - Eu^{\pm}$ on $K \subset \Omega_h^{\pm}$, this term is only non-zero on $K \cap \Omega_\delta$, the region with width $\mathcal{O}(h^2)$. Then by the Lemma 4.2.4 and the stability of E in H^1 and H^2 norm. We have $\|u - Eu^{\pm}\|_{0,K} \lesssim h^2 \|u\|_{H^2(\Omega^+ \cup \Omega^-)}$ and $\|\nabla(u - Eu^{\pm})\|_{0,K} \lesssim h\|u\|_{H^2(\Omega^+ \cup \Omega^-)}$.

Combine the results of the last two paragraphs give the desire estimate. □

With the interpolation error estimate, the approximation of the discrete solutions can be obtained (e.g. [29, 17]). The procedure is standard but lengthy, we shall present the result and skip the proof here.

Theorem 4.2.8 (Approximation of the linear finite element on body-fitted meshes). *Let $u \in H^2(\Omega^+ \cup \Omega^-)$ be the solution of (4.3), and $u_h \in \bar{V}_h$ be the solution of (4.7), then*

$$\|\nabla(u - u_h)\|_{0,\Omega} \lesssim h\|u\|_{H^2(\Omega^+ \cup \Omega^-)}.$$

4.2.2 Virtual Element Methods

In this section we review the virtual element method (VEM) approach on the interface problem [5, 7, 16]. We will only use the linear VEM space. The mesh is generated in similar way as in the body-fitted FEM. Let $\mathcal{K}_h$ be the interface-fitted polygonal mesh generated by algorithm in Section 2 of [16]. In 2D, given a polygonal mesh, the interface-fitted mesh is simply separating interface polygons into two by the cut points. We start our review of VEM by some definitions [5, 7].

Definition 4.2.9 (Local linear VEM space)**.** *Given a star-shaped polygon K, the local linear VEM space $U_h(K)$ is defined by $U_h(K) = \{u \in H^1(K) | \Delta u = 0, u|_e \in \mathbb{P}^1(e), u|_{\partial K} \in C^0(\partial K)\}$. That is, a harmonic function that is continuous and piece-wise linear polynomial on the boundary.*

Definition 4.2.10 (Global VEM space)**.** *The global virtual element space can be defined as*

$$U_h := \{u \in H^1(\Omega), u|_K \in \bar{U}_h(K) \forall K \in \mathcal{T}_h\}$$

Definition 4.2.11 (Degrees of Freedom of VEM space)**.** *The degrees of freedom of $u_h \in U_h$ are the values of the function at vertices $x \in \mathcal{N}_h(K)$ in the mesh $\mathcal{T}_h$.*

Definition 4.2.12. *For $u \in H^1(\Omega) \cap C(\Omega)$, the interpolation function u_I is defined by the unique $u_I \in \bar{U}_h$ (proof of uniqueness and existence can be found at [5]) that has identical degrees of freedom as u.*

The interpolation operator on VEM space satisfies standard optimal error estimates property provided the mesh is uniform star-shaped [8, 15, 12].

Lemma 4.2.13 (Interpolation Error Estimate)**.** *Given a polygon K, and let h_K be the* *diameter of K, we have*

$$\|\nabla(u - u_I)\|_{0,K} \lesssim h_K |u|_{2,K} \tag{4.8}$$

where the constant in the inequality depends on the star-shaped constant of K.

Notice that the polygon mesh $\mathcal{K}_h$, obtained by connecting cut points may contain anisotropic elements K so that the interpolation error estimate (4.8) on K is useless. Fortunately we shall only use (4.8) on shape regular polygons.

Before defining the discrete bilinear form, it is necessary to define a local H^1-projection operator.

Definition 4.2.14 (Local H^1 Projection)**.** *Let K be a polygon and $u \in H^1(K)$, we define the projection $\Pi_K u \in \mathbb{P}_1(K)$ the linear polynomial that satisfies $(\nabla u, \nabla q)_K = (\nabla \Pi_K u, \nabla q)_K$ for all $q \in \mathbb{P}_1(K)$. In addition, $\sum_{x\in\mathcal{V}(K)} u(x) = \sum_{x\in\mathcal{V}(K)} \Pi_K u(x)$. That is, the average value of the function across the vertices of K remains unchanged after projection.*

The projection operator can be computed without explicit formula of the basis function of local space, but only need the degrees of freedom of the function [5, 7].

Now for the interface problem we define

$$a_h^K(u_h, v_h) := (\beta_h \nabla \Pi_K u_h, \nabla \Pi_K v_h)_K + S_K(u_h - \Pi_K u_h, v_h - \Pi_K v_h)$$

for a suitable choice of the stabilization S_K. The global bilinear form then is defined as $a_h^{\text{VEM}}(u_h, v_h) = \sum_K a_h^K(u_h, v_h)$ and $|||\cdot|||^2 = \sum_{K\in\mathcal{T}_h} a_h^K(\cdot,\cdot)$ is the corresponding energy norm.

Stabilization $S_{K,\beta}$ has to be chosen in the way that, the following conditions are satisfied.

Assumption 4.2.15 (VEM stabilization term assumption)**.** *The $S_{K,\beta}$ has to be chosen in* *the way that,* $a_h^K(p, v_h)$ *satisfies*

- *Consistency:* $a_h^K(p, v_h) = a^K(p, v_h)$ *for all* $v_h \in \bar{V}_h(K)$ *and* $p \in \mathbb{P}_1(K)$
- *Stability:* $a_h^K(v_h, v_h) \simeq a^K(v_h, v_h)$ *for all* $v_h \in \bar{V}_h(K)$

The discrete problem is: find $u_h \in \bar{V}_h$ such that

$$a_h^{\mathrm{VEM}}(u_h, v_h) = (f_h, v_h) \quad \forall v_h \in \bar{V}_h. \tag{4.9}$$

If these assumptions hold, the error of the discrete solution can be bounded by standard interpolation and projection error estimates [5].

Lemma 4.2.16 (VEM error estimate [5])**.** *Suppose Assumptions 4.2.15 holds. Let* u_h *be the solution of* (4.9)*. Then we have the error bound*

$$|||u - u_h||| \lesssim |||u - u_I||| + \Big(\sum_K |||u - \Pi_K u|||_K^2\Big)^{1/2} + \|f - f_h\|_{\bar{V}_h'} \tag{4.10}$$

More detail and higher order cases can be found on, for example, [5, 7, 17]. The analysis involves proving each term converges with optimal order. Different choices of the stability term S_K and the proof of norm equivalence (4.10) with respect to each choice of S_K, can be found on [5, 15, 8].

4.3 Immersed Finite Element Methods

In this section, we review another approach, the immersed finite element method (IFE), of solving the elliptic interface problem.

In IFE methods [31, 26], the mesh is not fitted into the interface Γ. Instead, the local space on interface triangles is changed to fit the behavior of the solution across the interface.

We shall review the IFE method that can be found on [31].

Definition 4.3.1 (Local IFE Space [31])**.** *Given a triangle K and a curved interface Γ. Assume Γ intersects ∂K at D and E (on the different edges of K). Let $L = \overline{DE}$ and L cut K into $K^{\pm}$. Let $n^{\pm}$ be the outward normal vector of $K^{\pm}$ on the edge L. Denote the restriction of u to $K^{\pm}$ by $u^{\pm}$. We define $u \in \tilde{\mathbb{P}}_1(K)$ if and only if*

1. $u^{\pm} \in \mathbb{P}_1(K^{\pm})$
2. $u^+ = u^-$ *on* L
3. $\beta^+ \partial_{n^+} u^+ + \beta^- \partial_{n^-} u^- = 0$ *on* L

Note that as $u \in \tilde{\mathbb{P}}_1(K)$ is piecewise linear, the continuity of the function value can be ensured by that on the two cut points: $u^+(D) = u^-(D)$, $u^+(E) = u^-(E)$. And the flux condition can be imposed at any point on L.

Definition 4.3.2 (Global IFE Space)**.** *For a triangulation $\mathcal{T}_h$ of Ω, define $u \in V_h^{\mathrm{IF}}$ if u is continuous on non-interface edge and is in $\tilde{\mathbb{P}}_1(K)$ for each interface element K and $\mathbb{P}_1(K)$ otherwise.*

Notice the function value at the cut points are not degrees of freedom, but are determined elementwise, and might be different in two elements sharing the cut point. Thus in general V_h^{IF} is not a subspace of H^1. The discontinuity across the cut points make the penalization term in the bilinear form necessary.

Let ϕ_z^{IF} be the nodal basis function of the global IFE space, that is, $\phi_z^{\mathrm{IF}} \in V_h^{\mathrm{IF}}$, $\phi_z^{\mathrm{IF}}(z) = 1$ and $\phi_z^{\mathrm{IF}}(x) = 0$ for $x \in \mathcal{N}(K), x \neq z$. Then, the piecewise-defined nodal interpolation $I_h^{\mathrm{IF}}(\cdot)$

for any $v \in C^0(\Omega)$ is as follows:

$$I_h^{\mathrm{IF}} v(x)\Big|_K := \sum_{z \in \mathcal{N}(K)} v(z)\phi_z^{\mathrm{IF}}(x). \tag{4.11}$$

We need the following approximation property for the interpolation operator.

Lemma 4.3.3 (Interpolation Error Estimate [31, 24])**.** *For $u \in H^2(\Omega^+ \cup \Omega^-) \cap H_0^1(\Omega)$ satisfying the interface jump condition (equation* (4.2)*), there exists a constant C, independent of cut points location, such that*

$$\left\|u - I_h^{\mathrm{IF}} u\right\|_{L^2(\Omega)} + h \left(\sum_{T \in \mathcal{T}_h} \left\|u - I_h^{\mathrm{IF}} u\right\|_{H^1(T)}^2 \right)^{\frac{1}{2}} \leq Ch^2 \|u\|_{H^2(\Omega^+ \cup \Omega^-)}. \tag{4.12}$$

The bilinear form of partially penalized IFE approach in [31] is: for arbitrary parameters ϵ, $\alpha > 0$, and $\sigma_e^\circ \geq 0$

$$\begin{aligned} &a_h^{\mathrm{IF}}(u_h, v_h) := (\beta_h \nabla u_h, \nabla v_h)_\Omega \\ &- \sum_{\mathcal{E}_\Gamma} \int_e (\{\beta \nabla u_h \cdot n_e\}[v_h] + \epsilon\{\beta \nabla v_h \cdot n_e\}[u_h]) \\ &+ \sum_{\mathcal{E}_\Gamma} \int_e \frac{\sigma_e^\circ}{|e|^\alpha}[u_h][v_h] \end{aligned} \tag{4.13}$$

where the set of all edges with cut points is denoted $\mathcal{E}_\Gamma$ and the standard notation for jumps $[\cdot]$ and averages $\{\cdot\}$ are used. That is,

$$\{v\}_e := \frac{v|_{T_1} + v|_{T_2}}{2}, \quad \text{and } [v]_e := v|_{T_1} n_1 + v|_{T_2} n_2,$$

where e is the edge share by T_1 and T_2. The discretization in [31] is: find the $u_h \in V_h^{\mathrm{IF}}$ such that

$$a_h^{\mathrm{IF}}(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h^{\mathrm{IF}}(\Omega). \tag{4.14}$$

With this formulation the approximation property of u_h can be achieved provides $u \in H^2(\Omega^+ \cup \Omega^-)$,

Theorem 4.3.4 (Convergence analysis of the discrete solution [24])**.** *Let u be the solution of equation* (4.3) *and u_h be the solution of equation* (4.13)*, then provided $u \in H^2(\Omega^+ \cup \Omega^-)$*

$$\|u - u_h\|_h \lesssim h\|u\|_{H^2(\Omega^+ \cup \Omega^-)} \tag{4.15}$$

where $\| \cdot \|_h$ is the discrete energy norm defined by the bilinear form (4.13).

In [31], the estimate required H^3-norm is due to the usage of IFE trace theorem ([27]) on the H^2-norm on the edge, this is later removed in [24]. In our approach we seek to keep this advantage and simplify the formulation further.

In the end of this section we briefly review the result in [26]. It is shown that by using the non-conforming IFE space $\tilde{V}_h^{IF}$ constructed in [27] with some modifications on the penalty term, we can have a contrast independent estimate, assuming only H^2 regularity.

The discretization of partially penalized IFE approach in [26] is: Find the $u_h \in \tilde{V}_h^{\mathrm{IF}}$ such that

$$\begin{aligned} & a_h(u_h, v_h) := (\beta \nabla u, \nabla v)_\Omega \\ & - \sum_{\mathcal{E}_\Gamma} \int_e (\{\beta \nabla u\}[v] + \{\beta \nabla v\}[u]) \\ & + \sum_{\mathcal{E}_\Gamma} (\frac{\gamma}{|e^-|} \int_{e^-} [u][v] + \frac{\gamma}{|e^+|} \int_{e^+} [u][v]) \\ & + \sum_{\mathcal{E}_\Gamma} (\frac{\gamma}{|e^-|} \int_{e^-} [\nabla u][\nabla v] + \frac{\gamma}{|e^+|} \int_{e^+} [\nabla u][\nabla v]) = (f, v) \quad \forall v \in \bar{V}_h^{\mathrm{IF}}(\Omega) \end{aligned} \tag{4.16}$$

for a penalty parameter $\gamma > 0$.

Theorem 4.3.5 (Convergence analysis of the discrete solution [26])**.** *Let u be the solution of equation* (4.3) *and u_h be the solution of equation* (4.16)*, then provided $u \in \tilde{H}^2(\Omega^+ \cup \Omega^-)$*

$$
\begin{aligned}
\|u - u_h\|_h \lesssim & h[\sqrt{\beta^-}(\|Du\|_{L^2(\Omega^-)} + \|D^2u\|_{L^2(\Omega^-)}) + \\
& \sqrt{\beta^+}(\|Du\|_{L^2(\Omega^+)} + \|D^2u\|_{L^2(\Omega^+)})]
\end{aligned}
$$

where $\| \cdot \|_h$ is the energy norm defined by the bilinear form (4.16).

Chapter 5

Virtual Element Methods on Elliptic Interface Problems

In this chapter we present our main result of virtual element method of solving the elliptic interface problem. We not only create a more concise formulation and convergence proof of partially penalized IFE method, but also brings a connection between various methods reviewed in earlier chapters.

5.1 Preliminary

In this section, some definitions and notation are introduced, and certain existing results, which are essential to our error analysis, are reviewed as well.

Let $\mathcal{T}_h = \{K\}$ be a shape regular triangulation of the domain Ω that may not be fitted to the interface. A triangle K is defined as an interface triangle if $\text{meas}(K \cap \Omega^+) > 0$ and $\text{meas}(K \cap \Omega^-) > 0$, where $\text{meas}(\cdot)$ denotes the measure of a domain; otherwise K is called a non-interface element. Throughout the chapter we make the assumption 4.2.1 on the triangulation $\mathcal{T}_h$, which can be satisfied if $\mathcal{T}_h$ is fine enough [19, 21] provided that $\Gamma \in C^{1,1}$. Let $H^k(D)$ $k \geq 0$ be the standard Hilbert space on a domain D with the norm $\|\cdot\|_{H^k(D)}$ and the semi-norm $|\cdot|_{H^k(D)}$. Due to the discontinuity of the coefficient β, the solution to (4.3) is not in $H^2(\Omega)$ globally. Under the setting introduced in section 4.1 that $f \in L^2(\Omega)$ and $\Gamma \in C^{1,1}$, it can be shown that (see e.g. [28, 29, 19]) the solution $u \in H^2(\Omega^+ \cup \Omega^-)$ and

$$\|u\|_{H^2(\Omega^+\cup\Omega^-)} \leq C_{\beta^\pm}\|f\|_{L^2(\Omega)}, \tag{5.1}$$

where, for $k > 1$,

$$H^k(\Omega^+ \cup \Omega^-) = \left\{u \in H^1(\Omega) \text{ and } u^\pm \in H^k(\Omega^\pm)\right\}$$

and the piecewise H^k–norm is defined by $\|u\|^2_{H^k(\Omega^+\cup\Omega^-)} = \|u\|^2_{H^k(\Omega^+)} + \|u\|^2_{H^k(\Omega^-)}$ for any $u \in H^k(\Omega^+ \cup \Omega^-)$. Next, $u^\pm_E := Eu^\pm \in H^2(\Omega)$ denotes a smooth Sobolev extension that is bounded in the H^2-norm (see e.g., [1]). If there is no danger of confusion, in the following discussion, we shall employ a simple notation for the norms: $\|\cdot\|_{k,D} = \|\cdot\|_{H^k(D)}$ and $\|\cdot\|_{k,D^-\cup D^+} = \|\cdot\|_{H^k(D^-\cup D^+)}$, and the semi-norms similarly.

For a non-interface element K, the local finite element space is simply defined as the linear polynomial space $\mathbb{P}_1(K)$ where the standard Lagrange elements are used. If $K \in \mathcal{T}_h$ is an interface triangle (see Figure 5.1), D and E denote the intersection points of the interface and edges of K, and we let $\Gamma^K_h = DE$. In addition, we let $\mathcal{E}_K$ be the collection of cut segments from the original edges of K, for example $\mathcal{E}_K = \{BC, CE, EA, AD, DB\}$ for the interface element K in Figure 5.1. In other words, we treat K as pentagon.

We define the union of cut segments Γ^K_h of all the interface elements as the approximated interface Γ_h. Γ_h also separates the original domain Ω into two subdomains $\Omega^\pm_h$, in which the $\pm$ are determined by the area overlap with $\Omega^\pm$. Define $\beta_h = \beta^\pm$ on $\Omega^\pm_h$. For each interface triangle K, K_δ is the subset of K such that $\beta \neq \beta_h$ (i.e. mismatch region). Using Figure 5.1 as an example, without loss of generality, $K^+ := \Delta ADE$ and K^- the quadrilateral complement formed by $BCED$, and the relevant definitions and proofs follow similarly when $\pm$ swaps.

In the rest of this chapter, all constants in $\lesssim$ are β dependent but cut point location independent unless stated otherwise.

The stabilization term $S_K(\cdot,\cdot)$ in (4.5) will be defined using a broken 1/2-seminorm on the boundary of each element, which has the sharpest estimate among other choices of stabilization. Let e be a line segment, and for an admissable function w, we define

$$|w|_{1/2,e}^2 := \int_e \int_e \frac{|w(x) - w(y)|^2}{|x-y|^2} \,\mathrm{d}x\,\mathrm{d}y.$$

By direct calculation, 1/2 semi-norm of any linear function on a straight line segment is equivalent to its (weighted) tangential derivative. Specifically, let w be a linear function defined on a line segment e with the endpoints a_e and b_e, then

$$|w|_{1/2,e} = |w(b_e) - w(a_e)| = h_e^{1/2} |w|_{1,e}.$$

Then the broken 1/2-seminorm on ∂K can be defined as,

$$|w|_{1/2,\mathcal{E}_K}^2 := \sum_{e\in\mathcal{E}_K} |w|_{1/2,e}^2. \tag{5.2}$$

Now we review some fundamental estimates that are crucial for our analysis. The following result can be found in [17, 29] which we can use to estimate the mismatch between β and β_h in the following section.

We will also use the following trace theorems and Poincaré inequality.

Theorem 5.1.1 (Poincare inequality [14])**.** *Let P be a polygon with the number of edges* *uniformly bounded,* $\int_{\partial P} w \, \mathrm{d}s = 0$*, then we have*

$$\|w\|_{0,\partial P} \lesssim h_P^{1/2} |w|_{1/2,\mathcal{E}_P},$$

where $\mathcal{E}_P$ *denotes the collection of edges of* P*. In addition, the constant in this inequality is independent of star-shaped constant.*

5.2 Virtual Immersed Spaces

In this section, we first briefly review the linear virtual element space and the IFE space defined on interface elements as well as their properties. Then we describe the associated projection and interpolation operators.

5.2.1 Spaces

For each interface element K, we begin with a virtual element space that involves the interface information:

$$\begin{aligned} V_h(K) = \{ v \ : \ \nabla \cdot (\beta_h \nabla v) = 0, \quad v|_e \in \mathbb{P}_1(e), \ \forall e \in \mathcal{E}_K, \quad v \in C^1(\partial K) \\ v \text{ satisfies the jump conditions on } \Gamma_h^K \}. \end{aligned} \tag{5.3}$$

Clearly $V_h(K) \subseteq H^1(K)$. This space can be understood as a generalization of the usual linear virtual space in the literature [5, 7] to the case of discontinuous coefficients. Then the global space is defined as

$$V_h = \{ v \in H_0^1(\Omega) : v|_K \in V_h(K) \text{ if } K \in \mathcal{T}_h^i \text{ and } v|_K \in \mathbb{P}_1(K) \text{ if } K \in \mathcal{T}_h^n \} \tag{5.4}$$

which is a H^1-conforming space.

However, due to jump conditions involved in (5.3), the standard linear polynomial space $\mathbb{P}_1(K)$ is not an appropriate choice onto which the virtual space (5.3) is projected in a manner of the usual virtual element method [5, 7], since the jump information will be missed. As the linear IFE space consists of piecewise linear polynomials satisfying the jump conditions on Γ_h^K, naturally it can be used as a computable space to project the virtual space (5.3) in computation.

So let us review the linear IFE space. Consider the approximate jump conditions to (4.2) defined on the segment Γ_h^K:

$$\nabla v^+ \cdot \bar{\mathbf{t}} = \nabla v^- \cdot \bar{\mathbf{t}}, \quad \beta^+ \nabla v^+ \cdot \bar{\mathbf{n}} = \beta^- \nabla v^- \cdot \bar{\mathbf{n}}, \tag{5.5}$$

where $\bar{\mathbf{t}}$ and $\bar{\mathbf{n}}$ are the unit tangential vector and normal vectors to Γ_h^K, respectively.

The local IFE space on K is then defined as

$$V_h^{\mathrm{IF}}(K) := \{v_h|_{K^\pm} \in \mathbb{P}_1(K^\pm) : \ v_h \text{ satisfies}(5.5)\} \tag{5.6}$$

It can be immediately proved that the IFE space (5.6) has the dimension three since the jump conditions in (5.5) defines a bijective mapping from one side to another. Besides the dimension, it also shares some other nice properties as the standard linear FE space. For example, the following trace theorem for IFE functions can be found in [22].

Theorem 5.2.1 (Trace theorem for IFE function)**.** *Let e be an edge of K. Then $\forall w \in V_h^{\mathrm{IF}}(K)$, there holds*

- $|e|^{1/2}\|\beta_h w\|_{0,e} \lesssim \frac{\beta^-}{\sqrt{\beta^+}}\|\beta_h \nabla w\|_{0,K}$
- $h^{1/2}\|\beta_h \nabla w\|_{0,\Gamma_K} \lesssim \frac{\beta^-}{\sqrt{\beta^+}}\|\beta_h \nabla w\|_{0,K}$

where the hidden constant is independent to β and cut point location.

It can be shown that the IFE space $V_h^{\mathrm{IF}}(K)$ is unisolvent by the nodal values at vertices of K [21], which gives the Lagrange type IFE shape functions. The nodal value degrees of freedom (DoFs) are then widely used in the IFE literature [21, 24, 31] for both analysis and computation. However, the proof of the unisolvence with respect to nodal DoFs is in general very technical and relies on mesh assumption [25], and the unisolvence may not hold for some problems [23]. However we highlight that both the analysis and implementation of the proposed method do not rely on the nodal value DoFs of the IFE space as it only serves to projecting the underling virtual space. Roughly speaking, the usual nodal IFE shape functions will be replaced by $\Pi_K\phi_{i,h}$, $i = 1, 2, 3, 4, 5$, where $\phi_{i,h}$ are the shape functions of the virtual space associated with the element nodes and interface-cutting points and Π_K is the projection operator defined in (5.9) below. This is one of the major difference of the proposed method from those classical penalty-type IFE works.

In order to compute the projection, one only needs to find a basis of the gradient space of $V_h^{\mathrm{IF}}(K)$ to perform projection of which the explicit form can be written as

$$\mathbf{v}_{h,i} \in \nabla V_h^{IF}(K), \text{ with } \mathbf{v}_{h,i}^- = \mathbf{e}_i \text{ in } K_h^- \text{ and } \mathbf{v}_{h,i}^+ = M\mathbf{e}_i \text{ in } K_h^+, \tag{5.7}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ can be any two linearly independent vectors in $\mathbb{R}^2$, and M is a matrix involving the jump information

$$M = \begin{bmatrix} n_2^2 + \rho n_1^2 & (\rho - 1)n_1 n_2 \\ (\rho - 1)n_1 n_2 & n_1^2 + \rho n_2^2 \end{bmatrix} \tag{5.8}$$

where $\bar{\mathbf{n}} = [n_1, n_2]$, $\bar{\mathbf{t}} = [t_1, t_2]$ and $\rho = \beta^-/\beta^+$. Here the matrix is constructed according to the jump conditions in (5.5). Then each shape function in the virtual element space $V_h(K)$ is projected to the IFE space by these two basis functions through the projection operator described below. It is an advantage in both computation and analysis since it circumvents the issue of the existence of IFE shape functions and makes the implementation more easily.

5.2.2 Projection and Interpolation

For the proposed immersed virtual element method, both the error analysis and computation rely on a weighted Galerkin projection $\Pi_K : H^1(K) \to V_h^{\mathrm{IF}}(K)$:

$$\begin{aligned} &(\beta_h \nabla \Pi_K u, \nabla v_h)_K = (\beta_h \nabla u, \nabla v_h)_K, \quad \forall v_h \in V_h^{\mathrm{IF}}(K), \\ \text{and} \quad &\int_{\partial K} \beta_h^{1/2}(u_h - \Pi_K u_h)\, \mathrm{d}s = 0 \end{aligned} \tag{5.9}$$

This projection exactly mimics the the usual one used in the VEM literature, and the only difference is the non-smooth coefficient β_h.

Similar to standard projection, the approximation result of Π_K follows from the smallest distance property and the approximation of another interpolation operator for IFE functions such as those in [21, 26]. But the analysis in the next Section shows that we need the approximation of each polynomial component of Π_K on the whole element K, which needs some special treatment. For this purpose, we need to use a quasi-IFE interpolation operator introduced in [26] as an intermediate tool. So let us provide its interpolation for readers' sake. For any interface element K, we first define its patch as

$$\omega_K := \bigcup_{T \in \mathcal{T}_h, \overline{K} \cap \overline{T} \neq \emptyset} T.$$

For $u \in H^2(\Omega^- \cup \Omega^+)$ satisfying the jump condition (4.2) with the extension $u_E^{\pm}$, we let $J_K u_E^{\pm}$ be L^2–projection of $u_E^{\pm}$ to $\mathbb{P}_1(\omega_K)$ and let x_0 be the middle point of Γ_h^K. Still, we let $\bar{\mathbf{n}}$ and $\bar{\mathbf{t}}$ be a fixed unit normal and tangential vector to Γ_h^K, respectively. Then, the quasi-IFE interpolation $I_K^{\mathrm{IF}} u$ is given by the following definition

$$\begin{aligned} I_K^{\mathrm{IF},-} u(x_0) &= I_K^{\mathrm{IF},+} u(x_0) = J_K u_E^{+}(x_0), \\ \nabla I_K^{\mathrm{IF},-} u(x_0) \cdot \bar{\mathbf{t}} &= \nabla I_K^{\mathrm{IF},+} u(x_0) \cdot \bar{\mathbf{t}} = \nabla J_K u_E^{+}(x_0) \cdot \bar{\mathbf{t}}, \\ \beta^- \nabla I_K^{\mathrm{IF},-} u(x_0) \cdot \bar{\mathbf{n}} &= \beta^+ \nabla I_K^{\mathrm{IF},+} u(x_0) \cdot \bar{\mathbf{n}} = \beta^- \nabla J_K u_E^{-}(x_0) \cdot \bar{\mathbf{n}}, \end{aligned} \tag{5.10}$$

where $I_K^{\mathrm{IF},\pm} u$ denote the two polynomial components of $I_K^{\mathrm{IF}} u$. It is important to note that these two polynomials can be naturally defined and used on the whole patch ω_K. Moreover, these two polynomials have the desired optimal approximation to their corresponding functions $u_E^{\pm}$ also on the whole patch which is given by the lemma below. This crucial property serves as the key to in our analysis.

Lemma 5.2.2 (Quasi IFE interpolation error estimate [26])**.** *For $u \in H^2(\Omega^- \cup \Omega^+)$ satisfying the jump conditions* (4.2)*, there holds*

$$|u_E^{\pm} - I_K^{\mathrm{IF},\pm} u|_{1,\omega_K} \lesssim h_K (\|u_E^{+}\|_{2,D} + \|u_E^{-}\|_{2,D}). \tag{5.11}$$

In the following discussion, without causing confusion, for any subdomain $D \subseteq \Omega$, we denote

$$\|u_E^{\pm}\|_{2,D} := \|u_E^{+}\|_{2,D} + \|u_E^{-}\|_{2,D}.$$

A similar estimate for $\Pi_K^{\pm}$ can be established on the whole patch ω_K where, again, $\Pi_K^{\pm}$ denote the two polynomial components from the definition. The analysis needs to employ the quasi interpolation $I_K^{IF,\star}$ as the bridge which is postponed to the next section.

5.3 A New Formulation and The Error Equation

In this section, we first present a new method to solve the elliptic interface problem based on the IFE method based and the virtual element method.

5.3.1 A Virtual Immersed Scheme

The key idea is to use the virtual element space defined on the background mesh for approximation and project the virtual functions to the IFE spaces on interface elements for projections as the immersed elements described above satisfy the jump conditions thus offering sufficient approximation locally. Consequently, we define the local discrete bilinear form on an interface element K as: $a_h^K(\cdot,\cdot) : H^1(K) \times H^1(K) \to \mathbb{R}$ where

$$a_h^K(u_h, v_h) := (\beta_h \nabla \Pi_K u_h, \nabla \Pi_K v_h)_K + S_K(u_h - \Pi_K u_h, v_h - \Pi_K v_h). \tag{5.12}$$

The stabilization term above $S_K(\cdot,\cdot)$ is

$$S_K(Y, Z) := \sum_{e \in \mathcal{E}_K} \beta_e (Y, Z)_{1/2,e}$$

where $Y = u_h - \Pi_K u_h$ and $Z = v_h - \Pi_K v_h$, $\beta_e = \beta^{\pm}$ depends on $e \subset K^{\pm}$, and

$$(Y, Z)_{1/2,e} := \int_e \int_e \frac{(Y(x) - Y(y))(Z(x) - Z(y))}{|z - y|^2} \, \mathrm{d}x \, \mathrm{d}y.$$

Since both Y and Z are linear functions on each e, a simple formulation of the stabilization term can be obtained:

$$S_K(Y, Z) = \sum_{e \in \mathcal{E}_K} \beta_e (Y(b_e) - Y(a_e))(Z(b_e) - Z(a_e)). \tag{5.13}$$

For each non-interface element, Π_K is simply the identity operator and S_K vanishes since the stabilization is applied on $(I - \Pi_K)$. Therefore, the bilinear form on non-interface elements reduces to

$$a_h^K(u_h, v_h) := (\beta_h \nabla u_h, \nabla v_h)_K.$$

As a result, some key estimates on non-interface elements fall into the standard FEM regime, as such, the results on these elements will be omitted, and the focus is on the interface elements. In the rest of this section, on each element K, the notions of Π_K and S_K, regardless of being interface element or not, are adopted to maintain a consistent and concise set of notations.

The proposed virtual immersed scheme is to find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h) = \sum_{K \in \mathcal{T}_h} (f, \Pi_K v_h)_K. \tag{5.14}$$

There are several major differences of the proposed virtual immersed scheme above from the classic penalty-type IFE scheme [31]. First, the proposed scheme does not require those edge terms originated from the integration parts, but only the stabilization term. As a result, the discretization is parameter-free, and yields a symmetric system which can be solved by fast linear solvers. Second, it does not need the interaction between two neighbor elements such that the computation is more parallelizable. However, compared with the classical IFE, there are more DoFs locally on each interface element, of which the extra are associated with the cutting points.

5.3.2 Error representation

Now we proceed to derive an error equation for the numerical solution u_h. Given each $u \in H^2(\Omega^- \cup \Omega^+)$, denote u_I as the interpolation of u in V_h. Namely, it is the standard linear

Lagrange interpolant on non-interface elements, but a special non-polynomial function on interface elements. Due to the conforming property of the virtual elements, there always holds $u_I \in H^1(\Omega)$. To estimate the error sourcing from different terms, we shall derive an error equation for $u_h - u_I$ under the bilinear form induced norm $\|\!|\cdot|\!\|^2 := a_h(\cdot,\cdot)$.

Lemma 5.3.1 (Error equation)**.** *Let $u \in H^2(\Omega^- \cup \Omega^+)$ satisfy the jump condition* (4.2)*, and denote $v_h = u_h - u_I$, then the following identity holds*

$$
\begin{aligned}
\|\!|v_h|\!\|^2 = &\sum_{K\in\mathcal{T}_h} (\beta_h \nabla \Pi_K (u - u_I), \nabla \Pi_K v_h)_K \\
&+ (\beta_h \nabla (u - \Pi_K u)\cdot \mathbf{n}, v_h - \Pi_K v_h)_{\partial K} - S_K(u_I - \Pi_K u_I, v_h - \Pi_K v_h) \\
&+ ((\beta - \beta_h)\nabla u, \nabla \Pi_K v_h)_K.
\end{aligned}
\tag{5.15}
$$

Proof. We start by using the equation $-\nabla\cdot(\beta\nabla u) = f$ in $\Omega^-\cup\Omega^+$ to obtain

$$
\begin{aligned}
\|\!|v_h|\!\|^2 =& a_h(u_h - u_I, v_h) = a_h(u_h, v_h) - a_h(u_I, v_h) \\
=& \sum_{K\in\mathcal{T}_h} (f, \Pi_K v_h)_K - a_h(u_I, v_h) \\
=& \sum_{K\in\mathcal{T}_h} (-\nabla\cdot(\beta\nabla u), \Pi_K v_h)_K - a_h(u_I, v_h) \\
=& \sum_{K\in\mathcal{T}_h} [\underbrace{(\beta\nabla u, \nabla\Pi_K v_h)_K}_{(I)} - \underbrace{(\beta\nabla u\cdot\mathbf{n}, \Pi_K v_h)_{\partial K}}_{(II)}] - a_h(u_I, v_h)
\end{aligned}
\tag{5.16}
$$

where in the last identity we have also used the integration by parts on each subelement $K^\pm$, the flux jump conditions of u and the continuity of $\Pi_K v_h$ on K. For the term (I) in (5.16), using the definition of Π_K we have

$$
\begin{aligned}
(I) &= (\beta_h\nabla u, \nabla\Pi_K v_h)_K + ((\beta - \beta_h)\nabla u, \nabla\Pi_K v_h)_K \\
&= (\beta_h\nabla\Pi_K u, \nabla\Pi_K v_h)_K + ((\beta-\beta_h)\nabla u, \nabla\Pi_K v_h)_K.
\end{aligned}
\tag{5.17}
$$

For the term (II), since it is defined on ∂K, β exactly matches β_h, and thus we obtain

$$(II) = (\beta_h \nabla u \cdot \mathbf{n}, \Pi_K v_h)_{\partial K} = (\beta_h \nabla u \cdot \mathbf{n}, \Pi_K v_h - v_h)_{\partial K}. \tag{5.18}$$

where in the second identity we have used $v_h = u_h - u_I$ being continuous across each edge as it is in the virtual element space V_h. Using integration by parts on the subelements $K_h^{\pm}$, the flux jump conditions of the IFE functions on Γ_h^K, $v_h - \Pi_K v_h$ be continuous across Γ_h^K, and definition the projection Π_K in (5.9), we have

$$\begin{aligned}
(\beta_h \nabla \Pi_K u \cdot \mathbf{n}, v_h - \Pi_K v_h)_{\partial K} &= \sum_{s=\pm} (\beta_h \nabla \Pi_K u \cdot \mathbf{n}, v_h - \Pi_K v_h)_{\partial K_h^s} \\
&= \sum_{s=\pm} (\beta_h \nabla \Pi_K u, \nabla (v_h - \Pi_K v_h))_{K_h} \\
&= 0.
\end{aligned} \tag{5.19}$$

Thus, (5.18) further becomes

$$(II) = (\beta_h \nabla (u - \Pi_K u) \cdot \mathbf{n}, \Pi_K v_h - v_h)_{\partial K}. \tag{5.20}$$

Putting (5.17) and (5.20) into (5.16), and using the format of $a_h(u_I, v_h)$, we obtain the desired result.

□

In the derivation above, there are two steps involving integration by parts in which the one in (5.16) is for the exact solution u with respect to the subelements $K^{\pm}$ and another one in (5.19) is for virtual and IFE functions with respect to the subelements $K_h^{\pm}$. The different manners are according to their corresponding jump conditions imposed on Γ or Γ_h^K such that those extra terms occurring on Γ or Γ_h^K can be cancelled.

5.4 Error Estimates

In this section, we proceed to estimate the solution errors. Based on the error equation in Lemma 5.3.1, we first get the bound of the error $u_h - u_I$.

Theorem 5.4.1 (A priori error bound)**.** *Let $u \in H^2(\Omega^- \cup \Omega^+)$ satisfy the jump conditions in* (4.2)*, then the following estimate holds*

$$\begin{aligned} \|\!|v_h|\!\| \lesssim \sum_{K\in\mathcal{T}_h} [\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K} \\ + h_K^{1/2}\|\beta_h^{1/2}\nabla(u-\Pi_K u)\cdot\mathbf{n}\|_{0,\partial K} + |\beta_h^{1/2}(u_I-\Pi_K u_I)|_{1/2,\mathcal{E}_K} \\ + \|\beta_{\max}^{1/2}\nabla u\|_{0,K_\delta}]. \end{aligned} \tag{5.21}$$

Proof. Note that $\beta \neq \beta_h$ only on K_δ. So for the error equation in Lemma 5.3.1, applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\!|v_h|\!\|^2 \leq \sum_{K\in\mathcal{T}_h} \Big(\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K}\|\beta_h^{1/2}\nabla\Pi_K v_h\|_{0,K} \\ + \|\beta_h^{1/2}\nabla(u-\Pi_K u)\cdot\mathbf{n}\|_{0,\partial K}\|\beta_h^{1/2}(v_h-\Pi_K v_h)\|_{0,\partial K} \\ + |\beta_h^{1/2}(u_I-\Pi_K u_I)|_{1/2,\mathcal{E}_K}|\beta_h^{1/2}(v_h-\Pi_K v_h)|_{1/2,\mathcal{E}_K} \\ + \|\beta_{\max}^{1/2}\nabla u\|_{0,K_\delta}\|\beta_{\max}^{1/2}\nabla\Pi_K v_h\|_{0,K}\Big). \end{aligned} \tag{5.22}$$

In the bound above, it is clear that $\|\beta_h^{1/2}\nabla\Pi_K v_h\|_{0,K}$ and $|\beta_h^{1/2}(v_h-\Pi_K v_h)|_{1/2,\mathcal{E}_K}$ are bounded above by $\|\!|v_h|\!\|$, and $\|\beta_{\max}^{1/2}\nabla\Pi_K v_h\|_{0,K}$ is also bounded above by $\|\!|v_h|\!\|$ with a β dependent constant.

Then it remains to estimate the second term in (5.22). Note that $\int_{\partial K}\beta_h^{1/2}(v_h-\Pi_K v_h)ds = 0$. So by Theorem **??**, there holds

$$\|\beta_h^{1/2}(v_h-\Pi_K v_h)\|_{0,\partial K} \lesssim h_K^{1/2}|\beta_h^{1/2}(v_h-\Pi_K v_h)|_{1/2,\mathcal{E}_K} \lesssim h_K^{1/2}\|\!|v_h|\!\|.$$

Combining the estimates above and cancelling out a $|||v_h|||$ from the left hand side, we get the desired a priori estimate. □

Now our task is to estimate each term in the right hand side of the error bound (5.21). Before getting into the estimate, we emphasize that the set $\mathcal{E}_K$ consists of the edges formed by element vertices and cut points. So in the following discussion, for each edge $e \in \mathcal{E}_K$ that connects the element vertices and cut points, we will use $\hat{e}$ to denote the extension of e to the actual edge of the triangle K (e.g. $e = \overline{AD}$ to $\hat{e} = \overline{AB}$). Now, let us first dervie the estimate of the first term in the right hand side of the error bound in (5.21).

Lemma 5.4.2. *Let $u \in H^2(\Omega^- \cup \Omega^+)$ satisfy the jump conditions in (4.2), then on each interface element K there holds*

$$\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K} \lesssim h_K\|u_E^{\pm}\|_{2,\omega_K} \tag{5.23}$$

Proof. By the definition of projection, we immediately have

$$\begin{aligned}\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K}^2 =&(\beta_h\nabla\Pi_K(u-u_I), \nabla\Pi_K(u-u_I))_K\\ =&(\beta_h\nabla\Pi_K(u-u_I), \nabla(u-u_I))_K.\end{aligned}$$

Using integration by parts on the subelements $K_h^{\pm}$, $\Pi_k(u-u_I)$ satisfying the jump condition on Γ_K and $u-u_I$ being H^1, we have

$$\begin{aligned}\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K}^2 =&(\beta_h\nabla\Pi_K(u-u_I)\cdot\mathbf{n}, u-u_I)_{\partial K}\\ \leq&\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\cdot\mathbf{n}\|_{0,\partial K}\|\beta_h^{1/2}(u-u_I)\|_{0,\partial K}.\end{aligned} \tag{5.24}$$

For each edge on ∂K, applying the IFE trace inequality in (5.2.1), we obtain

$$\begin{aligned}\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\cdot\mathbf{n}\|_{0,e} \leq& \|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\cdot\mathbf{n}\|_{0,\hat{e}}\\ \lesssim& h_K^{-1/2}\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K}.\end{aligned} \tag{5.25}$$

Putting (5.25) into (5.24) and cancelling out the term $\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K}$ leads to

$$\|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K} \lesssim h_K^{-1/2}\|\beta_h^{1/2}(u-u_I)\|_{0,\partial K}. \tag{5.26}$$

So it remains to estimate the right-hand side above. Notice β_h is constant on each edge in $\mathcal{E}_K$. Without loss of generality, we focus on an edge $e \subseteq K^+$. Then by the interpolation estimate on this edge, we have

$$\|\beta_h^{1/2}(u-u_I)\|_{0,e} \lesssim h_e^{3/2}|u|_{3/2,e} \lesssim h_K^{3/2}|u_E^+|_{3/2,\hat{e}} \lesssim h_K^{3/2}|u_E^+|_{2,K} \tag{5.27}$$

where in the last inequality, we have also applied the trace inequality in [14, Lemma 6.2]. Putting (5.27) into (5.26) gives the desired estimate on this edge. Similar arguments apply to the case $e \subset \partial K^-$ which together finishes the proof.

□

To estimate the second and third term in the right-hand side of the error bound (5.21), we need to establish the estimate for Π_K, particularly its every polynomial component $\Pi_K^{\pm}$ on the whole element K.

Lemma 5.4.3. *For $u \in H^2(\Omega^- \cup \Omega^+)$ satisfying the jump conditions* (4.2), *then on each interface element K there holds*

$$|u_E^{\pm} - \Pi_K^{\pm}u|_{1,K} \lesssim h_K\left(\|u_E^+\|_{2,\omega_K} + \|u_E^-\|_{2,\omega_K}\right) + |u_E^+|_{1,K_\delta} + |u_E^-|_{1,K_\delta}. \tag{5.28}$$

Proof. By the jump conditions on Γ_h^K and employing the matrix in (5.8), we first have the following identity for gradients of an IFE function $v_h \in V_h^{IF}(K)$:

$$\nabla v_h^+ = M\nabla v_h^-.$$

It clearly shows

$$\|\nabla v_h^+\| \simeq \|\nabla v_h^-\| \tag{5.29}$$

where $\|\cdot\|$ are just Euclidean norms for vectors, and the hidden constant depends on β. Now let us move to the estimate in (5.28), and without loss of generality, we only discuss the $+$ piece. We have the following trivial split

$$|u_E^+ - \Pi_K^+ u|_{1,K} \lesssim \underbrace{|u_E^+ - \Pi_K^+ u|_{1,K_h^+}}_{(I)} + \underbrace{|u_E^+ - \Pi_K^+ u|_{1,K_h^-}}_{(II)}. \tag{5.30}$$

The estimate for (I) follows from inserting u

$$\begin{aligned} |u_E^+ - \Pi_K^+ u|_{1,K_h^+} &\lesssim |u - \Pi_K^+ u|_{1,K_h^+} + |u - u_E^+|_{1,K_h^+} \\ &\lesssim |u - I_h^{IF,+} u|_{1,K} + |u|_{1,K_\delta} \\ &\lesssim h_K(\|u_E^+\|_{2,\omega_K} + \|u_E^-\|_{2,\omega_K}) + |u_E^+|_{1,K_\delta} + |u_E^-|_{1,K_\delta} \end{aligned} \tag{5.31}$$

where in the second inequality we have used the smallest distance property for Π_K under energy norm which is equivalent to the $|\cdot|_{1,K}$ norm. The more difficult one is (II) as we need to analyze it on K_h^-. The idea is to insert the quasi IFE interpolation I_h^{IF} as it already admits the desired estimate for each component:

$$|u_E^+ - \Pi_K^+ u|_{1,K_h^-} \leq \underbrace{|u_E^+ - I_h^{\mathrm{IF},+} u|_{1,K_h^-}}_{(IIa)} + \underbrace{|I_h^{IF,+} u - \Pi_K^+ u|_{1,K_h^-}}_{(IIb)}. \tag{5.32}$$

(IIa) immediately follows from the approximation capabilities of I_h^{IF} in Lemma 5.2.2. For (IIb), noticing $I_h^{IF,+} u - \Pi_K^+ u$ is one polynomial component of the IFE function $I_h^{IF} u - \Pi_K u$.

So applying the equivalence in (5.29) together with the triangular inequality, we obtain

$$
\begin{aligned}
(IIb) &\lesssim |I_h^{IF,-}u - \Pi_K^- u|_{1,K_h^-} \\
&\lesssim |I_h^{IF,-}u - u|_{1,K_h^-} + |u - \Pi_K^- u|_{1,K_h^-} \\
&\lesssim h_K(\|u_E^+\|_{2,\omega_K} + \|u_E^-\|_{2,\omega_K}) + |u_E^+|_{1,K_\delta} + |u_E^-|_{1,K_\delta}
\end{aligned} \tag{5.33}
$$

where the argument for the second and third inequalities are similar to (5.31). Putting (5.31)-(5.33) into (5.30), we have the desired result. □

Next, we estimate the second term in the right-hand side of the error bound (5.21).

Lemma 5.4.4. *For $u \in H^2(\Omega^- \cup \Omega^+)$ satisfying the jump conditions (4.2), then on each interface element K there holds*

$$
\|\beta_h^{1/2}\nabla(u - \Pi_K u)\cdot \mathbf{n}\|_{0,\partial K} \lesssim h_K^{1/2}\|u_E^\pm\|_{2,\omega_K} + h_K^{-1/2}|u_E^\pm|_{1,K_\delta}. \tag{5.34}
$$

Proof. Without loss of generality, we only consider + side. Given an edge $e \in \mathcal{E}_K$ with $e \subseteq K_h^+$ and its extension $\hat{e}$ as an edge of K, we apply the trace inequality to obtain

$$
\begin{aligned}
\|\beta_h\nabla(u - \Pi_K u)\cdot\mathbf{n}\|_{0,e} &\leq (\beta^+)^{1/2}\|\nabla(u_E^+ - \Pi_K^+ u)\cdot\mathbf{n}\|_{0,\hat{e}} \\
&\lesssim h_K^{-1/2}|u_E^+ - \Pi_K^+ u|_{1,K} + h_K^{1/2}|u_E^+|_{2,K}
\end{aligned}
$$

which yields the desired result by Lemma 5.4.3.

□

Then we estimate the third term in the right-hand side of the error bound (5.21).

Lemma 5.4.5. *For $u \in H^2(\Omega^- \cup \Omega^+)$ satisfying the jump conditions (4.2), then on each* *interface element K there holds*

$$|\beta_h^{1/2}(u_I - \Pi_K u_I)|_{1/2,\mathcal{E}_K} \lesssim h_K \|u_E^{\pm}\|_{2,\omega_K} + |u_E^{\pm}|_{1,K_\delta}. \tag{5.35}$$

Proof. It suffices to establish an edge-wise bound $|\cdot|_{1/2,e}$ by (5.2). For each edge, since β_h is a constant, we have

$$|\beta_h^{1/2}(u_I - \Pi_K u_I)|_{1/2,e} \lesssim \underbrace{|u_I - \Pi_K u|_{1/2,e}}_{(I)} + \underbrace{|\Pi_K(u - u_I)|_{1/2,e}}_{(II)}.$$

In the following discussion, without loss of generality we only consider the $e \subseteq K_h^+$. For (I), since it is linear on e, and u and u_I match at the end points a_e and b_e of e, we obtain

$$\begin{aligned}(I) = &\left|(u_I - \Pi_K^+ u)|_{a_e}^{b_e}\right| = \left|(u - \Pi_K^+ u)|_{a_e}^{b_e}\right| \\ = &\left|\int_e \partial_e(u - \Pi_K^+ u)\,\mathrm{d}s\right| \leq h_e^{1/2}\left\|\partial_e(u - \Pi_K^+ u)\right\|_{0,e} \\ \leq &\, h_e^{1/2}|u - \Pi_K^+ u|_{1,e}.\end{aligned} \tag{5.36}$$

Replacing u by its extension u_E^+ and recalling that $\Pi_K^+ u$ is a polynomial being trivially used on the whole element K, we apply the standard trace inequality and Lemma 5.4.3 to get

$$\begin{aligned}(I) &\leq h_K^{1/2}|u_E^+ - \Pi_K^+ u|_{1,\hat{e}} \lesssim |u_E^+ - \Pi_K^+ u|_{1,K} \\ &\lesssim h_K\|u_E^{\pm}\|_{2,\omega_K} + |u_E^{\pm}|_{1,K_\delta}.\end{aligned} \tag{5.37}$$

For (II), applying the trace inequality for IFE functions in Theorem 5.2.1 and Lemma 5.4.2, we obtain

$$
\begin{aligned}
(II) = &\left|\Pi_K(u-u_I)|_{a_e}^{b_e}\right| = \left|\int_e \partial_e \Pi_K(u-u_I)\,\mathrm{d}s\right| \\
\leq & h_e^{1/2}|\Pi_K(u-u_I)|_{1,\hat{e}} \lesssim h_K^{-1/2} h_e^{1/2} |\Pi_K(u-u_I)|_{1,K} \\
\leq & h_K \|u_E^{\pm}\|_{2,\omega_K}.
\end{aligned} \tag{5.38}
$$

Here we note that $\Pi_K(u-u_I)$ is a piecewise polynomial used on the whole element and thus the standard trace inequality is not applicable; instead we need to use the trace inequalities for IFE functions in Theorem 5.2.1. Combining the estimates of (I) and (II), we have the desired result. □

Remark 5.4.6. *In the analysis of classical VEM on anisotropic elements [14], the main difficulty is to obtain an error bound that is independent of element anisotropy such as shrinking elements. We highlight that one of the key obstacles for anisotropic analysis is the failure of the standard trace inequalities as the height supporting an edge may very small. For example for the present situation, in the estimation of* (5.25) *and* (5.38)*, the standard trace inequality can not be applied directly to each polynomial on each subelement as it may shrink, and thus the hidden constant may not be uniform with respect h anymore. So the estimation for VEM generally requires some special analysis techniques such as the Poincaré inequality on an anisotropic cut element developed in [14]. However, we note that these special treatments are not needed in the proposed method and analysis since the IFE functions even as piecewise polynomial do admit the trace inequalities on interface elements, and the constants are independent of cut points as shown in Theorem 5.2.1. These trace inequalities actually significantly simplify the analysis as it is more close the the analysis on isotropic elements.*

Combining the results of Lemma 5.4.2, 5.4.4 and 5.4.5 and the error bound in Theorem 5.4.1, we achieve the following conclusion.

Theorem 5.4.7. *Let $u \in H^2(\Omega^- \cup \Omega^+)$ satisfy the jump conditions and u_h the IFE-VEM solution, then the following estimate holds*

$$\|\!|u - u_h\|\!| \lesssim h\|u\|_{2,\Omega^-\cup\Omega^+}. \tag{5.39}$$

Proof. Triangle inequality yields $\|\!|u-u_h\|\!| \leq \|\!|u-u_I\|\!| + \|\!|u_I-u_h\|\!|$. For $\|\!|u_I-u_h\|\!|$, combining the results of Lemma 5.4.2, 5.4.4 and 5.4.5 and the error bound in Theorem 5.4.1, we have

$$\begin{aligned} \|\!|u_I - u_h\|\!| &\lesssim \sum_{K\in\mathcal{T}_h^n} h_K\|u\|_{2,K} + \sum_{K\in\mathcal{T}_h^i} \left(h_K\|u_E^{\pm}\|_{2,\omega_K} + |u_E^{\pm}|_{1,K_\delta}\right) \\ &\lesssim h\|u_E^{\pm}\|_{2,\Omega} \lesssim h\|u\|_{2,\Omega^-\cup\Omega^+}. \end{aligned} \tag{5.40}$$

where we have used the finite overlapping property of ω_K and the strip argument in Lemma **??** to control $|u|_{1,K_\delta}$ and finally the boundedness for Sobolev extensions.

Then we proceed to estimate $\|\!|u - u_I\|\!|$. Since it is trivial on non-interface elements, we only need to estimate it on interface elements. By the triangular inequality, we have

$$\|\!|u - u_I\|\!| \lesssim \sum_{K\in\mathcal{T}_h^i} \|\beta_h^{1/2}\nabla\Pi_K(u-u_I)\|_{0,K} + |u-u_I|_{1/2,\mathcal{E}_K} \tag{5.41}$$

The first term can be handled by Lemmas 5.4.2. For the third term, given $e \in \mathcal{E}_K$ and without loss of generality assuming it is K_h^+, by the interpolation estimate in 1D and the standard trace inequality on K, we have

$$|u - u_I|_{1/2,e} \lesssim h_e|u|_{3/2,e} \lesssim h_e|u_E^+|_{3/2,\hat{e}} \lesssim h_K\|u_E^+\|_{2,K} \tag{5.42}$$

where $\hat{e}$ is the extension of e. Putting (5.42) to (5.41) and applying the boundedness for Sobolev extensions, we have the desired result.

□

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