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Summary. - The theorem for distributions in the angles θ, ϕ is extended to the third Euler angle, Ψ, whereupon it becomes equivalent to the rotational invariance of the responsible interactions. An interpretation of half-angle terms in the general rate formula is given.

1. Introduction

A formally complete representation of the content of rotational invariance is given by the Wigner-Eckart theorem. Although this theorem refers directly only to matrix elements between eigenstates of the total angular momentum J and its z component M, it may nevertheless be applied to other matrix elements, by introducing transformation matrices to and from a J, M representation.

However, when one set of states is given in a J, M representation and the other is given in terms of Euler angles and rotational invariants, the content of rotational invariance may be developed very directly. The most striking result is a restriction on the dependence on the Euler angles of the matrix elements; for fixed J, there appear only a finite number of standard functions. Such restrictions have long been known,[1, 2] but the arguments are simplified when presented for helicity eigenstates. The discussion here will aim at generality, in presenting results involving all three Euler angles.

2. Euler Angles

Euler angles are familiar from the discussion of a rigid body in classical mechanics. One imagines the body placed in some standard or fiduciary orientation with respect to a right-handed Cartesian reference frame. The body is rotated first through angle Ψ about the z axis, by which is meant a counterclockwise or positive rotation in the xy plane, and which is formally equivalent to a corresponding negative rotation of the reference frame; then through angle θ about the y axis; and finally through angle ϕ about the z axis. The entire operation will be designated $R(\phi\theta\Psi)$. The same operation can be applied to a fiduciary quantum-mechanical state, to generate a rotationally invariant family of states as the Euler angles ϕ, θ, Ψ are allowed to vary.

If the original state be decomposed into its irreducible parts according to the covering group of the rotation group, i. e., into eigenstates of total angular momentum J, it is seen that a rotation of 2π about any axis restores the components of integral J, and restores those of half-odd-integral J except for reversal of sign. Therefore, as ϕ, θ, Ψ vary over a doubled Euler angle domain, the state in all cases varies over all its rotational images. The usual convention for Euler angle domain is $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \Psi < 2\pi$, and an appropriate doubled range may be obtained by, e. g., allowing either ϕ or Ψ the range $[0, 4\pi)$.

More particularly, consider a state of n particles of definite spin, the ith particle with momentum $\underset{\sim}{p}_i$, and with helicity (spin component along the direction of $\underset{\sim}{p}_i$) λ_i, with total momentum $\sum_{i=1}^{n} \underset{\sim}{p}_i = \underset{\sim}{0}$. Such a state may be specified by a list of rotational invariants: the λ_i, the $\underset{\sim}{p}_i \cdot \underset{\sim}{p}_j$ -- the latter being of course highly redundant for large n, and any specifications of the types of particle involved, augmented by three Euler angles. The latter may be assigned as follows: Ψ to specify the orientation of the figure of n-1 of the momenta about a distinguished one, and θ, ϕ to specify the direction of the distinguished momentum. Such assignments are particularly convenient if one wishes to describe the angular distribution of one particular final-state particle, although other assignments could be equally interesting if one is interested in the entire figure of n momenta.

Exact phase conventions are easily fixed.[3] Pick creation operators for particles at rest, spin-quantized on the z axis, with Condon-Shortley conventions for the infinitesimal rotation operators J_x, J_y. Then apply an active Lorentz transformation to produce a creation operator for a particle of momentum p along the z axis, and finally rotate with the operator $R(\phi_p, \theta_p, 0)$ to produce the creation operator appropriate to an arbitrary direction. $R(\phi_p, \theta_p, -\phi_p)$ is an alternate choice used by JW, equally applicable here; any definite convention for Ψ_p will do. Our fiducial state is produced by n successive creation operators of this form acting on a vacuum state, and the general state is obtained by applying $R(\phi\theta\Psi)$ to the entire fiducial state.

Thus we have produced a rather general example of how coördinates may be so specified as to yield a system of states labelled exclusively by a list β of rotational invariants or "scalars" and Euler angles, ϕ, θ, Ψ. Since the scalars don't enter actively in our arguments, no detailed discussion of their redundancy in particular examples will be presented, but redundancy involving the Euler angles will claim some of our attention.

For $n \geq 3$ particles, we have, except in very special cases which we will not discuss, a configuration which requires all three angles for the description of its orientation. Thus, θ, ϕ may be used as spherical angles for the momentum of largest absolute value, and Ψ as the angle from the plane of longitude through that momentum to the plane determined by the two largest momenta; Ψ is then the angle east of south of the second momentum's projection on a sphere at the point θ, ϕ determined by the first. There is, then, no linear dependence among the states given by the ϕ, θ, Ψ of the Euler angle domain, but the states given by ϕ, θ, Ψ in the other half of a doubled domain are either $\pm$ the corresponding states, obtained from them by a 2π rotation, according as the number of fermions is even or odd. This redundancy is not necessarily present for states where the number of fermions is not sharp.

For a two-body state of particles of definite spin and helicity, there is much more redundancy, so that in this discussion, it appears as a degenerate case. In fact, the Ψ rotation leads only to a phase factor. Since the absolute value of the momentum is not in this case available for distinction of particles, there is even further redundancy for the case of two identical particles of the same helicity, for which a complete but independent set of states is obtained by cutting the Euler angle domain in half.

3. The General Theorem

The matrix element of a rotationally invariant operator S between a state $|J\,M\,\alpha\rangle$ specified by the total angular momentum J, its z component M, and a list α of rotational invariants, and a state $|\phi\theta\psi\beta\rangle$ specified by Euler angles and a list β of rotational invariants, is always of form

$$\begin{aligned}\langle\phi\theta\psi\beta|S|JM\alpha\rangle &= \Sigma_{M'}\,(\beta|JM'\alpha)e^{iM'\psi}d_{M'M}{}^{J}(-\theta)e^{iM\phi} \\ &= \Sigma_{M'}\,(-)^{M-M'}(\beta|JM'\alpha)\,D_{-M,-M'}{}^{J}(\phi\theta\psi),\end{aligned} \quad (1)$$

where the $(\beta|JM\alpha)$ are the values of the matrix elements for all Euler angles zero, and the d and D functions are the matrices for active rotations defined in JW; namely,

$$R(\phi\theta\psi) = e^{-iJ_z\phi}e^{-iJ_y\theta}e^{-iJ_z\psi}, \quad (2)$$

$$R(\phi\theta\psi)\,|JM\rangle = \sum_{M'} D_{M'M}{}^{J}(\phi\theta\psi)\,|JM'\rangle, \quad (3)$$

$$e^{-iJ_y\theta}|JM\rangle = \sum_{M'} d_{M'M}{}^{J}(\theta)\,|JM'\rangle, \quad (4)$$

so that

$$D_{M'M}{}^{J}(\phi\theta\psi) = e^{-iM'\phi}d_{M'M}{}^{J}(\theta)e^{-iM\psi}. \quad (5)$$

The result (1) is obtained as follows. $|\phi\theta\psi\beta\rangle = R(\phi\theta\psi)\,|000\beta\rangle$, so that $\langle\phi\theta\psi\beta| = \langle 000\beta|\,R^{-1}(\phi\theta\psi)$. The rotation operator thus extracted is commuted with the rotationally invariant operator S, and then in the form

$$R^{-1}(\phi\theta\psi) = e^{iJ_z\psi}e^{iJ_y\theta}e^{iJ_z\phi}$$

is applied via Equ. (4) to $|JM\alpha\rangle$, to obtain (1). The second form is obtained by means of the identity

$$d_{M'M}{}^{J}(\theta) = (-)^{2J+M+M'}d_{M'M}{}^{J}(-\theta) \quad (6)$$

supplemented by the identity

$$d_{M'M}{}^{J}(\theta) = (-)^{M'-M}d_{MM'}{}^{J}(\theta) \quad (7)$$

of JW, Equ. (A1).

The importance of the result lies in the fact that only a small number of functions of the Euler angles is involved, for fixed J. Such qualitative remarks will also be seen to hold for appropriate rates.

4. Relations Among the Generators

It is almost obvious that (1) not only follows from rotational invariance of S, but also implies rotational invariance. More precisely, we inquire whether the generators $(\beta|JM\alpha)$ of the matrix element according to (1) may be chosen as independent complex numbers, or what relations they must satisfy, to assure rotational invariance and a consistent definition of the matrix elements.

Suppose, first, that the states $|\phi\theta\Psi\beta\rangle$ are linearly independent, so that (1) may be used as a definition of its left-hand side for arbitrary complex numbers $(\beta|JM\alpha)$ on the right. Then it is easy to prove that the matrix so defined is rotationally invariant. Explicitly, we ask

$$\langle\phi_1\theta_1\Psi_1\beta|R^{-1}(\phi_2\theta_2\Psi_2)S|JM\alpha\rangle \overset{?}{=} \langle\phi_1\theta_1\Psi_1\beta|S\sum_{M'}D^{-1}{}_{M'M}{}^{J}(\phi_2\theta_2\Psi_2)|JM'\alpha\rangle. \qquad (8)$$

The l. h. s. is equal to $\langle\Phi\Theta\Psi\beta|S|JM\alpha\rangle$, where Φ, Θ, Ψ are the angles achieved by the successive operation $R(\phi_2\theta_2\Psi_2)R(\phi_1\theta_1\Psi_1) = R(\Phi\Theta\Psi)$. By (1), it follows that the l. h. s. is equal to

$$\langle 000\beta|S\sum_{M'}D^{-1}{}_{M'M}{}^{J}(\Phi\Theta\Psi)|JM'\alpha\rangle$$
$$= \langle 000\beta|S\sum_{M'M''}D^{-1}{}_{M'M''}{}^{J}(\phi_1\theta_1\Psi_1)D^{-1}{}_{M''M}{}^{J}(\phi_2\theta_2\Psi_2)|JM'\alpha\rangle.$$

But this is also the result obtained by applying (1) directly to the r. h. s. of (8).

We will now examine cases where the $|\phi\theta\Psi\beta\rangle$ are not linearly independent. We will always imagine redundant values of β to be cast out, so that we address ourselves only to redundancy involving the Euler angle variables. Of course, both sides of (1) are periodic with any doubled Euler angle domain for period, so that the general restriction of (1) as a definition to a single doubled Euler angle domain involves no restriction on the generators.

The first material restrictions we discuss arise in cases of states of n particles of definite spin, where the states for $\phi\Theta\Psi$ in a doubled Euler angle domain depend on those in an ordinary Euler angle domain. Equ. (1) may still be used as a definition of its l. h. s. if restricted to an ordinary Euler angle domain, provided that there are no further redundancies. If the number of fermions is even, then the l. h. s. reproduces itself in the other half of a doubled Euler angle domain, but the r. h. s. does not, unless it contains no term of half-odd-integral J. If the number of fermions is odd, the l. h. s. reproduces itself with a sign reversal, but the r. h. s. does not, unless it contains no term of integral J. The restrictions on the generators $(\beta|JM\alpha)$ are, therefore, that all those with half-odd-integral or integral J vanish, respectively. If the $|\phi\Theta\Psi\beta\rangle$ restricted to an ordinary Euler angle domain are linearly independent, there are no further relations implied by rotational invariance.

For the case of n = 2 particles with definite spins and helicities, we noted in §2 that there is further redundancy, namely, that Ψ is a superfluous coördinate. If there is no redundancy beyond this, (1) can be taken as a definition of its l. h. s. for $\Psi = 0$. For rotational invariance, it must however hold for all Ψ. If the fiducial state has particles of helicities λ_1, λ_2 and absolute value of momentum p moving respectively in the $\pm z$ directions, we write the fiducial state as $|000p\lambda_1\lambda_2\rangle$. The Ψ dependence of the l. h. s. is, then,

$$\langle \phi\theta\Psi p\lambda_1\lambda_2|S|JM\alpha\rangle = e^{i(\lambda_1-\lambda_2)\Psi}\langle \phi\theta 0 p\lambda_1\lambda_2|S|JM\alpha\rangle.$$

The r. h. s. of (1) is a superposition of several terms with Ψ dependence $e^{iM'\Psi}$, so that (1) remains true for all $\Psi \neq 0$ if and only if

$$(p\lambda_1\lambda_2|JM'\alpha) = 0 \text{ when } M' \neq \lambda_1 - \lambda_2, \tag{9}$$

a relation which has the obvious meaning of conservation of the z component of angular momentum in transitions to the fiducial state.

There is still further redundancy for n = 2 identical particles of the same helicity $\lambda_1 = \lambda_2 = \lambda$. In the Appendix, it is shown that

$$|0\pi 0 p\lambda\lambda\rangle = |000p\lambda\lambda\rangle. \tag{10}$$

All the further redundancy is obtained by applying a rotation to both sides of (10). Since (1) is rotationally invariant, it is necessary to satisfy the identification implied by (10) only for one example of identified states. All further relations among the generators must therefore follow from the requirement

$$\langle 0\pi 0 p\lambda\lambda|S|J0\alpha\rangle = \langle 000p\lambda\lambda|S|J0\alpha\rangle \equiv (p\lambda\lambda|J0\alpha). \tag{11}$$

The l. h. s. of (11) is, by (1), equal to $(p\lambda\lambda|J0\alpha)d_{00}{}^{J}(-\pi)$. But $d_{00}{}^{J}(-\pi) = (-)^{J}$, which with (11) shows that the relations are precisely that

$$(p\lambda\lambda|J0\alpha) = 0 \text{ unless } J \text{ is even}, \tag{12}$$

a result which appears in JW after Equ. (47).

If S is to commute with operators P other than rotations, there will be further restrictions among the generators. The additional relations all follow from

$$\langle 000\beta|P^{-1}SP|JM\alpha\rangle = (\beta|JM\alpha), \tag{13}$$

provided $P^{-1}SP$ is rotationally invariant, which will be true if P commutes with rotations. The case of parity is such an example, but these relations usually involve the scalars β in such a way that they may be satisfied by a restriction of the list β of scalars. In general, it is only for $n \leq 3$ particles and all helicities zero that the parity image is equal to a rotate of the original state, up to a phase factor. These relations will not be worked out; for the case of $n = 2$, see JW.

5. Rates

The matrix elements of Equ. (1) will be introduced into the general formula,

$$R_{FI}(\phi\theta\psi) = \sum_{\alpha\beta\alpha'\beta' JKm_1m_2} \langle\phi\theta\psi\beta|S|Jm_1\alpha\rangle\langle\phi\theta\psi\beta'|S|K,-m_2,\alpha'\rangle^* \cdot\rho^I_{J,m_1,\alpha;K,-m_2,\alpha'}\,\rho^F_{\beta',\beta'} \tag{14}$$

for a rate per $2\pi\chi$ volume of phase space, or an absolute rate if conservation δ functions are kept in the matrix elements, from an initial mixture given by the density matrix ρ^I to a final mixture with sharp values for the Euler angles $\phi\theta\psi$, but which may otherwise be a mixture, so that the final mixture is given by the operator $|\phi\theta\psi\beta'\rangle\rho^F_{\beta',\beta}\langle\phi\theta\psi\beta|$. The use of an initial mixture is too well known to deserve comment; the use of a final mixture corresponds to an experiment where several linearly independent system states may correlate with the same indication of a measuring apparatus.

From (1),

$$R_{FI}(\phi\theta\psi) = \sum_{\alpha\beta\alpha'\beta' JKm_1m_2m_1'm_2'} \rho^I_{J,m_1,\alpha;K,-m_2,\alpha'}\rho^F_{\beta',\beta}(\beta|Jm_1'\alpha)(\beta'|K,-m_2',\alpha')^* \cdot e^{i(m_1'+m_2')\psi} d^J_{m_1',m_1}(-\theta)\,d^K_{-m_2',-m_2}(-\theta)\,e^{i(m_1+m_2)\phi}.$$

By means of Equ. (32) of JW, the product of d functions may be written as a sum of d functions,

$$d_{m_1',m_1}{}^J(-\theta)\,d_{-m_2',-m_2}{}^K(-\theta)$$

$$= \sum_L c^{J\ K\ L}_{m_1'\ m_2'\ m_1'+m_2'}\ c^{J\ K\ L}_{m_1\ m_2\ m_1+m_2}\ (-)^{m_2-m_2'}\ d_{m_1'+m_2',m_1+m_2}{}^L(-\theta),$$

where the c's are Clebsch-Gordan coefficients in obvious notation.

By using (6) and Equ. (A1) of JW,

$$d_{-m',-m}{}^L(\theta) = d_{m,m'}{}^L(\theta), \tag{15}$$

one obtains

$$R_{FI}(\phi\theta\psi) = \sum_{\alpha\beta\alpha'\beta'JKLm_1m_2mm_1'm_2'm'} \rho^I{}_{J,m_1,\alpha;K,-m_2,\alpha'}\rho^F{}_{\beta',\beta}(\beta|Jm_1'\alpha)$$
$$\cdot(\beta'|K,-m_2',\alpha')^*(-)^{m_1+m_1'}c^{J\ K\ L}_{m_1'\ m_2'\ -m'}\ c^{J\ K\ L}_{m_1\ m_2\ -m}\ e^{-im\phi}d_{m,m'}{}^L(\theta)e^{-im'\psi}, \tag{16}$$

which has been brought to the standard form of an expansion in the $D_{m,m'}{}^L(\phi\theta\psi)$, as may be seen by comparison with (5). This formula is useful when $\rho^I{}_{J,m_1,\alpha;K,-m_2,\alpha'}(\beta|Jm_1'\alpha)(\beta'|K,-m_2',\alpha')^*$ is large only for a few J, K, as then Equ. (16) involves only a small number of standard functions of the Euler angles, although in general, (16) is only a formal expansion in the complete set of functions $D_{m,m'}{}^L(\phi\theta\psi)$. In fact, the Clebsch-Gordan coefficients vanish, unless

$|J - K| \leq L \leq J + K$ and $m_1 + m_2 + m = 0$, $m_1' + m_2' + m' = 0$.

If the initial mixture has a definite value for J, then $L \leq 2J$. If also all $m_1 = -m_2 = M$, then $m = m' = 0$, so that the basic functions reduce to Legendre polynomials of degree up to 2J, as $d_{00}{}^L(\theta) = P_L(\theta)$. A form with $m = m' = 0$ is also obviously obtained if one is not interested in the ϕ and ψ distributions, and therefore integrates the rate over these angles. If one is interested in the detailed angular distribution of one particle, one may choose its momentum to define θ, ϕ, and integrate over ψ, thereby obtaining $m' = 0$ and an expansion in spherical harmonics, since the

$d_{m,0}^{L}(\theta)$ are proportional to associated Legendre polynomials, $d_{m,0}^{L}(\theta) = [4\pi/(2L+1)]^{\frac{1}{2}} \mathcal{P}_{Lm}(\theta)$.[4] If one wishes to sum over final state helicities or over initial state helicities subject to definite value of total angular momentum and z component thereof, one may indicate this in the density matrices ρ^{I}, ρ^{F}, for such helicities are included in the lists α, β of scalars, or one may sum several particularized rates (16); either way, it is clear that the form of (16) remains unchanged by such summations. Other scalars, as mutual angles between momenta, and absolute values of momenta, may also be summed over, to obtain, e. g., a formula of form (16) for the angular distribution of one particle in a final state, irrespective of other final-state variables.

Inequalities among the coefficients consequent on positiveness of the density matrices will not be explicitly displayed. For a partial discussion, see references 5, 6.

In order to obtain a rate for a definite polarization or definite polarizations other than definite helicities, the helicity states must be appropriately superposed. If, e. g., the final-state helicity quantum number λ is involved in this way, then we may use λ as one of the scalars in the list β, λ' in β', and incorporate $\ell_{\lambda'}^{*} \ell_{\lambda}$ in $\rho^{F}_{\beta',\beta}$, where ℓ_{λ} is the coefficient of the λ helicity state in the final polarization eigenstate. If the ℓ_{λ} are constants, a definite relation of polarization to orientation is implied, which is nevertheless more general than that given by pure helicity eigenstates, whereas the Euler-angle dependence of (16) is unaltered. Thus, if the ℓ_{λ} device with constant ℓ_{λ} is used to give a transverse polarization in the $\phi = 0$ or x direction to a particle moving in the +z direction in the fiducial state $\phi = \theta = \Psi = 0$, then the general state with $\Psi = 0$ has this particle with a transverse polarization pointing due south, and for $\Psi \neq 0$ the polarization is still transverse, but is oriented Ψ east of due south. If the polarization is required to have other dependence on the Euler angles $\phi\theta\Psi$, then the ℓ_λ will be functions of the Euler angles, so that the Euler-angle dependence of (16) will no longer be that explicitly displayed in (16). For an analogous explicit expression, the ℓ_λ and $\ell_{\lambda'}{}^*$ would have to be analyzed into D functions, which would when multiplied by each other and by the explicit D functions in (16) give rise again to a sum of D functions.

Finally, there is the very important case of the 2-particle initial state, with sharp momenta along the z axis, and sharp helicities λ_1, λ_2, the first associated with the particle moving in the +z direction. In the notation and normalization of JW, who suppress the index p, JW (20, 22, 24), this state is

$$|00\lambda_1\lambda_2\rangle = \sum_J [(2J+1)/4\pi]^{\frac{1}{2}} |JM\lambda_1\lambda_2\rangle,$$
$$M = \lambda_1 - \lambda_2,$$

which corresponds to the use of $(4\pi)^{-1}(2J+1)^{\frac{1}{2}}(2K+1)^{\frac{1}{2}}$ for ρ^I and $m_1 = -m_2 = \lambda_1 - \lambda_2$, $m = 0$, in Equ. (16).

6. Interpretation of Half-Odd-Integral D Functions in the Rate

The formal expression (16) admits nonzero terms with half-odd-integral L, when ρ^I terms with one of J, K integral and the other half-odd-integral occur, provided that also ρ^F include such states β that $(\beta|Jm_1'\alpha)$ of both integral and half-odd-integral J don't vanish, or more precisely, that $S^\dagger|\beta')\rho^F_{\beta',\beta}(\beta|S$ and ρ^I have elements between common integral and half-odd-integral J states. If we confine ourselves to states formed by creation of fermions and bosons into vacuum, this requirement means that, when (16) describes a transition probability between pure states, both initial and final states must be superpositions of states with even and odd numbers of fermions, with a definite relative phase γ between such components of a state. Since a density matrix is used to represent the correlations of a limited system with external systems or "measuring devices" in cases where the relevant matrix elements involve only limited system variables, the use of density matrices of the type necessary to yield half-odd-integral L will arise only if there exist interactions involving such phases γ. Such interactions are not known; the content of the half-odd-integral L terms of (16) is that their existence would be equivalent to an independent physical meaning for all the coördinates in a doubled Euler angle domain. The remainder of this section is devoted to the elaboration of a rather fanciful example, to make this point clear.

Let $a^\dagger$ be a creation operator for a spinless particle at rest, and let $b^\dagger$ be a creation operator for a spin-$\frac{1}{2}$ particle at rest, with z component of spin $\frac{1}{2}$. Then

$$\begin{aligned} & R(\alpha\beta\gamma)2^{-\frac{1}{2}}(a^\dagger + b^\dagger)(R(\alpha\beta\gamma))^{-1} \\ &= 2^{-\frac{1}{2}}(a^\dagger + \sum_M e^{-i\alpha M} d_{M,\frac{1}{2}}^{\frac{1}{2}}(\beta) e^{-i\gamma/2}\, b_M^\dagger) \\ &= 2^{-\frac{1}{2}}(a^\dagger + \sum_M D_{M,\frac{1}{2}}^{\frac{1}{2}}(\alpha\beta\gamma) b_M^\dagger) \\ &\equiv q^\dagger(\alpha\beta\gamma), \end{aligned} \qquad (17)$$

where $b_M^\dagger$ is the creation operator for the fermion at rest with z component of spin M, may be regarded as the creation operator for a particle of mixed spin at rest, with its polarization pointing in the direction β, α and with γ specifying the relative phase between the fermion and boson components. For the corresponding mixed particles to appear as physical entities with well-defined internal phase γ, it is necessary that the masses of boson and fermion components be sufficiently close that the mass difference not imply too rapid a change of γ, and that an interaction involving mixed creation and annihilation operators, $q^{\dagger}$ and q, occur. Thus, the decay of a heavy mixed particle into two spinless bosons and a light mixed particle could be described by formula (16), with $\phi\theta\psi$ having the usual interpretation as Euler angles for a 3-body final state. Let θ, ϕ be the polar angles for the final mixed particle, let ψ specify the orientation of the decay plane, and let us assume detectors sensitive to the helicity of the fermion component and the internal phase of the mixed final particle; at each angle θ, ϕ, for definite fermion helicity, there is still a 2-dimensional space of mixed-particle states, and we assume a detector sensitive to a particular one of these dimensions. Such a detector could distinguish between a state and its 2π-rotate, because the change of internal phase renders the 2π-rotate orthogonal to the unrotated state. Therefore the result (16) for the rate, involving half angles in this case, is perfectly reasonable.

A detector sensitive to internal phase is also necessary to prepare the initial state. Such a detector could be imagined to be fashioned from mixed particles itself, provided that there be a mutual interaction between mixed particles depending on the difference of internal phase of the interacting particles, so that internal phases could be calibrated relative those of the mixed particles in the detector.

A simpler "detection" could be imagined if the detection interactions are allowed to violate rotational invariance, although the transition operator S is not; e. g., the lifetimes of mixed particles could be taken dependent on internal phase.

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The general theorem for the distribution in polar angles θ, ϕ of one particle from the decay of an eigenstate of J and M, quoted to me by Prof. T. D. Lee, proved very valuable in simplifying model calculations of angular distributions of neutrons from mu capture. Conversations with Dr. J. V. Lepore on the angular distribution of a pion from antiproton annihilation were also stimulating.

Appendix

This is a verification that

$$|0\pi 0 p\lambda\lambda\rangle = |000p\lambda\lambda\rangle, \tag{10}$$

in the case of two identical particles, or see JW, p. 419.

$$|000p\lambda\lambda\rangle = L_z(p)a_\lambda^\dagger L_z(-p)L_z(-p)a_{-\lambda}^\dagger L_z(p)|vac\rangle, \tag{18}$$

where $a_\lambda^\dagger$ creates one particle of momentum $\underset{\sim}{0}$, z component of spin λ, total spin s, and $L_z(p)$ is a Lorentz transformation along the z axis leading to a +z component p of momentum for a single particle originally at rest.

$$|0\pi 0p\lambda\lambda\rangle = R(0\pi 0)|000p\lambda\lambda\rangle, \tag{19}$$

and we abbreviate $R(0\pi 0) = R_y(\pi)$. Clearly,

$$R_y(\pi)L_z(p)a_\lambda^\dagger|vac\rangle = aL_z(-p)a_{-\lambda}^\dagger|vac\rangle, \tag{20}$$

$$R_y(\pi)L_z(-p)a_{-\lambda}^\dagger|vac\rangle = bL_z(p)a_\lambda^\dagger|vac\rangle, \tag{21}$$

where a and b are phase factors. By applying $R_y(\pi)$ to both sides of (20) and using (21), we find

$$R_y(2\pi)L_z(p)a_\lambda^\dagger |vac\rangle = abL_z(p)a_\lambda^\dagger |vac\rangle,$$

whereupon the fact that $R_y(2\pi)$ acts as $\pm$ 1 on a state of even or odd total angular momentum, respectively, gives us

$$ab = (-)^{2s}. \tag{22}$$

From (20, 21) and standard q-number theory conventions,

$$R_y(\pi)L_z(p)a_\lambda^\dagger L_z(-p)R_y(-\pi) = aL_z(-p)a_{-\lambda}^\dagger L_z(p), \tag{23}$$

$$R_y(\pi)L_z(-p)a_{-\lambda}^\dagger L_z(p)R_y(-\pi) = bL_z(p)a_\lambda^\dagger L_z(-p). \tag{24}$$

Explicitly, (19, 18) yield

$$|0\pi 0p\lambda\lambda\rangle = R_y(\pi)L_z(p)a_\lambda^\dagger L_z(-p)R_y(-\pi)R_y(\pi)L_z(-p)a_{-\lambda}^\dagger L_z(p)R_y(-\pi)|vac\rangle, \tag{25}$$

so that by (23, 24),

$$|0\pi 0p\lambda\lambda\rangle = ab\left[L_z(-p)a_{-\lambda}^\dagger L_z(p)\right]\left[L_z(p)a_\lambda^\dagger L_z(-p)\right]|vac\rangle. \tag{26}$$

If the operators in brackets could be interchanged, we see by (18) that $ab|000p\lambda\lambda\rangle$ would be obtained. According to the spin-statistics relation, this can be done if we introduce a factor $(-)^{2s}$. By (22), this cancels ab, yielding (10).

References

1 C. N. Yang, Phys. Rev. 74, 764 (1948)

2 P. Morrison, in Segre's "Experimental Nuclear Physics", Wiley, 1953, Vol. II, Part VI, §1B.

3 M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959), hereafter designated JW

4 JW, p. 426

5 T. D. Lee and C. N. Yang, Phys. Rev. 109, 1755 (1958)

6 M. Tausner, Columbia University doctoral dissertation