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A SINGULAR SOLUTION OF THE CAPILLARY EQUATION, I: EXISTENCE

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A singular solution of the capillary equation, I: existence *

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It is known, as a special case of a more general theorem [1], that if the constant κ is non-negative, every isolated singularity of a single valued solution of the capillary equation

$$\text{(1)} \qquad \operatorname{div} Tu = \kappa u \ , \quad Tu = \frac{1}{W}\nabla u \ , \quad W = \sqrt{1 + |\nabla u|^2} \ ,$$

is removable. If $\kappa < 0$ the proof of the general theorem fails; in this case, little is known about the behavior of solutions, presumably owing to the failure of the maximum principle.[(1)]

We show here that if $\kappa < 0$, (1) admits, in any number $n \geq 2$ of dimensions, a rotationally symmetric solution $U(r)$, $r^2 = \sum_1^n x_i^2$, with a non-removable isolated singularity at $r = 0$. In contrast to the behavior of solutions of linear elliptic equations (e.g., the Laplace equation), $U(r)$ has the same order of growth in r for every $n \geq 2$; more precisely, $U(r) \sim \frac{n-1}{\kappa r}$ as $r \to 0$. In the paper [2] directly following this one, we show that no other growth is possible for a symmetric singular solution.

The solution $U(r)$ is apparently related to a limiting behavior of pendent water drops; it is in that connection we first encountered it. For a discussion of these properties we refer the reader to our earlier paper [3] and to our forthcoming paper [4]. The solution was also encountered independently in a computational study by Huh [5].

1. For solutions depending only on distance r from the origin, (1) becomes, after a formal transformation,

$$\left(\frac{r^{n-1}\, u_r}{\sqrt{1+u_r^2}}\right)_r = -(n-1)\, r^{n-1}\, u \tag{2}$$

We seek, for $0 < r \le r_o$, a solution in the form

$$u(r) = -\frac{1}{r} + \frac{n+3}{2(n-1)}\, r^3 + \alpha_o(r)\, r^3 \tag{3}$$

with $\lim_{r\to 0} \alpha_o(r) = 0.$

We introduce a vector operator $\underline{\underline{T}}\alpha$, with $\underline{T} = (T_o, T_1)$ defined as follows: set $\underline{\alpha} = (\alpha_o, \alpha_1)$, and

$$\mathcal{F}(\underline{\alpha}(r); r) \equiv \frac{2}{r^3} - 3\,\frac{n+3}{n-1}\, r + \frac{(n+2)(n+6)}{4}\,\alpha_o r + 3(n+5)\alpha_1 r + \frac{n-1}{r^3}\,\alpha_o - \frac{n-1}{r}\,\gamma_1(1+\gamma_1^2) - (n-1)\gamma_o(1+\gamma_1^2)^{3/2} \tag{4}$$

with

$$\gamma_o(\alpha_o(r); r) \equiv -\frac{1}{r} + \frac{1}{2}\,\frac{n+3}{n-1}\, r^3 + \alpha_o(r)\, r^3 \tag{5_o}$$

$$\gamma_1(\alpha_1(r); r) \equiv \frac{1}{r^2} + \frac{3}{2}\,\frac{n+3}{n-1}\, r^2 + 3\alpha_1(r)\, r^2 \;. \tag{5_1}$$

We set

$$\mathcal{S}[f(\cdot)] = \int_0^r f(\tau)\, \sin\left[\sqrt{\frac{n-1}{2}}\left(\frac{1}{\tau^2} - \frac{1}{r^2}\right)\right] d\tau \tag{6}$$

$$\mathcal{C}[f(\cdot)] = \int_0^r f(\tau)\, \cos\left[\sqrt{\frac{n-1}{2}}\left(\frac{1}{\tau^2} - \frac{1}{r^2}\right)\right] d\tau \tag{7}$$

and we write, for any function f(r),

(8) $$^{(q)}f(r) = r^{\frac{n+q}{2}} f(r) \quad .$$

We then set

(9_o) $$T_o\underline{\alpha} = \frac{1}{\sqrt{n-1}}\, r^{-\frac{n+8}{2}} \mathcal{J}[^{(8)}\mathcal{F}(\underline{\alpha}(\cdot);\cdot)]$$

(9_1) $$T_1\underline{\alpha} = -\frac{n+2}{6} T_o\underline{\alpha} + \frac{1}{3} r^{-\frac{n+12}{2}} \mathcal{G}[^{(8)}\mathcal{F}(\underline{\alpha}(\cdot);\cdot)] \quad .$$

One verifies formally that for functions $u(r) \in C^{(2)}$ defined by (3), the relations (2) and

(10) $$\frac{d}{dr}\gamma_o(r) = \gamma_1(r)$$

are equivalent to the relation

(11) $$\underline{\alpha} = \underline{T\alpha} \quad .$$

2. Let $p_o(r)$ be a nondecreasing continuous function defined on the interval $[0,1]$, and satisfying

(12) $$p(0) = 0 \ , \quad \lim \frac{r^4}{p_o(r)} = 0 \quad .$$

We introduce the space $\mathcal{A}$ of continuous vector valued functions $\underline{\alpha}(r)$ on an interval $I_o = [0,r_o]$; we set

(13) $$\|\underline{\alpha}\| = \max_{r\in I_o} |\alpha_o(r)|p_o^{-1}(r) + \max_{r\in I_o} |\alpha_1(r)|$$

and we denote by $\mathcal{A}_M$ the ball $\|\underline{\alpha}\| < M$.

We assert first that <u>if r_o is sufficiently small</u>, <u>depending on</u> M, <u>then</u> <u>$T\alpha$</u> <u>maps</u> <u>$\mathcal{A}_M$ into itself</u>. To prove this, we use the relations, valid for $n+k > -6$,

(14) $$\mathcal{A}[{}^{(k)}1] = \frac{1}{\sqrt{n-1}}\,{}^{(k+6)}1 - \frac{(n+k+6)(n+k+10)}{4(n-1)}\mathcal{A}[{}^{(k+8)}1]$$

(15) $$\mathcal{B}[{}^{(k)}1] = \frac{n+k+6}{2\sqrt{n-1}}\,\mathcal{A}[{}^{(k+4)}1]$$

From (3,4,5) we obtain, for small r,

(16) $$\mathcal{F}(\underline{\alpha};r) = \lambda r + P(\underline{\alpha})r + Q(\underline{\alpha};r)$$

where

(17) $$\lambda = -\frac{3}{8}\frac{(5n+11)(n+7)}{n-1}$$

(18) $$P(\underline{\alpha}) = \frac{(n+2)(n-18)}{4}\alpha_o - 9(n-1)\alpha_o\alpha_1$$

(19) $$|Q(\underline{\alpha};r)| < A(\underline{\alpha})\, r^5$$

(20) $$\left|\frac{\partial Q(\underline{\alpha};r)}{\partial\alpha_j}\right| < B(\underline{\alpha})\, r^5 \quad , \quad j = 0,1 \quad ,$$

the functions A and B being uniformly bounded in any ball $\|\underline{\alpha}\| < M < \infty$.

From (14,15) we find

(21) $$\mathcal{A}[{}^{(10)}\lambda] = \frac{1}{\sqrt{n-1}}\lambda\, r^{\frac{n+16}{2}} + O(r^{\frac{n+24}{2}})$$

(22) $$\mathcal{B}[{}^{(10)}\lambda] = \frac{n+16}{2(n-1)}\lambda\, r^{\frac{n+20}{2}} + O(r^{\frac{n+28}{2}})$$

for small r.

From (6,7) follow directly

(23) $$|\mathcal{A}[{}^{(8)}Q(\underline{\alpha}(\cdot);\cdot)]| < C\, r^{\frac{n+20}{2}}$$

(24) $$|\mathcal{B}[{}^{(8)}Q(\underline{\alpha}(\cdot);\cdot)]| < C\, r^{\frac{n+20}{2}}$$

for a constant C depending only on the bound for A in (19). Similarly, the inequality $|\alpha_o| < C_o p_o(r)$ implies

$$(25_o) \qquad |\mathcal{A}[{}^{(10)}\alpha_o(\cdot)]| < \frac{2C_o}{n+12}\, r^{\frac{n+12}{2}}\, p_o(r)$$

$$(25_1) \qquad |\mathcal{A}[{}^{(10)}\alpha_o(\cdot)\alpha_1(\cdot)]| < \frac{2C_o}{n+12}\, r^{\frac{n+12}{2}}\, \|\underline{\alpha}\| p_o(r)$$

with analogous inequalities for $\mathcal{C}$.

We place these estimates in (9) to obtain, for $\underline{\alpha} \in \mathcal{A}_M$,

$$(26_o) \qquad |T_o\underline{\alpha}|p_o^{-1} < C(r^2 + r^4 p_o^{-1})$$

$$(26_1) \qquad |T_1\underline{\alpha}| < C(r^4 + r^2 p_o) + \frac{(n+2)|n-18| + 36(n-1)M}{6(n+12)} M\, p_o(r)$$

where C depends only on M. On the interval $0 \leq r \leq r_o$, the ball $\mathcal{A}_M$ is then mapped interior to a ball of radius

$$R(M) = C(r_o^2 + \varepsilon(r_o))(1+p_o(r_o)) + \frac{(n+2)|n-18| + 36(n-1)M}{6(n+12)} M\, p_o(r_o)$$

with

$$\varepsilon(r_o) = \max_{r \leq r_o} \frac{r^4}{p_o(r)},$$

hence into itself for sufficiently small r_o, depending only on M and on $p_o(r)$.

3. Using the same procedure, we may estimate $\|\underline{T\alpha} - \underline{T\beta}\|$, for any $\underline{\alpha}$, $\underline{\beta}$ in $\mathcal{A}_M$. We obtain now the estimates

$$(27_o) \qquad |T_o\underline{\alpha} - T_o\underline{\beta}|p_o^{-1} < C\|\underline{\alpha} - \underline{\beta}\|\, r_o^2$$

$$(27_1) \qquad |T_1\underline{\alpha} - \underline{T}_1\underline{\beta}| < C\|\underline{\alpha} - \underline{\beta}\|\, p_o(r_o)$$

the constant C depending now on a bound for B in (20). It follows that <u>for sufficiently small</u> r_o, <u>the mapping</u> (9) <u>contracts the ball</u> $\|\underline{\alpha}\| < M$. We conclude <u>there exists a unique fixed point</u> $\underline{A}(r) \in \mathcal{A}_M$, <u>and hence a corresponding solution</u> U(r) <u>of</u> (1) <u>in the indicated form</u> (3), <u>in some deleted neighbourhood of the origin</u>.

4. It follows immediately from the representation (11) that the solution $\underline{A}(r)$, sought originally in a class of functions

supposed only continuous, is in fact an analytic function of the (complex) variable r, and that for real r it has a zero of fourth order at $r = 0$. In particular, the sum of the first two terms on the right in (3) is asymptotic to $U(r)$ at $r = 0$.

5. We may use the representation (11) to obtain an asymptotic development for $U(r)$ in powers of r, at $r = 0$. To do so, we set

(28) $$a_3 = \frac{n+3}{2(n-1)} \quad , \qquad U^{(3)} = -\frac{1}{r} + a_3 r^3 \quad .$$

Then, as just observed,

$$\lim_{r\to 0} \frac{1}{r^3} (U(r) - U^{(3)}(r)) = 0 \quad .$$

We now note the decomposition

(29) $$Q(\underline{\alpha};r) = Q^{(5)}(\underline{\alpha}) r^5 + O(r^9)$$

where $Q^{(5)}(\underline{\alpha})$ is a polynomial in α_o, α_1. In particular, $Q^{(5)}(0)$ is a constant, and we may apply (14, 15) to the term $Q^{(5)}(0)r^5$. We may thus compute

(30) $$T_o[0] = -\frac{3}{8} \frac{(5n+11)(n+7)}{(n-1)^2} r^4 + O(r^8)$$

and we set, accordingly,

(31) $$a_7 = -\frac{3}{8} \frac{(5n+11)(n+7)}{(n-1)^2}$$

$$U^{(7)} = -\frac{1}{r} + a_3 r^3 + a_7 r^7 \quad .$$

On the other hand, using (29), we obtain after a single iteration in (11),

$$T_o \underline{A} = -\frac{3}{8}\frac{(5n+11)(n+7)}{(n-1)^2} r^4 + O(r^8) .$$

Thus, in particular,

$$\lim_{r\to 0} \frac{1}{r^7}(U(r) - U^7(r)) = 0 .$$

Continuing, we compute

$$T_o[a_7 r^7] = a_{11} r^8 + O(r^{12})$$

(32) $$r^3 T_o[a_7 r^7 + \dots + a_{2n-1} r^{2n-1}] =$$

$$= a_{11} r^{11} + \dots + a_{2n+3} r^{2n+3} + O(r^{2n+7})$$

and demonstrate, using successive decompositions of $Q(\underline{a};r)$, that the development is asymptotic.

The procedure is formal but tedious if the exact values of the coefficients are desired. For the (physical) case $n = 2$, we obtain, starting with $a_3 = \frac{5}{2}$, the successive numbers

$$\frac{5}{2}, -\frac{567}{8}, \frac{123149}{16}, -\frac{212466731}{128}, \dots .$$

It seems unlikely that the expansion is convergent.

6. The method, as is known, assures the uniqueness of the solution in the class considered. We study the uniqueness question further in the paper directly following this one, where we show that if $u(r)$ is a solution of (2) with an isolated singularity at $r = 0$, then either the singularity is removable or $u(r)$ has (up to a change in sign) the asymptotic form

$$u(r) = -\frac{1}{r} + O(r) \quad .$$

An improvement of this result to an estimate of the form (3) would thus establish the solution constructed here as the unique solution of (2) with a (non-removable) isolated singularity at the origin.

7. The singular solution U(r) can be obtained constructively by iteration, as the successive iterates of an arbitrary function in $\mathcal{A}_M$ will converge to $\underline{A}(r)$ on I_o. From a numerical point of view, it is preferable to use the first few terms of the asymptotic expansion as initial conditions for a numerical solution of the equation. The procedure appears to be stable with respect to small perturbations of the data, and yields the function shown in figure 1. See also Huh [5], who encountered the solution from this point of view.

The numerical result, together with independent information on asymptotic behavior of "pendent drop" solutions [3,4], leads us to the conjecture that the solution U(r) can be continued as a single valued solution of (2) for all positive r. It seems unlikely that the solution would be observed as a stable physical configuration for r large, cf the remarks in [6, § 2.2].

8. We note for reference the crucial importance for the existence proof, of the circumstance that α_o is a factor in both terms of $P(\underline{\alpha})$.

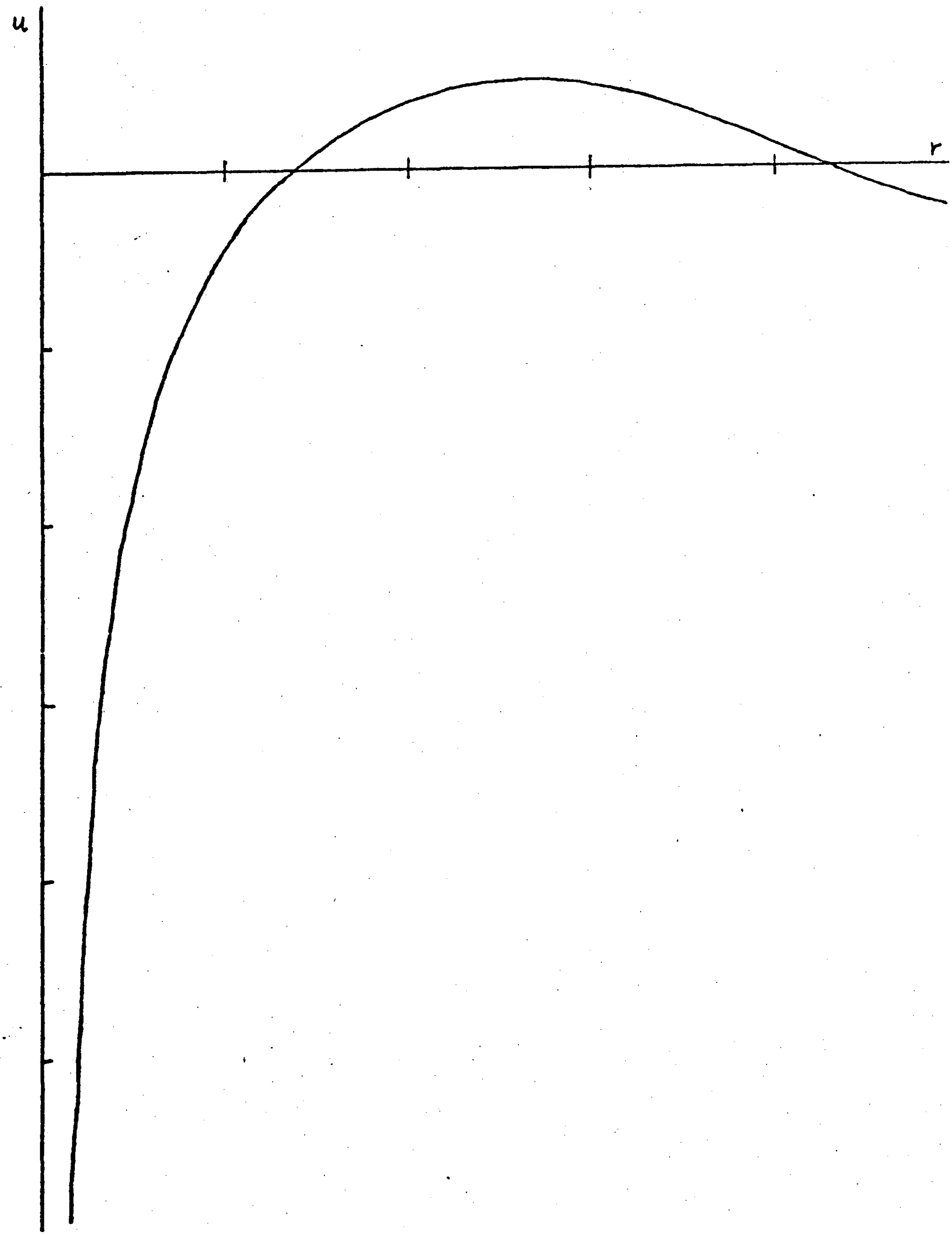

Figure 1

Footnotes

(1) p. 1 See however a remarkable paper of Wente [7] with regard to a related parametric problem.

(2) p. 7 We wish to thank G. Sod for carrying out part of the calculation.

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