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Applications of the Intersection Theory of Singular Varieties

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by

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Abstract

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We develop tools for computing invariants of singular varieties and apply them to the classical theory of nodal curves and the complexity analysis of non-convex optimization problems.

The first result provides a method for computing the Segre class of a closed embedding $X \to Y$ in terms of the Segre classes of X and Y in an ambient space Z. This method is used to extend the classical Riemann-Kempf formula to the case of nodal curves.

Next we focus on techniques for computing the ED degree of a complex projective variety associated to an optimization problem. As a first application we consider the problem of scene reconstruction and find a degree 3 polynomial that computes the ED degree of the multiview variety as a function of the number of cameras. Our second application concerns the problem of weighted low rank approximation. We provide a characterization of the weight matrices for which the weighted 1-rank approximation problem has maximal ED degree.

To Inbar

Contents

Acknowledgments

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Chapter 1

Introduction

Before continuing to the main body of this thesis, we'll motivate each chapter and highlight our main results.

In chapter 2 we investigate a property of Segre classes which we call *cancellation*. This concept is introduced in section 1.2.2, together with a quick introduction to Segre classes themselves.

The basic situation of interest is when we have a sequence af embeddings of schemes

$$X \xrightarrow{i} Y \xrightarrow{j} Z$$

where i is a closed embedding and j is a regular embedding and we are interested in computing the Segre class $s(X, Y)$. If Z is a simple ambient space such as $\mathbb{P}^n$, it is useful to have a *cancellation* formula such as

$$s(X, Y) = c(N_Y Z|_X) \cap s(X, Z)$$

. This formula holds when all three schemes are smooth but, as we will see in section 1.2.2, there are simple examples in which it fails when Y is not smooth.

Our main contribution in section 2.1 is to identify a class of map j for which the cancellation formula holds.

Definition. Let k be a field of characteristic 0 and let Y and Z be varieties over k. We call a map $Y \xrightarrow{f} Z$ a *tubular regular embedding* if for any closed point $y = \mathrm{Spec}(k(y)) \to Y$, there exists a smooth map $P \to Y$, a section $Y \xrightarrow{s} P$, and an isomorphism φ

$$\begin{array}{ccc} & & \hat{Z} \\ & \overset{\hat{f}}{\nearrow} & \downarrow \varphi \;\cong \\ \hat{Y} & \underset{\hat{s}}{\searrow} & \\ & & \hat{P} \end{array}$$

where $\hat{Y}$, $\hat{Z}$ and $\hat{P}$ are the formal completions of Y, Z and P along the points y, $f(y)$ and $s(y)$ respectively.

Theorem. *Let k be a field of characteristic zero, and let X, Y and Z be varieties over k. Suppose that we have a closed embedding $X \to Y$ and a tubular regular embedding $Y \xrightarrow{f} Z$. Finally, suppose that Y and Z are equi-dimensional. Then,*

$$s(X, Y) = c(N_Y Z|_X) \cap s(X, Z)$$

After proving the cancellation theorem, in section 2.2 we use it to extend the classical Riemann-Kempf formula to arbitrary nodal curves.

Theorem. *Let k be a algebraically closed field of characteristic 0 and let X be a projective integral curve of arithmetic genus p over k with at worst planar singularities. Let x be a k-point of P_{p-1-d} corresponding to a rank-1 torsion free sheaf $\mathcal{I}$ and set $r = h^1(X, \mathcal{I}) - 1$ (so that $(\mathcal{A}^d_\omega)^{-1}(x) \cong \mathbb{P}^r_k$). Let W_d be the scheme theoretic image of $\mathcal{A}^d_\omega$.*

Then

$$\begin{aligned}\mathrm{mult}_x W_d &= \mathrm{mult}_x P_{p-1-d} \cdot (1+h)^{p-d+r} \cap [(\mathcal{A}^d_\omega)^{-1}(p)] \\ &= \mathrm{mult}_x P_{p-1-d} \cdot \binom{p-d+r}{r}\end{aligned}$$

where h is the first Chern class of the canonical bundle on $\mathbb{P}^r_k$. In particular, for $d = p-1$ we obtain a generalization of the Riemann singularity theorem.

In chapter 3 we develop techniques for computing the *Euclidean Distance Degree* as defined in [8]. As explained in section 3.1, the complexity of many non-convex optimization problems can be measured in terms of the ED degree of an associated complex projective variety.

The chapter starts with an introduction to the ED degree and the various types of characteristic classes used throughout the chapter.

Our first contribution in this chapter is a formula relating the hard to compute ED degree to the classical *polar classes*:

Proposition. *Let $X \subset \mathbb{P}(V)$ be a projective variety, let $\overline{X} \xrightarrow{\pi} X$ be a Nash blowup, let $\mathcal{E}_{\overline{X}}$ be the modified Euclidean normal bundle of X, let $L := \mathcal{O}_{\mathcal{E}_{\overline{X}}}(1)$ denote the tautological bundle associated to $\mathbb{P}(\mathcal{E}_{\overline{X}})$, and let B denote the base locus of*

$$\mathrm{ed}_Q : \mathbb{P}(\mathcal{E}_{\overline{X}}) \dashrightarrow \mathbb{P}(V)$$

Then,

$$\deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}})) \geq 0$$

and

$$\sum_i \delta_i(X) - \mathrm{ED}(X) = \deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}}))$$

Next we focus our attention on two specific optimization problems.

In section we compute the complexity of a problem in computer vision called *3D triangulation* or *scene reconstruction*. As explained in section 3.3.1, the basic idea is that we want have multiple camera images of a 3D object and we would like to use them to compute the 3D position of the object. The associated complex variety in this case is the *multiview variety*, and we obtain a closed form solution for the ED degree of this variety as a function of the number of cameras which addresses the projective version of a conjecture posed in [8].

Theorem. *The ED degree of the multiview variety with N cameras is*

$$p(N) = 6N^3 - 15N^2 + 11N - 4$$

where $N \geq 3$ is the number of cameras.

Finally, in section 3.4 we study the problem of *weighted low rank approximation*. This is a version of the classical low rank matrix approximation problem and is motivated in section 3.4.1.

The general setup is that we are given an $n \times m$ matrix M, and an $n \times m$ *weight matrix* W. The goal is to find a low rank $n \times m$ matrix X which approximates the matrix M using a Frobenious norm that is weighted by W. When all of the entries of W are equal to 1, this reduces to the standard low rank approximation problem which can be solved in polynomial time. On the other hand, for generic W the problem is NP hard.

Our main contribution in this section is to find an explicit condition on the weight matrix W which guarentees that the corresponding rank 1 approximation problem has the highest possible ED degree.

Theorem. *Let M_1 denote the variety of $n \times n$ matrices with rank 1. Let W be an $n \times n$ matrix which defines a non-degenerate quadratic form on $\mathcal{C}^n \otimes \mathcal{C}^n$.*

Then,

$$\mathrm{ED}_W(M_1) \leq \sum \delta_i(X)$$

and equality holds if and only if all of the minors of W have maximal rank.

1.1 Conventions and Notation

In this paper, we will generally follow the conventions in [12]. In particular, all of our schemes will be of finite type over a field, and we define a variety to be an integral scheme.

The one exception is with the definition of the projective scheme $\mathbb{P}(\mathcal{F})$ associated to a sheaf $\mathcal{F}$ on a scheme X. In order to be consistent with [2], we define

$$\mathbb{P}(\mathcal{F}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}(\mathcal{F}))$$

For example, an epimorphism of coherent sheaves $u : \mathcal{E} \to \mathcal{F}$ on a variety J induces a closed immersion

$$\mathbb{P}(\mathcal{F}) \xrightarrow{q} \mathbb{P}(\mathcal{E})$$

such that $q^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

1.2 Segre Classes

Let X be a proper subvariety of a variety Y. The *Segre class* $s(X, Y) \in A_\bullet(Y)$ is an invariant of the embedding of X in Y which has great theoretical and enumerative importance.

From a theoretical viewpoint, the Segre class provides a way of extending fundamental notions such as the multiplicity of a closed point $x \in X$ and the Chern class of the normal bundle of a smooth embedding $X \to Y$ to arbitrary closed embeddings. Furthoremore, as we will discuss in section 1.2.2, it extends these notions in a way that is functorial under both proper birational maps and flat maps.

For these reasons, the Segre class is a cornerstone of Fulton's intersection theory as evidenced for example in [12, Proposition 6.1].

In addition to it's theoretical importance, the Segre class shows up in a wide range of enumerative calculations and formulas. We will see one example of this in section 1.2.2 and chapter 2 where the functoriality of the Segre class allows us to transform a multiplicity calculation into an easier Chern class computation.

Another source of examples comes from the key role that the Segre class of the base locus plays when performing blowups. See for instance the "Blow up Formula" [12, Theorem 6.7].

Finally, Segre classes tend to show up in formulas related to degrees of varieties which are defined in terms of spans of points. One nice example of this is [11, Proposition 4.2] in which the degree of the secant variety of a Veronese embedding of the plane is given in terms of the degree of the Segre class of a certain vector bundle on the Hilbert scheme of points. We will see another example of this in Proposition 3.2.6.1.

In this expository section we will provide an introduction to Segre classes following [12, Chapter 4], and work out examples that will provide intuition for some of the material in later chapters.

1.2.1 Definitions

We start by defining the Segre class of a cone, and then proceed to the Segre class of a subscheme.

Let X be a scheme. A *cone* over X is a scheme of the form

$$C = \underline{\mathrm{Spec}}(S^\bullet) \to X$$

where $S^\bullet$ is a graded $\mathcal{O}_X$ algebra satisfying:

- The map $\mathcal{O}_X \to S^0$ is surjective.
- The module S^1 is coherent.
- $S^\bullet$ is generated by S^1.

The *projective completion* of a cone C is defined to be:

$$P(C \oplus 1) = \underline{\mathrm{Proj}}(S^\bullet[z])$$

with projection $P(C \oplus 1) \xrightarrow{q} X$ and canonical bundle $\mathcal{O}(1)$.

The *Segre class* of a cone C, denoted $s(C)$, is the class in $A_\bullet$ defined by:

$$s(C) = q_* \left(\sum_{i>=0} c_1(\mathcal{O}(1))^i \cap [P(C \oplus 1)] \right)$$

The relationship with Chern classes comes from the following proposition, which you can apply to your favorite definition of the Chern class of a vector bundle.

Proposition 1.2.1.1. *[12, Proposition 4.1] If $E \to X$ is a vector bundle on a scheme X then*

$$s(E) = c(E)^{-1} \cap [X]$$

Note that if C is a pure dimensional cone none of the irreducible components of C have an empty projectivization, it is not necessary to pass to the projective completion of C. Rather, in this case by [12, Example 4.1.2] we have

$$s(C) = q_* \left(\sum_{i>=0} c_1(\mathcal{O}(1))^i \cap [P(C)] \right)$$

which follows from the fact that $c_1(\mathcal{O}(1)) \cap [P(C \oplus 1)] = [P(C)]$.

We now turn to the definition of the Segre class of a closed subscheme. Let $X \subset Y$ be a closed subscheme with ideal sheaf $\mathcal{I}$. The *normal cone to X in Y* is defined by

$$C_X Y = \underline{\mathrm{Spec}}_X \left(\sum_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}) \right)$$

The *Segre class of X in Y*, denoted by $s(X, Y)$, is defined to be the Segre class of the cone $C_X Y$:

$$s(X, Y) = s(C_X Y) \in A_\bullet(X)$$

By proposition 1.2.1.1, if $X \to Y$ is a regular embedding with normal bundle $N_X Y$ then $s(X, Y) = c(N_X Y)^{-1} \cap [X]$.

1.2.2 Basic Properties

In this section we record some basic properties of Segre classes which will be important later on.

The most important of these is the following functoriality statement.

Proposition 1.2.2.1. *[12, Proposition 4.2] Let $Y' \to Y$ be a morphism of pure dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme and $g : X' \to X$ the induced morphism.*

1. *If f is proper, Y irreducible and f maps each irreducible component of Y' onto Y, then*

$$g_*(s(X', Y')) = \deg(Y'/Y)s(X, Y)$$

2. *If f is flat, then*

$$g^*(s(X, Y)) = s(X', Y')$$

The following relationship between Segre classes and blowups is useful in practice.

Proposition 1.2.2.2. *[19, Corollary 4.2.2]et X be a proper closed subscheme of a variety Y. Let $\tilde{Y}$ be the blow-up of Y along X, $\tilde{X} = P(C)$ the exceptional divisor and $\eta : \tilde{X} \to X$ the projection. Then,*

$$\begin{aligned} s(X, Y) &= \sum_{k\geq 1}(-1)^{k-1}\eta_*(\tilde{X}^k) \\ &= \sum_{i\geq 0}(c_1(\mathcal{O}(1))^i \cap [P(C)]) \end{aligned}$$

Another important property of the Segre class is that it can be used to compute the multiplicity of a singular point. More generally, Fulton [12, Section 4.3] defines the multiplicity of an irreducible subvariety $X \subset Y$, denoted $e_X Y$, to be the coefficient of $[X]$ in $s(X, Y)$.

If $X = P$ is a point and $C = C_P Y$ is the tangent cone, then from the definition of the Segre class we see that

$$e_P Y = \deg s(X, Y) = \deg[P(C)]$$

which shows that $e_P Y$ agrees with the classical definition of multiplicity in terms of the degree of the tangent cone.

Finally, we discuss a phenomenon that we call *cancellation* and will investigate more carefully in chapter 2.

Let X be a closed subscheme of Y, let E be a vector bundle on Y and let let $Y \xrightarrow{s} E$ be the zero section. Then it is not hard to show ([12, Example 4.2.7]) that

$$s(X, Y) = c(E|_X) \cap s(X, E)$$

We call such a phenomenon *cancellation* as it allows us to compute the Segre class of X in Y by starting with the Segre class of X in E, and then *cancelling out* the part of $s(X, E)$ coming from the embedding of Y in E.

More generally, we say that cancellation holds for a sequence of embeddings

$$X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} Z$$

where i is a closed embedding and j is a regular embedding if

$$s(X, Y) = c(N_Y Z|_Z) \cap s(X, Z)$$

However, this kind of cancellation does not always hold, and in fact fails in the following simple example ([12, Example 4.2.8]). Consider the case where $Z = \mathbb{P}^2$, $Y \xrightarrow{j}$ is a nodal curve and $X \xrightarrow{i} Y$ is the node. Then, $s(X,Y) = 2[X]$ since X is a point of multiplicity 2. Similarly, $s(X,Z) = [X]$ since X is a smooth point of Z. In addition, since X is a point, $c(N_Y Z|X)$ is trivial.

Therefore,

$$s(X,Y) = 2[X] \neq [X] = c(N_Y Z|X) \cap s(X,Z)$$

In chapter 2 we will show that there exists a larger class of embeddings $Y \xrightarrow{j} Z$ for which cancellation holds.

1.2.3 Riemann-Kempf for Smooth Curves

In this section we work through example [12, Example 4.3.2] which in addition to providing motivation for chapter 2, is a beautiful demonstration of the power of Fulton's intersection theory.

We start by recalling the setup and statement of the Riemann-Kempf formula.

Let X be a projective non-singular curve of genus g over an algebraically closed field k. Let $X^{(n)}$ denote the n-th symmetric power of the curve and let J_X denote the Jacobian of X. Recall the the Jacobian is defined to be the moduli space of degree zero line bundles on X. We choose a point $p \in X$.

After choosing a point $p \in X$, for every integer d, we have an Abel-Jacobi map

$$A^d : X^{(d)} \to J_X$$

which takes a degree d divisor D to the line bundle $\mathcal{O}(D - d \cdot p)$.

Let W_d denote the scheme theoretic image of A^d. In particular, W_{g-1} is the theta divisor.

Theorem 1.2.3.1 (Riemann-Kempf formula, smooth case). *Let D be a degree d divisor. Then the multiplicity of W_d at $A^d(D)$ is*

$$\binom{g-d+r}{r}$$

where $r+1$ is the dimension of the linear system $|D| \cong \mathbb{P}^r_k$.

In particular, when $d = g-1$, the multiplicity is r. This is known as the Riemann singularity theorem.

In [12, p. 4.3.2], Fulton gives a concise proof of this theorem using elementary facts about Segre classes, together with the following properties of the Abel-Jacobi map which are well known (and implied by theorem 2.2.1.6 and lemma **??**):

1. The scheme theoretic fibers of A^d are the linear systems $|D| \cong \mathbb{P}^r_k$.

2. If $d \geq 2g - 1$ then A^d makes $X^{(d)}$ a projective bundle over J_X of relative dimension $d - g$.

3. If $1 \leq d \leq g$ then A^d maps $X^{(d)}$ birationally onto it's image.

Let D be a degree d divisor and r an integer such that $|D| \cong \mathbb{P}^r$. Fulton obtains the theorem as a corollary of the following more general fact:

$$s(|D|, X^{(d)}) = (1+h)^{g-d+r} \cap [|D|] \tag{1.2.3.1}$$

where h is the first Chern class of the canonical line bundle on $|D| = \mathbb{P}^r$. This implies the theorem since the degree of the pushforward of this Segre class is the multiplicity of $A^d(|D|)$ in W_d, and the degree of $(1+h)^{g-d+r} \cap [\mathbb{P}^r]$ is precisely:

$$\binom{g-d+r}{r}$$

The proof of equation 1.2.3.1 is done by reduction to the case where $d >> 0$.

Let d be an arbitrary integer. In order to reduce to the case where d is large, we choose some integer s such that $d + s \geq 2g - 1$. Furthermore, by choosing a point p in X, we obtain an embedding

$$X^{(d)} \hookrightarrow X^{(d+s)}$$

by sending a degree d divisor D to $D + sp$. The reduction from the degree $d + s$ case to the degree d case is motivated by the following commutative diagram with a left (but not right) fibered square:

$$\begin{array}{ccccc} |D| & \longrightarrow & X^{(d)} & \longrightarrow & X^{(d+s)} \\ \downarrow & & \downarrow{\scriptstyle A^d} & & \downarrow{\scriptstyle A^{d+s}} \\ A^d(|D|) & \longrightarrow & W_d & \longrightarrow & J_X \end{array}$$

Proof when d is large:

When $d \geq 2g-1$, then A^d is a smooth map and in particular it is flat. Therefore, since J_X is smooth,

$$\begin{aligned} s(|D|, C^{(d)}) &= (A^d)^* s(A^d(D), J_X) \\ &= (A^d)^*(\text{mult}_{A^d} \cdot J_X[A^d(D)]) \\ &= \text{mult}_{A^d} \cdot J_X[|D|] = [|D|] \end{aligned} \tag{1.2.3.2}$$

Calculation of $s(|D|, X^{(d+s)})$ using $s(|D+sp|, X^{(d+s)})$ when $d+s$ is large:

By equation 1.2.3.2, we know that

$$s(|D+sp|, X^{(d+s)}) = [|D+sp|] = [\mathbb{P}^{d+s-g}]$$

Since $|D| = \mathbb{P}^r$, we have the following composition of closed embeddings

$$|D| = \mathbb{P}^r \to \mathbb{P}^{d+s-g} \to X^{(d+s)}$$

and since everything is smooth, from the conormal sequence we can deduce that

$$\begin{aligned} [|D|] &= s(|D+sp|, X^{(d+s)})|_{|D|} \\ &= c(N_{|D|}|D+sp|) \cap s(|D|, X^{(d+s)}) \\ &= (1+h)^{d+s-g-r} \cap s(|D|, X^{(d+s)}) \end{aligned} \tag{1.2.3.3}$$

where h is the first Chern class of the canonical bundle on $|D| \cong \mathbb{P}^r$

Calculation of $s(|D|, X^{(d)})$ using $s(|D|, X^{(d+s)})$ when $d+s$ is large:

Finally, we consider the sequence of embeddings

$$|D| \to X^{(d)} \to X^{(d+s)}$$

It is easy to check that $c(N_{X^{(d)}}X^{(d+s)}|_{|D|}) = (1+h)^s$. Therefore, by a version of cancellation of Segre classes which can be explicitly shown to hold in this case (or even more easily, by proposition 2.1.1.1),

$$\begin{aligned} s(|D|, X^{(d)}) &= c(N_{X^{(d)}}X^{(d+s)}|_{|D|}) \cap s(|D|, X^{(d+s)}) \\ &= \frac{(1+h)^s}{(1+h)^{d+s-g-r}} \cap [|D|] \\ &= (1+h)^{g-d+r} \cap [|D|] \end{aligned}$$

Chapter 2

Cancellation of Segre Classes and the Riemann-Kempf Formula

In [12, Chapter 4], Fulton defines the notion of the Segre class $s(X, Y) \in A_*X$ of a closed embedding of schemes $X \to Y$ over a field k. The Segre class allows us to measure the way in which X sits inside Y, and is functorial for sufficiently nice maps ([12, Proposition 4.2]). One important case is the embedding of a closed point, for which the Segre class gives us its multiplicity.

Suppose we have an embedding $X \to Y$ and the schemes in question embed into a simpler space Z. For example, X and Y could be projective schemes and $Z = \mathbb{P}^n_k$. In this setup, it is natural to ask whether we can calculate $s(X, Y)$, assuming that we understand $s(X, Z)$ and $s(Y, Z)$. In other words, can we deduce intersection theoretic invariants of the possibly complicated embedding $X \to Y$, from the hopefully simpler embeddings into Z?

Fulton [12, Example 4.2.7] provides the answer to this question in the case where the map $Y \to Z$ is the zero section of a vector bundle. More precisely, he shows that if X is a closed subscheme of Y, E is a vector bundle on Y and Y is embedded into E as the zero section then,

$$s(X, Y) = c(E|_X) \cap s(X, E)$$

Inspired by this example, one might conjecture that a similar formula holds in the more general setting described above in which the map $Y \to Z$ is not necessarily the zero section of a vector bundle E over Y. Unfortunately, this is false in extremely simple cases, even when $Y \to Z$ is a regular embedding. See [12, Example 4.2.8] for an example.

Despite this failure in the general case, it is intuitive that the statement should hold when the embedding of Y into Z "looks like" a zero section of a vector bundle. For instance, it seems plausible that we could replace the condition of $Y \to Z$ being a zero section, with the condition that $Y \to Z$ has some sort of tubular neighborhood.

As a preliminary result along these lines, we show in proposition 2.1.1.1 that an analogous formula holds when both Y and Z are smooth.

The primary result of this paper is an extension of [12, Example 4.2.7] to the situation in which the hypothesis of that example is satisfied only formally locally at each k point. We formalize this situation in the following definition which is inspired by [9, Definition 3.3].

Definition 2.0.3.1. Let k be a field of characteristic 0 and let Y and Z be varieties over k. We call a map $Y \xrightarrow{f} Z$ a *tubular regular embedding* if for any closed point $y = \mathrm{Spec}(k(y)) \to Y$, there exists a smooth map $P \to Y$, a section $Y \xrightarrow{s} P$, and an isomorphism φ

$$\begin{array}{ccc} & & \hat{Z} \\ & \nearrow^{\hat{f}} & \cong \downarrow \varphi \\ \hat{Y} & & \\ & \searrow_{\hat{s}} & \\ & & \hat{P} \end{array}$$

where $\hat{Y}$, $\hat{Z}$ and $\hat{P}$ are the formal completions of Y, Z and P along the points y, $f(y)$ and $s(y)$ respectively.

This property of f seems to capture the notion of a tubular neighborhood, as we can formally locally view it as a section of a vector bundle. We can now state our generalization of example [12, Example 4.2.7].

Theorem 2.0.3.2. *Let k be a field of characteristic zero, and let X, Y and Z be varieties over k. Suppose that we have a closed embedding $X \to Y$ and a tubular regular embedding $Y \xrightarrow{f} Z$. Finally, suppose that Y and Z are equi-dimensional. Then,*

$$s(X, Y) = c(N_Y Z|_X) \cap s(X, Z)$$

We call it a "cancellation theorem" since intuitively, it tells us that in order to calculate $s(X, Y)$, we can first calculate $s(X, Z)$, and then "cancel out" the contribution of the embedding $Y \to Z$.

The theorem can be useful in practice, since there exist algorithms which have been implemented in Macaulay2 that can compute the Segre class of an embedding $X \to \mathbb{P}^n_k$ [10]. Our theorem allows us to use this algorithm in order to compute $s(X, Y)$ when both X and Y are subschemes of $\mathbb{P}^n$ and the embedding $Y \to \mathbb{P}^n$ satisfies our condition. In particular, proposition 2.1.1.1 allows us to compute $s(X, Y)$ when Y is a smooth intersection of hypersurfaces in $\mathbb{P}^n_k$ since we then know $c(N_Y \mathbb{P}^n_k)$ as well.

Theorem 2.0.3.2 is particularly useful when the spaces Y and Z represent functors and the embedding $Y \to Z$ corresponds to a natural transformation of these functors. In this case, the formal neighborhoods pro-represent local deformation functors which are typically easy to describe.

Indeed, as the main application of this paper, we will use theorem 2.0.3.2 in order to extend the Riemann Kempf formula to the case of general integral curves. See [12, Example 4.3.2] for the classical statement of this formula.

As we will discuss in section 2.2.1, P_{p-1-d} denotes the compactified Picard scheme and $\mathcal{A}_\omega$ denotes the Abel-Jacobi map.

Corollary 2.2.2.1. *(*Riemann Kempf for planar curves*) Let k be a algebraically closed field of characteristic* 0 *and let X be a projective integral curve of arithmetic genus p over k with at worst planar singularities. Let x be a k-point of P_{p-1-d} corresponding to a rank-1 torsion free sheaf $\mathcal{I}$ and set $r = h^1(X, \mathcal{I}) - 1$ (so that $(\mathcal{A}^d_\omega)^{-1}(x) \cong \mathbb{P}^r_k$). Let W_d be the scheme theoretic image of $\mathcal{A}^d_\omega$.*

Then

$$\begin{aligned}\mathrm{mult}_x W_d &= \mathrm{mult}_x P_{p-1-d} \cdot (1+h)^{p-d+r} \cap [(\mathcal{A}^d_\omega)^{-1}(p)] \\ &= \mathrm{mult}_x P_{p-1-d} \cdot \binom{p-d+r}{r}\end{aligned}$$

where h is the first Chern class of the canonical bundle on $\mathbb{P}^r_k$. In particular, for $d = p-1$ we obtain a generalization of the Riemann singularity theorem.

In particular, this provides an affirmative answer to conjecture [7, Equality 0.1].

In section 2.2.2, we use theorem 2.0.3.2 to study the change in various Segre classes as we vary the degree d. This allows us to deduce facts about the Abel Jacobi map in small degrees from the simpler Abel Jacobi maps in higher degrees and ultimately prove theorem 2.2.2.1.

2.1 Proof of the Cancellation Theorem

In this section we prove theorem 2.0.3.2. We start by proving it in the special case where Y is smooth. We then use Artin approximation and Hironaka's functorial resolution of singularities to deduce the general case.

2.1.1 The Smooth case

We will now prove the following special case of the cancellation theorem.

Proposition 2.1.1.1. *Let X be a finite type k-scheme let Y and Z be smooth k-schemes. Suppose that we have a closed embedding $X \to Y$ and a regular embedding $Y \to Z$. Then,*

$$s(X, Y) = c(N_Y Z|_X) \cap s(X, Z)$$

It seems to be easier to prove the following slightly stronger statement.

Proposition 2.1.1.2. *Let $X \to Y$ be a closed embedding of k schemes. Let Z_1 and Z_2 be smooth k schemes, and suppose we have regular embeddings $Y \to Z_1$ and $Y \to Z_2$. Then*

$$c(N_Y Z_1|_X) \cap s(X, Z_1) = c(N_Y Z_2|_X) \cap s(X, Z_2)$$

Remark 2.1.1.1. The class in question is reminiscent of Fulton's canonical class [12, Example 4.2.6], and indeed, the proof of the equality is similar. For the purposes of this paper we will only use the proposition when Y is also smooth, in which case the asserted equality follows almost immediately from the independence of Fulton's class. Indeed, in [12, Example 4.2.6] Fulton proves the equality

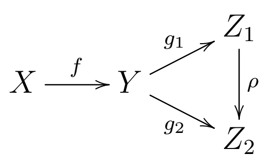
$$c(TZ_1|_X) \cap s(X, Z_1) = c(TZ_2|_X) \cap s(X, Z_2)$$

multiplying both sides by $c(Y)^{-1}$ gives the equality stated in the proposition.

Proof. Let $X \xrightarrow{f} Y$ be a closed embedding of k-schemes, Z_1 and Z_2 smooth k-schemes and $Y \xrightarrow{g_i} Z_i$ regular embeddings. We want to show that

$$c(N_Y Z_1) \cap s(X, Z_1) = c(N_Y Z_2) \cap s(X, Z_2)$$

As in [12, Example 4.2.6], since Z_1 and Z_2 are dominated by the smooth scheme $Z_1 \times Z_2$, by replacing Z_1 with this product and replacing g_1 with the induced map, we can assume without loss of generality that there exists a smooth map $Z_1 \xrightarrow{\rho} Z_2$ such that the following diagram commutes:



Now, by [12, Example 4.2.6], we have the following short exact sequences of cones (in the sense of [12, Example 4.1.6]).

- $0 \to g_1^* T_\rho \to C_Y Z_1 \to C_Y Z_2 \to 0$
- $0 \to (g_1 \circ f)^* T_\rho \to C_X Z_1 \to C_X Z_2 \to 0$

Note that in this case, $C_Y Z_i = N_Y Z_i$. By pulling back the first sequence of bundles, we obtain:

$$0 \to (g_1 \circ f)^* T_\rho \to f^* N_Y Z_1 \to f^* N_Y Z_2 \to 0$$

Therefore, by [12, Example 4.1.6] and the definition of the Segre class we have:

- $s(f^* N_Y Z_2) = c((g_1 \circ f)^* T_\rho) \cap s(f^* N_Y Z_1)$
- $s(X, Z_2) = c((g_1 \circ f)^* T_\rho) \cap s(X, Z_1)$

Since by definition the Chern class is the inverse of the Segre class, the first equality implies that

$$c((g_1 \circ f)^* T_\rho) = c(f^* N_Y Z_1) \cap s(f^* N_Y Z_2)$$

By using this equality to replace $c((g_1 \circ f)^* T_\rho)$ the second equation listed above, we get the result. □

We can now easily deduce proposition 2.1.1.1. Recall that by example [12, Example 4.2.7] the proposition holds when $Y \to Z$ is the zero section of a vector bundle. The result now follows by applying proposition 2.1.1.2 to the case where $Y \to Z_1$ is the map $Y \to Z$ in proposition 2.1.1.1, and $Y \to Z_2$ is the zero section of the bundle $N_Y Z$. Note that the assumption in proposition 2.1.1.1 that Y is smooth is necessary in order for the total space of the bundle $N_Y Z$ to be a smooth scheme.

2.1.2 Reduction to the Smooth Case

We will now prove theorem 2.0.3.2 by a reduction to the smooth case which was covered in the previous section. As usual in intersection theory, it would be enough to dominate the map f in theorem 2.0.3.2 by a map of smooth schemes. In other words, we want to find smooth schemes M and N which fit into the following fiber diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & Z \end{array}$$

such that the vertical maps are proper birational maps. We formalize this in the following lemma.

Lemma 2.1.2.1. *Let $X \to Y$ be a closed embedding and $Y \xrightarrow{f} Z$ a regular embedding where X, Y and Z are k-schemes and Y and Z are equi-dimensional. Let $\tilde{Y} \xrightarrow{\tilde{f}} \tilde{Z}$ be a regular embedding of smooth k-schemes together with a fiber diagram*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Z} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

such that h and g are proper and birational. Then

$$s(X, Y) = c(N_Y Z|_X) \cap s(X, Z)$$

Proof. Consider the extended fiber diagram

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Z} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & Y & \xrightarrow{f} & Z \end{array}$$

By [12, Proposition 4.2], we know that $\pi_* s(\tilde{X}, \tilde{Y}) = s(X, Y)$ and that $\pi_* s(\tilde{X}, \tilde{Z}) = s(X, Z)$. Furthermore, by proposition 2.1.1.1, we know that

$$s(\tilde{X}, \tilde{Y}) = c(N_{\tilde{Y}} \tilde{Z}|_{\tilde{X}}) \cap s(\tilde{X}, \tilde{Z})$$

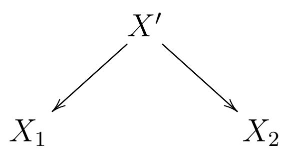
So by proper base change, it suffices to show that in the current scenario, $h^* N_Y Z \cong N_{\tilde{Y}} \tilde{Z}$.

For this, note that for any cartesian diagram as in the statement of the lemma where the horizontal maps are closed embeddings, the natural map of conormal sheaves $h^*(N_Y^\vee Z) \to N_{\tilde{Y}}^\vee \tilde{Z}$ is a surjection. In our case, both of the conormal sheaves are locally free and of equal rank, so the surjection is an isomorphism. □

We now use a combination of Artin approximation and Hironaka's functorial resolution of singularities to produce a map $\tilde{Y} \to \tilde{Z}$ which satisfies the requirements of the lemma.

For convenience, we state the relevant proposition from [5].

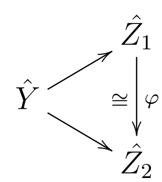
Theorem 2.1.2.1. *([5, Corollary 2.6]) Let S be a scheme of finite type over an excellent Dedekind domain, let X_1 and X_2 be finite type S-schemes and let $x_i \in X_i$ be points. If the complete local rings $\hat{\mathcal{O}}_{X_i,x_i}$ are $\mathcal{O}_S$-isomorphic, then X_1 and X_2 are locally isomorphic for the etale topology. By this we mean that there exists a diagram of etale maps*



and a point $x' \in X'$ that maps to x_1 and x_2 respectively and induces an isomorphism of fraction fields.

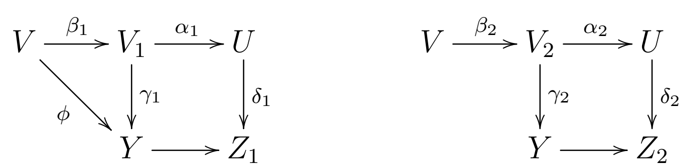
We will need a slight generalization of this theorem. The letters denoting the schemes have been modified in order to clarify the application to our current question.

Lemma 2.1.2.2. *Let $Y \to Z_1$ and $Y \to Z_2$ be maps of finite type S-schemes and let $y \in Y$ be a point that maps to $z_i \in Z_i$. Suppose there exists an isomorphism φ of formal neighborhoods:*



where $\hat{Y}$ and $\hat{Z}_i$ are the formal completions of Y and Z_i and along the points y and z_i respectively. Then the maps $Y \to Z_1$ and $Y \to Z_2$ have a common etale neighborhood.

By this we mean that we have k-schemes U and V together with the following two diagrams:



such that β_i, γ_i, δ_i and $\phi = \gamma_1\beta_1$ are etale, $\alpha_1\beta_1 = \alpha_2\beta_2$, and V is an etale neighborhood of y.

Proof. By Artin approximation (theorem 2.1.2.1), there exists a scheme U and maps δ_i satisfying the conditions in the theorem. We denote by u the point in U mapping to z_1 and z_2. We also define the schemes $V_i = Y \times_{Z_i} U$ together with the points $v_i = (y, u) \in V_i$.

Note that the two natural maps from V_1 to Z_2, namely, $V_1 \to Y \to Z_1$ and $V_1 \to U \to Z_2$ induce isomorphic maps between the formal neighborhoods of v_1 and z_2. Therefore, after replacing V_1 by an etale cover we can assume without loss of generality that there is a map $V_1 \to V_2$ over U that induces an isomorphism between the formal neighborhoods of v_1 and v_2.

The lemma then follows from a second application of Artin approximation which is used to realize this equivalence of formal neighborhoods by the etale maps $V \xrightarrow{\beta_i} V_i$. □

We will now combine this lemma with Hironaka's functorial resolution of singularities in order to achieve the conditions in lemma 2.1.2.1. This will conclude the proof of theorem 2.0.3.2.

Before starting we'll recall the precise statement of functorial resolutions from [18]. Note that we are only stating the subset of the theorem that we'll be using.

Theorem 2.1.2.2. *([18, Theorem 3.36]) Let k be a field with characteristic zero. Then there exists an assignment* **BR** *from finite type k-schemes to schemes such that:*

- $\mathbf{BR}(X) \to X$ *may be constructed from X by a finite sequence of blowups.*
- $\mathbf{BR}(X)$ *is smooth.*
- **BR** *commutes with pullbacks along smooth morphisms.*

Note that the map $\mathbf{BR}(X) \to X$ is part of the data of $\mathbf{BR}(X)$ as it is simply the composition of blowups. See [18] for details on the construction.

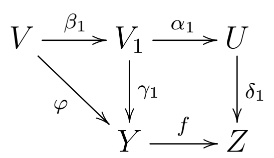
We now apply this theorem to our situation.

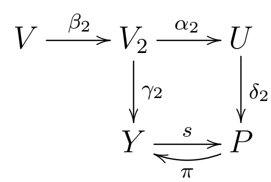
Proposition 2.1.2.1. *Let k be a field of characteristic zero, and let X, Y and Z be varieties over k. Let $X \to Y$ be a closed embedding and $Y \xrightarrow{f} Z$ a tubular regular embedding (cf. 2.0.3.1). Then there is a regular embedding $\tilde{Y} \xrightarrow{\tilde{f}} \tilde{Z}$ of smooth schemes over k together with a fiber diagram*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Z} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

such that h and g are proper and birational.

Proof. By lemma 2.1.2.2, for each point $y \in Y$ we have k-schemes U and V together with the following two diagrams:





such that all vertical and diagonal arrows are etale, $\alpha_1\beta_1 = \alpha_2\beta_2$, and V is an etale neighborhood of y.

Now, since π is smooth, by functorial resolution of singularities (theorem 2.1.2.2) we get that

$$s^*\mathbf{BR}(P) = s^*\pi^*\mathbf{BR}(Y) = \mathbf{BR}(Y)$$

By combining this with another application of the resolution of singularities theorem for the vertical etale maps, we obtain the following chain of equalities:

$$\begin{aligned} (f\varphi)^*\mathbf{BR}(Z) &= (\delta_1\alpha_1\beta_1)^*\mathbf{BR}(Z) \\ &= (\alpha_1\beta_1)^*\mathbf{BR}(U) \\ &= (\alpha_2\beta_2)^*\mathbf{BR}(U) \\ &= (\delta_2\alpha_2\beta_2)^*\mathbf{BR}(P) \\ &= (s\gamma_2\beta_2)^*\mathbf{BR}(P) \\ &= \mathbf{BR}(V) \end{aligned}$$

Therefore, the schemes $\tilde{Z} = \mathbf{BR}(Z)$ and $\tilde{Y} = Y \times_Z \tilde{Z}$ satisfy the requirements of the proposition. To see this, note that by theorem 2.1.2.2, Z is smooth and $\mathbf{BR} \to Z$ is a proper birational map. Furthermore, by the above chain of inequalities, we see that $\tilde{Y}$ is etale locally smooth, and that the map $\tilde{Y} \to Y$ is birational etale locally on the base. It is also proper since proper maps are preserved by base change. □

Theorem 2.0.3.2 now follows immediately from this proposition combined with lemma 2.1.2.1.

2.2 The Riemann Kempf Formula

2.2.1 Preliminaries

We'll start by recalling the definition of the compactified Picard scheme in [2], together with some auxiliary notation. Readers familiar with [2] are encouraged to skip to this section.

The discussion in [2] takes place in the general setting of a proper morphism $X \xrightarrow{f} S$, whereas we are primarily concerned with integral curves over a field. Nevertheless, we'll start by describing the general case since it isn't any more difficult than the curve case, and it will allow us to provide cleaner arguments later on.

Let $X \xrightarrow{f} S$ be a finitely presented morphism of schemes and let $\mathcal{F}$ be a locally finitely presented $\mathcal{O}_X$-module. Altman and Kleiman define a collection of functors associated to such data, which are related to one another by the Abel-Jacobi map.

The first one is the familiar

$$\mathbf{Quot}_{(\mathcal{F}/X/S)} : \boldsymbol{Sch}_S \to \boldsymbol{Set}$$

which maps an S-scheme T to the set of locally finitely presented T-flat quotients of $\mathcal{F}_T$ whose support is proper and finitely presented over T.

In addition, if ϕ is a polynomial in $\mathbb{Q}[x]$, we define

$$\mathbf{Quot}^{\phi}_{(\mathcal{F}/X/S)} \subset \mathbf{Quot}_{(\mathcal{F}/X/S)}$$

to be the set of such quotients with Hilbert polynomial ϕ on each fiber.

Definition 2.2.1.1. [2, Definition 2.5] Let $X \xrightarrow{f} S$ be a finitely presented morphism of schemes, $\mathcal{F}$ a locally finitely presented $\mathcal{O}_X$-module, and $\mathcal{G}$ an S-flat quotient of $\mathcal{F}$. Define the *pseudo-ideal* $\mathcal{I}(\mathcal{G})$ as the kernel of $\mathcal{F} \to \mathcal{G} \to 0$.

Remark 2.2.1.1. We require $\mathcal{G}$ to be flat so that the formation of the pseudo-ideal commutes with base change.

Given a line bundle $\mathcal{L}$ on a curve, we have the notion of a linear system which is defined to be the collection of effective divisors D such that $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$. In [2, Definition 4.1], Altman and Kleiman generalize this as follows. Let $X \to S$ and $\mathcal{F}$ be as before, and let $\mathcal{I}$ be a finitely presented $\mathcal{O}_X$-module. We define the functor

$$\mathbf{LinSyst}_{(\mathcal{I},\mathcal{F})} \subset \mathbf{Quot}_{(\mathcal{F}/X/S)}$$

to be the functor which maps an S-scheme T to the set of quotients $\mathcal{G} \in \mathbf{Quot}_{(\mathcal{F}/X/S)}$ such that there exists a line bundle $\mathcal{N}$ on T and an isomorphism

$$\mathcal{I}(\mathcal{G}) \cong \mathcal{I} \otimes \mathcal{N}$$

One of the key observations in [2] is that the linear systems functor is representable by a very natural projective scheme on S. This is precisely analogous to the fact that the linear system of a line bundle on a curve can be described as the projective space associated to the module of the line bundle's global sections.

In fact, the same thing is true in this case when $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{I}$ is an ideal of $\mathcal{O}_X$. For arbitrary $\mathcal{F}$ and $\mathcal{I}$, we generalize the notion of global sections via the following combination of lemma and definition.

Lemma 2.2.1.1. *([2, Section 1.1]) Let $X \to S$ be a finitely presented proper morphism of schemes and let $\mathcal{I}$ and $\mathcal{F}$ be two locally finitely presented $\mathcal{O}_X$-modules with $\mathcal{F}$ flat over S. Then, there exists a locally finitely presented $\mathcal{O}_S$-module $H(\mathcal{I}, \mathcal{F})$ and an element $h(\mathcal{I}, \mathcal{F})$ of $\mathrm{Hom}_X(\mathcal{I}, \mathcal{F} \otimes_{\mathcal{O}_S} H(\mathcal{I}, \mathcal{F}))$ which together represent the covariant functor:*

$$\mathcal{M} \mapsto \mathrm{Hom}_X(\mathcal{I}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M})$$

from the category of quasi-coherent $\mathcal{O}_S$-modules to ***Set****.*

Remark 2.2.1.2. It's easy to see that $H(\mathcal{I}, \mathcal{J})$ is covariant in $\mathcal{I}$ and contravariant in $\mathcal{J}$. Furthermore, it is right exact in each entry. (See [2, page 55])

We can now rigorously state the observation we mentioned above.

Theorem 2.2.1.1. *[2, Theorem 4.2] Let $X \xrightarrow{f} S$ be a proper finitely presented morphism of schemes and let $\mathcal{F}$ and $\mathcal{I}$ be two finitely presented $\mathcal{O}_X$-modules. Assume that $\mathcal{F}$ is S-flat and that, for each S-scheme T for which $\mathcal{I}_T$ is T-flat, the canonical map*

$$\mathcal{O}_T^\times \to (f_T)_* \mathbf{Isom}_{X_T}(\mathcal{I}_T, \mathcal{I}_T)$$

is an isomorphism.

Then, the functor $\mathbf{LinSyst}_{(\mathcal{I},\mathcal{F})}$ is representable by an open subscheme U of $\mathbb{P}(H(\mathcal{I}, \mathcal{F}))$ such that the inclusion is quasi-compact. Moreover, U is equal to $\mathbb{P}(H(\mathcal{I}, \mathcal{F}))$ if and only if, for each geometric point s of S, every non-zero $\mathcal{O}_{X(s)}$-homomorphism $\mathcal{I}(s) \to \mathcal{F}(s)$ is an injection.

The proof of this theorem is purely formal, and involves writing down the natural functor represented by $\mathbb{P}(H(\mathcal{I}, \mathcal{F}))$ and unraveling the definition of $H(\mathcal{I}, \mathcal{F})$.

In light of this role played by $H(\mathcal{I}, \mathcal{F})$, the following theorem is crucial.

Theorem 2.2.1.2. *[2, Theorem 1.3] Let $X \to S$ be a finitely presented, proper morphism of schemes, and let $\mathcal{I}$ and $\mathcal{F}$ be locally finitely presented S-flat $\mathcal{O}_X$-modules. Assume,*

$$\mathrm{Ext}^1_{X(s)}(\mathcal{I}(s), \mathcal{F}(s)) = 0$$

for some point s in S. Then there exists an open neighborhood U of s such that $H(\mathcal{I}, \mathcal{F})|_U$ is locally free with finite rank.

This tells us that in good conditions, the linear systems functor is representable by a projective bundle.

In order to define the Abel Jacobi map, we'll need to be more selective in the types of quotients that we allow. This is done by requiring that the pseudo-ideal of our quotients satisfy the following condition.

Definition 2.2.1.2. [2, Definition 5.1] Let $X \xrightarrow{f} S$ be a morphism of schemes and let $\mathcal{I}$ be an $\mathcal{O}_X$-module. Then $\mathcal{I}$ will be called *simple over* S or S*-simple* if $\mathcal{I}$ is locally finitely presented and flat over S and if for every S scheme T, the canonical map

$$\mathcal{O}_T \to (f_T)_* \mathbf{Hom}_{X_T}(\mathcal{I}_T, \mathcal{I}_T)$$

is an isomorphism.

The following stronger property will also play a central role.

Definition 2.2.1.3. [2, Lemma 3.1] Let X be a geometrically integral scheme over a field k and let $\mathcal{I}$ be a coherent $\mathcal{O}_X$ module. We call $\mathcal{I}$ a *rank-1, torsion-free sheaf* if it is torsion free and generically isomorphic to $\mathcal{O}_X$.

There is also the relative version.

Definition 2.2.1.4. [2, Definition 5.10] Let $X \xrightarrow{f} S$ be a finitely presented morphism with integral geometric fibers. We call an $\mathcal{O}_X$-module $\mathcal{I}$ *relatively rank-1, torsion-free* over S if it is locally finitely presented and flat over S and if for each geometric point s in S, the fiber $\mathcal{I}(s)$ is a rank-1, torsion-free $\mathcal{O}_{X_s}$-module.

We can now define the functor which is the target of the Abel-Jacobi map.

Definition 2.2.1.5. [2, Definition 5.5] Let $X \xrightarrow{f} S$ be a morphism of schemes. We define the functor

$$\mathbf{Spl}_{(X/S)} : \boldsymbol{Sch}_S \to \boldsymbol{Set}$$

to be the functor which assigns to each S-scheme T, the set of equivalence classes of T-simple $\mathcal{O}_{X_T}$-modules, where two modules $\mathcal{I}$ and $\mathcal{J}$ are defined to be equivalent if there exists a line bundle $\mathcal{N}$ on T such that

$$\mathcal{I} \otimes_{\mathcal{O}_S} N \cong \mathcal{J}$$

Similarly, we define the *compactified Picard functor* to be the subfunctor

$$\overline{\mathbf{Pic}}_{(X/S)} \subset \mathbf{Spl}_{(X/S)}$$

which maps an S-scheme T to the classes in $\mathbf{Spl}_{(X/S)}(T)$ which are represented by a relatively rank-1 torsion-free $\mathcal{O}_{X_T}$-module.

As usual, we define $\mathbf{Spl}_{(X/S)(\text{ét})}$ and $\overline{\mathbf{Pic}}_{(X/S)(\text{ét})}$ to be the associated sheaves in the étale topology. However, this distinction will not play a role in our application. Just like with the **Quot** functor, after fixing a very ample line bundle on X, for a given polynomial ϕ we denote by $\mathbf{Spl}^{\phi}_{(X/S)(\text{ét})}$ and $\overline{\mathbf{Pic}}^{\phi}_{(X/S)(\text{ét})}$ the subfunctors defined by the additional condition that the $\mathcal{O}_{X_T}$ module $\mathcal{I}$ have a Hilbert polynomial ϕ.

Similarly, the source of the Abel-Jacobi map is defined as follow.

Definition 2.2.1.6. [2, Definition 5.14] Let $X \xrightarrow{f} S$ be a finitely presented proper morphism of schemes and let $\mathcal{F}$ be a locally finitely presented $\mathcal{O}_X$-module. We define the functor

$$\mathbf{Smp}_{(\mathcal{F}/X/S)} \subset \mathbf{Quot}_{(\mathcal{F}/X/S)}$$

to be the functor which assigns to each S-scheme T, the quotients $\mathcal{G} \in \mathbf{Quot}_{(\mathcal{F}/X/S)}$ whose pseudo-ideals $\mathcal{I}(\mathcal{G})$ are simple over S.

Finally, we can define the Abel-Jacobi map.

Definition 2.2.1.7. [2, Section 5.16] Let $X \xrightarrow{f} S$ be a proper, finitely presented morphism of schemes, and let $\mathcal{F}$ be a locally finitely presented $\mathcal{O}_X$-module. We define the *Abel-Jacobi map associated to* $\mathcal{F}$ to be the map of functors:

$$\mathcal{A}_{(\mathcal{F}/X/S)} : \mathbf{Smp}_{(\mathcal{F}/X/S)} \to \mathbf{Spl}_{(X/S)(\text{ét})}$$

which takes a quotient $\mathcal{G}$ of $\mathcal{F}$ to the equivalence class of its pseudo-ideal $\mathcal{I}(\mathcal{G})$.

Note that when X is a smooth curve over a field and $\mathcal{F} = \mathcal{O}_X$, this is the standard Abel-Jacobi map of a curve. As in that case, the fibers of the Abel-Jacobi map are naturally linear systems in the following sense.

Theorem 2.2.1.3. *[2, Lemma 5.17, Theorem 5.18] Let $X \xrightarrow{f} S$ be a proper, finitely presented morphism of schemes with integral geometric fibers, and let $\mathcal{F}$ be an S-flat, locally finitely presented $\mathcal{O}_X$-module. Let T be an S-scheme and let $\mathcal{I}$ be a T-simple $\mathcal{O}_{X_T}$-module. Furthermore, suppose that the geometric fibers of f are integral and that $\mathcal{I}$ and $\mathcal{F}$ are relatively rank-1 torsion-free. Then we have the following commutative diagram with a right Cartesian square:*

$$\begin{array}{ccccc} \mathbb{P}(H(\mathcal{I},\mathcal{F}_T)) & \xrightarrow{\cong} & \mathbf{LinSyst}_{(\mathcal{I},\mathcal{F}_T)} & \longrightarrow & \mathbf{Smp}_{(\mathcal{F}/X/S)} \\ & \searrow & \downarrow & & \downarrow \mathcal{A}_{\mathcal{F}} \\ & & T & \xrightarrow{\tau_{\mathcal{I}}} & \mathbf{Spl}_{(X/S)(\acute{e}t)} \end{array}$$

where $\mathbb{P}(H(\mathcal{I},\mathcal{F}_T))$ and T stand for the respective functors of points and $\tau_{\mathcal{I}}$ is the map of functors taking a T-scheme Y to the Y-simple ideal $\mathcal{I}_Y$ on $Y \times_S X$.

In particular, for every geometric point t of $\mathbf{Spl}_{(X/S)(\acute{e}t)}$, if $\mathcal{I}$ is a representing $\mathcal{O}_{X_t}$-module then

$$\dim(\mathcal{A}_{\mathcal{F}}^{-1}(t)) = \dim_{k(t)}(\mathrm{Hom}_{X(t)}(H(\mathcal{I},\mathcal{F}), k(t))) - 1 = \dim_{k(t)}(\mathrm{Hom}_{X(t)}(\mathcal{I},\mathcal{F})) - 1$$

provided that if there exists a non-zero map from $\mathcal{I}$ to $\mathcal{F}(t)$ then there exists an injective map from $\mathcal{I}$ to $\mathcal{F}(t)$.

This follows almost immediately from theorem 2.2.1.1.

The following theorem is a fairly straightforward corollary as well.

Theorem 2.2.1.4. *[2, Theorem 5.20, Remark 5.21] Let $X \xrightarrow{f} S$ be a finitely presented, proper morphism of schemes, whose geometric fibers are integral. Let $\mathcal{F}$ be a relatively rank-1 torsion-free $\mathcal{O}_X$-module. Suppose that P represents $\overline{\mathbf{Pic}}_{(X/S)(\acute{e}t)}$. Then:*

1. *The restriction of $\mathcal{A}_{\mathcal{F}}|_P$ is proper and finitely presented.*
2. *Assume that f is projective and that the fibers $X(s)$ (resp. $\mathcal{F}(s)$) all have the same Hilbert polynomial ϕ (resp. ψ). Then for each polynomial θ, the restriction $\mathcal{A}|_{P^\theta}$ is strongly projective. This means that it factors as a closed embedding into a projective bundle over P^θ, followed by the projection.*
3. *Assume there exists a universal $\mathcal{O}_{X_P}$-module $\mathcal{I}$ such that $(P, \mathcal{I})$ represents $\overline{\mathbf{Pic}}_{(X/S)(\acute{e}t)}$. Then $\mathcal{A}_{\mathcal{F}}|_P$ is equal to the structure map of $\mathbb{P}(H(\mathcal{I}, \mathcal{F}_P))$ over P. Furthermore, this condition holds if the smooth locus of X/S admits a section.*

Note that condition in part 3 of the theorem implies in particular that $\overline{\mathbf{Pic}}_{(X/S)}$ is already an étale sheaf and so $\overline{\mathbf{Pic}}_{(X/S)} = \overline{\mathbf{Pic}}_{(X/S)(\text{ét})}$

From the discussion up to this point we can see that if we forget the gnarly issue of representability, the structure of the Abel-Jacobi map as a natural transformation of functors is fairly transparent and requires minimal machinery. Furthermore, nothing up to this point depended on special facts about curves. Since our current application will only require access to this functorial description, using this setup will allow us to obtain cleaner statements and proofs.

We now narrow our focus to the case where $X \xrightarrow{f} S$ is a family of integral projective curves. In other words, f is a flat, locally projective, finitely presented morphism of schemes whose geometric fibers are integral curves.

The advantage of this is that Riemann Roch now allows us to satisfy the conditions of theorem 2.2.1.2 fairly easily. Combined with theorem 2.2.1.4, this will force the Abel Jacobi map to be the structure map of a projective bundle in sufficiently general situations. Another benefit is that we now have the following representability result.

Theorem 2.2.1.5. *[2, Theorem 8.1] Let $X \to S$ be a locally projective, finitely presented, flat morphism of schemes whose geometric fibers are integral curves. Then, $\overline{\mathbf{Pic}}_{(X/S)(\acute{e}t)}$ is represented by a disjoint union $P = \coprod P_n$ of S-schemes, $P_n = \overline{\mathrm{Pic}}^n_{(X/S)(\acute{e}t)}$, and P_n parametrizes the rank-1 torsion-free sheaves with Euler characteristic n on the fibers of X/S.*

Furthermore, in this case, if $\mathcal{F}$ is a locally finitely presented S-flat $\mathcal{O}_X$-module, then we have [2, Equation 8.2.1]:

$$\mathbf{Smp}_{(\mathcal{F}/X/S)} = \mathbf{Quot}_{(\mathcal{F}/X/S)}$$

Therefore, in this case the restriction of the Abel-Jacobi map to the compactified Picard scheme takes the following more familiar form:

$$\mathcal{A}_{\mathcal{F}} : \mathrm{Quot}_{(\mathcal{F}/X/S)} \to P = \overline{\mathrm{Pic}}_{(X/S)(\text{ét})}$$

Where $\mathrm{Quot}_{(\mathcal{F}/X/S)}$ is the scheme representing the functor $\mathbf{Quot}_{(\mathcal{F}/X/S)}$.

In addition, if $\chi(\mathcal{F}(s))$ is independent of the point s in S, the map decomposes as

$$\mathcal{A}^d_{\mathcal{F}} : \mathrm{Quot}^d_{(\mathcal{F}/X/S)} \to P_n$$

where $n = \chi(\mathcal{F}(s)) - d$. For example, this happens if $\mathcal{F}$ is the relative dualizing sheaf ω of X/S and the fibers of f have the same arithmetic genus.

The final result that we will need now follows fairly easily from Riemann-Roch combined with theorems 2.2.1.2 and 2.2.1.4 as we mentioned earlier.

Theorem 2.2.1.6. *[2, Theorem 8.4] Let $X \to S$ be a flat, finitely presented, locally projective morphism whose geometric fibers are integral curves with the same geometric genus p. Let ω denote the relative dualizing sheaf. Fix an integer $d \geq 2p-1$. Then the d-th part of the Abel-Jacobi map*

$$\mathcal{A}^d_{\omega} : \mathrm{Quot}^d_{(\omega/X/S)} \to P_{p-1-d}$$

is smooth with relative dimension $d - p$.

Note that for any line bundle $\mathcal{L}$ on X and any integer n, tensoring with $\mathcal{L}$ defines an isomorphism

$$\nu_{\mathcal{L}} : P_n \to P_m$$

where $m = n + \chi(\mathcal{L})$

2.2.2 The Riemann Kempf Formula for Integral Curves

In this section we prove a generalization of the Riemann Kempf formula for integral curves. The notation here is taken from section 2.2.1. In particular, P_d will denote $\overline{\mathrm{Pic}}^d_{(X/S)(\text{ét})}$ and

$$\mathcal{A}^d_{\omega} : \mathrm{Quot}^d_{(\omega/X/S)} \to P_{p-1-d}$$

is the d-th part of the Abel-Jacobi map.

Theorem 2.2.2.1. *(*Riemann Kempf for integral curves*) Let k be an algebraically closed field of characteristic 0 and let X/k an integral projective k-scheme with arithmetic genus p.*

Let d be an integer, Z be an equi-dimensional subscheme of P_{p-1-d}, $\mathrm{Quot}^d_{(\omega/X/k)Z}$ the pullback of $\mathrm{Quot}^d_{(\omega/X/k)}$ and $\mathcal{A}^d_{\omega Z}$ the pullback of $\mathcal{A}^d_{\omega}$.

Let x be a k point of Z corresponding to a rank-1 torsion free sheaf $\mathcal{I}$ and set $r = h^1(X, \mathcal{I}) - 1$ (so that $(\mathcal{A}^d_{\omega})^{-1}(x) \cong \mathbb{P}^r_k$).

Then,

$$\begin{aligned} s((\mathcal{A}^d_{\omega Z})^{-1}(x), \mathrm{Quot}^d_{(\omega/X/k)Z}) &= \mathrm{mult}_x Z \cdot (1+h)^{p-d+r} \cap [(\mathcal{A}^d_{\omega Z})^{-1}(x)] \\ &= \mathrm{mult}_x Z \cdot \binom{p-d+r}{r} \end{aligned}$$

where h is the first Chern class of the canonical bundle on $\mathbb{P}^r_k$.

Remark 2.2.2.1. The condition that Z be equi-dimensional comes from the conditions of proposition [12, p. 4.2]. In general, proper pushforward of Segre classes in ill defined for schemes with components of varying dimensions.

When X is an integral curve with at worst planar singularities, both the Picard schemes and the Quot schemes that we are considering are integral ([3, Corollary 7]). This allows us obtain the following strengthening of the theorem.

Corollary 2.2.2.1. *(*Riemann Kempf for planar curves*) Let k be a algebraically closed field of characteristic* 0 *and let X be a projective integral curve of arithmetic genus p over k with at worst planar singularities. Let x be a k-point of P_{p-1-d} corresponding to a rank-1 torsion free sheaf $\mathcal{I}$ and set $r = h^1(X, \mathcal{I}) - 1$ (so that $(\mathcal{A}^d_\omega)^{-1}(x) \cong \mathbb{P}^r_k$). As before, let W_d be the scheme theoretic image of $\mathcal{A}^d_\omega$.*

Then

$$\begin{aligned} \mathrm{mult}_x W_d &= \mathrm{mult}_x P_{p-1-d} \cdot (1+h)^{p-d+r} \cap [(\mathcal{A}^d_\omega)^{-1}(p)] \\ &= \mathrm{mult}_x P_{p-1-d} \cdot \binom{p-d+r}{r} \end{aligned}$$

where h is the first Chern class of the canonical bundle on $\mathbb{P}^r_k$. In particular, for $d = p-1$ we obtain the Riemann singularity theorem.

Proof. By [3, Corollary 7], P_{p-1-d} and $\mathrm{Quot}^d_{(\omega/X/k)}$ are integral. The corollary then follows immediately from the functoriality of Segre classes and by applying theorem 2.2.2.1 to the case where $Z = P_{p-1-d}$. □

Note that in the case where $d = p-1$, corollary 2.2.2.1 recovers a theorem proved by S. Casalania-Martin and J. Kass [7, Theorem A].

We now turn to the proof of theorem 2.2.2.1.

We will follow the strategy of Fulton's proof in example [12, Example 4.3.2]. As in that example, the key step will be to reduce the general case to case where d is large. To this end, for a fixed d, choose s such that $d + s > 2p - 1$. For each i and $1 \leq j \leq s$ we will define natural maps of functors

$$\mathbf{Quot}^i_{(\omega/X/k)} \xrightarrow{q_j} \mathbf{Quot}^{i+1}_{(\omega/X/k)} \tag{2.2.2.1}$$

. The maps q_j all increase the degree by one so the degree i will be implied by the context.

Let $x_1, \ldots, x_s$ be distinct k-points in the smooth locus of X and let $\mathcal{I}_{x_i}$ denote their ideal sheaves. For each i and $1 \leq j \leq s$ we define the map

$$\mathbf{Quot}^{i}_{(\omega/X/k)} \xrightarrow{q_j} \mathbf{Quot}^{i+1}_{(\omega/X/k)}$$

as follows. Let T be an S scheme. Then our map will send a quotient

$$0 \to \mathcal{I}(G) \to \omega \to \mathcal{G} \to 0$$

in $\mathbf{Quot}^{i}_{(\omega/X/k)}(T)$ to the quotient

$$0 \to \mathcal{I}(G) \otimes \mathcal{I}_{x_j} \to \omega \to \mathcal{G}' \to 0$$

in $\mathbf{Quot}^{i+1}_{(\omega/X/k)}(T)$ where $\mathcal{G}'$ is defined to be the quotient. One way to verify that this is in fact a morphism of functors is to note that X has an affine cover

$$X = (X \setminus x_j) \coprod X^{\mathrm{sm}}$$

such that on the first chart $\mathcal{I}_{x_j}$ is trivial, and on the second chart both $\mathcal{I}_{x_j}$ and ω are line bundles.

Remark 2.2.2.2. These morphisms are related by the following fiber diagram

$$\begin{array}{ccc} \mathbf{Quot}^{i}_{(\omega/X/k)} & \xrightarrow{q_l} & \mathbf{Quot}^{i+1}_{(\omega/X/k)} \\ \downarrow q_j & & \downarrow q_j \\ \mathbf{Quot}^{i+1}_{(\omega/X/k)} & \xrightarrow{q_l} & \mathbf{Quot}^{i+2}_{(\omega/X/k)} \end{array}$$

for any i and $1 \leq j, k \leq s$.

As we'll soon see, the maps q_j are closed embeddings so at the level of the schemes representing these functors, this implies that composing the maps q_l and q_j is equivalent to taking the intersection of the images of these maps.

A key ingredient in the proof of theorem 2.2.2.1 is that the maps q_j are tubular regular embeddings (lemma 2.2.3.3). This fact will allow us to transfer intersection theoretic data between Quot schemes of different degrees.

Proof of theorem 2.2.2.1:
As we mentioned above, we will follow Fulton's strategy from [12, Example 4.3.2] of first proving the theorem in the case where d is large, and then proving it in general by quantifying what changes as we pass from lower to higher degrees.

Proof when d is large:

When $d \geq 2p-1$, then by theorem 2.2.1.6, $\mathcal{A}^d_\omega$ is a smooth map and in particular it is flat. Therefore,

$$s((\mathcal{A}^d_{\omega Z})^{-1}(x), \mathrm{Quot}^d_{(\omega/X/k)Z}) = (\mathcal{A}^d_{\omega Z})^* s(x, Z) = \mathrm{mult}_x Z \cdot [(\mathcal{A}^d_{\omega Z})^{-1}(x)]$$

Calculation of $s((\mathcal{A}^d_{\omega Z})^{-1}(x), \mathrm{Quot}^{d+s}_{(\omega/X/k)Z})$ using $s((\mathcal{A}^{d+s}_{\omega Z})^{-1}(x), \mathrm{Quot}^{d+s}_{(\omega/X/k)Z})$ when $d+s$ is large:

By theorem 2.2.1.3, we know that

$$(\mathcal{A}^d_{\omega Z})^{-1}(x) \cong \mathbb{P}^r_k$$

for some r and together with theorem 2.2.1.6 we know that

$$(\mathcal{A}^{d+s}_{\omega Z})^{-1}(x) \cong \mathbb{P}^{d+s-p}_k$$

For the rest of this step we will identify the fibers with these projective spaces. By the previous step, we know that

$$s(\mathbb{P}^{d+s-p}_k, \mathrm{Quot}^{d+s-p}_{(\omega/X/k)Z}) = \mathrm{mult}_x Z \cdot [\mathbb{P}^{d+s-p}_k]$$

Furthermore, we have the following composition of closed embeddings

$$\mathbb{P}^r \xrightarrow{i} \mathbb{P}^{d+s-p} \to \mathrm{Quot}^{d+s}_{(\omega/X/S)Z}$$

and by lemma 2.2.3.4 the first embedding is regular and the Chern class of its normal bundle is

$$(1+h)^{d+s-p-r}$$

where h is the first Chern class of the canonical line bundle on $\mathbb{P}^r_k$. Therefore, by lemma 2.2.3.7

$$\begin{aligned}\mathrm{mult}_x Z \cdot [\mathbb{P}^r] &= i^* s(\mathbb{P}^{d+s-p}_k, \mathrm{Quot}^{d+s}_{(\omega/X/S)Z}) \\ &= (1+h)^{d+s-p-r} \cap s(\mathbb{P}^r_k, \mathrm{Quot}^{d+s}_{(\omega/X/S)Z})\end{aligned}$$

Calculation of $s((\mathcal{A}^d_{\omega Z})^{-1}(x), \mathrm{Quot}^{d}_{(\omega/X/k)Z})$ using $s((\mathcal{A}^{d}_{\omega Z})^{-1}(x), \mathrm{Quot}^{d+s}_{(\omega/X/k)Z})$ when $d+s$ is large:

As before, we have

$$(\mathcal{A}^d_{\omega Z})^{-1}(x) \cong \mathbb{P}^r_k$$

Consider the sequence of embeddings

$$\mathbb{P}^r_k \to \mathrm{Quot}^d_{(\omega/X/k)Z} \xrightarrow{q} \mathrm{Quot}^{d+s}_{(\omega/X/k)Z}$$

where q is equal to the composition $q_1 \circ q_2 \cdots \circ q_s$. By lemma 2.2.3.5 combined with remark 2.2.2.2, the restriction of the normal bundle of the embedding q to $\mathbb{P}^r_k$ is $(1+h)^s$. Therefore, by lemma 2.2.3.3 and our cancellation theorem for Segre classes (theorem 2.0.3.2)

$$s(\mathbb{P}^r_k, \mathrm{Quot}^d_{(\omega/X/k)Z}) = (1+h)^s \cap s(\mathbb{P}^r, \mathrm{Quot}^{d+s}_{(\omega/X/k)Z}) =$$

$$\mathrm{mult}_x Z \cdot \frac{(1+h)^s}{(1+h)^{d+s-p-r}} \cap [\mathbb{P}^r_k] = \mathrm{mult}_x Z \cdot (1+h)^{p-d+r} \cap [\mathbb{P}^r_k]$$

□

2.2.3 A Collection of Technical Lemmas

The primary goal of this section is to prove that the maps q_i defined in equation 2.2.2.1 are tubular regular embeddings (definition 2.0.3.1). We also prove the additional lemmas which were used in the proof of theorem 2.2.2.1.

As preparation we first present a fairly well known deformation theory result which we prove here for completeness.

Lemma 2.2.3.1. *([14, Example 14.3]) Let k be an algebraically closed field, let $\mathcal{F}$ be a functor*

$$\mathcal{F} : \boldsymbol{Sch}_k \to \boldsymbol{Set}$$

and let X_0 be an element of $\mathcal{F}(\mathrm{Spec}(k))$. We can define a local functor

$$\mathbf{F} : \boldsymbol{C} \to \boldsymbol{Set}$$

where $\boldsymbol{C}$ is the category of finitely generated local Artin rings over k with residue field k, by sending an element $A \in \boldsymbol{C}$ to the subset of $\mathcal{F}(\mathrm{Spec}(A))$ consisting of those elements $X \in \mathcal{F}(\mathrm{Spec}(A))$ that reduce to $X_0 \in \mathcal{F}(\mathrm{Spec}(k))$ under the natural pull-back morphism. We call this the functor of local deformations.

If $\mathcal{F}$ is representable by a scheme M and a family $\mathcal{I} \in \mathcal{F}(M)$ such that X_0 corresponds to the point x_0 in M, then $\mathbf{F}$ is pro-representable by the complete local ring $\hat{\mathcal{O}}_{M,x_0}$

All of the terms in this lemma are defined in [14, cpt. 14].

Lemma 2.2.3.2. *Let $x = [\omega \twoheadrightarrow \mathcal{G}]$ be an element of $\mathrm{Quot}^d_{(\omega/X/k)}$ and let $\mathbf{F}$ be the local deformation functor of x. Suppose that the support of $\mathcal{G}$ is the disjoint union of finite k-schemes Z_1 and Z_2 which have lengths d_1 and d_2 respectively. Let $\mathcal{G}_i$ be the subsheaf of $\mathcal{G}$ supported at Z_i and let $x_i = [\omega \twoheadrightarrow \mathcal{G}_i] \in \mathrm{Quot}^{d_i}_{(\omega/X/k)}$ be the associated quotients. Finally, let $\mathbf{F}_i$ be the local deformation functor of x_i. Then,*

$$\mathbf{F} \cong \mathbf{F}_1 \times \mathbf{F}_2$$

Proof. We start be constructing a map

$$\phi : \mathbf{F}_1 \times \mathbf{F}_2 \to \mathbf{F}$$

Let A be a local Artin ring over k and let $x_i = \omega \twoheadrightarrow \mathcal{G}'_i$ be elements of $\mathbf{F}_i(A)$ for $i = 1, 2$. By definition, $\mathcal{G}'_i$ is a sheaf on $X \times_k \mathrm{Spec}(A)$ which is flat over A and the special fiber $(\omega \twoheadrightarrow \mathcal{G}'_i)_0$ is isomorphic to $\omega \twoheadrightarrow \mathcal{G}_i$.

Since the pullback map from X_0 to $X \times_k \mathrm{Spec}(A)$ is a homeomorphism, the support of $\mathcal{G}'_1$ is contained in an open set U_1 which is disjoint from the support of $\mathcal{G}_2$ and vice versa. Therefore, we can define $\mathcal{G}'$ to be the sheaf on $X \times_k \mathrm{Spec}(A)$ whose restriction to U_i is equal to $\mathcal{G}'_i$. We similarly can define the map $x = \omega \twoheadrightarrow \mathcal{G}'$.

As $\mathcal{G}'$ is flat over A and $\omega \twoheadrightarrow \mathcal{G}'$ restricts to $\omega \twoheadrightarrow \mathcal{G}$ on X_0, we can define

$$\phi(x_1, x_2) = x$$

We now construct a map

$$\psi : \mathbf{F} \to \mathbf{F}_1 \times \mathbf{F}_2$$

which will be the inverse of ϕ. As the construction is similar, we will provide less details than we did for the definition of ϕ.

Let A be a local Artin ring over k and let $x = \omega \twoheadrightarrow \mathcal{G}'$ be an element of $\mathbf{F}(A)$.

Since $(\mathcal{G}')_0$ is isomorphic to $\mathcal{G}$, the support of $\mathcal{G}'$ is the disjoint union of two schemes Z_1' and Z_2' which restrict to Z_1 and Z_2 over the special fiber. Let $x_i = \omega \twoheadrightarrow \mathcal{G}_i'$ be the map obtained by restricting x to Z_i. We can now define $\psi(x)$ as

$$\psi(x) = (x_1, x_2)$$

It is easy to see that ψ is the inverse of ϕ. □

Lemma 2.2.3.3. *For any integer i and $1 \leq j \leq s$, the morphism* $\mathrm{Quot}^i_{(\omega/X/k)} \xrightarrow{q_j} \mathrm{Quot}^{i+1}_{(\omega/X/k)}$ *induced by the corresponding map of functors is a tubular regular embedding (definition 2.0.3.1) of codimension one.*

Proof. To facilitate the notation, we will denote $\mathrm{Quot}^l_{(\omega/X/k)}$ by Q^l. Furthermore, we will prove the lemma for $j = 1$ and the rest of the lemma will follow by iterating this special case. We will also denote x_1 by x.

Let $u = [\omega \hookrightarrow \mathcal{O}_Z]$ be a k point of Q^i and let $v = [\omega \hookrightarrow \mathcal{O}_{(Z\cup x)}]$ be the image of u under the map q_1. Note that $Z \cup x$ is well defined since x is a smooth point of X.

We have to prove that there exists an isomorphism φ making the following diagram commute

$$\begin{array}{ccc} & & \mathrm{Spec}(\hat{\mathcal{O}}_{\mathrm{Q}^2,v}) \\ & \overset{\hat{q}_1}{\nearrow} & \\ \mathrm{Spec}(\hat{\mathcal{O}}_{\mathrm{Q}^1,u}) & & \cong \downarrow \varphi \\ & \underset{s}{\searrow} & \\ & & \mathrm{Spec}(\hat{\mathcal{O}}_{\mathrm{Q}^2,u}[[t]]) \end{array}$$

where s is the zero section.

By lemma 2.2.3.1, $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathrm{Q}^1,u})$ pro-represents a local deformation functor $\mathbf{F}_1$ of u at Q^1 and $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathrm{Q}^2,v})$ pro-represents a local deformation functor $\mathbf{F}_2$ of v at Q^2.

Let Z^{sm} and Z^{sing} be the restrictions of Z to the smooth and singular locus of X respectively. The schemes $(Z \cup x)^{\mathrm{sing}}$ and $(Z \cup x)^{\mathrm{sm}}$ are defined analogously.

Let $\mathbf{G}_1^{\mathrm{sing}}$ and $\mathbf{G}_1^{\mathrm{sm}}$ be the local deformation functors of the restriction of u to Z^{sing} and Z^{sm} respectively. By lemma 2.2.3.2,

$$\mathbf{F}_1 \cong \mathbf{G}_1^{\mathrm{sing}} \times \mathbf{G}_1^{\mathrm{sm}}$$

By applying the same construction to v we obtain the functors $\mathbf{G}_2^{\mathrm{sing}}$ and $\mathbf{G}_2^{\mathrm{sm}}$ and the decomposition

$$\mathbf{F}_2 \cong \mathbf{G}_2^{\mathrm{sing}} \times \mathbf{G}_2^{\mathrm{sm}}$$

Now, since $x \in X^{\mathrm{sm}}$, the map q_1 induces an isomorphism between $\mathbf{G}_1^{\mathrm{sing}}$ and $\mathbf{G}_2^{\mathrm{sing}}$. Therefore, it suffices to prove the existence of the isomorphism φ on the restriction of q_1 to $\mathbf{G}_1^{\mathrm{sm}}$.

Let $\mathbf{G}_x$ be the local deformation functor of the point x in X. Since x is a smooth point, $\mathbf{G}_x$ is pro-representable by $\hat{\mathbb{A}}_k^1$ so it will be enough to show that there exists an isomorphism φ that makes the following diagram commute:

$$\begin{array}{ccc} & & \mathbf{G}_2^{\mathrm{sm}} \\ & \overset{q_1}{\nearrow} & \cong \downarrow \varphi \\ \mathbf{G}_1^{\mathrm{sm}} & \underset{s}{\searrow} & \\ & & \mathbf{G}_1^{\mathrm{sm}} \times \mathbf{G}_x \end{array}$$

However, since we are now working on the smooth locus of X, this follows immediately from the analogous fact about the attaching map between Hilbert schemes of points on a smooth curve which is stated in [Stacks, Tag 0B9G]. □

We continue to discuss the maps q_i from equation 2.2.2.1. Recall that before defining these maps, we fixed an integer d, a large integer s and defined a map q_j for each $1 \leq j \leq s$.

Lemma 2.2.3.4. *For each integer i and $1 \leq j \leq s$, let $\mathrm{Quot}^i_{(\omega/X/k)} \xrightarrow{q_j} \mathrm{Quot}^{i+1}_{(\omega/X/k)}$ be the morphism defined in equation 2.2.2.1. Let $p_i = [\mathcal{I}]$ be a k-point of P_{p-i-1} and let $p_{i+1} = [\mathcal{I} \otimes \mathcal{I}_{x_j}]$ denote the corresponding element of P_{p-i-2}. Let*

$$\mathbb{P}_k^{r_i} \cong (\mathcal{A}_\omega^{d+i})^{-1}(p_i) \xrightarrow{u} (\mathcal{A}_\omega^{d+i+1})^{-1}(p_{i+1}) \cong \mathbb{P}_k^{r_{i+1}}$$

denote the map on fibers induced by q_j where r_i and r_{i+1} are defined to be the dimensions of the fibers. Then, u is a degree one embedding of projective spaces.

The proof of this lemma will be provided shortly. We also used the following similar looking lemma. Let $\mathcal{J}_i$ denote the tautological rank-1 torsion-free sheaf on $X \times P_{p-i-1}$. Recall that by theorem 2.2.1.2 and theorem 2.2.1.4, for each i, $\mathrm{Quot}^i_{(\omega/X/k)}$ is a projective scheme over P_{p-i-1} of the form $\mathbb{P}(H(\mathcal{J}_i, \omega))$ and the Abel Jacobi map is the structure map. Furthermore, by our choice of s, $H(\mathcal{I}_{d+s-1}, \omega)$ and $H(\mathcal{I}_{d+s}, \omega)$ are locally free.

Lemma 2.2.3.5. *For each $1 \leq j \leq s$, the map $\mathrm{Quot}^{d+s-1}_{(\omega/X/k)} \xrightarrow{q_j} \mathrm{Quot}^{d+s}_{(\omega/X/k)}$ embeds the projective bundle $\mathrm{Quot}^{d+s-1}_{(\omega/X/k)} \cong \mathbb{P}(H(\mathcal{J}_{d+s-1}, \omega))$ as a degree one Cartier divisor in the projective bundle $\mathrm{Quot}^{d+s}_{(\omega/X/k)} \cong \mathbb{P}(H(\mathcal{J}_{d+s}, \omega))$ and is cut out by a section of $\mathcal{O}_{\mathbb{P}(H(\mathcal{I}_{d+s}, \omega))}(1)$.*

In the interest of economy and clarity, we will prove both of these lemmas as special cases of the following general statement about linear systems.

Lemma 2.2.3.6. *Let l be a positive integer and $1 \leq j \leq s$. Let T be a P_{p-i-1} scheme and denote the induced rank-1 torsion-free sheaf on $X \times_k T$ by $\mathcal{I}$. Consider the map $\mathbf{Quot}^{l}_{(\omega/X/S)} \xrightarrow{q_j} \mathbf{Quot}^{l+1}_{(\omega/X/S)}$ and the induced map on the linear systems as shown in the following diagram of Cartesian squares:*

$$\begin{array}{ccccccc}
\mathbb{P}(H(\mathcal{I},\omega)) & \xrightarrow{\cong} & \mathbf{LinSyst}_{(\mathcal{I},\omega)} & \longrightarrow & \mathbf{Quot}^{i}_{(\omega/X/S)} & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle q_j} & & \\
\mathbb{P}(H(\mathcal{I}\otimes\mathcal{I}_{x_j},\omega)) & \xrightarrow{\cong} & \mathbf{LinSyst}_{(\mathcal{I}\otimes\mathcal{I}_{x_j},\omega)} & \longrightarrow & \mathbf{Quot}^{i+1}_{(\omega/X/S)} & \xrightarrow{\cong} & \mathbf{Quot}^{i+1}_{(\omega/X/S)} \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathcal{A}^{i+1}_{\omega}} \\
 & & T & \xrightarrow{\mu_{\mathcal{I}}} & P_{p-i-1} & \xrightarrow[\cong]{\nu_{\mathcal{I}_{x_j}}} & P_{p-i-2}
\end{array}$$

Then, the map on linear systems is induced by the canonical map

$$f : H(\mathcal{I}\otimes\mathcal{I}_{x_j},\omega) \to H(\mathcal{I},\omega)$$

coming from the natural map $\mathcal{I}\otimes\mathcal{I}_{x_j} \to \mathcal{I}$, and f is a surjection.

Proof. The first statement follows from unraveling the identification of linear systems with projective space in theorem 2.2.1.3. Since this is done in detail in the proof of this lemma in [3, p. 5.17], we won't reproduce it here.

We now show that the map

$$H(\mathcal{I}\otimes\mathcal{I}_{x_j},\omega) \to H(\mathcal{I},\omega)$$

is a surjection.

Consider the ideal sheaf sequence of $x_j \times_k T$ on $X \times_k T$:

$$0 \to \mathcal{I}_{x_j} \to \mathcal{O}_{X\times_k T} \to \mathcal{O}_{x_j\times T} \to 0$$

Note that we are considering $\mathcal{I}_{x_j}$ as an $\mathcal{O}_{X\times_k T}$-module in the natural way.

By tensoring this sequence with $\mathcal{I}$ we obtain the exact sequence

$$\mathcal{I}\otimes_{\mathcal{O}_{X\times T}}\mathcal{I}_{x_j} \to \mathcal{I} \to \mathcal{I}\otimes_{\mathcal{O}_{X\times T}}\mathcal{O}_{x_j\times T} \to 0$$

In fact, this sequence is also exact on the left. To see this, it suffices to prove the exactness on the complement of $x_j \times_k T$ and on $X^{\mathrm{sm}} \times_k T$. On the complement of $x_j \times_k T$ the map $\mathcal{I}\otimes_{\mathcal{O}_{X\times T}}\mathcal{I}_{x_j} \to \mathcal{I}$ is isomorphic to the identity map. On the other hand, on $X^{\mathrm{sm}} \times_k T$ the sheaf $\mathcal{I}$ is a locally free so tensoring with it is exact.

Let us denote the cokernel $\mathcal{I}\otimes_{\mathcal{O}_{X\times T}}\mathcal{O}_{x_j\times T}$ by τ. We thus have the short exact sequence

$$0 \to \mathcal{I}\otimes_{\mathcal{O}_{X\times T}}\mathcal{I}_{x_j} \to \mathcal{I} \to \tau \to 0 \tag{2.2.3.1}$$

Now, as mentioned in 2.2.1.2, the functor $H(\bullet, \omega)$ is covariant and right exact. By applying it to the sequence 2.2.3.1 we obtain the exact sequence

$$H(\mathcal{I} \otimes \mathcal{I}_{x_j}, \omega) \to H(\mathcal{I}, \omega) \to H(\tau, \omega) \to 0$$

Finally, note that since τ is torsion in each fiber of $X \times_k T$ over T, for any T-module M we have

$$\mathrm{Hom}_{X \times T}(\tau, \omega \otimes M) = 0$$

By the definition of $H(\tau, \omega)$, this implies that $H(\tau, \omega) = 0$ as required. □

We can now quickly deduce lemmas 2.2.3.4 and 2.2.3.5.

Proof of lemma 2.2.3.4: We apply lemma 2.2.3.6 in the case where T is the k point p_i. □

Proof of lemma 2.2.3.5: We apply lemma 2.2.3.6 in the case where T is P_{p-d-s} with the universal rank-1 torsion-free sheaf $\mathcal{J}_{d+s-1}$. By the choice of s, both $H(\mathcal{J}_{d+s-1}, \omega)$ and $H(\mathcal{J}_{d+s}, \omega)$ are locally free. Furthermore, by lemma 2.2.3.3, the embedding is of codimension one. □

We also need the following technical result about the Segre class of a subscheme of a fiber in a fiber bundle.

Lemma 2.2.3.7. *Let $X \to B$ be a Zariski-locally trivial fibration of k-varieties. Let $b \in B$ be a k point and let $i : Y \hookrightarrow X_b$ be a regular embedding. Then*

$$s(Y, X) = s(Y, X_b) \cap i^* s(X_b, X)$$

Proof. Since all of the Segre classes in question only depend on a Zariski open neighborhood of X_b in X, we can assume without loss of generality that $X \cong F \times B$ where $F = X_b$ and that the map π is the projection onto the second factor.

We will now show the following equality of cones:

$$C_Y X = C_Y F \oplus i^* C_F X$$

where we are using the same definition of the sum of cones as in example [12, Example 4.1.5]. Since $C_Y F$ is a vector bundle, the lemma will follow from the statement in that example.

To prove this claim about cones, we will explicitly exhibit the structure sheaf of $C_Y X$ as the tensor product of the structure sheaves of $C_Y F$ and $i^* C_F X$.

Let $\mathfrak{m}$ denote the maximal ideal defining b and let $\mathcal{I} \subset \mathcal{O}_F$ denote the ideal sheaf of Y in F.

We also define the following sheaves of graded $\mathcal{O}_Y$ algebras

$$\mathcal{R}_{Y/F} = \bigoplus_{i=0}^{\infty}(\mathcal{I}^i/\mathcal{I}^{i+1})$$
$$\mathcal{R}_{F/X} = \bigoplus_{i=0}^{\infty}(\mathfrak{m}^i/\mathfrak{m}^{i+1})$$
$$\mathcal{R}_{Y/X} = \bigoplus_{i=0}^{\infty}((\mathfrak{m}+\mathcal{I})^i/(\mathfrak{m}+\mathcal{I})^{i+1})$$

With this notation, $C_Y F = \underline{\mathrm{Spec}}_Y(\mathcal{R}_{Y/F})$, $i^*C_F X = \underline{\mathrm{Spec}}_Y(\mathcal{R}_{Y/F})$ and $C_Y X = \underline{\mathrm{Spec}}_Y(\mathcal{R}_{Y/X})$.

Consider the following bilinear map of graded $\mathcal{O}_Y$ algebras:

$$b : \mathcal{R}_{Y/F} \times \mathcal{R}_{F/X} \to \mathcal{R}_{Y/X}$$

which is defined by multiplication in $\mathcal{R}_{Y/X}$. We claim that with respect to this map $\mathcal{R}_{Y/X}$ has the universal property of $\mathcal{R}_{F/X} \otimes_{\mathcal{O}_Y} \mathcal{R}_{Y/F}$.

Indeed, let $f : \mathcal{R}_{Y/F} \times \mathcal{R}_{F/X} \to M$ be a bilinear map of $\mathcal{O}_Y$-modules. We must show that there exists a unique morphism $f' : \mathcal{R}_{Y/X} \to M$ such that $f = f' \circ b$.

We will show the existence of f as the uniqueness is clear. Let $a{\cdot}b$ be an element of $(\mathfrak{m}{+}\mathcal{I})^n$ where $a \in \mathfrak{m}^i$, $b \in \mathcal{I}^j$ and $i + j$=n. We define $f'(ab) = f(a, b)$. To show that this is well defined, we must show that if $ab \in (\mathfrak{m}{+}\mathcal{I})^{n+1}$ then $f'(ab) = 0$. Suppose that $ab \in (\mathfrak{m}{+}\mathcal{I})^{n+1}$. Then, either $a \in \mathfrak{m}^{i+1}$ or $b \in \mathcal{I}^{j+1}$, both of which imply that $f'(ab) = f(a, b) = 0$. □

Chapter 3

Euclidean Distance Degree

3.1 Introduction

Many engineering problems can be broken down into two fundemantal steps. The first one it to formulate a model of the system of interest which depends on a collection of unknown parameters. Next, you solve for the parameters based on observed data. Typically, each data point will impose a (polynomial) constraint on the parameters, and the hope is that with enough data points it is possible to uniquely determine the unknown coefficients.

In practice, no model is perfect and it may not be possible to solve for the coefficients precisely. Instead, one tries to find a collection of parameters that best explain the data.

As a concrete example, suppose you have a collection of n news articles, each of which comes from either the sports or finance sections. However, the articles are unfortunately uncategorized, and you would like to categorize them in an automatic fashion.

One technique for doing this is *Latent Semantic Indexing* (LSI).

The underlying assumption of this method is that the probability of a word appearing in a document depends only on the documents topic. More precisely, we make the stronger assumption that the i-th word carries a length 2 vector u_i and the j-th document has an associated length vector v_i such that the number of times that the i-th word appears in the j-th document is $u_i^T v_j$. Intuitively, u_i records how likely the i-th word is to appear in each topic, and v_j represents the relevance of each topic to the j-th document. This description is a model for the number of times each word appears in each document. The unknown parameters are the vectors u_i and v_j.

To applying LSI we first create a $v \times n$ matrix M where v is the size of the vocabulary, i.e, the number of distinct words in the set of documents. The value of M_{ij} is equal to the number of times that the i-th word appears in the j-th document. The matrix M records the real world observations.

To compute the parameters of the model, we factor this matrix as:

$$M = U \times V$$

where U is a $v \times 2$ matrix and V is a $2 \times n$ matrix.

In this case, we can find U and V by solving n^2 equations that are polynomial in the $4n$ entries of U and V.

However, our models assumptions are clearly an oversimplification, and we can not hope to literally solve this system. Instead, we try to find U and V which can best explain M, without doing so perfectly. This is achieved by minimizing the following loss function:

$$l(U, V) = ||U \times V - M||^2$$

Another example comes from the field of computer vision. Under the pinhole camera model, the map from the 3D world to the 2D image can be represented in projective coordinates by a 3×4 matrix P. Furthermore, this matrix is determined by the orientation of the camera and it's position. The orientation can be represented by a 3×3 orthogonal matrix R and the position by a length 3 column vector T. There are also a collection of *intrinsic* parameters that can be collected in a 3×3 matrix C. The assumption of the pinhole model implies that

$$P = C \times [R^{-1}|R^{-1}T]$$

Now, suppose that we have an image taken by a camera with known intrinsic parameters and we would like to determine the position and orientation of the camera when the image was taken. A standard approach is to identify objects in the image and measure their (projective) 3D world positions. We thus obtain a list of pairs (X_i, y_i) where X_i is a length 4 world vector and y_i is a length 3 vector representing the projetive coordinates of the image of X_i under the camera.

Theoretically, we could find R and T by solving the equations:

$$C \times [R^{-1}T|R^{-1}T] \times X_i = y_i$$

As before, this is a system of equations which is polynomial in the coefficients fo T and R^{-1}. However, the pinhole model is only a rough approximation of how a camera works and even if it were correct, there is no sense in which one can measure the location of a 3D object with infinite precision.

Therefore, one typically finds R and T by minimizing the following loss function:

$$l(R, T) = \sum_i ||C \times [R^{-1}T|R^{-1}T] \times X_i - y_i||^2$$

The common theme in these type of models is that one has a set of parameters $W\mathbb{R}^n$, and a function $f(W, X)$ which maps an input data point $X\mathbb{R}^m$ to an output $y = f(W, X) \in \mathbb{R}^k$. Given a collection of data (X_i, y_i), $1 \leq l$, we find W by minimizing the loss function:

$$l(W) = \sum_{i=1}^{l} ||f(W, X_i) - y_i||^2$$

In practice, $f(W, X)$ is frequently a polynomial function. This means that we can think of the data points X_i as defining an algebraic map

$$f(W) : \mathbb{R}^n \to \mathbb{R}^{k \cdot l}$$

which sends $W \in \mathbb{R}^n$ to the concatenation of the vectors $f(W, X_i)$ for $1 \leq i \leq l$. The image of this map defines a variety $V \subset \mathbb{R}^{k \cdot l}$. Let $y \in \mathbb{R}^{k \cdot l}$ be the concatenation of the vectors y_i, $1 \leq i \leq l$. Assuming a perfect model and data, y would be a point in V. In practice, y is close to V but not contained in it.

With this perspective, minimizing the loss function $l(W)$ is equivalent to finding a point v on the variety $V \subset \mathbb{R}^{k \cdot l}$ which minimizes the Euclidean distance to y. Another way of saying this is that we try to minimize the function

$$l : \mathbb{R}^{k \cdot l} \to \mathbb{R}$$

define by

$$l(y') = ||y' - y||^2$$

on the variety V.

Now that we've described the basic setup of the optimization problem, we can discuss the basic approach to solving it. In the general case, solving for W (or equivalently, a point on V) involves solving a non-convex optimization problem. A popular method for solving these types of problems is to start at a random vector on V, and iteratively update it via gradient descent. The process terminates when we reach a local minimum of $l(v)$ on V.

If there is only one such local minimum, then we can terminate the process. Otherwise, we can run the gradient descent procedure multiple times and take the local minimum with the smallest loss value. If we cared about finding the absolute minimum, we would have to run gradient decsent until all local minima were found. In this case, the number of iterations would be bounded below by the number of local minima.

In practice, one usually runs the procedure for a fixed number of times. The likelihood of finding the absolute minima in this manner is also determined by the overall number of local minima.

The upshot is that the number of local minima of $l(y')$ on V is a reasonable gauge for the difficulty of solving the given optimization problem. However, even calculating the number of local minima is not usually feasible in practice. Instead, it's easier to compute the upper bound given by the number of critical points over the complex numbers.

This brings us to the key observation behind the definition of the Euclidean distance degree.

Observation. *The difficulty of many optimization problems can be estimated by computing the number of critical points of the Euclidean distance function on an associated complex projective variety.*

This is the motivation behind the definition *Euclidean Distance Degree* (ED degree) which was first defined in [8] and which can be found in 3.2.6.1. Roughly speaking, the ED degree of a complex projective variety X is equal to the number of critical points of the Euclidean distance function $l(y') = ||y - y'||^2$ for a generic choice of y.

The goal of this chapter is to first develop machinery that is useful for computing the ED degree of varieties that come up in practice, and then apply it to two problems of interest.

The main issue that one encounters when computing the ED degree of a variety X associated to an optimization problem is that X is not typically smooth. This means that standard characteristic class computations are not sufficient. On the otherhand, in many cases X does come with a resolution of singularities that is somehow natural with respect to the problem it came from. Therefore, our strategy will be to develop methods for relating the ED degree of a singular variety X to the ED degree of a resolution $\tilde{X}$.

The structure of this chapter is as follows.

In section 3.2 we recall some standard types of characteristic classes that one can define for singular varieties. Next we define the Euclidean Distance degree, and develop the theory in a way that will be useful for the applications that follow.

The relevance of the charateristic classes theory comes from proposition 3.2.6.1, which relates the ED degree of a projective variety to it's polar classes and a certain Segre class.

Proposition. *Let $X \subset \mathbb{P}(V)$ be a projective variety, let $\overline{X} \xrightarrow{\pi} X$ be a Nash blowup, let $\mathcal{E}_{\overline{X}}$ be the modified Euclidean normal bundle of X, let $L := \mathcal{O}_{\mathcal{E}_{\overline{X}}}(1)$ denote the tautological bundle associated to $\mathbb{P}(\mathcal{E}_{\overline{X}})$, and let B denote the base locus of*

$$\mathrm{ed}_Q : \mathbb{P}(\mathcal{E}_{\overline{X}}) \dashrightarrow \mathbb{P}(V)$$

Then,

$$\deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}})) \geq 0$$

and

$$\sum_i \delta_i(X) - \mathrm{ED}(X) = \deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}}))$$

By proposition 3.2.3.1, this can reduce the problem of computing the ED degree to one of computing Mather classes.

Finally, it turns out to be easier to relate the Schwartz-Macpherson classes of X to those of a blowup $\tilde{X}$ than it is to relate the Mather classes. For this reason, we introduce the theory of Schwartz-Macpherson classes and Euler obstructions in section 3.2.4 and give some simple examples of how they can be used to compute Mather classes of singular varieties in section 3.2.5.

In section 3.3 we apply this theory to a problem in computer vision called 3D scene reconstruction. The objective of scene reconstruction is to recover the 3D geometry of a scene based of pictures of the scene that were taken from different positions and angles. A basic question in this field is how the difficulty of the problem varies with the number of pictures being used.

The complex projective variety associated to this optimization problem is the *Multiview Variety*. In section 3.3.3 we construct a resolution of this variety, and in the remainder of section 3.3 we use this resolution of compute the ED degree of the multiview variety as a function of the number of cameras involved.

In section 3.4 we focus on the problem of *weighted low rank approximation* which is a version of low rank approximation in which the Euclidean norm being used is not uniform in the entries of the matrix. This type of problem arrises when not all of the entries of the matrix we approximating are known with the same degree of certainty. An extreme version of this is when some of the entries are not known at all and is known as *matrix completion*.

We focus on the question of how the difficulty of weighted low rank approximation depends on the weight matrix. It is well known that the problem has a small number of critical points when the weight matrix is rank one. Our main result of section 3.4 is to determine for which weight matrices the number of critical points is maximal.

3.2 Characteristic Classes of Singular Varieties

3.2.1 Polar Classes

Let $X \hookrightarrow \mathbb{P}(V)$ be a projective variety with dimension r.

We will start with a quick but precise definition of the polar classes of X, and later explain different points of view.

Recall that the Gauss map is a rational map

$$\phi : X \dashrightarrow G(r+1, V^*)$$

The value of ϕ at a smooth point $x \in X$ is the $r+1$ dimensional quotient $V^* \twoheadrightarrow W_x^*$ such that the induced r dimensional linear space $\mathbb{P}(W_x) \hookrightarrow \mathbb{P}(V)$ is tangent to X at x.

To resolve this map, let $\overline{X} \hookrightarrow X \times G(r+1, V^*)$ be the closure of the graph of ϕ. The projection onto the first factor gives us a map $\overline{X} \xrightarrow{\overline{\pi}} X$ which is called the *Nash Blowup* of X. The projection onto the second factor gives a map $\overline{X} \xrightarrow{\overline{\phi}} G(r+1, V^*)$. Together this gives us the following commutative diagram

$$\begin{array}{ccc} \overline{X} & & \\ \downarrow{\scriptstyle \overline{\pi}} & \searrow^{\overline{\phi}} & \\ X & \overset{\phi}{\dashrightarrow} & G(r+1, V^*) \end{array}$$

The pullback of the universal quotient of $G(r+1, V^*)$ via $\overline{\phi}$ gives us a rank $r+1$ quotient

$$V^*_{\overline{X}} \twoheadrightarrow \mathcal{P}$$

We define the k-th polar class of X to be

$$[M_k(X)] = \overline{\pi}_* c_k(\mathcal{P})$$

It is also possible to define the bundle $\mathcal{P}$ in terms of the sheaf of principle parts of X. Recall that there is a map

$$V_X^* \to \mathrm{P}_X^1(L)$$

where L is the very ample line bundle corresponding to the embedding of X in $\mathbb{P}(V)$. This map is a surjection on the smooth locus of X, and is precisely the pullback of tautological quotient on $G(r+1, V^*)$ along the Gauss map ϕ.

This means that the quotient $V_{\overline{X}} \twoheadrightarrow \mathcal{P}$ on $\overline{X}$ extends the quotient $V_{X^{\mathrm{sm}}} \twoheadrightarrow \mathcal{P}^1_{X^{\mathrm{sm}}}(L)$ on X^{sm}.

It is not hard to show that if $p : Z \to X$ is any birational map such that there is a rank r quotient $V_Z \twoheadrightarrow \mathcal{P}$ extending $V_{X^{\mathrm{sm}}} \twoheadrightarrow \mathcal{P}^1_{X^{\mathrm{sm}}}(L)$ then $p_* c_k(\mathcal{P})$ is equal to the k-th polar class. The idea is that the map p will have to factor through $\overline{\pi}$, and then the result follows by the push pull formulas for Chern classes.

3.2.2 Mather Classes

The definition of Mather classes is somewhat similar to the above definition of Polar classes, but subtly different.

We start with an alternative construction of the Nash blowup. Let $G(r, \Omega^1_X) \to X$ be the relative Grassmanian of r quotients of Ω^1_X. There is a rational section $\psi : X \dashrightarrow G(r, \Omega^1_X)$ which is defined on the smooth locus of X. In fact, since Ω^1_X is itelf locally free with rank r, the map is given by the "identity" quotient.

Let $\tilde{X}$ be the closure of this section in the Grassmanian. As before, we have the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow{\scriptstyle \tilde{\pi}} & \searrow^{\tilde{\psi}} & \\ X & \overset{\psi}{\dashrightarrow} & G(r, \Omega^1_X) \end{array}$$

By pulling back the tautological quotient on $G(r, \Omega^1_X)$ via $\tilde{\psi}$ we get rank r locally free sheaf Ω on $\tilde{X}$ and and a quotient

$$\tilde{\pi}^* \Omega^1_X \twoheadrightarrow \Omega$$

.

We define the *Mather class* of X to be

$$c^M(X) = \tilde{\pi}_*(c(\Omega^*))$$

Similarly to what happens for polar classes, if $p : Z \to X$ is any birational map with a rank r quotient

$$p^* \Omega^1_X \to \Omega$$

then Ω can be used to compute the Mather class of X.

3.2.3 Relationship of Polar and Mather Classes

In this section we will see that Polar classes and Mather classes contain the same information, in the sense that one is easily computable from the other.

To see this, we will show that $\tilde{X}$ admits a rank $r+1$ quotient $V^*_{\tilde{X}} \twoheadrightarrow \mathcal{P}$ which extends $V^*_X \to \mathcal{P}^1_X(L)$. Since $\mathcal{P}$ can be used to compute polar classes, we can reduce the problem to relating Ω to $\mathcal{P}$.

In order to construct $\mathcal{P}$, let $i : X \hookrightarrow \mathbb{P}(V)$ be the projective embedding and consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ((i \circ \tilde{\pi})^* \Omega^1_{\mathbb{P}(V)}) \otimes L & \longrightarrow & V^*_{\tilde{X}} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & ((\tilde{\pi})^* \Omega^1_X) \otimes L & \longrightarrow & \tilde{\pi}^* \mathcal{P}^1_X(L) & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

If we pushout the bottom extension by $\tilde{\pi}^* \Omega^1_X \twoheadrightarrow \Omega$ then we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ((i \circ \tilde{\pi})^* \Omega^1_{\mathbb{P}(V)}) \otimes L & \longrightarrow & V^*_{\tilde{X}} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & ((\tilde{\pi})^* \Omega^1_X) \otimes L & \longrightarrow & \tilde{\pi}^* \mathcal{P}^1_X(L) & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega \otimes L & \longrightarrow & \mathcal{P} & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

This implies that the polar classes of X are given by

$$[M_k(X)] = \tilde{\pi}_* c_k(\mathcal{P})$$

On the otherhand, the Mather classes are given by

$$c^M(X) = \tilde{\pi}_* c(\Omega)$$

So using the exact sequence

$$0 \to \Omega \otimes L \to \mathcal{P} \to L \to 0$$

It is easy to prove the following formula

Proposition 3.2.3.1. *[23, Theorem 3]*

$$[M_k(X)] = \sum_{i=0}^{k} (-1)^{k-i} \binom{r+1-k+i}{i} c_1(L)^i c^M_{k-i}(X)$$

By a similar construction, there is a rank $r+1$ quotient $V^*_{G(r,\Omega^1_X)} \twoheadrightarrow Q$ on $G(r,\Omega^1_X)$ which induces the following map over X:

$$\begin{array}{ccc} G(r,\Omega^1_X) & \longrightarrow & X \times G(r+1,V) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

It is not hard to show that this map induces an isomorphism between the maps $\overline{\pi} : \overline{X} \to X$ and $\tilde{\pi} : \tilde{X} \to X$. For this reason we call both of these maps Nash blowups of X.

3.2.4 The Schwartz-Macpherson Class

The Schwartz-Macpherson class, originally defined in [20], is another way to assign a characteristic class to a variety which has the advantage of being functorial with respect to proper maps. In this section we will give a brief introduction to this class, mostly following [17].

In order to define this class, we must first discuss the constructible functions functor and the Euler obstruction.

The *constructible functions functor* is a functor **CF** from the category of varieties with proper maps to the category of groups. If X is a variety, then $\mathbf{CF}(X)$ is defined to be the group of integer valued constructible functions on X.

We now define the behavior of **CF** under proper morphisms. Let $f : X \to Y$ be a proper morphism. In order to define the map

$$\mathbf{CF}(f) : \mathbf{CF}(X) \to \mathbf{CF}(Y)$$

it suffices to specify the value it takes on the basis of indicator functions $\mathbb{1}_W$ when W is a closed subvariety of X. Let $W \subset X$ be a closed subvariety and $y \in Y$ a closed point. We define:

$$\mathbf{CF}(\mathbb{1}_W)(y) = \chi(f^{-1}(y) \cap W)$$

where χ is the topological Euler characteristic in the compactly supported homology.

The *Schwatrz-Macpherson class*, denoted by c^{SM}, is the unique natural transformation from the constructible functions functor to the Chow functor

$$c^{SM} : \mathbf{CF} \to \mathrm{A}_*$$

which sends the identity function $\mathbb{1}_X$ of a smooth variety X to the standard Chern class $c(X) \cap [X] \in \mathrm{A}_*(X)$.

In order to explicitely define c^{SM}, we will first introduce a different basis of the group of constructible functions called Euler obstructions. Let X be a variety. We will now define a constructible function Eu_X on X called the *Euler obstruction*.

Following [17], let $\tilde{X}$ be the Nash blowup of X as above. Recall that there is a rank $\dim(X)$ locally free sheaf Ω and a quotient:

$$\tilde{\pi}^*\Omega^1_X \twoheadrightarrow \Omega$$

. Let $T = \Omega^*$. For any point $p \in X$ define the Euler obstruction to be:

$$\mathrm{Eu}_X(p) = \int_{\tilde{\pi}^{-1}(p)} c(T|_{\tilde{\pi}^{-1}(p)}) \cap s(\tilde{\pi}^{-1}(p), \tilde{X})$$

.

Lemma 3.2.4.1. *[17, Lemma 4] Let X be a variety. Then, the function Eu_X is a constructible function.*

The following lemma is crucial for the definition of the Schwatrz-Macpherson class.

Lemma 3.2.4.2. *[17, Lemma 5] Let X be a variety. As W ranges over the closed subvarieties of X, the functions Eu_W form a basis for $\mathbf{CF}(X)$.*

By the previous lemma, it suffices to define c^{SM} on the constructible functions Eu_W, where W is a variety. In that case, we define:

$$c^{SM}(\mathrm{Eu}_W) = c^M(W)$$

where $c^M(W)$ is the Mather class defined above.

Proposition 3.2.4.1. *[17, Proposition 2] The map of functors*

$$c^{SM} : \mathbf{CF} \to \mathrm{A}_*$$

is a natural transformation.

By an abuse of notation, we will write $c^{SM}(X)$ as a shorthand for $c^{SM}(\mathbb{1}_X)$.

3.2.5 Examples

In this section we will work out a couple of example showing the interactions between the characteristic classes defined above.

3.2.5.1 Cusp Curve

Let V be a three dimensional vector space and let C be the cusp curve $(y^2z - x^3) \subset \mathbb{P}(V)$. To compute $c^M(C)$ we must first find a birational map $\tilde{\pi} : \tilde{C} \to C$ together with a rank 1 quotient $\tilde{\pi}^*\Omega^1_C \twoheadrightarrow \Omega$.

The first step is to consider the normalization map

$$\tilde{\pi} : \mathbb{P}^1 \to C$$

which is defined by $[s : t] \mapsto [s^2t : s^3 : t^3]$.

We have the standard contanget sequence on $\mathbb{P}^1$:

$$0 \to \tilde{\pi}^*\Omega^1_C \xrightarrow{\alpha} \Omega^1_{\mathbb{P}^1} \to \Omega_{\tilde{\pi}} \to 0$$

By [24, lem 1.1], the image of α is locally free iff the Fitting ideal of the cokernel $F^0(\Omega_{\tilde{\pi}})$ is locally free. In this case the Fitting ideal is the ideal of a (possibly reduced) point on a smooth curve so the image of α is indeed locally free. We will denote this image by Ω. Clearly we have an exact sequence

$$0 \to \Omega \to \Omega^1_{\mathbb{P}^1} \to \Omega_{\tilde{\pi}} \to 0$$

By the definition of the Mather class we have:

$$c_1^M(C) = \tilde{\pi}_* c_1(\Omega^*) = \tilde{\pi}_*(c_1(\Omega^*_{\mathbb{P}^1}) + c_1(\Omega_{\tilde{\pi}})) = \tilde{\pi}_*(c_1(\mathbb{P}^1) + c_1(\Omega_{\tilde{\pi}}))$$

By direct calculation, it is not hard to see that $c_1(\Omega_{\tilde{\pi}}) = h$ which means that

$$\text{deg}c^M(C) = 2 + 1 = 3$$

We can also calculate $c^M(C)$ using Chern-Macpherson classes. For this, recall the the value of the Euler function Eu_C at the cusp point p is equal to the multiplicity at that point which is 2. Therefore,

$$\mathbb{1}_C = \text{Eu}_C - \text{Eu}_p$$

which implies that

$$c(C) = c^M(C) - [p]$$

so $\text{deg}c(C) = 3 - 1 = 2$ which is consistent with $\chi(C) = 2$.

In addition, since $\tilde{\pi}_*(\mathbb{1}_{\mathbb{P}^1}) = \mathbb{1}_C$ we know that

$$\text{deg}c^M(C) = \text{deg}c(C) + 1 = \text{deg}c(\mathbb{P}^1) + 1 = 2 + 1 = 3$$

3.2.5.2 Nodal Curve

Let C be a nodal curve in $\mathbb{P}^2$. This is very similar to the cusp case the normalization is also $\mathbb{P}^1$. The only difference is that now $\Omega_{\tilde{\pi}} = 0$ so

$$\text{deg}(c^M(C)) = \text{deg}\tilde{\pi}_* c(\mathbb{P}^1) = 2$$

As before, this implies that

$$\text{deg}c(C) = \text{deg}c^M(C) - 1 = 1$$

which is consistent with $\chi(C) = 1$.

Also in this case we can compute $c^M(C)$ using only the fact that

$$\mathbb{1}_C = \text{Eu}_C - \text{Eu}_p$$

Indeed, this equality implies that

$$\tilde{\pi}_* \mathbb{1}_{\mathbb{P}^1} = \text{Eu}_C$$

so $c^M(C) = \tilde{\pi}_* c(\mathbb{P}^1)$.

3.2.5.3 Tangential Variety of the Twisted Cubic

The tangential variety of the twisted cubic C is a degree 4 singular surface $i : S \hookrightarrow \mathbb{P}^3$. By a Macaulay2 computation, the pushforward of its polar class to $\mathbb{P}^3$ is $3h^2 + 4h$.

We will now compute this class using purely topological considerations regarding Euler obstructions. First of all, note that we have a birational map

$$\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to S$$

which takes $([f], [g])$ to $[f^2 g]$ where f and g are degree one polynomials in two variables. In particular, if h is the generator of $A^\bullet(\mathbb{P}^3)$ and g_1 and g_2 are generators for $A^\bullet(\mathbb{P}^1 \times \mathbb{P}^1)$ then

$$\pi^* h = 2g_1 + g_2$$

Note that the restriction of π to the diagonal is the usual map from $\mathbb{P}^1$ to the twisted cubic C.

To compute the Mather classes of S, we will first try to express $\mathbb{1}_S$ in terms of Euler obstructions on S. Since the singular locus of S is the smooth curve C, it is reasonable to expect that

$$\mathbb{1}_S = \mathrm{Eu}_S - (\mathrm{Eu}_S(p) - 1)\mathrm{Eu}_C$$

where p is some point in C.

Locally at p, S looks like the product of a line with a cusp curve. So it is reasonable to assume that $\mathrm{Eu}_S(p) = 2$ which means that

$$\mathbb{1}_S = \mathrm{Eu}_S - \mathrm{Eu}_C$$

Now, since the preimage of any point in S under the map π is a single point, we have

$$\pi_* c(\mathbb{P}^1 \times \mathbb{P}^1) = c(\mathbb{1}_S) = c^M(S) - c^M(C)$$

It is easy to see that $c(\mathbb{P}^1 \times \mathbb{P}^1) = 1 + 2(g_1 + g_2) + 4g_1 g_2$. Therefore,

$$\begin{aligned}
\deg \pi_* c_0(\mathbb{P}^1 \times \mathbb{P}^1) &= \deg((2g_1 + g_2)^2 = 4 \\
\deg \pi_* c_1(\mathbb{P}^1 \times \mathbb{P}^1) &= \deg((2g_1 + g_2)(2(g_1 + g_2))) = 2\deg(3g_1 g_2) = 6 \\
\deg \pi_* c_2(\mathbb{P}^1 \times \mathbb{P}^1) &= \deg(4g_1 g_2) = 4
\end{aligned}$$

Together, we've shown that

$$i_* \pi_* c(\mathbb{P}^1 \times \mathbb{P}^2) = 4h + 6h^2 + 4h^3$$

In addition, since $\chi(C) = 2$ we have

$$i_* c^M(C) = i_* c(C) = 3h^2 + 2h^3$$

Putting everything together we have obtained the Mather class of S:

$$i_* c^M(S) = 4h + 9h^2 + 6h^3$$

Finally, we can use the formula from the previous section relating Mather classes to polar classes to obtain the polar classes of S. Explicitly,

$$\begin{aligned}
[M_0] =& c_0^M(S) = 4h \\
[M_1] =& -c_1^M(S) + 3hc_0^M(S) = -9h^2 + 3h \cdot 4h = 3h^2 \\
[M_2] =& c_2^M(S) - 2h \cdot c_1^M(S) + 3h^2 \cdot c_0^M(S) = 6h^3 - 18h^3 + 12h^3 = 0
\end{aligned}$$

3.2.6 The Euclidean Distance Degree

3.2.6.1 Definitions

In this section we recall the definition of the Euclidean distance degree from [**EDD**]. We will present it in a way that will be conventient for the remainder of the paper.

Let V be a complex vector space, let $Q(v)$ be a non-degenerate quadratic form on V, and let $\langle , \rangle$ denote the induced bilinear form. Also, let Q be the hypersurface in $\mathbb{P}(V)$ defined by $Q(v)$.

If $W \subset V$ is a subspace, then we will denote it's orthogonal complement with respect to $\langle , \rangle$ by $W^\perp$.

Now let $X \subset \mathbb{P}(V)$ be a subvariety of $\mathbb{P}(V)$ and let $u \in V$ be a general point. Following [**EDD**], the *Euclidean distance degree* of X is defined to be the number of smooth points $x \in C(X)$ such that

$$u - x \in T_x C(X)^\perp$$

Note that this is equal to the number of smooth points $[x] \in X$ for which there exists a scalar $\lambda \in \mathcal{C}^\times$ such that

$$u - \lambda x \in T_x C(X)^\perp$$

Finally, since u is general, we can assume that $u \notin T_x C(X)^\perp$ which means that this in turn is equal to the number of smooth points $[x] \in X$ such that

$$u \in \mathrm{Span}(T_x C(X)^\perp, x)$$

Now, we define the orthogonal complement to $\mathbb{T}_{[x]}(X)$ to be:

$$\mathbb{T}_{[x]}(X)^\perp := \mathbb{P}(\mathrm{Span}(T_x C(X)^\perp, x)) \subset \mathbb{P}(V)$$

We can now present our definition of the Euclidean distance degree.

Definition 3.2.6.1. Let $X \subset \mathbb{P}(V)$ be a projective variety. Let $[u] \in \mathbb{P}(V)$ be a general point. We define the *Euclidean distance degree* of X, denoted by $\mathrm{ED}(X)$, to be the number of smooth points $[x] \in X$ for which

$$[u] \in \mathbb{T}_{[x]}(X)^\perp$$

If it is not clear which quadric we are using, the ED degree with respect to a given quadric Q will be denoted by $\mathrm{ED}_Q(X)$.

3.2.6.2 The Euclidean Normal Bundle, Smooth Case

In this section we assume that $X \subset \mathbb{P}(V)$ is a projective variety with codimension d. Later we will see how to extend the results of this section to the general case.

Let $P^1_X(1)$ be the sheaf of principal parts of X. There is a natural map

$$\phi : V^*_X \to P^1_X(1)$$

which is a surjection since X is smooth. We will denote the kernel by $\mathcal{K}_X$. The sheaf $\mathcal{K}_X$ is locally free with rank d.

The *Euclidean normal bundle* to X is defined to be:

$$\mathcal{E}_X := \mathcal{K}_X \oplus \mathcal{O}_X(-1)$$

By composing the embedding $\mathcal{K}_X \hookrightarrow V^*_X$ with the isomophism $V^* \cong V$ induced by Q we obtain a map $\mathcal{K}_X \hookrightarrow V_X$. Together with the natural map $\mathcal{O}_X(-1) \hookrightarrow V_X$, we obtain a map

$$\psi : \mathcal{E}_X \to V_X$$

It is easy to see that for any point $[x] \in X$,

$$\mathbb{T}_{[x]}(X)^\perp \cong \psi(\mathbb{P}(\mathcal{E}_{X,[x]})) \subset \mathbb{P}(V)$$

Furthermore, the morphism ψ induces a rational map

$$\mathbb{P}(\mathcal{E}_X) \dashrightarrow \mathbb{P}(V)$$

which is defined over $X \setminus Q$.

The following lemma now follows immediately from the definition of the Euclidean distance degree.

Lemma 3.2.6.1. *Let $X \subset \mathbb{P}(V)$ be a smooth projective variety. Then, the Euclidean distance degree of X is equal to the degree of the natural rational map*

$$\mathbb{P}(\mathcal{E}_X) \dashrightarrow \mathbb{P}(V)$$

3.2.6.3 The Modified Euclidean Normal Bundle

We now extend lemma 3.2.6.1 to the case of a general projective variety $X \subset \mathbb{P}(V)$.

The idea is to modify X by a birational map in a way that allows us to extend $\mathcal{K}_X|_{X^{\mathrm{sm}}}$ to a vector bundle.

More formally, we will modify X by a Nash blowup in the following sense.

Definition 3.2.6.2. Let $X \subset \mathbb{P}(V)$ be a projective variety and let $\pi : \overline{X} \to X$ be a birational map which is an isomorphism over X^{sm}.

We say that $\overline{X}$ is a *Nash Blowup* of X if there exists a locally free quotient

$$V^*_{\overline{X}} \twoheadrightarrow \mathcal{P}$$

on $\overline{X}$ that extends the surjection

$$\pi^* V^*_{X^{\mathrm{sm}}} \twoheadrightarrow \pi^* P^1_{X^{\mathrm{sm}}}$$

We will call the bundle $\mathcal{P}$ the *modified bundle of principle parts.*

Remark 3.2.6.1. Let $X \subset \mathbb{P}(V)$ be a variety of dimension d The map Gauss map

$$\mathcal{G}_X : X \dashrightarrow G(d, V)$$

is the map defined by

$$V^*_X \to \mathcal{P}^1(X)$$

which is a surjection on the smooth locus of X. Therefore, a Nash blowup of X is the same thing as a birational extension of the Gauss map of X.

We can use the modified bundle of principle parts to define a modified Euclidean normal bundle.

Definition 3.2.6.3. Let $X \subset \mathbb{P}(V)$ be a projective variety, let $\overline{X} \to X$ be a Nash blowup, and let $\mathcal{P}_{\overline{X}}$ be the modified bundle of principal parts.

Let $\mathcal{K}_{\overline{X}}$ be the kernel of the surjection $V^*_{\overline{X}} \to \mathcal{P}_{\overline{X}}$. The *modified Euclidean normal bundle* is defined to be

$$\mathcal{E}_{\overline{X}} := \mathcal{K}_{\overline{X}} \oplus \mathcal{O}_{\overline{X}}(-1)$$

By composing the embedding $\mathcal{K} \to V^*_{\overline{X}}$ with the isomorphism $V^* \cong V$ induced by $Q(v)$, we obtain a natural map

$$\psi_Q : \mathcal{E}_{\overline{X}} \to V_{\overline{X}}$$

Suppose that $\pi : \overline{X} \to X$ is a Nash blowup. We will now see that we can use the modified Euclidean normal bundle $\mathcal{E}_{\overline{X}}$ to compute the Euclidean distance degree of X.

Lemma 3.2.6.2. *Let $X \subset \mathbb{P}(V)$ be a projective variety, let $Q \subset \mathbb{P}(V)$ be a non-degenerate quadric, let $\pi : \overline{X} \to X$ be a Nash blowup and let $\mathcal{E}_{\overline{X}}$ be the modified Euclidean normal bundle. Then, the Euclidean distance degree of X is equal to the degree of the rational map*

$$\mathrm{ed}_Q : \mathbb{P}(\mathcal{E}_{\overline{X}}) \dashrightarrow \mathbb{P}(V)$$

induced by the natural map ψ_Q.

Proof. Indeed, over the smooth locus of X, π is an isomorphism and $\mathcal{E}$ agrees with $\pi^*\mathcal{E}_X$. So the following lemma follows immediately from lemma 3.2.6.1. □

3.2.6.4 For which quadrics is the ED degree maximal?

We can use lemma 3.2.6.2 to analyze the dependence of the ED degree on the quadric Q. Set $n = \dim(V)$.

Proposition 3.2.6.1. *Let $X \subset \mathbb{P}(V)$ be a projective variety, let $\overline{X} \xrightarrow{\pi} X$ be a Nash blowup, let $\mathcal{E}_{\overline{X}}$ be the modified Euclidean normal bundle of X, let $L := \mathcal{O}_{\mathcal{E}_{\overline{X}}}(1)$ denote the tautological bundle associated to $\mathbb{P}(\mathcal{E}_{\overline{X}})$, and let B denote the base locus of*

$$\mathrm{ed}_Q : \mathbb{P}(\mathcal{E}_{\overline{X}}) \dashrightarrow \mathbb{P}(V)$$

Then,

$$\deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}})) \geq 0$$

and

$$\sum_i \delta_i(X) - \mathrm{ED}(X) = \deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}}))$$

Proof. By lemma 3.2.6.2 and proposition [12, Proposition 4.3],

$$\mathrm{ED}(X) = \deg(c_1(L)^n) - \deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}))$$

By the construction of the Euclidean bundle, it's easy to see that L is globally generated. Therefore, by lemma 3.2.6.3 below,

$$\deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{X}})) \geq 0$$

Furthermore, by the definition of the Segre class, we know that $\deg(s(\mathcal{E}_{\overline{X}})) = \deg(c_1(L)^n)$, and it is well known that $\deg(s(\mathcal{E}_{\overline{X}}))$ is equal to the sum of the polar classes $\delta_i(X)$ of X. □

Lemma 3.2.6.3. *Let X be a variety and set $n := \dim X$. Let L be a line bundle on X and let $V \subset \mathrm{H}^0(X, L)$ be a subspace of it's global sections such that $\dim V \leq n$. Let $B \subset X$ be the base locus of V.*

If $L|_B$ is globally generated then

$$\deg(c(L)^n s(B, X)) \geq 0$$

If in addition $L|_B$ is ample them

$$\deg(c(L)^n s(B, X)) > 0$$

Proof. Let $D_i := V(s_i)$. Since $\dim V \leq n$, there are sections $s_1, \ldots, s_n \in V$ such that

$$B = \bigcap_{i=1}^{n} D_i$$

Therefore, using the notation of [12, Chapter 9],

$$(D_1 \cdot \ldots \cdot D_n \cdot X)^B = c(L)^n s(B, X)$$

The lemma now follows from [12, Theorem 12.2]. □

Remark.

1. Note that the base locus B is determined by the quadric Q, but L is independent of Q.

2. Proposition 3.2.6.1 gives an alternative proof of theorem [**EDD**] which states that

$$\mathrm{ED}(X) \leq \sum \delta_i(X)$$

3.3 The Multiview Variety

3.3.1 Introduction

Suppose that a collection of cameras are used to generate images of a scene. The problem of *triangulation* is to deduce the world coordinates of an object from its position in each of the camera images. If we assume that the image points are given with infinite precision, then two cameras suffice to determine the world point. However, due to the many sources of noise in real images such as pixelization and distortion, there typically will not be an exact solution and we will instead try to find a world point whose picture is "as close as possible" to the image points.

More precisely, suppose the cameras are $C_1, \dots, C_N$ and the image points are $p_1, \dots, p_N \in \mathbb{R}^2$. The goal is to find a world point $q \in \mathbb{R}^3$ that minimizes the least squares error

$$\mathrm{error}(q) = \sum_{i=1}^{N} (C_i(q) - p_i)^2.$$

One application is the problem of reconstructing the 3D structure of a tourist attraction based on millions of online pictures. It is difficult to obtain the precise configuration of any single camera, so it would not make sense to use only a small subset of them and disregard the rest. A better approach is to solve an optimization problem which incorporates as many of the cameras as possible. This technique was used in [1] to reconstruct the entire city of Rome from two million online images.

Since the camera function $C_i : \mathbb{R}^3 \to \mathbb{R}^2$ is not linear, the standard method for solving the triangulation problem is to first find the critical points of $\mathrm{error}(q)$ (e.g, with gradient descent), and then select the one with the smallest error. In order to gauge the difficulty of this problem, it is important to be able to predict the number of critical points that we expect to find for a given configuration of cameras.

The goal of this paper is to give an explicit expression for the number of critical points of $\mathrm{error}(q)$ as a function of the number of cameras N. In fact, we compute this expression for a variation of the problem in which we allow the world points to take complex values, and we allow these points to be in the projective space $\mathbb{P}^3_{\mathcal{C}}$ as opposed to the affine space $\mathcal{C}^3$. Our main result is that the number of critical points of $\mathrm{error}(q)$ is polynomial in the number of cameras.

Theorem 3.3.1.1. *The number of critical points of* $\mathrm{error}(q)$ *on* $\mathbb{P}^3_{\mathcal{C}}$ *is equal to*

$$p(N) = 6N^3 - 15N^2 + 11N - 4$$

where $N \geq 3$ *is the number of cameras.*

By the argument in the proof of theorem 3.3.5.2, the polynomial $p(N)$ is an upper bound on the number of critical points in the complex affine version of the problem. One can solve the original real version by first finding these complex affine points, and then discarding the ones that are not in $\mathbb{R}^3$.

In [25], a detailed investigation of the Lagrange multiplier equations which define the complex affine critical points is used to compute the number of such points for $N \leq 7$. Based on these results, it was conjectured in [8, Conjecture 3.4] that the number of points should grow as the following polynomial:

$$q(N) = \frac{9}{2}N^3 - \frac{21}{2}N^2 + 8N - 4.$$

We note that our upper bound $p(N)$ is fairly close.

In order to compute the number $p(N)$, we take a slightly different perspective on the function $\mathrm{error}(q)$. By combining the cameras $C_i : \mathbb{R}^3 \to \mathbb{R}^2$ we obtain a rational map

$$\phi : \mathbb{R}^3 \to \mathbb{R}^{2N}.$$

After passing to the complex numbers and taking the projective closure we obtain a rational map

$$\phi : \mathbb{P}^3_{\mathcal{C}} \to \mathbb{P}^{2N}_{\mathcal{C}}.$$

The image of this map is a three-dimensional variety $MV_N \subset \mathbb{P}^{2N}$ which is known as the *multiview variety*. We can now interpret the error function $\mathrm{error}(q)$ as measuring the distance between a point $q \in \mathbb{P}^{2N}$ and MV_N. With this formulation, the number of critical points is known as the *Euclidean distance degree* of the variety MV_N. The notion of ED degree was introduced in [8], and the authors remark in [8, ex 3.3] that the triangulation problem was their original motivation for this concept.

In particular, by using results from [8] we prove in section 3.3.5 that this number can be computed in terms of the Chern-Mather class $c^M(MV_N)$. In general, the Chern-Mather class only provides an upper bound on the ED degree, but in the proof of theorem 3.3.5.2 we show that this inequality can be promoted to an equality for reasons specific to the multiview variety. One advantage of this approach is that it depends only on the geometric properties of MV_N, and not on the specific features of the defining equations. Another advantage is that it reduces most of the difficulty to local calculations on MV_N.

One common way of calculating the Chern-Mather class of a singular variety X is to first find a resolution

$$\tilde{X} \xrightarrow{f} X$$

and then analyze the singularities of f in order to compare the Chern class $c(\tilde{X})$ to the Chern-Mather class $c^M(X)$

In our situation, it is natural to build a resolution of MV_N by resolving the rational map ϕ. In section 3.3.3, we construct such a resolution

$$\tilde{\phi} : \tilde{\mathbb{P}^3} \to MV_N$$

and calculate its Chow ring and Chern class.

In order to compare the Chern class of $\tilde{\mathbb{P}^3}$ to the Chern-Mather class of MV_N, we use the theory of *higher discriminants* which was introduced in [21]. One aspect of this theory is that it specifies which parts of the singular locus of X we need to understand in order to relate $c(\tilde{X})$ to $c^M(X)$. A precise statement is given in proposition 3.3.4.1.

As we show in proposition 3.3.4.2, the higher discriminants of $\tilde{\phi}$ are surprisingly nice. Specifically, it turns out that in order to calculate $c^M(MV_N)$, we only have to compute the Euler obstruction of a single point $x \in MV_N$.

Moreover, in section 3.3.5.2 we show that after intersecting MV_N with a hyperplane at x, the resulting surface singularity (S, x) is *taut*. In particular, the Euler obstruction $\mathrm{Eu}_{MV_N}(x)$ is determined by the resolution graph of x in S. This allows us to use the enumerative properties of $\tilde{\mathbb{P}}^3$ that are worked out in section 3.3.3 to compute $\mathrm{Eu}_{MV_N}(x)$.

In the final section, we put these pieces together and obtain the polynomial $p(N)$.

3.3.2 Definitions and notation

Let P be a 3×4 matrix with values in $\mathbb{R}$. We consider each row l as an affine function on $\mathbb{R}^3$. Explicitly, l sends a vector $v = (x, y, z)$ to the dot product of l and $(x, y, z, 1)$. We denote these functions by f, g and h.

The matrix P defines a rational map $\phi_P : \mathbb{R}^3 \dashrightarrow \mathbb{R}^2$:

$$v \mapsto (f(v)/h(v), g(v)/h(v))$$

which corresponds to the operation of mapping the "world coordinates" $\mathbb{R}^3$ to the "image coordinates" $\mathbb{R}^2$. In other words, it describes the process of taking a picture of the world with a camera whose parameters are encoded in P.

It is not hard to prove that this description of a camera is equivalent to the pinhole camera model. In particular, the camera has a position called the *camera center* and is pointing in a certain direction. The plane defined by the camera center and direction is called the *principal plane*. It turns out that with the above notation, the principal plane is the plane defined by the ideal (h), and the camera center is the point defined by (f, g, h). For the purposes of this paper, this observation will be taken as a definition.

Now, suppose that we have a collection of cameras $P_1, \ldots, P_N$. By taking a picture of the world with each of the cameras, we obtain a rational map:

$$\phi_{P_1} \times \cdots \times \phi_{P_N} : \mathbb{R}^3 \dashrightarrow \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \cong \mathbb{R}^{2N}$$

This map clearly extends to the complex numbers, giving us a rational map from $\mathcal{C}^3 \dashrightarrow \mathcal{C}^{2N}$. Furthermore, by clearing the denominators in the definition of the maps ϕ_{P_i} we obtain a rational map

$$\phi : \mathbb{P}^3_{\mathcal{C}} \dashrightarrow \mathbb{P}^{2N}_{\mathcal{C}}$$

defined by

$$\phi\left([x : y : z : w]\right) = \left(f_1 h_2 \dots h_N : g_1 h_2 \dots h_N : \cdots : h_1 \dots h_{N-1} g_N : h_1 \dots h_N\right). \qquad (3.3.2.1)$$

The scheme theoretic image of this map is called the *multiview variety* associated to the cameras $P_1, \dots, P_N$.

Example 3.3.2.1. Consider the following three cameras:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The associated rational map is

$$\phi([x : y : z : w]) = [(x-w)xz : (z-w)xz : (y-w)yz : (z-w)yz : (x-w)xy : (y-w)xy : xyz]).$$

We say that a collection of cameras is in *general position* if the hyperplanes defined by the linear functions $\{f_1, g_1, h_1, \dots, f_N, g_N, h_N\}$ associated to the rows of the camera matrices are in general position.

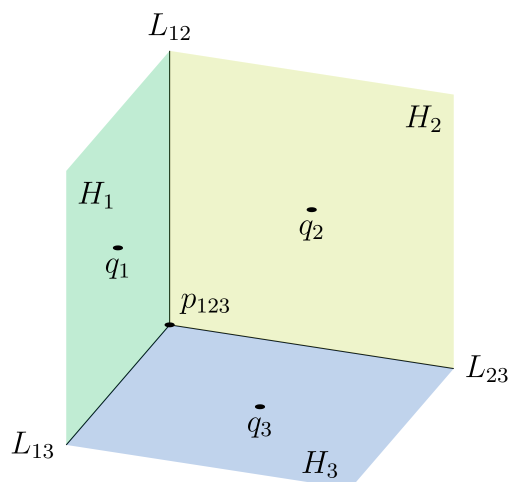


Figure 3.1: Schematic of three cameras

Finally, we will use the following notation throughout the paper (see figure 3.1). The principal plane of the i-th camera will be denoted by H_i and the center of the i-th camera will be denoted by q_i. Also, we define $L_{ij} = H_i \cap H_j$ for all $1 \le i < j \le N$, and $p_{ijk} = H_i \cap H_j \cap H_k$ for all $1 \le i < j < k \le N$.

3.3.3 A resolution of the multiview variety

In this section we describe a resolution of the multiview variety associated to $N \geq 3$ cameras in general position. It is obtained as an iterated blow up along smooth centers. We then apply standard theorems to compute a presentation of the Chow ring of the resolution, and identify a couple of important ring elements.

Let $P_1, \ldots, P_N$ be camera matrices for a collection of N cameras in general position, and let

$$\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{2N}$$

be the corresponding rational map. We denote the associated multiview variety by $MV_N \subset \mathbb{P}^{2N}$.

Proposition 3.3.3.1. *The base locus B of ϕ is the reduced scheme supported on the union of the camera centers $q_1, \ldots, q_N$ and the lines $L_{ij} = H_i \cap H_j$ for all $1 \leq i < j \leq N$.*

Proof. It can be seen directly from the equations of ϕ (equation 3.3.2.1) that B is supported on the camera centers union the lines L_{ij}. We will show that the scheme structure of B is the reduced structure on this set. By a strategic choice of coordinates on $\mathbb{P}^3$, we can assume that $h_1 = x$, $h_2 = y$ and $h_3 = z$.

We now analyze the scheme structure of B in a neighborhood of the point $p_{123} = (x, y, z)$. First of all, recall that the i-th camera contributes the two equations $f_i \cdot \prod_{j \neq i} h_j$ and $g_i \cdot \prod_{j \neq i} h_j$ to the ideal of B.

By our genericity assumptions, all of the f_i's, all of the g_i's, and h_i for $i \geq 4$ are invertible in some Zariski neighborhood of p_{123}. This implies that in a neighborhood of p_{123}, the ideal of B has the form:

$$(xy, xz, yz).$$

Thus, the ideal defined by this scheme is reduced and supported on the coordinate axes. The same argument shows that all of the lines L_{ij} in the base locus have the reduced scheme structure. A similar argument implies the points q_i are reduced. □

3.3.3.1 Constructing a resolution of ϕ

In this section we construct a resolution of MV_N in two stages. First, we blow up $\mathbb{P}^3$ at the points $q_1, \ldots, q_N$ and at the points p_{ijk} for all $1 \leq i < j < k \leq N$. This gives us a map

$$b_1 : Y_1 \to \mathbb{P}^3.$$

Let $\tilde{L}_{ij} \subset Y_1$ denote the proper transform of L_{ij}. Note that these proper transforms are disjoint lines in Y_1.

For the second step, we blow up each of the lines $\tilde{L}_{ij}$ and obtain a resolution

$$b_2 : Y_2 \to Y_1$$

Let us denote Y_2 by $\tilde{\mathbb{P}}^3$, and denote the composition $b_1 \circ b_2$ by π. Since the pullback of the base locus $\pi^{-1}(B)$ is a Cartier divisor on $\tilde{\mathbb{P}}^3$, there exists a canonical map $\tilde{\mathbb{P}}^3 \xrightarrow{\psi} \mathrm{Bl}_B\mathbb{P}^3$ which fits into the following diagram:

$$\begin{array}{ccccc} \tilde{\mathbb{P}}^3 & \xrightarrow{\psi} & \mathrm{Bl}_B\mathbb{P}^3 & & \\ & \searrow^{\pi} & \downarrow b & \searrow^{\mathrm{Bl}_B\phi} & \\ & & \mathbb{P}^3 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^{2N} \end{array}$$

were b is the blowup map and $\mathrm{Bl}_B\phi$ is the resolution of the rational map ϕ.

Finally, we define $\tilde{\phi} = \mathrm{Bl}_B\phi \circ \psi$. Since $\tilde{\mathbb{P}}^3$ is smooth, we thus obtain the following resolution of MV_N:

$$\begin{array}{ccc} \tilde{\mathbb{P}}^3 & & \\ \downarrow \pi & \searrow^{\tilde{\phi}} & \\ \mathbb{P}^3 & \overset{\phi}{\dashrightarrow} & MV_N \subset \mathbb{P}^{2N} \end{array}$$

By an abuse of notation, we will sometimes think of $\tilde{\phi}$ as a map to $\mathbb{P}^{2N}$, and other times as a map to MV_N.

3.3.3.2 The Chow ring of $\tilde{\mathbb{P}}^3$

Since $\tilde{\mathbb{P}}^3$ is an iterated blowup of $\mathbb{P}^3$ along smooth centers, we can use standard theorems to compute its Chow ring. We will use a statement in [16] which we state here for convenience.

Theorem 3.3.3.1. *[16, Appendix, Thm. 1] Let $X \xrightarrow{i} Y$ be a closed embedding of smooth schemes. Let d denote the codimension of X in Y. Let $\tilde{Y}$ be the blowup of Y along X and let $\tilde{X}$ denote the exceptional divisor. Suppose the map $i^* : \mathrm{A}^\bullet(Y) \to \mathrm{A}^\bullet(X)$ is surjective. Then, $\mathrm{A}^\bullet(\tilde{Y})$ is isomorphic to*

$$\frac{\mathrm{A}^\bullet(Y)[T]}{(P(T), T \cdot \ker(i^*))}$$

where $P(T) = P_{X/Y}(T) \in \mathrm{A}^\bullet(Y)[T]$ is a degree d polynomial whose constant term is $[X]$, and whose restriction to X is the Chern polynomial of $N_{X/Y}$. In other words,

$$i^* P_{X/Y}(T) = T^d + c_1(N_{X/Y})T^{d-1} + \cdots + c_{d-1}(N_{X/Y})T + c_d(N_{X/Y}).$$

The isomorphism is induced by the map $f^* : \mathrm{A}^\bullet(Y) \to \mathrm{A}^\bullet(\tilde{Y})$, and by sending $-T$ to the class of the exceptional divisor.

The polynomial $P_{X/Y}$ is called the *Poincaré polynomial of X in Y*.

By applying theorem 3.3.3.1 first to $Y_1 \xrightarrow{b_1} \mathbb{P}^3$ and then to $\tilde{\mathbb{P}}^3 = Y_2 \xrightarrow{b_2} Y_1$ we find that $\mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$ is a quotient of the polynomial algebra

$$A = \mathbb{Z}[\{h\} \cup \{Q_i\}_{1 \le i \le N} \cup \{P_{ijk}\}_{1 \le i < j < k \le N} \cup \{T_{ij}\}_{1 \le i < j \le N}].$$

The meaning of the generators is as follows. Let $\tilde{q}_i \in \tilde{\mathbb{P}}^3$ denote the exceptional divisor of the camera center q_i, let $\tilde{p}_{ijk} \in \tilde{\mathbb{P}}^3$ the exceptional divisor of the point p_{ijk}, and let $\tilde{L}_{ij} \subset \tilde{\mathbb{P}}^3$ the exceptional divisor of the line L_{ij}.

Then, we have the following identities in $\mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$:

$$[\tilde{q}_i] = -Q_i, \qquad [\tilde{p}_{ijk}] = -P_{ijk}, \qquad [\tilde{L}_{ij}] = -T_{ij}.$$

In the next section, we will need to evaluate the degree map

$$\deg : \mathrm{A}^3(\tilde{\mathbb{P}}^3) \to \mathbb{Z}.$$

Since $\tilde{\mathbb{P}}^3$ is irreducible, $\mathrm{A}^3(\tilde{\mathbb{P}}^3)$ has rank one. In addition, $\deg(h^3) = 1$. This means that calculating the degree map is equivalent to expressing every monomial $\alpha \in \mathrm{A}^3(\tilde{\mathbb{P}}^3)$ as a multiple of h^3:

$$\alpha = \deg(\alpha) \cdot h^3.$$

To simplify the calculation, note that product of two generators that correspond to disjoint subschemes of $\tilde{\mathbb{P}}^3$ is zero. For example, $Q_i \cdot P_{jkl} = 0$ for all i, j, k and l.

Thus, the main difficulty is dealing with self intersections such as T_{ij}^3. In order to deal with these, we will calculate the Poincaré polynomials of $q_i \in \mathbb{P}^3$, $p_{ijk} \in \mathbb{P}^3$ and $L_{ij} \subset Y_1$. By theorem 3.3.3.1, this will give us relations involving the self intersections, which in this case turn out to suffice for the degree calculation.

Since $q_i \subset \mathbb{P}^3$ is a point, its Poincaré polynomial is

$$P_{q_i/\mathbb{P}^3}(Q_i) = Q_i^3 + h^3,$$

and similarly,

$$P_{p_{ijk}/\mathbb{P}^3}(P_{ijk}) = P_{ijk}^3 + h^3.$$

Finally, note that $L_{ij} \subset Y_1$ is a line that passes through $N-2$ blown up points. We deduce from this that

$$P_{L_{ij}/Y_1}(T_{ij}) = T_{ij}^2 - 2(N-3)hT_{ij} + h^2 + \sum_{k \notin \{i,j\}} P_{ijk}^2.$$

Putting this all together the relations defining $\mathrm{A}^3(\tilde{\mathbb{P}}^3)$ are:

- h^4
- $h^3 + Q_i^3$ for all $1 \le i \le N$
- $h^3 + P_{ijk}^3$ for all $1 \le i < j < k \le N$
- hQ_i for all $1 \le i \le N$
- hP_{ijk} for all $1 \le i < j < k \le N$

- $P_{ijk}Q_l$ for all $1 \le l \le N$ and $1 \le i < j < k \le N$
- $T_{ij}^2 - 2(N-3)hT_{ij} + h^2 + \sum_{k \notin \{i,j\}} P_{ijk}^2$ for all $1 \le i < j \le N$
- $T_{ij}(h + P_{ijk})$ for all $1 \le i < j \le N$ and $k \notin \{i,j\}$
- $T_{ij}(P_{ijk} - P_{ijl})$ for all $1 \le i < j \le N$ and $k < l \notin \{i,j\}$
- $T_{ij}P_{abc}$ for all $1 \le i < j \le N$ and $1 \le a < b < c \le N$ such that $\{i,j\} \nsubseteq \{a,b,c\}$
- $T_{ij}Q_k$ for all $1 \le i < j \le N$ and $1 \le k \le N$.
- $T_{ij}T_{kl}$ for all $1 \le i < j \le N$ and $1 \le k < l \le N$ such that $\{i,j\} \neq \{k,l\}$.

3.3.3.3 The Chern class of the resolution

In this section we compute $c(\tilde{\mathbb{P}}^3)$ as an element of $\mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$ and find its pushforward to $\mathbb{P}^{2N}$ (proposition 3.3.3.4). Our main tool will be the following proposition.

Proposition 3.3.3.2. *[**F**] Let Y be a smooth scheme and $X \subset Y$ be a closed smooth subscheme with codimension d. Consider the following blowup diagram.*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array}$$

Suppose that $c_k(N_{X/Y}) = i^* c_k$ for some $c_k \in \mathrm{A}^k(Y)$, and that $c(X) = i^*\alpha$ for some $\alpha \in \mathrm{A}^\bullet(Y)$. Let $\eta = c_1(\mathcal{O}_{\tilde{Y}}(\tilde{X}))$. Then,

$$c(\tilde{Y}) - f^*c(Y) = f^*(\alpha) \cdot \beta$$

where

$$\beta = (1+\eta)\sum_{i=0}^{d}(1-\eta)^i f^* c_{d-i} - \sum_{i=0}^{d} f^* c_{d-i}.$$

One takeaway of this proposition is that the Chern class of the blowup along a disjoint union of subvarieties is obtained by summing over contributions from the individual components.

Proposition 3.3.3.3. *The Chern class of the resolution $\tilde{\mathbb{P}}^3$ is equal to*

$$c(\tilde{\mathbb{P}}^3) = (1+h)^4 + \sum_{1 \le i \le N} \alpha_i + \sum_{1 \le i < j \le N} \beta_{ij} + \sum_{1 \le i < j < k \le N} \gamma_{ijk}$$

where

$$\begin{aligned}\alpha_i &= (1-Q_i)(1+Q_i)^3 - 1,\\ \beta_{ij} &= (1+h)^2 \cdot [(1-T_{ij})((1+T_{ij})(-2(N-3)h) + (1+T_{ij})^2) - (1-2(N-3)h)],\\ \gamma_{ijk} &= (1-P_{ijk})(1+P_{ijk})^3 - 1.\end{aligned}$$

Proof. Our strategy will be to use proposition 3.3.3.2 to compute the contributions to the Chern class of each of the varieties that are blown up during the construction of $\tilde{\mathbb{P}}^3$.

We first apply proposition 3.3.3.2 to the situation where $Y = \mathbb{P}^3$ and $X = q_i$ for some i. In this case, we can take $c_0 = 1$, $c_k = 0$ for $k > 0$ and $\alpha = 1$. By proposition 3.3.3.2 the blowup at q_i will contribute

$$\alpha_i = (1-Q_i)(1+Q_i)^3 - 1.$$

Similarly, γ_{ijk} represents the contribution from the blowup of the point p_{ijk}.

Finally, we compute the contribution from the blowup along a line $f : L_{ij} \hookrightarrow Y_1$. Since L_{ij} passes through $N-2$ of the blown up points in Y_1, a quick calculation shows that we can take $c_0 = 1$, $c_1 = -2(N-3)h$, and the rest to be zero. In addition, since $L_{ij} \cong \mathbb{P}^1$, we can take $\alpha = (1+h)^2$. This implies that the contribution coming from L_{ij} is

$$\beta_{ij} = (1+h)^2 \cdot [(1-T_{ij})((1-\eta)(-2(N-3)h) + (1+T_{ij})^2) - (1-2(N-3)h)].$$

□

We now compute the pullback of $c_1(\mathcal{O}_{\mathbb{P}^{2N}}(1))$ in $\mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$ along the map $\tilde{\phi}$.

Lemma 3.3.3.1. *The pullback of* $c_1(\mathcal{O}_{\mathbb{P}^{2N}}(1))$ *to* $\tilde{\mathbb{P}}^3$ *is*

$$\tilde{\phi}^*(c_1(\mathcal{O}_{\mathbb{P}^{2N}}(1))) \cap [\tilde{\mathbb{P}}^3] = N \cdot h + 2 \cdot \sum_{1 \le i < j < k \le N} P_{ijk} + \sum_{1 \le i \le N} Q_i + \sum_{1 \le i < j \le N} T_{ij}.$$

Proof. It is well known (e.g [**F**]) that if L is a line bundle on X, $V \subset \mathrm{H}^0(X, L)$ is a linear system and

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow \pi & \searrow f & \\ X & \dashrightarrow & \mathbb{P}(V^*) \end{array}$$

is the induced resolution, then

$$f^*\mathcal{O}(1) = \pi^*(L) \otimes \mathcal{O}(-E)$$

where $E \subset \tilde{X}$ is the exceptional divisor.

In our case, one can show by a local calculation that the preimage in $\tilde{\mathbb{P}}^3$ of the base locus B has class

$$c_1(\mathcal{O}(-E)) \cap [\tilde{\mathbb{P}}^3] = 2 \cdot \sum_{1\le i<j<k\le N} P_{ijk} + \sum_{1\le i\le N} Q_i + \sum_{1\le i<j\le N} T_{ij},$$

so that $c_1(\tilde{\phi}^*(\mathcal{O}(1))) = c_1(\pi^*\mathcal{O}(N)) + c_1(\mathcal{O}(-E))$ gives the stated expression. □

We can now compute the pushforward $\tilde{\phi}_* c(\tilde{\mathbb{P}}^3)$ as an element of the Chow ring of $\mathbb{P}^{2N}$.

Proposition 3.3.3.4. *The pushforward to $\mathbb{P}^{2N}$ of $c(\tilde{\mathbb{P}}^3)$ is*

$$\begin{aligned}\tilde{\phi}_* c(\tilde{\mathbb{P}}^3) = &\left(N^3 - (4+N)\binom{N}{2} - N - 2\binom{N}{3}\right)[\mathbb{P}^3] + \left(4N^2 - 2\binom{N}{3} - 6\binom{N}{2} - 2N\right)[\mathbb{P}^2] \\ &+ \left(6N + (N-4)\binom{N}{2}\right)[\mathbb{P}^1] + \left(4 + 2N + 2\binom{N}{3} + 2\binom{N}{2}\right)[\mathbb{P}^0]\end{aligned}$$

Proof. Since we have already calculated $c(\tilde{\mathbb{P}}^3) \in \mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$ and $\tilde{\phi}^*(c_1(\mathcal{O}_{\mathbb{P}^{2N}}(1))) \in \mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$, the calculation of $\tilde{\phi}_* c(\tilde{\mathbb{P}}^3)$ is reduced to calculating the degrees of the intersections

$$\pi^*(c_1(\mathcal{O}_{\mathbb{P}^{2N}}(1)))^k \cap c(\tilde{\mathbb{P}}^3)$$

for $0 \le k \le 3$. Using the relations in $\mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$ that we described in section 3.3.3.2, the result follows by a direct calculation. □

3.3.4 Higher discriminants

Higher discriminants, introduced in [21], provide a framework in which to study the singularities of a map. In particular, we will use them to understand how the Chern class of $\tilde{\mathbb{P}}^3$ computed above pushes forward along $\tilde{\phi}$. We now recall the definitions from [21], and phrase them in a way that will be easiest to use in our context.

Definition 3.3.4.1. Let $f : Y \to X$ be a map of smooth manifolds. The *i-th higher discriminant of the map f* is the locus of points $x \in X$ such that for every $i-1$ dimensional subspace $V \subset T_x X$, there exists a point $y \in f^{-1}(x)$ such that:

$$\langle V, f_* T_y Y \rangle \neq T_x X$$

We denote the i-th higher discriminant by $\Delta^i(f)$.

For example, a point $x \in X$ is in $\Delta^1(f)$ if and only if it is a critical value of f. Indeed, according to the definition this happens exactly when there is a point $y \in f^{-1}(x)$ whose Jacobian

$$J(f)_y : T_y Y \to T_x X$$

is not surjective.

On the other extreme, $x \in \Delta^{\dim(X)}(f)$ if and only if for every codimension one subspace $V \subset T_x X$, there exists a point $y \in f^{-1}(x)$ that satisfies:

$$f_* T_y Y \subset V.$$

It is instructive to consider the blow down map: $f : Y = \mathrm{Bl}_p \mathbb{P}^2 \to \mathbb{P}^2$. For every point $y \in E_p = f^{-1}(p)$, $f_* T_y Y$ is one dimensional. This means that $p \in \Delta^1(f)$. In addition, it is not hard to see that for every one dimensional subspace $V \subset T_p \mathbb{P}^2$, there is a point $y \in E_p$ such that $f_* T_y Y = V$. This implies that $p \in \Delta^2(f)$.

Lemma 3.3.4.1. *[21, Rem. 3] Let $Y \to X$ be a proper map of smooth manifolds. Then all of the higher discriminants of f are closed, and we have the following stratification of X:*

$$\Delta^{\dim(X)}(f) \subset \cdots \subset \Delta^2(f) \subset \Delta^1(f) \subset X.$$

Furthermore,

$$\mathrm{codim}(\Delta^i(X)) \geq i.$$

The significance of the higher discriminants is that they tell us which strata appear when writing $f_* \mathbb{1}_Y$ in the basis of Euler obstruction functions on X. For background on Euler obstructions we recommend [20].

Proposition 3.3.4.1. *[21, Cor. 3.3] Let $f : Y \to X$ be a proper map of complex varieties. Let $\{\Delta^{i,\alpha}\}$ be the codimension i components of $\Delta^i(f)$. Then,*

$$f_* \mathbb{1}_Y = \sum \eta^{i,\alpha} \mathrm{Eu}_{\Delta^{i,\alpha}}$$

for some integers $\eta^{i,\alpha}$.

3.3.4.1 Higher discriminants of the resolution $\tilde{\phi}$

In this section we describe the higher discriminants of the map

$$\tilde{\phi} : \tilde{\mathbb{P}}^3 \to MV_N \subset \mathbb{P}^{2N}.$$

Since the definition of higher discriminants assumes that the source and target are smooth, in this section we consider $\tilde{\phi}$ as a map to $\mathbb{P}^{2N}$.

Let $X_i \cong \mathbb{P}^1 \subset MV_N$ denote the image of the proper transform of the principal plane of the i-th camera. The restriction of $\tilde{\phi}$ to the complement of the preimage of the X_i's is an isomorphism, which means that the set theoretic singular locus of $\tilde{\phi}$ is contained in the disjoint union $\amalg_i X_i$.

The following proposition describes the higher discriminants of $\tilde{\phi}$.

Proposition 3.3.4.2. *The higher discriminants of $\tilde{\phi}$ are given as follows:*

- $\Delta^{2N-3}(\tilde{\phi}) = \Delta^{2N-2}(\tilde{\phi}) = MV_N$
- $\Delta^{2N-1}(\tilde{\phi}) = \amalg_i X_i$
- $\Delta^{2N}(\tilde{\phi}) = \emptyset$

To prove this proposition, we use the following lemma, which follows almost immediately from the definition of the higher discriminants.

Lemma 3.3.4.2. *Let $f : Y \to X$ be a map of smooth complex algebraic varieties. Let $C \subset X$ be a smooth curve. Suppose that the restriction of f to $f^{-1}(C)$ has no critical values. Then $C \cap \Delta^{\dim(X)} = \emptyset$.*

Proof. Since the restriction of f to $f^{-1}(C)$ has no critical values, for every point $x \in C$ and every point $y \in f^{-1}(x)$ the one dimensional space $T_xC \subset T_xX$ is contained in f_*T_yY. Therefore, if $V \subset T_xX$ is any vector space complement to T_xC, then f_*T_yY is *not* contained in V. By definition, this implies that $x \notin \Delta^{\dim(X)}(X)$. □

We apply this lemma to each of the $\mathbb{P}^1$'s $X_i \subset \mathbb{P}^{2N}$. Let $f : Y \to \mathbb{P}^1 \cong X_i$ denote the restriction of $\tilde{\phi}$ to X_i. Then Y is isomorphic to the blowup of $\mathbb{P}^2$ at $1 + \binom{N-1}{2}$ points: $q = q_i$ and p_{ijk} for $j, k \neq i$.

The map f is obtained as follows. First, let

$$g : \mathrm{Bl}_q\mathbb{P}^2 \to \mathbb{P}^1$$

be the resolution of the projection away from q. Then, let

$$h : \mathrm{Bl}_{q,p_{ijk}}(\mathbb{P}^2) \to \mathrm{Bl}_p(\mathbb{P}^2)$$

be the blowup along all of the points p_{ijk} for $j, k \neq i$.

Finally, we claim that $f \cong g \circ h$. In particular, f has no critical values. According to the lemma, this proves proposition 3.3.4.2.

3.3.5 The Chern-Mather class of the multiview variety

In this section we compute the Chern-Mather class of MV_N using the theory of higher discriminants. We then use the result to determine the ED degree of MV_N.

3.3.5.1 The basic setup

By propositions 3.3.4.1 and 3.3.4.2, there exists and integer α such that

$$\tilde{\phi}_*(\mathbb{1}_{\tilde{\mathbb{P}}^3}) = \mathrm{Eu}_{MV_N} + \alpha \cdot \sum_{i=1}^{N} \mathrm{Eu}_{X_i}. \tag{3.3.5.1}$$

At a general point $x \in X_i$, the Euler characteristic of the fiber is $\chi(\tilde{\phi}^{-1}(x)) = \chi(\mathbb{P}^1) = 2$ and $\mathrm{Eu}_{X_i}(x) = 1$. This implies that

$$2 = \mathrm{Eu}_{MV_N}(x) + \alpha \Rightarrow \alpha = 2 - \mathrm{Eu}_{MV_N}(x).$$

For the moment, suppose we knew the Euler obstruction $\mathrm{Eu}_{MV_N}(x)$. Then, by taking the Chern-Schwartz-MacPherson class (see [20]) of both sides of equation 3.3.5.1 and recalling that $X_i \cong \mathbb{P}^1$ we obtain

$$\tilde{\phi}_*(c(\tilde{\mathbb{P}}^3)) = c^M(MV_N) + (2 - \mathrm{Eu}_{MV_N}(x))c^M(\mathbb{P}^1). \tag{3.3.5.2}$$

Since we have already calculated $\tilde{\phi}_*(c(\tilde{\mathbb{P}}^3))$ for all N, this would give us the Chern-Mather class of the multiview variety MV_N.

3.3.5.2 Calculating $\mathrm{Eu}_{MV_N}(x)$

To compute $\mathrm{Eu}_{MV_N}(x)$, first note that we can intersect MV_N with a general hypersurface H passing through x. As a result, we obtain a surface singularity:

$$x \in S = MV_N \cap H.$$

By a well known theorem about Euler obstructions (see [6, Sec. 3]),

$$\mathrm{Eu}_{MV_N}(x) = \mathrm{Eu}_S(x).$$

Now, suppose we restrict the resolution $\tilde{\phi}$ to S.

Lemma 3.3.5.1. *$\tilde{\phi}|_S$ is a resolution of S such that the preimage of x is a rational curve with self intersection $-(N-1)$.*

Proof. Let E be the preimage of x. Note that E is the proper transform of a line in the principal plane of the i-th camera. To compute the self intersection of E in $\tilde{S} = \tilde{\phi}^{-1}(S)$ consider the following embeddings:

$$E \overset{i}{\hookrightarrow} \tilde{S} \overset{j}{\hookrightarrow} \tilde{\mathbb{P}}^3.$$

By the Whitney sum formula, we have

$$(ji)_*(c(N_{E/\tilde{S}})) = (ji)_*c(N_{E/\tilde{P}^3}) \cap \tilde{\phi}^*(\mathcal{O}_{\mathbb{P}^{2N}}(-1)).$$

As we have already computed $\tilde{\phi}^*(\mathcal{O}_{\mathbb{P}^{2N}}(1)) \in \mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$, we just have to calculate $(ji)_*c(N_{E/\tilde{P}^3})$. By intersecting E with the generators of $\mathrm{A}^2(\tilde{\mathbb{P}}^3)$ we find

$$[E] = h^2 + Q_i^2 + h\sum_{j \neq i} T_{ij}.$$

Using this identity together with our presentation of $\mathrm{A}^\bullet(\tilde{\mathbb{P}}^3)$ gives

$$(ji)_*c(N_{E/\tilde{\mathbb{P}}^3}) = [E] - (N-1)h^3.$$

Plugging everything into the Whitney sum formula shows that the degree of $c(N_{E/S})$ is $-(N-1)$, which completes the proof. □

We now show that this self intersection number determines the Euler obstruction $\mathrm{Eu}_S(x)$.

Lemma 3.3.5.2. *With $x \in S$ the isolated singularity as above,* $\mathrm{Eu}_S(x) = 3 - N$.

Proof. Recall ([19]) that a singularity germ (X,x) is *taut* if the analytic type of (X, x) is determined by the resolution graph of some resolution of singularities. By [19, p. 2.2] the vertex of the cone over the rational normal curve with degree n is taut. Let us denote this singularity by $(X_n, 0)$. Since this singularity has a resolution in which the exceptional divisor is a $\mathbb{P}^1$ with self intersection $-n$, the resolution graph is a single vertex with weight $(0, -n)$. It follows that *any* singularity with this resolution graph is analytically equivalent to $(X_n, 0)$.

In particular, by lemma 3.3.5.1, (S, x) is analytically equivalent to $(X_{N-1}, 0)$ so the Euler obstruction $\mathrm{Eu}_S(x)$ is equal to the Euler obstruction $\mathrm{Eu}_{X_{N-1}}(0)$. By [4, p. 3.17], the latter is equal to $3 - N$. □

In conclusion, $\mathrm{Eu}_{MV_N}(x) = 3 - N$, so equation 3.3.5.2 becomes

$$\tilde{\phi}_*(c(\tilde{\mathbb{P}}^3)) = c^M(MV_N) + (N-1)c^M(\mathbb{P}^1)$$

By plugging in our calculation of $\tilde{\phi}_*(c(\tilde{\mathbb{P}}^3))$ we obtain $c^M(MV_N)$.

Theorem 3.3.5.1. *The Chern-Mather class of the multiview variety of N cameras in general position is $\sum_{i=0}^{3} c_i^M(MV_N)$ where*

- $c_0^M(MV_N) = 4 + 4N - 2N^2 + 2\binom{N}{3} + 2\binom{N}{2}$
- $c_1^M(MV_N) = 7N - N^2 + (N-4)\binom{N}{2}$
- $c_2^M(MV_N) = 4N^2 - 2\binom{N}{3} - 6\binom{N}{2} - 2N$
- $c_3^M(MV_N) = N^3 - (4+N)\binom{N}{2} - N - 2\binom{N}{3}$

and $c_i^M(MV_N) = \int c^M(MV_N) \cap [\mathbb{P}^{2N-i}]$.

3.3.5.3 The ED degree of the multiview variety

As a corollary of theorem 3.3.5.1, we can compute the Euclidean distance degree of MV_N.

Theorem 3.3.5.2. *The ED degree of the multiview variety of $N \geq 3$ cameras in general position is equal to*

$$ED(MV_N) = 6N^3 - 15N^2 + 11N - 4.$$

Proof. We can use the formula in [4] to express the sum of the polar degrees of MV_N in terms of the Chern-Mather classes. Using this formula gives:

$$\sum \delta_i(MV_N) = 6N^3 - 15N^2 + 11N - 4.$$

Now, by the proof of [8, p. 6.11], if X is an affine cone, then the ED degree of $\overline{X}_v$ is equal to the sum of the polar classes of $\overline{X}_v$ for a general translate X_v of X.

Suppose MV_N is the multiview variety associated to the camera matrices $P_1, \ldots, P_N$. Recall that $MV_N \subset \mathbb{P}^{2N}$ is the projective closure of a subvariety of $\mathcal{C}^{2N}$ which we will call X. Let $(v_1, v_2, \ldots, v_{2N-1}, v_{2N}) \in \mathcal{C}^{2N}$ be a vector. We will now show that $\overline{X}_v$ is multiview variety associated to a different collection of cameras. Indeed, let M_i be the matrix

$$M_i = \begin{pmatrix} 1 & 0 & v_{2i-1} \\ 0 & 1 & v_{2i} \\ 0 & 0 & 1 \end{pmatrix}$$

for $1 \leq i \leq N$. Then, the variety $\overline{X}_v$ is the multiview variety associated to the cameras $M_i \cdot P_i$ for $1 \leq i \leq N$.

In conclusion, there exists a general configuration of cameras such that the ED degree of the associated multiview variety MV_N is equal to the sum of the polar classes of MV_N. □

3.4 Weighted Low Rank Approximation

3.4.1 Introduction

Many methods for extracting structure from data can be reduced to the problem of approximating a given matrix with a low rank matrix. Examples include PCA, factor analysis, and Latent Semantic Analysis which was explained in the introduction to chapter 3.

In the standard formulation of the problem, we are given an $n \times m$ matrix X and an integer $r << n, m$ and would like to find an $n \times m$ matrix M with rank at most r that minimizes the loss function:

$$l(M) = ||M - X||^2$$

where the norm on the right is the Frobenious norm.

In another formulation, the goal is to find an $n \times r$ matrix U and $r \times m$ matrix V which minimize the loss:

$$l(U, V) = ||(U \times V) - X||^2$$

If one thinks of the rows of X as distinct measurements of m quantities, then the decomposition $U \times V$ can be thought of as explaining the variation among the m observed quantities using only $r << m$ underlying factors.

The problem of *weighted* low rank approximation arrises when not all of the observation in X should be given equal importance. In this case, in addition to the observation matrix X we have an additional $n \times m$ weight matrix W and we would like to minimize the weighted loss:

$$l_W(M) = ||M - X||_W^2 = \sum_i \sum_j W_{ij}((M_{ij} - X_{ij})^2$$

with the constraint that $\mathrm{rk} M \leq r$.

An common example of such a situation is when some of the entires of X have not been observed at all. In this case, W will be a binary matrix in which the entries corresponding to unobserved data are set to 0 and the rest are set to 1.

Another situation is when the method used to collect the data returns both a value and a confidence level. In this case, it is natural to weigh each entry in X proportionally to the confidence. For example, consider the *structure from motion* problem in which we have a collection of 2D observations of n points on an object from m cameras and we would like to recover the 3D position and orientation of the object. This problem can be reduced to a rank 4 approximation problem for the $2n \times m$ matrix X whose i column is equal to the conncatentation of the image coordinates of all of the points using the j-th camera [15]. In this case, some entries of X will be missing due to occlusion. Furthermore, if the image points are obtained by running object recognition software, then not all points will be identified with the same degree of accuracy.

We now consider the complexity of WLRA.

A first result in this direction is the *Eckart-Young* theorem which implies that when all of the entries of W are equal to 1, the problem can be solved by finding a singular value decomposition. This implies that this version of WLRA can be solved in polynomial time. It is fairly easy to see that the same is true more generally when W has rank 1.

Conversely, [13] show that the general weighted rank 1 approximation problem is NP-hard. Moreover, they show that even finding a rank one X for which $l_W(X)$ is less than a perscribed constant is NP-hard.

This raises the question which motivates this section

Question. How does the difficulty of the weighted low rank approximation problem depend on the weight matrix W?

In keeping with the perspective of this chapter, we formalize this question in terms of an ED degree calculation.

Let $V := \mathcal{C}^n \otimes \mathcal{C}^n$ be the vector space of $n \times n$ matrices and let $M_r \subset \mathbb{P}(V)$ denote the projective variety of rank r matrices.

For a given weight matrix W with no zero entries, we define an associated quadratic form Q_W on V by

$$Q_W(N) = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} W_{i,j} \cdot N_{i,j}$$

Note that we can express the number of critical points of l_W in terms of the Euclidean distance degree of M_r with respect to the quadratic form Q_W. We can therefore rephrase our motivating question as follows:

Question 3.4.1.1. How does the Euclidean distance degree of M_r with respect to Q_W depend on the weight matrix W?

The dichotomy of rank one W versus generic W persists in this version of the question as well. The Eckart-Young theorem implies that when $\mathrm{rk}W = 1$, weighted Euclidean distance function from a generic point in V has $\binom{n}{r}$ critical points on M_r. In particular, for $r = 1$, the number of critical points grows linearly with n.

However, when W is generic theorem [22, Theorem 3.4] shows that the number of critical points grows superlinearly.

Our main theorem in this section provides an explicit classification of the weight matrices for which the rank one approximation problem has the largest number of critical points.

Theorem 3.4.1.1. *Let M_1 denote the variety of $n \times n$ matrices with rank 1. Let W be an $n \times n$ matrix which defines a non-degenerate quadratic form on $\mathcal{C}^n \otimes \mathcal{C}^n$.*

Then,

$$\mathrm{ED}_W(M_1) \leq \sum \delta_i(X)$$

and equality holds if and only if all of the minors of W have maximal rank.

One can think of theorem 3.4.1 as providing an explicit characterization of the quadrics which are generic in the context of the ED degree of M_1.

The remainder of this section will proceed as follows. In 3.4.2 we construct an explicit Nash blowup of the variety of rank at most r matrices.

In 3.4.3 we'll use this to get a handle on the Euclidean normal bundle which will allow us to compute the base locus that shows up in proposition 3.2.6.1. Finally, we prove theorem by finding a necessary and sufficient condition for the base locus to be empty.

3.4.2 A Nash Blowup of M_r

In order to analyze the Euclidean distance degree of $M_r \subset \mathbb{P}(V)$ with respect to a quadric Q_W, it will be useful to have a reasonable Nash blowup of M_r so that we can take advantage of lemma 3.2.6.2.

We will first define the blowup $\overline{M}_r \to M$, and we will then define a map $\overline{M}_r \to G(\dim(M_r), V)$ that extends the Gauss map $\mathcal{G}_{M_r}$.

To motivate the construction, recall that one can define the Gauss map

$$\mathcal{G}_{M_r} : M_r \dashrightarrow G(\dim(M_r), V)$$

as follows. Let $A \in M_r$ be a matrix with maximal rank. Then,

$$\mathcal{G}_{M_r}(A) := [\{B | B(\ker(A)) \subset \mathrm{im} A\} \subset V] \in G(\dim(M_r), V)$$

With this in mind, we define the variety $\overline{M}_r$ as:

$$\overline{M}_r := \{(L, A, N) | L \subset \ker(A),\ \mathrm{im}(A) \subset N\} \subset G(n-r, n) \times M_r \times G(r, n)$$

The variety $\overline{M}_r$ fits into the following diagram

$$\begin{array}{ccc} \overline{M}_r & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\mathcal{G}_r} & \\ M_r & \overset{\mathcal{G}_{M_r}}{\dashrightarrow} & G(\dim(M_r), V) \end{array}$$

where the map $\mathcal{G}_r$ is defined as

$$\mathcal{G}_r((L, A, N)) := [\{B | B(L) \subset N\} \subset V] \in G(\dim(M_r), V)$$

From our previous description of $\mathcal{G}_{M_r}$ it is clear that the diagram commutes. This shows that $\overline{M}_r$ is indeed a Nash blowup of M_r.

3.4.3 Weighted rank 1 approximation

We will now consider our motivating question 3.4.1.1 in the case where $r = 1$. Our contribution here will be to identify the weight matrices W for which the ED degree of M_1 is maximal.

It will be easier to analyze the $r = n - 1$ case, and deduce the $r = 1$ case by a duality theorem that we'll discuss later.

To this end, we will first describe the modified Euclidean normal bundle $\mathcal{E}_{\overline{M}_{n-1}}$ on $\overline{M}_{n-1}$. We will find weight matrices W for which the base locus of

$$\mathrm{ed}_W : \mathbb{P}(\mathcal{E}_{\overline{M}_{n-1}}) \dashrightarrow \mathbb{P}(V)$$

is empty. It will then follow from proposition 3.2.6.1 that $\mathrm{ED}(X)$ is maximal for these weight matrices. Finally, we use results from [12, Chapter 12] to prove that the ED degree is not maximal if the base locus is non-empty.

3.4.3.1 The Base Locus of ed_W

Let $\mathcal{E}_{\overline{M}_{n-1}}$ denote the modified Euclidean normal bundle of M_{n-1}, which is a vector bundle on $\overline{M}_{n-1}$. In this case,

$$\overline{M}_{n-1} \subset \mathbb{P}^1 \times M_{n-1} \times G(n-1, \mathcal{C}^n)$$

In what follows, if $N \subset \mathcal{C}^n$ is an $n-1$ dimensional subspace then we will use α_N to denote the column vector defining a linear function vanishing on N which is defined up to a scalar. Similarly, if $L \subset \mathcal{C}^n$ is a one dimensional subspace then v_L will denote a column vector in L, also defined up to a scalar.

In addition, we will use $\odot$ to denote the element-wise product of two matrices.

Proposition 3.4.3.1. *Let*

$$f : \mathbb{P}(\mathcal{E}_{\overline{M}_{n-1}}) \to \overline{M}_{n-1}$$

denote the natural projection. Let B denote the base locus of the map

$$\mathrm{ed}_W : \mathbb{P}(\mathcal{E}_{\overline{M}_{n-1}}) \dashrightarrow \mathbb{P}(V)$$

.

A point $p = (L, A, N) \in \overline{M}_{n-1}$ is in $f(B)$ iff

$$A \sim W^{\odot -1} \odot (v_L(\alpha_N)^T)$$

Proof. Recall that the Nash blowup $\overline{M}_{n-1}$ comes with a modified bundle of principle parts $\mathcal{P}_{\overline{M}_{n-1}}$ and a surjection

$$V^*_{\overline{M}_{n-1}} \twoheadrightarrow \mathcal{P}_{\overline{M}_{n-1}}$$

By our construction of $\overline{M}_{n-1}$, we have a natural inclusion

$$\mathcal{P}^*_{\overline{M}_{n-1}} \subset V_{\overline{M}_{n-1}}$$

where at a point $p = (L, A, N)$,

$$\mathcal{P}^*_{\overline{M}_{n-1}}|_p = \{B | B(L) \subset N\} \subset V_{\overline{M}_{n-1}}$$

The surjection $V^*_{\overline{M}_{n-1}} \twoheadrightarrow \mathcal{P}_{\overline{M}_{n-1}}$ is simply the dual of this inclusion.

From this it is easy to see that at a point p, the kernel

$$\mathcal{K} \hookrightarrow V^*_{\overline{M}_{n-1}}$$

is equal to the span of the $n \times n$ matrix $v_L(\alpha_N)^T$.

Finally, following the construction in section 3.2.6.3, we see that at a point $p = (L, A, N)$, the image of the map

$$\psi : \mathcal{E} \to V_{\overline{M}_{n-1}}$$

is equal to the subspace of V spanned by $W^{\odot -1} \odot (v_L(\alpha_N)^T)$ and A. From this we see that p will be in $f(B)$ iff $W^{\odot -1} \odot (v_L(\alpha_N)^T)$ is a scalar multiple of A. □

3.4.3.2 Generic Rank $n-1$ Approximation

We will now use proposition 3.4.3.1 to find a set of weight matrices W for which $\mathrm{ED}(M_{n-1})$ is maximal. This will follow from the following lemma in linear algebra.

Lemma 3.4.3.1. *Let M be an $n \times n$ matrix with rank r. Let $\underline{a}$ and $\underline{b}$ be column vectors such that*

$$\underline{a}^T M = M\underline{b} = 0$$

Set $N = \underline{ab}^T$ and let W be an $n \times n$ matrix.

If $W \odot N = M$ then there exists an $r+1 \times r+1$ minor m of M such that

$$\mathrm{rk}(m(W)) = \mathrm{rk}(m(M))$$

Proof. Let $D_{\underline{v}}$ denote the diagonal matrix with diagonal $\underline{v}$. Then by the assumption

$$W \odot N = D_{\underline{a}} W D_{\underline{b}}$$

Now, suppose for contradiction that $\underline{a}_1 = \cdots = \underline{a}_{n-r} = 0$. Then, since $\underline{a}^T M = 0$, the last r rows of M are linearly dependent which means that the first $n-r$ rows of M are not all 0. But in this case, the first $n-r$ rows of N, and therefore also $W \odot N$, are equal to zero which contradicts the assumption that $W \odot N = M$. We conclude that there exist $r+1$ indices $i_1, \ldots, i_{r+1}$ such that $\underline{a}_{i_k} \neq 0$.

Similarly, there exist $r+1$ indices $j_1, \ldots, j_{r+1}$ such that $\underline{b}_{j_k} \neq 0$.

Now, let m denote the minors with rows $i_1, \ldots, i_{r+1}$ and columns $j_1, \ldots, j_{r+1}$. For ease of notation, we define $\underline{a}' = [a_{i_1}, \ldots, a_{i_{r+1}}]^T$ and $\underline{b}' = [b_{j_1}, \ldots, b_{j_{r+1}}]^T$.

Then, as $D_{\underline{a}'}$ and $D_{\underline{b}'}$ are invertible we have

$$m(W) = D_{\underline{a}'}^{-1} m(M) D_{\underline{b}'}^{-1}$$

which concludes the proof. □

Lemma 3.4.3.2. *Let W be an $n \times n$ matrix. The following conditions are equivalent.*

1. *There exists an $n \times n$ matrix M with rank r and column vectors $\underline{a}$ and $\underline{b}$ such that*

$$\underline{a}^T M = M\underline{b} = 0$$

   and $W \odot (\underline{ab}^T) = M$.

2. *There exists an $r+1 \times r+1$ minor or W with rank r.*

Proof. The direction ($1 \Rightarrow 2$) follows from lemma 3.4.3.1.

We will now prove ($2 \Rightarrow 1$). Without loss of generality, suppose that the top left $r{+}1{\times}r{+}1$ minor of W has rank r. Let $\underline{u} = [u_1, \ldots, u_{r+1}, 0, \ldots, 0]^T$ and $\underline{v} = [v_1, \ldots, v_{r+1}, 0, \ldots, 0]^T$ satisfy

$$\underline{u}^T W = W\underline{v} = \underline{0}$$

Choose $\underline{a}$ and $\underline{b}$ such that

$$\underline{a} \odot \underline{a} = \underline{u}$$
$$\underline{b} \odot \underline{b} = \underline{v}$$

Then, it is easy to see that $M := W \odot (\underline{a}^T \underline{b})$ and the column vectors $\underline{a}$ and $\underline{b}$ satisfy the conditions of 1. □

The following proposition now follows immediately from proposition 3.4.3.1 and lemma 3.4.3.2.

Proposition 3.4.3.2. *Let $\mathcal{E}_{\overline{M}_{n-1}}$ be the modified Euclidean normal bundle of M_{n-1}. Let W be an $n \times n$ matrix representing a non-degenerate quadratic form on $V = \mathcal{C}^n \otimes \mathcal{C}^n$.*

Then, the base locus of

$$\mathrm{ed}_W : \mathbb{P}(\mathcal{E}_{\overline{M}_{n-1}}) \dashrightarrow \mathbb{P}(V)$$

is empty if and only if all of the minors of $W^{\odot -1}$ have maximal rank.

We can now prove an $r = n-1$ version of theorem 3.4.1.

Theorem 3.4.3.1. *Let M_{n-1} denote the variety of $n \times n$ matrices with rank $n-1$. Let W be an $n \times n$ matrix which defines a non-degenerate quadratic form on $\mathcal{C}^n \otimes \mathcal{C}^n$.*

Then,

$$\mathrm{ED}_W(M_{n-1}) \leq \sum \delta_i(M_{n-1})$$

and equality holds if and only if all of the minors of $W^{\odot -1}$ have maximal rank.

Proof. Let L denote the tautological bundle $\mathcal{O}_{\mathcal{E}_{\overline{M}_{n-1}}}(1)$ and let B denote the base locus of ed_W.

By proposition 3.2.6.1,

$$\sum_i \delta_i(M_{n-1}) - \mathrm{ED}(M_{n-1}) = \deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{M}_{n-1}}))$$

so $\mathrm{ED}_W(M_{n-1}) = \sum \delta_i(M_{n-1})$ iff

$$\alpha := \deg(c(L)^n \cap s(B, \mathbb{P}(\mathcal{E}_{\overline{M}_{n-1}})) = 0.$$

If we assume that all of the minors of W have maximal rank, then by proposition 3.4.3.2, B is empty and so $\alpha = 0$.

For the other direction, we need to show that if B is not empty then $\alpha > 0$. By lemma 3.2.6.3, it suffices to prove that $L|_B$ is globally generated and ample. By definition, this is equivalent to showing that $\mathcal{E}^*_{\overline{M}_{n-1}}$ has those properties.

To show this, we recall the definition of $\mathcal{E}_{\overline{M}_{n-1}}$. For ease of notation, let $X := \overline{M}_{n-1}$. Then,

$$\mathcal{E}_X = \mathcal{K}_X \oplus \mathcal{O}_X(-1)$$

It is clear that $\mathcal{O}_X(-1)^*$ is globally generated and ample, so we will focus on $\mathcal{K}_X^*$. The canonical surjection

$$V_X^* \twoheadrightarrow \mathcal{K}_X^*$$

shows that $\mathcal{K}_X^*$ is globally generated.

To show that $\mathcal{K}_X^*|_B$ is ample, we will show that it can be used to define a closed embedding of B into projective space. First of all, we can use the standard identification $G(n-1, \mathcal{C}^n) \cong \mathbb{P}^n$ to embed M_{n-1} into a product of projective spaces as follows:

$$i : \overline{M}_{n-1} \hookrightarrow \mathbb{P}^1 \times M_{n-1} \times \mathbb{P}^1$$

. Furthermore, let π_{13} denote the projection from $\mathbb{P}^1 \times M_{n-1} \times \mathbb{P}^1$ onto $\mathbb{P}^1 \times \mathbb{P}^1$. Under this identification,

$$\mathcal{K}_X^* \cong (\pi \circ i)^* \mathcal{O}(1,1)$$

. Finally, the restriction of $\pi \circ i$ to B is a closed embedding. This concludes the proof that $\mathcal{K}^*$ is ample. □

3.4.3.3 Generic Rank 1 Approximation

In this section we finally prove theorem 3.4.1. In fact, it follows immediately from theorem 3.4.3.1 and the following duality theorem.

Proposition 3.4.3.3. *[22, Proposition 2.2] Let U be a generic $n \times n$ matrix, let W be a weight matrix and fix an integer $1 \leq r \leq n$. There there is a bijection between the critical points of*

- $Q(X) = \sum_{i,j} w_{ij}(x_{ij} - u_{ij})^2$ *on the affine variety of corank r matrices X*
- $Q_{\text{dual}}(Y) = \sum_{i,j} \frac{1}{w_{ij}}(y_{ij} - w_{ij}u_{ij})$ *on the affine variety of rank r matrices Y.*

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