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Temporal logic specifications for hybrid dynamical systems

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TEMPORAL LOGIC SPECIFICATIONS FOR HYBRID DYNAMICAL SYSTEMS

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by

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List of Symbols

$\mathbb{N}$	The set of natural numbers including zero, i.e., $\{0, 1, 2, \ldots\}$
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{R}$	The set of all real numbers
$\mathbb{B}$	The closed unit ball
$\mathbb{R}_{\geq 0}$	The set of all non-negative real numbers
$\overline{\mathrm{con}}(M)$	The closure of the convex hull of a set M
$\overline{M}$	The closure of a set $M \subset \mathbb{R}^n$ defined by the intersection of all closed sets containing M
$\mathrm{int}M$	The set of interior points of M
∂M	The set of boundary points of a set M
$T_M(x)$	The tangent cone of a closed set M at a point $x \in M$
$\lvert x \rvert$	The Euclidean vector norm $\lvert x \rvert := \sqrt{x^\top x}$
$\lvert x \rvert_M$	The distance from points $x \in \mathbb{R}^n$ to a closed set $M \subset \mathbb{R}^n$, i.e., $\lvert x \rvert_M = \inf_{\xi \in M} \lvert x - \xi \rvert$
dom M	The domain of $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, i.e., dom $M = \{x \in \mathbb{R}^m : M(x) \neq 0\}$

rge M	The range of $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, i.e., rge $M = \{y \in \mathbb{R}^m : y \in M(x), x \in \mathbb{R}^m\}$
$L_V(r)$	The r-sublevel set of V, i.e., $L_V(r) = \{x \in \text{dom } V : V(x) \leq r\}$
ceil(x)	The smallest integer upper bound for $x \in \mathbb{R}$
$U(M)$	An open neighborhood around a set $M \subset \mathbb{R}^n$
$\mathcal{C}^1$	The set of continuously differentiable functions

Abstract

Temporal logic specifications for hybrid dynamical systems

by

Hyejin Han

This dissertation focuses on developing tools for certifying temporal logic properties in hybrid dynamical systems that combine continuous and discrete dynamics. In particular, operators, semantics, characterizations, and solution-independent conditions to guarantee temporal logic specifications for hybrid dynamical systems are presented. Hybrid dynamical systems are given in terms of differential inclusions – capturing the continuous dynamics – and difference inclusions – capturing the discrete dynamics or events – with constraints. State trajectories (or solutions) to such systems are parameterized by a hybrid notion of time. Characterizations of temporal logic formulas in terms of dynamical properties of hybrid systems are presented – in particular, forward invariance, conditional invariance, and finite time attractivity. These characterizations are exploited to formulate sufficient conditions assuring the satisfaction of temporal logic formulas — when possible, these conditions do not involve solution information. Notions for specifying dynamical properties of systems with robustness to perturbations are proposed. Characterizations of basic signal temporal logic formulas are presented. Combining the results for formulas with a single operator, ways to certify more complex formulas are pointed out, in particular, via a decomposition using a finite state automaton. An object grasping application and academic examples are given to illustrate the results.

Dedicated to Mom, Dad, and Keunhwan.

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Chapter 1

Introduction

1.1 Background

High-level languages are useful in formulating specifications for dynamical systems that go beyond classical asymptotic stability, where convergence to the desired point or set is typically certified to occur in the limit, that is, over an infinitely long time horizon; see, e.g., [1–3]. Temporal logic employs operators and logic to define formulas that the solutions (or executions) to the systems should satisfy after some finite time, or during a particular amount of bounded time. In particular, temporal logic can be efficiently employed to determine safety and liveness type properties. Safety-type properties typically guarantee that the state remains in a particular set, while liveness-type properties guarantee that the state of system reaches a specific set in finite time. Such specifications are given in terms of a language that employs logical and temporal connectives (or operators) applied to propositions and their combinations. For certain classes of dynamical systems, verification of these properties can be performed using model checking tools. For instance, the question of whether a safety-type specification is satisfied can be answered by finding an execution that violates the specification in finite time.

Linear temporal logic (LTL), as introduced in [4], permits to formulate specifications that involve temporal properties of computer programs; see also [5]. Numerous contributions pertaining to modeling, analysis, design, and verification of LTL specifications for dynamical systems have appeared in the literature in recent years. Without attempting to present a thorough review of the very many articles in such topic, it should be noted that in [6], the authors employ temporal logic to solve a problem involving multiple mobile robots. In their setting, the robots are modeled by continuous-time systems with second-order dynamics and the proposed temporal logic specifications model reachability, invariance, sequencing, and obstacle avoidance. Similar approaches but for dynamical systems given in discrete time, which are more amenable to computational tools, such as model checking, have also been pursued in the literature. In [7], the authors propose mixed integer linear programming and quadratic programming tools for the design of algorithms required to satisfy LTL specifications for dynamical systems with both continuous-valued and discrete-valued states, which are called *mixed logic dynamical systems.* This broad class of systems are expressive enough to be able to model discontinuous and (hybrid, in the sense of having states that take on continuous, and on discrete values) piecewise discrete-time linear systems. In [8], for discrete-time nonlinear systems with continuous-valued and discrete-valued states, the authors formulate optimization problems related to trajectory generation with linear temporal logic specifications for which mixed integer linear programming tools are applied. In [9], the design of controllers to satisfy alternating-time temporal logic (ATL*), which is an expressive branching-time logic that allows for quantification over control strategies, is pursued using barrier and Lyapunov functions for a class of continuous-time systems. More recently, using similar programming tools, in [10], tools to design reactive controllers for mixed logical dynamical systems so as to satisfy high-level specifications given in the language of metric temporal logic are proposed. Promising extensions of these techniques to the case of specifications that need to hold over pre-specified bounded horizons, called Signal Temporal Logic (STL), have been recently pursued in several articles; see, e.g., [11], to just list a few.

Metric temporal logic (MTL) [12] and signal temporal logic (STL) [13] are extensions of LTL that provide a "measure" of how robustly the specifications are satisfied. In [14], MTL is used to analyze the robustness of continuous-time signals. The proposed approach provides a robustness degree in terms of the bound on the perturbation that a signal can tolerate for the given specifications to still be satisfied. In [15], STL is employed for the verification of hybrid dynamical systems. The authors propose robustness measures that indicate how far a given trajectory stands, in space and time, from satisfying a given STL formula.

1.2 Motivation

Tools for the systematic study of temporal logic properties in dynamical systems that have solutions (or executions) changing continuously over intervals of ordinary continuous time and, at certain time instances, having jumps in their continuous-value and discrete-valued states, such as the frameworks proposed in [16–20], are much less developed. A hybrid system $\mathcal{H} = (C, F, D, G)$ exhibiting such behavior can be described as follows [20]:

$$\begin{aligned} \dot{x} &\in F(x) && x \in C \\ x^+ &\in G(x) && x \in D \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state. The map $F : \mathcal{X} \rightrightarrows \mathbb{R}^n$ is a set-valued map and denotes the flow map capturing the continuous dynamics on the flow set C, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map and defines the jump map capturing the discrete dynamics on the jump set D. Throughout the dissertation, we assume $C \subset \operatorname{dom} F$, and $D \subset \operatorname{dom} G$.

A canonical academic example of a hybrid system is the well-known bouncing ball system, which has infinitely many events over a bounded ordinary time horizon (i.e., Zeno) at time instances that are not pre-specified and actually depend on the initial condition of the system; see, e.g., [21]. Another canonical example is the dynamical system resulting from controlling the temperature of a room with a logic controller, in which the jumps of the logic variables in the controller occur when the temperature hits certain thresholds. In such hybrid dynamical systems, the study of temporal logic using discretization-based approaches may not be fitting as, in principle, the time at which a jump occurs is not known a priori and these times are likely to occur aperiodically. Though results enabling the reasoning of continuously changing systems and signals using discrete-time methods are available in the literature (see, e.g., [14]), the sampling effect may prevent one from being able to guarantee that the properties certified for the discretization extend to the actual continuous time process.

Consider the temperature control model described as

$$\dot{z} = -z + z_0 + q z_{\triangle}. \tag{1.2}$$

The variable q denotes the state of the heater, i.e., $q = 1$ when the heater is on and $q = 0$ when the heater is off. The state z is the room temperature. The constants z_0 and $z_{\triangle}$ denote the room temperature when the heater is off and the capacity of the heater to raise the temperature, respectively. We observe that the system constantly measures and controls the room temperature z to maintain it within $[z_{\min}, z_{\max}]$, where $z_{\min}$ and $z_{\max}$ are the minimum and the maximum of the room temperature, respectively. This system requirement can be expressed in terms of an LTL formula f involving the always operator $\Box$ as follows:

$$f := \Box p$$

where p is true when the temperature is in the desired range and is false otherwise.

Since robustness is a widely studied property of dynamical systems, we are also interested in certifying LTL properties for hybrid systems with robustness to perturbations.

For example, consider the temperature control system in (1.2). While the system controls the temperature in the given range $[z_{\min}, z_{\max}]$, we need to carefully verify such requirements under the presence of unknown perturbations. Indeed, in the presence of perturbations, one would want to relax the specification to $z \in [z_{\min} - \varepsilon, z_{\max} + \varepsilon]$, where $\varepsilon > 0$ quantifies the approximation error in the satisfaction of the original specification.

To specify and verify such property with robustness to perturbations, we propose a notion allowing for approximate satisfaction of LTL formulas. For the temperature control problem, the proposed notions allow the temperature to start from the given range $[z_{\min}, z_{\max}]$ but to remain in the larger range $[z_{\min} - \varepsilon, z_{\max} + \varepsilon]$. We refer to this notion as ε-approximate satisfaction of $\Box p$, and we characterize it in terms of conditional invariance.

Slight modifications in temporal logic specifications to achieve the satisfaction of the given specifications has been presented in [14, 15] while using the quantitative measure of the satisfaction of the specifications. The difference in our approach is that our notions do not use the quantitative measure of the satisfaction of the specifications reflecting the robustness of the satisfaction with multi-valued and quantitative temporal logics. Our approach focuses on how to certify LTL properties of systems under the presence of perturbations.

The proposed notion for approximate satisfaction of LTL formulas is extended to the case when perturbations are explicitly considered in the model. For the temperature control problem, we can include the disturbance w in the system in (1.2). The system with disturbance w is given by

$$\dot{z} = -z + z_0 + q z_{\triangle} + w. \tag{1.3}$$

Since the perturbation is modeled, the specification to satisfy is $z \in [z_{\min} - \varepsilon, z_{\max} + \varepsilon]$, where $\varepsilon > 0$, rather than $z \in [z_{\min}, z_{\max}]$. To specify and verify such property, we propose a notion for robust approximate satisfaction of LTL formulas. The proposed notion allows the temperature starting from the given range $[z_{\min}, z_{\max}]$ to remain in the larger range $[z_{\min} - \varepsilon, z_{\max} + \varepsilon]$ under the presence of disturbance w. We refer to this notion as robust ε-approximate satisfaction of $\Box p$, and we characterize it in terms of robust conditional invariance.

In this dissertation, ε-approximate and robust ε-approximate satisfaction of LTL specifications are introduced to specify dynamical properties of hybrid dynamical systems with robustness to perturbations. Such notions allow to analyze dynamical properties of hybrid dynamical systems with robustness to perturbations such as unmodeled dynamics, measurement noise, and exogenous disturbances. We propose an approach of specifying robustness properties using LTL and certifying such properties without requiring the computation of the systems' solutions, but rather, guaranteeing the existence of barrier function candidates or Lyapunov-like functions, along with properties of the data defining the system.

Furthermore, as an extension of LTL, we consider signal temporal logic(STL) for specifying and certifying properties of hybrid systems. The difference between LTL and STL is that STL provides a framework for reasoning about temporal properties over time with numerical predicates. Consider the continuous-time dynamics of the thermostat model in (1.2). For example, we consider a simple requirement as follows: during the first 60 seconds, the temperature always stay within $[z_{\min}, z_{\max}]$. Then, this requirement can be expressed in terms of a STL formula with the same proposition p above

$$\Box_{[0,60]} p.$$

In this dissertation, inspired by the ideas in [11, 22, 23] for STL in continuous-time and discrete-time systems, we introduce STL for hybrid systems. We propose sufficient conditions that guarantee STL properties of hybrid systems employing results on LTL for hybrid systems.

1.3 Contributions

In this dissertation, tools that permit guaranteeing high-level specifications for solutions to hybrid dynamical systems without requiring the computation of the solutions themselves or discretization of the dynamics, but rather, guaranteeing properties of the data defining the system and the existence of Lyapunov-like functions are presented.

We consider a broad class of hybrid dynamical systems, in which the state vector may include physical and continuous-valued variables, logic and discrete-valued variables, timers, memory states, and others; solutions may not be unique and may not necessarily exist for arbitrary long hybrid time (namely, solutions may not be complete); and solutions may exhibit Zeno behavior. In particular, as in [20], a hybrid dynamical system is defined by a flow map, which is given by a set-valued map governing the continuous change of the state variables, a flow set, which is a subset of the state space on which solutions are allowed to evolve continuously, a jump map, which is also given by a set-valued map governing the discrete change of the variables, and a jump set, which defines the set of points where jumps can occur. Along with the state space, these four objects define the data of a hybrid dynamical system.

For this broad class of hybrid dynamical systems, characterizations of formulas involving one temporal operator and atomic propositions are presented in terms of dynamical properties of hybrid systems, in particular, forward pre-invariance and finite time attractivity. These notions are used to formulate sufficient conditions for the satisfaction of basic temporal logic formulas. More precisely, we show that the specifications using the always operator can be guaranteed to hold under mild conditions on the data of the hybrid system when a forward invariance property of an appropriately defined set holds. To arrive to such conditions, we present sufficient conditions for forward (pre-)invariance of closed sets in hybrid dynamical systems that extend those in [24]. To derive conditions that certify that formulas using the eventually operator hold, we generate results to certify finite time attractivity of sets in hybrid dynamical systems, for which we exploit and extend the ideas used to certify finite time stability of hybrid dynamical systems in [25]. Furthermore, our (mostly solution-independent) approach allows us to provide an estimate of the (hybrid) time it takes for a temporal specification to be satisfied, with the estimate only depending on a Lyapunov function and the initial condition of the solution being considered. Moreover, we introduce sufficient conditions for certain formulas that combine more than one temporal operator, which combine our conditions for the individual temporal operators.

While many of our results do not require computing solutions to the hybrid dynamical system, which is a key advantage when compared to methods for continuous-time, discrete-time, and mixed logic dynamical systems cited above and the method for hybrid traces in [26], the price to pay when using the results in this paper is finding a certificate for finite time attractivity, which is in terms of a Lyapunov function. It should be noted that though our conditions are weaker than those in [9], finding such functions might be challenging at times. However, the same complexity is present in Lyapunov methods for certifying asymptotic stability of a point or a set [27], or for employing continuously differentiable barrier certificates and Lyapunov functions to certify temporal logic constraints for continuous-time systems. On the other hand, it should be noted that the framework for hybrid dynamical systems considered here is such that, under mild conditions, in addition to enabling a converse theorem for asymptotic stability, has robustness properties to small perturbations, which may permit extending the results in this paper to the case under perturbations; see [20, Chapters 6 and 7].

Furthermore, we propose notions to specify and verify temporal logic specifications under the presence of perturbations. To this end, following [28], we first build the set $K := \{x \in \mathbb{R}^n : p(x) = \texttt{True}\}$ with a given proposition p. Then, by relaxing the set K, we collect additional points that are ε-close to the set K. The relaxed set is denoted by K^ε, i.e., $K^\varepsilon := K + \varepsilon\mathbb{B}$ where $\mathbb{B}$ is the closed unit ball. Using the set K^ε, we introduce notions encoding ε-approximate satisfaction and robust ε-approximate satisfaction of temporal logic specifications. As a first step, we focus on basic temporal logic formulas involving the always and the eventually operators to define the notions allowing for ε-approximate satisfaction and robust ε-approximate satisfaction. For example, consider the satisfaction of the formula $\Box p$ from the set set K.

a) ε-approximate satisfaction of the formula $\square p$ from the set K implies that each solution, starting from the set K, stays in the set K^ε.

b) Robust ε-approximate satisfaction of the formula $\square p$ from the set K implies that each solution, starting from the set K, stays in the set K^ε under the presence of perturbations.

Moreover, equivalence relationships are established between the ε-approximate satisfaction notions, conditional invariance, and FTA with robustness to perturbations. For this purpose, we introduce robust conditional invariance and robust FTA for hybrid systems that extend conditional invariance and FTA, respectively, to the case with perturbations. Moreover, due to the equivalence relationships, sufficient conditions that guarantee ε-approximate and robust ε-approximate satisfaction of the given formulas are formulated. In particular, we propose sufficient conditions using barrier functions and Lyapunov-like functions tailored to robust conditional invariance and robust FTA for hybrid systems, by extending the results in [25, 28, 29].

We also propose characterizations of the satisfaction of the STL formulas involving the *always* ($\square_{\mathcal{I}}$) operator and the *eventually* ($\diamond_{\mathcal{I}}$) operator. To characterize the behavior of solutions to hybrid systems $\mathcal{H}$ during the given time $\mathcal{I}$, we formulate the considered STL formulas as an LTL formula involving the strong until operators ($\mathcal{U}_s$) using the auxiliary system, denoted $\mathcal{H}_\tau$. Finally, we present sufficient conditions that guarantee the satisfaction of the STL formulas with the *always* ($\square_{\mathcal{I}}$) operator and the *eventually* ($\diamond_{\mathcal{I}}$) operator.

In addition, sufficient conditions for temporal logic formulas that have more than one operator are presented in more detail than in [30]. In particular, we show how to derive conditions for formulas that have more than one operator by combining the conditions for formulas that have one operator. Additionally, a discussion on the decomposition of temporal logic formulas using finite state automata is included.

1.4 Organization

The contents of this dissertation are organized into following chapters.

Chapter 2: Preliminaries

In this chapter, the hybrid systems framework, basic properties used throughout this dissertation are presented.

Chapter 3: Linear temporal logic for hybrid systems

In this chapter, linear temporal logic (LTL) for hybrid dynamical systems is introduced. We define operators and specify properties of hybrid systems with LTL formulas. For hybrid systems, nominal satisfaction of atomic propositions and LTL formulas are presented. Furthermore, we introduce notions capturing approximate satisfaction and robust approximate satisfaction of LTL formulas, which extend the nominal satisfaction of LTL formulas to certify satisfaction of LTL formulas robustly.

Chapter 4: Characterizations of temporal operators using dynamical properties

In this chapter, we present equivalence characterizations for the satisfaction of LTL formulas involving one temporal operator. The satisfaction of the formula is assured by guaranteeing particular properties of sets such as invariance and finite time attractivity notions for hybrid systems.

Chapter 5: Sufficient conditions guaranteeing the satisfaction of temporal formulas for hybrid systems

In this chapter, sufficient conditions guaranteeing the nominal satisfaction of temporal formulas are proposed. Sufficient conditions for each temporal operator use either a certificate for finite-time convergence in terms of a Lyapunov function, or the data of the hybrid system and the set of points where the proposition is true to satisfy conditions for invariance using Lyapunov-like functions or barrier functions.

Chapter 6: Sufficient conditions guaranteeing the satisfaction of temporal formulas for hybrid systems under perturbations

In this chapter, sufficient conditions guaranteeing the satisfaction of temporal formulas for hybrid systems under perturbations are proposed. Notions such that ε-approximate and robust ε-approximate satisfaction of LTL formulas in Chapter 3 allow to analyze dynamical properties of hybrid systems with robustness to perturbations. Then, using Lyapunov-like functions or barrier functions, we derive sufficient conditions that guarantee the satisfaction of temporal formulas for hybrid systems under perturbations.

Chapter 7: Sufficient conditions for LTL formulas combining operators

In this chapter, sufficient conditions for LTL formulas that combine more than one operator. Using the sufficient conditions for LTL formulas that involve a single temporal operator in Chapter 5, conditions for LTL formulas that combine more than one operator are given by compositions of the conditions for LTL having a single temporal operator.

Chapter 8: Signal temporal logic for hybrid dynamical systems

In this chapter, signal temporal logic (STL) for hybrid dynamical systems is introduced to specify and verify dynamical properties of hybrid systems. Equivalence relationships between the satisfaction of STL formulas and the satisfaction of LTL formulas involving the strong until operators are proposed. Using the equivalence relationship, sufficient conditions that guarantee the satisfaction of STL formulas are proposed.

Chapter 9: Object grasping using multiple agents

In this chapter, a networked hybrid system which is described by a multi-agent system that consists of multiple subsystems is proposed to solve object grasping problems. Each subsystem, a hybrid closed-loop system corresponding to each agent, is commanded by an individual feedback controller which are coordinated by a supervisory controller.

Chapter 10: Conclusion and future directions

In this chapter, the results in this dissertation are summarized and potential future directions are discussed.

Chapter 2

Preliminaries

2.1 Models of hybrid dynamical systems

In this dissertation, hybrid systems $\mathcal{H}$ modeled as in (1.1) are mainly considered. A solution ϕ to $\mathcal{H}$ is parameterized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the ordinary time variable, j is the discrete jump variable, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{N} := \{0, 1, 2, \ldots\}$. The domain dom $\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ of ϕ is a hybrid time domain if for every $(T, J) \in$ dom ϕ, the set dom $\phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J}(I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution is given by a hybrid arc. A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j = \{t : (t, j) \in E\}$.

Definition 2.1 (Concept of solution to $\mathcal{H}$)**.** *A hybrid arc ϕ : dom $\phi \to \mathbb{R}^n$ is a* solution *to $\mathcal{H}$ if*

(S0) $\phi(0, 0) \in \overline{C} \cup D$;

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in dom\ \phi\}$ has nonempty interior, $t \mapsto \phi(t, j)$ is locally absolutely continuous and

$$\begin{aligned} &\phi(t, j) \in C && \textit{for all } t \in \textit{int}(I^j), \\ &\dot{\phi}(t, j) \in F(\phi(t, j)) && \textit{for almost all } t \in I^j; \end{aligned}$$

(S2) for all $(t, j) \in dom\ \phi$ such that $(t, j+1) \in dom\ \phi$,

$$\phi(t, j) \in D, \qquad \phi(t, j+1) \in G(\phi(t, j)).$$

A solution ϕ to $\mathcal{H}$ is said to be maximal if there is no solution ϕ' to $\mathcal{H}$ such that $\phi(t, j) = \phi'(t, j)$ for all $(t, j) \in \text{dom } \phi$ with dom ϕ a proper subset of dom ϕ'. It is said to be nontrivial if dom ϕ contains at least two elements. A solution ϕ is said to be complete if its domain is unbounded. It is Zeno if it is complete and $\sup_t \text{dom } \phi < \infty$. It is eventually discrete if $T = \sup_t \text{dom } \phi < \infty$ and $\text{dom } \phi \cap (\{T\} \times \mathbb{N})$ contains at least two elements. It is genuinely Zeno if it is Zeno, but not eventually discrete. See [20] for more details about hybrid dynamical systems.

For convenience, we define the range of a solution ϕ to $\mathcal{H}$ as $\text{rge } \phi = \{\phi(t, j) : (t, j) \in \text{dom } \phi\}$. We use $\mathcal{S}_{\mathcal{H}}(x)$ to denote the set of maximal solutions to $\mathcal{H}$ starting from $x \in \overline{C} \cup D$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{R}(\mathcal{A})$ denotes the (infinite-horizon) reachable set from $\mathcal{A}$; i.e., $\mathcal{R}(\mathcal{A}) := \{\phi(t, j) : \phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{A}), (t, j) \in \text{dom } \phi\}$.

Definition 2.2 (Hybrid basic conditions)**.** *A hybrid system $\mathcal{H} = (C, F, D, G)$ is said to satisfy the* hybrid basic conditions *if the following hold:*

- *C and D are closed;*
- *The flow map F is outer semicontinuous and locally bounded relative to C, $C \subset dom\ F$, and $F(x)$ is convex for each $x \in C$;*

- *The jump map G is outer semicontinuous and locally bounded relative to D, and $D \subset dom\ G$.*

When $\mathcal{H}$ satisfies these mild regularity properties, then $\mathcal{H}$ is said to be *well-posed* [20, Theorem 6.30]. Throughout the paper, we assume that $\mathcal{H}$ satisfies the hybrid basic conditions.

Perturbations affecting hybrid systems can be considered to model the continuous and discrete dynamics of hybrid systems. The general form of a hybrid system $\mathcal{H}$ with a state perturbation w is denoted by $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$, is written as

$$\begin{aligned} \dot{x} \ &\in F_w(x, w) \qquad && (x, w) \in C_w \\ x^+ &\in G_w(x, w) \qquad && (x, w) \in D_w. \end{aligned} \tag{2.1}$$

The solution concept for $\mathcal{H}$ in (1.1) can be extended to a solution pair (ϕ, w) for $\mathcal{H}_w$.

A pair (ϕ, w), consisting of a hybrid arc ϕ and a state perturbation $w \in \mathcal{W}$ with dom ϕ = dom w(= dom (ϕ, w)), is a solution pair to $\mathcal{H}_w$ in (2.1) if $(\phi(0,0), w(0,0)) \in \overline{C}_w \cup D_w$, and

(S1) For all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \text{dom } \phi\}$ has nonempty interior,

$$\begin{aligned} &(\phi(t,j), w(t,j)) \in C_w && \text{for all } t \in \text{int } I^j \\ &\tfrac{d\phi}{dt}(t,j) \in F_w(\phi(t,j), w(t,j)) && \text{for almost all } t \in I^j \end{aligned}$$

(S2) For all $(t, j) \in \text{dom } \phi$ such that $(t, j+1) \in \text{dom } \phi$,

$$(\phi(t,j), w(t,j)) \in D_w, \;\; \phi(t, j+1) \in G_w(\phi(t,j), w(t,j)).$$

In addition, a solution pair (ϕ, w) to $\mathcal{H}_w$ is said to be nontrivial if dom (ϕ, w) contains at least two points; complete if dom (ϕ, w) is unbounded; maximal if there does not exist another $(\phi, w)'$ such that (ϕ, w) is a truncation of $(\phi, w)'$ to some proper subset of dom $(\phi, w)'$.

Given a set $K \subset \mathbb{R}^n$, we also define the set of all maximal solution pairs (ϕ, w) to $\mathcal{H}_w$ with $\phi(0,0) \in K$ as $\mathcal{S}_{\mathcal{H}_w}(K)$. The state perturbation w is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto w(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t : (t,j) \in \text{dom } w\}$. Additionally, when $w(t,j) = 0$ for every $(t,j) \in \text{dom } w$, this means that $\mathcal{H}_w$ reduces to the nominal hybrid system $\mathcal{H}$ as in (1.1).

For convenience, we denote $\Pi(C_w) := \{x \in \mathbb{R}^n : \exists w \in \mathcal{W} \text{ s.t. } (x,w) \in C_w\}$ and $\Pi(D_w) := \{x \in \mathbb{R}^n : \exists w \in \mathcal{W} \text{ s.t. } (x,w) \in D_w\}$. Throughout the paper, we assume that $C_w \subset \text{dom } F_w$ and $D_w \subset \text{dom } G_w$.

Furthermore, given a function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we define a perturbed hybrid system $\mathcal{H}_\rho$ on $\mathbb{R}^n$ capturing the effect of perturbations w with worst case size given by ρ as in [20, Definition 6.27]. This model of $\mathcal{H}_\rho$ is given as follows:

$$\begin{aligned} \dot{x} &\in F_\rho(x) & x &\in C_\rho \\ x^+ &\in G_\rho(x) & x &\in D_\rho, \end{aligned} \tag{2.2}$$

where $C_\rho := \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap \Pi(C_w) \neq \emptyset\}$, $F_\rho(x) := \overline{\text{con}} F((x + \rho(x)\mathbb{B}) \cap \Pi(C_w)) + \rho(x)\mathbb{B}$ for all $x \in \mathbb{R}^n$; $D_\rho := \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap \Pi(D_w) \neq \emptyset\}$, $G_\rho := \{\nu \in \mathbb{R}^n : \nu \in g + \rho(g)\mathbb{B}, g \in G((x + \rho(x)\mathbb{B}) \cap \Pi(D_w)\}$ for all $x \in \mathbb{R}^n$. This perturbed system is used to analyze robustness of a hybrid system having perturbations w of size ρ.

2.2 Invariance and attractivity notions

In the following, we introduce the invariance and attractivity notions that are used in this dissertation.

Definition 2.3 (Forward (pre-)invariance)**.** *A set $K \subset \mathbb{R}^n$ is said to be* forward pre-invariant *for $\mathcal{H}$ if, for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$, rge $\phi \subset K$. The set K is said to be* forward invariant *for $\mathcal{H}$ if it is forward pre-invariant for $\mathcal{H}$ and, for every $x \in K$, every solution $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ is complete.*

Definition 2.4 (Conditional invariance)**.** *Given sets $K \subset \mathbb{R}^n$ and $\mathcal{X}_o \subset K$, the set K is said to be* conditionally invariant *with respect to the set $\mathcal{X}_o$ for $\mathcal{H}$ if, for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_o)$, rge $\phi \subset K$.*

Remark 2.5. *Note that when $\mathcal{X}_o = K$, conditional invariance of K with respect to $\mathcal{X}_o$ is equivalent to forward pre-invariance of K.*

Definition 2.6 (Safety)**.** *A hybrid system $\mathcal{H}$ is said to be safe with respect to $(\mathcal{X}_o, \mathcal{X}_u)$, where $\mathcal{X}_u \subset \mathbb{R}^n$ denotes the unsafe set and $\mathcal{X}_o \subset \mathbb{R}^n \setminus \mathcal{X}_u$ denotes the initial set, if each solution ϕ to $\mathcal{H}$ from $\mathcal{X}_o$ satisfies rge $\phi \subset \mathbb{R}^n \setminus \mathcal{X}_u$.*

Remark 2.7. *Following the notion of safety in [31], conditional invariance of K with respect to $\mathcal{X}_o$ is equivalent to safety with respect to $(\mathcal{X}_o, \mathcal{X}_u)$, with $\mathcal{X}_u := \mathbb{R}^n \backslash K$ defining the region of the state space that the solutions to $\mathcal{H}$ must avoid when starting from the set of initial conditions $\mathcal{X}_o$.*

In the following, inspired by the ideas in [32] for continuous-time systems, we introduce eventual conditional invariance for hybrid systems $\mathcal{H} = (C, F, D, G)$.

Definition 2.8 (Eventual conditional invariance)**.** *Given sets $\mathcal{O} \subset \overline{C} \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$, the set $\mathcal{A}$ is said to be* eventually conditionally invariant *with respect to*

$\mathcal{O}$ for $\mathcal{H}$ if, for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, there exists a hybrid time $(t^\star, j^\star) \in \mathrm{dom}\ \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \mathrm{dom}\ \phi$ such that $t + j \geq t^\star + j^\star$.

Since $\mathcal{H}$ can have maximal solutions that are not complete, we introduce the following notion which, compared to Definition 2.8, requires that only the complete solutions to $\mathcal{H}$ must reach the set $\mathcal{A}$ in finite hybrid time.

Definition 2.9 (Pre-eventual conditional invariance). *Given sets $\mathcal{O} \subset \overline{C} \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$, the set $\mathcal{A}$ is said to be* pre-eventually conditionally invariant *with respect to $\mathcal{O}$ for $\mathcal{H}$ if, for each complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, there exists a hybrid time $(t^\star, j^\star) \in \mathrm{dom}\ \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \mathrm{dom}\ \phi$ such that $t+j \geq t^\star+j^\star$.*

Definition 2.10 (Finite-time attractivity). *Given sets $\mathcal{O} \subset \overline{C} \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$ such that $\mathcal{A}$ is closed, the set $\mathcal{A}$ is said to be* finite-time attractive (FTA) *with respect to $\mathcal{O}$ for $\mathcal{H}$ if, for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, $\mathcal{T}_{\mathcal{A}}(\phi) < \infty$.*

2.3 Nonsmooth Lyapunov functions

For a hybrid system $\mathcal{H} = (C, F, D, G)$, let $V : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathbb{R}^n$ and locally Lipschitz on a neighborhood of C. The generalized gradient of V at $x \in C$, denoted by $\partial V(x)$, is a closed, convex and nonempty set equal to the convex hull of all limits of the sequence $\nabla V(x_i)$, where x_i is any sequence converging to x while x avoids an arbitrary set of measure zero containing all the points at which V is not differentiable. As V is locally Lipschitz, ∇V exists almost everywhere. The generalized directional derivative of V at x in the direction of v can be presented as follows [33]:

$$V^\circ(x, v) = \max_{\zeta \in \partial V(x)} \langle \zeta, v \rangle. \tag{2.3}$$

In addition, for any solution $t \mapsto \phi(t,0)$ to $\dot{x} \in F(x)$,

$$\tfrac{d}{dt}V(\phi(t,0)) \le V^{\circ}(\phi(t,0), \dot{\phi}(t,0)) \tag{2.4}$$

for almost all t in the domain of definition of ϕ, where $\frac{d}{dt}V(\phi(t,0))$ is understood in the standard sense since V is locally Lipschitz.

To bound the increase of the function V along solutions to a hybrid system $\mathcal{H}$, we define the function $u_C : \mathbb{R}^n \to [-\infty, +\infty)$ as follows [34]:

$$u_C(x) := \begin{cases} \max\limits_{v \in F(x)} \max\limits_{\zeta \in \partial V(x)} \langle \zeta, v \rangle & x \in C \\ -\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

In particular, for any solution ϕ to $\mathcal{H}$ and any t where $\frac{d}{dt}V(\phi(t,j))$ exists, we have

$$\tfrac{d}{dt}V(\phi(t,j)) \le u_C(\phi(t,j)). \tag{2.6}$$

Furthermore, in order to bound the change in V after jumps, we define the following quantity:

$$u_D(x) := \begin{cases} \max\limits_{\zeta \in G(x)} V(\zeta) - V(x) & x \in D \\ -\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

Then, for any solution ϕ to $\mathcal{H}$ and for any $(t_{j+1}, j), (t_{j+1}, j+1) \in \operatorname{dom} \phi$, it follows that

$$V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \le u_D(\phi(t_{j+1}, j)). \tag{2.8}$$

Note that when F is a single-valued map, $u_C(x) = V^{\circ}(x, F(x))$ for each $x \in C$. When G is a single-valued map, $u_D(x) = V(G(x)) - V(x)$ for each $x \in D$.

Chapter 3

Linear temporal logic for hybrid dynamical systems

Linear Temporal Logic (LTL) provides a framework to specify desired properties such as *safety*, i.e., “something bad never happens”, and *liveness*, i.e., “something good eventually happens”. In this section, for a given hybrid system $\mathcal{H}$, we define operators and specify properties of $\mathcal{H}$ with LTL formulas [35]. We first introduce atomic propositions.

Definition 3.1 (Atomic Proposition)**.** *An atomic proposition p is a statement on the system state x that, for each x, p is either* `True` *(1 or $\top$) or* `False` *(0 or $\perp$).*

A proposition p will be treated as a single-valued function of x, that is, it will be a function $x \mapsto p(x)$. The set of all possible atomic propositions will be denoted by $\mathcal{P}$.

Logical and temporal operators are defined as follows.

Definition 3.2 (Logic Operators)**.**

- $\neg$ *is the* negation *operator*

- $\vee$ *is the* disjunction *operator*

- $\wedge$ *is the* conjunction *operator*

- $\Rightarrow$ *is the* implication *operator*

- $\Leftrightarrow$ *is the* equivalence *operator*

Definition 3.3 (Temporal Operators)**.**

- $\bigcirc$ *is the* next *operator*

- $\Diamond$ *is the* eventually *operator*

- $\Box$ *is the* always *operator*

- $\mathcal{U}_s$ *is the strong* until *operator*

- $\mathcal{U}_w$ *is the weak* until *operator*

For example, consider the object grasping problem in the previous chapter. Consider the situation where a vehicle reaches a target point on the object. Such a behavior can be expressed in terms of an LTL formula involving the eventually ($\Diamond$) operator; namely,

$$\Diamond p$$

where the atomic proposition p is defined as true when the vehicle reaches the target.

As an additional example, in the autonomous navigation problem, consider a vehicle navigates its environment without colliding with obstacles while approaching a target and it eventually reaches the target. Similarly, define the atomic proposition p to be true when the vehicle reaches the target; and define the atomic proposition q to be true when the vehicle stays in the safe area. Then, the given requirement can be expressed in terms of an LTL formula involving the strong until ($\mathcal{U}_s$) operator; namely,

$$q\mathcal{U}_s p.$$

Given a hybrid system $\mathcal{H}$, the semantics of LTL are defined as follows. For simplicity, we consider the case of no inputs and state-dependent atomic propositions. When a proposition p is `True` at $(t,j) \in \operatorname{dom}\phi$, i.e., $p(\phi(t,j)) = 1$, it is denoted by

$$\phi(t,j) \Vdash p, \tag{3.1}$$

whereas if p is `False` at $(t,j) \in \operatorname{dom}\phi$, it is written as

$$\phi(t,j) \nVdash p. \tag{3.2}$$

An LTL formula is a sentence that consists of atomic propositions and operators of LTL. An LTL formula f being satisfied by a solution $(t,j) \mapsto \phi(t,j)$ at some time (t,j) is denoted by

$$(\phi,(t,j)) \vDash f, \tag{3.3}$$

while f not satisfied by a solution $(t,j) \mapsto \phi(t,j)$ at some time (t,j) is denoted by[1]

$$(\phi,(t,j)) \nvDash f. \tag{3.4}$$

Let $p,q \in \mathcal{P}$ be atomic propositions. The semantics of LTL are defined as follows: given a solution ϕ to $\mathcal{H}$ and $(t,j) \in \operatorname{dom}\phi$

$$(\phi,(t,j)) \vDash p \;\Leftrightarrow\; \phi(t,j) \Vdash p \tag{3.5a}$$

[1]Note that to be compatible with the literature, instead of $\Vdash$, we use $\vDash$ for a formula.

$$(\phi,(t,j)) \vDash \neg p \;\Leftrightarrow\; (\phi,(t,j)) \nvDash p \tag{3.5b}$$

$$(\phi,(t,j)) \vDash p \vee q \;\Leftrightarrow\; (\phi,(t,j)) \vDash p \text{ or } (\phi,(t,j)) \vDash q \tag{3.5c}$$

$$(\phi,(t,j)) \vDash \bigcirc p \;\Leftrightarrow\; (t,j+1) \in \text{dom } \phi \text{ and } (\phi,(t,j+1)) \vDash p \tag{3.5d}$$

$$\begin{aligned}(\phi,(t,j)) \vDash p\mathcal{U}_s q \;\Leftrightarrow\; & \exists (t',j') \in \text{dom } \phi, t'+j' \geq t+j \text{ s.t. } (\phi,(t',j')) \vDash q, \\ & \text{and } \forall (t'',j'') \in \text{dom } \phi \text{ s.t. } t+j \leq t''+j'' < t'+j', (\phi,(t'',j'')) \vDash p\end{aligned} \tag{3.5e}$$

$$\begin{aligned}(\phi,(t,j)) \vDash p\mathcal{U}_w q \;\Leftrightarrow\; & (\phi,(t',j')) \vDash p \;\; \forall (t',j') \in \text{dom } \phi \text{ s.t. } t'+j' \geq t+j \\ & \text{or } (\phi,(t,j)) \vDash p\mathcal{U}_s q\end{aligned} \tag{3.5f}$$

$$(\phi,(t,j)) \vDash p \wedge q \;\Leftrightarrow\; (\phi,(t,j)) \vDash p \text{ and } (\phi,(t,j)) \vDash q \tag{3.5g}$$

$$(\phi,(t,j)) \vDash \Box p \;\Leftrightarrow\; (\phi,(t',j')) \vDash p \quad \forall\, t'+j' \geq t+j,\ (t',j') \in \text{dom } \phi \tag{3.5h}$$

$$(\phi,(t,j)) \vDash \Diamond p \;\Leftrightarrow\; \exists (t',j') \in \text{dom } \phi, t'+j' \geq t+j \text{ s.t. } (\phi,(t',j')) \vDash p. \tag{3.5i}$$

The same semantics of LTL are used for formulas. For example, with a given formula f, $\bigcirc f$ is satisfied by ϕ at $(t,j) \in \text{dom } \phi$ when $(t,j+1) \in \text{dom } \phi$ and $(\phi,(t,j+1))$ satisfies f.

Given an atomic proposition p, we introduce the following set where p is satisfied:

$$K := \{x \in \mathbb{R}^n : p(x) = 1\}. \tag{3.6}$$

The satisfaction of a given LTL formula is assured by guaranteeing particular properties of the solutions to $\mathcal{H}$ relative to the set K.

Definition 3.4 (Nominal Satisfaction of Propositions)**.** *Given an atomic proposition p, let the set K be given as in* (3.6)*. The atomic proposition p is satisfied by a solution ϕ to $\mathcal{H}$ at $(t,j) \in$ dom ϕ if $p(\phi(t,j)) = 1$; i.e.,*

$$(\phi,(t,j)) \vDash p \;\Leftrightarrow\; \phi(t,j) \in K.$$

Definition 3.5 (Nominal Satisfaction of Formulas)**.** *Given an LTL formula f, the formula f is satisfied by a solution ϕ to $\mathcal{H}$ at $(t, j) \in \operatorname{dom} \phi$ if $(\phi, (t, j)) \vDash f$. Moreover, the formula f is satisfied by $\mathcal{H}$ at $(t, j) = (0, 0)$ if each solution ϕ to $\mathcal{H}$ satisfies $(\phi, (0, 0)) \vDash f$.*

In the following, we introduce a notion of approximate satisfaction of propositions which is used to define notions of approximate satisfaction of LTL formulas. First, given an atomic proposition p and $\varepsilon > 0$, we define the set K^ε given by

$$K^\varepsilon := K + \varepsilon\mathbb{B}, \tag{3.7}$$

where K is given as in (3.6). The set K^ε is exploited to capture approximate satisfaction of the proposition p since it collects points in $\mathbb{R}^n$ that are ε-close to the set K.

Definition 3.6. *(Approximate Satisfaction of Propositions). Given an atomic proposition p, let the set K be as in* (3.6)*. The proposition p is ε-approximately satisfied by a solution ϕ to $\mathcal{H}$ at $(t, j) \in \operatorname{dom} \phi$ with $\varepsilon > 0$ if*

$$\phi(t, j) \in K^\varepsilon.$$

Remark 3.7. *Given an atomic proposition p, one can also define a proposition p^ε from the set K^ε in* (3.7) *that is satisfied (exactly) by a solution ϕ to $\mathcal{H}$ at $(t, j) \in \operatorname{dom} \phi$ when $\phi(t, j) \in K^\varepsilon$. Specifically, once we build the set K^ε with $\varepsilon > 0$ as in* (3.7)*, p^ε can be defined as $p^\varepsilon(x) = 1$ for every $x \in K^\varepsilon$, and $p^\varepsilon(x) = 0$ otherwise.*

Next, given an atomic proposition p, we present notions capturing approximate satisfaction of the formulas $\Box p$ and $\Diamond p$ using the set K^ε in (3.7).

Definition 3.8. *(Approximate Satisfaction of $\Box p$). Given an atomic proposition p, let $\varepsilon > 0$ and the set K^ε be as in* (3.7)*. The formula $f = \Box p$ is ε-approximately satisfied by $\mathcal{H}$ at $(t, j) = (0, 0)$ if each solution ϕ to $\mathcal{H}$ satisfies $\phi(t, j) \in K^\varepsilon$ for each $(t, j) \in dom\ \phi$.*

We are also interested in $\Box p$ being satisfied only for solutions starting from K. The following notion requires that each solution ϕ to $\mathcal{H}$ starting from $K \subset K^\varepsilon$ stays in K^ε. As we show later in this section, this notion corresponds to conditional invariance.

Definition 3.9. *(Approximate Satisfaction of $\Box p$ from K). Given an atomic proposition p, let $\varepsilon > 0$ and the sets K and K^ε be as in* (3.6) *and* (3.7)*, respectively. The formula $f = \Box p$ is ε-approximately satisfied by $\mathcal{H}$ from the set K at $(t, j) = (0, 0)$ if each solution ϕ to $\mathcal{H}$ with $\phi(0, 0) \in K$ satisfies $\phi(t, j) \in K^\varepsilon$ for each $(t, j) \in dom\ \phi$.*

The following academic example illustrates the case when the formula $\Box p$ is not satisfied by a solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$, but $\Box p$ is ε-approximately satisfied by a solution ϕ to $\mathcal{H}$.

Example 3.10. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}$ given by*

$$F(x) := \begin{cases} -1 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 0 \end{cases}$$

for all $x \in C := \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x \geq 1\}$ and

$$G(x) := \begin{cases} -2 & \text{if } x \in [0, 1/2) \\ \delta & \text{if } x \in [1/2, 1) \\ 1/2 + \delta & \text{if } x = 1 \end{cases}$$

for all $x \in D := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, *where* $0 < \delta < 1/2$. *Let an atomic proposition* p *given by*

$$p(x) = \begin{cases} 1 & \text{if } x \in (-2, 2) \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Pick a solution ϕ *satisfying* $\phi(0,0) = 1.5$, *which implies that* $\phi(0,0)$ *satisfies* p. *The solution* ϕ *from* $\phi(0,0)$ *flows initially and it is such that* $\phi(t,0)$ *satisfies* p *for all* $t \leq t_1$ *when* $\phi(t_1,0) = 1$. *Since* $\phi(t_1,0) = 1$ *is in the jump set, the solution* ϕ *jumps to* $1/2 + \delta$. *And, the solution* ϕ *satisfies* p *after the first jump; i.e.,* $(\phi,(t_1,1)) \vDash p$. *Since the solution* ϕ *is still in the jump set after the first jump, the solution* ϕ *jumps to* δ *and* $\phi(t_1,2)$ *satisfies* p*; i.e.,* $(\phi,(t_1,2)) \vDash p$. *Since* $\phi(t_1,2)$ *is in the jump set, the solution* ϕ *jumps to* $\phi(t_1,3) = -2$. *Thus, the solution* ϕ *does not satisfy* p *after the third jump. This example shows that the formula* $\Box p$ *is not satisfied by the solution* ϕ *at* $(t,j) = (0,0)$.

On the other hand, with the atomic proposition p *in* (3.8), *consider* $\varepsilon > 0$ *and the sets* K *and* K^ε *as in* (3.6) *and* (3.7), *respectively. Then, we have that* $K^\varepsilon = (-2-\varepsilon, 2+\varepsilon)$. *Note that* p *is* ε*-approximately satisfied by a solution* ϕ *to* $\mathcal{H}$ *at* $(t,j) \in \text{dom}\,\phi$ *when* $\phi(t,j) \in K^\varepsilon$. *Now, pick a solution* ϕ *satisfying* $\phi(0,0) = 1.5$, *which implies* $\phi(0,0) \in K$. *The solution* ϕ *from* $\phi(0,0)$ *flows initially so that* $\phi(t,0) \in K^\varepsilon$ *for all* $t \leq t_1$ *when* $\phi(t_1,0) = 1$. *Since* $\phi(t_1,0) = 1$ *is in the jump set, the solution* ϕ *jumps to* $1/2 + \delta$. *The solution* ϕ *is in* K^ε *after the first jump; i.e.,* $\phi(t_1,1) \in K^\varepsilon$. *Since the solution* ϕ *is still in the jump set after the first jump, the solution* ϕ *jumps to* δ. *The solution* ϕ *stays in* K^ε *after the second jump; i.e.,* $\phi(t_1,2) \in K^\varepsilon$. *Since* $\phi(t_1,2)$ *is in the jump set, the solution* ϕ *jumps to* -2. *Here,* $\phi(t_1,3)$ *stays in* K^ε *and the solution* ϕ *flows after the third jump. Then, the solution* ϕ *stays in* K^ε *by the flow map. Therefore, the solution*

ϕ satisfying $\phi(0,0) \in K$ stays in K^ε for all future time; namely, the formula $\Box p$ is ε-approximately satisfied by the solution ϕ from K at $(t,j) = (0,0)$. $\triangle$

Next, we introduce a notion capturing approximate satisfaction of the formula $\Diamond p$.

Definition 3.11. *(Approximate Satisfaction of $\Diamond p$). Given an atomic proposition p, let $\varepsilon > 0$ and the set K^ε be as in* (3.7)*. The formula $f = \Diamond p$ is ε-approximately satisfied by $\mathcal{H}$ at $(t,j) = (0,0)$ if for each solution ϕ to $\mathcal{H}$, there exists $(t,j) \in$ dom ϕ such that $\phi(t,j) \in K^\varepsilon$.*

In the following, given an atomic proposition p, we present notions capturing ε-approximate satisfaction of the formulas $\Box p$ and $\Diamond p$ with robustness to perturbations w using the set K^ε in (3.7). First, we introduce a notion capturing robust ε-approximate satisfaction of the formula $\Box p$.

Definition 3.12. *(Robust Approximate Satisfaction of $\Box p$). Given an atomic proposition p, let $\varepsilon > 0$ and the set K^ε be as in* (3.7)*. The formula $f = \Box p$ is robustly ε-approximately satisfied by $\mathcal{H}$ at $(t,j) = (0,0)$ if each solution pair (ϕ, w) to $\mathcal{H}_w$ satisfies $\phi(t,j) \in K^\varepsilon$ for every $(t,j) \in$ dom ϕ.*

The following notion is a refinement of the notion capturing approximate satisfaction of $\Box p$ from the set K in Definition 3.9.

Definition 3.13. *(Robust Approximate Satisfaction of $\Box p$ from K). Given an atomic proposition p, let $\varepsilon > 0$, and the sets K and K^ε be as in* (3.6) *and* (3.7)*, respectively. The formula $f = \Box p$ is robustly ε-approximately satisfied by $\mathcal{H}$ from K at $(t,j) = (0,0)$ if each solution pair (ϕ, w) to $\mathcal{H}_w$ with $\phi(0,0) \in K$ satisfies $\phi(t,j) \in K^\varepsilon$ for every $(t,j) \in$ dom ϕ.*

Next, we introduce a notion capturing robust ε-approximate satisfaction of the formula $\Diamond p$.

Definition 3.14. *(Robust Approximate Satisfaction of $\Diamond p$). Given an atomic proposition p, let $\varepsilon > 0$ and the set K^ε be as in* (3.7)*. The formula $f = \Diamond p$ is robustly ε-approximately satisfied by $\mathcal{H}$ at $(t, j) = (0, 0)$ if for each solution pair (ϕ, w) to $\mathcal{H}_w$, there exists $(t, j) \in$ dom (ϕ, w) such that $\phi(t, j) \in K^\varepsilon$.*

Chapter 4

Characterizations of temporal operators using dynamical properties

In this chapter, we present basic necessary and sufficient conditions for the satisfaction of LTL formulas involving one temporal operator such as *always* ($\Box$), *eventually* ($\Diamond$), *next* ($\bigcirc$), and *until* ($\mathcal{U}_s$,$\mathcal{U}_w$). Given an atomic proposition p, we build the set K as in (3.6) on which p is satisfied. Then, the satisfaction of the formula is assured by guaranteeing particular properties of the solutions to the hybrid system relative to the set K.

4.1 Characterization of $\Box$ via forward invariance

According to the definition of the $\Box$ operator, given an atomic proposition p, a solution $(t, j) \mapsto \phi(t, j)$ to a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the formula

$$f = \Box p \tag{4.1}$$

at (t, j) when we have that $\phi(t', j')$ satisfies p for all $t' + j' \geq t + j$ such that $(t', j') \in \text{dom } \phi$.

Using the set K in (3.6), to characterize that every solution ϕ to $\mathcal{H}$ satisfies f in (4.1) at each $(t, j) \in \text{dom } \phi$, each solution starting in K needs to stay in K for all time. For this purpose, we recall the definition of forward pre-invariance and then present necessary and sufficient conditions guaranteeing f in (4.1).

Definition 4.1 (Forward pre-invariance)**.** *Consider a hybrid system $\mathcal{H}$. A set $K \subset \mathbb{R}^n$ is said to be forward pre-invariant for $\mathcal{H}$ if every solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ satisfies rge $\phi \subset K$.*

Furthermore, we are also interested in f in (4.1) being satisfied at some $(t, j) \in \text{dom } \phi$ (not necessarily at $(t, j) = (0, 0)$). For this purpose, we define the following notion.

Definition 4.2 (Eventually forward pre-invariance)**.** *Consider a hybrid system $\mathcal{H}$. A set $K \subset \mathbb{R}^n$ is said to be eventually forward pre-invariant for $\mathcal{H}$ if*[1] *for every solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$, there exists $(t, j) \in$ dom ϕ such that $\phi(t', j') \in K$ for all $(t', j') \in$ dom ϕ such that $t' + j' \geq t + j$.*

Theorem 4.3 ($\Box p$ via Forward pre-invariance)**.** *Given an atomic proposition p, the formula $f = \Box p$ is satisfied for every maximal solution ϕ to a hybrid system $\mathcal{H}$ at $(t, j) = (0, 0)$ with $\phi(0, 0) \Vdash p$ if and only if the set K in* (3.6) *is* forward pre-invariant *for $\mathcal{H}$.*

Proof. ($\Rightarrow$) Since $\Box p$ is satisfied for all solutions ϕ at $(t, j) = (0, 0)$ and $\phi(0, 0)$ satisfies p, we have that every solution ϕ to $\mathcal{H}$ satisfies that $\phi(t, j) \in K = \{x \in \mathbb{R}^n : p(x) = 1\}$ for all $(t, j) \in \text{dom } \phi$. This implies that K is forward pre-invariant

[1]A notion that does not insist on the solutions starting from K can also be formulated, but it would be a departure from a forward invariance notion since such a notion would hold for solutions that do not start from K.

via the definition of forward pre-invariance of the set K in Definition 4.1; namely, $\operatorname{rge} \phi \subset K$.

($\Leftarrow$) Since the set K is forward pre-invariant, each solution ϕ that starts in K stays in K. That is, $\phi(0,0)$ satisfies p and each solution ϕ at (t,j) in the domain of each solution satisfies p. This implies that $f = \Box p$ is satisfied for every solution ϕ to $\mathcal{H}$ at $(t,j) = (0,0)$ with $\phi(0,0) \Vdash p$. $\square$

Theorem 4.4 ($\Box p$ via Eventual forward pre-invariance)**.** *Given an atomic proposition p, the formula $f = \Box p$ is satisfied for every maximal solution ϕ to a hybrid system $\mathcal{H}$ at some $(t,j) \in dom\ \phi$ with $\phi(0,0) \Vdash p$ if and only if the set K in* (3.6) *is* eventually forward pre-invariant *for $\mathcal{H}$.*

Proof. ($\Rightarrow$) By the definition of $\Box$ and the definition of solutions to $\mathcal{H}$, since every solution ϕ to $\mathcal{H}$ starting from K satisfies $\Box p$ at some $(t,j) \in \operatorname{dom} \phi$, $\phi(t',j')$ satisfies p for all $(t',j') \in \operatorname{dom} \phi$ such that $t'+j' \geq t+j$; and thus, $\phi(t',j') \in K$ for all (t',j') such that $t'+j' \geq t+j$. This implies that K is forward pre-invariant after $(t,j) \in \operatorname{dom} \phi$. Then, we conclude that K is eventually forward pre-invariant for $\mathcal{H}$ via the definition of eventually forward pre-invariance of the set K in Definition 4.2.

($\Leftarrow$) Since the set K is eventually forward pre-invariant, for each solution ϕ that starts from K, there exists $(t,j) \in \operatorname{dom} \phi$ such that $\phi(t',j') \in K$ for all $(t',j') \in \operatorname{dom} \phi$ such that $t'+j' \geq t+j$. This implies that $\phi(0,0)$ satisfies p and such solution ϕ satisfies p at each $(t',j') \in \operatorname{dom} \phi$ such that $t'+j' \geq t+j$. Therefore, we conclude that $f = \Box p$ is satisfied for every solution ϕ to $\mathcal{H}$ at $(t,j) \in \operatorname{dom} \phi$ with $\phi(0,0) \Vdash p$. $\square$

Note that when K in (3.6) is not forward pre-invariant for $\mathcal{H}$, $\Box p$ is not satisfied for all solutions ϕ to $\mathcal{H}$ at every $(t,j) \in \operatorname{dom} \phi$ with $\phi(0,0) \Vdash p$. The following example shows the case when $\Box p$ is not satisfied for a solution ϕ to $\mathcal{H}$ at every $(t,j) \in \operatorname{dom} \phi$ with $\phi(0,0) \Vdash p$.

Example 4.5. *Let an atomic proposition p given by*

$$\begin{aligned} p(x) &= 1 \qquad && \text{if } x \in [0,1] \\ p(x) &= 0 \qquad && \text{otherwise.} \end{aligned} \tag{4.2}$$

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ with the state $x \in \mathbb{R}$ given by

$$\begin{aligned} F(x) &:= 0 \qquad && \forall x \in C := \left[0, \tfrac{1}{2}\right] \\ G(x) &:= \begin{cases} 2 & \text{if } x = 1 \\ 0 & \text{if } x = 2 \end{cases} \qquad && \forall x \in D := \{1\} \cup \{2\}. \end{aligned} \tag{4.3}$$

Now, pick $\phi(0,0) = 1$ so that $\phi(0,0)$ satisfies p. A solution ϕ from $\phi(0,0)$ does not satisfy p after the first jump; i.e., $\phi(0,1) \nVdash p$; however, $\phi(0,1)$ is still in the jump set D so that it jumps to 0, and it satisfies p after the second jump; i.e., $\phi(0,2) \Vdash p$. Furthermore, the solution ϕ flows after the second jump so that $\phi(t,2)$ satisfies p for every $t \geq 0$. On the other hand, there exists another solution that starts from 1 and stays flowing there for all furture time; hence, it satisfies f. This example shows that $\Box p$ is not satisfied for all solutions ϕ to $\mathcal{H}$ at every $(t,j) \in \operatorname{dom} \phi$ when $K = \{x \in \mathbb{R}^n : p(x) = 1\}$ is not forward pre-invariant.

4.2 Characterization of $\Diamond$ via finite time attractivity

A solution $(t,j) \mapsto \phi(t,j)$ to a hybrid system $\mathcal{H}$ satisfies the formula

$$f = \Diamond p \tag{4.4}$$

at $(t, j) \in \text{dom } \phi$ when there exists $(t', j') \in \text{dom } \phi$ such that $t' + j' \geq t + j$, and $\phi(t', j')$ satisfies p. The same set K in (3.6) is used in this section.

To guarantee that every solution ϕ to $\mathcal{H}$ satisfies f in (4.4) at each $(t, j) \in \text{dom } \phi$, the distance of each solution to K should become zero at some finite $(t, j) \in \text{dom } \phi$ so that ϕ reaches K. Related to this property, we recall the definition of finite time attractivity (FTA) for hybrid systems and then present necessary and sufficient conditions guaranteeing the formula f in (4.4). In this definition, the amount of hybrid time required for a solution ϕ to converge to the set K is captured by a settling-time function $\mathcal{T}_K$ whose argument is the solution ϕ and its output is a positive number determining the time to converge to K. More precisely, given ϕ, $\mathcal{T}_K(\phi) := \inf\{t + j : \phi(t, j) \in K\}$ is the time to reach K. Below, given $x \in \mathbb{R}^n$ and a nonempty set $K \subset \mathbb{R}^n$, $|x|_K := \inf_{y \in K} |x - y|$. We use $\nearrow$ to denote the limit from below.

Definition 4.6 (Finite-time attractivity)**.** *A closed set K is said to be finite time attractive (FTA) for $\mathcal{H}$ with respect to $\mathcal{O} \subset \overline{C} \cup D$ if for every solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, $\sup_{(t,j) \in dom\ \phi} t + j \geq \mathcal{T}_K(\phi)$, and*

$$\lim_{(t,j) \in dom\ \phi : t+j \nearrow \mathcal{T}_K(\phi)} |\phi(t, j)|_K = 0. \tag{4.5}$$

Furthermore, the set K is said to be finite time attractive (FTA) for $\mathcal{H}$ if so it is with respect to $\overline{C} \cup D$.

Theorem 4.7 ($\Diamond p$ via FTA)**.** *Given an atomic proposition p, the formula $f = \Diamond p$ is satisfied for every solution ϕ to a hybrid system $\mathcal{H}$ at $(t, j) = (0, 0)$ if and only if the closed set K in* (3.6) *is* FTA *for $\mathcal{H}$.*

Proof. ($\Rightarrow$) Since $\Diamond p$ is satisfied for every solution ϕ to a hybrid system $\mathcal{H}$ at $(t, j) = (0, 0)$, there exists $(t', j') \in \text{dom } \phi$ such that $t' + j' \geq 0$ and $\phi(t', j') \in$

$K = \{x \in \mathbb{R}^n : p(x) = 1\}$. In fact, $\phi(t', j') \in K$ implies $|\phi(t', j')|_K = 0$ and $t' + j' = \mathcal{T}_K(\phi)$; that is,

$$\lim_{(t,j)\in \mathrm{dom}\ \phi : t+j \nearrow \mathcal{T}(\phi)} |\phi(t,j)|_K = 0$$

with $\mathcal{T}_K(\phi) = t' + j'$. This implies that K is FTA via the definition of FTA of the set K in Definition 4.6.

($\Leftarrow$) Since the closed set K is FTA for $\mathcal{H}$, each solution ϕ to $\mathcal{H}$ satisfies

$$\lim_{(t,j)\in \mathrm{dom}\ \phi : t+j \nearrow \mathcal{T}_K(\phi)} |\phi(t,j)|_K = 0$$

and $\sup_{(t,j)\in \mathrm{dom}\ \phi} t + j \geq \mathcal{T}_K(\phi)$, where $\mathcal{T}_K(\phi) = t' + j'$ for some $(t', j') \in \mathrm{dom}\ \phi$. Indeed, by the definition of the set K, its closedness, and the (local) absolute continuity of ϕ (along with the continuity of the distance function to the set K), there exists $(t', j') \in \mathrm{dom}\ \phi$ such that $\phi(t', j')$ satisfies p. This implies that $f = \Diamond p$ is satisfied for every solution ϕ to a hybrid system $\mathcal{H}$ at $(t, j) = (0, 0)$. $\square$

4.3 Characterization of $\bigcirc$ via properties of the data of $\mathcal{H}$

A solution $(t, j) \mapsto \phi(t, j)$ to a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the formula

$$f = \bigcirc p \tag{4.6}$$

when we have that $\phi(t, j + 1)$ satisfies p for each $(t, j) \in \mathrm{dom}\ \phi$. Here, the same set K in (3.6) is used. To guarantee that every solution ϕ to $\mathcal{H}$ satisfies f in (4.6) at each $(t, j) \in \mathrm{dom}\ \phi$, each solution needs to jump to the set K at the next hybrid time; i.e., $\phi(t, j+1) \in K$.

Theorem 4.8 ($\bigcirc p$ via the data of $\mathcal{H}$)**.** *Given an atomic proposition p, let the set K be as in* (3.6)*. The formula $f = \bigcirc p$ is satisfied for all maximal solutions ϕ to $\mathcal{H}$ at each $(t, j) \in$ dom ϕ if and only if all of the following properties hold simultaneously:*

a) each nontrivial solution ϕ to $\mathcal{H}$ is such that $\phi(0, 0) \in D$; and

b) no flows of $\mathcal{H}$ are possible from any $x \in C$; and

c) $G(D) \subset K \cap D$; and

d) $\overline{C} \subset D$.

Proof. ($\Rightarrow$) Suppose that $\bigcirc p$ is satisfied for all solutions to $\mathcal{H}$. We need to show that a), b), c), and d) hold. By definition of $\bigcirc$ and definition of solutions to $\mathcal{H}$, since every solution ϕ to $\mathcal{H}$ satisfies $\bigcirc p$, $\phi(0, 0) \in D$ and $\phi(0, 1) \in K$ for every $\phi(0, 0) \in \overline{C} \cup D$. In fact, if $\overline{C} \setminus D$ were not to be empty, then there would exist a (trivial) solution ϕ with $\phi(0, 0) \notin D$, so $\bigcirc p$ would not hold since $(0, 1) \notin$ dom ϕ. Hence, $\overline{C} \subset D$ and $\phi(0, 0) \in D$ hold; and thus, items a) and d) hold. Next, we show that item b) holds. Proceeding by contradiction, if flow is possible from a point $x \in C$, then there exists a solution ϕ to $\mathcal{H}$ with $\phi(0, 0) = x$ and there exists $\varepsilon > 0$ such that $[0, \varepsilon) \times \{0\} \subset$ dom ϕ. Since $x \in D$ due to $\overline{C} \subset D$, $\phi(0, 0) \in D$. However, $(0, 1) \notin$ dom ϕ since $[0, \varepsilon) \times \{0\} \subset$ dom ϕ. This is a contradiction, and thus, item b) holds. Finally, we show that item c) holds. By definition of $\bigcirc$, since every solution ϕ to $\mathcal{H}$ satisfies $\bigcirc p$, then $(t, j+1) \in$ dom ϕ and $\phi(t, j+1) \in K$ for each $(t, j) \in$ dom ϕ. By definition of solutions, it implies that for each $(t, j) \in$ dom ϕ, $\phi(t, j) = \xi \in D$ and $G(\xi) \subset K$. Hence, item c) holds.

($\Leftarrow$) Note that $\phi(0,0) \in D$ and $(0,1) \in \text{dom } \phi$ by items a) and b). Then, by item c), $G(\phi(0,0)) \subset K$ since $\phi(0,0) \in D$. Furthermore, for each $(t,j) \in \text{dom } \phi$ such that $\phi(t,j) \in \overline{C} \cup D$, no flows are possible from $\phi(t,j)$ by items b) and d). Thus, $(t, j+1) \in \text{dom } \phi$ and $\phi(t, j+1) \in K$ by item c). Therefore, $f = \bigcirc p$ is satisfied for every solution ϕ to $\mathcal{H}$. $\square$

4.4 Characterization of $\mathcal{U}$ via invariance and attractivity notions

The until operator has *strong* and *weak* versions, named as strong until ($\mathcal{U}_s$) and weak until ($\mathcal{U}_w$); see, e.g., [36]. Given two propositions p and q, the satisfaction of the formula $p\mathcal{U}_s q$ implies that p is true until q happens to be true, and q must become true eventually. For the weak version, the satisfaction of the formula $p\mathcal{U}_w q$ implies that p is true until q happens to be true; however, q is not required to become true if p is true forever. In other words, according to the definition of the $\mathcal{U}_s$ operator, a solution $(t,j) \mapsto \phi(t,j)$ to a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the formula

$$f = p\,\mathcal{U}_s\,q \tag{4.7}$$

at $(t,j) \in \text{dom } \phi$ when there exists $(t', j') \in \text{dom } \phi$ such that $t' + j' \geq t + j$ and $\phi(t', j')$ satisfies q; and $\phi(t'', j'')$ satisfies p for all $(t'', j'') \in \text{dom } \phi$ such that $t + j \leq t'' + j'' < t' + j'$.

According to the definition of the $\mathcal{U}_w$ operator, a solution $(t,j) \mapsto \phi(t,j)$ to a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the formula

$$f = p\,\mathcal{U}_w\,q \tag{4.8}$$

at $(t, j) \in \text{dom}\ \phi$ when $f = p\mathcal{U}_s q$ is satisfied at $(t, j) \in \text{dom}\ \phi$; or $\phi(t', j')$ satisfies p for all $(t', j') \in \text{dom}\ \phi$ such that $t' + j' \geq t + j$.

The set of points in $\mathbb{R}^n$ satisfying an atomic proposition p or an atomic proposition q are respectively given by

$$P = \{x \in \mathbb{R}^n : p(x) = 1\} \ \text{ and } \ Q = \{x \in \mathbb{R}^n : q(x) = 1\}, \tag{4.9}$$

and we assume that the sets P and Q are closed and $P \subset C \cup D$.

With the sets P and Q as in (4.9), when a solution ϕ to $\mathcal{H}$ satisfies $p\mathcal{U}_w q$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$, we have the following cases:

1) the solution ϕ starts and remains in the set P for all hybrid time; or

2) the solution ϕ starts and remains in the set P up to when it reaches Q.

3) the solution ϕ starts from the set Q.

Remarkably, these properties can be assured using the conditional invariance notion in Definition 2.4. In fact, notice that based on items 1) - 3), the solution needs to either remain in P or remain in $P \cup Q$ for some time. Such a property coincides with conditional invariance of $P \cup Q$ with respect to $P \setminus Q$ for the following auxiliary system $\mathcal{H}_w$: given a closed set Q and a hybrid system $\mathcal{H} = (C, F, D, G)$, we consider the system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ given by

$$\begin{aligned} F_w(x) &:= F(x) && \forall x \in C_w := C \setminus Q \\ G_w(x) &:= \begin{cases} x & \text{if } x \in Q \\ G(x) & \text{otherwise} \end{cases} && \forall x \in D_w := D \cup Q. \end{aligned} \tag{4.10}$$

The intuition behind the construction of the system $\mathcal{H}_w$ is as follows: the system $\mathcal{H}_w$ is used to characterize the behavior of the system $\mathcal{H}$ outside the set Q. Indeed, the solutions to $\mathcal{H}$ are the solutions to $\mathcal{H}_w$ (and vice versa) up to when they reach (if they do) the set Q. By Definition of the weak until, when $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$, it follows that, the solutions to $\mathcal{H}_w$ starting from the set $P \backslash Q$ stay in $P \cup Q$. Hence, by guaranteeing conditional invariance of $P \cup Q$ with respect to $P \backslash Q$ for $\mathcal{H}_w$, we establish that every solution to $\mathcal{H}$ starting from $P \cup Q$ satisfies $p\mathcal{U}_w q$ for $\mathcal{H}$. Alternatively, the satisfaction of $p\mathcal{U}_w q$ for $\mathcal{H}$ can be guaranteed by conditional invariance of $P \cup Q$ with respect to $P \cup Q$ (namely, forward invariance of $P \cup Q$) for $\mathcal{H}$.

Example 4.9 (Timer). *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a constantly evolving timer with the state $x \in \mathbb{R}$ and*

$$\begin{aligned} F(x) &:= 1 && \forall x \in C := [0, 1], \\ G(x) &:= 0 && \forall x \in D := [1, +\infty). \end{aligned}$$

Define the atomic propositions p and q as

$$p(x) := \begin{cases} 1 & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ 0 & \text{otherwise} \end{cases}$$

$$q(x) := \begin{cases} 1 & \text{if } x \in [1, +\infty) \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}^n$. The sets P and Q in (4.9) *and the system $\mathcal{H}_w$ in* (4.10) *are given by $Q = D$, $P = [\frac{1}{2}, 1]$, and*

$$\begin{aligned} F_w(x) &:= 1 && \forall x \in C_w := [0, 1), \\ G_w(x) &:= x && \forall x \in D_w := D = Q. \end{aligned}$$

We notice that each solution to $\mathcal{H}_w$ starting from $P \backslash Q = [\frac{1}{2}, 1)$ flows in P and *reaches $x = 1 \in Q$. Once a solution reaches $x = 1$, it jumps according to the jump map $G_w(x) = x$ and stays at $\{1\} \in Q$ by jumping since it cannot flow back to $P\backslash Q$. Hence, the solutions to $\mathcal{H}_w$ starting from $P\backslash Q$ never leave the set $P \cup Q$, which implies that the set $P \cup Q$ is conditionally invariant with respect to $P\backslash Q$ for $\mathcal{H}_w$. Note that conditional invariance of $P \cup Q$ with respect to $P\backslash Q$ does not hold for $\mathcal{H}$ since once a solution to $\mathcal{H}$ reaches Q, it jumps outside $P \cup Q$. As a consequence, the formula $f = p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$ since the solutions to $\mathcal{H}$ starting from $P\backslash Q$ remain in P until reaching the jump set $D = Q$.* $\triangle$

Theorem 4.10 ($p\mathcal{U}_w q$ via Conditional Invariance)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given two atomic propositions p and q, let the sets P and Q be given as in* (4.9) *and let the system $\mathcal{H}_w$ be as in* (4.10)*. The formula $f = p\mathcal{U}_w q$ is satisfied for every maximal solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$ if $P \cup Q$ is conditionally invariant with respect to $P \setminus Q$ for $\mathcal{H}_w$.*

Proof. Suppose that $P \cup Q$ is conditionally invariant with respect to $P\backslash Q$ for $\mathcal{H}_w$. We show that, for each solution ϕ to $\mathcal{H}$ such that $\phi(0, 0) \in P\backslash Q$, ϕ stays in $P \cup Q$ for all $(t, j) \in \operatorname{dom} \phi$ such that $t + j \leq \mathcal{T}_Q(\phi)$. Indeed, let ψ be a maximal solution to $\mathcal{H}_w$ such that $\psi(t, j) = \phi(t, j)$ for all $(t, j) \in \operatorname{dom} \phi$ such that $t + j \leq \mathcal{T}_Q(\phi)$; such a solution ψ to $\mathcal{H}_w$ always exists since the systems $\mathcal{H}$ and $\mathcal{H}_w$ share the same data outside the set Q. Furthermore, since $P \cup Q$ is conditionally invariant with respect to $P\backslash Q$ for $\mathcal{H}_w$, we conclude that $\psi(t, j) \in P \cup Q$ for all $(t, j) \in \operatorname{dom} \psi$. Therefore, $\phi(t, j) \in P \cup Q$ for all $(t, j) \in \operatorname{dom} \phi$ such that $t + j \leq \mathcal{T}_Q(\phi)$, which completes the proof. $\square$

The bouncing ball example in [20, Example 1.1] illustrates Theorem 4.10.

Example 4.11 (Bouncing Ball)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a ball bouncing vertically on the ground, with the state $x = (x_1, x_2) \in \mathbb{R}^2$*

and the data

$$
\begin{aligned}
F(x) &:= \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} && \forall x \in C := \{x \in \mathbb{R}^2 : x_1 \geq 0\} \\
G(x) &:= \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} && \forall x \in D := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\},
\end{aligned}
$$

where x_1 *denotes the height above the surface and* x_2 *is the vertical velocity. The parameter* $\gamma > 0$ *is the gravity coefficient and* $\lambda \in (0,1)$ *is the restitution coefficient. Let* $\varepsilon > 0$ *and define atomic propositions* p *and* q *such that*

$$
p(x) := \begin{cases} 1 & \textit{if } \ x_1 \in [0, \varepsilon] \ \textit{ and } \ x_2 \leq 0 \\ 0 & \textit{otherwise,} \end{cases}
$$

and

$$
q(x) := \begin{cases} 1 & \textit{if } \ x_1 \geq 0 \ \textit{ and } \ x_2 \geq 0 \\ 0 & \textit{otherwise.} \end{cases}
$$

The sets P *and* Q *in* (4.9) *and the data of* $\mathcal{H}_w$ *in* (4.10) *are given by* $P = [0, \varepsilon] \times \mathbb{R}_{\leq 0}$, $Q = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, *and*

$$
\begin{aligned}
F_w(x) &= F(x) && \forall x \in C_w \\
G_w(x) &= \begin{cases} x & \textit{if } \ x \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \\ G(x) & \textit{if } \ x \in \{0\} \times \mathbb{R}_{\leq 0} \end{cases} && \forall x \in D_w,
\end{aligned}
$$

where $D_w = (\{0\} \times \mathbb{R}_{<0}) \cup (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ *and* $C_w = \mathbb{R}_{\geq 0} \times \mathbb{R}_{<0}$. *Note that each solution to* $\mathcal{H}_w$ *from* $P \backslash Q$ *either flows in* P *and reaches* Q *after jumping from* $\{0\} \times \mathbb{R}_{<0} \subset D_w$ *or directly jumps from* $\{0\} \times \mathbb{R}_{<0}$ *towards* Q. *Once a solution reaches a point* $x \in Q$ *after a jump, it jumps according to the jump map* $G_w(x) = x$. *Note that each solution to* $\mathcal{H}_w$ *from* $(0, \infty) \times \{0\} \subset Q$ *can only flow. Hence,* *every solution to $\mathcal{H}_w$ starting from $P \backslash Q$ never leaves the set $P \cup Q$, implying that the set $P \cup Q$ is conditionally invariant with respect to $P \backslash Q$ for $\mathcal{H}_w$. Then, using Theorem 4.10, we conclude that the formula $p \mathcal{U}_w q$ is satisfied for $\mathcal{H}$.* △

Next, we consider the definition of the strong until operator. With the same sets P and Q in (4.9), to assure that a solution ϕ to $\mathcal{H}$ satisfies $p\mathcal{U}_s q$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$,

1) the solution ϕ starts and remains in the set P until reaching the set Q in finite hybrid time; or

2) the solution ϕ starts from the set Q.

Therefore, the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$ requires, additionally, that every solution ϕ reaches Q in finite hybrid time. When the set $P \cup Q$ is conditionally invariant with respect to $P \backslash Q$ for $\mathcal{H}_w$, this property is guaranteed by Q being eventually conditionally invariant with respect to $P \cup Q$ for the auxiliary hybrid system $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ given by

$$\begin{aligned} &F_s(x) := F(x) && x \in C_s := (C \backslash Q) \cap P \\ &G_s(x) := \begin{cases} x & \text{if } x \in Q \\ G(x) & \text{otherwise} \end{cases} && x \in D_s := (D \cap P) \cup Q. \end{aligned} \tag{4.11}$$

The hybrid system $\mathcal{H}_s$ is just the restriction of $\mathcal{H}_w$ in (4.10) to $P \cup Q$. It is easy to see that $C_s := C_w \cap (P \cup Q)$ and $D_s := D_w \cap (P \cup Q)$. As a result, when, additionally, Q is eventually conditionally invariant with respect to $P \cup Q$ for $\mathcal{H}_s$, each solution to $\mathcal{H}_s$ starting from $P \backslash Q$ reaches Q in finite hybrid time. Since the solutions to $\mathcal{H}_s$ are the solutions to $\mathcal{H}_w$ (and vice versa) up to when they reach Q, and the solutions to $\mathcal{H}_w$ are the solutions to $\mathcal{H}$ (and vice versa) up to when they reach Q, each solution to $\mathcal{H}$ starting from $P \backslash Q$ reaches Q in finite time and remains in P until it reaches Q, which implies the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}$. Alternatively, the satisfaction of $p\mathcal{U}_s q$ for $\mathcal{H}_w$ can be guaranteed by using FTA of Q with respect to $P \cup Q$ for $\mathcal{H}_s$ instead of eventual conditional invariance of Q with respect to $P \cup Q$ for $\mathcal{H}_s$.

Theorem 4.12 ($p\mathcal{U}_s q$ via $p\mathcal{U}_w q$ + Eventual Conditional Invariance)**.** *Consider a hybrid system* $\mathcal{H} = (C, F, D, G)$*. Given two atomic propositions* p *and* q*, let the sets* P *and* Q *be given as in* (4.9) *and let the system* $\mathcal{H}_s$ *be given as in* (4.11)*. The formula* $f = p\mathcal{U}_s q$ *is satisfied for every solution* ϕ *to* $\mathcal{H}$ *at* $(t, j) = (0, 0)$ *with* $(\phi, (0, 0)) \vDash p \vee q$ *if*

1) *the formula* $p\mathcal{U}_w q$ *is satisfied for every solution* ϕ *to* $\mathcal{H}$ *at* $(t, j) = (0, 0)$ *with* $(\phi, (0, 0)) \vDash p \vee q$ *(see Theorem 4.10); and*

2) *the set* Q *is eventually conditionally invariant with respect to* $P \cup Q$ *for* $\mathcal{H}_s$*.*

Proof. By definition of $\mathcal{H}_s$, if the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$ by item 1), each solution to $\mathcal{H}_s$ starting from $P \backslash Q$ remains in the set $P \cup Q$. Furthermore, when additionally Q is ECI with respect to $P \cup Q$ for $\mathcal{H}_s$, each maximal solution to $\mathcal{H}_s$ starting from $P \backslash Q$ remains in the set $P \cup Q$ and reaches the set Q in finite hybrid time. The proof is completed if we show that each maximal solution ϕ to $\mathcal{H}$ starting from $P \backslash Q$ stays in $P \cup Q$ for all $(t, j) \in \operatorname{dom} \phi$ such that $t + j \leq \mathcal{T}_Q(\phi)$, and $\mathcal{T}_Q(\phi) < \infty$. To this end, let ϕ be a maximal solution to $\mathcal{H}$ starting from $P \backslash Q$. By item 1), ϕ remains in $P \backslash Q$ up to when it reaches Q (if that ever happens). Next, since both $\mathcal{H}$ and $\mathcal{H}_s$ share the same data on $P \backslash Q$, there always exists a solution ψ to $\mathcal{H}_s$ such that $\psi(t, j) = \phi(t, j)$ for all $(t, j) \in \operatorname{dom} \phi$ provided that $t + j \leq \mathcal{T}_Q(\phi) = \mathcal{T}_Q(\psi)$. Furthermore, by item 2), we know that $\mathcal{T}_Q(\psi) = \mathcal{T}_Q(\phi) < \infty$. Then, since we already know that $\psi(t, j) \in P \cup Q$ for all $(t, j) \in \operatorname{dom} \psi$ by item 1), we conclude that $\phi(t, j) = \psi(t, j) \in P \cup Q$ for all $(t, j) \in \operatorname{dom} \phi$ provided that $t + j \leq \mathcal{T}_Q(\phi) = \mathcal{T}_Q(\psi)$; and thus, the proof is completed. □

The following example illustrates Theorem 4.12.

Example 4.13 (Thermostat)**.** *Consider a hybrid system* $\mathcal{H} = (C, F, D, G)$ *modeling a controlled thermostat system. The variable* h *denotes the state of the heater, i.e.,* $h = 1$ *implies the heater is on and* $h = 0$ *implies the heater is off. The variable* z *is the room temperature,* z_0 *denotes the room temperature when the heater is off, and* $z_\triangle$ *denotes the capacity of the heater to raise the temperature such that* $z_0 < z_{\min} < z_{\max} < z_0 + z_\triangle$*. The system* $\mathcal{H}$ *with the state* $x := (h, z) \in \{0, 1\} \times \mathbb{R}$ *is given by*

$$\begin{aligned} F(x) &:= \begin{bmatrix} 0 & -z + z_0 + z_\triangle h \end{bmatrix}^\top & \forall x \in C := C_0 \cup C_1 \\ G(x) &:= \begin{bmatrix} 1 - h & z \end{bmatrix}^\top & \forall x \in D := D_0 \cup D_1, \end{aligned}$$

where $C_0 := \{x \in \mathbb{R}^2 : h = 0, z \geq z_{\min}\}$*,* $C_1 := \{x \in \mathbb{R}^2 : h = 1, z \leq z_{\max}\}$*,* $D_0 := \{x \in \mathbb{R}^2 : h = 0, z \leq z_{\min}\}$*, and* $D_1 := \{x \in \mathbb{R}^2 : h = 1, z \geq z_{\max}\}$*. Define the atomic propositions* p *and* q *as*

$$p(x) := \begin{cases} 1 & \text{if } x \in \{1\} \times (-\infty, z_{\max}] \\ 0 & \text{otherwise,} \end{cases}$$
$$q(x) := \begin{cases} 1 & \text{if } x \in \{0\} \times [z_{\min}, +\infty) \\ 0 & \text{otherwise,} \end{cases}$$

for each $x \in \mathbb{R}^n$*. The sets* P *and* Q *in* (4.9) *are given by* $P = \{1\} \times (-\infty, z_{\max}]$*,* $Q = \{0\} \times [z_{\min}, +\infty)$*. The system* $\mathcal{H}_w$ *is given as in* (4.10)*, where* $C_w = \{1\} \times (-\infty, z_{\max}] (= P)$ *and* $D_w = (\{0\} \times \mathbb{R}) \cup (\{1\} \times [z_{\max}, +\infty))$*. Then, the system* $\mathcal{H}_s$ *is given as in* (4.11) *with* $C_s = C_w (= P)$ *and* $D_s = (\{0\} \times [z_{\min}, +\infty)) \cup \{(1, z_{\max})\}$*. Note that each solution to* $\mathcal{H}_s$ *from* $P \backslash Q (= P)$ *flows in* P *and reaches* Q *after jumping from* $\{(1, z_{\max})\} \in D_s$*. Once a solution reaches a point* $x \in Q$*, it jumps according to the jump map* $G_s(x) = x$ *and stays in* Q *by jumping, which implies* Q

is eventually conditionally invariant with respect to $P \cup Q$ for $\mathcal{H}_s$. Furthermore, each solution to $\mathcal{H}_w$ starting from $P \backslash Q (= P)$ flows in P and reaches Q for the first time by jumping from $\{(1, z_{\max})\}$ to $\{(0, z_{\max})\}$. Once a solution to $\mathcal{H}_w$ lands on $\{(0, z_{\max})\}$, it jumps according to the jump map $G_w(x) = x$ and stays in Q by jumping. Hence, each solution to $\mathcal{H}_w$ starting from $P \backslash Q$ does not leave the set $P \cup Q$, which implies that the set $P \cup Q$ is conditionally invariant with respect to $P \backslash Q$ for $\mathcal{H}_w$; and thus, using Theorem 4.10, we conclude that the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. As a result, using Theorem 4.12, we conclude that the formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$. $\triangle$

The following result characterizes the satisfaction of $p\mathcal{U}_s q$ using FTA for hybrid systems in addition to the satisfaction of $p\mathcal{U}_w q$.

Theorem 4.14 ($p\mathcal{U}_s q$ via $p\mathcal{U}_w q$ + FTA)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4.9) *such that the set Q is closed and let the data of $\mathcal{H}_s$ be given in* (4.11)*. The formula $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$ if and only if*

1) the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$; and

2) the set Q is FTA with respect to $P \cup Q$ for $\mathcal{H}_s$.

Proof. Suppose that $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$. By definition of $p\mathcal{U}_s q$, we conclude that $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. Next, we show that the set Q is FTA with respect to $P \cup Q$ for $\mathcal{H}_s$. To do so, we consider a maximal solution ϕ to $\mathcal{H}_s$ starting from $P \cup Q$. In particular, each maximal solution to $\mathcal{H}_s$ starting from Q, the solution stays in Q by construction of $\mathcal{H}_s$. Hence, we consider a maximal solution ϕ to $\mathcal{H}_s$ starting from $P \setminus Q$. Since $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$, the solution ϕ either reaches the set Q in finite time or remains in $P \backslash Q$. To exclude the latter case, we show that when ϕ remains in $P\backslash Q$, then ϕ is a maximal solution to $\mathcal{H}$. Indeed, assume the existence of a solution ψ to $\mathcal{H}$ that is a nontrivial extension of ϕ; namely, there exists $I \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $I \neq \emptyset$ and $\text{dom } \psi = \text{dom } \phi \cup I$. Note that $\psi(\text{dom } \phi) = \phi(\text{dom } \phi) \subset P\backslash Q$. Also, since ψ must remain in $P\backslash Q$ up to when it reaches Q, we can choose I such that $\psi(\text{dom } \phi \cup I) \subset P\backslash Q$. Hence, ψ is a solution to $\mathcal{H}_s$, which contradicts the fact that ϕ is a maximal solution to $\mathcal{H}_s$. Furthermore, since $p\mathcal{U}_s q$ is satisfied for $\mathcal{H}$, we conclude that ϕ, being a maximal solution to $\mathcal{H}$, must reach Q in finite hybrid time.

Now, suppose that the formula $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. This implies that each maximal solution ϕ to $\mathcal{H}$ remains in $P\backslash Q$ for all hybrid time; otherwise, ϕ remains in $P\backslash Q$ up to when it reaches Q in finite hybrid time. To exclude the first scenario, we note that when ϕ remains in $P\backslash Q$ for all hybrid time, it follows that ϕ is also a maximal solution to $\mathcal{H}_s$. However, by item 2), the maximal solutions to $\mathcal{H}_s$ must reach Q. $\square$

Chapter 5

Sufficient conditions guaranteeing the satisfaction of temporal formulas for hybrid systems

5.1 Sufficient Conditions for $\Box p$

In the following, we present sufficient conditions guaranteeing $f = \Box p$. Due to the equivalence we provide in Section 4.1, any sufficient condition that guarantees the needed invariance property of the set guarantees the satisfaction of the formula $\Box p$. For example, in [24,37], such invariance property for hybrid systems is studied as follows:

- Forward pre-invariance of a set in [24, Theorem 4.3];
- Forward pre-invariance of a subset of the sublevel sets of a Lyapunov function in [24, Theorem 5.1];
- Forward pre-invariance of a set defined by a barrier function in [37, Theorem 1].

By exploiting the results and the ideas in [37], the conditions given below provide sufficient conditions to verify that $\mathcal{H}$ is such that every solution ϕ to $\mathcal{H}$ with $\phi(0,0) \Vdash p$ satisfies $f = \Box p$. Below, the concept of tangent cone of a set is used; see [20, Definition 5.12]. The tangent cone at a point $x \in \mathbb{R}^n$ of a set $K \subset \mathbb{R}^n$, denoted $T_K(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exists $x_i \in K$, $\tau_i > 0$ with $x_i \to x$ and $\tau_i \searrow 0$ such that $w = \frac{x_i - x}{\tau_i}$. For a set $K \subset \mathbb{R}^n$, $U(K)$ denotes any open neighborhood of K and ∂K denotes its boundary. Furthermore, the notion of barrier function candidate with respect to K for $\mathcal{H}$ is given as follows [37]:

Definition 5.1 (Barrier Function Candidate)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. A function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a barrier function candidate with respect to K for $\mathcal{H}$ if*

$$\begin{cases} B(x) \leq 0 & \forall x \in K \\ B(x) > 0 & \forall x \in (C \cup D \cup G(D)) \setminus K. \end{cases} \tag{5.1}$$

Assumption 5.2. *The flow map F is outer semicontinuous, nonempty, and locally bounded with convex values on C. Furthermore, the jump map G is nonempty on D.*

Theorem 5.3. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying Assumption 5.2. Given an atomic proposition p, suppose the set K in* (3.6) *is closed and $K \subset C \cup D$. Then, the formula $f = \Box p$ is satisfied for all solutions ϕ to $\mathcal{H}$ (and for all $(t, j) \in \text{dom } \phi$) with $\phi(0,0) \Vdash p$ if there exists a barrier function candidate B with respect to K for $\mathcal{H}$ as in* (5.1) *that is continuously differentiable and the following properties hold:*

1) $\langle \nabla B(x), \eta \rangle \leq 0$ for all $x \in C \cap (U(\partial K) \setminus K)$ and all $\eta \in F(x) \cap T_C(x)$.

2) $B(\eta) \leq 0$ for all $x \in D \cap K$ and all $\eta \in G(x)$.

3) $G(D \cap K) \subset C \cup D$.

Proof. Under conditions 1)–3), we conclude that the set K in (3.6) is forward pre-invariant for $\mathcal{H}$ using [37, Theorem 1]. Then, by Theorem 4.3, the formula $f = \Box p$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $\phi(0, 0) \Vdash p$ since the set K is forward pre-invariant for $\mathcal{H}$. Moreover, this property at $(t, j) = (0, 0)$ implies $\phi(t, j) \Vdash p$ at each $(t, j) \in \text{dom } \phi$; and thus, the formula $f = \Box p$ is satisfied for each solution ϕ to $\mathcal{H}$ and at each $(t, j) \in \text{dom } \phi$ with $\phi(0, 0) \Vdash p$. $\Box$

Remark 5.4. *Note that $\Box p$ is satisfied for all solutions ϕ to $\mathcal{H}$ if $\phi(0, 0) \Vdash p$ and $\phi(t, j) \Vdash p$ for all future hybrid time $(t, j) \in$ dom ϕ. Under the conditions in Theorem 5.3, solutions with $\phi(0, 0) \nVdash p$ may satisfy p after some time if ϕ reaches the set K in* (3.6) *in finite time. Convergence to such set in finite hybrid time is presented in the next section.*

Next, the bouncing ball example in [20, Example 1.1] illustrates Theorem 5.3.

Example 5.5. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a ball bouncing vertically on the ground, with state $x = (x_1, x_2) \in \mathbb{R}^2$ and the data given by*

$$\begin{aligned} F(x) &:= \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \forall x \in C := \{x \in \mathbb{R}^n : x_1 \geq 0\}, \\ G(x) &:= \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} \quad \forall x \in D := \{x \in \mathbb{R}^n : x_1 = 0, x_2 \leq 0\}, \end{aligned} \tag{5.2}$$

where x_1 denotes the height above the surface and x_2 is the vertical velocity. The parameter $\gamma > 0$ is the gravity coefficient and $\lambda \in [0, 1]$ is the restitution coefficient. Every maximal solution to this system is Zeno. Define an atomic proposition p as follows: for every $x \in \mathbb{R}^n$, $p(x) = 1$ when $x \in C \cup D$ and

$2\gamma x_1 + (x_2 - 1)(x_2 + 1) \leq 0$; $p(x) = 0$ *otherwise. Let* K *be given as in* (3.6)*. Then, we observe that the closed set* K *is the sublevel set where the total energy of the ball is less than or equal to* $1/2$*. The function* $B(x) := 2\gamma x_1 + (x_2 - 1)(x_2 + 1)$ *is a barrier function candidate since* $B(x) \leq 0$ *for all* $x \in K$ *and* $B(x) > 0$ *otherwise. Then, we have* $\langle \nabla B(x), F(x) \rangle = 0$ *for each* $x \in C$*; and thus, condition 1) in Theorem 5.3 is satisfied. Moreover, we have* $B(G(x)) = 2\gamma x_1 + \lambda^2 x_2^2 - 1 \leq 0$ *for every* $x \in D \cap K$ *since* $\lambda \in [0, 1]$*; and thus, condition 2) in Theorem 5.3 is satisfied. Finally, since* $G(D) = \{0\} \times \mathbb{R}_{\geq 0} \subset C \cup D$*, condition 3) in Theorem 5.3 is satisfied. Therefore, via Theroem 5.3, the formula* $f = \Box p$ *is satisfied for each solution* ϕ *to* $\mathcal{H}$ *from* K *and at each* $(t, j) \in$ *dom* ϕ*.* △

Example 5.6. *Consider a hybrid system* $\mathcal{H} = (C, F, D, G)$ *modeling a constantly evolving timer system with the state* $x = (\tau, h) \in \mathcal{X} := [0, \infty) \times \{0, 1\}$ *given by*

$$\begin{aligned} F(x) &:= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \forall x \in C := \{x \in \mathcal{X} : 0 \leq \tau \leq T\}, \\ G(x) &:= \begin{bmatrix} 0 \\ 1 - h \end{bmatrix} \quad \forall x \in D := \{x \in \mathcal{X} : \tau \geq T\}, \end{aligned} \tag{5.3}$$

where τ *denotes a timer variable,* h *is a logic variable, and* T *is the period of the timer. Moreover, for each* $x \in \mathcal{X}$ *such that* $0 \leq \tau \leq T$*,* $p(x) = 1$*; otherwise,* $p(x) = 0$*. Let* K *be given as in* (3.6)*. Consider the barrier function candidate* $B(x) := \tau - T$*. We notice that* $C \cap (U(\partial K) \setminus K) = \emptyset$*; and thus, condition 1) in Theorem 5.3 is trivially satisfied. Moreover, we have* $B(G(x)) = -T \leq 0$ *for every* $x \in D$*; and thus, condition 2) in Theorem 5.3 is satisfied. Furthermore, since* $G(D) = \{0\} \times \{0, 1\} \subset C \cup D$*, condition 3) in Theorem 5.3 is satisfied. Therefore, via Theorem 5.3, the formula* $f = \Box p$ *is satisfied for each solution* ϕ *to* $\mathcal{H}$ *and at each* $(t, j) \in$ *dom* ϕ*.* △

5.2 Sufficient Conditions for $\diamond p$

In the following, we present sufficient conditions guaranteeing the formula $f = \diamond p$. Due to the equivalence we provide in Section 4.2, any sufficient condition that guarantees the FTA property of the set K in (3.6) guarantees the satisfaction of the formula $\diamond p$. In that sense, we observe that the results on finite-time stability (FTS) for a set for hybrid systems in [25] and the results on recurrence for a set for hybrid systems in [38] can be applied to derive sufficient conditions guaranteeing the desired FTA property. In the following, by exploiting the results and the ideas in [25], sufficient conditions are proposed to verify that $\mathcal{H}$ is such that every solution ϕ to $\mathcal{H}$ satisfies $f = \diamond p$; see Appendix B for more details about sufficient conditions for FTA.

As stated above, the satisfaction of the formula $f = \diamond p$ is assured by conditions that guarantee that the set K in (3.6) is FTA for $\mathcal{H}$, where

$$p(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

In the following, we propose sufficient conditions to satisfy the formula $f = \diamond p$. Below, the function $V : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n$ and locally Lipschitz on a neighborhood of C. Using Clarke generalized derivative, we define the functions u_C and u_D as follows: $u_C(x) := \max_{v \in F(x)} \max_{\zeta \in \partial V(x)} \langle \zeta, v \rangle$ for each $x \in C$, and $-\infty$ otherwise; $u_D(x) := \max_{\zeta \in G(x)} V(\zeta) - V(x)$ for each $x \in D$, and $-\infty$ otherwise, where ∂V is the generalized gradient of V in the sense of Clarke; see Section 2.3 for more details. Moreover, a function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, denoted by $\alpha \in \mathcal{K}$, if it is zero at zero, continuous, and strictly increasing and α is a class-$\mathcal{K}_\infty$ function, denoted by $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded. Given a real number $s \in \mathbb{R}$, $\mathrm{ceil}(s)$ denotes the smallest integer upper bound for

s.

Theorem 5.7. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given an atomic proposition p, let the set K in (3.6) is closed. Suppose there exists an open set*[1] *$\mathcal{N}$ that defines an open neighborhood of K such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathbb{R}^n$. Then, if*

1) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c_1 > 0, c_2 \in [0, 1)$ such that

1.1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D)$ such that $p(x) = 0$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies

$$\tfrac{V^{1-c_2}(x)}{c_1(1-c_2)} \leq \sup_{(t,j)\in dom\,\phi} t; \tag{5.5}$$

1.2) the function V is positive definite with respect to K and

1.2a) for each $x \in C \cap \mathcal{N}$ and $p(x) = 0$, $u_C(x) + c_1 V^{c_2}(x) \leq 0$;

1.2b) for each $x \in D \cap \mathcal{N}$ and $p(x) = 0$, $u_D(x) \leq 0$.

or

2) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c > 0$ such that

2.1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D)$ such that $p(x) = 0$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies

$$ceil\left(\tfrac{V(x)}{c}\right) \leq \sup_{(t,j)\in dom\,\phi} j; \tag{5.6}$$

2.2) the function V is positive definite with respect to K and

2.2a) for each $x \in C \cap \mathcal{N}$ and $p(x) = 0$, $u_C(x) \leq 0$;

2.2b) for each $x \in D \cap \mathcal{N}$ and $p(x) = 0$, $u_D(x) \leq -\min\{c, V(x)\}$.

[1] A set $\mathcal{N}$ can be chosen as $\mathcal{N} = \mathbb{R}^n$ for the global version of FTA.

hold, then, the formula $f = \Diamond p$ *is satisfied for every solution* ϕ *to* $\mathcal{H}$ *from* $L_V(r) \cap (\overline{C} \cup D)$ *at* $(t,j) = (0,0)$ *where* $L_V(r) = \{x \in \mathbb{R}^n : V(x) \leq r\}$, $r \in [0, \infty]$, *is a compact sublevel set of* V *contained in* $\mathcal{N}$. *Moreover, for each* $\phi \in \mathcal{S}_{\mathcal{H}}(L_V(r) \cap (\overline{C} \cup D))$, *defining* $\xi = \phi(0,0)$, *the first time* $(t', j') \in \operatorname{dom} \phi$ *such that* $\phi(t', j') \Vdash p$ *satisfies*

$$t' + j' = \mathcal{T}(\phi), \tag{5.7}$$

and an upper bound on that hybrid time is given as follows:

a) if 1) holds, then $\mathcal{T}$ *is upper bounded by* $\mathcal{T}^\star(\xi) + \mathcal{J}^\star(\phi)$, *where* $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$ *and* $\mathcal{J}^\star(\phi)$ *is such that* $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\phi)) \in \operatorname{dom} \phi$.

b) if 2) holds, then $\mathcal{T}$ *is upper bounded by* $\mathcal{T}^\star(\phi) + \mathcal{J}^\star(\xi)$, *where* $\mathcal{J}^\star(\xi) = \operatorname{ceil}\left(\frac{V(\xi)}{c}\right)$ *and* $\mathcal{T}^\star(\phi)$ *is such that* $(\mathcal{T}^\star(\phi), \mathcal{J}^\star(\xi)) \in \operatorname{dom} \phi$ *and* $(\mathcal{T}^\star(\phi), \mathcal{J}^\star(\xi) - 1) \in \operatorname{dom} \phi$.

Proof. Note that the set K is closed and collects the set of points such that p is satisfied. Now we show that the conditions in Proposition B.1 or Proposition B.3 hold for K.

- Item 1) implies that for every $x \in \mathcal{N} \cap (\overline{C} \cup D) \setminus K$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies (5.5); and the function V is positive definite with respect to K; and $u_C(x) + c_1 V^{c_2}(x) \leq 0$ for every $x \in (C \cap \mathcal{N}) \setminus K$ and $u_D(x) \leq 0$ for all $x \in (D \cap \mathcal{N}) \setminus K$. Thus, Proposition B.1 applies.

- Item 2) implies that for every $x \in \mathcal{N} \cap (\overline{C} \cup D) \setminus K$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies (5.6); and the function V is positive definite with respect to K; and $u_C(x) \leq 0$ for every $x \in (C \cap \mathcal{N}) \setminus K$ and $u_D(x) \leq -\min\{c, V(x)\}$ for every $x \in (D \cap \mathcal{N}) \setminus K$. Thus, Proposition B.3 applies.

Therefore, K is FTA for $\mathcal{H}$ if item 1) or 2) holds. Then, by Theorem 4.7, the formula $f = \Diamond p$ is satisfied for all solutions to $\mathcal{H}$ at $(t, j) = (0, 0)$. $\square$

Note that the conditions about the supremum over the hybrid time of a solution in (5.5) and (5.6) are due to not insisting on completeness of maximal solutions. When every maximal solution is complete, these conditions hold automatically. See Remark 5.9 for more details.

Remark 5.8. *Under condition 1.2) or 2.2) in Theorem 5.7, given a solution ϕ to $\mathcal{H}$, there exists some time $(t', j') \in \operatorname{dom} \phi$ such that ϕ satisfies p. Furthermore, we have this satisfaction in finite time (t', j'), obtained by the settling-time function $\mathcal{T}$, for which an upper bound depends on the Lyapunov function and the solution only. Note that a settling-time function $\mathcal{T}$ does not need to be computed. However, we provide an estimate of when convergence happens using an upper bound that depends on V and the constants involved in items 1) and 2) only.*

Remark 5.9. *Note that conditions* (5.5) *and* (5.6) *hold for free for complete solutions unbounded in t or/and j in their domain. Moreover, maximal solutions are complete when the conditions in [20, Proposition 2.10 or Proposition 6.10] hold. Specifically, if maximal solutions ϕ are complete with $\operatorname{dom} \phi$ unbounded in its t component, then* (5.5) *holds automatically; and, if the solutions are complete with $\operatorname{dom} \phi$ unbounded in its j component, then* (5.6) *holds automatically.*

Remark 5.10. *Item 1) in Theorem 5.7 characterizes the situation when the formula $f = \Diamond p$ is being satisfied for all solutions ϕ to $\mathcal{H}$ due to the strict decrease of a Lyapunov function during flows. Item 2) in Theorem 5.7 provides conditions for f to be satisfied for all solutions ϕ to $\mathcal{H}$ due to a Lyapunov function strictly decreasing at jumps. Finally, we can combine the properties in item 1) and item 2) to arrive to strict Lyapunov conditions for verifying that $\mathcal{H}$ is such that every ϕ satisfies f at $(t, j) = (0, 0)$; see Proposition B.4.*

Remark 5.11. *Based on the definition of recurrence for sets in [38, Definition 1], the recurrence property could be used for certifying the formula* $\Diamond p$*. When the set* K *that collects the set of points such that* p *is satisfied is globally recurrent for a given hybrid system* $\mathcal{H} = (C, F, D, G)$*, for each complete solution* $\phi \in \mathcal{S}_{\mathcal{H}}(C \cup D)$*, there exists* $(t, j) \in dom\ \phi$ *such that* $\phi(t, j) \in K$*; namely, it implies that* ϕ *satisfies* p *at* $(t, j) \in dom\ \phi$*. In [38], robustness of recurrence and equivalence between the uniform and non-uniform notions are established for open and bounded sets. We observe that the recurrence property is studied with respect to open sets. Therefore, once we have an open, bounded set that collects the set of points satisfying* p*, we can employ the recurrence property to verify that* $\Diamond p$ *is satisfied. Furthermore, we can use the results on robustness of recurrence presented in [38] to derive the satisfaction of the formula* $\Diamond p$ *with robustness.*

In the following examples, the item 1) in Theorem 5.7 is exercised.

Example 5.12. *Inspired from [25, Example 3.3], consider a hybrid system* $\mathcal{H} = (C, F, D, G)$ *with state* $x = (z, \tau) \in \mathbb{R} \times [0, 1]$ *given by*[2]

$$\begin{aligned} F(x) &:= \begin{bmatrix} -k|z|^{\alpha} sgn(z) \\ 1 \end{bmatrix} \qquad \forall x \in C := \mathbb{R} \times [0, 1], \\ G(x) &:= \begin{bmatrix} -z \\ 0 \end{bmatrix} \qquad\qquad\quad \forall x \in D := \mathbb{R} \times \{1\}, \end{aligned} \tag{5.8}$$

where $\alpha \in (0, 1)$ *and* $k > 0$*. Consider the function* $V : \mathbb{R} \times [0, 1] \to \mathbb{R}_{\geq 0}$ *given by* $V(x) = \frac{1}{2} z^2$ *for each* $x \in C$*. Moreover, each* $x \in C$ *satisfies* p *only when* $x \in \{0\} \times [0, 1]$*. Now we consider the set* $K = \{x \in C : p(x) = 1\}$*. We have that,*

[2]The function $\mathrm{sgn} : \mathbb{R} \to \{-1, 1\}$ is defined as $\mathrm{sgn}(x) = 1$ if $x \geq 0$, and $\mathrm{sgn}(x) = -1$ otherwise.

for each $x \in C \setminus K$,

$$\langle \nabla V(x), F(x) \rangle = -k|z|^{1+\alpha} = -2^{\frac{1+\alpha}{2}} k V(x)^{\frac{1+\alpha}{2}}.$$

Furthermore, for all $x \in D \setminus K$, $V(G(x)) - V(x) = 0$. *Therefore, condition 1.2) in Theorem 5.7 is satisfied with* $\mathcal{N} = \mathbb{R} \times [0,1]$, $c_1 = 2^{\frac{1+\alpha}{2}} k > 0$ *and* $c_2 = \frac{1+\alpha}{2} \in (0,1)$. *By applying [20, Proposition 6.10], condition 1.1) in Theorem 5.7 holds since every maximal solution to* $\mathcal{H}$ *is complete with its domain of definition unbounded in the* t *direction. Thus, the formula* $f = \Diamond p$ *is satisfied for all solutions* ϕ *to* $\mathcal{H}$ *at* $(t,j) = (0,0)$. △

Next, the bouncing ball example in Example 5.5 illustrates Lyapunov conditions for verifying that $\Diamond p$ is satisfied for all solutions to $\mathcal{H}$ at $(t,j) = (0,0)$.

Example 5.13. *Consider* $\mathcal{H} = (C, F, D, G)$ *in Example 5.5. Define an atomic proposition* p *as follows: for each* $x \in \mathbb{R}^n$, $p(x) = 1$ *when* $x_2 \leq 0$, *and* $p(x) = 0$ *otherwise. With* K *in* (3.6) *and* $\mathcal{N} = \mathbb{R}^n$, *let* $V(x) = |x_2|$ *for all* $x \in \mathbb{R}^n$. *This function is continuously differentiable on the open set* $\mathbb{R}^n \setminus (\mathbb{R} \times \{0\})$ *and it is Lipschitz on* $\mathbb{R}^n$. *It follows that*

$$\langle \nabla V(x), F(x) \rangle = -\gamma \qquad \forall x \in (C \cap \mathcal{N}) \setminus K,$$

and $u_C(x) + c_1 V^{c_2}(x) \leq 0$ *holds with* $c_1 = \gamma$ *and* $c_2 = 0$. *For each* $x \in (D \cap \mathcal{N}) \setminus K$,

$$V(G(x)) = -\lambda x_2 = |-\lambda x_2| = \lambda |x_2| = \lambda V(x),$$

and $u_D(x) = V(G(x)) - V(x) = \lambda V(x) - V(x) = -(1-\lambda)V(x)$. *Thus, condition 1.2) in Theorem 5.7 is satisfied since* $(D \cap \mathcal{N}) \setminus K = \emptyset$. *Note that by applying [20, Proposition 6.10], every maximal solution is complete and condition 1.1)* *in Theorem 5.7 holds with the chosen constants c_1 and c_2 due to the properties of the hybrid time domain of each maximal solution. Therefore, the formula $f = \Diamond p$ is satisfied for all maximal solutions to $\mathcal{H}$ at $(t, j) = (0, 0)$. Since every solution from K, after some time, jumps from K and then converges to K again in finite time, we have that $f = \Diamond p$ holds for every (t, j) in the domain of each solution.* $\triangle$

Note that Theorem 5.7 guarantees that $\Diamond p$ is satisfied for all solutions ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$. These conditions can be extended to guarantee that $\Diamond p$ is satisfied for all (t, j) in the domain of any solution if the set K is forward pre-invariant or when only jumps are allowed from points in K and the jump map maps points in K into $\mathcal{N}$.

Theorem 5.14. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given an atomic proposition p, let the set K in* (3.6) *is closed. Assume that there exists an open set $\mathcal{N}$ that defines an open neighborhood of K such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathbb{R}^n$. Then, if there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c, c_1 > 0$, $c_2 \in [0, 1)$, such that each $\phi \in \mathcal{S}_{\mathcal{H}}(L_V(r) \cap (C \cup D))$ is complete, $G(D \cap K) \subset L_V(r) \cap (C \cup D)$, and at least one among items 1.2) and 2.2) in Theorem 5.7 holds, then, the formula $f = \Diamond p$ is satisfied for every solution ϕ to $\mathcal{H}$ from $L_V(r) \cap (C \cup D)$ and for all (t, j) in the domain of each solution, where $L_V(r) = \{x \in \mathbb{R}^n : V(x) \leq r\}$, $r \in [0, \infty]$ is a compact sublevel set of V contained in $\mathcal{N}$.*

Proof. The set K is closed and collects the points such that p is satisfied. We first show the case when item 1.2) in Theorem 5.7 holds. Since each solution $\phi \in \mathcal{S}_{\mathcal{H}}(L_V(r) \cap (C \cup D))$ is complete, this implies that there exists $(t_1, j_1) \in \operatorname{dom} \phi$ such that

$$\lim_{t+j \nearrow t_1+j_1} |\phi(t, j)|_K = 0.$$

If there exists $(t_2, j_2) \in \operatorname{dom} \phi$ such that $\phi(t_2, j_2) \notin K$, then ϕ left K by jumping since condition 1.2a) in Theorem 5.7 does not allow flowing out of K. However, if that is the case, then $\phi(t_2, j_2) \in L_V(r) \cap (C \cup D)$ since $G(D \cap K) \subset L_V(r) \cap (C \cup D)$; and then, due to completeness of ϕ, there exists (t_3, j_3) such that $\lim_{t+j \nearrow t_3+j_3} |\phi(t,j)|_K = 0$. Thus, proceeding in this way for all hybrid time instant that the solution leaves K, condition 1) in Theorem 5.7 holds and for every (t, j) in the domain of each solution ϕ. Therefore, the formula $f = \Diamond p$ is satisfied for all solutions to $\mathcal{H}$ from $L_V(r) \cap (C \cup D)$ for every (t, j) in the domain of each solution. The proof for the cases when item 2.2) in Theorem 5.7 holds follows similarly. $\square$

The following example about the firefly model in [39, Example 25] illustrates Theorem 5.14.

Example 5.15. *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ modeling two impulsive oscillators capturing the dynamics of two fireflies. This system has the state $x = (x_1, x_2) \in \mathbb{R}^2$ and the data given by*

$$\begin{aligned} F(x) &:= \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} & \forall x \in C := [0,1] \times [0,1], \\ G(x) &:= \begin{bmatrix} g((1+\tilde{\varepsilon})x_1) \\ g((1+\tilde{\varepsilon})x_2) \end{bmatrix} & \forall x \in D := \{x \in C : \max\{x_1, x_2\} = 1\}, \end{aligned} \tag{5.9}$$

where $\gamma > 0$ and the parameter $\tilde{\varepsilon} > 0$ denotes the effect on the timer of a firefly when the timer of the other firefly expires, and the set-valued map g is given by $g(z) = z$ when $z < 1$; $g(z) = 0$ when $z > 1$; $g(z) = \{0, 1\}$ when $z = 1$. Define p as follows: for each $x \in \mathbb{R}^2$, $p(x) = 1$ when $x \in C$ and $x_1 = x_2$, and $p(x) = 0$ otherwise. Then, the set K is $\{x \in C : p(x) = 1\}$. Let $k = \frac{\tilde{\varepsilon}}{2+\tilde{\varepsilon}}$ and note that

$\frac{1+\tilde{\varepsilon}}{2+\tilde{\varepsilon}} = \frac{1+k}{2}$. *Define*

$$V(x) := \min\{|x_1 - x_2|, 1 + k - |x_1 - x_2|\}$$

for all $x \in \mathcal{X} := \{x \in \mathbb{R}^2 : V(x) < \frac{1+k}{2}\} = \{x \in \mathbb{R}^2 : |x_1 - x_2| \neq \frac{1+k}{2}\}$. *This function is continuously differentiable on the open set* $\mathcal{X} \setminus K$ *and it is Lipschitz on* $\mathcal{X}$. *Let* $m^\star = \frac{1+k}{2}$ *and* $m \in (0, m^\star)$. *Consider* $C_m = C \cap M$ *and* $D_m = D \cap M$, *where* $M := \{z \in C \cup D : V(x) \leq m\}$. *By the definition of* V, *it follows that*

$$\langle \nabla V(x), F(x) \rangle = 0 \qquad \forall x \in C_m \setminus K.$$

We now consider $x \in D_m \setminus K$. *Since* V *is symmetric, without loss of generality, consider* $x = (1, x_2) \in D_m \setminus K$ *where* $x_2 \in [0,1] \setminus \{\frac{1}{2+\tilde{\varepsilon}}\}$.[3] *Then, we obtain*

$$\begin{aligned} V(x) &= \min\{1 - x_2, k + x_2\}, \\ V(G(x)) &= \min\{g((1+\tilde{\varepsilon})x_2), 1 + k - g((1+\tilde{\varepsilon})x_2)\}. \end{aligned}$$

When $g((1+\tilde{\varepsilon})x_2) = 0$, *it follows that* $V(G(x)) = 0$. *When* $g((1+\tilde{\varepsilon})x_2) = (1+\tilde{\varepsilon})x_2$, *there are two cases:*

a) $x_2 < \frac{1}{2+\tilde{\varepsilon}}$, $V(x) = k + x_2 > (1+\tilde{\varepsilon})x_2 \geq V(G(x))$;

b) $x_2 > \frac{1}{2+\tilde{\varepsilon}}$, $V(x) = 1 - x_2 \geq V(G(x))$.

Thus, $V(G(x)) - V(x) \leq 0$ *for all* $x \in D_m \setminus K$. *By applying [39, Proposition 6.10], every maximal solution to the hybrid system* $\mathcal{H}_m = (C_m, F, D_m, G)$ *is complete. Moreover, given* $\tilde{\varepsilon} > 0$, *for* $\varepsilon = \frac{\tilde{\varepsilon}}{1+\tilde{\varepsilon}}$ *and* m *such that* $(K + \varepsilon\mathbb{B}) \cap C \subset C_m$, *we have that for all* $x \in D_m \cap (K + \varepsilon\mathbb{B})$, $G(x) = 0 \in K$. *Therefore, it follows from*

[3]Since $\left(1, \frac{1}{2+\tilde{\varepsilon}}\right) \in \{x \in \mathbb{R}^n : V(x) = \frac{1+k}{2}\}$.

Theorem 5.14 that the formula $f = \diamond p$ is satisfied for every solution ϕ to $\mathcal{H}$ from $\mathcal{N} := \{x \in C \cup D : V(x) < m\}$ for all (t, j) in the domain of each solution. $\triangle$

5.3 Sufficient Conditions for $\bigcirc p$

Theorem 5.16. *Given an atomic proposition p, let the set K be as in* (3.6). *The formula $f = \bigcirc p$ is satisfied for all solutions ϕ to $\mathcal{H}$ at each $(t, j) \in dom\ \phi$ if the properties a), b), and c) in Theorem 4.8 hold simultaneously.*

Remark 5.17. *By the definition of* next *operator, one could consider that the flow set C is empty to specify $\bigcirc p$ for all solutions ϕ to $\mathcal{H}$. Under this assumption, $\mathcal{H}$ reduces to a discrete-time system.*

The following example illustrates the sufficient conditions in Theorem 5.16 to guarantee the satisfaction of $\bigcirc p$.

Example 5.18. *Let a hybrid system $\mathcal{H} = (C, F, D, G)$ with the state $x \in \mathbb{R}$ and data given by*

$$D := \mathbb{R}, \quad G(x) := sgn(x), \tag{5.10}$$

C is empty, and the flow map F is arbitrary. The function $sgn(x)$ is defined in Example 5.12, and $p(x) = 1$ if $|x| = 1$. Let $K := \{-1, 1\}$. By using the map G, for every $x \in D \cap K$, $G(x) \in K$; for every $x \in D \setminus K$, $G(x) \in K$. Therefore, the formula $f = \bigcirc p$ is satisfied for all solutions to $\mathcal{H}$.

5.4 Sufficient Conditions for $p\mathcal{U}q$

In this section, sufficient conditions that guarantee the satisfaction of the formulas $p\mathcal{U}_w q$ and $p\mathcal{U}_s q$ are presented by employing sufficient conditions that guarantee the needed invariance and attractivity properties of the sets.

First, we present sufficient conditions that guarantee the satisfaction of the formula $p\mathcal{U}_w q$ by using the sufficient conditions for conditional invariance in Proposition C.2.

Theorem 5.19 ($p\mathcal{U}_w q$)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4.9) *such that P and Q are closed and $P \subset C \cup D$. Then, the formula $f = p\mathcal{U}_w q$ is satisfied for all solutions ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$ if there exists a $\mathcal{C}^1$ barrier function candidate B with respect to the sets $(P \setminus Q, \mathbb{R}^n \setminus (P \cup Q))$ for $\mathcal{H}$ as in* (C.3) *such that $K := \{x \in C \cup D \cup Q : B(x) \leq 0\}$ is closed and the following hold:*

1) $\langle \nabla B(x), \eta \rangle \leq 0$ for all $x \in (C \backslash Q) \cap (U(\partial K) \backslash K)$ and all $\eta \in F(x) \cap T_{C \backslash Q}(x)$.

2) $B(\eta) \leq 0$ for all $x \in K \cap (D \backslash Q)$ and all $\eta \in G(x)$.

3) $G(x) \subset C \cup D \cup Q$ for all $x \in K \cap (D \backslash Q)$.

Proof. Let the system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ be as in (4.10). Since $K = \{x \in C \cup D \cup Q : B(x) \leq 0\}$ and the barrier function candidate B satisfies $B(x) \leq 0$ for all $x \in P \backslash Q$ and $B(x) > 0$ for all $x \in (C \cup D) \backslash (P \cup Q) = (C \cup D \cup Q) \backslash (P \cup Q)$, we conclude that B is a barrier candidate with respect to $(P \backslash Q, \mathbb{R}^n \backslash (P \cup Q))$ for $\mathcal{H}_w$ in (4.10). Furthermore, item 1) implies that $\langle \nabla B(x), \eta \rangle \leq 0$ for all $x \in (U(\partial K) \backslash K) \cap C_w$ and all $\eta \in F(x) \cap T_{C_w}(x)$. Item 2) implies that $B(\eta) \leq 0$ for all $x \in K \cap (D \backslash Q)$ and all $\eta \in G_w(x)$. Furthermore, when $x \in K \cap Q$, $G_w(x) = x$ and $B(x) \leq 0$. Hence, $B(\eta) \leq 0$ for all $x \in K \cap D_w$ and all $\eta \in G_w(x)$. Item 3) implies that $G_w(K \cap (D \backslash Q)) \subset C_w \cup D_w$. Furthermore, $G_w(K \cap Q) \subset K \cap Q \subset C_w \cup D_w$. Hence, $G_w(K \cap D_w) \subset C_w \cup D_w$. Thus, using item 1) in Proposition C.2 with $\mathcal{O}$ and $\mathcal{X}_u$ therein replace by $P \backslash Q$ and $\mathbb{R}^n \backslash (P \cup Q)$, respectively, we conclude that $P \cup Q$ is conditionally inavariant with respect to $P \backslash Q$ for $\mathcal{H}_w$. Hence, using Theorem 4.10, we conclude that $p\mathcal{U}_w q$ is satisfied for $\mathcal{H}$. $\square$

The following example illustrates Theorem 5.19.

Example 5.20 (Bouncing Ball)**.** *Consider the bouncing ball example in Example 5.5 to confirm the conclusions therein using Theorem 5.19. Consider the barrier function candidate $B(x) := x_1 - \varepsilon$ with $\varepsilon > 0$. Indeed, B is a barrier function candidate with respect to $(P \backslash Q, \mathbb{R}^n \backslash (P \cup Q))$ for $\mathcal{H}$ since $B(x) \leq 0$ for all $x \in P \backslash Q = [0, \varepsilon] \times \mathbb{R}_{<0}$ and $B(x) > 0$ for all $x \in (C \cup D) \backslash (P \cup Q) = (\varepsilon, \infty) \times \mathbb{R}_{<0}$. Furthermore, for all $x \in C \backslash Q = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}$, we have $\langle \nabla B(x), F(x) \rangle = x_2 \leq 0$; hence, item 1) holds. Furthermore, for all $x \in K \cap D = D$, $B(G(x)) = B(x) \leq 0$; hence, item 2) holds. Finally, for all $x \in D$, $G(x) \in \{0\} \times \mathbb{R}_{\geq 0} \subset C$; hence, item 3) holds. As a consequence, using Theorem 5.19, we conclude that the formula $f = p\mathcal{U}_w q$ is satisfied for all solutions ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$.*

In the following, we present sufficient conditions that guarantee the satisfaction of the formula $p\mathcal{U}_s q$ by using sufficient conditions for ECI for hybrid systems.

Theorem 5.21 ($p\mathcal{U}_s q$ using Eventual Conditional Invariance)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Let the system $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ be as in (4.11). Given atomic propositions p and q, let the sets P and Q be as in (4.9) such that P and Q are closed and $P \subset \overline{C} \cup D$. Then, the formula $f = p\mathcal{U}_s q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$.*

2) *There exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a constant $r_1 > 0$ such that the following hold:*

2.1) $\langle \nabla v(x), \eta \rangle \leq f_c(v(x))$ *for all* $\eta \in F(x) \cap T_{\overline{C}_s}(x)$ *and for all* $x \in \overline{C}_s$*;*

2.2) $v(\eta) \leq v(x)$ *for all* $\eta \in G(x)$ *and for all* $x \in D \cap P$*;*

2.3) The solutions to

$$\dot{y} = f_c(y) \tag{5.11}$$

starting from $v(P \setminus Q)$ *converge to* $(-\infty, r_1)$ *in finite time.*

3) There exist a $\mathcal{C}^1$ *function* $w : \mathbb{R}^n \to \mathbb{R}$*,* $f_d : \mathbb{R} \to \mathbb{R}$ *which is nondecreasing, and a constant* $r_2 > 0$ *such that the following hold:*

3.1) $\langle \nabla w(x), \eta \rangle \leq 0$ *for all* $\eta \in F(x) \cap T_{\overline{C}_s}(x)$ *and all* $x \in \overline{C}_s$*;*

3.2) $w(\eta) \leq f_d(w(x))$ *for all* $\eta \in G(x)$ *and all* $x \in D \cap P$*;*

3.3) The solutions to

$$z^+ = f_d(z) \tag{5.12}$$

starting from $w(P \setminus Q)$ *converge to* $(-\infty, r_2)$ *in finite time.*

4) One of the following conditions holds:

4a) Each complete solution to $\mathcal{H}$ *starting from* $P \setminus Q$ *is eventually continuous and, with* r_1 *coming from item 2),*

$$S_1 := \{x \in C \cap (P \cup Q) : v(x) < r_1\} \subset Q. \tag{5.13}$$

4b) Each complete solution to $\mathcal{H}$ *starting from* $P \setminus Q$ *is eventually discrete and, with* r_2 *coming from item 3),*

$$S_2 := \{x \in D \cap (P \cup Q) : w(x) < r_2\} \subset Q. \tag{5.14}$$

4c) Each complete solution to $\mathcal{H}$ *starting from* $P \setminus Q$ *is eventually continu-*

ous, eventually discrete, or has a hybrid time domain that is unbounded in both the t and the j direction and, with r_1 and r_2 coming from item 2) and item 3) respectively, (5.13) *and* (5.14) *hold.*

4d) With r_1 and r_2 coming from item 2) and item 3) respectively, (5.13) *and* (5.14) *hold, and $G(S_2) \cap \overline{C}_s \subset S_1$.*

5) No maximal solution starting from P has a finite time escape within $(P \backslash Q) \cap C$

6) Every maximal solution from $(P \cap \partial C) \setminus (D \cup Q)$ is nontrivial.

Proof. By item 1), every maximal solution to $\mathcal{H}$ from $P \cup Q$ satisfies $p\mathcal{U}_w q$. It remains to show that every maximal solution to $\mathcal{H}$ starting from $P \backslash Q$ also satisfies $p\mathcal{U}_s q$, as solutions from Q already satisfy it. To this end, note that each maximal solution ϕ to $\mathcal{H}$ from $P \backslash Q$ satisfy one of the following conditions:

a) ϕ reaches Q in finite hybrid time;

b) ϕ is not complete and does not reach Q in finite hybrid time; or

c) ϕ is complete and does not reach Q in finite hybrid time.

In the rest of the proof, we show that ϕ can only satisfy case a). First, we show that case b) is not possible due to items 5) and 6) using contradiction. That is, suppose ϕ is not complete and never reach Q; in particular, dom ϕ is bounded. Let $(T, J) = \sup \operatorname{dom} \phi$. Due to the fact that ϕ never reach Q and since ϕ satisfies $p\mathcal{U}_w q$, we conclude that ϕ remains in $P \backslash Q$. Moreover, under item 5), the maximal solution ϕ does not have a finite escape time inside $(P \backslash Q) \cap C$, which implies that $(T, J) \in \operatorname{dom} \phi$. Now, by the definition of solutions to $\mathcal{H}$, $\phi(T, J) \in \overline{C} \cup D$. First, let $\phi(T, J) \in D$. In this case, ϕ can be extended via a jump. Next, let $\phi(T, J) \in \overline{C} \backslash D$. In this case, when $\phi(T, J) \in \operatorname{int}(C) \backslash D$, we use (SA) to conclude that ϕ can be extended via flow; and for the case when $\phi(T, J) \in \partial C \backslash D$, we use item 6) to conclude that ϕ can be extended via flow. Therefore, if $(T, J) \in \operatorname{dom} \phi$, then ϕ can be extended via flow or a jump. This contradicts maximality of ϕ; and thus, case b) is not possible.

Next, we show that case c) is not possible due to items 2)-4) using contradiction. Suppose that items 2), 3), and 4a) hold. Suppose that there exists a complete solution ϕ to $\mathcal{H}$ that does not reach Q in finite hybrid time. By construction of $\mathcal{H}_s$ and since $\mathcal{H}$ satisfies $p\mathcal{U}_w q$, we claim that ϕ is also a maximal solution to $\mathcal{H}_s$. However, using the arguments in a) in the proof of Theorem C.5, there must exist $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in S_1 \subset Q$ for all $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq t^\star + j^\star$. This implies that ϕ must reach Q in finite hybrid time via flow. Next, suppose that items 2), 3), and 4b) hold. Proceeding as when 4a) holds, we claim that ϕ is also a maximal solution to $\mathcal{H}_s$. Using the arguments in b) in the proof of Theorem C.5, there exists $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in S_2 \subset Q$ for all $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq t^\star + j^\star$. This implies that ϕ must reach Q in finite hybrid time by jumps. Similarly, suppose that items 2) and 3) hold and either item 4c) or item 4d) holds. Using the claim that ϕ is a maximal solution to $\mathcal{H}_s$ and the arguments in the proof of Theorem C.5, we conclude that there exists $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in S_1 \cup S_2 \subset Q$ for all $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq t^\star + j^\star$. This implies that ϕ must reach Q in finite hybrid time via flow or jumps. Therefore, we conclude that case c) is not possible. $\square$

Theorem 5.22 (Strong Until using Eventual Conditional Invariance via Flows)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4.9) *such that $P \subset \overline{C} \cup D$. Then, the formula $f = p\mathcal{U}_s q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$ if the following hold:*

1) The formula $p\mathcal{U}_w q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t,j)=(0,0)$ with $(\phi,(0,0)) \vDash p \vee q$.

2) There exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a constant $r_1 > 0$ such that the following hold:

2.1) $\langle \nabla v(x), \eta \rangle \leq f_c(v(x))$ for all $\eta \in F(x) \cap T_C(x)$ and for all $x \in (C \cap P) \setminus Q$;

2.2) $v(\eta) \leq v(x)$ for all $\eta \in G(x)$ and all $x \in D \cap P$;

2.3) $S_1 := \{x \in C \cap (P \cup Q) : v(x) < r_1\} \subset Q$ and the solutions y to (5.11) starting from $v(P \setminus Q)$ converge to $(-\infty, r_1)$ in finite time.

3) For each solution $\phi \in \mathcal{S}_{\mathcal{H}}(P \setminus Q)$, there exists a solution y to (5.11) starting from $v(\phi(0,0))$ such that there exists $t^\star \in \mathbb{R}_{\geq 0}$ satisfying:

$$t^\star \leq \sup\{t : (t,j) \in \mathit{dom}\ \phi\}, \quad y(t) \in (-\infty, r_1] \qquad \forall t \geq t^\star. \tag{5.15}$$

Proof. Consider system $\mathcal{H}_s$ introduced in (4.11). Using item 1), we conclude that a maximal solution ψ to $\mathcal{H}$ starting from $P \backslash Q$ either remains in $P \backslash Q$ for all time, otherwise, ψ remains in $P \backslash Q$ up to when it reaches the set Q. Hence, each maximal solution ϕ to $\mathcal{H}_s$ starting from $P \backslash Q$ remains in $P \cup Q$. In particular, either ϕ reaches Q in finite time, or ϕ remains in $P \backslash Q$. To exclude the latter case, we show that, when ϕ remains in $P \backslash Q$, ϕ must be a maximal solution to $\mathcal{H}$. Indeed, assume the existence of a solution ψ to $\mathcal{H}$ which is a nontrivial extension of ϕ; namely, there exists $I \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $I \neq \emptyset$ and $\text{dom}\ \phi \cup I = \text{dom}\ \psi$. Note that $\psi(\text{dom}\ \phi) = \phi(\text{dom}\ \phi) \subset P \backslash Q$. Also, since ψ must remain in $P \backslash Q$ up to when it reaches Q, we can choose I such that $\psi(\text{dom}\ \phi \cup I) \subset P \backslash Q$. Hence, ψ is a solution to $\mathcal{H}_s$, which contradicts the fact that ϕ is a maximal solution to

$\mathcal{H}_m^{'}$. Next, using item 3), we conclude the existence of a solution y to $\dot{y} = f_c(y)$ starting from $v(\phi(0,0))$ such that, for some $t^\star \geq 0$, (5.15) holds. Combining the latter fact to item 2) and using Theorem C.15 for $\mathcal{H}_s$, we conclude that ϕ must reach Q in finite time. Hence, Q is eventual conditional invariant with respect to $P \backslash Q$ for $\mathcal{H}_s$. Finally, the proof is completed using Theorem 4.12. $\square$

Theorem 5.23 ($p\mathcal{U}_s q$ using Eventual Conditional Invariance via Jumps)**.** *Consider a hybrid system*

$\mathcal{H} = (C, F, D, G)$. *Given atomic propositions p and q, let the sets P and Q be as in* (4.9) *such that $P \subset \overline{C} \cup D$. Then, the formula $f = p\mathcal{U}_s q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$.*

2) *There exist a $\mathcal{C}^1$ function $w : \mathbb{R}^n \to \mathbb{R}$, $f_d : \mathbb{R} \to \mathbb{R}$ which is nondecreasing, and a constant $r_2 > 0$ such that the following hold:*

   2.1) *$\langle \nabla w(x), \eta \rangle \leq 0$ for all $\eta \in F(x) \cap T_C(x)$ and all $x \in (C \cap P) \setminus Q$;*

   2.2) *$w(\eta) \leq f_d(w(x))$ for all $\eta \in G(x)$ and all $x \in D \cap P$;*

   2.3) *$S_2 := \{x \in D \cap (P \cup Q) : w(x) < r_2\} \subset Q$ and the solutions to* (5.12) *starting from $w(P \setminus Q)$ converge to $(-\infty, r_2)$ in finite time.*

3) *For each solution $\phi \in \mathcal{S}_{\mathcal{H}}(P \setminus Q)$, there exists a solution z to* (5.12) *starting from $v(\phi(0, 0))$ such that there exists $j^\star \in \mathbb{N}$ satisfying:*

$$j^\star \leq \sup\{j : (t, j) \in dom\ \phi\}, \quad z(j) \in (-\infty, r_2] \qquad \forall j \geq j^\star.$$

Proof. The proof follows the exact same steps used to prove Theorem 5.22 while using Theorem C.16 instead of Theorem C.15. $\square$

Next, we employ the conditions for pre-ECI in Theorem C.13 for hybrid systems when we know the lengths of the flow interval between each successive jumps approximately.

Theorem 5.24 ($p\mathcal{U}_s q$ using Eventual Conditional Invariance under Approximate Flow Lengths)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given atomic propositions p and q, let the sets P and Q be as in* (4.9) *such that $P \subset \overline{C} \cup D$. Let a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ be a closed set of approximate flow lengths of the solutions to $\mathcal{H}$ starting from $P \setminus Q$ as in* (C.9)*, and let $\tau_M := \sup \mathcal{I}$. Then, the formula $f = p\mathcal{U}_s q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$ if the following hold:*

1) *The formula $p\mathcal{U}_w q$ is satisfied for each solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ with $(\phi, (0, 0)) \vDash p \vee q$.*

2) *There exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a function $f_d : \mathbb{R} \to \mathbb{R}$ which is nondecreasing such that*

$$\begin{aligned} &\langle \nabla v(x), \eta \rangle \leq f_c(v(x)) && \forall \eta \in F(x) \cap T_C(x),\ \forall x \in (C \cap P) \setminus Q, \\ &v(\eta) \leq f_d(v(x)) && \forall \eta \in G(x),\ \forall x \in D \cap P. \end{aligned}$$

3) *There exists a constant $r > 0$ such that*

$$S := \{x \in (C \cup D) \cap (P \cup Q) : v(x) < r\} \subset Q.$$

4) *The solutions to the reduced hybrid system $\mathcal{H}_r$ starting from $v(P \setminus Q) \times \{0\}$*

converge to $(-\infty, r] \times \mathbb{R}_{\geq 0}$ *in finite time, where*

$$\mathcal{H}_r : \begin{cases} \begin{bmatrix} \dot{y} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} f_c(y) \\ 1 \end{bmatrix} & (y, \tau) \in \mathbb{R} \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0}), \\ \begin{bmatrix} y^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} f_d(x) \\ 0 \end{bmatrix} & (y, \tau) \in \mathbb{R} \times \mathcal{I}. \end{cases}$$

5) No maximal solution starting from P *has a finite time escape within* $P \cap (C \setminus Q)$ *and every maximal solution from* $(P \cap \partial C) \setminus (D \cup Q)$ *is nontrivial.*

Proof. Consider the system $\mathcal{H}_s$ introduced in (4.11). Using Theorem C.13 for $\mathcal{H}_s$ under items 2), 3), and 4), we conclude that Q is pre-ECI with respect to $P \backslash Q$ for $\mathcal{H}_s$. The rest of the proof follows using the same steps in the proof of Theorem 5.21. $\square$

Chapter 6

Sufficient conditions guaranteeing the satisfaction of temporal formulas for hybrid systems under perturbations

6.1 Approximate satisfaction of LTL formulas

First, we provide equivalent characterizations of approximate satisfaction of $\Box p$. For this purpose, we recall a notion of conditional invariance for hybrid systems in Definition 2.4.

Proposition 6.1. *(Approximate Satisfaction of $\Box p$ from K). Given an atomic proposition p, let $\varepsilon > 0$ and the sets K and K^ε be as in* (3.6) *and* (3.7)*, respectively. The formula $f = \Box p$ is ε-approximately satisfied by $\mathcal{H}$ from K at $(t, j) = (0, 0)$ if and only if the set K^ε is conditionally invariant with respect to the set K for $\mathcal{H}$.*

Next, we present an equivalent characterization of approximate satisfaction of

$\diamond p$, in terms of finite time attractivity, defined as follows. Below, the amount of hybrid time required for a solution ϕ to converge to the set M is captured by a settling-time function $\mathcal{T}_M$ whose argument is the solution ϕ and its output is a positive number determining the amount of (hybrid) time needed to converge to M; i.e., given ϕ, $\mathcal{T}_M(\phi) := \inf\{t + j : \phi(t, j) \in M\}$. Below, the arrow $\nearrow$ is used to denote the limit from below.

Definition 6.2. *(Finite Time Attractivity). A closed set $M \subset \mathbb{R}^n$ is said to be finite time attractive (FTA) for a hybrid system $\mathcal{H}$ with respect to $\mathcal{O} \subset \overline{C} \cup D$ if for every solution ϕ to $\mathcal{H}$ with $\phi(0,0) \in \mathcal{O}$, $\sup_{(t,j)\in dom\ \phi} t+j \geq \mathcal{T}_M(\phi)$, and*

$$\lim_{(t,j)\in dom\ \phi : t+j \nearrow \mathcal{T}_M(\phi)} |\phi(t,j)|_M = 0. \tag{6.1}$$

Furthermore, the set M is said to be FTA for $\mathcal{H}$ if so it is with respect to $\overline{C} \cup D$.

The following result is immediate.

Proposition 6.3. *(Approximate Satisfaction of $\diamond p$). Given an atomic proposition p, let $\varepsilon > 0$ and the set K and K^ε be as in (3.6) and (3.7), respectively. The formula $f = \diamond p$ is ε-approximately satisfied by $\mathcal{H}$ at $(t, j) = (0, 0)$ if and only if the set K^ε is FTA for $\mathcal{H}$.*

Due to the equivalence presented in Proposition 6.1, sufficient conditions that guarantee the approximate satisfaction of the formula $f = \square p$ from the set K are proposed by employing sufficient conditions to guarantee conditional invariance of the set K^ε with respect to the set K as in [29, Theorem 3.2]. Below, the concept of the tangent cone[1] to a set is used; see [20, Definition 5.12]. The tangent cone

[1] This tangent cone is also known as the contingent cone, or the Bouligand tangent cone.

at a point $x \in \mathbb{R}^n$ of a set $K \subset \mathbb{R}^n$ is given by

$$T_K(x) := \Big\{ v \in \mathbb{R}^n : \lim_{h \to 0^+} \inf \tfrac{|x+hv|_K}{h} = 0 \Big\}.$$

Given sets $\mathcal{X}_o, \mathcal{X}_u \subset \mathbb{R}^n$ with $\mathcal{X}_o \cap \mathcal{X}_u = \emptyset$, we introduce the notion of a barrier function candidate with respect to $(\mathcal{X}_o, \mathcal{X}_u)$ for $\mathcal{H}$.

Definition 6.4. *(Barrier Function Candidate for $\mathcal{H}$). Consider $\mathcal{H} = (C, F, D, G)$. Given sets $\mathcal{X}_o, \mathcal{X}_u \subset \mathbb{R}^n$ with $\mathcal{X}_o \cap \mathcal{X}_u = \emptyset$, a function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a barrier function candidate with respect to $(\mathcal{X}_o, \mathcal{X}_u)$ for $\mathcal{H}$ if*

$$\begin{cases} B(x) \leq 0 & \forall x \in \mathcal{X}_o \\ B(x) > 0 & \forall x \in (C \cup D) \cap \mathcal{X}_u. \end{cases} \tag{6.2}$$

Theorem 6.5. *(Approximate Satisfaction of $\Box p$ from K). Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given an atomic proposition p, let $\varepsilon > 0$ and the sets K and K^ε be as in* (3.6) *and* (3.7)*, respectively. Then, the formula $f = \Box p$ is ε-approximately satisfied by $\mathcal{H}$ from K if there exists a $\mathcal{C}^1$ barrier function candidate B with respect to $(K, \mathbb{R}^n \setminus K^\varepsilon)$ for $\mathcal{H}$ as in* (6.2) *such that $L := \{x \in C \cup D : B(x) \leq 0\}$ is closed and the following hold:*

1) $\langle \nabla B(x), \eta \rangle \leq 0$ for all $x \in (U(\partial L) \setminus L) \cap C$ and for all $\eta \in F(x) \cap T_C(x)$;

2) $B(\eta) \leq 0$ for all $x \in L \cap D$ and for all $\eta \in G(x)$;

3) $G(x) \subset C \cup D$ for all $x \in L \cap D$.

Due to the equivalence in Proposition 6.3, sufficient conditions to guarantee the approximate satisfaction of the formula $f = \Diamond p$ is proposed by using sufficient conditions to guarantee FTA property of sets similar to those in [28, Theorem 5.7] with the set K replaced by the set K^ε as in (3.7).

Theorem 6.6. *(Approximate Satisfaction of $\Diamond p$). Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given an atomic proposition p, let $\varepsilon > 0$ and the set K^ε be as in (3.7). Suppose there exists an open set $\mathcal{N}$ that defines an open neighborhood of K^ε such that $\mathcal{N} \subset \mathbb{R}^n$. Suppose that there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$ such that $\mathcal{O} \subset L_V(r) \cap (\overline{C} \cup D)$ where $L_V(r) = \{x \in \mathbb{R}^n : V(x) \leq r\}, r \in [0, \infty]$, is a sublevel set of V contained in $\mathcal{N}$. Suppose that for all $x \in D$ such that $x \in \mathcal{N}$, $G(x) \in \mathcal{N}$. Then, the formula $f = \Diamond p$ is ε-approximately satisfied by $\mathcal{H}$ with $\phi(0,0) \in \mathcal{O}$ at $(t, j) = (0, 0)$ if*

1) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and constants $c_1 > 0, c_2 \in [0, 1)$ such that

1.1) for every $x \in (\mathcal{N} \cap (\overline{C} \cup D)) \setminus K^\varepsilon$, each solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x$ satisfies $\frac{V^{1-c_2}(x)}{c_1(1-c_2)} \leq \sup_{(t,j) \in dom\, \phi} t$;

1.2) the function V is positive definite with respect to K^ε and

1.2a) for each $x \in (C \cap \mathcal{N}) \setminus K^\varepsilon$, $u_C(x) + c_1 V^{c_2}(x) \leq 0$;

1.2b) for each $x \in (D \cap \mathcal{N}) \setminus K^\varepsilon$, $u_D(x) \leq 0$; or

2) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and a constant $c > 0$ such that

2.1) for every $x \in (\mathcal{N} \cap (\overline{C} \cup D)) \setminus K^\varepsilon$, each solution ϕ to $\mathcal{H}$ with $\phi(0,0) = x$ satisfies $ceil\left(\frac{V(x)}{c}\right) \leq \sup_{(t,j) \in dom\, \phi} j$;

2.2) the function V is positive definite with respect to K^ε and

2.2a) for each for each $x \in (C \cap \mathcal{N}) \setminus K^\varepsilon$, $u_C(x) \leq 0$;

2.2b) for each for each $x \in (D \cap \mathcal{N}) \setminus K^\varepsilon$, $u_D(x) \leq -\min\{c, V(x)\}$.

Moreover, for each solution ϕ to $\mathcal{H}$ with $\phi(0,0) \in \mathcal{O}$, defining $\xi = \phi(0,0)$, the first time $(t', j') \in dom\ \phi$ such that $\phi(t', j') \in K^\varepsilon$ satisfies

$$t' + j' = \mathcal{T}_{K^\varepsilon}(\phi),$$

and an upper bound on that hybrid time is given as follows:

a) if 1) holds, then $\mathcal{T}_{K^\varepsilon}$ is upper bounded by $\mathcal{T}^\star(\xi) + \mathcal{J}^\star(\phi)$, where $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$ and $\mathcal{J}^\star(\phi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\phi)) \in dom\ \phi$.

b) if 2) holds, then $\mathcal{T}_{K^\varepsilon}$ is upper bounded by $\mathcal{T}^\star(\phi) + \mathcal{J}^\star(\xi)$, where $\mathcal{J}^\star(\xi) = ceil(\frac{V(\xi)}{c})$ and $\mathcal{T}^\star(\phi)$ is such that $(\mathcal{T}^\star(\phi), \mathcal{J}^\star(\xi)) \in dom\ \phi$ and $(\mathcal{T}^\star(\phi), \mathcal{J}^\star(\xi) - 1) \in dom\ \phi$.

6.2 Robust approximate satisfaction of LTL formulas

First, we provide equivalent characterizations of robust ε-approximate satisfaction of $\Box p$, in terms of robust conditional invariance properties for hybrid systems. For this purpose, we introduce robust conditional invariance.

Definition 6.7. *(Robust Conditional Invariance). Given two sets $K^\varepsilon, K \subset \mathbb{R}^n$ such that $K \subset K^\varepsilon$, the set K^ε is said to be robustly conditionally invariant with respect to the set $K \subset K^\varepsilon$ for $\mathcal{H}_w$ if, for each solution pair (ϕ, w) to $\mathcal{H}_w$ with $\phi(0,0) \in K$, $\phi(t,j) \in K^\varepsilon$ for every $(t,j) \in dom\ \phi$.*

The following result is immediate.

Proposition 6.8. *(Robust Approximate Satisfaction of $\Box p$ from K). Given an atomic proposition p, let $\varepsilon > 0$ and the sets K and K^ε be as in (3.6) and (3.7),* *respectively. The formula $f = \Box p$ is robustly ε-approximately satisfied by $\mathcal{H}$ from K at $(t,j) = (0,0)$ if and only if the set K^ε is robustly conditionally invariant with respect to the set K for $\mathcal{H}_w$.*

The following example illustrates a situation when robust approximate satisfaction of $\Box p$ is needed.

Example 6.9. *(Bouncing Ball with Perturbations). Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a ball bouncing vertically on the ground, with the state $x = (x_1, x_2) \in \mathbb{R}^2$ and the data*

$$F(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \qquad \forall x \in C := \{x \in \mathbb{R}^2 : x_1 \geq 0\},$$
$$G(x) := \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} \qquad \forall x \in D := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\},$$

where x_1 is the height above the surface and x_2 denotes the vertical velocity. The parameter $\gamma > 0$ is the gravity coefficient and $\lambda \in [0,1]$ is the restitution coefficient. Here, we include uncertainties at impacts with the ground and in the velocity of the ball. The hybrid system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ with a state perturbation $w = (w_1, w_2) \in \mathbb{R}^2$ is given by

$$\mathcal{H}_w \begin{cases} \dot{x} = F_w(x, w_1) & (x, w_1) \in C_w \\ x^+ = G_w(x, w_2) & (x, w_2) \in D_w, \end{cases} \tag{6.3}$$

where the flow and jump maps are

$$F_w(x, w_1) := \begin{bmatrix} x_2 - w_1 \\ -\gamma \end{bmatrix}, \quad G_w(x, w_2) := \begin{bmatrix} 0 \\ -w_2 x_2 \end{bmatrix}$$

and the flow and jump sets are

$$C_w := \{(x, w_1) \in \mathbb{R}^2 \times \mathbb{R} : x_1 \geq 0, w_1 \in [0, w_{\max})\},$$
$$D_w := \{(x, w_2) \in \mathbb{R}^2 \times \mathbb{R} : x_1 = 0, x_2 \leq 0, w_2 \in (0, 1)\}.$$

The disturbances w_1 and w_2 satisfy $w_1 \in [0, w_{\max})$ with $w_{\max} > 0$ and $w_2 \in (0, 1)$. Define an atomic proposition p as follows: for each $x \in \mathbb{R}^2$, $p(x) = 1$ if $x \in \Pi(C_w) \cup \Pi(D_w)$ and $2\gamma x_1 + (x_2 - 1)(x_2 + 1) \leq 0$; $p(x) = 0$ otherwise. Let K be given as in (3.6), which collects the set of points such that p is satisfied. The set K is the sublevel set where the total energy of the ball is less than or equal to $1/2$. By constructing K^ε as in (3.7) with $\varepsilon = w_{\max}$, we have that for each $x \in K^\varepsilon$, $2\gamma x_1 + (x_2 - 1)(x_2 + 1) \leq \varepsilon$. We observe that , without disturbances, all solutions starting from the set K remain in the set K; thus, the set K is forward invariant with $w \equiv 0$. However, in the presence of the disturbance $w_1 \in (0, w_{\max})$, solutions starting from the set K may leave the set K due to the effect of w_1. In the meanwhile, by the construction of the set K^ε, we observe that all solutions starting from the set K remain in the set K^ε even if such solutions leave the set K. Hence, to specify and verify such dynamical property for $\mathcal{H}_w$, we need a notion of robust approximate satisfaction of the formula $\Box p$. $\triangle$

Due to the equivalence presented in Proposition 6.8, any sufficient condition that guarantees robust conditional invariance of K^ε with respect to the set K guarantee robust approximate satisfaction of the formula $\Box p$. Inspired by the results for nominal conditional invariance in [29], we propose sufficient conditions for robust conditional invariance using barrier functions. First, the notion of barrier function candidate B for $\mathcal{H}_w$ is formulated. This notion extends the one in [29] to the case with disturbances w.

Definition 6.10. *(Barrier Function Candidate for $\mathcal{H}_w$). Consider $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$.*

Given sets $\mathcal{X}_o, \mathcal{X}_u \subset \mathbb{R}^n$ *with* $\mathcal{X}_o \cap \mathcal{X}_u = \emptyset$*, a function* $B : \mathbb{R}^n \to \mathbb{R}$ *is said to be a barrier function candidate with respect to* $(\mathcal{X}_o, \mathcal{X}_u)$ *for* $\mathcal{H}_w$ *if*

$$\begin{cases} B(x) \leq 0 & \forall x \in \mathcal{X}_o \\ B(x) > 0 & \forall x \in (\Pi(C_w) \cup \Pi(D_w)) \cap \mathcal{X}_u. \end{cases} \tag{6.4}$$

A barrier function candidate B is defined as a scalar function of the state variables x which is nonpositive on the set of initial conditions $\mathcal{X}_o$ and strictly positive on the unsafe set $\mathcal{X}_u$. This barrier function candidate B is exploited to verify invariance properties for hybrid systems with a perturbation w. Similar to the results on conditional invariance in [29], the following is assumed:

(A1) The flow map F_w is outer semicontinuous[2], nonempty, and locally bounded with convex images on C_w. Furthermore, the jump map G_w is nonempty on D_w.

First, we introduce sufficient conditions for robust conditional invariance using barrier functions.

Proposition 6.11. *(Robust Conditional Invariance). Consider a hybrid system* $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ *satisfying (A1). Let the sets* $\mathcal{X}_o$ *and* $\mathcal{X}_u$ *be such that* $\mathcal{X}_o, \mathbb{R}^n \setminus \mathcal{X}_u \subset \Pi(C_w) \cup \Pi(D_w)$*. The set* $\mathbb{R}^n \setminus \mathcal{X}_u$ *is robustly conditionally invariant with respect to* $\mathcal{X}_o$ *for* $\mathcal{H}_w$ *if there exists a* $\mathcal{C}^1$ *barrier function candidate* B *with respect to* $(\mathcal{X}_o, \mathcal{X}_u)$ *for* $\mathcal{H}_w$ *as in* (6.4) *such that* $L := \{x \in \Pi(C_w) \cup \Pi(D_w) : B(x) \leq 0\}$ *is closed and the following hold:*

1) $\langle \nabla B(x), \eta \rangle \leq 0$ *for all* $(x, w) \in ((U(\partial L) \setminus L) \times \mathcal{W}) \cap C_w$ *and for all* $\eta \in F_w(x, w) \cap T_{\Pi(C_w)}(x)$*; and*

[2] A set-valued map $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^{\infty}$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in M(x_i)$ converging to a point y, it holds that $y \in M(x)$; see [20, Definition 5.9].

2) $B(\eta) \leq 0$ for all $(x, w) \in (L \times \mathcal{W}) \cap D_w$ and for all $\eta \in G_w(x, w)$; and

3) $G_w(x, w) \subset \Pi(C_w) \cup \Pi(D_w)$ for all $(x, w) \in (L \times \mathcal{W}) \cap D_w$.

We have the following result.

Theorem 6.12. *(Robust Approximate Satisfaction of $\Box p$ from K). Consider $\mathcal{H} = (C, F, D, G)$ and $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ satisfying (A1). Given an atomic proposition p, let $\varepsilon > 0$ and the sets K and K^ε be as in* (3.6) *and* (3.7)*, respectively. Then, the formula $f = \Box p$ is robustly ε-approximately satisfied by $\mathcal{H}$ from K if there exists a $\mathcal{C}^1$ barrier function candidate B with respect to $(K, \mathbb{R}^n \setminus K^\varepsilon)$ for $\mathcal{H}_w$ as in* (6.4) *such that $L := \{x \in \Pi(C_w) \cup \Pi(D_w) : B(x) \leq 0\}$ is closed and the following hold:*

1) $\langle \nabla B(x), \eta \rangle \leq 0$ for all $(x, w) \in ((U(\partial L) \setminus L) \times \mathcal{W}) \cap C_w$ and for all $\eta \in F_w(x, w) \cap T_{\Pi(C_w)}(x)$; and

2) $B(\eta) \leq 0$ for all $(x, w) \in (L \times \mathcal{W}) \cap D_w$ and for all $\eta \in G_w(x, w)$; and

3) $G_w(x, w) \subset \Pi(C_w) \cup \Pi(D_w)$ for all $(x, w) \in (L \times \mathcal{W}) \cap D_w$.

In the following, we characterize robust ε-approximate satisfaction of the formula $\Diamond p$ via a new robust FTA property for hybrid systems. Below, the amount of hybrid time required for a solution pair (ϕ, w) for the convergence of ϕ to the set M is captured by a settling-time function $\mathcal{T}_M$, whose argument is the solution pair (ϕ, w) and its output is a positive number determining the time to converge to M; i.e., given ϕ, $\mathcal{T}_M(\phi, w) := \inf\{t + j : \phi(t, j) \in M\}$ is the time to reach K.

Definition 6.13. *(Robust Finite Time Attractivity) A closed set $M \subset \mathbb{R}^n$ is said to be robustly finite-time attractive (FTA) for $\mathcal{H}_w$ with respect to $\mathcal{O} \subset \overline{\Pi(C_w)} \cup \Pi(D_w)$*

if for every solution pair (ϕ, w) *to* $\mathcal{H}_w$ *with* $\phi(0,0) \in \mathcal{O}$, $\sup_{(t,j)\in dom\ \phi} t + j \geq \mathcal{T}_M(\phi, w)$, *and*

$$\lim_{(t,j)\in dom\ \phi : t+j \nearrow \mathcal{T}_M(\phi,w)} |\phi(t,j)|_M = 0. \tag{6.5}$$

Furthermore, the set M *is said to be robustly FTA for* $\mathcal{H}_w$ *if so it is with respect to* $\overline{\Pi(C_w)} \cup \Pi(D_w)$.

The following result is immediate.

Proposition 6.14. *(Robust Approximate Satisfaction of* $\Diamond p$*). Given an atomic proposition* p*, let* $\varepsilon > 0$ *and the set* K *and* K^ε *be as in* (3.6) *and* (3.7)*, respectively. The formula* $f = \Diamond p$ *is robustly* ε*-approximately satisfied by* $\mathcal{H}$ *at* $(t,j) = (0,0)$ *if and only if the set* K^ε *is robustly FTA for* $\mathcal{H}_w$.

Due to the equivalence presented in Proposition 6.14, any sufficient condition that guarantees robust finite time attractivity of K^ε guarantee robust approximate satisfaction of the formula $\Diamond p$. Similar to Theorem 6.6, using sufficient conditions for nominal finite time attractivity, we extends the one in Theorem 6.6 to the case with a perturbation w for sufficient conditions that guarantee robust approximate satisfaction of $\Diamond p$.

Theorem 6.15. *(Robust Approximate Satisfaction of* $\Diamond p$*). Consider a hybrid system* $\mathcal{H} = (C, F, D, G)$*. Given an atomic proposition* p*, let* $\varepsilon > 0$ *and the set* K^ε *be as in* (3.7)*. Suppose there exists an open set* $\mathcal{N}$ *that defines an open neighborhood of* K^ε *such that* $\mathcal{N} \subset \mathbb{R}^n$*. Suppose that there exists a continuous function* $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$ *such that* $\mathcal{O} \subset L_V(r) \cap \overline{C} \cup D$ *where* $L_V(r) = \{x \in \mathbb{R}^n : V(x) \leq r\}, r \in [0, \infty]$*, is a sublevel set of* V *contained in* $\mathcal{N}$*. Suppose that for all* $x \in D$ *such that* $x \in \mathcal{N}$*,* $G(x) \in \mathcal{N}$*. Then, the formula* $f = \Diamond p$ *is robustly* ε*-approximately satisfied by* $\mathcal{H}$ *with* $\phi(0,0) \in \mathcal{O}$ *if*

1) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$*, locally Lipschitz on an open neighborhood of* $\Pi(C_w) \cap \mathcal{N}$*, and constants* $c_1 > 0, c_2 \in [0, 1)$ *such that*

1.1) for every $x \in (\mathcal{N} \cap (\overline{\Pi(C_w)} \cup \Pi(D_w))) \setminus K^\varepsilon$*, each solution pair* (ϕ, w) *to* $\mathcal{H}_w$ *with* $\phi(0,0) = x$ *satisfies* $\frac{V^{1-c_2}(x)}{c_1(1-c_2)} \leq \sup_{(t,j)\in dom\ \phi} t$*;*

1.2) the function V *is positive definite with respect to* K^ε *and*

1.2a) for each $x \in (\Pi(C_w) \cap \mathcal{N}) \setminus K^\varepsilon$*,* $u_C(x) + c_1 V^{c_2}(x) \leq 0$ *;*

1.2b) for each $x \in (\Pi(D_w) \cap \mathcal{N}) \setminus K^\varepsilon$*,* $u_D(x) \leq 0$*;*

or

2) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$*, locally Lipschitz on an open neighborhood of* $\Pi(C_w) \cap \mathcal{N}$*, and* $c > 0$ *such that*

2.1) for every $x \in (\mathcal{N} \cap (\overline{\Pi(C_w)} \cup \Pi(D_w))) \setminus K^\varepsilon$*, each solution pair* (ϕ, w) *to* $\mathcal{H}_w$ *with* $\phi(0,0) = x$ *satisfies* $ceil\left(\frac{V(x)}{c}\right) \leq \sup_{(t,j)\in dom\ \phi} j$*;*

2.2) the function V *is positive definite with respect to* K^ε *and*

2.2a) for each for each $x \in (\Pi(C_w) \cap \mathcal{N}) \setminus K^\varepsilon$*,* $u_C(x) \leq 0$*;*

2.2b) for each for each $x \in (\Pi(D_w) \cap \mathcal{N}) \setminus K^\varepsilon$*,* $u_D(x) \leq -\min\{c, V(x)\}$*.*

Moreover, for each solution pair (ϕ, w) *to* $\mathcal{H}_w$ *with* $\phi(0,0) \in \mathcal{O}$*, defining* $\xi = \phi(0,0)$*, the first time* $(t', j') \in dom\ (\phi, w)$ *such that* $\phi(t', j') \in K^\varepsilon$ *satisfies*

$$t' + j' = \mathcal{T}_{K^\varepsilon}(\phi, w),$$

and an upper bound on that hybrid time is given as follows:

a) if 1) holds, then $\mathcal{T}_{K^\varepsilon}$ *is upper bounded by* $\mathcal{T}^\star(\xi) + \mathcal{J}^\star(\phi)$*, where* $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$ *and* $\mathcal{J}^\star(\phi)$ *is such that* $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\phi)) \in dom\ (\phi, w)$*.*

b) if 2) holds, then $\mathcal{T}_{K^\varepsilon}$ *is upper bounded by* $\mathcal{T}^\star(\phi) + \mathcal{J}^\star(\xi)$*, where* $\mathcal{J}^\star(\xi) = ceil(\frac{V(\xi)}{c})$ *and* $\mathcal{T}^\star(\phi)$ *is such that* $(\mathcal{T}^\star(\phi), \mathcal{J}^\star(\xi)) \in dom\ (\phi, w)$ *and* $(\mathcal{T}^\star(\phi), \mathcal{J}^\star(\xi) - 1) \in dom\ (\phi, w)$*.*

Chapter 7

Sufficient conditions for LTL formulas combining operators

In the previous chapters, we provide sufficient conditions for formulas that involve a single temporal operator. Table 7.1 summarizes the conditions for each temporal operator. As indicated therein, all that is needed is either a certificate for finite-time convergence in terms of a Lyapunov function, or the data of the hybrid system and the set of points where the proposition is true to satisfy conditions for invariance. The latter can be actually certified using Lyapunov-like functions or barrier functions as in [24], which for space reasons is not pursued here.

Moreover, the case of logic operators can be treated similarly by using intersections, unions, and complements of the sets where the propositions hold. For instance, sufficient conditions for $\Box(p \wedge q)$ can immediately be derived from the sufficient conditions already given in Chapter 4 with $\{x \in \mathbb{R}^n : p(x) = 1\} \cap \{x \in \mathbb{R}^n : q(x) = 1\}$ in place of $\{x \in \mathbb{R}^n : p(x) = 1\}$.

The following sections present sufficient conditions for formulas that combine more than one operator. The conditions therein are given by compositions of the conditions in Table 7.1.

	Sufficient Conditions
$\Box p$	a) Barrier function for forward pre-invariance
$\Diamond p$	b) Lyapunov function for FTA
$p\mathcal{U}q$	c) Combination of a) and b)
$\bigcirc p$	d) $G(D) \subset D \cap \{x \in \mathbb{R}^n : p(x) = 1\}$

Table 7.1: Sufficient conditions for $\Box$, $\Diamond$, $\mathcal{U}$, $\bigcirc$

7.1 Conditions for $\Diamond\Box$

Corollary 7.1. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and an atomic proposition p. Suppose C is closed relative to $\mathbb{R}^n$, and*

- *Given the atomic proposition p, the set $\{x \in \mathbb{R}^n : p(x) = 1\}$ is closed;*
- *The map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded relative to $\{x \in C : p(x) = 1\}$, and $F(x)$ is convex for every $x \in \{x \in C : p(x) = 1\}$. The map F is locally Lipschitz on $\{x \in C : p(x) = 1\}$; and*
- *There exists an open set $\mathcal{N}$ that defines an open neighborhood of $\{x \in \mathbb{R}^n : p(x) = 1\}$ such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathbb{R}^n$.*

Then, the formula $f = \Diamond\Box p$ is satisfied for all solutions ϕ to $\mathcal{H}$ for all $(t, j) \in \operatorname{dom} \phi$ if the following properties hold:

1) Conditions 1), 2), and 3) in Theorem 5.3 hold; and

2) Condition 1) or condition 2) in Theorem 5.7 holds.

Alternatively, sufficient conditions to guarantee the formula $\Diamond\Box p$ can be obtained by strengthening the Lyapunov conditions in Theorem 5.7.

Corollary 7.2. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and an atomic proposition p. Suppose*

- *Given the atomic proposition p, the set $\{x \in \mathbb{R}^n : p(x) = 1\}$ is closed; and*
- *There exists an open set $\mathcal{N}$ that defines an open neighborhood of $\{x \in \mathbb{R}^n : p(x) = 1\}$ such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathbb{R}^n$.*

Then, the formula $f = \Diamond\Box p$ is satisfied for all solutions ϕ to $\mathcal{H}$ that remain in a compact subset of $\mathcal{N}$ for all $(t, j) \in dom\ \phi$ if the following properties hold:

1) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c, c_1 > 0$, $c_2 \in [0, 1)$ such that

1.1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D)$ such that $p(x) = 0$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies $\frac{V^{1-c_2}(x)}{c_1(1-c_2)} \leq \sup_{(t,j)\in dom\ \phi} t$ and $ceil\left(\frac{V(x)}{c}\right) \leq \sup_{(t,j)\in dom\ \phi} j$;

1.2) the function V is positive definite with respect to K and

1.2a) for each $x \in C \cap \mathcal{N}$, $u_C(x) + c_1 V^{c_2}(x) \leq 0$;

1.2b) for each $x \in D \cap \mathcal{N}$, $u_D(x) \leq -\min\{c, V(x)\}$.

Corollary 7.2 imposes bounds on 1.2a) and 1.2b) for each point where flow and jump is possible, respectively, rather than only when p is not true. Such conditions further guarantee invariance of $\{x \in \mathbb{R}^n : p(x) = 1\}$.

A similar estimate for the time to converge as in Theorem 5.7 holds. Condition 1) in Corollary 7.1 can be alternatively guaranteed with a Lyapunov-like/barrier function as in [24].

Corollary 7.2 requires strict Lyapunov functions, but nonstrict versions as in Theorem 5.7 can be similarly stated.

7.2 Conditions for $\Box\Diamond$

Sufficient conditions to guarantee the formula $f = \Box\Diamond p$ are given by those in Theorem 5.14.

7.3 Conditions for $\square(p\mathcal{U}_s q)$

Sufficient conditions to guarantee the formula $f = \square(p\mathcal{U}_s q)$ are already given by those in Theorems 5.21, 5.22, 5.23, and 5.24.

7.4 Conditions for $p\mathcal{U}_s\square q$

The formula $f = p\mathcal{U}_s\square q$ can be certified by applying one of Theorems 5.21–5.24 and Corollary 7.2 with p therein replaced by q.

Corollary 7.3. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Suppose C is closed in $\mathbb{R}^n$ and*

- *There exists an open set $\mathcal{N}$ defining an open neighborhood of $\{x \in \mathbb{R}^n : q(x) = 1\}$ such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathbb{R}^n$;*
- *The map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded relative to $\{x \in C : p(x) = 1\}$, and $F(x)$ is convex for every $\{x \in C : p(x) = 1\}$. Additionally, the map F is locally Lipschitz on $\{x \in C : p(x) = 1\}$.*

Then, the formula $f = p\mathcal{U}_s\square q$ is satisfied for every solution ϕ to $\mathcal{H}$ if

1) *one of Theorems 5.21–5.24 holds; and*

2) *condition 1.2) in Corollary 7.2 with p therein replaced by q holds.*

7.5 Decomposition of general formulas using finite state automata

In certain cases, formulas that combine more than one operator can be decomposed into simpler formulas for which our results for formulas with a single operator can be applied. To decompose a general formula combining into several formulas with a single operator, one can employ the finite state automaton (FSA) representation of an LTL formula [40–42]. Following [42, Chapter 2], a particular fragment of LTL, called syntactically co-safe LTL (scLTL), is considered so that each formula f over a set of observations can always be translated into an FSA. An LTL formula belongs to the scLTL fragment if it contains only temporal operators $\diamond$, $\bigcirc$, $\mathcal{U}$, and it is written in positive normal form: the negation operator $\neg$ occurs only in front of atomic propositions. Next, given an LTL formula f in the scLTL fragment, we outline the process of constructing an FSA, which we denote A_f, and specify properties of a hybrid system $\mathcal{H}$ with A_f. We first introduce the FSA representation of LTL formulas that belongs to the scLTL fragment.

Definition 7.4 (Finite State Automaton)**.** *Given an scLTL formula f, a finite state automaton (FSA) is given by the tuple $A_f = (S, s_0, O, \delta, S_F)$, where*

- *S is a finite set of states,*
- *$s_0 \in S$ is the initial state,*
- *O is a finite set of observations,*
- *$\delta : S \times O \to S$ is a transition function,*[1]
- *$S_F \subseteq S$ is the set of accepting (final) states.*

The semantics of an FSA are defined over finite words of observations (or inputs). A run of A_f over a word of observations $w_O = w_O(1)w_O(2)\ldots w_O(n)$ with $w_O(k) \in O$ for all $k = 1, \ldots, n$ is a sequence $w_S = w_S(1)w_S(2)\ldots w_S(n+1) \in S$ where $w_S(1) = s_0$ and $w_S(k+1) = \delta(w_S(k), w_O(k))$ for all $k = 1, \ldots, n$. The word

[1]When δ is set valued, namely, $\delta : S \times O \rightrightarrows S$ maps points in $S \times O$ to subsets of S, then A_f is said to be non-deterministic.

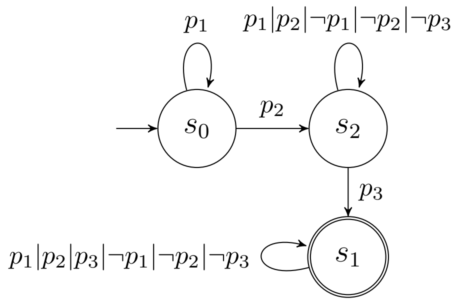


Figure 7.1: An example of an FSA representing the formula $f = \Diamond p_3 \wedge (p_1 \mathcal{U}_s p_2)$. The state s_0 is the initial state and s_1 is the final state. When several transitions are present between two states, one transition labeled by the set of all observations using the symbol $|$ as shown.

w_O is accepted by A_f if the corresponding run ends in an accepting automaton state; i.e., $w_S(n+1) \in S_F$.

With an FSA associated to a general formula f in the scLTL fragment, the tools presented in this paper for the satisfaction of basic formulas having one operator can be applied to certify f. For instance, the formula $f = \Diamond p_3 \wedge (p_1 \mathcal{U}_s p_2)$ has the following associated FSA: $A_f = (S, s_0, O, \delta, S_F)$, where

$$
\begin{aligned}
&S = \{s_0, s_1, s_2\}, \quad S_F = \{s_1\}, \quad O = \{p_1, p_2, p_3, \neg p_1, \neg p_2, \neg p_3\}, \\
&\delta(s, o) = \begin{cases} s_0 & \text{if } s = s_0,\, o = p_1, \\ s_2 & \text{if } s = s_0,\, o = p_2, \\ s_2 & \text{if } s = s_2,\, o \neq p_3, \\ s_1 & \text{if } s = s_2,\, o = p_3, \\ s_1 & \text{if } s = s_1. \end{cases} \qquad \forall (s, o) \in S \times O
\end{aligned}
\tag{7.1}
$$

This FSA is shown in Figure 7.1. As shown therein, the FSA state s is initially at s_0 and when s reaches the final state s_1, it implies that the given formula f is satisfied. As s starts at s_0, we must have that the initial observation o is either $o = p_1$ or $o = p_2$. If it is $o = p_1$, s remains at s_0, but if $o = p_2$, we have a transition from $s = s_0$ to $s = s_2$. Then, once s is at s_2, we have a transition of s from s_2 to s_1 if $o = p_3$. If $o \neq p_3$, s remains at s_2. In other words, the FSA captures the given formula as follows:

1) When s is at s_0, p_2 has to be eventually satisfied and p_1 has to be satisfied until p_2 is satisfied; i.e., $p_1 \mathcal{U}_s p_2$ is satisfied. Once p_2 is satisfied, s jumps to s_2.

2) When s is at s_2, p_3 needs to be eventually satisfied for f to be satisfied; i.e., $\Diamond p_3$ is satisfied. Additionally, once p_3 is satisfied, s jumps to s_1.

To apply our tools, by extending the ideas in [43], we build an augmented version of $\mathcal{H}$, denoted by $\mathcal{H}_A$, with state $(x, s) \in \mathbb{R}^n \times S$ and input $o \in O$ in which s transitions according to the FSA associated with the formula. Its input o, namely, the observation o, is determined by the propositions that are satisfied (or not). For example, when x is such that $p_1(x) = 1$ then $o = p_1$, while when $p_1(x) = 0$ then $o = \neg p_1$. Then, according to our tools, the satisfaction of the formula f is assured by the following conditions:

- Conditions in one of Theorems 5.21–5.24, with q therein replaced by p_2 and with $P = \{(x, s) \in \mathbb{R}^n \times S : p_1(x) = 1, s = s_0\}$ and $Q = \{(x, s) \in \mathbb{R}^n \times S : p_2(x) = 1, s = s_2\}$, are satisfied; and

- $K = \{(x, s) \in \mathbb{R}^n \times S : p_3(x) = 1, s = s_1\}$ is FTA for $\mathcal{H}_A$; namely, conditions in Theorem 5.7, with p therein replaced by p_3 and with set K just defined, are satisfied.

The methodology outlined above can be automated, and is part of current research.

Chapter 8

Signal temporal logic for hybrid dynamical systems

Signal temporal logic (STL) is a simple extension of Metric temporal logic (MTL) where real-valued variables are mapped to Boolean values via predicates. In this section, inspired by the ideas in [11,22,23] for continuous-time and discrete-time systems, we introduce STL for hybrid systems. For a given hybrid system $\mathcal{H}$, we define operators and specify properties of $\mathcal{H}$ with STL formulas.

In the following, the syntax of STL formula φ is defined recursively as follows:

$$\varphi ::= p \,|\, \neg\varphi \,|\, \varphi \vee \psi \,|\, \varphi \mathcal{U}_{[a,b]}, \tag{8.1}$$

where p is an atomic proposition $\mathbb{R}^n \to \{0,1\}$ and φ, ψ are STL formulas. The operators $\neg, \vee, \mathcal{U}$ are the *negation*, *disjunction*, *until* operator, respectively. One can also define operators other than the ones that are used for constructing the grammar. Given the operators *negation* and *disjunction*, the operators *conjunction* ($\wedge$), *implication* ($\Rightarrow$), *equivalency* ($\Leftrightarrow$) are defined as $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$, $\varphi \Rightarrow \psi = \neg\varphi \vee \psi$, $\varphi \Leftrightarrow \psi = (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ respectively. Furthermore, the operators *eventually* ($\Diamond$) and *always* ($\Box$) are defined as $\Diamond_{[a,b]}\varphi = \top\mathcal{U}_{[a,b]}\varphi$ and $\Box_{[a,b]}\varphi = \neg(\Diamond_{[a,b]}\neg\varphi)$, respectively.

A STL formula φ being satisfied by a solution $(t,j) \mapsto x(t,j)$ at some time (t,j) is denoted by

$$(x,(t,j)) \vDash \varphi. \tag{8.2}$$

Let p and q be atomic propositions. Given a solution x to $\mathcal{H}$, $(t,j) \in \mathrm{dom}\ x$, and $\mathcal{I} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, the semantics of STL are defined as

$$(\phi,(t,j)) \vDash \neg p \Leftrightarrow \neg((\phi,(t,j)) \vDash p) \tag{8.3a}$$

$$(\phi,(t,j)) \vDash p \vee q \Leftrightarrow (\phi,(t,j)) \vDash p \vee (\phi,(t,j)) \vDash q \tag{8.3b}$$

$$(\phi,(t,j)) \vDash p \wedge q \Leftrightarrow (\phi,(t,j)) \vDash p \wedge (\phi,(t,j)) \vDash q \tag{8.3c}$$

$$(\phi,(t,j)) \vDash \Diamond_{\mathcal{I}} p \Leftrightarrow \exists (t',j') \in \mathrm{dom}\ \phi : (\phi,(t',j')) \vDash p,\ (t',j') \in \{(t,j)\} + \mathcal{I} \tag{8.3d}$$

$$(\phi,(t,j)) \vDash \Box_{\mathcal{I}} p \Leftrightarrow \forall (t',j') \in \mathrm{dom}\ \phi : (\phi,(t',j')) \vDash p,\ (t',j') \in \{(t,j)\} + \mathcal{I} \tag{8.3e}$$

$$(\phi,(t,j)) \vDash p\mathcal{U}_{s,\mathcal{I}}q \Leftrightarrow \exists (t',j') \in \mathrm{dom}\ \phi : (t',j') \in \{(t,j)\} + \mathcal{I}, (\phi,(t',j')) \vDash q \tag{8.3f}$$

$$\text{and } \forall (t'',j'') \in \mathrm{dom}\ \phi \cap ([t,t'] \times \{j, j+1, \ldots j'\}) : (\phi,(t'',j'')) \vDash p$$

$$(\phi,(t,j)) \vDash p\mathcal{U}_{w,\mathcal{I}}q \Leftrightarrow \forall (t',j') \in \mathrm{dom}\ \phi : (t',j') \in \{(t,j)\} + \mathcal{I}, (\phi,(t',j')) \vDash p \tag{8.3g}$$

$$\text{or } (\phi,(t,j)) \vDash p\mathcal{U}_{s,\mathcal{I}}q.$$

The same semantics of STL are used for formulas. For example, given a STL formula f, when a solution x satisfies $\Diamond_{\mathcal{I}} f$ at $(t,j) \in \mathrm{dom}\ x$ if the formula f holds at some time $(t',j') \in \mathrm{dom}\ x$ such that $(t',j') \in \{(t,j)\} + \mathcal{I}$.

Note that the STL syntax reduces to that of LTL when it is untimed; i.e., $\mathcal{I} = \mathbb{R}_{\geq 0} \times \mathbb{N}$.

8.1 Characterizations of STL Formulas using Dynamical Properties

In the following, we first consider the continuous-time dynamics of $\mathcal{H}$ to illustrate our approach. Consider the continuous-time dynamics of $\mathcal{H}$ given by $\mathcal{H}_f = (C, F)$ as follows:

$$\dot{x} = F(x) \qquad\qquad x \in C \subset \mathbb{R}^n. \tag{8.4}$$

A solution $x : \mathrm{dom}\ x \to \mathbb{R}^n$ to (8.4) is given by a function $t \mapsto x(t)$ satisfying (8.4) for all $t \in \mathrm{dom}\ x$, where $\mathrm{dom}\ x \subset \mathbb{R}_{\geq 0}$ denotes the domain of definition of the solution x. Given an atomic proposition p, we define a set of points that satisfy p given by

$$P := \{x \in \mathbb{R}^n : p(x) = 1\}. \tag{8.5}$$

Consider a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ such that

$$\mathcal{I} := [T_{\min}, T_{\max}], \tag{8.6}$$

where $T_{\min} \geq 0$ and $T_{\max} \geq T_{\min}$.

8.1.1 Characterization of $\Box_{\mathcal{I}}$

With the set P in (8.5) and the set $\mathcal{I}$ in (8.6), given the system in (8.4) and an atomic proposition p, when the formula

$$f = \Box_{\mathcal{I}} p$$

is satisfied for every solution ϕ to (8.4) at $t = 0$, $\phi(t) \in P$ for all $t \in \mathcal{I}$. Here, we consider the following system with the state $(x, \tau) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ given by, with $\tau(0) = 0$,

$$\mathcal{H}_{f,\tau} : \begin{cases} \dot{x} = F(x) \\ \dot{\tau} = 1 \end{cases} \qquad (x, \tau) \in C \times \mathbb{R}_{\geq 0}. \tag{8.7}$$

Note that for each solution pair (φ, τ) to $\mathcal{H}_{f,\tau}$ with $\tau(0) = 0$, a solution component φ is a solution to (8.4). The intuition behind the construction of the system $\mathcal{H}_{f,\tau}$ in (8.7) is as follows. The system $\mathcal{H}_{f,\tau}$ is used to characterize the behavior of solutions ϕ to (8.4) while $t \in \mathcal{I}$. Indeed, with the evolving timer state τ from $\tau(0) = 0$, when $\tau \in \mathcal{I}$, this implies that $t \in \mathcal{I}$. We notice that to satisfy $\Box_{\mathcal{I}} p$ for each solution ϕ to (8.4) at $t = 0$, each solution ψ to $\mathcal{H}_{f,\tau}$ starting from $\mathbb{R}^n \times \{0\}$ satisfies the following properties:

- the solution ψ stays in $\mathbb{R}^n \times [0, T_{\min})$ until reaching $P \times [T_{\min}, T_{\max}]$; and
- once the solution ψ reaches $P \times [T_{\min}, T_{\max}]$, ψ stays in $P \times [T_{\min}, T_{\max}]$ until reaching $\mathbb{R}^n \times (T_{\max}, \infty)$.

The fact that each solution to $\mathcal{H}_{f,\tau}$ starting from $\mathbb{R}^n \times \{0\}$ stays in $P \times [T_{\min}, T_{\max}]$ implies that the solution stays in $P \times [T_{\min}, T_{\max}]$ for each $t \in [T_{\min}, T_{\max}]$; and thus, we conclude that each solution to (8.4) stays in P for each $t \in [T_{\min}, T_{\max}]$.

Hence, the satisfaction of $\Box_{\mathcal{I}} p$ for each solution to (8.4) at $t = 0$ is assured by guaranteeing particular properties of the solutions to $\mathcal{H}_{f,\tau}$ from $\mathbb{R}^n \times \{0\}$ as stated above. For this purpose, given an atomic proposition p and the state

$(x, \tau) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, we define atomic propositions p_a, p_b, and p_c as follows:

$$
\begin{aligned}
p_a(\tau) &:= \begin{cases} 1 & \text{if } \tau \in [0, T_{\min}) \\ 0 & \text{otherwise,} \end{cases} \\
p_b(x, \tau) &:= \begin{cases} 1 & \text{if } p(x) = 1,\ \tau \in [T_{\min}, T_{\max}] \\ 0 & \text{otherwise,} \end{cases} \\
p_c(\tau) &:= \begin{cases} 1 & \text{if } \tau \in (T_{\max}, +\infty) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{8.8}
$$

Theorem 8.1 ($\Box_{\mathcal{I}} p$)**.** *Consider a system in* (8.4)*. Given an atomic proposition p, let P be given as in* (8.5)*. Given a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$, let $T_{\min}$ and $T_{\max}$ be as in* (8.6)*. Let the system $\mathcal{H}_{f,\tau}$ be as in* (8.7) *and atomic propositions p_a, p_b, p_c be as in* (8.8)*. Then, the formula $f = \Box_{\mathcal{I}} p$ is satisfied for every solution to* (8.4) *at $t = 0$ if and only if the formula $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for every solution to $\mathcal{H}_{f,\tau}$ from $\mathbb{R}^n \times \{0\}$ at $t = 0$.*

Proof. ($\Rightarrow$) Suppose that $f = \Box_{\mathcal{I}} p$ is satisfied at $t = 0$ for every solution ϕ to (8.4). We consider a solution pair (ψ, τ) to (8.7) such that $\psi(0) = \phi(0)$ and $\tau(0) = 0$ and $\psi(t) = \phi(t)$ for all $t \in \operatorname{dom} \psi$. In fact, such a solution pair (ψ, τ) always exists since both the systems (8.4) and (8.7) have the same flow set and flow map. Moreover, we note that $\tau(t) = t$ for all $t \in \operatorname{dom} \psi$ since $\tau(0) = 0$. By definition of $\Box_{\mathcal{I}}$ operator, when $f = \Box_{\mathcal{I}} p$ is satisfied at $t = 0$ for every solution ϕ to (8.4), $\phi(t) \in P$ for all $t \in \mathcal{I} = [T_{\min}, T_{\max}]$, which implies (ψ, τ) satisfies

1) for every $t \in \operatorname{dom} \phi$ such that $t < T_{\min}$, $\tau(t) \in [0, T_{\min})$;

2) for every $t \in \operatorname{dom} \phi$ such that $t \in \mathcal{I} = [T_{\min}, T_{\max}]$, $\psi(t) \in P$ when $\tau(t) \in \mathcal{I}$;

3) for every $t \in \operatorname{dom} \phi$ such that $t > T_{\max}$, $\tau(t) \in (T_{max}, +\infty)$.

That is, items 1) and 2) implies that (ψ, τ) satisfies p_a until satisfying p_b; and items 2) and 3) implies that (ψ, τ) satisfies p_b until satisfying p_c. Thus, we conclude that $\tilde{f} = p_a \mathcal{U}_s(p_b \mathcal{U}_s p_c)$ is satisfied for every solution to (8.7) at $t = 0$ with $\tau(0) = 0$.

($\Leftarrow$) Suppose that $\tilde{f} = p_a \mathcal{U}_s(p_b \mathcal{U}_s p_c)$ is satisfied for every solution to (8.7) at $t = 0$ with $\tau(0) = 0$. We show that, for each solution ϕ to (8.4), ϕ stays in P for all $t \in \mathcal{I} = [T_{\min}, T_{\max}]$. Let (ψ, τ) be a solution pair to (8.7) such that $\tau(0) = 0$ and $\psi(t) = \phi(t)$ for all $t \in \operatorname{dom} \phi$. The solution component ψ is a solution ϕ to (8.4) since the systems (8.4) and (8.7) share the same flow set and flow map; and we note that $\tau(t) = t$ for all $t \in \operatorname{dom} \phi$. Since $\tilde{f} = p_a \mathcal{U}_s(p_b \mathcal{U}_s p_c)$ is satisfied for each solution (ψ, τ) to (8.7) at $t = 0$ with $\tau(0) = 0$, by definition of $\mathcal{U}_s$ operator,

- when (ψ, τ) does not satisfy $p_b \mathcal{U}_s p_c$, (ψ, τ) satisfy p_a, which implies that $\tau \in [0, T_{\min})$;
- when (ψ, τ) satisfies $p_b \mathcal{U}_s p_c$, (ψ, τ) satisfies p_b until satisfying p_c; namely, $(\psi, \tau) \in P \times \mathcal{I}$ until $\tau \in (T_{\max}, \infty)$.

Hence, we conclude that each solution ϕ to (8.4) such that $\phi(t) = \psi(t)$ satisfies p for all $t \in \mathcal{I}$, which implies that $f = \Box_{\mathcal{I}} p$ is satisfied for every solution ϕ to (8.4) at $t = 0$. $\square$

In the following, we propose characterization of the satisfaction of $\Box_{\mathcal{I}} p$ using conditions that guarantee the satisfaction of the strong until ($\mathcal{U}_s$) in Section 5.4.

Consider the system $\mathcal{H}_{f,\tau}$ in (8.7) and atomic propositions p_a, p_b, p_c in (8.8), let the sets P_a, P_b, and P_c be as in (8.5) while replacing p therein by the atomic propositions p_a, p_b, and p_c, respectively. Following (4.10), let the system $\mathcal{S}_m$ be given by

$$\mathcal{S}_m : \begin{cases} \dot{x} = F(x) \\ \dot{\tau} = 1 \end{cases} \qquad (x, \tau) \in (C \times \mathbb{R}_{\geq 0}) \setminus (P_b \cup P_c). \tag{8.9}$$

The system $\mathcal{S}_m$ is what we refer to as the modified version of (8.4), which is used to characterize the behavior of (8.4) outside the set $P_b \cup P_b$. Here, the solutions to (8.4) are the solutions to $\mathcal{S}_m$ (and vice versa) up to when they reach (if they do) the set $P_b \cup P_c$. Moreover, we consider the system $\mathcal{S}_m^{'}$ by following (4.11), as the restricted system of $\mathcal{S}_m$, given by

$$\mathcal{S}_m^{'} : \begin{cases} \dot{x} = F(x) \\ \dot{\tau} = 1 \end{cases} \qquad (x, \tau) \in ((C \times \mathbb{R}_{\geq 0}) \cap P_a) \setminus (P_b \cup P_c). \tag{8.10}$$

Using conditions in Section 5.4, the satisfaction of the formula $p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is verified.

Theorem 8.2. *Consider a system in* (8.4)*. Given an atomic proposition p and a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$, let $T_{\min}$ and $T_{\max}$ be as in* (8.6) *and let the system $\mathcal{H}_{f,\tau}$ be as in* (8.7) *and atomic proposition p_a, p_b, and p_c be as in* (8.8)*. Let the sets P_a, P_b, and P_c be as in* (8.5) *while replacing p therein by p_a, p_b, and p_c, respectively. Then, the formula $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for every solution to $\mathcal{H}_{f,\tau}$ starting from $\mathbb{R}^n \times \{0\}$ at $t = 0$ if*

1.a) $(P_a \cup P_b \cup P_c)$ is conditionally invariant with respect to P_a for $\mathcal{S}_m$; and

1.b) $(P_b \cup P_c)$ is eventually conditionally invariant with respect to P_a for $\mathcal{S}_m^{'}$ or $(P_b \cup P_c)$ is finite-time attractive with respect to P_a for $\mathcal{S}_m^{'}$.

and

2.a) $(P_b \cup P_c)$ is conditionally invariant with respect to P_b for $\mathcal{S}_m$; and

2.b) P_c is eventually conditionally invariant with respect to P_b for $\mathcal{S}_m^{'}$ or P_c is finite-time attractive with respect to P_b for $\mathcal{S}_m^{'}$.

Remark 8.3. *Given sets P_a, P_b and P_c which are defined by atomic proposition p_a, p_b, and p_c in* (8.8)*, when conditions in Theorem 8.2 hold, the formula $p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for every solution to $\mathcal{H}_{f,\tau}$ starting from $\mathbb{R}^n \times \{0\}$ at $t = 0$, which implies that $f = \Box_{\mathcal{I}} p$ is satisfied for every solution to* (8.4) *at $t = 0$.*

Furthermore, the satisfaction of the formula $f = \Box_{\mathcal{I}} p$ for each solution to (8.4) at each $t \geq 0$ implies that the set P in (8.5) is forward pre-invariant for (8.4) after $t' = t + T_{\min}$.

Corollary 8.4. *Consider the system $\mathcal{H}_f$ in* (8.4)*. Given an atomic proposition p, let P be given as in* (8.5)*. Given a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$, let $T_{\min}$ and $T_{\max}$ be as in* (8.6)*. Let the system $\mathcal{H}_{f,\tau}$ be as in* (8.7)*. Then, the formula $f = \Box_{\mathcal{I}} p$ is satisfied for every solution to* (8.4) *at each $t \geq 0$ if $P \times [T_{\min}, \infty)$ is eventually forward pre-invariant for $\mathcal{H}_{f,\tau}$.*

Proof. Suppose that $P \times [T_{\min}, \infty)$ is eventually forward pre-invariant for $\mathcal{H}_{f,\tau}$. We show that, at every $t \geq 0$, for each solution ϕ to $\mathcal{H}_f$, ϕ stays in P for all $t' \in [t + T_{\min}, t + T_{\max}]$. Let (ψ, τ) be a solution to $\mathcal{H}_{f,\tau}$ such that $\psi(t) = \phi(t)$ for all $t \geq 0$ with $\tau(0) = 0$; such a solution satisfies $\tau(t) = t$ since $\tau(0) = 0$. Hence, since $P \times [T_{\min}, \infty)$ is eventually forward pre-invariant for $\mathcal{H}_{f,\tau}$, we conclude that $\psi(t) \in P$ for all $t \in [T_{\min}, \infty)$ while $\tau(t) \in [T_{\min}, \infty)$. Therefore, at every $t \geq 0$, $\phi(t') \in P$ for all $t' \in [t + T_{\min}, t + T_{\max}]$, which completes the proof. $\square$

8.1.2 Characterization of $\Diamond_{\mathcal{I}}$

With the set P in (8.5), given the system in (8.4), an atomic proposition p and a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$, when the formula

$$f = \Diamond_{\mathcal{I}} p$$

is satisfied for every solution ϕ to (8.4) at $t = 0$, there exists $t \in \mathcal{I}$ such that $\phi(t) \in P$. The system $\mathcal{H}_{f,\tau}$ in (8.7) is used to characterize the behavior of solutions ϕ to (8.4) while $t \in \mathcal{I}$. We notice that to satisfy $\Diamond_{\mathcal{I}} p$ for each solution ϕ to (8.4) at $t = 0$, each solution ψ to $\mathcal{H}_{f,\tau}$ in (8.7) starting from $\mathbb{R}^n \times \{0\}$ satisfies the following properties, with $T_{\min} = \min \mathcal{I}$, $T_{\max} = \max \mathcal{I}$ and the set P in (8.5):

- the solution ψ stays in $\mathbb{R}^n \times [0, T_{\min})$ until reaching $\mathbb{R}^n \times [T_{\min}, T_{\max}]$; and
- once the solution ψ reaches $\mathbb{R}^n \times [T_{\min}, T_{\max}]$, ψ stays in $\mathbb{R}^n \times [T_{\min}, T_{\max}]$ until reaching $P \times [T_{\min}, T_{\max}]$.

Now, we redefine atomic propositions p_b and p_c in (8.8) as follows:

$$\begin{aligned} p_b(\tau) &:= \begin{cases} 1 & \text{if } \tau \in \mathcal{I} \\ 0 & \text{otherwise,} \end{cases} \\ p_c(x, \tau) &:= \begin{cases} 1 & \text{if } x \in P,\ \tau \in \mathcal{I} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.11}$$

Theorem 8.5 ($\Diamond_{\mathcal{I}} p$)**.** *Consider a system in* (8.4)*. Given an atomic proposition p, let P be given as in* (8.5)*. Let the system $\mathcal{H}_{f,\tau}$ be as in* (8.7)*. Given a set $\mathcal{I} \subset \mathbb{R}_{\geq 0}$, let $T_{\min}$ be as in* (8.6) *and let an atomic proposition p_a be as in* (8.8) *and atomic propositions p_b and p_c be as in* (8.11)*. Then, the formula $f = \Diamond_{\mathcal{I}} p$ is satisfied for every solution to* (8.4) *at $t = 0$ if and only if the formula $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for every solution to $\mathcal{H}_{f,\tau}$ from $\mathbb{R}^n \times \{0\}$ at $t = 0$.*

Proof. ($\Rightarrow$) Suppose that $f = \Diamond_{\mathcal{I}} p$ is satisfied at $t = 0$ for every solution x to (8.4). We consider a solution pair (ψ, τ) to (8.7) such that $\psi(0) = x(0)$ and $\tau(0) = 0$ and $\psi(t) = x(t)$ for all $t \in \operatorname{dom} \psi$. In fact, such a solution pair (ψ, τ) always exists since both the systems (8.4) and (8.7) have the same flow set and flow map. Moreover, we note that $\tau(t) = t$ for all $t \in \operatorname{dom} \psi$ since $\tau(0) = 0$. By definition of

$\diamond_{\mathcal{I}}$ operator, when $f = \diamond_{\mathcal{I}} p$ is satisfied at $t = 0$ for every solution x to (8.4), there exists $t \in \mathcal{I} = [T_{\min}, T_{\max}]$ such that $x(t) \in P$. This implies that (ψ, τ) satisfies

1) for every $t \in \text{dom } x$ such that $t < T_{\min}$, $\tau(t) \in [0, T_{\min})$;

2) for every $t \in \text{dom } x$ such that $t \in \mathcal{I}$, $\tau(t) \in \mathcal{I}$; and there exists $t' \in \mathcal{I}$ such that $x(t') \in P$ and $\tau(t') \in \mathcal{I}$.

This implies that (ψ, τ) satisfies p_a until satisfying p_b, and (ψ, τ) satisfies p_b until satisfying p_c. Thus, we conclude that $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for every solution to (8.7) at $t = 0$ with $\tau(0) = 0$.

($\Leftarrow$) Suppose that $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for every solution to (8.7) at $t = 0$ with $\tau(0) = 0$. We show that, for each solution x to (8.4), there exists $t \in \mathcal{I}$ such that $x(t) \in P$. Let (ψ, τ) be a solution pair to (8.7) such that $\tau(0) = 0$ and $\psi(t) = x(t)$ for all $t \in \text{dom } x$. The solution component ψ is a solution x to (8.4) since the systems (8.4) and (8.7) share the same flow set and flow map; and we note that $\tau(t) = t$ for all $t \in \text{dom } x$. Since $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for each solution (ψ, τ) to (8.7) at $t = 0$ with $\tau(0) = 0$, by definition of $\mathcal{U}_s$ operator,

- when (ψ, τ) does not satisfy $p_b \mathcal{U}_s p_c$, (ψ, τ) satisfy p_a, which implies that $\tau \in [0, T_{\min})$;

- when (ψ, τ) satisfies $p_b \mathcal{U}_s p_c$, (ψ, τ) satisfies p_b until satisfying p_c; namely, $(\psi, \tau) \in \mathbb{R}^n \times \mathcal{I}$ until $(x, \tau) \in P \times \mathcal{I}$.

Hence, we conclude that each solution x to (8.4) such that $x(t) = \psi(t)$ satisfies p for all $t \in \mathcal{I}$, which implies that $f = \diamond_{\mathcal{I}} p$ is satisfied at $t = 0$ for every solution x to (8.4). $\square$

Remark 8.6. *Given sets P_a, P_b and P_c which are defined by an atomic proposition p_a in* (8.8) *and atomic propositions p_b and p_c in* (8.11)*, when conditions in*

Theorem 8.2 hold, the formula $p_a \mathcal{U}_s(p_b \mathcal{U}_s p_c)$ is satisfied for every solution to $\mathcal{H}_{f,\tau}$ starting from $\mathbb{R}^n \times \{0\}$ at $t = 0$, which implies that $f = \Diamond_{\mathcal{I}} p$ is satisfied for every solution to (8.4) at $t = 0$.

8.1.3 In hybrid systems

In the following, we consider a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (1.1). Here, we consider the hybrid system $\mathcal{H}_\tau = (C_\tau, F_\tau, D_\tau, G_\tau)$ with the state $(x, \tau, k) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{N}$ given by

$$\begin{aligned} F_\tau(x, \tau, k) &:= \begin{bmatrix} F(x) \\ 1 \\ 0 \end{bmatrix} \quad \forall x \in C_\tau := C \times \mathbb{R}_{\geq 0} \times \mathbb{N} \\ G_\tau(x, \tau, k) &:= \begin{bmatrix} G(x) \\ \tau \\ k+1 \end{bmatrix} \quad \forall x \in D_\tau := D \times \mathbb{R}_{\geq 0} \times \mathbb{N}. \end{aligned} \tag{8.12}$$

Note that for each solution pair (φ, τ, k) to $\mathcal{H}_\tau$ with $\tau(0, 0) = 0$ and $k(0, 0) = 0$, the solution φ is a solution to $\mathcal{H}$. Consider a set $\mathcal{I} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that

$$\mathcal{I} := [T_{\min}, T_{\max}] \times \{J_{\min}, J_{\min} + 1, \ldots, J_{\max}\}, \tag{8.13}$$

where $T_{\min}, J_{\min} \geq 0$, $T_{\max} \geq T_{\min}$, and $J_{\max} \geq J_{\min}$. The satisfaction of $\square_{\mathcal{I}} p$ for each solution to $\mathcal{H}$ at $(t, j) = (0, 0)$ is assured by guaranteeing particular properties of the solutions to $\mathcal{H}_\tau$ from $\mathbb{R}^n \times \{0\} \times \{0\}$ as follows:

- the solution ψ to $\mathcal{H}_\tau$ stays in $\mathbb{R}^n \times [0, T_{\min}) \times \{0, 1, \ldots, J_{\min} - 1\}$ until reaching $P \times [T_{\min}, T_{\max}] \times \{J_{\min}, J_{\min} + 1, \ldots, J_{\max}\}$; and
- once ψ reaches $P \times [T_{\min}, T_{\max}] \times \{J_{\min}, J_{\min} + 1, \ldots, J_{\max}\}$, ψ stays in

$P \times [T_{\min}, T_{\max}] \times \{J_{\min}, J_{\min}+1, \ldots, J_{\max}\}$ until reaching $\mathbb{R}^n \times [T_{\max}, \infty) \times \{J_{\max}, J_{\max}+1, \ldots, \infty\}$.

Now, we redefine atomic propositions p_a, p_b, and p_c in (8.8) as follows:

$$\begin{aligned}
p_a(\tau, k) &:= \begin{cases} 1 & \text{if } \tau \in [0, T_{\min}),\ k \in \{0, 1, \ldots, J_{\min}-1\} \\ 0 & \text{otherwise,} \end{cases} \\
p_b(x, \tau, k) &:= \begin{cases} 1 & \text{if } x \in P, \tau \in [T_{\min}, T_{\max}], \\ & \quad k \in \{J_{\min}, J_{\min}+1, \ldots, J_{\max}\} \\ 0 & \text{otherwise,} \end{cases} \\
p_c(\tau, k) &:= \begin{cases} 1 & \text{if } \tau > T_{\max},\ k > J_{\max} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{8.14}$$

Theorem 8.7 ($\Box_{\mathcal{I}} p$)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Given an atomic proposition p, let P be given as in* (8.5)*. Given a set $\mathcal{I} \subset \mathbb{R}^n \times \mathbb{N}$, let $T_{\min}$, $T_{\max}$, $J_{\min}$, and $J_{\max}$ be as in* (8.13)*. Let the system $\mathcal{H}_\tau$ be as in* (8.12) *and atomic propositions p_a, p_b, and p_c be as in* (8.14)*. Then, the formula $f = \Box_{\mathcal{I}} p$ is satisfied for every solution to $\mathcal{H}$ at $(t, j) = (0, 0)$ if and only if*

- *the formula $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ is satisfied for each solution to $\mathcal{H}_\tau$ from $\mathbb{R}^n \times \{0\} \times \{0\}$ at $(t, j) = (0, 0)$.*

The following example illustrates Theorem 8.7.

Example 8.8 (Thermostat)**.** *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ with the state $x := (h, z) \in \{0, 1\} \times \mathbb{R}$ in Example 4.13. Following the formulation therein, a specification of interest is that the room temperature maintains within $[z_{\min}, z_{\max}]$ during the first 60 seconds while avoiding a lot of switching off and on the heater (i.e., less than three times), which is related to the satisfaction of $\Box_{\mathcal{I}} p$*

for $\mathcal{H}$ with the atomic proposition p as

$$p(x) := \begin{cases} 1 & \textit{if } x \in \{0,1\} \times [z_{\min}, z_{\max}] \\ 0 & \textit{otherwise,} \end{cases}$$

for each $x \in \mathbb{R}^n$ where $\mathcal{I} = [0, 60] \times \{0, 1, 2\}$; namely, $T_{\min}$, $T_{\max}$, $J_{\min}$, and $J_{\max}$ in (8.13) are 0, 60, 0, and 2, respectively. Then, the set P in (8.5) is given by $P = \{0, 1\} \times [z_{\min}, z_{\max}]$; and the system $\mathcal{H}_\tau$ is given as in (8.12). Since $T_{\min} = 0$, the propositions p_b and p_c in (8.14) are defined as

$$p_b(x, \tau, k) := \begin{cases} 1 & \textit{if } x \in P, \tau \in [0, 60],\ k \in \{0, 1, 2\} \\ 0 & \textit{otherwise,} \end{cases}$$

$$p_c(\tau, k) := \begin{cases} 1 & \textit{if } \tau > 60, k > 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Finally, the satisfaction of $\Box_{\mathcal{I}} p$ for $\mathcal{H}$ at $(t, j) = (0, 0)$ is translated into the satisfaction of the formula $p_b \mathcal{U}_s p_c$ for $\mathcal{H}$ from $(\{0, 1\} \times \mathbb{R}) \times \{0\} \times \{0\}$ at $(t, j) = (0, 0)$. $\triangle$

We redefine atomic proposition p_b and p_c in (8.14) as follows: atomic propositions p_a, p_b, p_c are given by

$$\begin{aligned} p_b(\tau, k) &:= \begin{cases} 1 & \text{if } \tau \in [T_{\min}, T_{\max}], \\ & \quad k \in \{J_{\min}, J_{\min} + 1, \dots, J_{\max}\} \\ 0 & \text{otherwise,} \end{cases} \\ p_c(x, \tau, k) &:= \begin{cases} 1 & \text{if } x \in P, \tau \in [T_{\min}, T_{\max}], \\ & \quad k \in \{J_{\min}, J_{\min} + 1, \dots, J_{\max}\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.15}$$

Theorem 8.9 ($\Diamond_{\mathcal{I}} p$)**.** *Consider a hybrid system* $\mathcal{H} = (C, F, D, G)$*. Given an atomic proposition* p*, let* P *be given as in* (8.5)*. Given a set* $\mathcal{I} \subset \mathbb{R}^n \times \mathbb{N}$*, let* $T_{\min}$*,* $T_{\max}$*,* $J_{\min}$*, and* $J_{\max}$ *be as in* (8.13)*. Let the system* $\mathcal{H}_\tau$ *be as in* (8.12)*. Let an atomic proposition* p_a *be as in* (8.14) *and atomic propositions* p_b, p_c *be as* (8.15)*. Then, the formula* $f = \Diamond_{\mathcal{I}} p$ *is satisfied for every solution to* $\mathcal{H}$ *at* $(t, j) = (0, 0)$ *if and only if*

- *the formula* $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ *is satisfied for each solution to* $\mathcal{H}_\tau$ *from* $\mathbb{R}^n \times \{0\} \times \{0\}$ *at* $(t, j) = (0, 0)$*.*

Corollary 8.10. *Using Theorem 8.2, the formula* $\tilde{f} = p_a \mathcal{U}_s (p_b \mathcal{U}_s p_c)$ *is satisfied for every solution to* $\mathcal{H}_\tau$ *starting from* $\mathbb{R}^n \times \{0\} \times \{0\}$ *at* $(t, j) = (0, 0)$ *if*

1.a) $(P_a \cup P_b \cup P_c)$ *is conditionally invariant with respect to* P_a *for* $\mathcal{H}_w$ *in* (4.10)*; and*

1.b) $(P_b \cup P_c)$ *is eventually conditionally invariant with respect to* P_a *for* $\mathcal{H}_s$ *in* (4.11) *or* $(P_b \cup P_c)$ *is finite-time attractive with respect to* P_a *for* $\mathcal{H}_s$ *in* (4.11)*.*

and

2.a) $(P_b \cup P_c)$ *is conditionally invariant with respect to* P_b *for* $\mathcal{H}_w$ *in* (4.10)*; and*

2.b) P_c *is eventually conditionally invariant with respect to* P_b *for* $\mathcal{H}_s$ *in* (4.11) *or* P_c *is finite-time attractive with respect to* P_b *for* $\mathcal{H}_s$ *in* (4.11)*.*

Chapter 9

Object grasping with multiple contact points

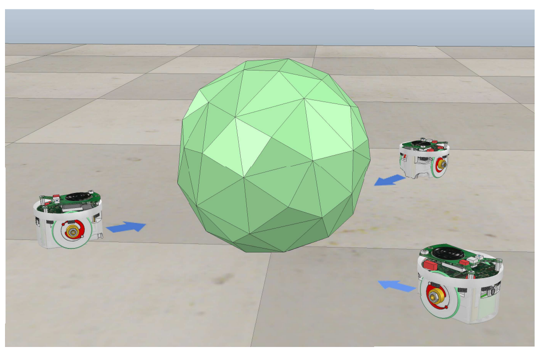


Figure 9.1: Grasping an object with multiple agents, in this case, given by ground vehicles. Each vehicle establishes contact at the desired locations to grasp the object, and after that, may steer the object to a different location. A hybrid control controller guarantees that the vehicles establish contact simultaneously without rebounding.

In this chapter, a grasping task involving multiple contacts is considered as an application of hybrid systems. Building from the control strategy in [44], a hybrid control approach is presented for grasping objects by multiple agents without rebounding. When multiple agents grasp an object cooperatively, the motion of the agents is constrained due to the geometrical and frictional conditions at the contact points. Each agent acting on an object of interest is controlled by a hybrid controller which includes a position controller, a force controller, and some logic to coordinate grasping. The proposed approach provides a method to steer the agents to the desired grasping positions on an object along the desired direction and to asymptotically exert the desired forces at each contact point.

In particular, we study the networked hybrid system which is described by a multi-agent system that consists of multiple subsystems. Each subsystem, a hybrid closed-loop system corresponding to each agent, is commanded by an individual feedback controller which are coordinated by a supervisory controller.

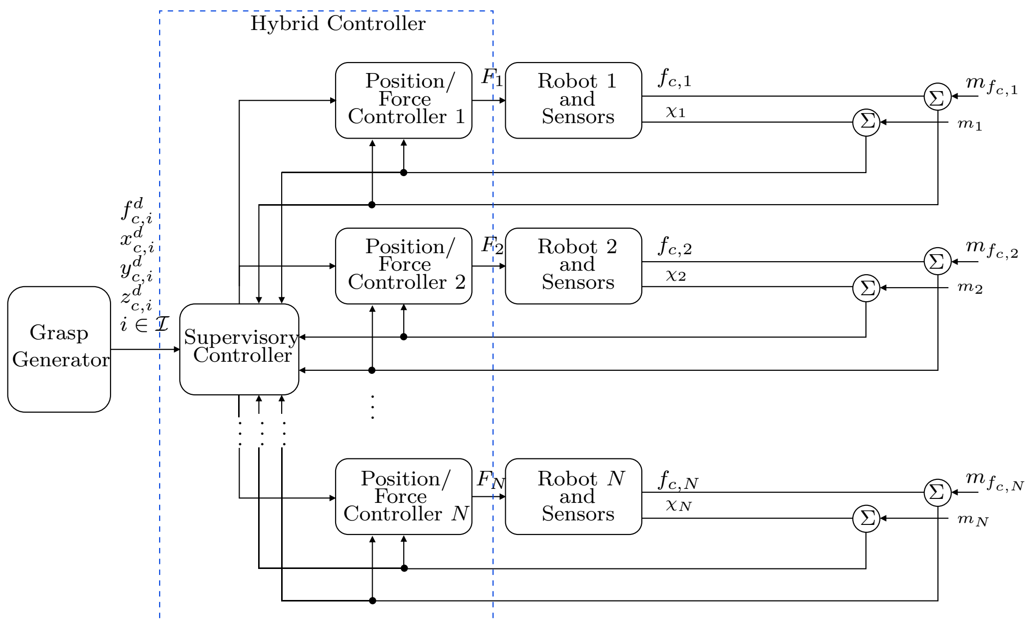


Figure 9.2: Example of networked hybrid systems. Hybrid control architecture with N individual position/force controllers, the supervisor, and the grasp generator. For each i, the signals $m_{f_{c,i}}$ and m_i are noises affecting the measurements of $f_{c,i}$ and z_i, respectively.

9.1 Object Grasping Problem

We consider the problem of grasping an object at N contact points using the same number of end effectors or agents evolving in space, as shown in Figure 9.1 for $N = 3$ agents evolving on the (x, y)-plane. Each contact point and associated force to exert, denoted $(x^d_{c,i}, y^d_{c,i}, z^d_{c,i})$ and $f^d_{c,i}$ for each $i \in \mathcal{I} := \{1, 2, \dots, N\}$, respectively, will serve as the reference to each of the agents. The problem to solve is as follows:

Problem: Given an object to grasp, N agents to achieve contact with the object, and, for each $i \in \mathcal{I}$, contact positions $(x^d_{c,i}, y^d_{c,i}, z^d_{c,i})$ and desired contact forces $f^d_{c,i}$, design an algorithm to guarantee that the N agents establish simultaneous contact at points nearby $(x^d_{c,i}, y^d_{c,i}, z^d_{c,i})$ without rebounding and, after that, asymptotically exert force $f^d_{c,i}$ for each $i \in \mathcal{I}$. □

The proposed approach to provide a solution to object grasping problem is to treat the grasping task involving multiple contact points as a multi-agent system in which each agent is commanded by an individual feedback controller, which, in turn, are coordinated by a (hybrid) supervisory algorithm. More precisely, a hybrid closed-loop system corresponding to agent i has state ξ_i and dynamics of the form

$$\mathcal{H}_i \left\{ \begin{array}{rcll} \dot{\xi}_i & = & F_i(\xi_i) & \xi_i \in C_i \\ \xi_i^+ & = & G_i(\xi_i) & \xi_i \in D_i \end{array} \right. \qquad i \in \mathcal{I}, \tag{9.1}$$

and the resulting system consists of N hybrid systems coordinated to perform the desired grasping task.

Figure 9.2 depicts the proposed control architecture. To solve the stated problem, we develop a hybrid controller supervising individual controllers in each agent, namely, position controllers and force controllers. Using the information provided by a grasp generator, the main task of each such controller is to first regulate position, so as to steer the agent to nearby the contact point simultaneously, and then regulate both position and force, so as to keep the vehicle nearby the contact point and exert the force needed to establish a stable grasp without rebounding. The supervisor employs the output of an algorithm providing the contact points and force needed to establish a stable grasp, which in Figure 9.2 corresponds to the grasp generator. Furthermore, when small perturbations are present in the system, which may trigger events of the supervisor at different time instances, contact with the object occur at times and at points that are nearby to those in the nominal conditions, and the resulting forces remain close to those determined by the grasp generator. The proposed logic does not incorporate avoidance strategies between the agents and the objects but that is part of future work.

9.1.1 Agents model

We consider agents with dynamics in joint space given by

$$\mathcal{M}(\mathfrak{q})\ddot{\mathfrak{q}} + \mathcal{C}(\mathfrak{q}, \dot{\mathfrak{q}})\dot{\mathfrak{q}} + \mathcal{N}(\mathfrak{q}, \dot{\mathfrak{q}}) = \mathcal{T} - \mathcal{J}^\top(\mathfrak{q}) f_c \tag{9.2}$$

where $\mathcal{M}$ is the manipulator inertia matrix, $\mathcal{C}$ is the Coriolis matrix, $\mathcal{N}$ includes gravity terms and other forces that act at the joints, $\mathcal{T}$ is the vector of the actuators torques, $\mathcal{J}$ is the Jacobian matrix relating the joint space velocity to the workspace velocity, and f_c is the vector of the contact forces due to the interaction between the manipulator and the environment. As the interest is in the interaction of N agents with an object, for each $i \in \mathcal{I}$, (9.2) is rewritten in the workspace coordinates $\chi_i = (x_i, y_i, z_i)$, after a coordinate transformation from $\mathfrak{q}$ to χ_i, which results in [45]

$$\widetilde{\mathcal{M}}_i \ddot{\chi}_i + \widetilde{\mathcal{C}}_i \dot{\chi}_i + \widetilde{\mathcal{N}}_i = F_i - f_{c,i} \tag{9.3}$$

where $\widetilde{\mathcal{M}}_i$, $\widetilde{\mathcal{C}}_i$, and $\widetilde{\mathcal{N}}_i$ (for simplicity their arguments are not included) are obtained from the matrices in joint space (namely, $\mathcal{M}_i$, $\mathcal{C}_i$, and $\mathcal{N}_i$; cf. (9.2)), and F_i is the vector of forces/torques applied at the end-effector of the i-th agent.

9.1.2 Object model

The surface of the object to grasp is assumed to be soft and so that both the tangent plane and the normal can be defined at the contact points – these points are obtained from an algorithm that calculates a stable grasp; see Section 9.1.4. The object is defined by the set of points given by

$$\mathcal{W} = \{\chi : s(\chi) \leq 0\}$$

where s is a function that is smooth enough and χ are the workspace coordinates.

9.1.3 Contact force model

To characterize the relationship between the bodies' penetration and the reaction force involved in the end effector-object interaction, we employ the so-called Kelvin-Voigt linear contact model. In such a model the viscoelastic material of the environment is described by the dynamics of a linear spring with stiffness k_c and damper coefficient b_c. The contact force $f_{c,i}$ is then given as follows [46]:

$$f_{c,i}(\chi_i, \dot{\chi}_i) = k_c \chi_{\ell,i} + b_c \dot{\chi}_{\ell,i} \tag{9.4}$$

when contact between the end-effector and the object occurs, and zero otherwise. The states $\chi_{\ell,i}$ and $\dot{\chi}_{\ell,i}$ are the compression distance and the compression velocity along the direction of contact, respectively – we use subindex ℓ to denote such local coordinates. Figure 9.3(a) depicts such local coordinates on the (x, y)-plane: red and green lines on the desired contact point refer to $x_{\ell,i}$-direction and $y_{\ell,i}$-direction, respectively.

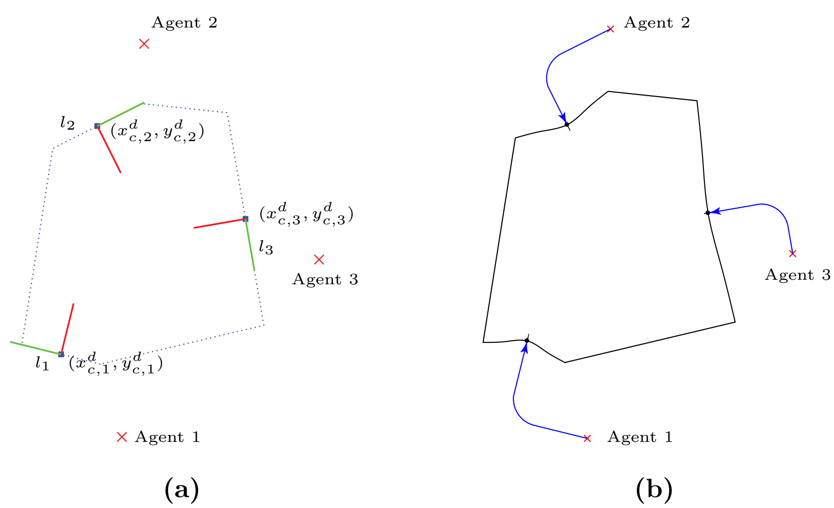


Figure 9.3: Example of grasping task with three agents (i.e., $N = 3$) on the (x, y)-plane. (a) For each agent i, the desired contact point $(x^d_{c,i}, y^d_{c,i})$ is given by the grasp generator. Red and green lines on the desired contact point refer to $x_{\ell,i}$-direction and $y_{\ell,i}$-direction, respectively. (b) Position trajectory of each agent until the contact force is stabilized.

9.1.4 Stable grasp generator

Given the object to grasp as defined in Section 9.1.2, we choose the contact points and forces in the workspace

$$(x^d_{c,i}, y^d_{c,i}, z^d_{c,i}) \in \mathbb{R}^3, \qquad f^d_{c,i} \in \mathbb{R}^3 \qquad \forall i \in \mathcal{I} \tag{9.5}$$

that guarantee a stable grasp.

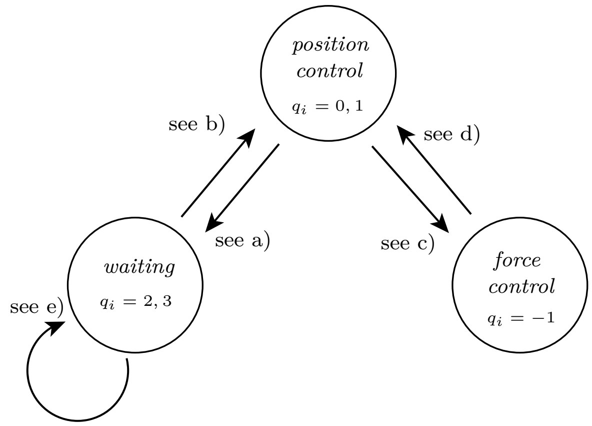


Figure 9.4: Modes of operation of the supervisory controller for agent i, $i \in \mathcal{I}$. The logic includes two different phases in *position control mode* and in *waiting mode* – see a)-e) list in Section 9.2.

The force balance equations for a grasped object subject to the contact force $f_{c,i}$ with a set of N contact points can be described as follows:

$$w = Wf \tag{9.6}$$

where w is the resulting wrench, $W = [W_1 \ \cdots \ W_N]$ such that

$$W_i = \begin{bmatrix} n_i & s_i & 0 \\ (x^d_{c,i}, y^d_{c,i}, z^d_{c,i}) \times n_i & (x^d_{c,i}, y^d_{c,i}, z^d_{c,i}) \times s_i & n_i \end{bmatrix}, \tag{9.7}$$

for each $i \in \mathcal{I}$; $f = (f^d_{c,1}, \ldots, f^d_{c,N})$, and n_i and s_i are the normal and tangent to the object at i-th contact point, respectively. A solution (9.5) to (9.6) with $w = 0$ defines a stable grasp, where G is determined by the contact force and object models.

There exist various approaches to optimize the placement of grasp points. In this paper, we use the method of maximizing grasp quality in [47]. Using the Ferrari-Canny metric, the most commonly used metric as a grasp quality evaluation, an optimal grasp minimizing the magnitudes has been chosen among multiple different grasps satisfying (9.5) with $w = 0$. We refer the reader to [48] for more details of the choice of the contact points and forces.

9.2 Hybrid Controller for Synchronized Grasping

The proposed controller is hybrid due to the combination of state variables that change continuously and, at times, jump [20,39]. To coordinate each of the agents, the hybrid controller implements a supervisory logic that employs a controllable decreasing timer variable $\tau_i \in \mathbb{R}_{\geq 0}$ and a logic variable $q_i \in Q := \{-1, 0, 1, 2, 3\}$ for each $i \in \mathcal{I}$ agent. The timer state is used to schedule the steering of the agents so as to make contact with the object simultaneously. The five possible values of the logic variable represent the different modes of operation and phases therein – these are defined in the enumerated list below; see also Figure 9.4.

Also for each agent, the hybrid controller includes a position controller and a force controller for the purposes of controlling the position of the agents and the force exerted to the object. The position controller steers the agent to contact. The force controller employs measurements of the contact force in the direction of motion.

To design these control algorithms, we follow the approaches in [49–51]. First, we design the following inner feedback loop that compensates for the internal and external forces of the manipulator, but certainly does not overcome the contact force:

$$F_i = u_i + \widetilde{\mathcal{C}}_i \dot{\chi}_i + \widetilde{\mathcal{N}}_i \tag{9.8}$$

where u_i is a new virtual control input. Then, as in [50], without loss of generality, we focus on the case in which the interaction between the agent and its environment occurs along a normal direction to the object, namely, the interaction between the agent and the object happens at a point on the line with that normal direction. We refer to this line as *the interaction line.* In this way, we further assume that the mass is unitary. Then, the dynamics of the agent along *the interaction line* is given as follows:

$$\ddot{\chi}_i = u_i - f_{c,i}(\chi_i, \dot{\chi}_i). \tag{9.9}$$

Using the contact force model in Section 9.1.3, once an agent reaches the surface of the object, the contact force is calculated based on the compression distance and velocity, respectively, for which, when focusing on the interaction between the agents and the object along the interaction line, results in $f_{c,i}$ in (9.4) being a scalar quantity given in local coordinates $x_{\ell,i}$ and $\dot{x}_{\ell,i}$; namely, when contact occurs,

$$f_{c,i}(\chi_i, \dot{\chi}_i) = k_c x_{\ell,i} + b_c \dot{x}_{\ell,i} \tag{9.10}$$

Note that the $x_{\ell,i}$-direction spans the interaction line as it is defined as the direction of the contact force; see Figure 9.3(a).

Given the contact points and forces in (9.5) from the grasp generator, and assuming that the agents start far enough away from the object, the proposed supervisory logic is as follows:

1) The position controller initially steers the agents to (nearby) the interaction line. During this phase the agent is in *position control mode*, for which $q_i = 0$.

2) When the agent position is close enough (characterized by the parameter $\varepsilon > 0$) to the interaction line, the agent enters into *waiting mode*, for which

$q_i = 2$. In this mode the agent holds its position.

3) When all agents are in *waiting mode*, all logic variables are set to value 3 and the travel time of each agent to make contact with the object is computed by solving the closed-loop system dynamics; see Section 9.3. Then, the appropriate waiting duration for each agent is calculated by resetting τ_i to a nonnegative value chosen to guarantee that all agents establish contact with the object simultaneously. The decreasing timer counts down as long as it is nonnegative. The logic variable q_i of the agents in this phase remains at 3.

4) When τ_i reaches zero, the logic variable is reset to $q_i = 1$, and the agent enters another *position control mode* phase, but now the agent is directly steered towards the object along the intersection line.

5) When the contact force $f_{c,i}$ is larger or equal than a certain threshold (denoted γ_2^i), we set $q_i = -1$ to put the agent in *force control mode*. The force controller is activated and the contact force is regulated to the magnitude of $f_{c,i}^d$. A switch back to the position controller is only possible when the contact force has decreased enough (characterized by a parameter γ_1^i, which is positive and strictly smaller than γ_2^i) – this hysteresis mechanism assures that rebounding does not occur and provides robustness to small perturbations; see [44] for more details.

As a result, all agents make contact with the object simultaneously and maintain contact at nearby the desired location without rebounds.

The logic outlined above can be modeled as a hybrid control algorithm given in terms of differential and difference equations with constraints. With such a model, the tools for stability analysis in [20, 39] are applied. The state of the control algorithm is given by

$$\eta = (q_1, \tau_1, q_2, \tau_2, \ldots, q_N, \tau_N)$$

and its input is

$$u_c = (\chi_1, f_{c,1}, \chi_2, f_{c,2}, \ldots, \chi_N, f_{c,N})$$

plus the measurement noise signal

$$m = (m_1, m_{f_{c,1}}, m_2, m_{f_{c,2}}, \ldots, m_N, m_{f_{c,N}})$$

as show in Figure 9.2. Next, we provide the differential and difference equations, along with the constraints, for each agent $i \in \mathcal{I}$. Below, l_i denotes the i-th interaction line.

Flows: The continuous change of the logic variable q_i is given by the trivial differential equation

$$\dot{q}_i = 0 \tag{9.11}$$

which always keeps the logic variables constant in the continuous-time regime. The timer variable τ_i continuously decrements itself according to the differential equation

$$\dot{\tau}_i = -1 \tag{9.12}$$

when $q_i = 3$ and $\tau_i \geq 0$ and, for any other values of q_i and τ_i, τ_i changes trivially according to

$$\dot{\tau}_i = 0 \tag{9.13}$$

Jumps: The jumps of the hybrid controller update the variables q_i and τ_i so as to implement the logic above. These updates are instantaneous and governed by the following difference equations with constraints:

a) From *position control mode* to *waiting mode*:

$$q_i^+ = 2, \qquad \tau_i^+ = \tau_i \tag{9.14}$$

when $q_i = 0$ and $\mathrm{dist}(l_i, \chi_i) \leq \varepsilon$

b) From *waiting mode* to *position control mode*:

b1) to steering toward the object (first phase):

$$q_i^+ = 1, \qquad \tau_i^+ = \tau_i \tag{9.15}$$

when $q_i = 3,\ \mathrm{dist}(l_i, \chi_i) \leq \varepsilon$ and $\tau_i \leq 0$,

b2) to steering back to nearby the line (second phase):

$$q_i^+ = 0, \qquad \tau_i^+ = \tau_i \tag{9.16}$$

when $q_i \in \{2, 3\}$ and $\mathrm{dist}(l_i, \chi_i) \geq \varepsilon$

c) From *position control mode* to *force control mode*:

$$q_i^+ = -1, \qquad \tau_i^+ = \tau_i \tag{9.17}$$

when $q_i = 1$ and $f_{c,i} \geq \gamma_2^i$

d) From *force control mode* to *position control mode*:

$$q_i^+ = 1, \qquad \tau_i^+ = \tau_i \tag{9.18}$$

when $q_i = -1$ and $f_{c,i} \leq \gamma_1^i$

e) From *waiting mode* (first phase) to *waiting mode* (second phase) when all agents are nearby their respective interaction line:

$$
\begin{aligned}
q_i^+ &= 3 \\
\tau_i^+ &= T_i(\chi_i)
\end{aligned}
\tag{9.19a}
$$

when

$$
q_k = 2 \text{ for all } k \in \mathcal{I} \tag{9.19b}
$$

where T_i is the waiting time.

The output of the hybrid controller assigns the virtual input u_i (see (9.8)) of the agents as follows:

$$
u_i = \begin{cases} \kappa_P(\chi_i) & \text{if } q_i \in \{0, 1\} \\ \kappa_F(\chi_i, f_{c,i}) & \text{if } q_i = -1 \\ (0, 0) & \text{if } q_i \in \{2, 3\} \end{cases} \tag{9.20}
$$

where κ_P is the position controller and κ_F is the force controller. Note that when the agent is in waiting mode, its input is identically zero so as to wait at the current location. The time to reach the surface of the object, namely, T_i, can be analytically computed once the position controller is designed. More details on how to design such feedback laws using Lyapunov theory are provided in Section 9.4.

9.3 Hybrid Closed-loop System for Synchronized Grasping

In the following, the state of each individual agent is denoted $\eta_i \in \mathbb{R}^3$ and represents position and velocity in the local frame.[1] For agent $i \in \mathcal{I}$, the states $\eta_{i,1}$ and $\eta_{i,2}$ are the position and velocity in the $x_{\ell,i}$-direction, and $\eta_{i,3}$ is the position in the $y_{\ell,i}$-direction. As stated in Section 9.2, the i-th hybrid controller employs τ_i and q_i to implement a supervisory logic. The logic therein leads to an i-th hybrid closed-loop system $\mathcal{H}_i = (C_i, F_i, D_i, G_i)$ as in (9.1) with state

$$\xi_i := (\eta_i, \tau_i, q_i) \in Z := \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \times Q$$

and dynamics given by

$$\left.\begin{array}{rcl} \dot{\eta}_{i,1} & = & \eta_{i,2} \\ \dot{\eta}_{i,2} & = & u_{x,i} \\ \dot{\eta}_{i,3} & = & u_{y,i} \\ \dot{\tau}_i & = & 0 \\ \dot{q}_i & = & 0 \end{array}\right\} \quad \text{if } \xi_i \in C_i^0 \cup C_i^1 \cup C_i^2 \cup C_i^3 \tag{9.21a}$$

$$\left.\begin{array}{rcl} \dot{\eta}_{i,1} & = & \eta_{i,2} \\ \dot{\eta}_{i,2} & = & 0 \\ \dot{\eta}_{i,3} & = & 0 \\ \dot{\tau}_i & = & -(q_i - 2) \\ \dot{q}_i & = & 0 \end{array}\right\} \quad \text{if } \xi_i \in C_i^4 \cup C_i^5 \tag{9.21b}$$

[1] For simplicity, we write it in local coordinates. The global coordinates case requires replacing η_i by $\Phi(\eta_i)$; see Appendix D.3 for more details.

and

$$
\begin{array}{llll}
\eta_i^+ = \eta_i, & \tau_i^+ = 0, & q_i^+ = -1 & \text{if } \xi_i \in D_i^0 \\
\eta_i^+ = \eta_i, & \tau_i^+ = 0, & q_i^+ = 1 & \text{if } \xi_i \in D_i^1 \cup D_i^5 \\
\eta_i^+ = \eta_i, & \tau_i^+ = 0, & q_i^+ = 0 & \text{if } \xi_i \in D_i^2 \\
\eta_i^+ = \eta_i, & \tau_i^+ = 0, & q_i^+ = 2 & \text{if } \xi_i \in D_i^3 \\
\eta_i^+ = \eta_i, & \tau_i^+ = T_i(\eta_i), & q_i^+ = 3 & \text{if } \xi_i \in D_i^4,
\end{array}
\tag{9.22}
$$

where $u_i := (u_{x,i}, u_{y,i})$ is given by

$$
u_{x,i} = \begin{cases} 0, & \text{if } y_{\ell,i} \text{ position is controlled} \\ k_p^i(x_{\ell,i}^d - \eta_{i,1}) - k_d^i \eta_{i,2}, & \text{if } x_{\ell,i} \text{ position is controlled} \\ k_f^i(|f_{c,i}^d| - f_{c,i}) + f_{c,i}, & \text{if force control is applied,} \end{cases}
$$

$$
u_{y,i} = \begin{cases} k_{p,y}^i(y_{\ell,i}^d - \eta_{i,3}), & \text{if } y_{\ell,i} \text{ position is controlled} \\ 0 & \text{otherwise.} \end{cases}
$$

The virtual control inputs $u_{x,i}$ and $u_{y,i}$, which result from applying (9.8) and decoupling horizontal and vertical motion, are assigned to position and force feedback controllers. Mass is assumed to be unitary for simplicity. The parameters k_p^i, k_d^i are the proportional and derivative feedback gains of the x_i position controller, respectively, and $k_{p,y}^i$ is the proportional feedback gain of the y_i position controller. The parameters $x_{\ell,i}^d$ and $y_{\ell,i}^d$ are the desired position in the $x_{\ell,i}$-direction and $y_{\ell,i}$-direction, repsectively. The parameter f_c^d is the desired contact force.

The flow set is $C_i := \{(\xi_i, \{\eta_{j,3}, q_j\}_{j \in \mathcal{I} \setminus \{i\}}) \in Z \times \mathbb{R}^{N-1} \times Q^{N-1} : \xi_i \in C_i^k \ \forall k \in \mathcal{K}, (\xi_i, \{\eta_{j,3}, q_j\}_{j \in \mathcal{I} \setminus \{i\}}) \in C_i^4\}$ where $\mathcal{K} := \{0, 1, 2, 3, 5\}$,

$$
C_i^0 := \{\xi_i \in Z : q_i = 1, \tau_i = 0, f_c(\eta_{i,1}, \eta_{i,2}) \le \gamma_2^i\}
$$

$$
C_i^1 := \{\xi_i \in Z : q_i = -1, \tau_i = 0, f_c(\eta_{i,1}, \eta_{i,2}) \ge \gamma_1^i\}
$$

$$
\begin{aligned}
C_i^2 &:= \{\xi_i \in Z : q_i = 0, \tau_i = 0, \eta_{i,1} \leq x_{\ell,i}^*, |\eta_{i,3} - y_{\ell,i}^d| \geq \varepsilon\} \\
C_i^3 &:= \{\xi_i \in Z : q_i = 1, |\eta_{i,3} - y_{\ell,i}^d| \leq \varepsilon, \forall j \in \mathcal{I}\} \\
C_i^4 &:= \{(\xi_i, \{\eta_{j,3}, q_j\}_{j\in\mathcal{I}\setminus\{i\}}) \in Z \times \mathbb{R}^{N-1} \times Q^{N-1} : q_i = 2, \\
&\quad |\eta_{i,3} - y_{\ell,i}^d| \leq \varepsilon, (q_j \neq 2 \wedge |\eta_{j,3} - y_{\ell,j}^d| \geq \varepsilon), \exists j \in \mathcal{I} \setminus \{i\}\} \\
C_i^5 &:= \{\xi_i \in Z : q_i = 3, \tau_i \geq 0, |\eta_{i,3} - y_{\ell,i}^d| \leq \varepsilon\}, \qquad (9.23)
\end{aligned}
$$

where γ_1^i, γ_2^i and ε are the thresholds. The controller parameter $x_{\ell,i}^*$ denotes the minimum position along the $x_{\ell,i}$-direction.[2] The jump set is $D_i := \{(\xi_i, \{\eta_{j,3}, q_j\}_{j\in\mathcal{I}\setminus\{i\}}) \in Z \times \mathbb{R}^{N-1} \times Q^{N-1} :\ \xi_i \in D_i^k\, \forall k \in \mathcal{K}, (\xi_i, \{\eta_{j,3}, q_j\}_{j\in\mathcal{I}\setminus\{i\}}) \in D_i^4\}$,

$$
\begin{aligned}
D_i^0 &:= \{\xi_i \in Z : q_i = 1, \tau_i = 0, f_c(\eta_{i,1}, \eta_{i,2}) \geq \gamma_2^i\} \\
D_i^1 &:= \{\xi_i \in Z : q_i = -1, \tau_i = 0, f_c(\eta_{i,1}, \eta_{i,2}) \leq \gamma_1^i\} \\
D_i^2 &:= \{\xi_i \in Z : q_i \neq 0, \tau_i = 0, \eta_{i,1} \leq x_{\ell,i}^*, |\eta_{i,3} - y_{\ell,i}^d| \geq \varepsilon\} \\
D_i^3 &:= \{\xi_i \in Z : q_i = 0, \tau_i = 0, |\eta_{i,3} - y_{\ell,i}^d| \leq \varepsilon\} \\
D_i^4 &:= \{(\xi_i, \{\eta_{j,3}, q_j\}_{j\in\mathcal{I}\setminus\{i\}}) \in Z \times \mathbb{R}^{N-1} \times Q^{N-1} : q_j = 2, \\
&\qquad\qquad |\eta_{j,3} - y_{\ell,j}^d| \leq \varepsilon, \forall j \in \mathcal{I}\} \\
D_i^5 &:= \{\xi_i \in Z : q_i = 3, \tau_i = 0, |\eta_{i,3} - y_{\ell,i}^d| \leq \varepsilon\}. \qquad (9.24)
\end{aligned}
$$

With these definitions, the flow map is defined for every $\xi_i \in C_i$, which is given by

$$
F_i(\xi_i) = \begin{cases} F_i^0(\xi_i) & \text{if } \xi_i \in C_i^0 \cup C_i^1 \cup C_i^2 \cup C_i^3 \\ F_i^1(\xi_i) & \text{if } \xi_i \in C_i^4 \cup C_i^5 \end{cases} \qquad (9.25a)
$$

[2]See Appendix D.1 for more details.

where

$$F_i^0(\xi_i) = [\eta_{i,2} \quad u_{x,i} \quad u_{y,i} \quad 0 \quad 0]^\top, \qquad\qquad (9.25b)$$
$$F_i^1(\xi_i) = [\eta_{i,2} \quad 0 \quad 0 \quad -(q_i - 2) \quad 0]^\top$$

and the jump map is given by

$$G_i(\xi_i) = \begin{cases} G_i^0(\xi_i) & \text{if } \xi_i \in D_i^0 \\ G_i^1(\xi_i) & \text{if } \xi_i \in D_i^1 \cup D_i^5 \\ G_i^2(\xi_i) & \text{if } \xi_i \in D_i^2 \\ G_i^3(\xi_i) & \text{if } \xi_i \in D_i^3 \\ G_i^4(\xi_i) & \text{if } \xi_i \in D_i^4 \end{cases} \qquad \forall \xi_i \in D_i \qquad (9.26a)$$

where

$$G_i^0(\xi_i) = \begin{bmatrix} \eta_i \\ 0 \\ -1 \end{bmatrix}, \quad G_i^1(\xi_i) = \begin{bmatrix} \eta_i \\ 0 \\ 1 \end{bmatrix}, \quad G_i^2(\xi_i) = \begin{bmatrix} \eta_i \\ 0 \\ 0 \end{bmatrix},$$
$$G_i^3(\xi_i) = \begin{bmatrix} \eta_i \\ 0 \\ 2 \end{bmatrix}, \quad G_i^4(\xi_i) = \begin{bmatrix} \eta_i \\ T_i(\eta_i) \\ 3 \end{bmatrix}. \qquad (9.26b)$$

9.4 Design

The state of each individual agent is denoted $\eta_i \in \mathbb{R}^3$ and represents position and velocity in the local frame.[3] For agent $i \in \mathcal{I}$, the states $\eta_{i,1}$ and $\eta_{i,2}$ are the position and velocity in the $x_{\ell,i}$-direction, and $\eta_{i,3}$ is the position in the $y_{\ell,i}$-

[3]For simplicity, we write it in local coordinates. The global coordinates case requires replacing η_i by $\Phi(\eta_i)$ where $\Phi(\eta_i)$ is a transformation involving both rotation and translation of the coordinates.

direction. As stated in Section 9.2, the i-th hybrid controller employs τ_i and q_i to implement a supervisory logic. The logic therein leads to an i-th hybrid closed-loop system $\mathcal{H}_i = (C_i, F_i, D_i, G_i)$ as in (9.1) with state $\xi_i := (\eta_i, \tau_i, q_i) \in Z := \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \times Q$. The controller parameters k_p^i, k_d^i are the proportional and derivative feedback gains of the $x_{\ell,i}$ position controller, respectively, and $k_{p,y}^i$ is the proportional feedback gain of the $y_{\ell,i}$ position controller. The parameter k_f^i is the proportional feedback gain of the force controller.

For each $i \in \mathcal{I}$, we define $\mathcal{A}_i := (x_{\ell,i}^F, 0, y_{\ell,i}^d)$ where $x_{\ell,i}^F = f_{c,i}^d / k_c$. Then, given parameters $k_c, b_c \in (0, +\infty)$ of the work environment and desired contact force $0 < f_{c,i}^d < \hat{f}_{c,i}$ where $\hat{f}_{c,i}$ is the maximum allowed force, one can always find

1. compact sets $K_{0,i}, K_{1,i}, K_{2,i} \subset \mathbb{R}^3$,

2. parameters $k_p^i, k_d^i, k_{p,y}^i, k_f^i, \gamma_1^i, \gamma_2^i, x_{\ell,i}^d$ of the hybrid controller

such that the set $\mathcal{A}_i \times \{0\} \times \{-1\}$ is locally asymptotically stable with basin of attraction containing $((K_{0,i} \times \{0\} \times \{1\}) \cup (K_{1,i} \times \{0\} \times \{0\}) \cup (K_{2,i} \times \{0\} \times \{-1\}))$ for $\mathcal{H}_i$.

In fact, a particular choice of these sets is

- $K_{0,i} = (L_{V_1}(r_1) \cap \{\eta_i \in \mathbb{R}^3 : \eta_{i,1} \leq 0\}) \cup (L_{V_2}(r_2) \cap \{\eta_i \in \mathbb{R}^3 : \eta_{i,1} \geq 0\})$ where $x_{\ell,i}^d, k_p^i, k_d^i > 0$, $V_1(\eta_{i,1}, \eta_{i,2}) = \frac{1}{2}a_1(\eta_{i,1} - x_{\ell,i}^d)^2 + \frac{1}{2}b_1\eta_{i,2}^2$ with a_1, b_1 satisfying $\frac{a_1}{b_1} = k_p^i$, and $V_2(\eta_{i,1}, \eta_{i,2}) = \frac{1}{2}a_2(\eta_{i,1} - x_{\ell,i}^P)^2 + \frac{1}{2}b_2\eta_{i,2}^2$ with a_2, b_2 satisfying $\frac{a_2}{b_2} = k_p^i + k_c$ where $x_{\ell,i}^P := \frac{k_p^i}{k_p^i + k_c} x_{\ell,i}^d$; r_1 and r_2 are the maximum value of the level set of V_1 and V_2, respectively, and the following conditions are satisfied: the r_1-level set of V_1 intersects the point $\eta_{i,1} = 0$, $\eta_{i,2} = \eta_{i,2}^* - \delta$ and $r_2 = \min\{r_2^a, r_2^b\}$, where the r_2^a-level set of V_2 is such that it crosses the intersection of the r_F-level set and the min γ_2^i line, and the r_2^b-level set of V_2 is such that it intersects the point $\eta_{i,1} = 0$, $\eta_{i,2} = \eta_{i,2}^* - \delta$, where $\eta_{i,2}^*$ is a bounded maximum value of the velocity with $\delta > 0$;

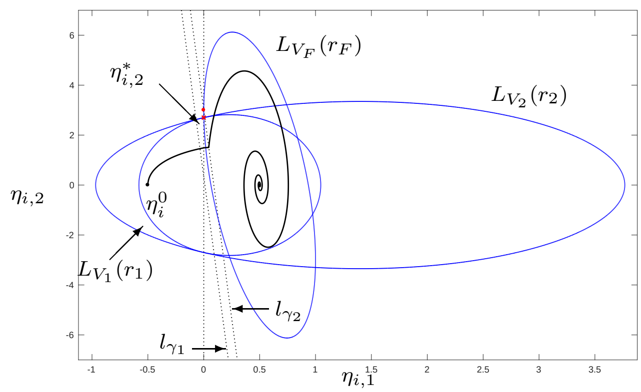


Figure 9.5: Example of sublevel sets of Lyapunov functions. $\eta_i^0 := (\eta_{i,1}^0, \eta_{i,2}^0)$ is the initial point. $\eta_{i,2}^*$ is the maximum impact velocity. The lines l_{γ_1} and l_{γ_2} are $l_{\gamma_1} := \{(\eta_{i,1}, \eta_{i,2}) : \eta_{i,2} = -\frac{k_c}{b_c}\eta_{i,1} + \frac{\gamma_1}{b_c}\}$ and $l_{\gamma_1} := \{(\eta_{i,1}, \eta_{i,2}) : \eta_{i,2} = -\frac{k_c}{b_c}\eta_{i,1} + \frac{\gamma_1}{b_c}\}$.

- $K_{1,i} = L_{V_3}(r_3)$ where $V_3(\eta_{i,3}) = \frac{1}{2}a_3(\eta_{i,3} - y_{\ell,i}^d)^2, a_3 > 0, k_{p,y}^i > 0$, and r_3 is the maximum value of the level set of V_3 when V_3 is at $\eta_{i,3} = |y_{\ell,i}^d| + d_i^* - \delta'$ where d_i^* is a maximum allowed distance $\text{dist}(l_i, \xi_i)$ with $\delta' > 0$;

- $K_{2,i} = L_{V_F}(r_F)$ where $V_F(\eta_{i,1}, \eta_{i,2}) := a(\eta_{i,1} - x_{\ell,i}^F)^2 + b\eta_{i,2}^2 + 2c(\eta_{i,1} - x_{\ell,i}^F)\eta_{i,2}$, and $P_F := \begin{bmatrix} a & c \\ c & d \end{bmatrix} = R\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}R^\top$, $p_1, p_2 > 0, R := \begin{bmatrix} -\sin\beta & -\cos\beta \\ \cos\beta & -\sin\beta \end{bmatrix}$, and $\beta := \arctan(-k_c/b_c)$, $k_f^i \in \Big(0, \frac{-2c^2k_c + abk_c + acb_c}{(bk_c - cb_c)^2}\Big)$; r_F is the maximum value of the level set of V_F when V_F is at $\eta_{i,1} = 0$, $\eta_{i,2} = \frac{c}{b}x_{\ell,i}^F$ where $x_{\ell,i}^F := f_{c,i}^d/k_c$; and

the parameters in 2) can be chosen as follows: $\gamma_{1,\min}^i = 0$, $\gamma_{1,\max}^i = x_{\ell,i}^F\Bigg(k_c - \sqrt{\frac{k_c^2 b - 2ck_c b_c + ab_c^2}{b}}\Bigg)$, $\gamma_{2,\min}^i = b_c\frac{c}{b}x_{\ell,i}^F$, $\gamma_{2,\max}^i = k_c \min\{\frac{k_p}{k_p+k_c}x_{\ell,i}^d, x_{\ell,i}^F\}$, $x_{\ell,i}^d \in [x_{\ell,i_{\min}}^d, +\infty]$ where $x_{\ell,i_{\min}}^d = x_{\ell,i}^F b_c \frac{c}{b}\frac{k_p+k_c}{k_p k_c}$. Since the hybrid closed-loop system satisfies the hybrid basic conditions (see [20]), we can find $\beta \in \mathcal{KL}$ such that for each $\epsilon > 0$ and each compact set $K_{i,0}, K_{i,1}, K_{i,2} \subset \mathbb{R}^3$ such that $((K_{0,i} \times \{0\} \times \{1\}) \cup$

$(K_{1,i} \times \{0\} \times \{0\}) \cup (K_{2,i} \times \{0\} \times \{-1\}))$ is a subset of the basin of attraction of $\mathcal{H}_i$, there exists $\delta^* > 0$ such that for each position and force measurement noise $m : \mathbb{R}_{\geq 0} \to \delta^*\mathbb{B}$, solutions ξ_i to $\mathcal{H}_i$ with noise m for initial conditions $z_i^0 \in ((K_{0,i} \times \{0\} \times \{1\}) \cup (K_{1,i} \times \{0\} \times \{0\}) \cup (K_{2,i} \times \{0\} \times \{-1\}))$ are such that the η_i component of the solutions satisfy $|\eta_i(t,j)|_{\mathcal{A}} \leq \beta(|\eta_i^0|_{\mathcal{A}}, t+j) + \epsilon$ for each $(t,j) \in \operatorname{dom} \xi_i$. Figure 9.5 shows an example of sublevel sets of Lyapunov functions above.

Theorem 9.1. *$\mathcal{H}_i$ satisfies the hybrid basic conditions, which are as follows:*

A1) C_i and D_i are closed sets in Z.

A2) F_i: $Z \to Z$ is continuous on C_i.

A3) G_i: $Z \to Z$ is an outer semicontinuous and locally bounded relative to D_i, and $D_i \subset domG_i$.

Proof. Condition (A1) is satisfied since C_i and D_i are closed. The flow map F_i in (D.7) is continuous on C_i, satisfying (A2). The jump map G_i in (D.8) is single valued on D_i and therefore it satisfies (A3). □

9.5 LTL specifications

In the following, we present how express system specifications as LTL formulas. For example, consider a horizontal position controller that is designed for controlling the horizontal position of the agent as follows:

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = -k_p\eta_1 - k_d\eta_2 + k_p x_\ell^d$$

where $k_p, k_d > 0$ are controller parameters. A specification of interest is that the position of the agent eventually reaches the equilibrium point $(x_\ell^d, 0)$ in finite time.

This can be expressed as

$$f = \Diamond p$$

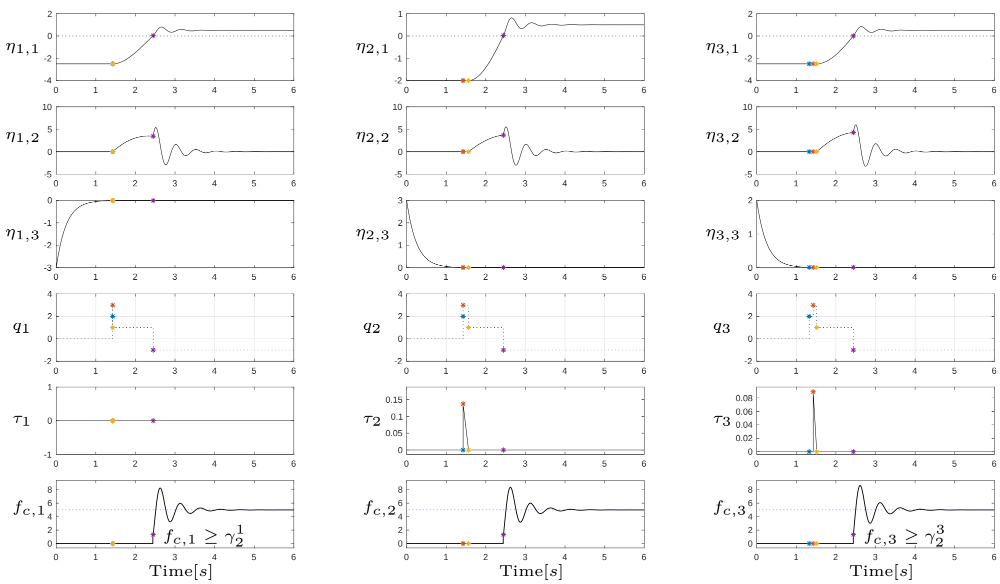
with the proposition p such that p is true when $(\eta_1, \eta_2) = (x_\ell^d, 0)$.

9.6 Nominal case

Now, we illustrate the design conditions above. We consider the task of grasping an object with multiple contact points. The proposed hybrid system is simulated with MATLAB using the Hybrid Equation (HyEQ) Toolbox [52]. The simulation results show how the proposed controller stabilizes the horizontal and vertical positions and ensures contact force regulation in the multi-agent systems. In the following simulation, we apply the proposed hybrid controller in Section 9.2 for $N = 3$ to grasp an object defined on the (x, y)-plane as a polygon with vertices (in clockwise order) given by $\{(-1.89, -4.95)$, $(-4.99, -4.22)$, $(-3.70, 3.26)$, $(-0.31, 4.96)$, $(2.94, 4.62)$, $(4.34, -3.48)\}$; see Figure 9.3. The parameters k_c and b_c are set to 10 and 0.3, respectively. For each $i \in \mathcal{I} := \{1, 2, 3\}$, the gains k_f^i, k_p^i and $k_{p,y}^i$ are set to 16.0, 2.0 and 4.0, respectively; the set of gains (k_d^1, k_d^2, k_d^3) is set to $(1, 0.5, 0.5)$. The desired contact forces $f_{c,i}^d$ obtained from the stable grasp generator are 4.39, 4.2 and 2.97, respectively, at each contact point. The thresholds γ_1^i and γ_2^i are chosen as 0.76 and 1.33, respectively. The thresholds ε and ε' are set to 0.01 and 0.05, respectively. The initial condition of each agent is $(\eta_{1,1}^0, \eta_{1,2}^0, \eta_{1,3}^0) = (-0.5, 0, 0)$, $(\eta_{2,1}^0, \eta_{2,2}^0, \eta_{2,3}^0) = (-0.5, 0, 1)$, $(\eta_{3,1}^0, \eta_{3,2}^0, \eta_{3,3}^0) = (1, 0, 0)$, respectively.

Figure 9.6 illustrates the closed-loop trajectory of the three agents obtained from the simulation. The vertical position controller is initially applied (i.e., $q_i = 0$). Approximately 1.5 seconds later, when $\text{dist}(l_i, \xi_i) = |\eta_{i,3} - y_{\ell,i}^d| \leq \varepsilon$, the horizontal position controller is applied (i.e., $q_i = 1$). When the contact force $f_{c,i} \geq \gamma_2^i$, the force controller is activated (i.e., $q_i = -1$). The contact force is regulated to $f_{c,i}^d$ by using the force controller. As the plots of the contact forces in Figure 9.6 indicate, the proposed hybrid controller guarantees that the agents do not bounce off the surface of the object after contact.

As shown in Figure 9.7, at approximately 2.5 seconds, all three agents make contact with the object (i.e., $\{x_{i,1}\}_{i=1}^3 = 0$) simultaneously. A movie of this simulation is available at `https://youtu.be/6B8m584u-g4`.



(a) Agent 1 **(b)** Agent 2 **(c)** Agent 3

Figure 9.6: Grasping task with three agents: Plots of state variables of agent 2 in its local frame.

Figure 9.8 shows the trajectories with respect to the local coordinates for different environments. After each agent makes contact with the object (i.e., $x_{\ell,i} = 0$), it maintains its contact with the object while the position of each agent is stabilized. Note that the different environment material stiffness k_c, the equilibrium point $x_{\ell,i}^F$ is changed according to $x_{\ell,i}^F = |f_{c,i}^d|/k_c$ so as to exert the desired contact force magnitudes of $f_{c,i}^d$.

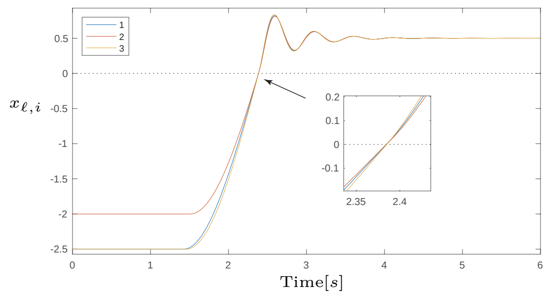


Figure 9.7: Plots of x_ℓ position corresponding to each agent.

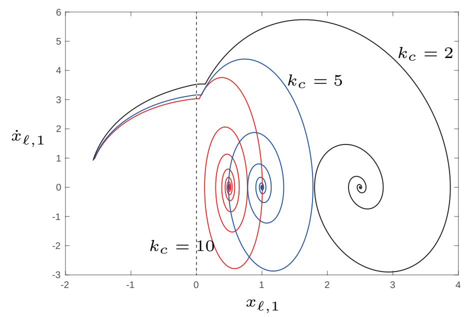
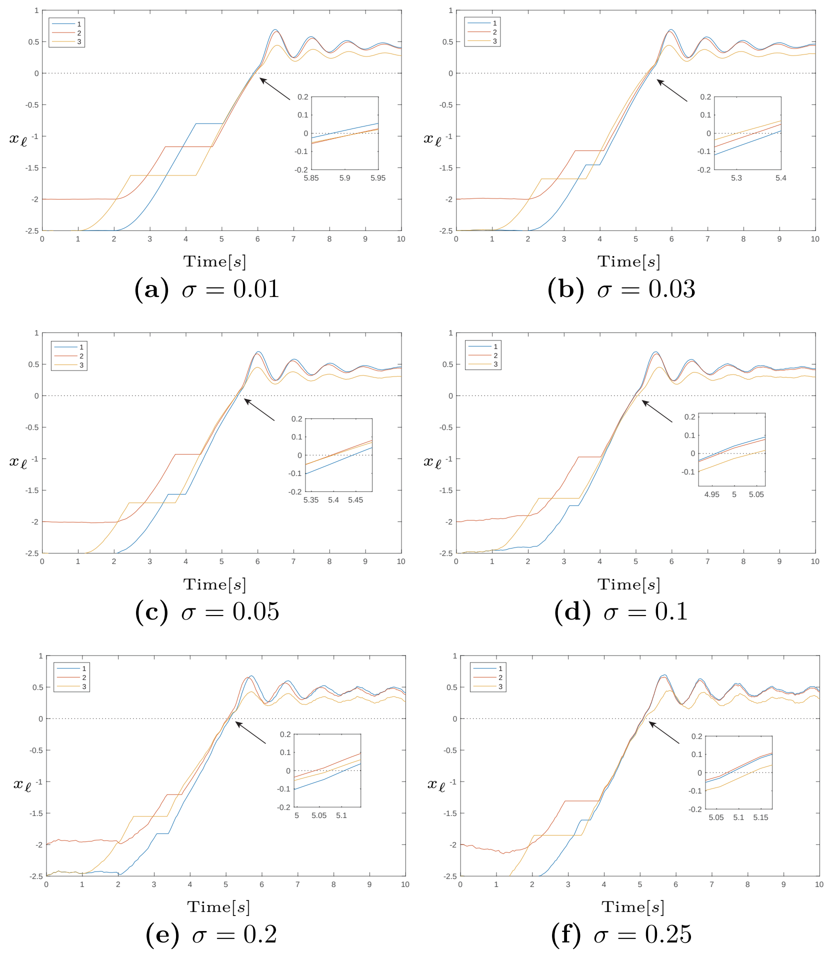


Figure 9.8: Position vs. velocity plots for agent 1 corresponding to different environment material stiffness k_c. The parameter k_c is set to 2, 5 and 10, respectively. Agent 1 does not bounce off the surface of the object after contact (i.e., $x_{\ell,1} = 0$).

9.7 Extension to agents having Dubins-like dynamics

Figure 9.9 shows the trajectories of the position $x_{\ell,i}$ corresponding to each agent with different values of noise. In this section, motivated by the wide applicability of Dubins-type models, we apply our hybrid controller to the car-type model given by

$$\begin{aligned} \dot{x}_{\ell,i} &= v_{\ell,i} \cos \theta_{\ell,i}, \quad \dot{y}_{\ell,i} = v_{\ell,i} \sin \theta_{\ell,i}, \\ \dot{v}_{\ell,i} &= u_{v,i} - f_{c,i}, \quad \dot{\theta}_{\ell,i} = u_{\theta,i} \end{aligned} \tag{9.27}$$



(a) $\sigma = 0.01$ **(b)** $\sigma = 0.03$

(c) $\sigma = 0.05$ **(d)** $\sigma = 0.1$

(e) $\sigma = 0.2$ **(f)** $\sigma = 0.25$

Figure 9.9: Plots of local position $x_{\ell,i}$ for each agent. Each case has a Gaussian noise with zero mean and variance of $\sigma = 0.01, 0.03, 0.05, 0.1, 0.2, 0.25$, respectively, as $f_{c,i}$ and m_i for each $i \in \mathcal{I}$.

where, for each $i \in \mathcal{I}$ agent, $(x_{\ell,i}, y_{\ell,i}) \in \mathbb{R}^2$ denotes planar position, $\theta_{\ell,i} \in \mathbb{R}$ denotes orientation and $v_{\ell,i} \in \mathbb{R}$ denotes the forward velocity, respectively; see, e.g., [53]. The inputs $u_{\theta,i}$ and $u_{v,i}$ are the angular velocity input and the acceleration input, respectively. The norm of the angular velocity input is upper bounded by the constant $\overline{u}_{\theta,i}$, which implies that the vehicle turns have a (nonzero) minimum turning radius. In other words, given an input signal $(u_{\theta,i}, u_{v,i})$, the resulting paths in the (x, y)-plane have bounded curvature.

In the following simulation, the initial condition of each agent $i \in \mathcal{I} := \{1, 2, 3\}$

is $(x^0_{\ell,1}, y^0_{\ell,1}, v^0_{\ell,1}, \theta^0_{\ell,1}) = (-2.5, -3, 1, \frac{\pi}{2})$, $(x^0_{\ell,2}, y^0_{\ell,2}, v^0_{\ell,2}, \theta^0_{\ell,2}) = (-2, 3, 1, -\frac{\pi}{2})$ and $(x^0_{\ell,3}, y^0_{\ell,3}, v^0_{\ell,3}, \theta^0_{\ell,3}) = (-2.5, 2, 1, -\frac{\pi}{2})$, respectively. A Gaussian noise with zero mean and variance of $\sigma = 0.01, 0.03, 0.05, 0.1, 0.2, 0.25$, respectively, define the noise signals $f_{c,i}$ and m_i for each $i \in \mathcal{I}$. The contact time t^i_c is changed under different noises, but mismatch of contact time is approximately 0.08 seconds in the worst case (Table 9.1).

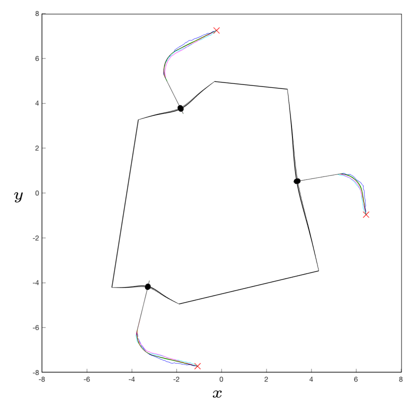


Figure 9.10: Plots of global position (x, y) for three agents both without noise and with different values of noise corresponding each case in Figure 9.9, such as $\sigma = 0.01$(green), 0.03(yellow), 0.05(magenta), 0.1(cyan), 0.2(blue), respectively.

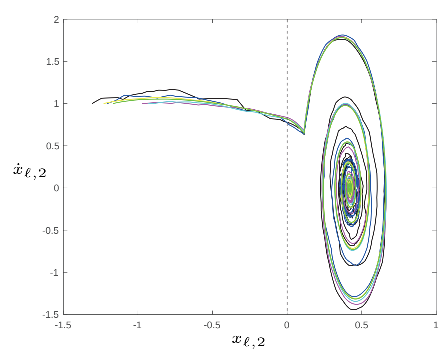


Figure 9.11: Position vs. velocity plots for agent 2 with disturbances: each case has a Gaussian noise with zero mean and variation of $\sigma = 0.01$(green), 0.03(yellow), 0.05(magenta), 0.1(cyan), 0.2(blue), 0.25(black), respectively.

Compared to the nominal case, mismatch of the equilibrium points is approxi-

σ	$\lvert t_c^1 - t_c^2 \rvert$	$\lvert t_c^1 - t_c^3 \rvert$	$\lvert t_c^2 - t_c^3 \rvert$
0	0	0	0
0.01	0.05	0.05	0
0.03	0.05	0.08	0.05
0.05	0.05	0.05	0
0.1	0	0.08	0.08
0.2	0.05	0.025	0.025
0.25	0.01	0.05	0.05

Table 9.1: Contact time of three agents.

mately 0.1 (in norm) in the worst case. Please go to `https://youtu.be/7VPnZj6a7Bo` to watch a movie of this simulation.

Chapter 10

Conclusion

In this dissertation, a hybrid controller for grasping tasks in a multi-agent system is introduced as an application of hybrid dynamical systems. The proposed hybrid controller supervises the position and force controllers for each agent, in order to steer the agent to the object while the contact force is regulated to avoid rebounding. Moreover, using timers and logic variables, multiple agents working on the object are synchronized so that they achieve a stable grasp.

On the other hand, temporal logic for hybrid systems is introduced to formulate specification for hybrid systems in high-level languages. For hybrid systems, notions encoding ε-approximate satisfaction and robust ε-approximate satisfaction of temporal logic specifications are proposed to specify and verify temporal logic specifications under the presence of perturbations. Our approach is proposed to establish relationship between the satisfaction of formulas having temporal logic operators and some of the invariance notions and finite-time attractivity (FTA) notions studied in control literature. In particular, (robust) forward invariance, (robust) conditional invariance, eventual conditional invariance, and (robust) finite time attractivity notions are revisited in the context of hybrid systems. Moreover, sufficient conditions certifying such dynamical properties are presented. As a consequence, sufficient conditions (not involving the computation of the systems' solutions) guaranteeing the satisfaction of temporal logic specifications (with robustness to perturbations) are proposed.

Furthermore, a relationship between signal temporal logic (STL) specifications linear temporal logic (LTL) formulas involving the strong until operators is established so that sufficient conditions that guarantee the satisfaction of basic STL specifications are derived.

There are many directions for future research towards developing tools for the formal verification and design of hybrid systems using LTL. By using LTL, we can guide the design of dynamical systems with constraints. A promising area is the use of learning techniques with formal specification expressed in temporal logic formulas.

Bibliography

[1] P. Tabuada and G. J. Pappas, "Linear time logic control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1862–1877, 2006.

[2] M. Kloetzer and C. Belta, "A fully automated framework for control of linear systems from temporal logic specifications," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 287–297, 2008.

[3] Y. Kwon and G. Agha, "LTLC: Linear temporal logic for control," *Hybrid Systems: Computation and Control*, pp. 316–329, 2008.

[4] A. Pnueli, "The temporal logic of programs," in *Foundations of Computer Science, 1977., 18th Annual Symposium on.* IEEE, 1977, pp. 46–57.

[5] A. Pnueli and Z. Manna, "The temporal logic of reactive and concurrent systems," *Springer*, vol. 16, p. 12, 1992.

[6] G. E. Fainekos, A. Girard, H. Kress-Gazit, and G. J. Pappas, "Temporal logic motion planning for dynamic robots," *Automatica*, vol. 45, no. 2, pp. 343–352, 2009.

[7] S. Karaman, R. G. Sanfelice, and E. Frazzoli, "Optimal control of mixed logical dynamical systems with linear temporal logic specifications," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on.* IEEE, 2008, pp. 2117–2122.

[8] E. M. Wolff, U. Topcu, and R. M. Murray, "Optimization-based trajectory generation with linear temporal logic specifications," in *Robotics and Automation (ICRA), 2014.* IEEE, 2014, pp. 5319–5325.

[9] R. Dimitrova and R. Majumdar, "Deductive control synthesis for alternating-time logics," in *Proceedings of the 14th International Conference on Embedded Software.* ACM, 2014, p. 14.

[10] S. Saha and A. A. Julius, "An MILP approach for real-time optimal controller synthesis with metric temporal logic specifications," in *American Control Conference (ACC), 2016.* IEEE, 2016, pp. 1105–1110.

[11] V. Raman, A. Donzé, D. Sadigh, R. M. Murray, and S. A. Seshia, "Reactive synthesis from signal temporal logic specifications," in *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*. ACM, 2015, pp. 239–248.

[12] R. Koymans, "Specifying real-time properties with metric temporal logic," *Real-time systems*, vol. 2, no. 4, pp. 255–299, 1990.

[13] O. Maler and D. Nickovic, "Monitoring temporal properties of continuous signals," in *Formal Techniques, Modelling and Analysis of Timed and Fault-Tolerant Systems*. Springer, 2004, pp. 152–166.

[14] G. E. Fainekos and G. J. Pappas, "Robustness of temporal logic specifications for continuous-time signals," *Theoretical Computer Science*, vol. 410, no. 42, pp. 4262–4291, 2009.

[15] A. Donzé and O. Maler, "Robust satisfaction of temporal logic over real-valued signals," in *International Conference on Formal Modeling and Analysis of Timed Systems*. Springer, 2010, pp. 92–106.

[16] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*. Lecture Notes in Control and Information Sciences, Springer, 2000.

[17] J. Lygeros, K. Johansson, S. Simić, J. Zhang, and S. S. Sastry, "Dynamical properties of hybrid automata," vol. 48, no. 1, pp. 2–17, 2003.

[18] P. Collins, "A trajectory-space approach to hybrid systems," 2004.

[19] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton University, 2006.

[20] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[21] K. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry, "On the regularization of Zeno hybrid automata," *Systems & Control Letters*, vol. 38, no. 3, pp. 141–150, 1999.

[22] A. Donzé, T. Ferrere, and O. Maler, "Efficient robust monitoring for stl," in *International Conference on Computer Aided Verification*. Springer, 2013, pp. 264–279.

[23] L. Lindemann and D. V. Dimarogonas, "Robust control for signal temporal logic specifications using discrete average space robustness," *Automatica*, vol. 101, pp. 377–387, 2019.

[24] J. Chai and R. G. Sanfelice, "Forward invariance of sets for hybrid dynamical systems (part i)," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2426–2441, 2018.

[25] Y. Li and R. G. Sanfelice, "Finite time stability of sets for hybrid dynamical systems," *Automatica*, vol. 100, pp. 200–211, 2019.

[26] A. Cimatti, M. Roveri, and S. Tonetta, "HRELTL: A temporal logic for hybrid systems," *Information and Computation*, vol. 245, pp. 54–71, 2015.

[27] H. K. Khalil, *Nonlinear systems*, 3rd ed. Prentice Hall, 2002.

[28] H. Han and R. G. Sanfelice, "Linear temporal logic for hybrid dynamical systems: Characterizations and sufficient conditions," *Nonlinear Analysis: Hybrid Systems*, vol. 36, p. 100865, 2020.

[29] M. Maghenem and R. G. Sanfelice, "Characterizations of safety in hybrid inclusions via barrier functions," in *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control (HSCC)*, 2019, pp. 109–118.

[30] H. Han and R. G. Sanfelice, "Sufficient conditions for temporal logic specifications in hybrid dynamical systems," vol. 51, no. 16. Elsevier, 2018, pp. 97–102.

[31] M. Maghenem and R. G. Sanfelice, "Characterization of safety and conditional invariance for nonlinear systems," in *2019 American Control Conference (ACC)*, July 2019, pp. 5039–5044.

[32] G. S. Ladde and S. Leela, "Analysis of invariant sets," *Annali di Matematica Pura ed Applicata*, vol. 94, no. 1, pp. 283–289, 1972.

[33] F. H. Clarke, *Optimization and nonsmooth analysis.* SIAM, 1990.

[34] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, 2007.

[35] R. G. Sanfelice, *Analysis and Design of Cyber-Physical Systems: A Hybrid Control Systems Approach.* CRC Press (to appear), 2015.

[36] C. Eisner, D. Fisman, and J. Havlicek, "A topological characterization of weakness," in *Proceedings of the twenty-fourth annual ACM symposium on Principles of distributed computing*, 2005, pp. 1–8.

[37] M. Maghenem and R. G. Sanfelice, "Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions," *Automatica*, vol. 124, p. 109328, 2021.

[38] A. Subbaraman and A. R. Teel, "On the equivalence between global recurrence and the existence of a smooth lyapunov function for hybrid systems," *Systems & Control Letters*, vol. 88, pp. 54–61, 2016.

[39] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems*, vol. 29, no. 2, 2009.

[40] P. Wolper, "Constructing automata from temporal logic formulas: A tutorial," in *School organized by the European Educational Forum.* Springer, 2000, pp. 261–277.

[41] T. Babiak, M. Křetínskỳ, V. Řehák, and J. Strejček, "Ltl to büchi automata translation: Fast and more deterministic," in *International Conference on Tools and Algorithms for the Construction and Analysis of Systems.* Springer, 2012, pp. 95–109.

[42] C. Belta, B. Yordanov, and E. A. Gol, *Formal methods for discrete-time dynamical systems.* Springer, 2017, vol. 89.

[43] A. Bisoffi and D. V. Dimarogonas, "A hybrid barrier certificate approach to satisfy linear temporal logic specifications," in *2018 Annual American Control Conference (ACC).* IEEE, 2018, pp. 634–639.

[44] R. Carloni, R. G. Sanfelice, A. R. Teel, and C. Melchiorri, "A hybrid control strategy for robust contact detection and force regulation," in *Proceedings of the American control conference*, 2007, pp. 1461–1466.

[45] J. Park and O. Khatib, "Robot multiple contact control," *Robotica*, vol. 26, no. 5, pp. 667–677, 2008.

[46] G. Gilardi and I. Sharf, "Literature survey of contact dynamics modelling," *Mechanism and machine theory*, vol. 37, no. 10, pp. 1213–1239, 2002.

[47] S. Liu and S. Carpin, "Fast grasp quality evaluation with partial convex hull computation," in *2015 IEEE International Conference on Robotics and Automation (ICRA).* IEEE, 2015, pp. 4279–4285.

[48] S. Carpin, S. Liu, J. Falco, and K. Van Wyk, "Multi-fingered robotic grasping: A primer," *arXiv preprint arXiv:1607.06620*, 2016.

[49] O. Khatib, "A unified approach for motion and force control of robot manipulators: The operational space formulation," *IEEE Journal on Robotics and Automation*, vol. 3, no. 1, pp. 43–53, 1987.

[50] T.-J. Tarn, Y. Wu, N. Xi, and A. Isidori, “Force regulation and contact transition control,” *IEEE Control Systems*, vol. 16, no. 1, pp. 32–40, 1996.

[51] M. C. Cavusoglu, J. Yan, and S. S. Sastry, “A hybrid system approach to contact stability and force control in robotic manipulators,” in *Intelligent Control, 1997. Proceedings of the 1997 IEEE International Symposium on.* IEEE, 1997, pp. 143–148.

[52] R. G. Sanfelice, D. Copp, and P. Nanez, “A toolbox for simulation of hybrid systems in matlab/simulink: Hybrid equations (hyeq) toolbox,” in *Proceedings of the 16th international conference on Hybrid systems: computation and control.* ACM, 2013, pp. 101–106.

[53] G. Walsh, D. Tilbury, S. Sastry, R. Murray, and J.-P. Laumond, “Stabilization of trajectories for systems with nonholonomic constraints,” *IEEE Transactions on Automatic Control*, vol. 39, no. 1, 1994.

[54] J. P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory.* Springer Science & Business Media, 2012, vol. 264.

[55] J. P. Aubin, *Viability Theory.* Cambridge, MA, USA: Birkhauser Boston Inc., 1991.

[56] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory.* Springer Science & Business Media, 2008, vol. 178.

[57] J.-P. Aubin and H. Frankowska, *Set-valued Analysis.* Springer Science & Business Media, 2009.

[58] S. Prajna and A. Jadbabaie, “Safety verification of hybrid systems using barrier certificates,” in *International Workshop on Hybrid Systems: Computation and Control.* Springer, 2004, pp. 477–492.

[59] J. Oehlerking, H. Burchardt, and O. Theel, “Fully automated stability verification for piecewise affine systems,” in *International Workshop on Hybrid Systems: Computation and Control.* Springer, 2007, pp. 741–745.

[60] P. Bernard and R. G. Sanfelice, “Observers for hybrid dynamical systems with linear maps and known jump times,” in *2018 IEEE Conference on Decision and Control (CDC).* IEEE, 2018, pp. 3140–3145.

Appendix A

Comparison Lemmas

The following result is a version of the well-known comparison Lemma that can be found in [27, Lemma 3.4].

Lemma A.1. *Consider the scalar system*

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0, \tag{A.1}$$

where for all $t \geq 0$ and all $u \in S \subset \mathbb{R}$, $f(t, u)$ is continuous in t and locally Lipschitz in u. Furthermore, let $[t_0, T)$ be the maximal interval, T can be infinity, of existence of the solution $u(t)$. Moreover, suppose that $u(t) \in S$ for all $t \in [t_0, T)$.

On the other hand, let $v(t)$ be a continuous function such that $v(t_0) \leq u_0$, $v(t) \in S$ for all $t \in [t_0, T)$, and its upper right-hand derivative $D^+v(t)$ satisfies the following differential inequality, for almost all $t \in [t_0, T)$,

$$D^+v(t) := \limsup_{s \to 0^+} \tfrac{v(t+s)-v(t)}{s} \leq f(t, v(t)). \tag{A.2}$$

Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Lemma A.2. *Assume that the function* $t \mapsto v(t)$ *in Lemma A.1 satisfies* $v(t) = v(x(t))$ *for all* $t \in [t_0, T)$ *with* $t \mapsto x(t)$ *a solution to the system* $\dot{x} \in F(x) \qquad x \in C \subset \mathbb{R}^n$, *and* $v \in \mathcal{C}^1$, *it follows that, for almost all* $t \in [t_0, T)$, $D^+v(t) = \dot{v}(t) = \langle \nabla v(x(t)), \dot{x}(t) \rangle$.

Proof. Since the solution x is absolutely continuous, it follows that $\dot{x}(t)$ exists for almost all $t \in [t_0, T)$. Furthermore, since $v \in \mathcal{C}^1$. Hence, $\dot{v}(t)$ exists for almost all $t \in [t_0, T)$. Let $t \in [t_0, T)$ such that $\dot{v}(t)$ exists, then, by definition of the time derivative, we conclude that

$$\dot{v}(t) = \lim_{s \to 0} \tfrac{v(t+s) - v(t)}{s} = \limsup_{s \to 0^+} \tfrac{v(t+s) - v(t)}{s} = D^+v(t).$$

Furthermore, using the classical chain rule for composition of differentiable functions, we conclude that $\dot{v}(t) = \langle \nabla v(x(t)), \dot{x}(t) \rangle$. $\square$

Lemma A.3. *Let* $x : [t_0, T) \to \mathbb{R}^n$ *be a solution to the following constrained differential inclusion* $\dot{x} \in F(x) \qquad x \in C \subset \mathbb{R}^n$. *Then, for almost all* $t \in [t_0, T)$, $\dot{x}(t) \in T_C(x(t))$.

Proof. Let $t \in [t_0, T)$ such that $\dot{x}(t)$ exists; thus, $\dot{x}(t) \in F(x(t, j))$. Furthermore, let a sequence $\{t_n\}_{n \in \mathbb{N}} \subset (t_0, T - t)$ such that $t_n \to 0$. That is, for $v_n(t) := (x(t_n) - x(t))/t_n$, we have $\lim_n v_n(t) = \dot{x}(t)$ and at the same time $x(t) + t_n v_n(t) = x(t_n) \in C$. Hence, using (C.2), we conclude that $\dot{x}(t) \in T_C(x(t))$. $\square$

Appendix B

Results on Finite Time Attractivity

In the following, we present sufficient conditions that guarantee FTA of a closed set K for a hybrid system $\mathcal{H}$; see [25]. First, Proposition B.1 characterizes the scenario where the distance of each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N})$ to K strictly decreases during flows, but is only non-increasing at jumps, and $\mathcal{N}$ is an open neighborhood of K.

Proposition B.1. *Let a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$ and a closed set $K \subset \mathcal{N} \subset \mathcal{X}$ with an open set $\mathcal{N}$ such that $G(\mathcal{N}) \subset \mathcal{N}$. The set K is FTA for $\mathcal{H}$ if there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c_1 > 0$, $c_2 \in [0, 1)$ such that*

1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D) \setminus K$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies

$$\tfrac{V^{1-c_2}(x)}{c_1(1-c_2)} \leq \sup_{(t,j) \in dom\, \phi} t,$$

2) there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_K) \leq V(x) \leq \alpha_2(|x|_K)$ for *all* $x \in (C \cup D \cup G(D)) \cap \mathcal{N}$ *and*

$$u_C(x) + c_1 V^{c_2}(x) \leq 0 \quad \forall x \in (C \cap \mathcal{N}) \setminus K \tag{B.1a}$$

$$u_D(x) \leq 0 \quad \forall x \in (D \cap \mathcal{N}) \setminus K, \tag{B.1b}$$

where the functions u_C *and* u_D *are defined in* (2.5) *and* (2.7)*, respectively.*

Furthermore, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ *with* $\xi = \phi(0,0)$*,*

a) the settling-time function $\mathcal{T}$ *satisfies* $\mathcal{T}(\phi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi)$ *where* $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$ *and* $\mathcal{J}^\star(\xi)$ *is such that* $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \operatorname{dom}\phi$*; and*

b) $|\phi(t,j)|_K = 0$ *for some* $(t,j) \in \operatorname{dom}\phi$ *such that* $t \geq \mathcal{T}^\star(\xi)$*.*

Proof. Let $\phi \in \mathcal{S}_{\mathcal{H}}$ with $\phi(0,0) = \xi \in \mathcal{N} \cap (\overline{C} \cup D)$ and rge $\phi \subset \mathcal{N}$. Pick any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} = t$ satisfy

$$\operatorname{dom}\phi \cap ([0,t] \times \{0,1,\ldots,j\}) = \bigcup_{i=0}^{j} ([t_i, t_{i+1}] \times \{i\}). \tag{B.2}$$

For each $i \in \{0,1,\ldots,j\}$ and almost all $s \in [t_i, t_{i+1}], \phi(s,i) \in (C \cap \mathcal{N}) \setminus K$. Using (2.6), the condition in (B.1a) implies that, for each $i \in \{0,1,\ldots,j\}$ and for almost all $s \in [t_i, t_{i+1}]$,

$$\tfrac{d}{ds} V(\phi(s,i)) \leq u_C(\phi(s,j)) \leq -c_1 V^{c_2}(\phi(s,i)), \tag{B.3}$$

which implies that

$$V^{-c_2}(\phi(s,i))\, dV(\phi(s,i)) \leq -c_1 ds. \tag{B.4}$$

Integrating over $[t_i, t_{i+1}]$ both sides of this inequality yields

$$\tfrac{1}{1-c_2}\left(V^{1-c_2}(\phi(t_{i+1}, i)) - V^{1-c_2}(\phi(t_i, i))\right) \leq -c_1(t_{i+1} - t_i). \tag{B.5}$$

Similarly, for each $i \in \{1, \ldots, j\}$, $\phi(t_i, i-1) \in (D \cap \mathcal{N}) \setminus K$ and

$$V(\phi(t_i, i)) - V(\phi(t_i, i-1)) \leq 0. \tag{B.6}$$

The two inequalities in (B.5) and (B.6) imply that, for each $(t, j) \in \text{dom } \phi$,

$$\tfrac{1}{1-c_2}\left(V^{1-c_2}(\phi(t, j)) - V^{1-c_2}(\xi)\right) \leq -c_1 t. \tag{B.7}$$

Using $G(\mathcal{N}) \subset \mathcal{N}$, the lower bound on the function V, and the fact that $c_2 \in (0, 1)$, we get

$$\alpha_1^{1-c_2}(|\phi(t, j)|_K) \leq V^{1-c_2}(\phi(t, j)) \leq V^{1-c_2}(\xi) - c_1(1 - c_2)t. \tag{B.8}$$

Then, it follows that

$$|\phi(t, j)|_K \leq \alpha^{-1}\left((V^{1-c_2}(\xi) - c_1(1 - c_2)t)^{\frac{1}{1-c_2}}\right). \tag{B.9}$$

Furthermore, an upper bound for the settling-time function can be computed as

$$\mathcal{T}(\phi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi), \tag{B.10}$$

where $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$, and $\mathcal{J}^\star(\xi)$ is chosen such $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \text{dom } \phi$. Note that $\mathcal{T}^\star(\xi) < \sup_{(t,j)\in\text{dom } \phi} t$ given by 1), the existence of $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \text{dom } \phi$ is guaranteed. $\square$

Remark B.2. *Condition 1) in Proposition B.1 guarantees that the domain of definition of the solutions to $\mathcal{H}$ are long enough to allow for the solution to converge to K. Condition* (B.1a) *guarantees finite time convergence of* $\lim_{t+j\to\mathcal{T}(\phi)} |\phi(t,j)|_K$ *to zero over a finite amount of ordinary time t (potentially with jumps within it). Finally, the upper bound on the settling-time function $\mathcal{T}$ depending on the Lyapunov function and the initial condition will be effectively exploited to estimate the amount of hybrid time it takes for a temporal specification to be satisfied.*

A dual version of Proposition B.1 is given next, namely, it pertains to the case, when the distance of a solution $\phi \in \mathcal{S}_\mathcal{H}$ to a closed set K strictly decreases at jumps.

Proposition B.3. *Let a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$ and a closed set $K \subset \mathcal{N} \subset \mathcal{X}$ with an open set $\mathcal{N}$ such that $G(\mathcal{N}) \subset \mathcal{N}$. The set K is FTA for $\mathcal{H}$ if there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c > 0$ such that*

1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D) \setminus K$, each $\phi \in \mathcal{S}_\mathcal{H}(x)$ satisfies

$$ceil\left(\tfrac{V(x)}{c}\right) \leq \sup_{(t,j)\in dom\,\phi} j;$$

2) there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with $\alpha_1(|x|_K) \leq V(x) \leq \alpha_2(|x|_K)$ for each $x \in (C \cup D \cup G(D)) \cap \mathcal{N}$ such that

$$u_C(x) \leq 0 \qquad \forall x \in (C \cap \mathcal{N}) \setminus K \tag{B.11a}$$

$$u_D(x) \leq -\min\{c, V(x)\} \quad \forall x \in (D \cap \mathcal{N}) \setminus K, \tag{B.11b}$$

where u_C and u_D are defined in (2.5) *and* (2.7)*, respectively.*

Moreover, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\xi = \phi(0,0)$,

a) the settling-time function $\mathcal{T}$ satisfies $\mathcal{T}(\phi) \leq \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi)$ where $\mathcal{J}^\star(\xi) = \text{ceil}\left(\frac{V(\xi)}{c}\right)$ and $\mathcal{T}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \text{dom}\,\phi$ and $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi) - 1) \in \text{dom}\,\phi$;

b) $|\phi(t,j)|_K = 0$ for some $(t,j) \in \text{dom}\,\phi$ such that $j \geq \mathcal{J}^\star(\xi)$.

Proof. Let $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ with $\xi \in \mathcal{N}$. Pick any $(t,j) \in \text{dom}\ \phi$ and let $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} = t$ satisfy

$$\text{dom}\ \phi \cap ([0,t] \times \{0,1,\ldots,j\}) = \bigcup_{i=0}^{j}([t_i, t_{i+1}] \times \{i\}). \tag{B.12}$$

For each $i \in \{0,1,\ldots,j\}$ and almost all $s \in [t_i, t_{i+1}]$, $\phi(s,i) \in C$. Using (2.6), the condition in (B.11b) implies that, for each $i \in \{0,1,\ldots,j\}$ and for almost all $s \in [t_i, t_{i+1}]$, $\frac{dV(\phi(s,i))}{ds} \leq 0$. Integrating over $[t_i, t_{i+1}]$ both sides of this inequality yields

$$V(\phi(t_{i+1}, i)) - V(\phi(t_i, i)) \leq 0. \tag{B.13}$$

Similarly, by using (2.8) and (B.11b), for each $i \in \{1,\ldots,j\}$, $\phi(t_i, i-1) \in D$ and

$$V(\phi(t_i, i)) - V(\phi(t_i, i-1)) \leq -\sum_{i=1}^{j} \min\{c, V(\phi(t_i, i-1))\}.$$

Using the lower bound on the function V and the fact that $c > 0$, we get

$$\alpha_1(|\phi(t,j)|_K) \leq V(\phi(t,j)) \leq V(\xi) - \sum_{i=1}^{j} \min\{c, V(\phi(t_i, i-1))\}.$$

Then, it follows that

$$|\phi(t,j)|_K \le \alpha_1^{-1}\left(V(\xi) - \sum_{i=1}^{j}\min\{c, V(\phi(t_i, i-1))\}\right).$$

Furthermore, an upper bound for the settling-time function can be computed as

$$\mathcal{T}(\phi) \le \mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi), \tag{B.14}$$

where $\mathcal{J}^\star(\xi) = \text{ceil}\left(\frac{V(\xi)}{c}\right)$ and $\mathcal{T}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)), (\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi) - 1) \in \text{dom } \phi$. Note that $\mathcal{J}^\star(\xi) < \sup_{(t,j)\in\text{dom } \phi} j$ given by 1), the existence of $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \text{dom } \phi$ is guaranteed. □

The following result combines the conditions in Proposition B.1 and in Proposition B.3. Its proof can be formulated by combining the arguments in the proofs of Proposition B.1 and Proposition B.3.

Proposition B.4. *Let a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$ and a closed set $K \subset \mathcal{N} \subset \mathcal{X}$ with an open set $\mathcal{N}$ such that $G(\mathcal{N}) \subset \mathcal{N}$. The set K is FTA for $\mathcal{H}$ if there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\ge 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c_1, c_3 > 0$, $c_2 \in [0, 1)$ such that item 1) in Proposition B.1 and item 1) in Proposition B.3 are satisfied, and there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_K) \le V(x) \le \alpha_2(|x|_K)$$

for all $x \in (C \cup D \cup G(D)) \cap \mathcal{N}$ and

$$u_C(x) + c_1 V^{c_2}(x) \le 0 \qquad \forall x \in (C \cap \mathcal{N}) \setminus K \tag{B.15a}$$

$$u_D(x) \leq -\min\{c_3, V(x)\} \quad \forall x \in (D \cap \mathcal{N}) \setminus K, \tag{B.15b}$$

where u_C and u_D are defined in (2.5) and (2.7), respectively. Furthermore, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{N} \cap (\overline{C} \cup D))$ with $\xi = \phi(0,0)$,

a) the settling-time function $\mathcal{T}$ satisfies $\mathcal{T}(\phi) \leq \min_{i\in\{1,2\}} \{\mathcal{T}_i^\star(\xi) + \mathcal{J}_i^\star(\xi)\}$ where $\mathcal{T}_1^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$, $\mathcal{J}_1^\star(\xi)$ is such that $(\mathcal{T}_1^\star(\xi), \mathcal{J}_1^\star(\xi)) \in dom\,\phi$, $\mathcal{J}_2^\star(\xi) = ceil\left(\frac{V(\xi)}{c_3}\right)$, and $\mathcal{T}_2^\star(\xi)$ is such that $(\mathcal{T}_2^\star(\xi), \mathcal{J}_2^\star(\xi)) \in dom\ \phi$ and $(\mathcal{T}_2^\star(\xi), \mathcal{J}_2^\star(\xi) - 1) \in dom\,\phi$;

b) $|\phi(t,j)|_K = 0$ for some $(t,j) \in dom\,\phi$ such that $t \geq \mathcal{T}_1^\star(\xi)$ or $j \geq \mathcal{J}_2^\star(\xi)$.

Appendix C

Results on Conditional Invariance and Eventual Conditional Invariance

The following results are valid for the general class of hybrid systems $\mathcal{H}$ satisfying the following mild assumption:

(SA) The system $\mathcal{H}$ is such that the set C is closed, F is outer semicontinuous and locally bounded with nonempty and convex values on C, and G has nonempty values on D.

The properties of F in (SA) are used in the literature of differential inclusions as mild requirements for existence of solutions from int$C$ plus adequate structural properties for the flows, see [54–56]. When F is single valued, the properties of F in (SA) reduce to simply continuity.

C.1 Sufficient Conditions for Conditional Invariance

First, we recall the sufficient conditions for invariance notions using a barrier function in [29, 37] for hybrid systems. Below, the concept of the tangent cone[1] to a set is used; see [20, Definition 5.12]. The tangent cone at a point $x \in \mathbb{R}^n$ of a set $C \subset \mathbb{R}^n$ given by

$$T_C(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_C}{h} = 0 \right\}. \tag{C.1}$$

We also recall the equivalence [57, Page 122]

$$\begin{aligned} v \in T_C(x) \Leftrightarrow \ & \exists \{h_i\}_{i \in \mathbb{N}} \to 0^+ \\ & \text{and } \{v_i\}_{i \in \mathbb{N}} \to v : x + h_i v_i \in C \quad \forall i \in \mathbb{N}. \end{aligned} \tag{C.2}$$

Furthermore, for the given two sets $\mathcal{O}, \mathcal{X}_u \subset \mathbb{R}^n$ with $\mathcal{O} \cap \mathcal{X}_u = \emptyset$, we recall from [29] the notion of a barrier function candidate with respect to $(\mathcal{O}, \mathcal{X}_u)$ for $\mathcal{H}$.

Definition C.1 (Barrier function candidate). *Consider $\mathcal{H} = (C, F, D, G)$. Given two sets $\mathcal{O}, \mathcal{X}_u \subset \mathbb{R}^n$ with $\mathcal{O} \cap \mathcal{X}_u = \emptyset$, a function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a barrier function candidate with respect to $(\mathcal{O}, \mathcal{X}_u)$ for $\mathcal{H}$ if*

$$\begin{cases} B(x) \le 0 & \forall x \in \mathcal{O} \\ B(x) > 0 & \forall x \in (C \cup D) \cap \mathcal{X}_u. \end{cases} \tag{C.3}$$

In the following, we recall a result on safety for hybrid systems [29, Theorem 3.2] to derive sufficient conditions for conditional invariance for hybrid systems.

[1]This tangent cone is also known as the contingent cone, or the Bouligand tangent cone.

Given two sets $\mathcal{O}$ and $\mathcal{X}_u$, the conditions given below provide sufficient conditions to verify that $\mathbb{R}^n \backslash \mathcal{X}_u$ is conditionally invariant with respect to $\mathcal{O}$ for $\mathcal{H}$.

Proposition C.2 (CI using barrier functions)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Let two sets $\mathcal{O}$ and $\mathcal{X}_u$ such that $\mathcal{O}$, $\mathbb{R}^n \backslash \mathcal{X}_u \subset C \cup D$. The set $\mathbb{R}^n \backslash \mathcal{X}_u$ is* CI *with respect to $\mathcal{O}$ for $\mathcal{H}$ if there exists a $\mathcal{C}^1$ barrier function candidate B with respect to $(\mathcal{O}, \mathcal{X}_u)$ for $\mathcal{H}$ as in* (C.3) *such that $K := \{x \in C \cup D : B(x) \leq 0\}$ is closed and the following hold:*

$$\begin{aligned}
&\langle \nabla B(x), \eta \rangle \leq 0 && \forall x \in (U(\partial K) \backslash K) \cap C, \forall \eta \in F(x) \cap T_C(x), \\
&B(\eta) \leq 0 && \forall x \in D \cap K, \forall \eta \in G(x), \\
&G(D \cap K) \subset C \cup D.
\end{aligned}$$

According to Remark 2.5, when $\mathcal{O} = \mathbb{R}^n \backslash \mathcal{X}_u$, CI of $\mathbb{R}^n \backslash \mathcal{X}_u$ with respect to $\mathcal{O}$ reduces to forward pre-invariance of the set $K := \mathcal{O}$. In the next statement, we recall from [37, Theorem 1 and Proposition 2] sufficient conditions for forward invariance using barrier functions.

Proposition C.3 (Forward invariance using barrier functions)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Let K be a closed set such that $K \subset C \cup D$. The set K is* forward pre-invariant *for $\mathcal{H}$ if there exists a $\mathcal{C}^1$ barrier function candidate B with respect to $(K, \mathbb{R}^n \backslash K)$ for $\mathcal{H}$ as in* (C.3) *such that the following hold:*

$$\begin{aligned}
&\langle \nabla B(x), \eta \rangle \leq 0 && \forall x \in (U(\partial K) \backslash K) \cap C, \forall \eta \in F(x) \cap T_C(x), \\
&B(\eta) \leq 0 && \forall x \in D \cap K, \forall \eta \in G(x), \\
&G(D \cap K) \subset C \cup D.
\end{aligned}$$

Furthermore, the set K is forward invariant *for $\mathcal{H}$ if the following additional conditions hold:*

- *No maximal solution to $\mathcal{H}$ starting from K has a finite time escape within $C \cap K$.*

- *Every maximal solution from $(\partial C \cap K) \backslash D$ is nontrivial.*

Remark C.4. *One can guarantee that the solutions to $\mathcal{H}$ do not have a finite escape time [2] inside the set $K \cap C$ when, for example, the set $K \cap C$ is compact or when the flow map F has a global linear growth on $K \cap C$. Furthermore, according to [37, Proposition 3], the existence of a nontrivial solution starting from each point in $(K \cap \partial C) \backslash D$ can be proved by verifying the following infinitesimal condition. $F(x) \cap T_{K \cap C}(x) \neq \emptyset$ for all $x \in U(x_o) \cap (K \cap \partial C)$ and for all $x_o \in (K \cap \partial C) \backslash D$.*

C.2 Sufficient Conditions for pre-Eventual Conditional Invariance

In the following, inspired by [32, Theorem 3.4], we propose sufficient conditions for pre-eventual conditional invariance for hybrid systems.

Theorem C.5 (Pre-eventual Conditional Invariance)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and sets $\mathcal{O} \subset C \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$. The set $\mathcal{A}$ is* pre-ECI *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if the following properties hold:*

1) There exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$ and a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$ such that

$$\text{1a)} \quad \langle \nabla v(x), \eta \rangle \leq f_c(v(x)) \qquad \forall x \in C,\ \forall \eta \in F(x) \cap T_C(x),$$

$$v(\eta) \leq v(x) \qquad \forall x \in D, \quad \forall \eta \in G(x);$$

[2] A solution has finite escape time inside a given set if the solution diverges while remaining inside the set within a bounded (hybrid) time domain; see [27, Chapter 3].

1b) there exists a constant $r_1 > 0$ such that the solutions to $\dot{y} = f_c(y)$, starting from $v(\mathcal{O})$, converge to $(-\infty, r_1)$ in finite time.[3]

2) There exist a $\mathcal{C}^1$ function $w : \mathbb{R}^n \to \mathbb{R}$ and a nondecreasing function[4] *$f_d : \mathbb{R} \to \mathbb{R}$ such that*

2a) $\langle \nabla w(x), \eta \rangle \leq 0 \qquad \forall x \in C,\ \forall \eta \in F(x) \cap T_C(x),$

$w(\eta) \leq f_d(w(x)) \qquad \forall x \in D,\ \forall \eta \in G(x);$

2b) there exists a constant $r_2 > 0$ such that the solutions to $z^+ = f_d(z)$, starting from $w(\mathcal{O})$, converge to $(-\infty, r_2)$ in finite time.

3) One of the following conditions holds:

3a) Each complete solution to $\mathcal{H}$ starting from $\mathcal{O}$ is eventually continuous and, with r_1 coming from item 1b),

$$S_1 := \{x \in C : v(x) < r_1\} \subset \mathcal{A}. \tag{C.4}$$

3b) Each complete solution to $\mathcal{H}$ starting from $\mathcal{O}$ is eventually discrete and, with r_2 coming from item 2b),

$$S_2 := \{x \in D : w(x) < r_2\} \subset \mathcal{A}. \tag{C.5}$$

3c) Each complete solution to $\mathcal{H}$ starting from $\mathcal{O}$ has a hybrid time domain that is unbounded in both the t and the j direction and with r_1 and r_2 coming from item 1b) and item 2b) respectively, (C.4) *and* (C.5) *hold.*

[3]The solutions to $\dot{y} = f_c(y)$ from $v(\mathcal{O})$ exist at least until they reach the set $(-\infty, r_1)$.
[4]A scalar-valued function f_d is said to be nondecreasing if for each $x \leq y$, $f_d(x) \leq f_d(y)$.

3d) With r_1 and r_2 coming from item 1b) and item 2b) respectively, (C.4) *and* (C.5) *hold, and $G(S_2) \cap C \subset S_1$.*

Proof. According to the definition of pre-ECI, we need to show that for each complete solution ϕ to $\mathcal{H}$ starting from $\mathcal{O}$, there exists $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $\phi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t + j \geq t^\star + j^\star$. Consider a complete solution ϕ to $\mathcal{H}$ starting from $\phi(0,0) \in \mathcal{O}$. Let y be the maximal solution to $\dot{y} = f_c(y)$ starting from $y(0) = v(\phi(0,0)) \in v(\mathcal{O})$ and let z be the complete solution to the system $z^+ = f_d(z)$ starting from $z(0) = w(\phi(0,0)) \in w(\mathcal{O})$.

First, if the solution ϕ initially flows, we use item 1a) to conclude, via the comparison lemma in Lemma A.1, Lemma A.2, and Lemma A.3 that $v(\phi(t,0)) \leq y(t)$ for all $t \in I^0$, where $I^0 := \{t \in \mathbb{R}_{\geq 0} : (t,0) \in \operatorname{dom} \phi\}$. To show this, we used the fact that[5] $I^0 \subset \operatorname{dom} y$. Furthermore, if the solution ϕ jumps initially, we conclude using item 1a) that $v(\phi(0,1)) \leq y(0)$. By extending this reasoning over the domain of ϕ, we conclude that $v(\phi(t,j)) \leq y(t)$ for all $(t,j) \in \operatorname{dom} \phi$.

On the other hand, using item 2a), we conclude that if ϕ initially jumps, then $w(\phi(0,1)) \leq f_d(w(\phi(0,0))) = f_d(z(0)) = z(1)$. Otherwise, when the solution ϕ initially flows, we conclude that $w(\phi(t,0)) \leq w(\phi(0,0)) = z(0)$ for all $t \in I^0$. Moreover, by extending this reasoning over the domain of ϕ and using the fact that f_d is nondecreasing, we conclude that $w(\phi(t,j)) \leq z(j)$ for all $(t,j) \in \operatorname{dom} \phi$. Indeed, it is easy to see that $w(\phi(t,j)) \leq w(\phi(t',j))$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t \geq t'$ and $(t',j) \in \operatorname{dom} \phi$; namely, the bound of the function w does not increase over the interval of flow of ϕ.

Since f_d is nondecreasing, the existence of $j_z \in \mathbb{N}$ such that $z(j_z) \in (-\infty, r_2)$

[5]When y has a finite-escape time $t_y > 0$, using item 1b), it follows that $\lim_{t \nearrow t_y} y(t) = -\infty$. Furthermore, since $v(\phi(t,0)) \leq y(t)$ for all $t \in \operatorname{dom} y \cap I^0$, then there must exist $t_\phi \in I^0$ with $t_\phi \leq t_y$ such that $\lim_{t \nearrow t_\phi} v(\phi(t,0)) = -\infty$. Hence, ϕ should escape to $-\infty$ no later than y.

(coming from item 2b)) implies that $z(j) \in (-\infty, r_2)$ for all $j \geq j_z$. Similarly, from the existence of $t_y \in \mathbb{R}_{\geq 0}$ such that $y(t_y) \in (-\infty, r_1)$ (in item 1b)), it follows that $y(t) \in (-\infty, r_1)$ for all $t \geq t_y$. Moreover, since the solution ϕ is complete, $(\{t_y\} \times \mathbb{N}) \cap \operatorname{dom} \phi \neq \emptyset$ or $(\mathbb{R}_{\geq 0} \times \{j_z\}) \cap \operatorname{dom} \phi \neq \emptyset$. Therefore, we conclude that $v(\phi(t,j)) < r_1$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t \geq t_y$ or $w(\phi(t,j)) < r_2$ for all $(t,j) \in \operatorname{dom} \phi$ such that $j \geq j_z$.

To complete the proof, we show that there exists $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $\phi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t + j \geq t^\star + j^\star$. To this end, we present the following cases depending on items 3a)-3d).

a) When the solution ϕ is complete and eventually continuous, it follows that, for some $\tilde{t} \in \mathbb{R}_{\geq 0}$, $\phi(t,j) \in C$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t \geq \tilde{t}$. Hence, we have $v(\phi(t,j)) < r_1$ and $\phi(t,j) \in C$ for all (t,j) such that $t \geq t^\star := \max\{t_y, \tilde{t}\}$; and thus, S_1 is nonempty. Then, if S_1 is a subset of $\mathcal{A}$, with $j^\star$ such that $(t^\star, j^\star) \in \operatorname{dom} \phi$, we have $\phi(t,j) \in S_1 \subset \mathcal{A}$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t + j \geq t^\star + j^\star$.

b) When the solution ϕ is complete and eventually discrete, it follows that, for some $\tilde{j} \in \mathbb{N}$, $\phi(t,j) \in D$ for all $(t,j) \in \operatorname{dom} \phi$ such that $j \geq \tilde{j}$. Hence, we have $w(\phi(t,j)) < r_2$ and $\phi(t,j) \in D$ for all (t,j) such that $j \geq j^\star := \max\{j_z, \tilde{j}\}$; and thus, S_2 is nonempty. Then, if S_2 is a subset of $\mathcal{A}$, with $t^\star$ such that $(t^\star, j^\star) \in \operatorname{dom} \phi$, we have $\phi(t,j) \in S_2 \subset \mathcal{A}$ for all $(t,j) \in \operatorname{dom} \phi$ such that $t + j \geq t^\star + j^\star$.

c) When the hybrid time domain of the solution ϕ achieves both an unbounded amount of flows and an unbounded number of jumps, we conclude that $(\{t_y\} \times \mathbb{N}) \cap \operatorname{dom} \phi \neq \emptyset$; hence, $v(\phi(t,j)) < r_1$ for all $t \geq t_y$ such that $(t,j) \in \operatorname{dom} \phi$. Also, $(\mathbb{R}_{\geq 0} \times \{j_z\}) \cap \operatorname{dom} \phi \neq \emptyset$; hence, $w(\phi(t,j)) < r_2$ for all $j \geq j_z$ such that

$(t, j) \in \text{dom } \phi$. As a result, S_1 and S_2 are nonempty. Then, if S_1 and S_2 are subsets of $\mathcal{A}$, with $(t^\star, j^\star) \in \text{dom } \phi$ such that $t^\star \geq t_y$ and $j^\star \geq j_z$, we conclude that $\phi(t, j) \in S_1 \cup S_2 \subset \mathcal{A}$ for all $(t, j) \in \text{dom } \phi$ such that $t + j \geq t^\star + j^\star$.

Then, when one among items 3a)-3c) holds, according to the arguments in a)-c), we have that every complete solution ϕ starting from $\mathcal{O}$ satisfies that there exists $(t^\star, j^\star) \in \text{dom } \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \text{dom } \phi$ such that $t + j \geq t^\star + j^\star$.

Next, suppose that item 3d) holds and the solution ϕ is genuinely Zeno, which implies that ϕ does not satisfy items 3a)-3c). In this case, the solution ϕ jumps infinitely many times on a bounded interval of ordinary time and it always flows after a finite number of jumps. Hence, due to the fact ϕ is genuinely Zeno, there exists $(\tilde{t}, \tilde{j}) \in \text{dom } \phi$ such that $\tilde{j} \geq j_z$ satisfying $w(\phi(\tilde{t}, \tilde{j})) < r_2$ and $\phi(\tilde{t}, \tilde{j}) \in D$, which, in turn, implies $\phi(\tilde{t}, \tilde{j}) \in S_2 \subset \mathcal{A}$. Note that $w(\phi(t, j)) < r_2$ for all $(t, j) \in \text{dom } \phi$ such that $j \geq \tilde{j} \geq j_z$. Moreover, using the fact that ϕ is genuinely Zeno, ϕ jumps to a point in C; namely, there exists $(\tilde{t}, \tilde{j}), (\tilde{t}, \tilde{j} + 1) \in \text{dom } \phi$ such that $\phi(\tilde{t}, \tilde{j}+1) \in C$. Now, according to item 3b), when $\phi(\tilde{t}, \tilde{j}+1) \in C$, $\phi(\tilde{t}, \tilde{j}+1) \in S_1$. Namely, the solution ϕ jumps to the set $S_1 \subset \mathcal{A}$ at $(\tilde{t}, \tilde{j})$, which implies that $v(\phi(\tilde{t}, \tilde{j}+1)) < r_1$. Now, we show that the solution ϕ, which jumps to S_1 at $(\tilde{t}, \tilde{j})$, satisfies $v(\phi(t, \tilde{j} + 1)) < r_1$ for all t such that $(t, \tilde{j} + 1) \in \text{dom } \phi$. Proceeding by contradiction, assume the existence of $t' > \tilde{t}$ such that $(t', \tilde{j} + 1) \in \text{dom } \phi$ (i.e., t' is in the interval of flow) and $v(\phi(t', \tilde{j} + 1)) \geq r_1$. Let y be a solution to $\dot{y} = f_c(y)$ starting from $v(\phi(\tilde{t}, \tilde{j} + 1)) < r_1$. Under item 1b), with a locally Lipschitz function f_c, the unique solution y has to remain in $(-\infty, r_1)$ once it reaches $(-\infty, r_1)$; otherwise, it contradicts uniqueness of solutions. Moreover, using the comparison lemma in Lemma A.1, Lemma A.2, and Lemma A.3, we have $v(\phi(t, \tilde{j}+1)) \leq y(t)$ for all $(t, \tilde{j}+1) \in \operatorname{dom} \phi$. Hence, we conclude that $v(\phi(t, \tilde{j}+1)) < r_1$ for all $t > \tilde{t}$ such that $(t, \tilde{j}+1) \in \operatorname{dom} \phi$, which contradicts the existence of $t' > \tilde{t}$ such that $(t', \tilde{j}+1) \in \operatorname{dom} \phi$ and $v(\phi(t', \tilde{j}+1)) \geq r_1$. Therefore, we conclude that $v(\phi(t, \tilde{j}+1)) < r_1$ for all t such that $(t, \tilde{j}+1) \in \operatorname{dom} \phi$; namely, $v(\phi(t, \tilde{j}+1)) < r_1$ for all t in the interval of flow. As a result, we conclude that $w(\phi(t, j)) < r_2$ for all $(t, j) \in \operatorname{dom} \phi$ such that $j \geq \tilde{j}$ and $v(\phi(t, j)) < r_1$ for all $(t, j) \in \operatorname{dom} \phi$ such that $t \geq \tilde{t}$; and thus, with $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $j^\star \geq \tilde{j}$ and $t^\star \geq \tilde{t}$, we conclude that $\phi(t, j) \in S_1 \cup S_2 \subset \mathcal{A}$ for all $(t, j) \in \operatorname{dom} \phi$ such that $t + j \geq t^\star + j^\star$. $\square$

The following example illustrates Theorem C.5.

Example C.6. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ with the state $x = (x_1, x_2) \in \mathbb{R}^2$ and the data*

$$F(x) := \begin{bmatrix} -x_1 - x_2 \\ x_1 - x_2 \end{bmatrix} \quad \forall x \in C := \{x \in \mathbb{R}^2 : x_1 \geq 0, x_1 \geq x_2\}$$

$$G(x) := \begin{bmatrix} -x_2/\sqrt{2} \\ -x_2/\sqrt{2} \end{bmatrix} \quad \forall x \in D := \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\}.$$

Consider the sets $\mathcal{O}$ and $\mathcal{A}$ given by $\mathcal{O} = [0,1] \times (-\infty, -1]$ and $\mathcal{A} = \mathbb{R}_{\geq 0} \times [-1/2, +\infty)$. Next, to conclude that the set $\mathcal{A}$ is pre-eventually conditionally invariant with respect to the set $\mathcal{O}$ for $\mathcal{H}$, we show that the conditions in Theorem C.5 are satisfied. Consider the functions $v(x) = |x|^2$ and $f_c(y) := -2y$. For all $x \in C$, $\langle \nabla v(x), F(x) \rangle = -2(x_1^2 + x_2^2) = f_c(v(x))$; and for all $x \in D$, $v(G(x)) = x_2^2 = v(x)$. Thus, item 1a) holds. Furthermore, we notice that $v(\mathcal{O}) = [1, +\infty)$ and that, for $r_1 = 1/2$, (C.4) holds. Finally, for the system $\dot{y} = f_c(y) = -2y$, it is easy to see that the solutions starting from $v(\mathcal{O}) = [1, +\infty)$

reach the set $(-\infty, 1/2)$ in finite time; hence, item 1b) is satisfied. On the other hand, consider the functions $w(x) = -x_2$ and $f_d(z) := z/2$ if $z \in w(D)$ and $f_d(z) = z$ otherwise. For all $x \in C$, $\langle \nabla w(x), x_1 - x_2 \rangle = x_2 - x_1 \leq 0$; and for all $x \in D$, $w(G(x)) = x_2/\sqrt{2} \leq f_d(-x_2) = -x_2/2$ since $x_2 \leq 0$. Hence, we conclude that item 2a) holds. Moreover, item 2b) holds for $r_2 = 1/2$ and the solutions to $z^+ = f_d(z)$ starting from $w(\mathcal{O}) = [1, +\infty)$ reach $(-\infty, 1/2)$. Finally, for all $x \in G(S_2) \cap C = \{x \in C : x_1 < \frac{1}{2\sqrt{2}}\}$, $v(x) < 1/2$. Hence, $G(S_2) \cap C \subset S_1$, which implies that item 3d) holds. $\triangle$

Remark C.7. *As illustrated in Example C.6, once we propose the candidate functions v and w, we find the functions f_c, f_d and the constants r_1, r_2 such that the conditions in Theorem C.5 hold. That is, for a particular expression of the data of the hybrid system and the sets $\mathcal{O}$ and $\mathcal{A}$, we can automate the process of generating the functions and parameters satisfying the conditions in Theorem C.5 as in [58, 59].*

Note that, it is possible to conclude pre-eventual conditional invariance of $\mathcal{A}$ with respect to $\mathcal{O}$ using only condition 1) (or only condition 2, respectively) in Theorem C.5 provided that we have the knowledge that the solutions from $\mathcal{O}$ can reach the set $\mathcal{A}$ only via flowing (or only jumping, respectively), as shown in the following result. Indeed, in many applications of hybrid systems, the state variable is composed of both continuous and discrete variables, see the thermostat hybrid model in Example 4.13. Furthermore, when the sets $\mathcal{O}$ and $\mathcal{A}$ are defined only in terms of the continuous state variables (respectively, only in terms of the discrete state variables), it is possible to conclude that the solutions from $\mathcal{O}$ reach the set $\mathcal{A}$ only by flowing (respectively, only by jumping).

Proposition C.8 (Pre-eventual Conditional Invariance via Flows)**.** *Consider a* *hybrid system $\mathcal{H} = (C, F, D, G)$ and sets $\mathcal{O} \subset C \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$. Then, the set $\mathcal{A}$ is* pre-eventually conditionally invariant *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if*

1) there exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a constant $r_1 > 0$ such that condition 1) in Theorem C.5 holds and, the set $S_1 := \{x \in C : v(x) < r_1\} \subset \mathcal{A}$; and

2) for each complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, the solution y to $\dot{y} = f_c(y)$ starting from $v(\phi(0,0))$ satisfies $y(t) \in (-\infty, r_1)$ for all $t \geq t^\star$, for some nonnegative $t^\star \leq \sup\{t : (t,j) \in dom\ \phi\}$; and

3) $G(\mathcal{A}) \cap D \subset \mathcal{A}$, or each solution from $\mathcal{O}$ is eventually continuous.

Proof. According to the definition of pre-eventual conditional invariance, we need to show that, for each complete solution ϕ to $\mathcal{H}$ starting from $\mathcal{O}$, there exists $(t^\star, j^\star) \in \text{dom}\ \phi$ such that $\phi(t,j) \in \mathcal{A}$ for all $(t,j) \in \text{dom}\ \phi$ such that $t+j \geq t^\star + j^\star$. Consider a complete solution ϕ to $\mathcal{H}$ starting from $\phi(0,0) \in \mathcal{O}$. Let y be a maximal solution to $\dot{y} = f_c(y)$ starting from $y(0) = v(\phi(0,0)) \in v(\mathcal{O})$. First, if the solution ϕ flows initially, using the comparison lemma in Lemma A.1, Lemma A.2, and Lemma A.3 under item 1a) in Proposition C.5, we conclude that $v(\phi(t,0)) \leq y(t)$ for all $t \in I^0$, where $I^0 := \{t \in \mathbb{R}_{\geq 0} : (t,0) \in \text{dom}\ \phi\}$. Furthermore, if the solution ϕ jumps initially, we conclude using item 1a) in Proposition C.5 that $v(\phi(0,1)) \leq y(0)$. By extending the latter reasoning along the domain of ϕ, we conclude that $v(\phi(t,j)) \leq y(t)$ for all $(t,j) \in \text{dom}\ \phi$. On the other hand, by item 2), there exists $t^\star \in \mathbb{R}_{\geq 0}$ such that $t^\star \leq \sup\{t : (t,j) \in \text{dom}\ \phi\}$ and $y(t) \in (-\infty, r_1)$ for all $t \geq t^\star$. This fact implies that $(\{t^\star\} \times \mathbb{N}) \cap \text{dom}\ \phi \neq \emptyset$; hence, $v(\phi(t,j)) < r_1$ for all $t \geq t^\star$ such that $(t,j) \in \text{dom}\ \phi$ and S_1 is nonempty. As a consequence, using the fact that S_1 is a subset of $\mathcal{A}$, we conclude that, for all $(t, j) \in \text{dom } \phi$ such that $t \geq t^\star$, we have $\phi(t, j) \in \mathcal{A}$ provided that $\phi(t, j) \in C$ and that ϕ reaches $\mathcal{A}$ as it flows after $t^\star$. Next, under item 3), if ϕ is eventually continuous, then ϕ remains in $\mathcal{A}$ by flowing; however, once ϕ reaches a point $x \in D \cap \mathcal{A}$, it jumps. However, according to item 3), it remains in the set $\mathcal{A}$ after the jump, which completes the proof. □

Proposition C.9 (Pre-eventual Conditional Invariance via Jumps)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and sets $\mathcal{O} \subset C \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$. Then, the set $\mathcal{A}$ is* pre-eventually conditionally invariant *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if*

1) there exist a $\mathcal{C}^1$ function $w : \mathbb{R}^n \to \mathbb{R}$, a nondecreasing function $f_d : \mathbb{R} \to \mathbb{R}$, and a constant $r_2 > 0$ such that condition 2) in Theorem C.5 holds and, the set $\widetilde{S}_2 := \{x \in C \cup D : w(x) < r_2\} \subset \mathcal{A}$; and

2) for each complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, the solution z to $z^+ = f_d(z)$ starting from $w(\phi(0, 0))$ satisfies $z(j) \in (-\infty, r_2)$ for all $j \geq j^\star$, for some nonnegative $j^\star \leq \sup\{j : (t, j) \in \text{dom } \phi\}$.

Proof. According to the definition of pre-eventual conditional invariance, we need to show that, for each complete solution ϕ to $\mathcal{H}$ starting from $\mathcal{O}$, there exists $(t^\star, j^\star) \in \text{dom } \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \text{dom } \phi$ such that $t + j \geq t^\star + j^\star$. Consider such a complete solution ϕ starting from $\phi(0, 0) \in \mathcal{O}$. Let z be a maximal solution to $z^+ = f_d(z)$ starting from $z(0) = w(\phi(0, 0)) \in w(\mathcal{O})$. First, if the solution ϕ jumps initially, then we conclude that $w(\phi(0, 1)) \leq z(1)$. Furthermore, if the solution ϕ initially flows, we conclude using item 2a) in Proposition C.5 that $w(\phi(t, 0)) \leq z(0)$ for all $t \in I^0$, where $I^0 := \{t \in \mathbb{R}_{\geq 0} : (t, 0) \in \text{dom } \phi\}$. By extending the latter reasoning along the domain of ϕ and using the fact that f_d is nondecreasing, we conclude that $w(\phi(t, j)) \leq z(j)$ for all $(t, j) \in \text{dom } \phi$. On the other hand, by item 2), there exists $j^\star \in \mathbb{N}$ such that $j^\star \leq \sup\{j : (t,j) \in \operatorname{dom} \phi\}$ and $z(j) \in (-\infty, r_2)$ for all $j \geq j^\star$. This fact implies the existence of $(t^\star, j^\star) \in (\mathbb{R}_{\geq 0} \times \{j^\star\}) \cap \operatorname{dom} \phi \neq \emptyset$ such that $w(\phi(t^\star, j^\star)) < r_2$. Hence, $\phi(t^\star, j^\star) \in \widetilde{S}_2$ and $w(\phi(t,j)) < r_2$ for all $(t,j) \in \operatorname{dom} \phi$ with $j \geq j^\star$ and $\widetilde{S}_2$ is nonempty. As a consequence, using the fact that $\widetilde{S}_2$ is a subset of $\mathcal{A}$, we conclude that, for all $(t,j) \in \operatorname{dom} \phi$ such that $j \geq j^\star$, it follows that $\phi(t,j) \in \mathcal{A}$, which completes the proof. $\square$

Remark C.10. *Note that condition 2) in Proposition C.8 holds for free for complete solutions starting from $\mathcal{O}$ for which the domain is unbounded in t. Similarly, condition 2) in Proposition C.9 holds for free for complete solutions starting from $\mathcal{O}$ for which the domain is unbounded in j. Moreover, maximal solutions are complete when the conditions in [20, Proposition 2.10 or Proposition 6.10] hold.*

Remark C.11. *In Theorem C.5, one could think of unifying conditions 1) and 2) as follows:*

$$\begin{aligned} &\langle \nabla v(x), \eta \rangle \leq f_c(v(x)) && \forall \eta \in F(x) \cap T_C(x), \forall x \in C, \\ &v(\eta) \leq f_d(v(x)) && \forall \eta \in G(x), \forall x \in D, \end{aligned} \tag{C.6}$$

where the functions f_c and f_d are defined in Theorem C.5. Furthermore, one could think of concluding pre-eventual conditional invariance of $\mathcal{A}$ with respect to $\mathcal{O}$ by showing that the set $(-\infty, r_1]$ is pre-eventually conditionally invariant with respect to $v(\mathcal{O})$ for the reduced system given by

$$\begin{aligned} \dot{y} &= f_c(y) && y \in v(C) \\ y^+ &= f_d(y) && y \in v(D). \end{aligned} \tag{C.7}$$

Such a comparison-based reasoning is very useful to analyze purely continuous-

time or purely discrete-time systems. In general, a key step for such a reasoning to hold consists in showing that (C.6) *and* (C.7) *imply that*

$$v(\phi(t,j)) \leq y(t,j) \qquad \forall (t,j) \in dom\ \phi. \tag{C.8}$$

However, (C.8) *does not necessarily hold under* (C.6) *and* (C.7) *due to the possible mismatch in the instant of jumps between the solutions ϕ to $\mathcal{H}$ and y to* (C.7)*. It holds, however, if we replace the inequalities in* (C.6) *by equalities (the latter is in general very restrictive). As a consequence, the comparison arguments, in general, do not extend directly to the context of hybrid systems. Hence, it would be interesting to investigate a general version of the comparison lemma for hybrid systems as it would simplify considerably the conditions in Theorem C.5.*

C.3 Sufficient Conditions for Pre-eventual Conditional Invariance using Approximate Flow Lengths

In this section, along the ideas discussed in Remark C.11 and inspired by the work in [60] for hybrid observers, we propose a new set of sufficient conditions for pre-eventual conditional invariance. Indeed, given two sets $\mathcal{O},\ \mathcal{A} \subset \mathbb{R}^n$, we assume the existence of a set $K \subset \mathcal{A}$ that is forward pre-invariant for $\mathcal{H}$. Furthermore, we assume that we know approximately the length of the flow interval, between each successive jumps, for all the solutions starting from $\mathcal{O}$ until they reach the set K. Below, we use $\mathrm{dom}_t\, \phi$ (respectively, $\mathrm{dom}_j\, \phi$) to denote the projection of $\mathrm{dom}\ \phi$ on the first (respectively, second) dimension, and we denote $T(\phi) = \sup \mathrm{dom}_t\, \phi$

and $J(\phi) = \sup \text{dom}_j\, \phi$. By $t_j(\phi)$, we denote the time stamp associated to the jump j uniquely characterized by $(t_j(\phi), j-1) \in \text{dom}\, \phi$ and $(t_j(\phi), j) \in \text{dom}\, \phi$.

Definition C.12 (Approximate flow lengths). *A closed set $\mathcal{I}_{\mathcal{O},K} \subset \mathbb{R}_{\geq 0}$ is said to be the set of approximate flow lengths for the solutions to $\mathcal{H}$ starting from the set $\mathcal{O}$ and remaining in $\mathbb{R}^n \backslash K$ if, for each such a solution, we have*

$$0 \leq t - t_j(\phi) \leq \sup \mathcal{I}_{\mathcal{O},K} \quad \forall (t,j) \in \textit{dom}\, \phi, \tag{C.9a}$$

$$t_{j+1}(\phi) - t_j(\phi) \in \mathcal{I}_{\mathcal{O},K} \quad \forall j \in \mathbb{N}_{>0} \textit{ if } J(\phi) = +\infty, \tag{C.9b}$$

$$\textit{or } \forall j \in \{1, \ldots, J(\phi) - 1\} \textit{ if } J(\phi) < +\infty.$$

The set $\mathcal{I}_{\mathcal{O},K} \subset \mathbb{R}_{\geq 0}$ contains the possible lengths of the flow intervals between successive jumps for the solutions starting from $\mathcal{O}$ and remaining in $\mathbb{R}^n \backslash K$. The role of (C.9a) is to bound the length of the intervals of flow which are not covered by (C.9b), namely possibly the first $[0, t_1(\phi)]$ and the last $\text{dom}_t\, \phi \cap [t_{J(\phi)}(\phi), +\infty]$ (when they are defined). The existence of a set $\mathcal{I}_{\mathcal{O},K} \subset \mathbb{R}_{\geq 0}$ is not a problem, even when $K = \emptyset$, since it can always be given by $\mathcal{I}_{\mathcal{O},K} = \mathbb{R}_{\geq 0}$. However, it has advantage when $\mathcal{I}_{\mathcal{O},K} \subset \mathbb{R}_{\geq 0}$ is selected as tight as possible, namely to have as much information about the duration of flow between successive jumps as possible so that we reduce the number of possible solutions.

Theorem C.13 (Pre-eventual Conditional Invariance under Approximate Flow Lengths). *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and three sets $\mathcal{O} \subset C \cup D$, $\mathcal{A} \subset \mathbb{R}^n$, and $K \subset \mathcal{A}$ such that K is forward pre-invariant for $\mathcal{H}$. Let the set $\mathcal{I}_{\mathcal{O},K}$ be as in Definition* (C.12) *and let $\tau_M := \sup \mathcal{I}_{\mathcal{O},K}$. Then, the set $\mathcal{A}$ is* pre-eventually conditionally invariant *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if*

1) there exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, *and a function $f_d : \mathbb{R} \to \mathbb{R}$ which is nondecreasing such that*

$$\langle \nabla v(x), \eta \rangle \leq f_c(v(x)) \quad \forall x \in C \backslash K, \forall \eta \in F(x) \cap T_C(x), \tag{C.10a}$$

$$v(\eta) \leq f_d(v(x)) \quad \forall x \in D \backslash K, \forall \eta \in G(x); \tag{C.10b}$$

2) there exists a constant $r > 0$ such that

$$S := \{x \in C \cup D : v(x) < r\} \subset \mathcal{A}; \tag{C.11}$$

3) Every solution to the reduced hybrid system $\mathcal{H}_r$ starting from $v(\mathcal{O}) \times \{0\}$ converge to $(-\infty, r] \times \mathbb{R}_{\geq 0}$ in finite time, where

$$\mathcal{H}_r : \begin{cases} \begin{bmatrix} \dot{y} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} f_c(y) \\ 1 \end{bmatrix} & (y, \tau) \in \mathbb{R} \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0}), \\ \begin{bmatrix} y^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} f_d(y) \\ 0 \end{bmatrix} & (y, \tau) \in \mathbb{R} \times \mathcal{I}_{\mathcal{O},K}. \end{cases} \tag{C.12}$$

Proof. According to the definition of pre-eventual conditional invariance, we need to show that for each complete solution ϕ to $\mathcal{H}$ starting from $\mathcal{O}$, there exists a hybrid time $(t^\star, j^\star) \in \text{dom } \phi$ such that $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \text{dom } \phi$ such that $t + j \geq t^\star + j^\star$. Without loss of generality, consider a complete solution ϕ to $\mathcal{H}$ starting from $\phi(0, 0) \in \mathcal{O}$ and remaining in the complement of K. Let (y, τ) be a maximal solution pair to the system in (C.12) such that $y(0, 0) = v(\phi(0, 0)) \in \mathbb{R}$ and $\tau(0, 0) = 0$. By definition of the set $\mathcal{I}_{\mathcal{O},K}$, we conclude the existence of a solution (y, τ) to $\mathcal{H}_r$ such that dom $(y, \tau) =$ dom $y =$ dom ϕ.

Now, we pick any $j \in \text{dom}_j\, \phi$ and we let $I^j := \{t \in \mathbb{R}_{\geq 0} : (t, j) \in \text{dom } \phi\}$.

Using Lemmas A.1, A.2, and A.3 under (C.10a), we conclude that $v(\phi(t,j)) \leq y(t,j)$ for all $t \in I^j$.

Furthermore, for any $j \in \operatorname{dom}_j \phi$ such that $(t, j-1), (t,j) \in \operatorname{dom} \phi$, using (C.10b), we conclude that $v(\phi(t,j)) \leq y(t,j)$ for all $(t,j) \in \operatorname{dom} \phi$. Since the solution y starting from $v(\phi(0,0))$ converges to $(-\infty, r]$ in finite time, we conclude that, there exists $(t^\star, j^\star) \in \operatorname{dom} \phi$, such that $v(\phi(t,j)) \in (-\infty, r]$ for all $(t,j) \in \operatorname{dom} \phi : t+j \geq t^\star + j^\star$. This implies, by item 2) and completeness of ϕ, the existence of $(t^\star, j^\star) \in \operatorname{dom} \phi$ such that $\phi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom} \phi : t+j \geq t^\star + j^\star$, which completes the proof. $\square$

The following example illustrates Theorem C.13.

Example C.14 (Bouncing ball). *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ in Example 5.5 with $\gamma = 1$ and $\lambda = 0.5$. Let the sets $\mathcal{O} := \{0\} \times [2,3]$ and $\mathcal{A} := [0,1] \times [-1,1]$. Furthermore, consider the set $K := \{x \in C \cup D : 2x_1 + x_2^2 \leq 1/2\} \subset \mathcal{A}$. Using Proposition C.2 with the barrier function candidate $B(x) := 2x_1 + x_2^2 - 1/2$, we conclude that the set K is forward pre-invariant for the considered hybrid system. Next, for $v(x) := 2x_1 + x_2^2$, we conclude that for $f_c(y) = 0$,* (C.10a) *holds. Furthermore, for $f_d(y) = y/4$,* (C.10b) *holds. Now, for $r = 1/2$, it is easy to see that* (C.11) *holds. Finally, to conclude pre-eventual conditional invariance of $\mathcal{A}$ with respect to $\mathcal{O}$ using Theorem C.13, it is enough to show that the set $\mathcal{I}_{\mathcal{O},K}$ is bounded. Indeed, using Proposition C.2 with the barrier function candidate given by $B_1(x) := 2x_1 + x_2^2 - 9$, we conclude that the zero sublevel set of B_1, which contains $\mathcal{O}$, is forward pre-invariant. Hence, $\mathcal{R}(\mathcal{O})$ is bounded. Furthermore, from every initial condition in $\overline{\mathcal{R}(\mathcal{O})}$, the unique maximal solution reaches the set D in finite time. Hence, the interval of flow of the solutions starting from $\mathcal{O}$ is uniformly bounded.* $\triangle$

C.4 Sufficient Conditions for Eventual Conditional Invariance

Theorem C.15 (Eventual Conditional Invariance via Flows)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and two sets $\mathcal{O} \subset C \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$. Suppose that there exist a $\mathcal{C}^1$ function $v : \mathbb{R}^n \to \mathbb{R}$, a locally Lipschitz function $f_c : \mathbb{R} \to \mathbb{R}$, and a constant $r_1 > 0$ such that the conditions in Proposition C.8 hold. Then, the set $\mathcal{A}$ is* eventually conditionally invariant *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if the following additional condition holds:*

- *for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, there exists a solution y to $\dot{y} = f_c(y)$ starting from $v(\phi(0,0))$ satisfying $y(t) \in (-\infty, r_1)$ for all $t \geq t^\star$ and for some nonnegative $t^\star \leq \sup\{t : (t,j) \in \mathit{dom}\ \phi\}$.*

Theorem C.16 (Eventual Conditional Invariance via Jumps)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ and two sets $\mathcal{O} \subset C \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$. Suppose that there exist a $\mathcal{C}^1$ function $w : \mathbb{R}^n \to \mathbb{R}$, a nondecreasing function $f_d : \mathbb{R} \to \mathbb{R}$, and a constant $r_2 > 0$ such that the conditions in Proposition C.9 hold. Then, the set $\mathcal{A}$ is* eventually conditionally invariant *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if the following additional condition holds:*

- *for each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{O})$, there exists a solution z to $z^+ = f_d(z)$ starting from $w(\phi(0,0))$ satisfying $z(j) \in (-\infty, r_2)$ for all $j \geq j^\star$ and for some nonnegative $j^\star \leq \sup\{j : (t,j) \in \mathit{dom}\ \phi\}$.*

Proof. The proof is the same as in Proposition C.9. □

Theorem C.17 (From pre to non-pre Eventual Conditional Invariance)**.** *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$. Consider two sets $\mathcal{O} \subset C \cup D$ and $\mathcal{A} \subset \mathbb{R}^n$*

such that $\mathcal{A}$ is pre-eventually conditionally invariant with respect to $\mathcal{O}$. Then, the set $\mathcal{A}$ is eventually conditionally invariant *with respect to the set $\mathcal{O}$ for $\mathcal{H}$ if the following property holds:*

($\star$) *There exists a set $S \subset C \cup D \cup \mathcal{A}$ such that $\mathcal{O} \cup \mathcal{A} \subset S$ and S is forward invariant for $\mathcal{H}_s = (C_s, F_s, D_s, G_s)$ in* (4.10) *with Q therein replaced by $\mathcal{A}$.*

Proof. When ($\star$) holds, since the set $\mathcal{A}$ is pre-eventually conditionally invariant with respect to the set $\mathcal{O}$ for $\mathcal{H}$, to complete the proof, it remains only to show that the solutions to $\mathcal{H}$ starting from $\mathcal{O}\backslash\mathcal{A}$ always reach the set $\mathcal{A}$. Proceeding by contradiction, assume the existence of a maximal solution ϕ to $\mathcal{H}$ starting from $\mathcal{O}\backslash\mathcal{A}$ that never reaches the set $\mathcal{A}$. We notice that each solution to $\mathcal{H}$ starting from $\mathcal{O}\backslash\mathcal{A}$ is a solution to $\mathcal{H}_s$ provided that it does not reach the set $\mathcal{A}$. Hence, since the set S is forward invariant for $\mathcal{H}_s$, we conclude that the solution ϕ is complete. The aforementioned fact contradicts the fact that $\mathcal{A}$ is pre-eventually conditionally invariant with respect to the set $\mathcal{O}$ for $\mathcal{H}$, which completes the proof. $\square$

Example C.18. *We reconsider the hybrid system in Example C.6. It is easy to see that the set $S := \mathcal{O} \cup \mathcal{A}$ is forward invariant for $\mathcal{H}_w$. Indeed, all the solutions to $\mathcal{H}_w$ starting from the set $\mathcal{O}$ flow in the set $\mathcal{O}$ until they reach the set $\mathcal{A}$. Since from $\mathcal{A}$, every solution is discrete, complete, and remains in $\mathcal{A}$, S is forward invariant.* $\triangle$

Appendix D

Hybrid Controller for Grasping

D.1 Hybrid Controller in Local Coordinates

For easiness of the exposition, we first introduce the hybrid controller supervising the position and force control in specific coordinates, then change coordinates to allow for arbitrary contact locations and force directions, and finally introduce the complete hybrid controller along with its main properties.

In this section, we introduce a (logic-based) hybrid controller using measurements of the contact force. With a specific coordinate system, the proposed approach enables transitions between the position and force controllers. For simplicity, we consider only one agent in local coordinates (as in [44]) but including vertical motion. The state of an individual agent $\eta := (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ is represented with respect to the local frame where the constraints are imposed by the object. The states η_1, η_2 and η_3 are the current horizontal position, the current horizontal velocity, and the current vertical position, respectively.

D.1.1 Position Controller

A position controller κ_P is designed for controlling the horizontal and vertical position of the agent. The horizontal position controller, denoted $\kappa_P^h(\eta, x_\ell^d)$, is composed of proportional/derivative control, and the vertical position controller, denoted $\kappa_P^v(\eta, y_\ell^d)$, is given by proportional control. The position controllers κ_P^h and κ_P^v stabilize the horizontal position and vertical position, respectively, and are given by

$$\kappa_P^h(\eta, x_\ell^d) = k_p(x_\ell^d - \eta_1) - k_d\eta_2 \tag{D.1a}$$

$$\kappa_P^v(\eta, y_\ell^d) = k_{p,y}(y_\ell^d - \eta_3). \tag{D.1b}$$

The parameters $x_\ell^d, y_\ell^d \in \mathbb{R}$ are the desired horizontal and vertical position when using the position controllers. The parameters k_p, $k_d \in \mathbb{R}$ are the proportional and derivative feedback gains of the horizontal position controller, and $k_{p,y} \in \mathbb{R}$ is the proportional feedback gain of the vertical position controller.

D.1.2 Force Controller

A force controller κ_F relies on the contact force. We assume that the mass is unitary. Proportional/feedforward control is used for the force controller as follows:

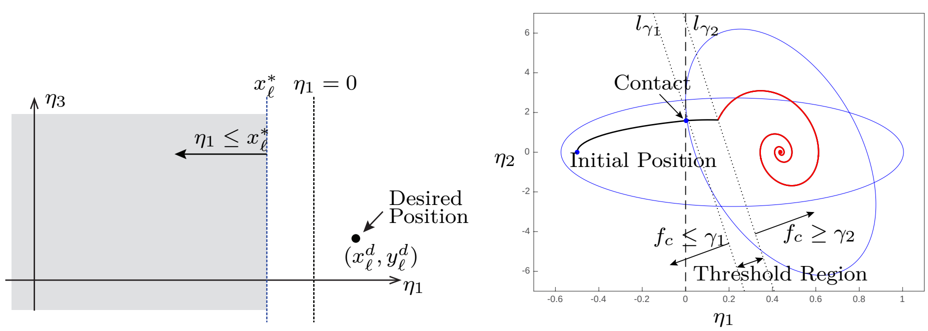
$$\kappa_F(\eta, f_c^d) = f_c(\eta_1, \eta_2) + k_f(f_c^d - f_c(\eta_1, \eta_2)) \tag{D.2}$$

where

$$f_c(\eta_1, \eta_2) = \begin{cases} k_c\eta_1 + b_c\eta_2 & \text{if} \quad \eta_1 \geq 0 \\ 0 & \text{if} \quad \eta_1 < 0 \end{cases}. \tag{D.3}$$

The parameter f_c^d is the desired set-point for the contact force $f_c(\eta_1, \eta_2)$. The desired contact force f_c^d satisfies $0 < f_c^d < \hat{f}_c$ where $\hat{f}_c$ is the maximum allowed force. The parameter $k_f \in \mathbb{R}$ is the proportional gain, and $k_c, b_c \in (0, +\infty)$ are the elastic and the viscous parameters of the contact, respectively. We consider the contact force due to the interaction between the robot and its environment occurs along a normal direction. The work environment can be defined along the horizontal direction, such as $W = \{\eta \in \mathbb{R}^3 : \eta_1 \geq 0\}$ with the surface $S = \{\eta \in \mathbb{R}^3 : \eta_1 = 0\}$.

D.1.3 Hybrid Controller Supervising κ_P and κ_F



(a) Example of position control.

(b) Example of switching between position/force control.

Figure D.1: (a) When the vertical position η_3 is controlled, the horizontal position η_1 is far enough from the surface of the work environment, $\eta_1 \leq x_\ell^*$. (b) When $f_c(\eta_1, \eta_2) \geq \gamma_2$, the position controller is switched to the force controller. Two boundary lines l_{γ_1}, l_{γ_2} correspond to threshold parameters γ_1, γ_2, respectively.

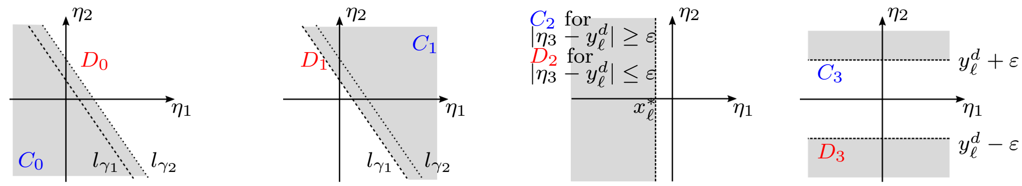
The main idea of the proposed approach is to stabilize the horizontal and vertical position, and to regulate the contact force by switching between the position and force control. The proposed control approach is implemented in a hybrid controller $\mathcal{H}_c$. At this point, the logic variable $q \in Q := \{-1, 0, 1\}$ are added to the state variable.

The logic variable q is used to switch between the position and force controllers, and the logic variable q makes the transitions between the horizontal and vertical controllers. Specifically, the position controller is activated and the force controller is deactivated when q is set to 0 or 1, whereas the position controller is deactivated and the force controller is activated when q is set to -1. Additionally, q is set to 0 when the vertical position controller is applied, whereas q is set to 1 when the horizontal position controller is applied.

Figure D.1 shows the basic concept of the proposed approach. As shown in Figure D.1(a), when the agent is controlled vertically, the horizontal position η_1 is far enough from the surface of the environment. In this way, when $q=0$, the horizontal position η_1 is restricted as $\eta_1 \leq x_\ell^*$ where $x_\ell^* \in \mathbb{R}_{<0}$ denotes the minimum horizontal distance from the surface of the environment $S = \{\eta \in \mathbb{R}^3 : \eta_1 = 0\}$. Note that $x_\ell^d > x_\ell^*$. Moreover, we define the region of vertical position control with the thresholds $\varepsilon \in \mathbb{R}_{>0}$. The vertical position controller can be applied (i.e., $q=0$) when $|\eta_3 - y_\ell^d| \geq \varepsilon$ and $\eta_1 \leq x_\ell^*$, whereas horizontal position control can be performed (i.e., $q=1$) when $|\eta_3 - y_\ell^d| \leq \varepsilon$.

Furthermore, when the agent establishes a contact with the work environment, the position controller is switched to the force controller according to the value of the contact force f_c relative to the hysteresis levels defined by the thresholds γ_1 and γ_2: The conditions $f_c(\eta_1, \eta_2) \leq \gamma_1$ and $f_c(\eta_1, \eta_2) \geq \gamma_2$ are used to determine which controller (κ_P^h or κ_F) is activated. As shown in Figure D.1(b), the threshold parameters γ_1 and γ_2 are used in the definition of the lines l_{γ_1} and l_{γ_2} as follows: $l_{\gamma_1} := \{(\eta_1, \eta_2) : \eta_2 = -\frac{k_c}{b_c}\eta_1 + \frac{\gamma_1}{b_c}\}$ and $l_{\gamma_2} := \{(\eta_1, \eta_2) : \eta_2 = -\frac{k_c}{b_c}\eta_1 + \frac{\gamma_2}{b_c}\}$.

While the horizontal position controller is applied (i.e., $q=1$), the supervisor switches the force controller on (i.e., $q^+ = -1$) if $f_c(\eta_1, \eta_2) \geq \gamma_2$. On the other hand, if $f_c(\eta_1, \eta_2) \leq \gamma_1$ while the force controller is applied (i.e., $q=-1$), the supervisor



(a) $\forall \eta_3 \in \mathbb{R}, q = 1$ **(b)** $\forall \eta_3 \in \mathbb{R}$, $q = -1$ **(c)** $q = 0, \eta_1 \leq x_\ell^*$ **(d)** $q = 1$

Figure D.2: Flow and jump sets of the hybrid controller.

selects the horizontal position controller (i.e., $q^+ = 1$).

The dynamics of the hybrid controller $\mathcal{H}_c$ has the update law for the logic variables q as follows.

Jumps:

- From horizontal position control to force control (i.e., $q^+ = -1$): when $q = 1$ and $f_c(\eta_1, \eta_2) \geq \gamma_2$, the logic variable q is mapped to -1.

- From force control to horizontal position control (i.e., $q^+ = 1$): when $q = -1$ and $f_c(\eta_1, \eta_2) \leq \gamma_1$, the logic variable q is mapped to 1.

- From vertical position control to horizontal position control (i.e., $q^+ = 1$): when $q = 0$ and $|\eta_3 - y_\ell^d| \leq \varepsilon$, the logic variable q is mapped to 1.

- From horizontal position control to vertical position control (i.e., $q^+ = 0$): when $q = 1$ and $|\eta_3 - y_\ell^d| \geq \varepsilon$, the logic variable q is mapped to 0.

Flows:

- $\dot{q} = 0$: when $q = 1$ and $f_c(\eta_1, \eta_2) \leq \gamma_2$; or when $q = -1$ and $f_c(\eta_1, \eta_2) \geq \gamma_1$; or when $q = 0$ and $|\eta_3 - y_\ell^d| \geq \varepsilon$; or when $q = 1$ and $|\eta_3 - y_\ell^d| \leq \varepsilon$, the logic variables remain constant.

The output of the hybrid controller is given by

$$u := (u_x, u_y) \tag{D.4a}$$

where

$$\begin{aligned} u_x &= \begin{cases} \kappa_P^h(\eta, x_\ell^d) & \text{if } q = 1 \\ \kappa_F(\eta, f_c^d) & \text{if } q = -1 \\ 0 & \text{otherwise,} \end{cases} \\ u_y &= \begin{cases} \kappa_P^v(\eta, y_\ell^d) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{D.4b}$$

The flow and jump sets, C and D with state

$$z := (\eta, q) \in X := \mathbb{R}^3 \times Q$$

are described in Figure D.2. The hybrid controller $\mathcal{H}_c$ in the closed-loop system, denoted by $\mathcal{H}_{cl}$ has continuous dynamics given by

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= u_x \\ \dot{\eta}_3 &= u_y \\ \dot{q} &= 0 \end{aligned}\right\} \quad z \in C \tag{D.5a}$$

where u_x and u_y are given in (D.4) and $C := C_0 \cup C_1 \cup C_2 \cup C_3 \subset X$ defines the flow set,

$$
\begin{aligned}
C_0 &:= \{z \in X : q = 1, f_c(\eta_1, \eta_2) \leq \gamma_2\} \\
C_1 &:= \{z \in X : q = -1, f_c(\eta_1, \eta_2) \geq \gamma_1\} \\
C_2 &:= \{z \in X : q = 0, \eta_1 \leq x_\ell^*, |\eta_3 - y_\ell^d| \geq \varepsilon\} \\
C_3 &:= \{z \in X : q = 1, |\eta_3 - y_\ell^d| \leq \varepsilon\}.
\end{aligned}
\tag{D.5b}
$$

The closed-loop system $\mathcal{H}_{cl}$ has jump dynamics given by

$$
\begin{aligned}
&\xi_\ell^+ = \xi_\ell, \quad q^+ = -1 \qquad z \in D_0 \\
&\xi_\ell^+ = \xi_\ell, \quad q^+ = 1 \qquad z \in D_1 \\
&\xi_\ell^+ = \xi_\ell, \quad q^+ = 1 \qquad z \in D_2 \\
&\xi_\ell^+ = \xi_\ell, \quad q^+ = 0 \qquad z \in D_3
\end{aligned}
\tag{D.6a}
$$

and the jump set is $D := D_0 \cup D_1 \cup D_2 \cup D_3 \subset X$,

$$
\begin{aligned}
D_0 &:= \{z \in X : q = 1, f_c(\eta_1, \eta_2) \geq \gamma_2\} \\
D_1 &:= \{z \in X : q = -1, f_c(\eta_1, \eta_2) \leq \gamma_1\} \\
D_2 &:= \{z \in X : q = 0, |\eta_3 - y_\ell^d| \leq \varepsilon\} \\
D_3 &:= \{z \in X : q = 1, |\eta_3 - y_\ell^d| \geq \varepsilon\}.
\end{aligned}
\tag{D.6b}
$$

With these definitions, the flow map and the jump map are defined as follows:

$$
F(z) = [\dot{\eta}_\ell \quad u_x \quad u_y \quad 0]^\top \qquad \forall z \in C \tag{D.7}
$$

where u_x and u_y are given in (D.4); and the jump map is given by

$$G(z) = \begin{cases} G_0(z) & \text{if } z \in D_0 \\ G_1(z) & \text{if } z \in D_1 \cup D_2 \quad \forall z \in D \\ G_2(z) & \text{if } z \in D_3 \end{cases} \tag{D.8a}$$

where

$$G_0(z) = (\eta, -1)\,, \quad G_1(z) = (\eta, 1)\,, \quad G_2(z) = (\eta, 0)\,. \tag{D.8b}$$

The resulting hybrid closed-loop system is given by

$$\mathcal{H}_{cl} \begin{cases} \dot{z} \;\; = F(z) & z \in C \\ z^+ = G(z) & z \in D. \end{cases} \tag{D.9}$$

Lemma D.1. *$\mathcal{H}_{cl}$ satisfies the hybrid basic conditions, which are as follows:*

A1) C and D are closed sets in X.

A2) F: $X \to X$ is continuous on C.

A3) G: $X \to X$ is an outer semicontinuous and locally bounded relative to D, and $D \subset domG$.

Proof. Condition (A1) is satisfied since C and D are closed. The flow map F in (D.7) is continuous on C, satisfying (A2). The jump map G in (D.8) is single valued on D and therefore it satisfies (A3). □

D.2 Stability of Hybrid Closed-loop System

In the following, the stability properties of the hybrid controller in the closed-loop system $\mathcal{H}_{cl}$ are revealed by using Lyapunov functions. The position controller κ_P and the force controller κ_F are designed for given parameters k_c, b_c of the work environment, desired contact force f_c^d, and controller parameters $k_p^h, k_d^h, k_p^v, k_f, x_\ell^d, \gamma_1, \gamma_2$. Before the stability result for $\mathcal{H}_{cl}$, we present a few technical lemmas.

Lemma D.2. *(Horizontal Position Controller) The closed-loop system with the horizontal position controller κ_P^h given by*

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = -k_p\,\eta_1 - k_d\,\eta_2 + k_p\,x_\ell^d \tag{D.10}$$

has the equilibrium point $(x_\ell^d, 0)$ globally asymptotically stable *where $x_\ell^d > 0$ and $k_p, k_d > 0$. Furthermore, a Lyapunov function certifying such property for system (D.10) is given by*

$$V_1(\eta_1, \eta_2) = \tfrac{1}{2}\,a_1\,(\eta_1 - x_\ell^d)^2 + \tfrac{1}{2}\,b_1\,\eta_2^2 \tag{D.11}$$

with a_1, b_1 satisfying $\frac{a_1}{b_1} = k_p$. Moreover, every solution to (D.10) starting from $(\eta_1^0, \eta_2^0) \in \mathbb{R}^2$ reaches the set $S_1 := \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 \geq 0\}$ in finite time. In particular, for every initial condition $(\eta_1^0, \eta_2^0) \in S_1^c := (\mathbb{R}^2 \setminus S_1) \cap \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_2 \geq 0\}$, every solution $t \mapsto (\eta_1(t), \eta_2(t))$ is such that $\eta_2(T) > 0$, where $T > 0$, is the time to reach S_1.

Proof. Note that the desired horizontal position x_ℓ^d is the steady state of the horizontal position. Let $e := \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \eta_1 - x_\ell^d \\ \eta_2 \end{bmatrix}$. A Lyapunov function V_1 :

$\mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be given by

$$V_1(e) = e^\top P_1\, e = \tfrac{1}{2} e^\top \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} e \\ = \tfrac{1}{2}\, a_1\, e_1^2 + \tfrac{1}{2}\, b_1\, e_2^2 \tag{D.12}$$

where $a_1, b_1 > 0$, and

$$f(e) := \begin{bmatrix} e_2 \\ -k_p\, e_1 - k_d\, e_2 \end{bmatrix}. \tag{D.13}$$

Therefore, it follows that, if $k_p, k_d > 0$ and $k_p = \frac{a_1}{b_1}$,

$$\langle \nabla V_1(e),\, f(e) \rangle = (a_1 - b_1\, k_p)\, e_1\, e_2 - b_1\, k_d\, e_2^2 \\ = -e^\top \begin{bmatrix} 0 & -a_1 + b_1\, k_p \\ 0 & b_1\, k_d \end{bmatrix} e \tag{D.14}$$

[44]. The equilibrium point $(x_\ell^d, 0)$ is said to be stable for the closed-loop system in (D.10). By Krasovskii-LaSalle's invariance principle, trajectories that stay in $\{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 \geq 0\}$ converges to the equilibrium point. Furthermore, to show that every solution starting from $(\eta_1^0, \eta_2^0) \in \mathbb{R}^2$ reaches the set $S_1 := \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 \geq 0\}$ in finite time. □

Lemma D.3. *(Horizontal Position Controller) The closed-loop system with the horizontal position controller κ_P^h given by*

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = -(k_p + k_c)\, \eta_1 + (-k_d - b_c)\, \eta_2 + k_p\, x_\ell^d \tag{D.15}$$

has the equilibrium point $(x_\ell^P, 0)$, $x_\ell^P := \frac{k_p}{k_p + k_c} x_\ell^d$ globally asymptotically stable *where $x_\ell^d > 0$ and $k_p, k_d > 0$. Moreover, a Lyapunov function certifying such* *property for system (D.15) is given by*

$$V_2(\eta_1, \eta_2) = \tfrac{1}{2}\, a_2\, (\eta_1 - x_\ell^P)^2 + \tfrac{1}{2}\, b_2\, \eta_2^2 \tag{D.16}$$

with a_2, b_2 *satisfying* $\frac{a_2}{b_2} = k_p + k_c$*. Moreover, every solution to (D.15) starting from* $(\eta_1^0, \eta_2^0) \in \mathbb{R}^2$ *reaches the set* $S_2 := \{(\eta_1, \eta_2) \in \mathbb{R}^2 : k_c\eta_1 + b_c\eta_2 \geq \gamma_2, \eta_1 \geq 0\}$ *in finite time. In particular, for every initial condition* $(\eta_1^0, \eta_2^0) \in S_2^c := (\mathbb{R}^2 \setminus S_2) \cap \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_2 \geq 0\}$*, every solution* $t \mapsto (\eta_1(t), \eta_2(t))$ *is such that* $\eta_2(T) > 0$*, where* $T > 0$*, is the time to reach* S_2*.*

Proof. Note that the steady state of the horizontal position is given by $x_\ell^P := \frac{k_p}{k_p+k_c} x_\ell^d$. Let $\bar{e} := \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} = \begin{bmatrix} \eta_1 - x_\ell^P \\ \eta_2 \end{bmatrix}$. A Lyapunov function $V_2 : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be given by

$$\begin{aligned} V_2(\bar{e}) = \bar{e}^\top P_2\, \bar{e} &= \frac{1}{2}\bar{e}^\top \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \bar{e} \\ &= \tfrac{1}{2}\, a_2\, \bar{e}_1^2 + \tfrac{1}{2}\, b_2\, \bar{e}_2^2 \end{aligned} \tag{D.17}$$

where $a_2, b_2 > 0$, and

$$f(\bar{e}) := \begin{bmatrix} \bar{e}_2 \\ -(k_p + k_c)\, \bar{e}_1 - (k_d + b_c)\, \bar{e}_2 \end{bmatrix}. \tag{D.18}$$

It follows that, if $k_p, k_d > 0$ and $\frac{a_2}{b_2} = k_p + k_c$,

$$\begin{aligned}\langle \nabla V_2(\bar{e}),\, f(\bar{e})\rangle &= (a_2 - b_2\,(k_p + k_c))\,\bar{e}_1\,\bar{e}_2 - b_2\,(k_d + b_c)\,\bar{e}_2^2 \\ &= -\bar{e}^\top \begin{bmatrix} 0 & -a_2 + b_2\,(k_p + k_c) \\ 0 & b_2\,(k_d + b_c) \end{bmatrix} \bar{e}\end{aligned} \tag{D.19}$$

[44]. The equilibrium point $(x_\ell^P, 0)$ is said to be stable for the closed-loop system in (D.15). By Krasovskii-LaSalle's invariance principle, trajectories that stay in $\{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 \geq 0\}$ converges to the equilibrium point. Furthermore, to show that every solution starting from $(\eta_1^0, \eta_2^0) \in \mathbb{R}^2$ reaches the set $S_2 := \{(\eta_1, \eta_2) \in \mathbb{R}^2 : k_c\eta_1 + b_c\eta_2 \geq \gamma_2, \eta_1 \geq 0\}$ in finite time. $\square$

Lemma D.4. *(Vertical Position Controller) The closed-loop system with the vertical position controller κ_P^v given by*

$$\dot{\eta}_3 = -k_{p,y}\eta_3 + k_{p,y}y_\ell^d \tag{D.20}$$

has the equilibrium point y_ℓ^d globally asymptotically stable *where $k_{p,y} > 0$. Furthermore, a Lyapunov function certifying such property for system (D.20) is given by*

$$V_3(\eta_3) = \frac{1}{2}\,a_3\,(\eta_3 - y_\ell^d)^2 \tag{D.21}$$

and $a_3 > 0$.

Proof. Note that the desired vertical position y_ℓ^d is the steady state of the vertical position. Denote $e_3 := y_\ell - y_\ell^d$, and $V_3 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be given by

$$V_3(e_3) = \frac{1}{2}\,a_3\,e_3^2 \tag{D.22}$$

where $a_3 > 0$. The continuous dynamic of e_3 is given by

$$\dot{e}_3 = -k_{p,y}\, e_3 \tag{D.23}$$

It follows that, if $k_{p,y} > 0$,

$$\langle \nabla V_3(e_3),\, \dot{e}_3 \rangle = -k_{p,y}\, a_3\, e_3^2. \tag{D.24}$$

Therefore, the equilibrium point y_ℓ^d is said to be asymptotically stable for the closed-loop system in (D.20). $\square$

Lemma D.5. *(Force Controller) The closed-loop system with the force controller κ_F given by*

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = -k_f k_c \eta_1 - k_f b_c \eta_2 + k_f f_c^d \tag{D.25}$$

has the equilibrium point $(x_\ell^F, 0)$, $x_\ell^F := \frac{f_c^d}{k_c}$ globally asymptotically stable *where $k_c, b_c > 0$, $k_f \in \left(0, \frac{-2c^2 k_c + abk_c + acb_c}{(bk_c - cb_c)^2}\right)$. A Lyapunov function for certifying such property for system (D.25) in coordinates $e_F := \begin{bmatrix} \eta_1 - x_\ell^F \\ \eta_2 \end{bmatrix}$ is given by*

$$V_F(e_F) = e_F^\top P_F e_F \tag{D.26a}$$

where

$$P_F := \begin{bmatrix} a & c \\ c & b \end{bmatrix} = R \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} R^\top, \tag{D.26b}$$

and $p_1, p_2 > 0$,

$$R := \begin{bmatrix} -\sin\beta & -\cos\beta \\ \cos\beta & -\sin\beta \end{bmatrix}, \quad \beta := \arctan\left(-\tfrac{k_c}{b_c}\right). \tag{D.26c}$$

Proof. Note that the steady state of the horizontal position is given by $x_\ell^F := \frac{f_c^d}{k_c}$. Denote $e_F := \begin{bmatrix} e_{F1} \\ e_{F2} \end{bmatrix} = \begin{bmatrix} \eta_1 - x_\ell^F \\ \eta_2 \end{bmatrix}$ and $p_1, p_2 > 0$. The diagonal matrix P_F corresponds to the matrix P_o rotated clockwise by $\beta = \arctan\left(-\frac{k_c}{b_c}\right)$ is given by

$$P_F := \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} -\sin\beta & -\cos\beta \\ \cos\beta & -\sin\beta \end{bmatrix} \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} -\sin\beta & \cos\beta \\ -\cos\beta & -\sin\beta \end{bmatrix}, \tag{D.27a}$$

where

$$\begin{aligned} a &:= p_1 \sin^2\beta + p_2 \cos^2\beta, \\ b &:= p_1 \cos^2\beta + p_2 \sin^2\beta, \\ c &:= (p_2 - p_1)\sin\beta\cos\beta, \quad p_2 < p_1. \end{aligned} \tag{D.27b}$$

Note that $\beta \in (-\frac{\pi}{2}, 0)$. It follows that

$$ab - c^2 > 0, \qquad \tfrac{b}{c} > \tfrac{b_c}{k_c}, \qquad \tfrac{a}{c} > \tfrac{k_c}{b_c}, \tag{D.28}$$

and $P_F = P_F^\top > 0$. Let $V_F : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be given by

$$\begin{aligned} V_F(e_F) &= e_F^\top P_F e_F = \tfrac{1}{2} e_F^\top \begin{bmatrix} a & c \\ c & b \end{bmatrix} e_F \\ &= \tfrac{1}{2}\, a\, e_{F1}^2 + \tfrac{1}{2}\, b\, e_{F2}^2 + c\, e_{F1}\, e_{F2} \end{aligned} \tag{D.29}$$

and

$$f(e_F) := \begin{bmatrix} e_2 \\ -k_f\, k_c\, e_1 - k_f\, b_c\, e_2 \end{bmatrix} \tag{D.30a}$$

where

$$k_c, b_c > 0, \quad k_f \in \left(0, \tfrac{-2c^2 k_c + abk_c + acb_c}{(bk_c - cb_c)^2}\right). \tag{D.30b}$$

It follows that

$$\begin{gathered} \langle \nabla V_F(e_F),\ f(e_F)\rangle = -\tfrac{1}{2} e_F^\top Q_F e_F, \\ Q_F = \begin{bmatrix} 2ck_c k_f & (bk_c + cb_c)k_f - a \\ (bk_c + cb_c)k_f - a & 2(bb_c k_f - c) \end{bmatrix}. \end{gathered} \tag{D.31}$$

We now check that Q_F is positive definite matrix. First, $2ck_c k_f$ is positive since $c, k_c, k_f > 0$. Then, note that the determinant of Q_F is given by

$$-(bk_c - cb_c)^2 k_f^2 + (-4c^2 k_c + 2abk_c + 2acb_c)k_f - a^2 \tag{D.32}$$

and the condition for the roots of this quadratic expressions to be real is

$$(-4c^2 k_c + 2bk_c a + 2cb_c a)^2 - 4a^2 (bk_c - cb_c)^2 \geq 0 \tag{D.33}$$

which holds true by $\frac{a}{c} > \frac{k_c}{b_c}$, then its two roots k_{f1}, k_{f2} are real and distinct. Furthermore, if $\frac{b}{c} > \frac{b_c}{k_c}$ then the largest rule, say $k_{f2} \in (0, \overline{k}_f)$, is positive where $\overline{k}_f = \frac{-2c^2 k_c + abk_c + acb_c}{(bk_c - cb_c)^2}$. Therefore, the equilibrium point $(x_\ell^F, 0)$ is asymptotically stable for the closed-loop system in (D.25) where $Q_F > 0$ such that $k_c, b_c > 0, k_f \in \left(0, \frac{-2c^2 k_c + abk_c + acb_c}{(bk_c - cb_c)^2}\right)$. □

The following result characterizes the compact set of initial conditions from where contact detection and contact force regulation are guaranteed, which are "subsets" of the basin of attraction of the closed-loop system.

Theorem D.6. *Denote* $\mathcal{A} := \{(x_\ell^F, 0, y_\ell^d)\}$*. Given parameters* $k_c, b_c \in (0, +\infty)$ *of the work environment and desired contact force* $0 < f_c^d < \hat{f}_c$*, there exist*

1) *Compact sets* $K_0, K_1, K_2 \subset \mathbb{R}^3$*,*

2) *Parameters* $k_p, k_d, k_f, \gamma_1, \gamma_2, x_\ell^d$ *of the hybrid controller such that* $\mathcal{A} \times \{-1\}$ *is locally asymptotically stable with basin of attraction containing* $((K_0 \times \{1\}) \cup (K_1 \times \{0\}) \cup (K_2 \times \{-1\})) \cap (C \cup D)$*.*

Remark D.7. *The set of initial conditions* K_0 *is such that, for every initial condition* $\eta^0 \in K_0$ *and for given parameters of the horizontal position controller, an agent can reach the surface of the work environment with a bounded value of the horizontal velocity. The set of initial conditions* K_1 *is such that, for every initial condition* $\eta^0 \in K_1$ *and for given parameters of the vertical position controller, the agent can approach the desired vertical position* $y_\ell^d \pm \varepsilon$*. The set of initial conditions* K_2 *can be estimated with the maximum level set of* $L_{V_F}(r_{max})$ *for* $\eta_1 \geq 0$*.*

Proof. We construct the compact set of initial conditions K from the Lyapunov functions in Lemmas D.2, D.3, D.4 and D.5. For every initial condition $\eta^0 \in K_0 \cup K_1$ and $q^0 \in Q$, by Lemmas D.2, D.3, D.4 and D.5, and the controller logic, every solution z to $\mathcal{H}_{cl}$ reaches the jump set D_0 in finite time. Moreover, when a solution reaches D_0 at the hybrid time (t, j), a solution $\eta(t, j) \in L_{V_F}(r_F)$ which is the maximum level set of the Lyapunov function V_F. Since $L_{V_F}(r_F) \times \{-1\} \subset C_1$, the flow set C_1 is forward invariant from the jumps at D_0. For every initial condition $\eta^0 \in K_2$ and $q^0 \in Q$, by definition of K_2 and the control logic, $\eta(t, j) \in L_{V_F}(r_F)$.

In addition, as every solution starting from K reaches the jump set D_0 in finite time and the flow set C_1 is forward invariant from the jumps at D_0, the solution stays in C_1 and converges to the equilibrium point. We can define the Lyapunov function V_F to show that the closed-loop system with the force controller guarantees Lyapunov stability of $\mathcal{A} \times \{-1\}$. The force controller which is defined as the system in (D.25) affects the states of horizontal position and velocity. Therefore, the states η_3 and q do not flow while the force controller is applied, and (η_3, q) is remained as it is initially. By Lemma D.5, the equilibrium point $(x_\ell^F, 0)$ is asymptotically stable for the closed-loop system in (D.25) provided that $Q_F > 0$. □

Lemma D.8. *From every point in $C \cup D$, there exists a solution and every maximal solution to $\mathcal{H}_{cl}$ is complete and bounded.*

Proof. The result follows from Proposition 6.10 in [39] using the following properties.

- For each point such that $z \in D$, the jump map satisfies $G(z) \subset C \cup D$. Therefore, (c) in Proposition 6.10 does not occur.
- Since z does not blow up in finite time, (b) in Proposition 6.10 does not occur.
- It is impossible for solutions with initial conditions $z(0, 0) \in C \cup D$ to escape $C \cup D$, all maximal solutions are complete and bounded.

□

D.3 Hybrid Controller in Global Coordinates

As explained in Section D.1, in the local coordinate system, the position of contact is defined as the origin and the direction of contact force is the horizontal direction according to the horizontal axis. In this section, we present matrix transformations to change the coordinates of the controller in Section D.1 so that contact is allowed in arbitrary locations, which we refer to is global coordinates.

A general transformation involves both rotation and translation of the coordinates; that is, the general transformation performs a rotation to make the axes of the two coordinates parallel, and then translates them. This relationship is described as

$$v_L = R(\angle v_L)\, v_G + v_{L/G} \tag{D.34}$$

where

$$v_{L/G} = R(\angle v_L)\,(p_G - p_L) \tag{D.35a}$$

$$R(\angle v_L) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{D.35b}$$

The subscripts L and G denote variables being in local coordinates and global coordinates, respectively. The variables $v_L, v_G \in \mathbb{R}^2$ are unitary vectors, and θ is the angle of v_G to match v_L direction. The vectors p_L and p_G denote the origin positions in the local and global coordinate systems with a specific coordinates, respectively.

The homogeneous transformation matrix combines both the rotation to global coordinates and the translation of the origin of local coordinates with respect to global coordinates. The vectors $v_G \in \mathbb{R}^2$ and $v_L \in \mathbb{R}^2$ are augmented by one and their dimensions of the relationship can be written in matrix form

$$\begin{bmatrix} v_L \\ 1 \end{bmatrix} = T_{L/G} \begin{bmatrix} v_G \\ 1 \end{bmatrix} \tag{D.36a}$$

where

$$T_{L/G} = \begin{bmatrix} R(\angle v_L) & v_{L/G} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & v_{L/G,1} \\ -\sin\theta & \cos\theta & v_{L/G,2} \\ 0 & 0 & 1 \end{bmatrix}. \tag{D.36b}$$

Now, we write the hybrid closed-loop system $\mathcal{H}_{cl}$, which is in the local coordinate system. For a single agent $\xi \in \mathbb{R}^3$ in the global coordinates defined by $\eta \in \mathbb{R}^3$, which are related by

$$\eta = \Phi(\xi) := \begin{bmatrix} \cos\theta\,\xi_1 + \sin\theta\,\xi_3 + v_{L/G,1} \\ \cos\theta\,\xi_2 + v_{L/G,1} \\ -\sin\theta\,\xi_1 + \cos\theta\,\xi_3 + v_{L/G,2} \end{bmatrix}. \tag{D.37}$$

We consider N hybrid systems on the (x, y)-plane, where the agent i has state η_i (in local coordinates) and ξ_i (in global coordinates). The output of the i-th controller is given by

$$u_i := (u_{x,i}, u_{y,i}) \tag{D.38a}$$

and

$$u_{x,i} = \begin{cases} \kappa_P^h(\Phi(\xi_i), x_{\ell,i}^d) & \text{if } q_i = 1 \\ \kappa_F(\Phi(\xi_i), f_{c,i}^d) & \text{if } q_i = -1 \\ 0 & \text{otherwise,} \end{cases} \tag{D.38b}$$

$$u_{y,i} = \begin{cases} \kappa_P^v(\Phi(\xi_i), y_{\ell,i}^d) & \text{if } q_i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The i-th hybrid controller leads to the closed-loop system $\mathcal{H}_i$ with state

$$z_i := (\xi_i, q_i) \in \; X := \mathbb{R}^3 \times Q$$

has continuous and discrete dynamics identical to the hybrid system defined in (D.9), and leads to the hybrid closed-loop system in local coordinates described as:

$$\mathcal{H}_{cl}^i \begin{cases} \dot{z}_i = F_i(z_i) & z_i \in C_i \\ z_i^+ = G_i(z_i) & z_i \in D_i \end{cases} \quad i \in \{1, 2, \ldots, N\} \tag{D.39}$$

where for each i, F_i and G_i are defined in (D.7) and (D.8), respectively; and C_i and D_i are given in (D.5b) and (D.6b), respectively. Moreover, the stability properties as in Section D.2 hold for the i-th hybrid system $\mathcal{H}_{cl}^i$. See [39] for more details.

Appendix E

List of Publications

Journal Articles

[1] Hyejin Han and Ricardo G. Sanfelice. Linear temporal logic for hybrid dynamical systems: Characterizations and sufficient conditions. *Nonlinear Analysis: Hybrid Systems*, 36:100865, 2020.

[2] Hyejin Han Mohamed Maghenem, and Ricardo G. Sanfelice. Certifying the LTL formula p until q in hybrid systems. *IEEE Transactions on Automatic Control*, 2021 (In preparation).

[3] Hyejin Han and Ricardo G. Sanfelice. Approximate and robust satisfaction of temporal logic specifications for hybrid dynamical systems under perturbations *Nonlinear Analysis: Hybrid Systems*, 2021 (In preparation).

Peer-Reviewed Conference Proceedings

[1] Hyejin Han, Mohamed Maghenem, and Ricardo G. Sanfelice. Sufficient conditions for satisfaction of formulas with until operators in hybrid systems. In *Proceedings of the 23rd International Conference on Hybrid Systems: Computation and Control (HSCC)*, pages 1–10, 2020.

[2] Hyejin Han and Ricardo G. Sanfelice. A hybrid control algorithm for object grasping using multiple agents. *Conference on Control Technology and Applications (CCTA)*, IEEE, 2018.

[3] Hyejin Han and Ricardo G. Sanfelice. Sufficient conditions for temporal logic specifications in hybrid dynamical systems. *Analysis and Design of Hybrid Systems (ADHS)*, 2018.