Title
Decreasing Marginal Value of Information Under Symmetric Loss

Journal
Decision Analysis, 10(3)

ISSN
1545-8490

Authors
Bikhchandani, Sushil
Mamer, John W

Publication Date
2013-09-01

DOI
10.1287/deca.2013.0274

Peer reviewed

DECISION ANALYSIS
Vol. 10, No. 3, September 2013, pp. 245–256
ISSN 1545-8490 (print) | ISSN 1545-8504 (online)
http://dx.doi.org/10.1287/deca.2013.0274

Decreasing Marginal Value of Information Under Symmetric Loss

Sushil Bikhchandani, John W. Mamer
Anderson School of Management, University of California, Los Angeles, Los Angeles, California 90095
{sushil.bikhchandani@anderson.ucla.edu, john.mamer@anderson.ucla.edu}

We investigate conditions under which the value of additional information is diminishing in a setting where the decision maker has access to multiple, identically-distributed, information signals. The signals are assumed to be independent conditional on an unknown payoff-relevant parameter. The decision maker minimizes a quadratic loss function. Quadratic losses arise in quality control, scoring rules, and other applications. We characterize two concepts of diminishing marginal value of information. The first is an ex ante concept, before any information is observed, and the second is an ex post concept, after observation of previous information signals. The former concept is useful for ex ante information acquisition decisions and the latter for sequential information acquisition.

Key words: value of information; concavity of value of information; quadratic loss function; Bayesian models
History: Received on July 16, 2012. Accepted by Rakesh Sarin on April 24, 2013, after 2 revisions.

1. Introduction

We consider a decision maker who may acquire multiple, identically-distributed, information signals. We are interested in conditions under which the value of an additional information signal diminishes in the number of signals already assessed. One's initial conjecture might be that diminishing marginal value of information is the norm rather than the exception, but it is well-known that the value of information need not obey the law of diminishing marginal returns. For instance, Radner and Stiglitz (1984) established that under certain conditions, the value of information is nonconcave.[1] Violations of diminishing returns to information are problematic in principal-agent models, quality control, demand estimation, and optimal experimentation.

Although we explicitly model a decision maker interested in a single project, decreasing marginal value of information is also useful when the choice is among multiple projects. Consider a decision maker who must choose among several projects (or alternatives). Before selecting a project, the decision maker may acquire information about one or more projects. Multiple, identically-distributed, information signals are available for each project. This is the ranking and selection problem introduced by Raiffa and Schlaifer (1968). Suppose further that the decision maker must allocate a budget to acquire information before any observation of information. If the return from the projects are independent, then diminishing marginal value of information for each project simplifies the information acquisition problem of the decision maker. Frazier and Powell (2010) derive information acquisition strategies in ranking and selection problems when the value of information is not diminishing.[2] In Bickel and Smith (2006), the value of information may exhibit nonconcavities in a dynamic information-gathering process. This complicates the problem of allocating a budget for acquiring information.

Because plausible general results on the value of information are not true, further enquiry in this area must impose restrictions on the decision problem. Radner and Stiglitz (1984) allowed for a general class of utility functions in their analysis. Throughout the

[1] See also Chade and Schlee (2002).

[2] Note that, among other differences, the payoff function for each alternative used in Frazier and Powell (2010) is different from our quadratic loss function.

paper, we restrict attention to a risk-neutral decision maker. The decision maker chooses one alternative from a set of available actions. The decision maker's payoff is symmetric in the "error"; that is, difference between the selected action and the optimal action under perfect information. Initially, we assume that the decision maker minimizes a quadratic loss (equivalently maximizes quadratic gain). Quadratic loss/gain is a natural choice as it is used in a variety of applications. Later, we allow for more general, symmetric, convex loss/gain functions under the assumption of normally-distributed uncertainty and information.

In addition to Radner and Stiglitz (1984), other work has obtained negative or counterintuitive results about the value of information. The value of information does not depend in a systematic way on the riskiness of the decision problem (Gould 1974), the risk aversion of the decision maker (Freixas and Kihlstrom 1984, Willinger 1989), or the wealth level of the decision maker (Hilton 1981). The dependence on the set of available actions is mixed: the value of information increases with the flexibility (in the sense of reversibility) but not necessarily with the range of available actions (Jones and Ostroy 1984).[3]

An individual's demand for information is derived from its use in a decision setting. Information does not directly enter the utility function; instead, it helps make better decisions under uncertainty. The derived nature of the demand for information is the cause of these negative results. Essentially, changes in the parameter (with respect to which a comparative static is desired) change the expected utility of the optimal information-based strategy *and* the expected utility of the optimal uninformed decision. This leads to a nonmonotonic effect on the change in the value of information.

We present the model and provide applications of quadratic loss and decreasing marginal value of information in §2. The underlying uncertainty is represented by a random variable $\tilde{\theta}$ and information signals about $\tilde{\theta}$ are $\tilde{X}_1, \tilde{X}_2, \ldots$. The information signals have decreasing marginal value if for all n the ex ante expected decrease in loss from observing $\tilde{X}_n$ (in addition to $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{n-1}$) is less than the expected decrease in loss from observing $\tilde{X}_{n+1}$ (in addition to $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$).

The main results are in §3, where we provide a characterization of decreasing marginal value of information in terms of the first and second moments of the posterior distribution of $\tilde{\theta}$. Let $\tilde{\mu}_n$ be the posterior mean and $\tilde{\sigma}_n^2$ the posterior variance of $\tilde{\theta}$ after observation of $\tilde{X}_1, \ldots, \tilde{X}_n$. Then the signals have decreasing marginal value if and only if $\mathrm{Var}[\tilde{\mu}_n]$ increases at a decreasing rate with n, or equivalently if and only if $\mathrm{E}[\tilde{\sigma}_n^2]$ decreases at a decreasing rate with n. We show that an important set of examples that have decreasing marginal value come from exponential families of distributions. These examples are characterized by the feature that $\tilde{\mu}_n$ is a linear function of a sufficient statistic of the signals.

In §4.1 we extend our results to an ex post notion of decreasing marginal value of information and in §4.2 to convex, symmetric loss functions. We conclude in §5.

2. The Model

A risk-neutral decision maker chooses an action d under uncertainty represented by a random variable $\tilde{\theta}$. The objective is to minimize the expected value of a quadratic loss function[4]

$$L(\tilde{\theta}, d) = (\tilde{\theta} - d)^2.$$

The cumulative probability distribution function of $\tilde{\theta}$ is $H(\theta)$ with support $(a, b) \subseteq \mathbb{R}$. The decision maker chooses an optimal action under imperfect information after observing identically-distributed information signals $\tilde{X}_1, \tilde{X}_2 \ldots$ that are conditionally independent given $\tilde{\theta}$. The conditional distribution of $\tilde{X}_i$ given $\tilde{\theta} = \theta$ is $F(x \mid \theta)$.[5]

Let $d_n(\tilde{X}_1, \ldots, \tilde{X}_n)$ denote an optimal action after observing n information signals. That is

$$d_n(\tilde{X}_1, \ldots, \tilde{X}_n) \in \underset{d \in [a, b]}{\arg\min}\, \mathrm{E}\big[(\tilde{\theta} - d)^2 \mid \tilde{X}_1, \ldots, \tilde{X}_n\big].$$

[3] A positive comparative statics result is obtained by Delquié (2008), who shows an increase in the value of information with a decrease in the intensity of prior preference for the optimal choice.

[4] Equivalently, one could think of maximizing the expected value of a gain function $G(\tilde{\theta}, d) = -L(\tilde{\theta}, d)$.

[5] We use $\tilde{R}$ for a random variable and r for its realization.

Define

$$L_n^*(\tilde{X}_1, \ldots, \tilde{X}_n) = \mathrm{E}\big[(\tilde{\theta} - d_n(\tilde{X}_1, \ldots, \tilde{X}_n))^2 \mid \tilde{X}_1, \ldots, \tilde{X}_n\big],$$
$$\bar{L}_n^* = \mathrm{E}\big[(\tilde{\theta} - d_n(\tilde{X}_1, \ldots, \tilde{X}_n))^2\big] = \mathrm{E}\big[L_n^*(\tilde{X}_1, \ldots, \tilde{X}_n)\big],$$

where $L_n^*(\tilde{X}_1, \ldots, \tilde{X}_n)$ is the random variable representing the minimum expected loss after observing n information signals and $\bar{L}_n^*$ is the minimum expected loss, computed ex ante before n information signals are observed.

Quadratic loss functions are commonly used in statistical decision theory (see DeGroot 1970, Chapter 7). A quadratic scoring rule induces truth telling when soliciting the opinion of an expert (see Winkler 1996 for a survey of scoring rules and Bickel 2007 for an assessment of different scoring rules). Quadratic loss functions play a prominent role in quality control. For instance, the Taguchi method introduces a loss function to describe the total cost of variation in product quality. Quadratic loss is commonly used in this application (see, for example, Ryan 1989).

We explore conditions under which the ex ante value of information signals has diminishing marginal returns. Recall that the decision maker is risk neutral and his objective is to minimize the expected value of the loss. Therefore, the *value of information* of information signals $\tilde{X}_1, \ldots, \tilde{X}_n$ is

$$\bar{L}_0^* - \bar{L}_n^*.$$

The *marginal value of information* of $\tilde{X}_n$, ex ante before observing $\tilde{X}_1, \ldots, \tilde{X}_{n-1}$, is

$$\bar{L}_{n-1}^* - \bar{L}_n^*.$$

Because additional information can always be ignored, the marginal value of $\tilde{X}_n$ is nonnegative.

Information signals $\tilde{X}_1, \tilde{X}_2, \ldots$ have *decreasing marginal value of information* (DMVOI) if

$$\bar{L}_{n-1}^* - \bar{L}_n^* \geq \bar{L}_n^* - \bar{L}_{n+1}^*, \quad n = 2, 3, \ldots. \tag{1}$$

DMVOI states that the minimum expected loss decreases with n at a decreasing rate. If each $\tilde{X}_i$ is either perfectly informative or completely uninformative about $\tilde{\theta}$ then DMVOI is satisfied. We rule out these trivial cases.

This is an ex ante notion of diminishing marginal value, appropriate for settings where the number of information signals to be acquired is determined before any signals are observed. See §4.1 for a definition of ex post decreasing marginal value of information, which is appropriate for sequential information acquisition decisions.

We provide a few applications of DMVOI. Observe that, as in the ranking and selection problem mentioned in §1, it is the information acquisition problem that is simplified by DMVOI.

- Consider a monopolist facing an uncertain demand curve $p = \tilde{\theta} - aq$, where p is the price, q is the quantity produced (the selected action), $a > 0$ is a constant, and $\tilde{\theta}$, the vertical intercept of the demand curve, is unknown. If the marginal cost of production is c per unit, the monopolist's profit is quadratic:

$$\Pi(\tilde{\theta}, q) = (\tilde{\theta} - aq)q - cq = -a\left[\frac{\tilde{\theta} - c}{2a} - q\right]^2 + \frac{(\tilde{\theta} - c)^2}{4a}.$$

In effect, the monopolist's optimal quantity q is his best estimate for $(\tilde{\theta} - c)/(2a)$. The monopolist can acquire several information signals $\tilde{X}_1, \tilde{X}_2, \ldots$ about $\tilde{\theta}$ before selecting q. If DMVOI is satisfied, then the monopolist's information-acquisition problem is well-behaved and has a unique optimum.

- The property of DMVOI makes the problem of calculating, ex ante, the optimal fixed sample size very easy. Suppose that the cost of obtaining n observations is c_n. We assume that c_n is increasing ($c_n > c_{n-1}$) and there are no economies of scale in obtaining observations ($c_n - c_{n-1}$ is nondecreasing). The optimal sample size minimizes $\bar{L}_n^* + c_n$ and is the largest n such that $\bar{L}_{n-1}^* - \bar{L}_n^* \geq c_n - c_{n-1}$.

- Another application is a principal-agent problem where the principal's gross profit is a function of a random variable $\tilde{\theta}$ and the agent has access to costly signals $\tilde{X}_1, \tilde{X}_2, \ldots$ about $\tilde{\theta}$. The principal cannot monitor the number of signals acquired by the agent. The agent acquires signals and reports an estimate d for $\tilde{\theta}$. If DMVOI is satisfied, then the principal can design a payment function for the agent that is quadratic in the difference between the realized $\tilde{\theta}$ and the agent's estimate d such that the number of signals gathered by the agent maximizes the principal's net profits, given the incentive constraints. Without DMVOI, the agent's information acquisition problem is not concave; it may not be possible to design an incentive scheme such that the agent selects the number of signals that maximize the principal's net payoff.

• Finally, consider a decision maker who can hire multiple experts. The experts are similar, and expert i can assess random variable $\tilde{X}_i$ that is informative about $\tilde{\theta}$. An appropriate scoring rule induces truth-telling by the experts. If DMVOI is satisfied for the decision maker, then the number of experts to be hired is easily determined and their reports merged via conditional expectation.

In this paper, we find it natural to focus on the problem of minimizing the expected value of a loss function and define the value of information as the decrease in expected loss brought about by acquisition of information. This is certainly not the only way to define the value of information or to frame the decision problem. One may frame the problem as that of maximizing expected utility, and measure the value of information as the certain amount that the decision maker would give up to acquire the information (the "buying price") or would require to relinquish the information (the "selling price" of information) (see for example Delquié 2008). For the situation of risk-neutral decision makers, these models yield equivalent specifications to ours.

3. Main Results

We obtain two characterizations of decreasing marginal value of information in Proposition 1 next. These characterizations make use of the mean and the variance of the random variable $\tilde{\theta}$ conditional on the information signals $\tilde{X}_1, \ldots, \tilde{X}_n$. Let

$$\tilde{\mu}_n = \mathrm{E}\big[\tilde{\theta} \,|\, \tilde{X}_1, \ldots, \tilde{X}_n\big], \tag{2}$$

$$\tilde{\sigma}_n^2 = \mathrm{Var}\big[\tilde{\theta} \,|\, \tilde{X}_1, \ldots, \tilde{X}_n\big] = \mathrm{E}\big[(\tilde{\theta} - \tilde{\mu}_n)^2 \,|\, \tilde{X}_1, \ldots, \tilde{X}_n\big]. \tag{3}$$

The dependence of $\tilde{\mu}_n$ and $\tilde{\sigma}_n^2$ on $\tilde{X}_1, \ldots, \tilde{X}_n$ is suppressed in the notation.

After observing $\tilde{X}_1, \ldots, \tilde{X}_n$, the expected loss minimizing action is

$$d_n(\tilde{X}_1, \ldots, \tilde{X}_n) = \tilde{\mu}_n.$$

See DeGroot (1970, p. 228) for a proof. Therefore, the minimum expected loss is

$$L_n(\tilde{X}_1, \ldots, \tilde{X}_n) = \tilde{\sigma}_n^2. \tag{4}$$

It is easy to show that $\mathrm{E}[\tilde{\sigma}_n^2]$ decreases with n and $\mathrm{Var}[\tilde{\mu}_n]$ increases with n for very general joint distributions of $(\tilde{\theta}, \tilde{X}_1, \tilde{X}_2, \ldots)$. The interpretation of $\mathrm{Var}[\tilde{\mu}_n]$ requires some care. When the decision maker has not observed any information signals, we have $\mu_0 = \mathrm{E}[\tilde{\theta}]$ and $\mathrm{Var}[\mu_0] = 0$, whereas after observing $\tilde{X}_1$ the posterior mean of $\tilde{\theta}$ is $\tilde{\mu}_1$, which is a random variable with positive variance. As n increases, $\mathrm{Var}[\tilde{\mu}_n]$ increases and approaches a limit as n goes to infinity.

The next proposition shows that DMVOI holds if and only if $\mathrm{E}[\tilde{\sigma}_n^2]$ decreases at a decreasing rate with n, or equivalently if and only if $\mathrm{Var}[\tilde{\mu}_n]$ increases at a decreasing rate with n.

Proposition 1. *The following are equivalent*:
(i) *Information signals $\tilde{X}_1, \ldots, \tilde{X}_n$ have DMVOI.*
(ii) *$E[\tilde{\sigma}_n^2]$ is decreasing in n at a decreasing rate.*
(iii) $\mathrm{Var}[\tilde{\mu}_n]$ *is increasing in n at a decreasing rate.*

Proof. (i) $\leftrightarrow$ (ii). Equation (4) implies that $\bar{L}_n^* = \mathrm{E}[\tilde{\sigma}_n^2]$. Therefore, (1) holds if and only if

$$\mathrm{E}[\tilde{\sigma}_{n-1}^2] - \mathrm{E}[\tilde{\sigma}_n^2] \geq \mathrm{E}[\tilde{\sigma}_n^2] - \mathrm{E}[\tilde{\sigma}_{n+1}^2] \geq 0.$$

(i) $\leftrightarrow$ (iii). From Ross (1983, p. 29),

$$\mathrm{Var}[\tilde{\theta}] = \mathrm{E}[\mathrm{Var}[\tilde{\theta} \,|\, \tilde{X}_1, \ldots, \tilde{X}_n]] + \mathrm{Var}[\mathrm{E}[\tilde{\theta} \,|\, \tilde{X}_1, \ldots, \tilde{X}_n]]. \tag{5}$$

Thus, for all n,

$$\begin{aligned} \mathrm{Var}[\tilde{\theta}] &= \mathrm{E}\big[\mathrm{E}[(\tilde{\theta} - \tilde{\mu}_n)^2 \,|\, \tilde{X}_1, \ldots, \tilde{X}_n]\big] + \mathrm{Var}[\tilde{\mu}_n] \\ &= \mathrm{E}\big[(\tilde{\theta} - \tilde{\mu}_n)^2\big] + \mathrm{Var}[\tilde{\mu}_n] \\ &= \bar{L}_n^* + \mathrm{Var}[\tilde{\mu}_n] \\ &= \bar{L}_{n-1}^* + \mathrm{Var}[\tilde{\mu}_{n-1}] \\ &\Rightarrow \quad \bar{L}_{n-1}^* - \bar{L}_n^* = \mathrm{Var}[\tilde{\mu}_n] - \mathrm{Var}[\tilde{\mu}_{n-1}]. \end{aligned}$$

The definition of DMVOI in Equation (1) implies that (i) if and only if (iii). $\square$

Equation (5) partitions the variance of $\tilde{\theta}$ into the expected posterior variance of $\tilde{\theta}$ and the variance of $\tilde{\mu}_n$, the optimal posterior action, both after observing $\tilde{X}_1, \ldots, \tilde{X}_n$. Because the expected posterior variance of $\tilde{\theta}$ is the expected posterior loss, this leads directly to the characterization in the proposition.

Example 1 (Normal–Normal).[6] If $\tilde{\theta} \sim \mathrm{N}(\tilde{\mu}_0, 1/\tau)$, $\tilde{X}_i = \tilde{\theta} + \tilde{\epsilon}_i$ where $\tilde{\epsilon}_i \sim \mathrm{N}(0, 1/\gamma)$, then $\tilde{\sigma}_n^2 = 1/(\tau + n\gamma)$

[6] See Winkler (1972) for a value of information problem with linear loss function for the distributions in this example.

(see De Groot 1970, p. 167). Thus, $\mathrm{E}[\tilde{\sigma}_n^2] = 1/(\tau + n\gamma)$ implies $\mathrm{E}[\tilde{\sigma}_{n-1}^2] - \mathrm{E}[\tilde{\sigma}_n^2] = \gamma/((\tau + n\gamma)(\tau + (n-1)\gamma)) \geq 0$ is decreasing in n. Hence, Proposition 1(ii) implies that DMVOI is satisfied. □

In this example, we used the characterization in Proposition 1(ii) to establish DMVOI. In Examples 2–4, it is convenient to use the characterization in Proposition 1(iii), i.e., show that the variance of the posterior mean is increasing at a decreasing rate, to establish DMVOI.

In each of the Examples 1–4, the posterior mean $\tilde{\mu}_n$ has the following linear form:

$$\tilde{\mu}_n = \mathrm{E}[\tilde{\theta} \mid \tilde{X}_1, \ldots, \tilde{X}_n] = \frac{x_0 + \sum_{i=1}^n h(\tilde{X}_i)}{n_0 + n} \tag{6}$$

where $h(X_i)$ is a function of X_i.[7] One may think of n_0 as representing the strength of the prior distribution of $\tilde{\theta}$ relative to n, the number of information signals. For example, in the multiple experts example discussed at the end of §2, n_0 is the number of experts who have already been consulted and n is the number who will be consulted.

With $\tilde{\mu}_n$ given in (6), we have

$$\mathrm{Var}[\tilde{\mu}_n] = \frac{\mathrm{Var}[\sum_{i=1}^n h(\tilde{X}_i)]}{(n_0 + n)^2}.$$

Using the fact that the $\tilde{X}_i$ are identically distributed and conditionally independent, the last expression can be simplified to

$$\begin{aligned}\mathrm{Var}[\tilde{\mu}_n] &= \frac{n\,\mathrm{Var}[h(\tilde{X}_i)] + n(n-1)\,\mathrm{Cov}[h(\tilde{X}_i), h(\tilde{X}_j)]}{(n_0+n)^2}\\ &= \frac{n + n(n-1)\rho}{(n_0+n)^2}\,\mathrm{Var}[h(\tilde{X}_i)],\end{aligned}$$

where $\mathrm{Corr}[h(\tilde{X}_i), h(\tilde{X}_j)]$ is denoted by ρ. Define

$$f(n) = \frac{n + n(n-1)\rho}{(n_0+n)^2}.$$

Proposition 2 gives a condition under which $f(n)$ is increasing and concave[8] in n. If this condition is satisfied in examples with $\tilde{\mu}_n$ of the form in (6), then Proposition 1(iii) implies DMVOI.

Proposition 2. *The function $f(n)$ is increasing and concave in n if $n_0 > 0$ and*

$$\frac{1}{2n_0+1} \leq \rho \leq \frac{2}{n_0+2}. \tag{7}$$

The proof of this (and of all subsequent propositions) is in the appendix.

We use Proposition 1(iii) and Proposition 2 to show that DMVOI is satisfied in Examples 2–4. With $h(\cdot)$ as implicitly defined in (6), we have $h(\tilde{X}_i) = \tilde{X}_i$ in Examples 2 and 3 and $h(\tilde{X}_i) = \tilde{X}_i^2$ in Example 4.

Example 2 (Beta-Bernoulli). Suppose $\tilde{\theta} \in [0,1]$ is distributed beta with parameters α and β and $\tilde{X}_i$ is distributed Bernoulli with parameter $\tilde{\theta}$. Then the posterior distribution of $\tilde{\theta} \mid \tilde{X}_1, \ldots, \tilde{X}_n$ is beta with parameters $\alpha_n = \alpha + \sum_{i=1}^n \tilde{X}_i$ and $\beta_n = \beta + n - \sum_{i=1}^n \tilde{X}_i$ (see De Groot 1970, p. 160). Thus,

$$\begin{aligned}\tilde{\mu}_n &= \frac{\alpha_n}{\alpha_n + \beta_n} = \frac{\alpha + \sum_{i=1}^n \tilde{X}_i}{\alpha + \beta + n},\\ \rho &= \mathrm{Corr}[\tilde{X}_i, \tilde{X}_j] = \frac{1}{\alpha + \beta + 1}.\end{aligned}$$

Then (6) implies that $x_0 = \alpha$, $n_0 = \alpha + \beta$, and (7) is satisfied for any pair $\alpha > 0$ and $\beta > 0$. Thus, Proposition 2 implies DMVOI. □

Example 3 (Gamma-Poisson). Suppose that $\tilde{\theta} \sim \Gamma(\alpha, \beta)$, $\tilde{X}_i$ is Poisson with unknown mean $\tilde{\theta}$. The posterior distribution of $\tilde{\theta} \mid \tilde{X}_1, \ldots, \tilde{X}_n$ is $\Gamma(\alpha_n, \beta_n)$, where $\alpha_n = \alpha + \sum_{i=1}^n \tilde{X}_i$ and $\beta_n = \beta + n$ (see De Groot 1970, p. 164). Thus,

$$\begin{aligned}\tilde{\mu}_n &= \frac{\alpha_n}{\beta_n} = \frac{\alpha + \sum_{i=1}^n \tilde{X}_i}{\beta + n},\\ \rho &= \mathrm{Corr}[\tilde{X}_i, \tilde{X}_j] = \frac{1}{\beta + 1}.\end{aligned}$$

Then (6) implies that $x_0 = \alpha$ and $n_0 = \beta$. It may be verified that (7) is satisfied for any $\beta > 0$. Thus, Proposition 2 implies DMVOI. □

Example 4 (Inverse Gamma-Normal). In Examples 1–3, the signals were unbiased estimates of $\tilde{\theta}$, i.e., $E[X_i \mid \tilde{\theta}] = \tilde{\theta}$. This is not the case in Example 4. Further, unlike the previous examples, the sufficient statistic

[7] As discussed in §3.1, Examples 1–4 are members of exponential families and, in particular, $\sum_i h(X_i)$ is a sufficient statistic for the information signals.

[8] It is convenient to treat n as a continuous variable keeping in mind that $f(n)$ has meaning in our context only when n is integer. Thus, if f is concave and increasing in n then

$$f(n) - f(n-1) \geq f(n+1) - f(n) \geq 0.$$

for the signals is not equal to the sum of the signals. That is, in applying Equation (6) to this example, $h(x)$ is equal to x^2 rather than x.

Suppose $\tilde{\theta}$ has an inverse gamma distribution

$$f(\theta \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}, \quad \theta > 0$$

where $\alpha > 2$ and $\beta > 0$. Then $\mathrm{E}[\tilde{\theta}] = \beta/(\alpha - 1)$ and $\mathrm{Var}(\tilde{\theta}) = \beta^2/[(\alpha-1)^2(\alpha-2)]$. An inverse gamma random variable is the reciprocal of a gamma random variable.

Next, let $\tilde{Z}_i$ be standard normal random variables that are pairwise independent and each $\tilde{Z}_i$ is independent of $\tilde{\theta}$. The signals are $X_i = \sqrt{\tilde{\theta}}\tilde{Z}_i$. Conditional on $\tilde{\theta}$, each X_i is normal with mean 0 and variance $\tilde{\theta}$.

It may be verified that

$$\tilde{\mu}_n = \frac{2\beta + \sum_{i=1}^n X_i^2}{2\alpha - 2 + n},$$

$$\rho = \mathrm{Corr}[\tilde{X}_i^2, \tilde{X}_j^2] = \frac{1}{2\alpha - 1}.$$

Thus, Equation (6) is satisfied with $x_0 = 2\beta$, $n_0 = 2\alpha - 2$, and $h(x) = x^2$, and condition (7) of Proposition 2 is also satisfied. Thus DMVOI holds. □

3.1. Exponential Family of Distributions

A distribution over $\tilde{\theta}$ and conditionally independent signals $\tilde{X}_i$ belongs to an exponential family if the conditional distribution of the signals given $\tilde{\theta}$ has the functional form

$$f(x_1, \ldots, x_n \mid \theta) = g(x_1, \ldots, x_n) e^{\eta(\theta) t(x_1, \ldots, x_n) - \gamma(\theta)}$$

for some functions $g(\cdot)$, $t(\cdot)$, $\eta(\cdot)$, and $\gamma(\cdot)$; observe that $t(\cdot)$ is a sufficient statistic for the signals $\tilde{X}_1, \ldots, \tilde{X}_n$. The support of the posterior distribution of $\tilde{\theta}$ does not change after observing any signals. See Bernardo and Smith (2000) for an extensive treatment of exponential families.

All of our examples of DMVOI belong to an exponential family. In Examples 1–3, $\sum_i \tilde{X}_i$ and in Example 4, $\sum_i \tilde{X}_i^2$, were sufficient statistics for $\tilde{X}_1, \ldots, \tilde{X}_n$. This is not accidental. A result in Diaconis and Ylvisaker (1979) states that under fairly general conditions members of an exponential family of distributions satisfy the following

$$\mathrm{E}[h(\tilde{X}_{n+1}) \mid \tilde{X}_1, \ldots, \tilde{X}_n] = \frac{x_0 + \sum_{i=1}^n h(\tilde{X}_i)}{n_0 + n} \qquad (8)$$

for some x_0 and $n_0 > 0$.[9] In addition, our Examples 1–4 satisfy

$$E[h(\tilde{X}) \mid \tilde{\theta}] = \tilde{\theta}. \qquad (9)$$

If (8) and (9) hold, then

$$\begin{aligned}
&\frac{x_0 + \sum_{i=1}^n h(\tilde{X}_i)}{n_0 + n} \\
&\quad = E[h(\tilde{X}_{n+1}) \mid \tilde{X}_1, \ldots, \tilde{X}_n] \\
&\quad = E[E[h(\tilde{X}_{n+1}) \mid \tilde{\theta}, \tilde{X}_1, \ldots, \tilde{X}_n] \mid \tilde{X}_1, \ldots, \tilde{X}_n] \\
&\quad = E[E[h(\tilde{X}_{n+1}) \mid \tilde{\theta}] \mid \tilde{X}_1, \ldots, \tilde{X}_n] \\
&\quad = E[\tilde{\theta} \mid \tilde{X}_1, \ldots, \tilde{X}_n].
\end{aligned}$$

The first equality follows from (8), the third equality follows from the conditional independence of the $\tilde{X}_i$ given $\tilde{\theta}$, and the fourth from (9). Consequently, $\tilde{\mu}_n$ is of the form (6) and, provided n_0 satisfies (7), DMVOI is satisfied.

Next, we present an example from the exponential family that does not satisfy DMVOI. This example puts to rest the plausible conjecture that a sufficient condition for DMVOI is conditional independence of the signals and membership of the exponential family. Observe that $\tilde{\mu}_n$ is not linear in the signals.

Example 5 (Bernoulli Signals). Assume that $\tilde{\theta} \in \{0, 1\}$ and the prior probability is $p = \Pr[\tilde{\theta} = 1]$. Conditionally i.i.d. signals $\tilde{X}_i$ take two possible values, l and h, with likelihoods

$$\Pr[\tilde{X}_i = h \mid \tilde{\theta} = 1] = \Pr[\tilde{X}_i = l \mid \tilde{\theta} = 0] = q > 0.5.$$

As the loss function is $L = (\tilde{\theta} - d)^2$ and $\tilde{\theta} \in \{0, 1\}$, the optimal action is $d(p_n) = p_n$ and $L_n^*(p_n) = p_n(1 - p_n)$, where the p_n is the probability belief (after observing n signals) that $\tilde{\theta} = 1$. A direct application of Bayes' rule establishes that p_n, hence μ_n, is nonlinear in the signals. Thus $\bar{L}_0^* = p(1-p) = \Pr[\tilde{\theta} = 1]\Pr[\tilde{\theta} = 0]$.

We show that DMVOI is violated when $p = 0.99$ and $q = 0.75$ or $q = 0.85$ or $q = 0.95$, i.e., when initially there is little uncertainty about $\tilde{\theta}$ and the information signal is relatively noisy compared to the prior belief.

[9] Note that $h(\tilde{X}_i)$ may be $k \geq 1$ dimensional in general. In our examples, $h(\tilde{X}_i)$ is one dimensional.

Table 1 **Marginal Value of Information (MVOI)**

MVOI of	$q = 0.75$	$q = 0.85$	$q = 0.95$
$\tilde{X}_1$	0.129×10^{-3}	0.363×10^{-3}	1.430×10^{-3}
$\tilde{X}_2$	0.278×10^{-3}	1.332×10^{-3}	5.562×10^{-3}

First, note that $\bar{L}_0^* = p(1-p) = 0.0099$ and

$$\begin{aligned}\bar{L}_1^* &= \Pr[\tilde{X}_1 = h]\Pr[\tilde{\theta} = 1 \mid \tilde{X}_1 = h]\Pr[\tilde{\theta} = 0 \mid \tilde{X}_1 = h] \\ &\quad + \Pr[\tilde{X}_1 = l]\Pr[\tilde{\theta} = 1 \mid \tilde{X}_1 = l]\Pr[\tilde{\theta} = 0 \mid \tilde{X}_1 = l], \\ \bar{L}_2^* &= \Pr[\tilde{X}_1\tilde{X}_2 = hh]\Pr[\tilde{\theta} = 1 \mid \tilde{X}_1\tilde{X}_2 = hh] \\ &\quad \cdot \Pr[\tilde{\theta} = 0 \mid \tilde{X}_1\tilde{X}_2 = hh] + 2\Pr[\tilde{X}_1\tilde{X}_2 = hl] \\ &\quad \cdot \Pr[\tilde{\theta} = 1 \mid \tilde{X}_1\tilde{X}_2 = hl]\Pr[\tilde{\theta} = 0 \mid \tilde{X}_1\tilde{X}_2 = hl] \\ &\quad + \Pr[\tilde{X}_1\tilde{X}_2 = ll]\Pr[\tilde{\theta} = 1 \mid \tilde{X}_1\tilde{X}_2 = ll] \\ &\quad \cdot \Pr[\tilde{\theta} = 0 \mid \tilde{X}_1\tilde{X}_2 = ll].\end{aligned}$$

The marginal value of $\tilde{X}_1$ is $\bar{L}_0^* - \bar{L}_1^*$ and the marginal value of $\tilde{X}_2$ is $\bar{L}_1^* - \bar{L}_2^*$. These are shown in Table 1 for $q = 0.75$, $q = 0.85$, and $q = 0.95$.

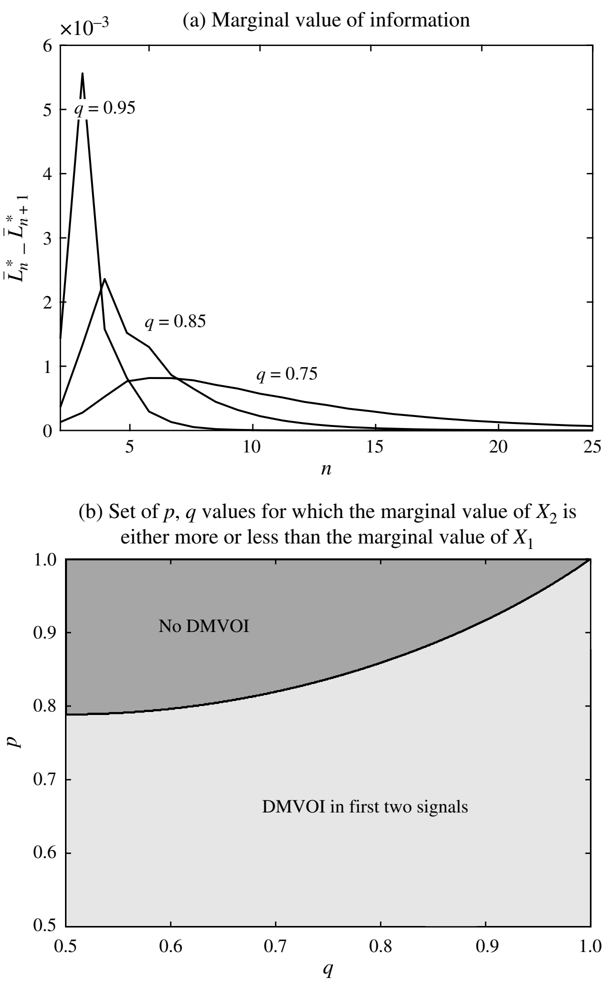
Observe that for each of the three numerical cases, $(p, q) = (0.99, 0.75)$, $(0.99, 0.85)$, and $(0.99, 0.95)$, DMVOI is violated as the marginal value of the second signal is greater than the marginal value of the first signal.[10] Figure 1(a) plots the marginal value of $\tilde{X}_n$ for $n = 1, 2, \ldots, 25$ for these three cases.

In Figure 1(b), the dark region is the set of (p, q) values for which DMVOI is not satisfied because the marginal value of the second signal is greater than the marginal value of the first signal. Essentially, if the prior p is close to one and the signals are not informative enough, then the first signal is not as valuable as the second signal.

In the lightly shaded region of Figure 1(b), the marginal value of signals is decreasing for the first two signals. Observe that in Figure 1(a) the marginal value as a function of n has a single peak; if this holds for all (p, q) and n values then DMVOI would be satisfied in the lightly shaded region of Figure 1(b). □

[10] There are other parameter values in this example (e.g., $p = 0.99$ and $q = 0.99$) for which the first signal has greater marginal value than the second.

Figure 1 **Example 5**



4. Extensions

We explore two extensions of our results. In §4.1, we define ex post DMVOI, a stronger concept of DMVOI that is suitable for sequential sampling applications. In §4.2, we investigate DMVOI with general convex, symmetric loss functions (quadratic loss being an example) under the assumption that the distribution of $(\tilde{\theta}, \tilde{X}_1, \ldots, \tilde{X}_n)$ is normal.

4.1. Ex Post DMVOI

The notion of decreasing marginal value of information investigated so far is an ex ante one. An ex post notion of diminishing marginal value would require that the value of information signal $\tilde{X}_{n+1}$ after observing $(\tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1}, \tilde{X}_n = x_n)$ is less than the value of information signal $\tilde{X}_n$ after observing

$(\tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1})$ for all $(x_1, \ldots, x_{n-1}, x_n)$ and for all n. We make this precise in the following.

Let

$$L_m^*(x_1, \ldots, x_k) \equiv \mathrm{E}[(\tilde{\theta} - \tilde{\mu}_m)^2 \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_k = x_k], \quad m \geq k \quad (10)$$

where $\tilde{\mu}_m = \mathrm{E}[\tilde{\theta} \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_k = x_k, \tilde{X}_{k+1} = x_{k+1}, \ldots, \tilde{X}_m = x_m]$.

The *ex post marginal value of information* of $\tilde{X}_n$, after observing $(\tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1})$, is

$$L_{n-1}^*(x_1, \ldots, x_{n-1}) - L_n^*(x_1, \ldots, x_{n-1}).$$

It is the expected reduction in loss from acquiring signal $\tilde{X}_n$ after having observed $(\tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1})$.

Information signals $\tilde{X}_1, \tilde{X}_2, \ldots$ have *ex post decreasing marginal value of information* (ex post DMVOI) if for all $(x_1, \ldots, x_{n-1}, x_n)$ and for all n,

$$\begin{aligned} &L_{n-1}^*(x_1, \ldots, x_{n-1}) - L_n^*(x_1, \ldots, x_{n-1}) \\ &\quad \geq L_n^*(x_1, \ldots, x_n) - L_{n+1}^*(x_1, \ldots, x_n). \quad (11) \end{aligned}$$

Note that both sides of (11) are nonnegative.

If after observing $(\tilde{X}_1, \ldots, \tilde{X}_{n-1}) = (x_1, \ldots, x_{n-1})$ the decision maker finds that the benefit of additional signal $\tilde{X}_n$ (the left-hand side of (11)) exceeds the cost of obtaining an additional signal, it is optimal to stop gathering information. No matter what $\tilde{X}_n$ might have been observed, it would not be optimal to pay to observe signal $\tilde{X}_{n+1}$ after observing $\tilde{X}_n$. In other words, the property of ex post DMVOI assures that the optimal policy for the sequential information acquisition problem is myopic. Thus, ex post DMVOI is appropriate when information is gathered sequentially. If, in the ranking and selection problem mentioned in §4.1, information acquisition decisions can be made sequentially, after observing some information signals about various alternatives, then ex post DMVOI simplifies the information acquisition problem.

Ex post DMVOI implies ex ante DMVOI. To see this, note that (11) implies that for all $x_1, \ldots, x_{n-1}$

$$\begin{aligned} &L_{n-1}^*(x_1, \ldots, x_{n-1}) - L_n^*(x_1, \ldots, x_{n-1}) \\ &\quad \geq \mathrm{E}[L_n^*(x_1, \ldots, x_{n-1}, \tilde{X}_n) - L_{n+1}^*(x_1, \ldots, x_{n-1}, \tilde{X}_n)] \\ &\quad = L_n^*(x_1, \ldots, x_{n-1}) - L_{n+1}^*(x_1, \ldots, x_{n-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathrm{E}[L_{n-1}^*(\tilde{X}_1, \ldots, \tilde{X}_{n-1}) - L_n^*(\tilde{X}_1, \ldots, \tilde{X}_{n-1})] \\ &\quad \geq \mathrm{E}[L_n^*(\tilde{X}_1, \ldots, \tilde{X}_{n-1}) - L_{n+1}^*(\tilde{X}_1, \ldots, \tilde{X}_{n-1})] \\ &\Longrightarrow \quad \bar{L}_{n-1}^* - \bar{L}_n^* \geq \bar{L}_n^* - \bar{L}_{n+1}^* \end{aligned}$$

which is (1).

Proposition 1 is adapted to ex post DMVOI in the next result. The proof is similar to the proof of Proposition 1 and relies on partitioning the posterior variances of $\tilde{\theta}$. Before stating the proposition, recall that $\tilde{\mu}_n$ and $\tilde{\sigma}_n^2$, defined in (2) and (3), are random variables that represent the posterior expected value of $\tilde{\theta}$ and the posterior variance of $\tilde{\theta}$, respectively.

Proposition 3. *The following are equivalent*:

(i) *Information signals* $\tilde{X}_1, \ldots, \tilde{X}_n$ *have ex post DMVOI.*

(ii) *For all* $(x_1, \ldots, x_n)$,

$$\begin{aligned} &E[\tilde{\sigma}_{n-1}^2 \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1}] \\ &\quad - E[\tilde{\sigma}_n^2 \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1}] \\ &\quad \geq E[\tilde{\sigma}_n^2 \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_n = x_n] \\ &\quad - E[\tilde{\sigma}_{n+1}^2 \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_n = x_n] \geq 0. \end{aligned}$$

(iii) *For all* $(x_1, \ldots, x_n)$,

$$\begin{aligned} &\mathrm{Var}[\tilde{\mu}_n \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_{n-1} = x_{n-1}] \\ &\quad \geq \mathrm{Var}[\tilde{\mu}_{n+1} \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_n = x_n]. \end{aligned}$$

Example 1, where $\tilde{\theta}$ is normally distributed and $\tilde{X}_i$ are normally distributed with mean $\tilde{\theta}$, satisfies ex post DMVOI. To see this, note that for this example $\mathrm{E}[\tilde{\sigma}_m^2] \equiv \mathrm{E}[\tilde{\sigma}_m^2 \mid \tilde{X}_1 = x_1, \ldots, \tilde{X}_k = x_k]$ and thus the condition in Proposition 3(ii) is equivalent to the condition in Proposition 1(ii).

4.2. Convex, Symmetric Loss Functions

We turn to sufficient conditions for DMVOI when the loss function is not quadratic. In particular, suppose that the loss function $L(\tilde{\theta}, d) = l(\tilde{\theta} - d)$ depends only on the difference between $\tilde{\theta}$, and the action d is convex and symmetric. It turns out that the distribution of $\tilde{\theta}$ and $\tilde{X}_1, \tilde{X}_2, \ldots$ in Example 1 satisfies DMVOI for this more general class of loss functions, as the next proposition shows.

Proposition 4. *Assume that*:

(i) *The loss function* $L(\theta, d) = l(\theta - d)$, *where* l: $\mathbb{R} \to \mathbb{R}_+$ *is convex with* $l(0) = 0$, $l(x) \geq 0$, *and* $l(x) = l(-x)$ *for all* x. *Further,* l *is a real analytic function and there exists* $M > 0$ *such that* $|l^{(i)}(0)| < M$ *where* $l^{(i)}(x)$ *denotes the ith derivative of* l *at* x. *Also,* $l^{(2i)}(0) \geq 0$.

(ii) *The information signals* $\tilde{X}_i$ *have a normal distribution with mean* $\tilde{\theta}$.

(iii) *The distribution of* $\tilde{\theta}$ *is normal*.

Then the optimal action after observing $\tilde{X}_1, \ldots, \tilde{X}_n$ *is* $\tilde{\mu}_n$, *and ex post DMVOI is satisfied*.

Loss functions that satisfy the hypotheses of Proposition 4 include

$$l(\theta - d) = (\theta - d)^{2k}, \quad k = 1, 2, \ldots \quad \text{and}$$

$$l(\theta - d) = \exp^{(\theta - d)^{2k}} - 1, \quad k = 1, 2, \ldots.$$

A sum of such l's satisfies the hypotheses of Proposition 4 as well.

5. Concluding Remarks

We investigate two concepts of diminishing marginal value of information, one ex ante and the other ex post, which are useful in information acquisition. The uncertainty is modeled as an unknown random variable $\tilde{\theta}$ about which conditionally independent information signals are valuable. For most of the paper, the decision maker's objective is to minimize a quadratic loss function. We provide two characterizations of ex ante DMVOI in Proposition 1. Ex ante DMVOI is equivalent to the requirement that expected posterior variance of $\tilde{\theta}$ after n signals is convex and decreasing in n, which in turn is equivalent to the requirement that the variance of the posterior expected value of $\tilde{\theta}$ after n signals is concave and increasing in n.

In Proposition 2, we establish that the following condition is sufficient for (ex ante) DMVOI: $\tilde{\mu}_n$, the conditional expectation of $\tilde{\theta}$ given n signals, is a linear function of a sufficient statistic of the signals (Equation (6)) and a parameter of the linear function satisfy a restriction (Equation (7)). Examples 1–4 satisfy the sufficient conditions of Proposition 2.[11] In addition, Examples 1–4 belong to the exponential family.

[11] We do not have an example that satisfies the linearity of $\tilde{\mu}_n$ that does not also exhibit DMVOI; in particular, we do not have an example that satisfies (6) but not (7).

Example 5 displays a member of an exponential family that does not satisfy DMVOI for some parameter values. It shows that membership of the exponential family is not enough to assure DMVOI. In this example, $\tilde{\mu}_n$ is not linear in a sufficient statistic of the signals. It is an open question whether linearity of the posterior mean is necessary for DMVOI to be satisfied for all parameter values.

Next, we considered two related versions of DMVOI. In §4.1, we characterized ex post DMVOI, a concept appropriate for sequential information acquisition. Ex post DMVOI implies ex ante DMVOI. In §4.2, we allowed for convex, symmetric loss functions; quadratic loss is an example of this class. The only example that we could find that satisfied these versions of DMVOI is the normal–normal case of Example 1. Whether there exists an example other than normal–normal that satisfies ex post DMVOI with convex, symmetric loss functions (for all parameter values of underlying distribution over $\tilde{\theta}$ and $\tilde{X}_i$) is another open question.

Our assumptions of quadratic loss, risk-neutrality, and conditionally independent signals limit the results of this paper. First, nonquadratic loss functions, such as logarithmic or spherical, should be considered.[12] Moreover, risk-aversion or general jointly distributed signals might modify the results in important ways. Relaxing these assumptions is the subject of future work.

Acknowledgments

We are grateful to the referees and the associate editor for many helpful comments that improved the paper.

Appendix

Proof of Proposition 2. It is clear that $f(n) \to \rho$ as $n \to \infty$. Thus $f(n)$ is increasing and concave in n iff $\rho - f(n)$ is decreasing and convex in n. To this end write

$$k(n) \equiv \rho - f(n) = \rho - \frac{n + n(n-1)\rho}{(n_0 + n)^2}$$
$$= \frac{(2\rho n_0 + \rho - 1)n + \rho n_0^2}{(n_0 + n)^2} \equiv \frac{an + b}{(n_0 + n)^2},$$

[12] Bickel (2007) shows several advantages of logarithmic loss functions.

where $a \equiv 2\rho n_0 + \rho - 1$ and $b \equiv \rho n_0^2$. Differentiating with respect to n gives

$$\begin{aligned}\frac{dk(n)}{dn} &= \frac{a(n_0+n)^2 - 2(an+b)(n_0+n)}{(n_0+n)^4}\\ &= \frac{a(n_0-n)-2b}{(n_0+n)^3}\\ &= \frac{(2\rho n_0+\rho-1)(n_0-n)-2\rho n_0^2}{(n_0+n)^3}\\ &= -\frac{n(2\rho n_0+\rho-1)+n_0(1-\rho)}{(n_0+n)^3}\\ &= -\frac{n(2n_0+1)(\rho-1/(2n_0+1))+n_0(1-\rho)}{(n_0+n)^3}.\end{aligned}$$

Because $n_0 > 0$ and $n \geq 0$, the denominator of this expression is positive. The left-hand side of (7) and $|\rho| \leq 1$ together imply that the numerator is positive. Thus, $k(n)$ is decreasing in n.

Differentiating $k(n)$ a second time yields

$$\begin{aligned}\frac{d^2k(n)}{dn^2} &= \frac{d}{dn}\left[\frac{a(n_0-n)-2b}{(n_0+n)^3}\right]\\ &\propto -a(n_0+n)^3 - 3(n_0+n)^2(a(n_0-n)-2b)\\ &\propto -a(n_0+n) - 3a(n_0-n)+6b\\ &\propto an - 2an_0 + 3b.\end{aligned}$$

Convexity of $k(n)$ requires $an+3b-2an_0 \geq 0$ for all n, which, setting $n=0$ amounts to $n_0 \leq 3b/2a$. We show that this is equivalent to the right-hand side of (7):

$$n_0 \leq \frac{3b}{2a} = \frac{3\rho n_0^2}{4\rho n_0 + 2\rho - 2}.$$

The left-hand side of (7) implies $4\rho n_0 + 2\rho - 2 \geq 0$. Thus, we can multiply through to obtain

$$\begin{aligned}&4\rho n_0^2 + 2n_0\rho - 2n_0 \leq 3\rho n_0^2\\ \Leftrightarrow\quad &\rho n_0 + 2\rho - 2 \leq 0\\ \Leftrightarrow\quad &\rho \leq \frac{2}{n_0+2}.\end{aligned}$$

Hence $f(n)$ is concave and increasing in n. $\square$

Proof of Proposition 3. (i) $\leftrightarrow$ (ii).

Equation (10) implies that

$$L_m^*(x_1,\ldots,x_k) = \mathrm{E}[\tilde{\sigma}_m^2 \mid \tilde{X}_1 = x_1,\ldots,\tilde{X}_k = x_k], \quad \forall\, m \geq k.$$

Applying this identity to (11) establishes that (i) if and only if (ii).

(i) $\leftrightarrow$ (iii). Analogous to (5), we have,

$$\begin{aligned}&\mathrm{Var}[\tilde{\theta} \mid \tilde{X}_1,\ldots,\tilde{X}_n] - \mathrm{E}\big[\mathrm{Var}[\tilde{\theta} \mid \tilde{X}_1,\ldots,\tilde{X}_{n+1}] \mid \tilde{X}_1,\ldots,\tilde{X}_n\big]\\ &\quad= \mathrm{Var}\big[\mathrm{E}[\tilde{\theta} \mid \tilde{X}_1,\ldots,\tilde{X}_{n+1}] \mid \tilde{X}_1,\ldots,\tilde{X}_n\big].\end{aligned}$$

Thus, for all n,

$$\begin{aligned}&L_n^*(\tilde{X}_1,\ldots,\tilde{X}_n) - L_{n+1}^*(\tilde{X}_1,\ldots,\tilde{X}_n)\\ &\quad= L_n^*(\tilde{X}_1,\ldots,\tilde{X}_n) - \mathrm{E}\big[L_{n+1}^*(\tilde{X}_1,\ldots,\tilde{X}_{n+1}) \mid \tilde{X}_1,\ldots,\tilde{X}_n\big]\\ &\quad= \mathrm{Var}[\tilde{\theta} \mid \tilde{X}_1,\ldots,\tilde{X}_n] - \mathrm{E}\big[\mathrm{Var}[\tilde{\theta} \mid \tilde{X}_1,\ldots,\tilde{X}_{n+1}] \mid \tilde{X}_1,\ldots,\tilde{X}_n\big]\\ &\quad= \mathrm{Var}\big[\mathrm{E}[\tilde{\theta} \mid \tilde{X}_1,\ldots,\tilde{X}_{n+1}] \mid \tilde{X}_1,\ldots,\tilde{X}_n\big]\\ &\quad= \mathrm{Var}\big[\tilde{\mu}_{n+1} \mid \tilde{X}_1,\ldots,\tilde{X}_n\big].\end{aligned}$$

The definition of ex post DMVOI in (11) implies that (i) if and only if (iii). $\square$

The proof of Proposition 4 requires the following result from the electrical engineering literature. It implies that under the hypothesis of this proposition, the expected-loss minimizing action is the posterior mean $\tilde{\mu}_n$. We provide a proof for completeness.

Lemma A (van Trees 2001, pp. 60–61). *Assume that*:

(i) *The loss function* $L(\theta, d) = l(\theta - d)$, *where* $l\colon \mathbb{R} \to \mathbb{R}_+$ *is convex with* $l(0) = 0$, $l(x) \geq 0$, *and* $l(x) = l(-x)$ *for all* x.

(ii) *The density* $f(\theta \mid \tilde{X}_1,\ldots,\tilde{X}_n)$ *exists and is unimodal and symmetric* (*about the mode*).

Then for any d

$$E[L(\tilde{\theta}, \tilde{\mu}_n) \mid \tilde{X}_1,\ldots,\tilde{X}_n] \leq E[L(\tilde{\theta}, d) \mid \tilde{X}_1,\ldots,\tilde{X}_n].$$

Proof. Let

$$\tilde{z} = \tilde{\theta} - \tilde{\mu}_n.$$

Let the density of $\tilde{z}$ be f_z (where we drop the dependence of f_z on $\tilde{X}_1,\ldots,\tilde{X}_n$ for simplicity). Then, hypothesis (ii) implies that

$$f_z(z) = f_z(-z).$$

Next,

$$\begin{aligned}\mathrm{E}[L(\tilde{\theta},\tilde{g}) \mid \tilde{X}_1 = x_1,\ldots,\tilde{X}_n = x_n] &= \int_{-\infty}^{\infty} l(\theta - d) f(\theta \mid x_1,\ldots,x_n)\,d\theta,\\ [\text{symmetric } l] &= \int_{-\infty}^{\infty} l(d-\theta) f(\theta \mid x_1,\ldots,x_n)\,d\theta,\\ [\text{change of variable}] &= \int_{-\infty}^{\infty} l(d-\mu_n - z) f_z(z)\,dz,\\ [\text{symmetric } l] &= \int_{-\infty}^{\infty} l(\mu_n - d + z) f_z(z)\,dz, \quad (12)\\ [\text{symmetric } f_z] &= \int_{-\infty}^{\infty} l(\mu_n - d - z) f_z(z)\,dz,\\ [\text{symmetric } l] &= \int_{-\infty}^{\infty} l(d - \mu_n + z) f_z(z)\,dz. \quad (13)\end{aligned}$$

Next, using (12) and (13) we have

$$\begin{aligned}&\mathrm{E}\big[L(\tilde{\theta}, d) \mid \tilde{X}_1 = x_1,\ldots,\tilde{X}_n = x_n\big]\\ &\quad= \int_{-\infty}^{\infty} \big[\tfrac{1}{2} l(\mu_n - d + z) + \tfrac{1}{2} l(d - \mu_n + z)\big] f_z(z)\,dz\\ &\quad\geq \int_{-\infty}^{\infty} l(z) f_z(z)\,dz\\ &\quad= \mathrm{E}\big[L(\tilde{\theta}, \tilde{\mu}_n) \mid \tilde{X}_1 = x_1,\ldots,\tilde{X}_n = x_n\big]\end{aligned}$$

where the inequality follows from the convexity of l. $\square$

Proof of Proposition 4. Because l is a real analytic function, it agrees with its Taylor series. Thus,

$$l(\theta-d) = l(0)+(\theta-d)l^{(1)}(0)+\frac{(\theta-d)^2}{2}l^{(2)}(0) + \frac{(\theta-d)^3}{3!}l^{(3)}(0)+\frac{(\theta-d)^4}{4!}l^{(4)}(0)+\cdots$$

or

$$l(\theta-d)=\sum_{i=1}^{\infty}\frac{(\theta-d)^i}{i!}l^{(i)}(0). \tag{14}$$

The distribution of $\tilde{X}_i$ is normal with mean $\tilde{\theta}$ and variance $1/\gamma$ and the distribution of $\tilde{\theta}$ is normal with mean μ and variance $1/\tau$. Conditional on observations $\tilde{X}_1,\ldots,\tilde{X}_n$, the posterior distribution of $\tilde{\theta}$ is normal with mean $\tilde{\mu}_n = (\tau\mu+\gamma\sum_i \tilde{X}_i)/(\tau+n\gamma)$ and variance $\tilde{\sigma}_n^2 = 1/(\tau+n\gamma)$. This distribution is unimodal and symmetric and Lemma A implies that $\tilde{\mu}_n$ is the optimal action. The challenge is to show that (11) is satisfied with this more general loss function. That is, with $\bar{L}_n^*(\tilde{X}_1,\ldots,\tilde{X}_n) = \mathrm{E}[l(\tilde{\theta}-\tilde{\mu}_n)\mid \tilde{X}_1,\ldots,\tilde{X}_n]$, etc. Hence, setting $d=\tilde{\mu}_n$ in (14) and taking expectations we have

$$\begin{aligned}
&\mathrm{E}[l(\tilde{\theta}-\tilde{\mu}_n)\mid \tilde{X}_1,\ldots,\tilde{X}_n]\\
&\quad=\mathrm{E}\left[\sum_{i=1}^{\infty}\frac{(\tilde{\theta}-\tilde{\mu}_n)^i}{i!}l^{(i)}(0)\,\middle|\,\tilde{X}_1,\ldots,\tilde{X}_n\right]\\
&\quad=\mathrm{E}\left[\lim_{N\to\infty}\sum_{i=1}^{N}\frac{(\tilde{\theta}-\tilde{\mu}_n)^i}{i!}l^{(i)}(0)\,\middle|\,\tilde{X}_1,\ldots,\tilde{X}_n\right]\\
&\quad=\sum_{i=1}^{\infty}\frac{l^{(i)}(0)}{i!}\mathrm{E}[(\tilde{\theta}-\tilde{\mu}_n)^i\mid \tilde{X}_1,\ldots,\tilde{X}_n].
\end{aligned}\tag{15}$$

Observe that $|l^{(i)}(0)|<M$ implies that for all N

$$\begin{aligned}
\left|\sum_{i=1}^{N}\frac{(\tilde{\theta}-\tilde{\mu}_n)^i}{i!}l^{(i)}(0)\right| &\le \sum_{i=1}^{N}\frac{|\tilde{\theta}-\tilde{\mu}_n|^i}{i!}|l^{(i)}(0)|\\
&\le \sum_{i=1}^{\infty}\frac{|\tilde{\theta}-\tilde{\mu}_n|^i}{i!}|l^{(i)}(0)| \le Me^{|\tilde{\theta}-\tilde{\mu}_n|}.
\end{aligned}$$

The rightmost expression mentioned previously has finite expectation. Thus, the interchange of limit and expectation in (15) is justified by the dominated convergence theorem.

Standard formulas establish that moments of odd order $\mathrm{E}[(\tilde{\theta}-\tilde{\mu}_n)^{2k-1}\mid \tilde{X}_1,\ldots,\tilde{X}_n]=0$ and even moments are

$$\begin{aligned}
\mathrm{E}[(\tilde{\theta}-\tilde{\mu}_n)^{2k}\mid \tilde{X}_1,\ldots,\tilde{X}_n] &= (\tilde{\sigma}_n)^{2k}(2k-1)!!\\
&=\left(\frac{1}{\tau+n\gamma}\right)^k(2k-1)!!
\end{aligned}$$

where $m!! = m(m-2)(m-4)\cdots(1)$. Thus, we can rewrite (15) as

$$\mathrm{E}\big[l(\tilde{\theta}-\tilde{\mu}_n)\mid \tilde{X}_1,\ldots,\tilde{X}_n\big]=\sum_{k=1}^{\infty}\left(\frac{1}{\tau+n\gamma}\right)^k\frac{(2k-1)!!}{(2k)!}l^{(2k)}(0)$$

which is well defined because $|l^{(2k)}(0)|<M$. Further, observe that $\mathrm{E}[l(\tilde{\theta}-\tilde{\mu}_n)\mid \tilde{X}_1,\ldots,\tilde{X}_n]$ does not depend on $\tilde{X}_1,\ldots,\tilde{X}_n$. The marginal value of the nth observation is

$$\begin{aligned}
&L_{n-1}^*(\tilde{X}_1,\ldots,\tilde{X}_{n-1})-L_n^*(\tilde{X}_1,\ldots,\tilde{X}_{n-1})\\
&\quad=\mathrm{E}\big[l(\tilde{\theta}-\tilde{\mu}_{n-1})\mid \tilde{X}_1,\ldots,\tilde{X}_{n-1}\big]-\mathrm{E}\big[l(\tilde{\theta}-\tilde{\mu}_n)\mid \tilde{X}_1,\ldots,\tilde{X}_{n-1}\big]\\
&\quad=\mathrm{E}\big[l(\tilde{\theta}-\tilde{\mu}_{n-1})\mid \tilde{X}_1,\ldots,\tilde{X}_{n-1}\big]\\
&\qquad-\mathrm{E}\big[\mathrm{E}[l(\tilde{\theta}-\tilde{\mu}_n)\mid \tilde{X}_1,\ldots,\tilde{X}_n]\tilde{X}_1,\ldots,\tilde{X}_{n-1}\big]\\
&\quad=\sum_{k=1}^{\infty}\left(\frac{1}{\tau+(n-1)\gamma}\right)^k\frac{(2k-1)!!}{(2k)!}l^{(2k)}(0)\\
&\qquad-\mathrm{E}\left[\sum_{k=1}^{\infty}\left(\frac{1}{\tau+n\gamma}\right)^k\frac{(2k-1)!!}{(2k)!}l^{(2k)}(0)\,\middle|\,\tilde{X}_1,\ldots,\tilde{X}_{n-1}\right]\\
&\quad=\sum_{k=1}^{\infty}\left[\left(\frac{1}{\tau+(n-1)\gamma}\right)^k-\sum_{k=1}^{\infty}\left(\frac{1}{\tau+n\gamma}\right)^k\right]\frac{(2k-1)!!}{(2k)!}l^{(2k)}(0).
\end{aligned}$$

The last expression is nonincreasing because $l^{(2k)}(0)\ge 0$. $\square$

References

Bernardo JM, Smith AFM (2000) *Bayesian Theory* (John Wiley & Sons, New York).

Bickel JE (2007) Some comparisons among quadratic, spherical, and logarithmic scoring rules. *Decision Anal.* 4(2):49–65.

Bickel JE, Smith JE (2006) Optimal sequential exploration: A binary learning model. *Decision Anal.* 3(1):16–32.

Chade H, Schlee EE (2002) Another look at the Radner-Stiglitz nonconcavity in the value of information. *J. Econom. Theory* 107(2):421–452.

DeGroot MH (1970) *Optimal Statistical Decisions* (McGraw Hill, New York).

Delquié P (2008) The value of information and intensity of preference. *Decision Anal.* 5(3):129–139.

Diaconis P, Ylvisaker D (1979) Conjugate priors for exponential families. *Ann. Statist.* 7(2):269–281.

Frazier PI, Powell WB (2010) Paradoxes in learning and the marginal value of information . *Decision Anal.* 7(4):378–403.

Freixas X, Kihlstrom RE (1984) Risk aversion and information demand. Boyer M, Kihlstrom RE, eds. *Bayesian Models in Economic Theory* (North-Holland, New York), 93–104.

Gould JP (1974) Risk, stochastic preference, and the value of information. *J. Econom. Theory* 8(1):64–84.

Hilton RW (1981) The determinants of information value: Synthesizing some general results. *Management Sci.* 27(1):57–64.

Jones RA, Ostroy JM (1984) Flexibility and uncertainty. *Rev. Econom. Stud.* 51(1):13–32.

Radner R, Stiglitz JE (1984) A nonconcavity in the value of information. Boyer M, Kihlstrom RE, eds. *Bayesian Models in Economic Theory* (North-Holland, New York), 33–52.

Raiffa H, Schlaifer R (1968) *Applied Statistical Decision Theory* (M.I.T. Press, Boston).

Ross S (1983) *Stochastic Processes* (John Wiley & Sons, New York).

Ryan TP (1989) *Statistical Methods for Quality Improvement* (John Wiley & Sons, New York).

van Trees HL (2001) *Detection, Estimation, and Modulation Theory, Part 1* (John Wiley & Sons, New York).

Willinger M (1989) Risk aversion and the value of information. *J. Risk and Insurance* 56(2):320–328.

Winkler RL (1972) *An Introduction to Bayesian Inference and Decision* (Holt, Rinehart and Winston, New York).

Winkler RL (1996) Scoring rules and the evaluation of probabilities. *Test* 5(1):1–60.

Sushil Bikhchandani has been at the UCLA Anderson School since 1985. He is interested in auctions, market institutions, herd behavior, and information economics. Dr. Bikhchandani teaches data and decisions in the MBA core and an elective on game theory and strategic behavior at UCLA. He is vice chair of the Anderson School.

John W. Mamer is a professor at the Anderson School of Management at the University of California, Los Angeles. He received a B.S. in mathematics and a B.A. in economics from the University of California, Davis, and M.S. and Ph.D. degrees from the Haas School at the University of California, Berkeley. His research interests include decision making under uncertainty, applied probability, game theory, and optimization. His work has appeared in *Management Science, Journal of Economic Theory, Mathematics of Operations Research, Naval Research Logistics*, and *INFORMS Journal on Computing*. His teaching responsibilities include probability and statistics, which he has taught to MBA students for the past 12 years.