University of California

Ernest O. Lawrence Radiation Laboratory

THE EVOLUTION OF A SYSTEM WITH A RANDOM HAMILTONIAN

Allan N. Kaufman

July 12, 1967

DISCLAIMER

Submitted to Physics of Fluids

UCRL-17673
Preprint

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

THE EVOLUTION OF A SYSTEM WITH A RANDOM HAMILTONIAN

Allan N. Kaufman

July 12, 1967

THE EVOLUTION OF A SYSTEM WITH A RANDOM HAMILTONIAN*

Allan N. Kaufman

Physics Department and Lawrence Radiation Laboratory
University of California
Berkeley, California

(To be submitted to Physics of Fluids)

July 12, 1967

ABSTRACT

An exact equation of evolution, with the structure of the diffusion equation, is derived for the phase-space probability density of a system whose Hamiltonian is random. No assumption need be made about the magnitude of the fluctuations nor about their time scale. In the limit of short correlation time, the equation reduces to the equation of Birmingham, Northrop, and Fälthammar. The reduction of the Fokker-Planck equation to diffusion form is also demonstrated.

---

* This research was supported in part by the United States Atomic Energy Commission.

Certain problems of physical interest can be represented as a system evolving under a random Hamiltonian, whose ensemble is independent of the state of the system. The simplest example is that of a charged particle acted upon by a random electromagnetic field. This example has been studied by Birmingham, Northrop, and Fälthammar,[1] for the case of particle diffusion in (α, β) - space, for fields conserving the magnetic moment and longitudinal action. They derived a diffusion equation for the evolution of the density in (α, β) - space, upon the assumption of small fluctuations of the fields. The present note drops this assumption, extending the work of BNF to arbitrary fluctuations and to a general Hamiltonian.

A system with Hamiltonian $H(\Gamma, t)$ and phase-orbit $\Gamma_S(t)$ has the phase-space density

$$\rho(\Gamma, t) = \delta[\Gamma - \Gamma_S(t)] , \tag{1}$$

which evolves as

$$\frac{\partial \rho}{\partial t} = - L \rho , \tag{2}$$

where

$$L \equiv \{ \, , H\} \equiv \dot{\Gamma} \, \partial_\Gamma \tag{3}$$

is the Liouville operator.[2] The symbol $\dot{\Gamma} \, \partial_\Gamma$ represents a scalar product over the phase space:

$$\dot{\Gamma}\, \partial_\Gamma \equiv \Sigma \cdot (\dot{q}\, \partial_q + \dot{p}\, \partial_p), \tag{4}$$

with summation over the f degrees of freedom of the system.

A randomness of $H(t)$ produces a randomness of $\Gamma_S(t)$ and correspondingly of $\rho(\Gamma, t)$. The randomness of $\Gamma_S(t)$ is (partially) characterized by the probability distribution in phase-space $P(\Gamma_S; t)$. This quantity is identical to the statistical mean of $\rho(\Gamma, t)$, since

$$\langle \rho \rangle\, (\Gamma, t) \equiv \int d\, \Gamma_S \;\; \delta(\Gamma - \Gamma_S)\, P(\Gamma_S;\, t) = P(\Gamma; t).$$

We shall derive the equation of evolution for $P(\Gamma;t)$.

Express ρ and L in terms of their means and their fluctuations about their respective means:

$$\begin{aligned} \rho &\equiv P + \delta\rho \\ L &\equiv \langle L \rangle + \delta L \end{aligned} \tag{5}$$

Substitute (5) into (2), and then take the mean of (2). The result is

$$dP/dt = - \langle \delta L\, \delta\rho \rangle\,, \tag{6}$$

where

$$d/dt \equiv (\partial/\partial t) + \langle L \rangle \tag{7}$$

is the convective derivative under the <u>mean</u> Hamiltonian. The difference between Eqs. (2) and (6) is

$$[(d/dt) + \delta L]\delta\rho = -\delta L\ P + \langle \delta L\ \delta\rho \rangle\ . \tag{8}$$

This equation was solved by BNF to first order in the fluctuations, but, since it is linear in $\delta\rho$, it can be solved exactly for $\delta\rho$ and thus for the quantity $\langle \delta L\ \delta\rho \rangle$ required in (6).

The exact equation of evolution is thus found to be

$$dP/dt = (1 - \langle \delta L\ G \rangle)^{-1} \langle \delta L\ G\ \delta L \rangle\ P\ , \tag{9}$$

where $G \equiv [(d/dt) + \delta L]^{-1}$. With the initial condition $\delta\rho(t = 0) = 0$, G takes the explicit form:

$$\begin{aligned} G\ A(t) &= \int_0^t dt'\ A(t') - \int_0^t dt'\ \delta L\ (t') \int_0^{t'} dt''\ A(t'') \\ &+ \int_0^t dt'\ \delta L\ (t') \int_0^{t'} dt''\ \delta L(t'') \int_0^{t''} dt'''\ A(t''') - \cdots; \end{aligned} \tag{10}$$

the time-integrals are convective, corresponding to (7). Formally, (10) may be written as

$$G = G_0(1 + \delta L\ G_0)^{-1}, \tag{11}$$

where $G_0 \equiv (d/dt)^{-1}$; i.e., $G_0\,A(t) = \int_0^t dt'\ A(t')$.

Using Eq. (11), and $\delta L \equiv \dot{\Gamma}\,\partial_\Gamma \equiv \partial_\Gamma\,\dot{\Gamma}$ (the latter identity following from Liouville's theorem), we may manipulate Eq. (9) into the form

$$\frac{dP}{dt} = \partial_\Gamma \;\; \mathcal{D}\, \partial_\Gamma\, P\, , \tag{12}$$

which has the formal structure of the standard diffusion equation. However, the "diffusivity" $\mathcal{D}$ is here still an integro (in time) - differential (in phase-space) operator:

$$\mathcal{D} \equiv [1 - M(1 + \partial_\Gamma M)^{-1}\, \partial_\Gamma]\, D\, , \tag{13}$$

with

$$D \equiv \langle \delta\dot{\Gamma}\; G\; \delta\dot{\Gamma} \rangle \tag{14}$$

and

$$M \equiv \langle \delta\dot{\Gamma}\; G_0\; \delta L\; G \rangle\, , \tag{15}$$

also operators. Note that D and $\mathcal{D}$ are second-rank tensors in the phase-space, while M and ∂_Γ are vectors.

The exact equation (12) becomes an actual diffusion equation, if one simultaneously makes two approximations: (a), one replaces $P(t - \tau)$, in the integrand of the time-integrals represented by G and G_0, by $P(t)$; (b), one keeps only the lowest-order (in $\delta\dot{\Gamma}$) contribution to $\mathcal{D}$:

$$\mathcal{D} \rightarrow D_0 \equiv \langle \delta\dot{\Gamma}\; G_0\; \delta\dot{\Gamma} \rangle\, , \tag{16}$$

where, because of the first approximation, G_0 now operates only on $\delta\dot{\Gamma}$ and not on P. This is the result obtained by BNF, for the case of one degree of freedom:

$$\frac{dP}{dt} = \partial_\Gamma D_0 \partial_\Gamma P . \tag{17}$$

The validity of these two approximations may be investigated by estimating the relative sizes of the respective leading correction terms. For (a), we obtain the condition

$$\tau_P >> \tau_\delta , \tag{18}$$

where τ_P is the characteristic evolution time for P, while τ_δ is the correlation time for the fluctuating Hamiltonian. This is the usual Markov assumption. From Eqs. (12) and (16), we may estimate

$$\tau_P \sim \sigma_\Gamma^2 D_0^{-1} \sim \sigma_\Gamma^2 |\delta\dot{\Gamma}|^{-2} \tau_\delta^{-1}, \tag{19}$$

where σ_Γ is a measure of the spread of P in phase-space. Substituting (19) into (18), we obtain the condition

$$\sigma_\Gamma >> |\delta\dot{\Gamma}| \tau_\delta \tag{20}$$

as equivalent to (18).

Turning now to (b), we have the condition

$$1 >> |\delta L\, G_0| \sim |\delta\dot{\Gamma}|\, \tau_\delta\, \sigma_\Gamma^{-1}, \tag{21}$$

which we see is identical to the previous condition (20). We conclude that the standard diffusion equation is valid for the evolution in phase-space of a system with random Hamiltonian, if only the Markov condition (18) is satisfied. From familiar examples of this equation, we know that $\tau_P \sim t$; thus the diffusion equation eventually becomes valid, after a time $t >> \tau_\delta$.

Finally, we demonstrate why the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\,\partial_\Gamma \frac{\langle \Delta\Gamma \rangle}{\Delta t} P + \partial_\Gamma \partial_\Gamma D_0 P , \tag{22}$$

which is valid under (18) and (21), reduces to the diffusion equation (17) for the present problem. We have

$$\Delta \Gamma[\Gamma(t),\, t,\, \Delta t] \equiv \int_t^{t+\Delta t} dt'\, \dot{\Gamma}\, [\Gamma(t'),\, t']. \tag{23}$$

In the argument of the integrand, we set $\Gamma(t') \equiv \langle \Gamma \rangle\, (t') + \delta\, \Gamma(t')$, where the mean is here conditional on $\Gamma(t)$. Expanding $\dot{\Gamma}$ to first order in $\delta\Gamma$, we obtain

$$\Delta\Gamma = \int_t^{t+\Delta t} dt' \,\{\dot{\Gamma}[\langle\Gamma\rangle(t'),\, t'] + \delta\Gamma(t')\, \partial_{\langle\Gamma\rangle}\, \dot{\Gamma}\, [\langle\Gamma\rangle(t'),\, t']\}.$$

Upon expressing $\delta\Gamma(t') \equiv \int_t^{t'} dt'' \; \delta\dot{\Gamma}(t'')$ herein, and then taking the mean, we find

$$\frac{\langle\Delta\Gamma\rangle}{\Delta t} = \langle\dot{\Gamma}\rangle + \partial_\Gamma D_0 , \tag{24}$$

to lowest order in Δt and to second order in $\delta\dot{\Gamma}$. Substituting (24) into (22), we obtain

$$\left(\frac{\partial}{\partial t} + \langle\dot{\Gamma}\rangle \partial_\Gamma\right) P = \partial_\Gamma D_0 \partial_\Gamma P, \tag{25}$$

which is Eq. (17).

It is a pleasure to acknowledge the benefit of discussions with R. Davidson, M. Kruskal, T. Northrop, and P. Schram.

FOOTNOTES AND REFERENCES

1. T. J. Birmingham, T. G. Northrop, and C.- G. Fälthammar, Charged Particle Diffusion by Violation of the Third Adiabatic Invariant, submitted to Physics of Fluids. This paper is here denoted by BNF.

2. We use the convention that an operator (like L or ∂_Γ) operates on <u>everything</u> to its right.