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SPECTRUM AND EIGENFUNCTIONS FOR A HAMILTONIAN WITH STOCHASTIC TRAJECTORIES

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SPECTRUM AND EIGENFUNCTIONS FOR A HAMILTONIAN WITH STOCHASTIC TRAJECTORIES*

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Abstract

Quantum stochasticity (the nature of wave functions and eigenvalues when the short-wave-limit Hamiltonian has stochastic trajectories) is studied for the two-dimensional Helmholtz equation with "stadium" boundary. The eigenvalue separations have a Wigner distribution (characteristic of a random Hamiltonian), in contrast to the clustering found for a separable equation. The eigenfunctions exhibit a random pattern for the nodal curves, with isotropic distribution of local wave-vectors.

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The current interest[1] in classical systems whose Hamiltonians have stochastic trajectories leads naturally to the question of how this stochasticity manifests itself in the corresponding quantum system. In a broader context, one may inquire into the nature of the solutions of wave equations (arising, e.g., in plasma physics, optics, acoustics, oceanography) whose ray trajectories (WKB solution, geometric optics) are stochastic.[2]

Studies in this area have considered either time-dependent Hamiltonians with one degree of freedom,[3-5] or time-independent Hamiltonians with two degrees of freedom. In the latter case, the work of Percival[6] and Pomphrey[7] indicates that the eigenvalues are sensitive to parameter variation, while Berry[8-10] and Tabor[10] and Zaslavskii[13] predict the following:

(1) The distribution of successive eigenvalue spacings is peaked about a finite value, as it is for a random matrix[11], rather than having its maximum at zero separation, which represents the clustering of eigenvalues characteristic of integrable Hamiltonians[10].

(2) The coarse-grained Wigner function (or local Fourier transform) for an eigenfunction is isotropic[8,9] in $\vec{k}$-space for any position in $\vec{x}$-space, in contrast to the ordered anisotropy characterizing an integrable Hamiltonian[8,12].

In this Letter we report our test of these two predictions. For the Hamiltonian to be studied, we choose a free particle (in two dimensions) confined in a stadium (or racetrack) boundary (see Fig. 1). This system is particularly simple classically[14], since it is stochastic for all nonzero values of the aspect ratio $\gamma \equiv a/R$ (a = half-length of straight side, R = radius of semicircle), with the degree of stochasticity increasing [see Fig. 4 of Ref. 14] from zero at $\gamma = 0$ (the circle) to a flat maximum near $\gamma = 1$ (the stadium of our Fig. 1).

The quantum problem[15] for a free particle is just the Helmholtz equation, $(\nabla^2 + k^2)\psi(\vec{x}) = 0$, with the energy eigenvalue $E = k^2$ for $\hbar^2/2m = 1$. The boundary condition $\psi = 0$ at the stadium "wall" is the same as for a vibrating membrane with clamped edge. To solve the Helmholtz equation numerically for its eigenvalues and eigenfunctions, at fixed aspect ratio, we use the algorithm of Lepore and Riddell[16]. For a reliability test, we use the circle ($\gamma = 0$) and the known mean density of eigenvalues[17] for $\gamma = 0$.

Because the Hamiltonian is invariant under reflection in x or y, we consider only the set of eigenfunctions of odd-odd parity, i.e., $\psi = 0$ at the boundary of the stadium-quadrant of Fig. 1. For nonzero aspect ratio, we adjust the absolute dimension to keep the quadrant area constant (at $\pi/4$), so that the asymptotic mean level spacing is independent of γ.

In Fig. 1 we exhibit a typical eigenfunction, corresponding to the eigenvalue $k = 50.158$, at $\gamma = 1$. The nodal curves are seen to be irregular in direction, verifying the second prediction of Berry. Their separation is roughly regular, representing the half-wave length π/k. There are no nodal crossings in the interior, since saddle points at the special value $\psi = 0$ would occur only at special γ values. We have not computed the coarse-grained Wigner function, since we feel that the qualitative question of local isotropy can be judged by eye.

The distribution of eigenvalue spacings ΔE is one statistical measure of the spectrum. Histograms are shown in Fig. 2 for the circle, and in Fig. 3 for the ($\gamma = 1$) stadium. They are seen to be strikingly different, in confirmation of the first prediction of Berry and Tabor. For the circle, the distribution is roughly exponential; small spacings are the most probable, the smallest found being $\Delta E = 0.003$ (!); large spacings (several times the mean) are also found. Hence the eigenvalue spectrum is highly clustered. For the stadium, on the other hand, small spacings are less probable, the smallest being $\Delta E = 1.69$; also large spacings are improbable. The spectrum exhibits apparent mutual repulsion of eigenvalues, as predicted by Zaslavskii[13], near the mean.

In conclusion, we have shown that the eigenvalue spectrum and eigenfunctions of a linear operator whose (short-wave-limit) rays are stochastic exhibit, respectively, mutual repulsion of neighboring eigenvalues and random directionality of nodal curves.

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References

1. For a clear and up-to-date review, see M. V. Berry, Regular and Irregular Motion, in Topics in Nonlinear Dynamics (S. Jorna, ed., AIP, New York 1978).
2. J.-M. Wersinger, E. Ott, and J.M. Finn, Phys. Fluids 21, 2263(1978).
3. G. Casati, B.V. Chirikov, J. Ford, and F.M. Izraelev, in Stochastic Behavior in Classical and Quantum Hamiltonian Systems (G. Casati and J. Ford, eds., Lecture Notes in Physics, Springer, New York 1979).
4. G.P. Berman and G.M. Zaslavskii, Physica 91A, 450 (1978).
5. M.V. Berry, N.L. Balazs, M. Tabor, and A. Voros, Quantum Maps, submitted to Ann. Phys. (NY) (Oct. 78).
6. I.C. Percival, Semiclassical Theory of Bound States, in Advances in Chemical Physics 36, 1(1977).
7. N. Pomphrey, J. Phys. B7, 1909 (1974).
8. M.V. Berry, Phil. Trans. Roy. Soc. (London) A287, 237(1977).
9. M.V. Berry, J. Phys. A10, 2083(1977).
10. M.V. Berry and M. Tabor, Proc. Roy. Soc. (in press). Level Clustering in the Regular Spectrum,
11. For a clear derivation, see A. Bohr and B. Mottelson, Nuclear Structure (Benjamin, New York, 1969), Appendix 2C.
12. J. Keller and S. Rubinow, Ann. Phys. (NY) 9, 24(1960).
13. G.M. Zaslavskii, Zh. Eksp. Teor. Fiz. 73, 2089 (1977) [Sov. Phys. J.E.T.P. 46, 1094 (1977)].
14. G. Benettin and J.-M. Strelcyn, Phys. Rev. A 17, 773(1978).

15. We are informed that G. Casati, I. Guarneri, and F. Valz-Gris are studying the same problem.

16. J.V. Lepore and R.J. Riddell, Jr., U. of Calif. Report LBL-3036 (1974).

17. H.P. Baltes and E.R. Hilf, Spectra of Finite Systems, (Bibliographisches Institut, Mannheim, 1976.)

Figure 1.

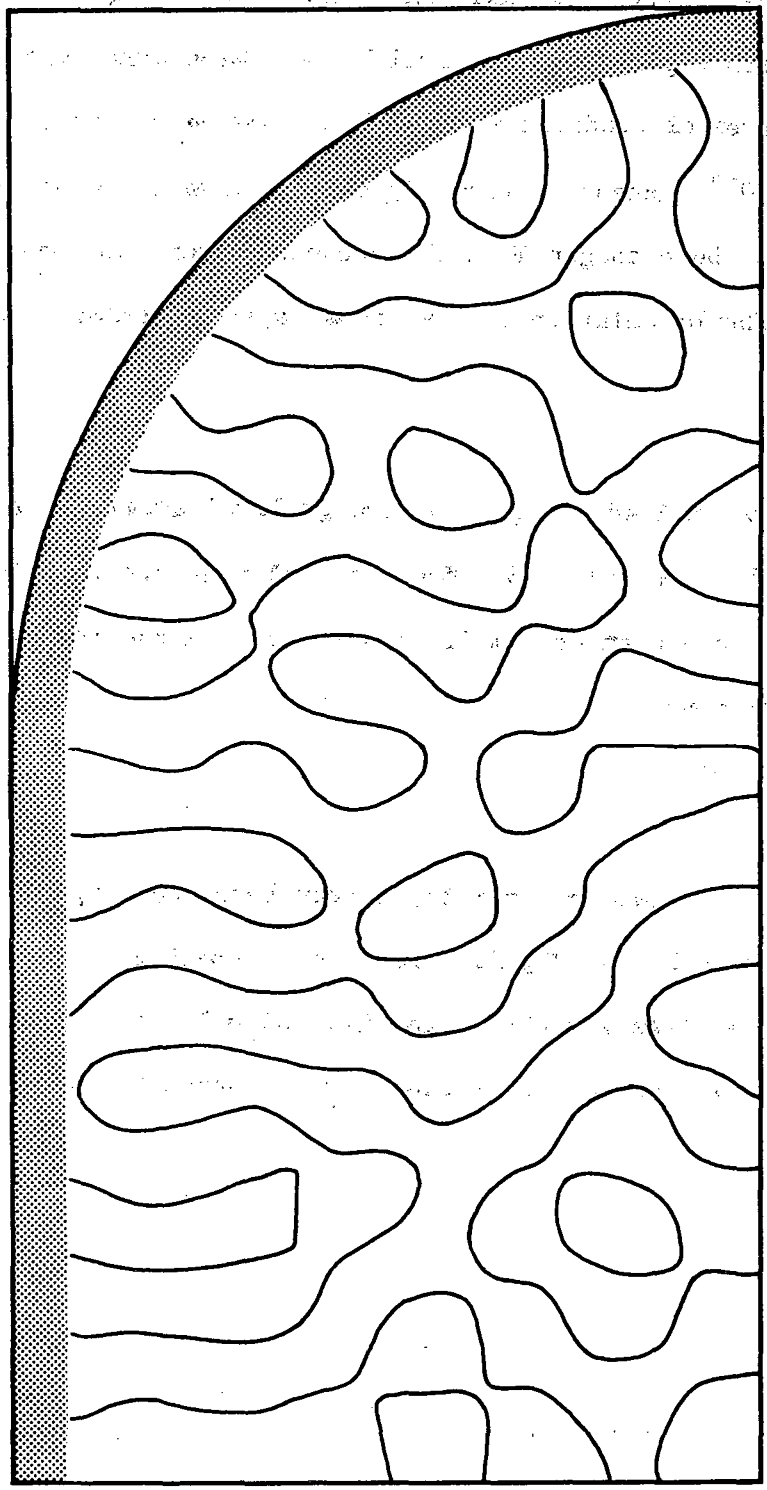
Nodal curves [$\psi(x,y) = 0$] for one quadrant of the (odd-odd parity) eigenfunction with eigenvalue $k = 50.158$, in the stadium with dimensions $a = R = 0.665$ (area of quadrant = $\pi/4$). The relative accuracy of the eigenfunction is $\sim 10^{-4}$, except in the stipled band along the boundary. The nodal curves must be orthogonal to the boundary; there are no crossings in the interior. The orientation of the curves appears quite random.

Figure 2.

Distribution of (odd-odd parity) energy level spacings, for the range $50 < k < 100$ ($2500 < E < 10{,}000$), for a circular boundary. The histogram bin size is 4. Note that the smallest spacings are the most frequent, indicating clustering.

Figure 3.

Distribution of (odd-odd parity) energy level spacings, for the range $50 < k < 70$ ($2500 < E < 4900$), for the $\gamma = 1$ stadium boundary. Bin size = 4. For $\Delta E < 4$, detailed histogram with $\Delta E = 1$ shows absence of separations with $\Delta E < 1$. Energy eigenvalues are computed to an absolute accuracy ± 0.2.



XBL 791-32

Figure 1

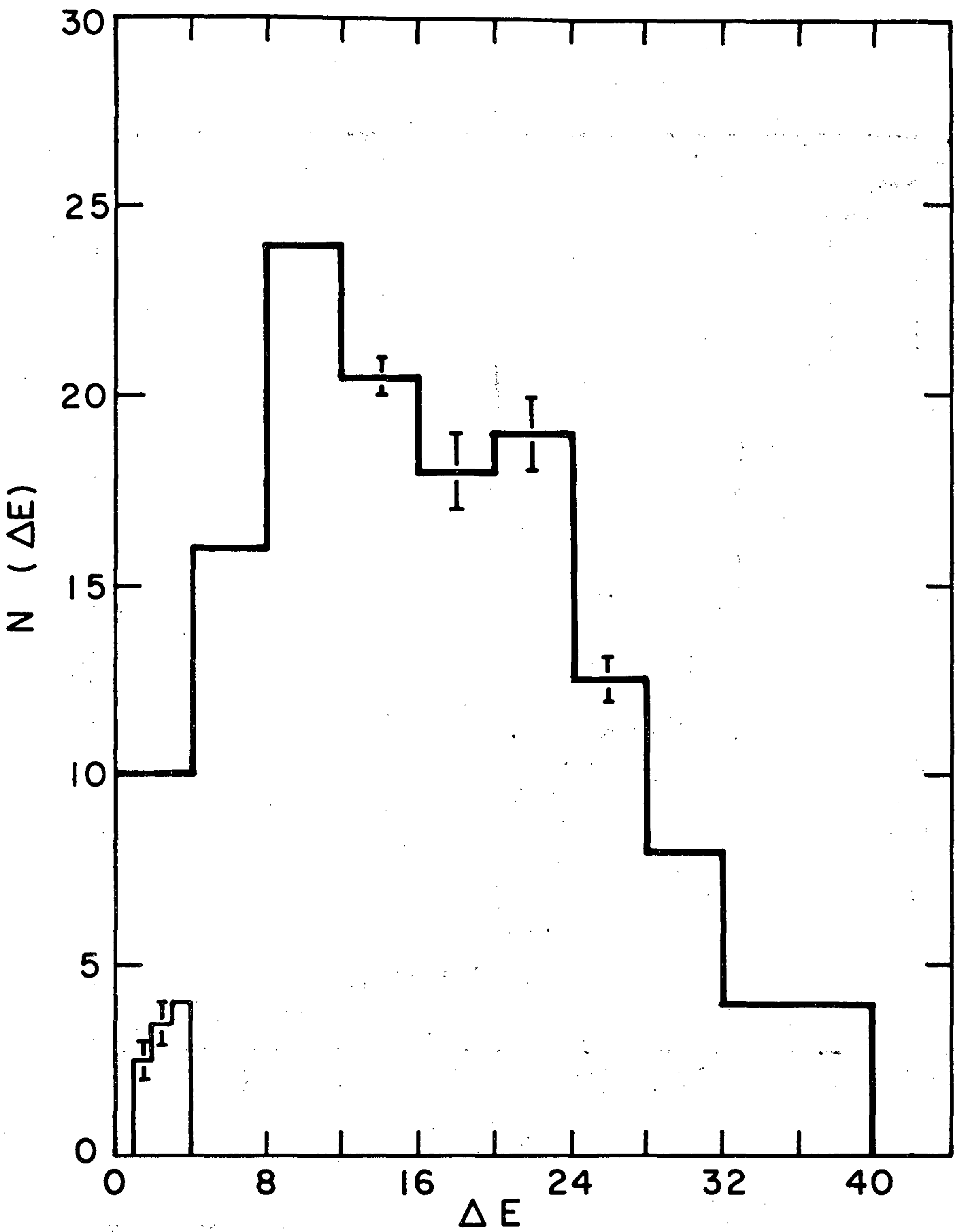


Figure 2

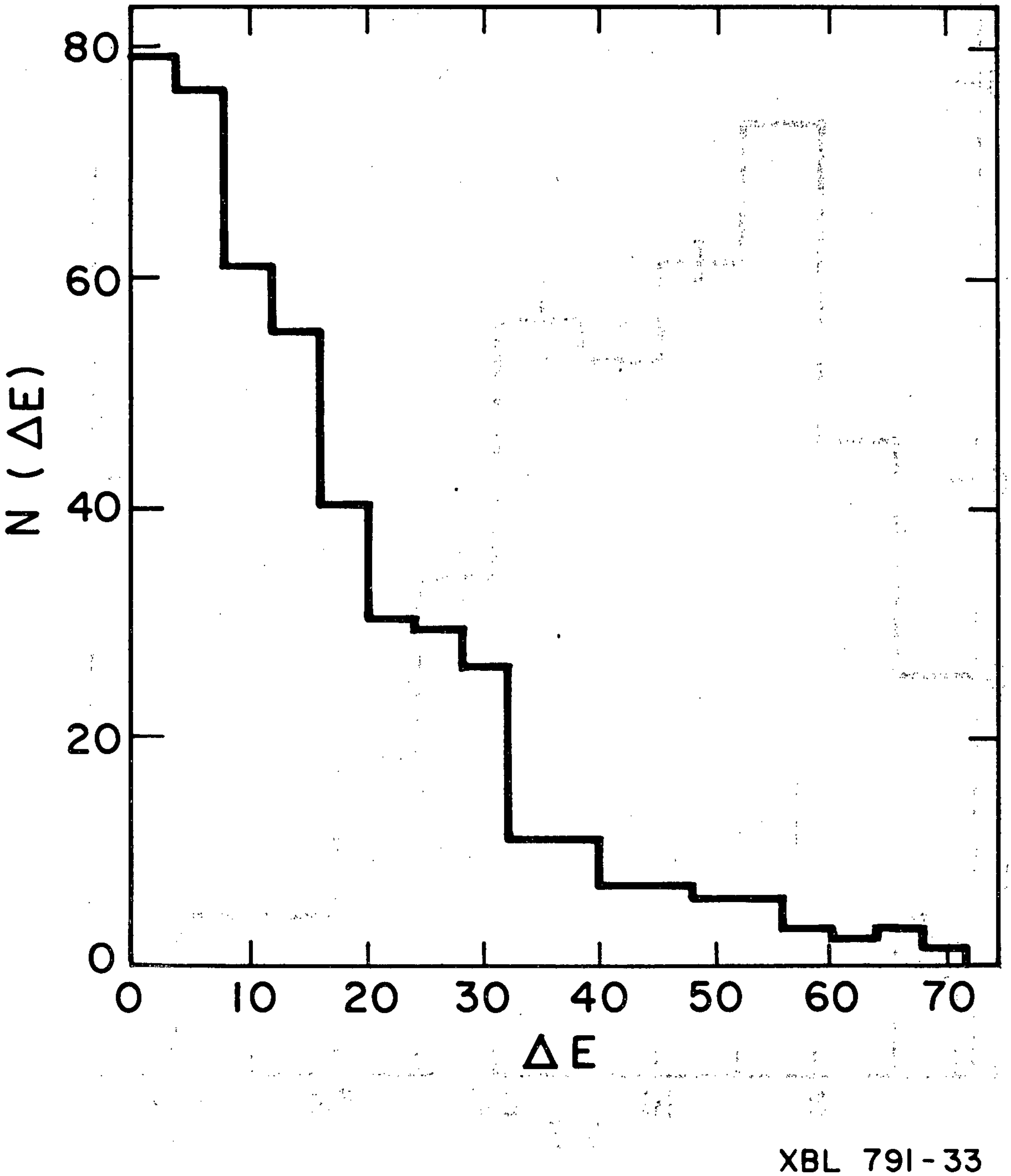


Figure 3

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