Title

Aspects of S-Duality

Author

Ju, Chao

Publication Date

2023

Aspects of S-Duality

by

Chao Ju

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Ori Ganor, Chair
Professor Yasunori Nomura
Professor Jelani Nelson

Fall 2023

Aspects of S-Duality

Abstract

Aspects of S-Duality

by

Chao Ju

Doctor of Philosophy in Physics

University of California, Berkeley

Professor Ori Ganor, Chair

Classically, the ground states of $\mathcal{N} = 4$ Super Yang-Mills Theory (SYM) on $\mathbb{R} \times S^3/\Gamma$ where Γ is one of the ADE subgroup of $SU(2)$ are flat Wilson lines winding around the ADE singularity. S-duality acts on this finite-dimensional ground state Hilbert space and its action is the same as the S operator in a certain dual Chern-Simons theory on T^2. The dual Chern-Simons theory arises out of the only long-range interaction in a string/M theory construction by considering a stack of D3(M5) branes on ADE singularity. This SYM/Chern-Simons duality is verified by matching the ground state Hilbert spaces of both theories and by comparing the S-duality operators of both theories.

To one-loop order, the SYM ground state degeneracy is exact. A detailed computation using the superconformal index shows that each classical SYM ground state acquires the same supersymmetric Casimir energy. S-duality maps the SYM ground state Wilson lines to ground state t' Hooft lines taking values in the Langlands dual group. The number of t' Hooft lines are shown to agree with that of the Wilson lines. In addition, the t' Hooft lines have the same supersymmetric Casimir energy as the corresponding Wilson lines. These two facts provide a ground state test of S-duality.

The SYM/Chern-Simons duality has an important extension to the class S theory obtained from compactifying M5 branes on a Riemann surface $\mathcal{R}$. The ground states of class S theory on $\mathbb{R} \times S^3/\Gamma$ are dual to the states of the dual Chern-Simons theory on $\mathcal{R}$. In particular, we uncover a surprising result that there is only one unique ground state for the conformal $\mathcal{N} = 2$ $SU(2)$ four-flavor theory on $\mathbb{R} \times S^3/\Gamma$. Finally, we apply the SYM/Chern-Simons duality to a non-Lagrangian class S theory and find that its ground states obey the fusion rule of the current algebra of the dual Chern-Simons theory.

To blonde.

Contents

List of Figures

List of Tables

Acknowledgments

First, I thank my parents for gunning the ignition of my universe, bringing me up, and letting me explore my interests freely. I thank my friends for tolerating my idiosyncrasies and bringing light into my life. I am proud to say that I have learned from each of you, and your success brings me the greatest joy in my life. I thank my advisor, Professor Ori Ganor, for educating me and taking me on an unforgettable adventure in two, three, four, five, six, seven, ten, and eleven dimensions in the past few years. I have matured into an independent researcher under his guidance. I thank Professor Yasunori Nomura for doing the reading course and answering many of my questions in the past few years. I thank Professor Jelani Nelson for kindly agreeing to be on my qualification and dissertation committee, and Professor Marjorie Shapiro for being on my qualification committee and quizzing me on supersymmetry. I thank Joelle Miles for providing administrative support. I also thank my collaborators, Emil Albrychiewicz and Andres Franco Valiente, for teaching me new things and offering new perspectives.

Because of Covid, I could not travel much in the past few years, so I would like to thank Pablo Neruda for bringing me on a world tour starting in the wild forest of Chile, Wole Soyinka for showing me west Africa, Virginia Woolf for taking me back to the lighthouse in Cornwall, Yasunari Kawabata for playing go with me on the road to snow country, Langston Hughes for his company on the train ride along the Mississippi... The list could go on and on, and I thank all the writers, dead and alive, who opened the keyhole of my eyes during my time in Berkeley.

Part I

Singular M5 Branes on T^2

Chapter 1

Friendly Introduction

In my junior year of college, I got interested in physics and decided to pursue a graduate education in physics. The decision to study physics was not made overnight. In hindsight, it was a confluence of influences built up over the years, random tidbits that do not make sense individually somehow converging to a coherent vision. The most prominent influences were the works by Stanislaw Lem, Cixin Liu, and Christopher Nolan. In particular, I wanted to answer questions such as “where do we come from”, “where are we going”, “what happens at the singularity of a black hole”, and “do we live in dimensions more than four”. Four and a half years later, I am still unable to answer any of these questions. Let us therefore consider a simpler concept: duality. A particular subclass of this concept called S-duality will be the main theme of this dissertation.

The concept of duality has appeared in many areas of science and humanities. For an example from the latter category, the concept of yin-yang features prominently in ancient Chinese cosmology, medicine, and philosophy. In one interpretation, there is a one-to-one map from what is weak (yin) to what is strong (yang). For an example from mathematics, we know that any probability value $p \in \mathbb{R}$ lies in the range $[0, 1]$. However [1], the axiomatic derivation of the range of p in fact permits two distinct but equivalent solutions: $p \in [0, 1]$ and $p \in [1, \infty)$. In the latter solution, “impossibility” corresponds to the value ∞ and “certainty” corresponds to the value 1. The map between these two ranges is simply an inversion $p \to 1/p$. We choose to work with the first range $[0, 1]$ because it is finite and therefore easier to work with.

The duality we will be interested in will come from physics, where there is an unambiguous definition. Let us look at an example of duality from string theory which is similar to the inversion duality in probability theory. Unlike point particles, strings perceive geometry very differently. To see this, consider a (possibly impossible) universe whose spatial geometry is a circle S^1 of radius R. Suppose that only closed strings exist in this universe. A closed string can tightly wind around the circle $m \in \mathbb{N}$ times, which results in an energy proportional to mR coming from the tension of the string. There is also kinetic energy coming from the motion of the string: it can rotate around the circle with some momentum. However, a circle has a closed topology, so the allowed momenta of the string are quantized in units of $1/R$. Overall, the energy of the closed string winding around the circle m times is

$$mR + \frac{n}{R} \tag{1.1}$$

where n is some integer characterizing how fast the string is moving. The above expression is not changed if we send R to $1/R$ and exchange m and n. This operation, in words, means that the energy does not change if we change the circle from a very small one (say R is small) to a very large one and exchange the winding number with the momentum number. This is more or less the essence behind T-duality in string theory. Closed strings cannot distinguish a small circle with radius R from a large circle with radius $1/R$! Particles, however, do see the difference between a small circle and a large circle. Unlike closed strings, particles cannot wind around the circle to create a mR term in the energy. Particles do not have any tension and they only carry momentum, so n/R is the only term in the expression for the particle energy. Therefore, there is no similar operation that sets equal the energy of a particle on a circle with radius R to that on a circle with radius $1/R$.

More precisely, for something to be called a duality in physics, it is not enough to match just one particular property such as energy. The "Hilbert space" should also match. The Hilbert space of a theory basically characterizes the particle spectrum of a theory. For example, a particle of spin 1/2 can be either spin up or spin down, creating a 2-dimensional Hilbert space. For two finite-dimensional Hilbert spaces to match, the only requirement is that they share the same dimension. If theory A is dual to theory B, there must be a one-to-one map between the Hilbert spaces of each theory. In the T-duality example, a more detailed analysis using string theory shows that there is indeed a match in Hilbert space between the theory on the circle with radius R and the theory on the circle with radius $1/R$ [2].

The particular kind of duality that will concern us in this dissertation is called S-duality which has roots in $\mathcal{N} = 4$ super Yang-Mills theory. Yang-Mills theory is a theory that describes the dynamics of quarks and gluons. It is a generalization of quantum electrodynamics that describes the dynamics of electrons and photons. Super Yang-Mills theory is a Yang-Mills theory with supersymmetry: each fermion (i.e. matter particle) in the theory has a bosonic partner (i.e. force carrier). $\mathcal{N} = 4$ super Yang-Mills theory is a supersymmetric Yang-Mills theory with 4 supercharges, those that turn fermions into bosons and vice versa. There is a coupling constant g in the theory which controls the strength of the interactions of gluons and quarks. For small g, quarks barely interacts with gluons, and they propagate more or less freely. If one is able to look at a quark when g is small, one would see more or less a localized, point-like blob. As we increase the value of g, the interaction gets stronger, so the blob becomes less point-like and more cloud-like. The blob will look dilated because of the interactions between the quark and the gluons. Virtual quarks and gluons fluctuate in and out of the vacuum to give the blob a cloud-like appearance. Excitations that look like this are called monopoles. In fact, the large g theory is more naturally described by monopoles, and there is a duality between the small g theory and the large g theory by exchanging the particles (quarks and gluons) with the monopoles. This is the basic idea behind S-duality: it is a strong-weak duality in the sense that a weakly-coupled theory is dual to a strongly-coupled theory. The map between the coupling constants turns out to be an inversion map $g \to 1/g$.

There is one thing that needs to be made more precise. In physics, it is well-known that coupling constants generically change value if we look at a theory from different scales. This is the idea behind renormalization group flow [3]. In computer science, people care about compressing data so that an image is readable but the memory used to store the image is reduced. This can be done, for example, by doing a singular value decomposition on the pixel data and discard those singular values that are small compared to the rest. In physics, if we want to understand the long-distance behavior of a system, we coarse-grain the system so that we forget about the short-distance degrees of freedom. However, this is different from data compression because in the process of coarse-graining, we do not just throw away the short-distance degrees of freedom like we do for the small singular values. Instead, we throw away the short-distance degrees of freedom while incorporating their effect on the system into the theory by demanding that the "partition function" of the theory remains unchanged in the process of coarse-graining. The partition function characterizes the Hilbert space of the theory and so it completely characterizes the theory. To illustrate this idea, the following contrived example involves a random theory with coupling constant g viewed from a short distance scale l. Suppose that the partition function Z has the form

$$Z \equiv \exp\left(ig(l)\int_l^\infty Q(l)\right) \tag{1.2}$$

where $Q(l)$ is some differential form depending on l and where the integration is done from the short distance scale l, signaling our ignorance of what is happening below the distance scale l. Now, suppose we want to look at the system at a larger length scale $l' > l$. We can break up the integral into two parts, one from l to l' and the other from l' to ∞:

$$Z = \exp\left(ig(l)\int_l^{l'} Q(l)\right)\exp\left(ig(l)\int_{l'}^\infty Q(l)\right) \tag{1.3}$$

The first term represents the effect of coarse-graining: we integrate over the details below the scale of interests l'. For someone interested in data compression, he would simply throw away the first term and keep the second term. However, in our setting, doing so would change Z and therefore change the whole story. This is the crucial difference between data compression and renormalization group flow. The former simply forgets the short-distance degrees of freedom while the latter is a consistent way of forgetting short-distance degrees of freedom (by keeping Z unchanged). We can rewrite the above expression into

$$Z = \exp\left(ig(l') \int_{l'}^{\infty} Q(l') + \text{other terms generated from coarse graining} \right) \tag{1.4}$$

where the other terms come from 1) the coarse-graining term (the first term in equation (1.3)), and 2) some potentially complicated terms by changing $Q(l)$ to $Q(l')$ in the integrand and $g(l)$ to $g(l')$. This suggests the following: coarse-graining generically changes the value of the coupling constants (in this case $g(l)$ to $g(l')$), and introduces other terms in the theory (they are called interaction terms in physics). This is the idea behind renormalization group flow: the coupling constants of a theory generically change as we look at the theory from different length scales. In addition, other interaction terms generically pop up in this process. As this example illustrates, both of these points are simply natural consequences of keeping the partition function Z unchanged as we coarse-grain.

We made a detour into renormalization group flow because we want to ask the following question. As seen earlier, S-duality maps the $\mathcal{N} = 4$ super Yang-Mills theory with coupling constant g to a different $\mathcal{N} = 4$ super Yang-Mills theory with coupling constant $1/g$. How do we know that such two theories are indeed different, that this change in the coupling constant is not something that results from renormalization group flow (coarse-graining) of the same theory? The answer is that the $\mathcal{N} = 4$ super Yang-Mills theory is conformal. A special feature of a conformal theory is that the theory is scale-invariant: it is the same no matter what distance scale we look at. We are free to zoom in and to zoom out. The coupling constants do not change under renormalization group flow. In nature, we can find similar scale-invariant systems in fractals. Therefore, $\mathcal{N} = 4$ super Yang-Mills theory with coupling constant g is truly different from the one with coupling constant $1/g$. There is no way the theory can flow from g to $1/g$ under coarse-graining because it is conformal.

The $\mathcal{N} = 4$ super Yang-Mills theory therefore has both supersymmetry and conformal symmetry. These two symmetry groups combine to create a larger symmetry group called superconformal symmetry. The more symmetry a system has, the more constrained the system is, and the solutions to the dynamics of a much constrained system are easier to spot than an unconstrained system. An example of how symmetry constrains the solutions comes from probability theory. Suppose that we want to write down a two-dimensional probability distribution that is invariant under rotation, then there can be many candidates as long as the distribution is a function of $\sqrt{x^2+y^2}$ only. However, if we demand further that the error along the x-direction is uncorrelated with the error along the y-direction (astronomers knew about this un-correlation, for example, a few hundred years ago when they tried to draw the celestial maps to locate planets), then there is only one unique solution: the two-dimensional Gaussian distribution. In our case, the superconformal group constrains the dynamics of the super Yang-Mills theory such that many quantities can be exactly computable, ones that do not depend on how small or large the coupling constant g is. In physics, for small coupling constant g, the usual way to compute things is by perturbation theory: the quantities that we want to compute are expanded in an asymptotic series as in $a_0 + a_1 g + a_2 g^2 + ...$ We compute each of the a_j and truncate the series at some point to get a perturbative answer. However, this method does not work when g is large, in which case we have no reason to expect that the first few terms in the asymptotic series are a good approximation to the quantity we want to compute. In fact, many of the quantities we compute will therefore depend on the value of the coupling constant g. Does there exist some computable quantities that are do not vary as g changes? These quantities are called "protected" and they indeed exist in theories with superconformal symmetry.

A particular protected quantity we will compute in this paper is called the superconformal index, a quantity that does not change as one varies the value of the coupling constant g. Here, we explain a simpler variant of this concept. We mentioned earlier that the action of supercharges (here we denote them as Q) on bosonic states (resp. fermionic states) turns them into fermion states (resp. bosonic states). This is the definition of supersymmetry. One can think of Q as a matrix and the states as vectors. It turns out that acting Q on a state does not change its energy. So when a bosonic state χ has a positive energy, $\psi = Q\chi$ will yield a fermionic state ψ having the same energy. Therefore, states having positive energy can be arranged in boson-fermion pairs. However, some states are annihilated by Q: that is, they lie in the nullspace of Q. Those states turn out to have zero energy. Because they are annihilated by Q, zero-energy states are not guaranteed to form perfect one-to-one boson-fermion pairing: we do not have a similar expression $\psi = Q\chi$ for a zero energy state χ because $Q\chi = 0$. In general, changing the coupling constant of the theory will cause the states in the theory to change energy. Zero energy states can be excited to gain energy, and positive energy states can lose energy to become zero energy states. Is there a quantity that is protected in this theory? Indeed, consider the quantity

$$\text{number of bosonic zero energy states} - \text{number of fermionic zero energy states} \quad (1.5)$$

The claim is that this quantity does not change as we vary the coupling constant. To see this, note that when a zero energy state gains an energy, it must be paired with a supersymmetric partner as argued earlier. However, all other states that have positive energy are already paired. Therefore, it must be that the zero energy states gain energy in *pairs* of fermions and bosons! Denote the number of zero energy states as (x, y) where x is the number of bosonic states and y is the number of fermionic states. An evolution of this quantity as we vary the coupling constant could be: $(3,5) \to (2,4) \to (1,3) \to (0,2)$. Each time, a pair of boson and fermion gains energy and leaves the zero energy subspace. However, the difference $x - y = -2$ stays constant. This is the basic idea behind the superconformal index. The key behind it is the perfect pairing between positive energy bosonic states and positive energy fermionic states. S-duality maps a weakly-coupled theory to a strongly-coupled theory. To compare both theories, we need some quantity that does not vary as the coupling constant changes. As we will see later in this dissertation, superconformal index comes in handy for this purpose because it is a protected quantity.

As mentioned earlier, when S-duality acts on the super Yang-Mills theory, it inverts the coupling constant and turns particles into monopoles. However, the Hilbert space of the super Yang-Mills theory is infinite-dimensional, and it is hard to find a representation of S-duality on this Hilbert space. It will be a blessing if we could isolate a particular part of the Hilbert space that is only finite-dimensional and that does not mix with the rest of the Hilbert space under the action of S-duality. The representation of the action of S-duality on this finite-dimensional subspace will therefore be finite-dimensional and should be easier to analyze. The way we achieve this is to put the super Yang-Mills theory on a singular geometry, of the form S^3/Γ. Here, S^3 is the three sphere, and Γ is some discrete group. This expression means that we identify two points on S^3 if they are related by some group element in Γ. An example of a singular space is $\mathbb{R}^2/\mathbb{Z}_2$, which results in a cone as we identify (x, y) with $(-x, -y)$. Once we put the super Yang-Mills theory on a singular geometry S^3/Γ, we can isolate the ground state Hilbert space which is finite dimensional: ground states are what is called the Wilson lines winding around the singularity. We would like to understand how S-duality acts on those ground state Wilson lines.

The action of S-duality turns out to be quite hard to compute aside from a few simple cases. To give a general expression for S (where S now represents the matrix of S-duality action), we use another duality called the holographic duality. A hologram is a 3D projection of 2D information. Holographic duality has its roots in gravitation, where people find that the entropy of a black hole scales not as its volume but as its area. This suggests that the black hole can be reconstructed from only its boundary information, and puts a universal constraint on the amount of information or data one is allowed to store in a region. Imagine we have a physical memory disk of a certain size, can we throw in as much data as we want? The current constraint on this is purely technological and has to do with material sciences, but the ultimate theoretical constraint comes from gravity: one can in principle throw in as much data as one wants before the disk turns into a black hole! This is analogous to the fact that the theoretical constraint on the computational speed of a computer comes from the Planck time $t \approx 5.39 \times 10^{-44}s$, the time a quantum state needs to jump to another quantum state, and yet we are nowhere near this limit due to limitations in technology. In a nutshell, the holographic duality gives a dual description of $\mathcal{N} = 4$ super Yang-Mills theory in terms of a gravitational theory (which turns out to be the IIB superstring theory). This duality deserves the name holography because the super Yang-Mills theory lives on a 4-dimensional spacetime whereas the gravity theory lives in the 5-dimensional bulk, mimicking the hologram. If we could somehow use the holographic duality to map the Wilson line ground states of the super Yang-Mills theory to some gravitational states that are easier to understand and are known how to transform under S-duality, we will have solved the problem of understanding how S acts on the Wilson line ground states. It turns out that the holographic dual of Wilson lines are states in a certain Chern-Simons theory, a topological theory that has no notion of length. The transformation of states in the Chern-Simons theory under the action of S is known, and this helps give a general formula of S for the super Yang-Mills theory. In the following sections, we will see precisely what kind of Chern-Simons theory arises under the holographic duality.

This concludes the non-expert introduction section of this dissertation. Besides S-duality, there are two other unfinished projects I did with Ori Ganor and Orion Ning. I decided against including these two projects in this dissertation because they do not fit into the broader scope of this dissertation.

We will now delve into the main body of the dissertation for technical details.

Chapter 2

The Action of S-Duality on Ground States of $\mathcal{N} = 4$ Super Yang-Mills Theory on S^3/Γ

2.1 Introduction

Supersymmetric Yang-Mills theory in four dimension has a varying degree of applications depending on the number of supercharges the theory has. $\mathcal{N} = 1$ SYM with flavor degrees of freedom (SQCD) is not only useful for phenomenological purposes [4, 5, 6, 7]; it also provides a fertile ground for concrete examples of dualities [8]. $\mathcal{N} = 2$ SYM provided the first realization of quark confinement via monopole condensation [9]. A whole class of $\mathcal{N} = 2$ theories can be engineered by putting M5 branes on Riemann surfaces, and the duality bewteen the theories can be analyzed using the geometry of the Riemann surface [10]. For a review, see [11]. $\mathcal{N} = 4$ SYM is a UV finite theory [4], and provides an explicit realization of holographic duality via the AdS/CFT correspondence [12]. It is this UV finite theory that will concern us in this paper.

$\mathcal{N} = 4$ Super Yang-Mills theory exhibits a strong-weak S-duality first proposed by Montonen and Olive [13]. There has been a number of deep tests and applications of S-duality (for example [14, 15, 16, 17, 18, 19]), and an "S-duality kernel" has been constructed in [20]. Nevertheless, for theories with non-abelian gauge groups how and why S-duality exactly works remains to be found.

The action of S-duality can be analyzed in a more controlled setting by studying the $\mathcal{N} = 4$ SYM theory with gauge group $U(q)$, $SU(q)$, or $SU(q)/\mathbb{Z}_q$, on $\mathbb{R}^4/\Gamma$, where Γ is one of the discrete subgroups of $SU(2)$ classified by simply laced ADE Coxeter-Dynkin diagrams [21]. The partition function of the theory, which depends on the boundary conditions at infinity, was calculated (among other things) three decades ago by Vafa and Witten [15], and they discovered that it matches a character of an affine Lie algebra at level q. This allowed Vafa and Witten to compute the action of S-duality on the system.

We denote any particular ADE diagram by $\mathcal{D}$, the corresponding finite subgroup by $\Gamma(\mathcal{D})$, and the associated Weyl group of the diagram by $W(\mathcal{D})$ (or just Γ and W, when $\mathcal{D}$ is understood from context). We also denote by $G(\mathcal{D})$ the simply connected Lie group associated with the Dynkin diagram $\mathcal{D}$ ($SU(r)$ for A_{r-1}, Spin$(2r)$ for D_r, and E_6, E_7, E_8 for the E-diagrams).

By taking Γ to be a subgroup of the first factor of $SU(2)\times SU(2)\cong \text{Spin}(4)$, we let it act as a finite group of rotations of $\mathbb{R}^4$ whose nontrivial elements have no fixed points but the origin, and when extended to act on 4d spinors, they preserve two linearly independent spinors. Γ thus defines an orbifold $\mathbb{R}^4/\Gamma$ on which a supersymmetric theory can be formulated, while preserving half the supersymmetry generators.

This type of orbifold played an important role in the original construction of the 6d $(2,0)$-theory [22]. The Lie group $G(\mathcal{D})$ was originally realized as a physical gauge group in this context in [22] as well.

To put a supersymmetric theory on $\mathbb{R}^4/\Gamma$ one needs to specify the behavior of the fields at the origin, and possibly additional degrees of freedom there. If the theory is also conformal, we can use the state-operator correspondence to map the theory on $\mathbb{R}^4/\Gamma$ to a theory on $\mathbb{R}\times(S^3/\Gamma)$, where the first factor can be interpreted as (Euclidean) time. Translations in $\mathbb{R}$ correspond to dilatations of the original $\mathbb{R}^4/\Gamma$. The question of what happens at the origin of $\mathbb{R}^4/\Gamma$ then corresponds to the question of the initial state $|i\rangle$ of the S^3/Γ theory at time $-\infty$. In the $\mathbb{R}^4/\Gamma$ formulation, $|i\rangle$ corresponds to an operator that is localized at the origin. In particular, ground states of the theory on S^3/Γ correspond to operators of conformal dimension $\Delta=0$ at the origin. They also commute with the supersymmetry generators (those that are invariant under Γ).

We specialize to the three choices of gauge group:

$$G = U(q), \qquad SU(q), \qquad SU(q)/\mathbb{Z}_q.$$

We wish to study the finite dimensional Hilbert space $\mathcal{H}_{\text{gs}}$ of ground states on S^3/Γ (for each gauge group) and how S-duality acts on it.

Let

$$\tau = \frac{4\pi i}{g^2}+\frac{\theta}{2\pi} \tag{2.1}$$

be the complex coupling constant of $\mathcal{N}=4$ SYM, taking values in the upper half plane. S-duality acts on it as $\tau\rightarrow -\frac{1}{\tau}$. At weak coupling, $g\rightarrow 0$, the Hilbert space is easy to describe. In this classical limit, a zero-energy configuration corresponds to setting the scalar fields and the gauge field strength to zero. The gauge field is then a flat connection, and gauge inequivalent flat connections are uniquely described by homomorphisms from $\pi_1(S^3/\Gamma) \cong \Gamma$ to G, up to conjugation. For $G = U(q)$, the various inequivalent homomorphisms Γ correspond to the inequivalent q-dimensional (not necessarily irreducible) unitary representations of Γ. The Hilbert space $\mathcal{H}_{\text{gs}}$ is constructed by assigning to each such representation a basis state.

S-duality maps the Hilbert space at τ to the Hilbert space at $-\frac{1}{\tau}$. At $\tau = i$ the two Hilbert spaces are the same, and S-duality becomes an isometry. But away from $\tau = i$, in order to ask "how does S-duality act?" we must first find a way to relate the Hilbert space at τ with the Hilbert space at $-\frac{1}{\tau}$. The Hilbert space forms a vector bundle of rank $\dim \mathcal{H}_{\text{gs}}$ over the τ-space, which is the upper half plane. If the Berry connection of this vector bundle is trivial, we can identify the Hilbert space at any two τ's by adiabatically changing τ (the path in τ-space does not matter). We can then look for an operator on the weakly coupled Hilbert space at $\tau = i\infty$ that represents S-duality. For $\mathcal{D}$ one of the D or E series of the ADE classification, $\Gamma(\mathcal{D})$ is nonabelian, and such an S-duality operator, therefore, will give us a clue on how S-duality acts on nonabelian gauge fields.

We will argue that the Berry connection is, at worst, a $U(1)$ connection. That means that starting with any state of $\mathcal{H}_{\text{gs}}(\tau)$ and taking τ adiabatically in a closed loop in τ-space, we end up with the same state, multiplied by a (Berry) phase $e^{i\phi}$ that only depends on the path in τ-space, but not on the state itself. The argument will be based on the AdS/CFT inspired solution to the S-duality operator that we propose below. The S-duality operator is then described up to an overall phase. In fact, we can describe the whole $SL_2(\mathbb{Z})$ action, generated by S, T. Because of the possibility of a $U(1)$ Berry connection, we are only proposing a *projective* representation of $SL_2(\mathbb{Z})$ on the Hilbert space.

The results of [15] can be interpreted as an equivalence between the Hilbert space of the ground states of the gauge theory on S^3/Γ and the Hilbert space of Chern-Simons theory on T^2 as in the diagram below.

$$\boxed{\begin{array}{ccc}
\left\{\begin{array}{c}\text{Hilbert space of}\\ \text{Ground states of } N=4 \text{ SYM}\\ \text{with gauge group } U(q) \text{ on } S^3/\Gamma(\mathcal{D})\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{Hilbert space of}\\ \text{Chern-Simons Theory on } T^2\\ \text{at level } q \text{ and gauge group } G(\mathcal{D})\end{array}\right\} \\
\\
\text{Gauge coupling} & \longleftrightarrow & \text{Complex structure of } T^2 \\
SL_2(\mathbb{Z}) \text{ duality} & \longleftrightarrow & \text{Mapping class group}
\end{array}} \tag{2.2}$$

This equivalence can be motivated by the appearance of a holomorphic WZW model in the low-energy description of the 6d $(2,0)$-theory of type A_{q-1} (associated with the low-energy behavior of q M5-branes) formulated on $(\mathbb{R}^4/\Gamma) \times \mathbb{R}^2$, as noted in [23, 24][1]

[1] We are grateful to Sergey Cherkis for pointing out one of these references.

Modifications to gauge groups $SU(q)$ and $SU(q)/\mathbb{Z}_q$ will be discussed later in our paper.

It is interesting that the Chern-Simons theory emerges in this duality. The 3-dimensional Chern-Simons theory is a remarkably rich tool that has applications in both mathematics and physics. For example, Witten showed that Chern-Simons theory is connected to the Jones polynomial and knot invariants [25]. It is also applied in the study of S-duality [26, 20], a subject related to the geometric Langlands program [19]. In terms of the more tangible physics applications, Chern-Simons theory has been used extensively to study physics on 2-dimensional surfaces. For example, it is used to endow particles in 2+1 dimensions with fractional statistics [27, 28], so that upon exchanging two identical particles the wave function of the two particles can end up with a phase different from ± 1. In addition, Laughlin explained fractional quantum hall effect by applying Chern-Simons theory[2] [30].

The fact that Chern-Simons theory is important for 2+1 dimensional physics is no accident. The Chern-Simons action

$$S = \frac{q}{4\pi}\int \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{2.3}$$

is the unique relativistically invariant action of the gauge field A that has only one derivative acting on A. In this paper, we take the convention that A is $\mathfrak{g}$ valued 1-form and the trace is taken in the representation such that q is a quantized positive integer. We take $\mathfrak{g}$ to be simply laced, namely $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$. The global property of the gauge group will not concern us in this paper.

The solution to the problem of finding the S-duality operator that we explore for $G = U(q)$ identifies $\mathcal{H}_{\mathrm{gs}}$ with the space of states of Chern-Simons theory on T^2 with the gauge group associated to the Lie algebra $\mathcal{D}$ at level q. The $SL_2(\mathbb{Z})$ action is the action of the mapping class group. We argue that that this identification can be understood as a mini-AdS/CFT correspondence. Indeed, the Chern-Simons Hilbert space can be constructed [31] as a W-invariant subspace of a representation of a certain Heisenberg algebra constructed from the weight lattice Λ_w and the root lattice Λ_r of $\mathcal{D}$. Upon quantizing the Chern-Simons theory on T^2, one gets a discrete set of states that can be identified with points in the set [31] (for Chern-Simons theory with simply laced gauge algebra)

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r} \tag{2.4}$$

This same Heisenberg algebra arises, as we will argue, in our setup, through the AdS/CFT correspondence [12], along the lines of Witten's work on the appearance of electric and magnetic fluxes via holography [32]. We describe this in detail in section 2.4.

[2]For a review of Chern-Simons theory and fractional quantum hall effect, see [29] and the references therein.

Assuming the identification (2.2), the Berry connection can be identified with that of Chern-Simons theory. The latter is nontrivial once properly regularized, but is a $U(1)$ connection.[3] Moreover, the holographic derivation of (2.2) implies an identification of the Hilbert space of ground states of $N=4$ SYM on S^3/Γ with the Hilbert space of the correpsonding Chern-Simons theory on T^2 tensored with a 1-dimensional Hilbert space associated with massive bosonic and fermionic modes. The latter carries its own Berry phase under a loop in τ-space. The perturbative argument at the end of section 2.2 suggests that the Berry phase of the 1-dimensional Hilbert space of those massive modes actually cancels the Berry phase of Chern-Simons theory.

More generally, by lifting the construction to 6d, we are motivated to conjecture that the space of ground states of a 4d $\mathcal{N}=2$ class-S theory [10] on S^3/Γ corresponds to the Hilbert space of Chern-Simons theory on a Riemann surface and the action of the group of dualities corresponds to the action of the mapping class group of the Riemann surface on the Hilbert space of Chern-Simons theory. Here, the 4d theory is constructed from the 6d $(2,0)$-theory of q M5-branes on a Riemann surface [10]. This will be the topic of chapter 5.

This dissertation is organized as follows. In section 2.2 of this chapter, we write out the bosonic part of the Lagrangian of $\mathcal{N}=4$ SYM and formulate the problem mentioned at the beginning of this section in a more detailed way. We briefly review the well-known McKay correspondence in section 2.3. In section 2.4, we present the proposed solution of our problem and motivate it with holography. We will see that, in the low energy limit, a dual Chern-Simons theory emerges naturally, and its quantization leads to the Heisenberg algebra. Section 2.2 contains most of the formalism and the main conjecture of this paper, and the claims made in (2.2) will also be generalized to $SU(q)$ and $SU(q)/\mathbb{Z}(q)$ there. In section 2.5 and section 2.6 we present evidence for the proposed duality. In section 2.5, we compute the action of S and T on the SYM side for $U(1)$ gauge group, and see that in a suitable basis it agrees with the action on the dual Chern-Simons side. In section 2.6, we count the number of ground states in both theories (with $U(q)$ gauge group for the SYM theory) and show that they agree not only in the large q limit but to all orders in q. In section 2.7, we sketch a counting argument for when the SYM side of the duality has $SU(q)$ gauge group instead of $U(q)$. In section 2.8, we give two numerical examples of S-duality matrix for non-abelian gauge theory. This chapter is based on the works done with Ori Ganor.

In chapter 3, we expand on section 2.7 of chapter 2 and give a detailed proof that the duality works for when the SYM has gauge group $SU(q)$. It turns out that the generating functions for counting $SU(q)$ ground state have interesting mathematical

[3] See, for example, section 7 of [33] for $U(1)$ gauge group. For $U(n)$ Chern-Simons theory, the Berry phase can be calculated by identifying the Hilbert space of $U(n)$ Chern-Simons theory on T^2, with a suitably chosen coupling for the diagonal $U(1)$, with the symmetric tensor product of the Hilbert space of $U(1)$ Chern-Simons theory on T^2 at the same coupling.

properties. The generating functions turn out to be Ehrhart polynomials [34], which are quasi-periodic polynomials that count rational points in polytopes. Because Ehrhart polynomials are closely connected to other branches of mathematics such as number theory and topology, we will formulate our proof in two different ways, one using Ehrhart polynomials and one using representation theory hinted at in section 2.7 of chapter 2.

The checks of the SYM-Chern Simons duality in chapter 2 and 3 are based on matching the dimension of the Hilbert space of both theories: Detailed calculations in these chapter will show that the dimension of the ground state Hilbert space of the SYM theory (i.e. the number of classically flat Wilson lines) matches that of the corresponding Chern-Simons theory. Can we go beyond this tree level result and check the duality at higher loops? In particular, we would like to understand if the ground state degeneracy for the flat Wilson lines is exact, and that quantum correction does not lift this degeneracy. In chapter 4, we will do a one-loop check of our duality using the superconformal index. We will see that, by computing the supersymmetric Casimir energy, the SYM ground state degeneracy is indeed exact up to one-loop. Since the superconformal index is a protected quantity, we will also use it to check that the ground state t' Hooft lines have the same ground state degeneracy as well as the same supersymmetric Casimir energy as the ground state Wilson lines. In addition, S-duality predicts that the number of ground state t' Hooft lines must equal that of the ground state Wilson lines. We will compute explicitly the number of ground state t' Hooft lines using the $SU(2)$ gauge group and check that the number agrees that of the Wilson lines. This combined with the supersymmetric Casimir energy will provide yet another test of S-duality.

As mentioned previously, the duality proposed here can be generalized so that the Chern-Simons theory is quantized on higher genus Riemann surfaces. The natural SYM dual is the class-S theory [10]. This important generalization will be explored in chapter 5.

2.2 The Problem

$\mathbb{R}^4$ is conformally equivalent to $S^3 \times \mathbb{R}$, with the metric

$$ds^2(\mathbb{R}^4) = dr^2 + r^2 d\Omega_3^2 \to ds^2(S^3 \times \mathbb{R}) = d\tau^2 + d\Omega_3^2 = \frac{1}{r^2} ds^2(\mathbb{R}^4), \qquad \tau = \log r.$$

This fact is used in the state-operator correspondence to match dimensions of operators of a CFT on $\mathbb{R}^4$ to energies of states of the CFT on S^3. We will be interested in studying $N = 4$ SYM. The conformally coupled Lagrangian is given by [4]

$$L = \frac{1}{4g^2} \operatorname{tr} \left\{ F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi^I D^\mu \Phi_I + [\Phi^I, \Phi^J][\Phi_J, \Phi_I] + \frac{1}{6} R \Phi^I \Phi_I + \text{fermions} \right\} \quad (2.5)$$

where g is the dimensionless coupling constant. The gauge group G will be $U(q)$, $SU(q)$, $SU(q)/\mathbb{Z}_q$ or other groups to be specified later.

Thanks to the coupling to the curvature R of S^3, the spectrum on S^3 is discrete. The ground state is unique, and corresponds to the identity operator. By supersymmetry, we can explore the ground states as $g \to 0$, where they correspond to solutions of $F=0$ up to gauge equivalence (i.e., flat connections). On S^3 all solutions to $F=0$ are gauge equivalent to $A=0$, and the solution is unique, corresponding to a unique ground state.

Now consider the case that the CFT is formulated on the cone $(S^3/\Gamma)\times\mathbb{R}$, where Γ is a finite subgroup of $SU(2)\times SU(2)$, the double-cover of the isometry group of S^3 (double-cover is needed since Γ acts on spinors). We will assume a trivial R-symmetry bundle, which means that acting on an R-charged (under the $SU(4)$ R-symmetry) field, say Φ^I, the action of Γ is defined by setting $\Phi^I(x)=\Phi^I(\gamma x)$ for all $\gamma\in\Gamma$, as opposed to a more complicated $\Phi^I(x)=\Lambda(\gamma)^I{}_J\Phi^J(\gamma x)$ for some nontrivial four dimensional unitary representation $\Lambda(\gamma)$ of Γ.

To preserve half of the supersymmetry, we need to ensure that Γ is embedded in one of the $SU(2)$ factors. In that case, under the state-operator correspondence, the problem is equivalent to studying $N=4$ SYM on an ALE space $\mathbb{R}^4/\Gamma$. By McKay correspondence, the possible finite Γ's, and hence the ALE spaces, are classified by simply-laced (ADE) Dynkin diagrams. As in the flat space, we can study the ground states as $g\to 0$, where they correspond to flat connections on S^3/Γ, i.e., solutions to $F=0$.

For gauge group G, a flat connection corresponds to a homomorphism $\rho:\pi_1(S_3/\Gamma)$, up to conjugation. Since S^3 is simply connected, the homomorphism reduces to $\rho:\Gamma$. That is, ρ has to satisfy $\rho(\gamma_1\gamma_2)=\rho(\gamma_1)\rho(\gamma_2)$, and we identify ρ and ρ' if $\rho'(\gamma)=\alpha^{-1}\rho(\gamma)\alpha$, for some fixed α. The connection is established by fixing a point $p\in S^3$ and identifying $\rho(\gamma)$ with the holonomy along a path in S^3 that connects p with $\gamma(p)$.

Denoting a basis for the ground states by $a,b,c,\dots$, we wish to understand the action of S-duality. In general, the Hilbert space forms a vector bundle over the space of coupling constants

$$\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2} \tag{2.6}$$

Assuming that this is a flat vector bundle, there is a unique way to map a state $|a\rangle_\tau$ defined over a coupling constant τ to a state $|a\rangle_{(-1/\tau)}$ defined over coupling constant $-1/\tau$. We can then ask what is the matrix element of the S-duality operator $\langle b|S|a\rangle$.

We can calculate the Berry phase by allowing $\tau=\tau_1+i\tau_2$ to vary with time t, and performing a path integral of the gauge theory on S^3/Γ, with initial condition at $t=-\infty$ corresponding to the flat connection associated with the state $|a\rangle$, and final condition at $t=\infty$ corresponding to $|b\rangle$. We are looking for a term in the low-energy effective action that corresponds to $\int(\cdots)\tau_1'(t)dt$. For example, a $U(1)$ Berry

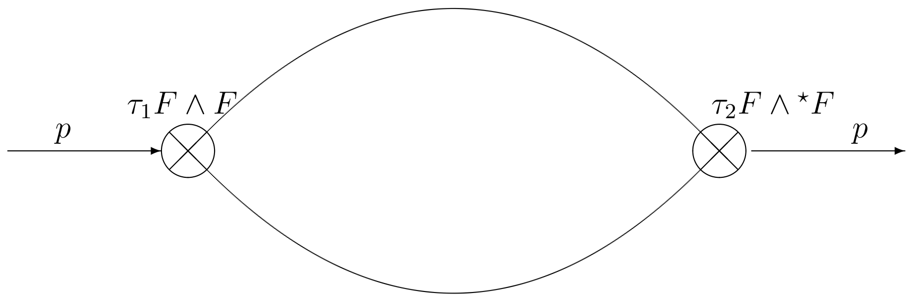

Figure 2.1: A Feynman diagram that calculates the evaluation of the Berry connection. All the lines are gluon propagators (or their SUSY partners) on $S^3\times\mathbb{R}$ in the presence of a background field A_a. The two $\otimes$ vertices correspond to insertions of $\widehat{\delta\tau_1}(\omega)$ times (the Fourier transform of) $F\wedge F$ and $\widehat{\delta\tau_2}(\omega)$ times (the Fourier transform of) $F\wedge{}^\star F$, both with momentum $p=(\omega,\vec{0})$.

connection with the $(2,\mathbb{R})$-invariant Berry curvature $d\tau_1\wedge d\tau_2/\tau_2^2$ would be read-off from an effective action of the form $\int d\tau_1/\tau_2^2$. Perturbatively, We can calculate the path integral diagrammatically. (See [35, 36] for a general recent discussion of Berry phases in QFTs.) One flat connection cannot transition into another perturbatively, so we can only get a potentially nonzero result if $a=b$. The diagrams are then standard Feynman diagrams in the background gauge field A_a. We introduce a 4-momentum $p^\mu=(\omega,\vec{0})$ and look for diagrams that contribute a term proportional to $\omega\widehat{\tau}_1(\omega)\widehat{\tau}_2(-\omega)$, where $\widehat{\tau}$ is the Fourier transform of τ. Such terms require insertions of $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ (at momentum p) and $F^{\alpha\beta}F_{\alpha\beta}$ at momentum $-p$. This is depicted in Figure 2.1.

Perturbatively in the gauge field A_a, it is easy to see by gauge invariance that the Berry curvature must vanish. This is because the Levi-Civita tensor $\epsilon^{\alpha\beta\gamma\delta}$ from the $F\wedge F$ insertion cannot contract with anything but the Chern-Simons form of the flat connection A_a on S^3/Γ. Any gauge invariant expression that contains the field strength $F_a=dA_a+A_a\wedge A_a$ vanishes, because the connection is flat. The gravitational Chern-Simons form of S^3/Γ vanishes as well. What is left is the exterior product of the Chern-Simons three form $\mathrm{CS}(A_a)$ and a 1-form. But the latter must be closed, for gauge invariance, and so must locally be of the form $df(\tau_1,\tau_2)$. This then reduces to a flat Berry connection df.

We conjecture that the vector bundle of ground states $|a\rangle$ over the fundamental domain of τ-space is flat also nonperturbatively, but we are willing to accept a phase ambiguity in $\langle b|S|a\rangle$, with an unknown phase that is independent of the states $|a\rangle$ and $|b\rangle$, as argued in section 2.1 based on the proposed solution.

We propose in section 2.4 that the ground state Hilbert space here is dual to the Hilbert space of a certain Chern-Simons theory. The duality also maps the S-duality operator from the SYM side to the action of S on the Chern-Simons theory which inverts the complex structure of the torus. In a suitable basis, the S-duality kernel is simply the S matrix that appears in Verlinde's formula [37].

We are particularly interested in the gauge groups $SU(q)$ and $SU(q)/\mathbb{Z}_q$ which are Langlands dual to each other. The S-duality on the SYM side should map these gauge groups into each other, and so the number of $SU(q)$ representations of Γ must be the same as the number of $SU(q)/\mathbb{Z}_q$ representations of Γ. On the dual Chern-Simons theory side, we will see in section 2.4 that the S-duality acts on the algebra in a natural way, exchanging the $SU(q)$ states with the $SU(q)/\mathbb{Z}_q$ states.

For the $U(q)$ SYM theory, the ground states are simply q dimensional unitary representations of Γ. Since Γ is a finite group, any finite dimensional representation is equivalent to some unitary representation [38]. The fundamental building blocks of finite dimensional representations are irreducible representations. Therefore, in the next section, we study the irreducible representations. It turns out that the type and the dimension of irreducible representations can be conveniently read off from the affine Dynkin diagram corresponding to Γ. This is the celebrated Mckay correspondence [21].

2.3 Review of the McKay Correspondence and the Representation of ADE Finite Groups

McKay classified all discrete subgroups Γ of $SU(2)$ and their irreducible representations using Dynkin diagrams [21]. The possible discrete subgroups Γ are $\mathbb{Z}_k$ (cyclic group of order k), Dic_n (binary dihedral group of order n), $2T$ (binary tetrahedral group), $2O$ (binary octahedral group), and $2I$ (binary icosahedral group). The last three groups are famously the double cover of the symmetry groups of the Platonic solids. Given a group Γ, let the number of irreducible representations of Γ be p, and let $\{V_i\}, i \in \{1, 2, ..., p\}$ be the set of finite dimensional irreducible representations. According to McKay, the irreducible representations $\{V_i\}$ have a quiver diagram representation. The diagram is composed of p nodes and links connecting the nodes. The i^{th} node represents the irreducible representation V_i. The nodes are connected by links as follows. Let V be the 2-dimensional defining (not necessarily irreducible) representation of Γ. There is a link connecting node V_i to node V_j if and only if V_j appears in the tensor product

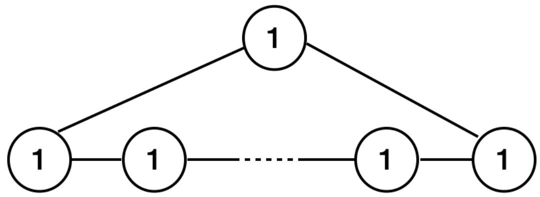

Figure 2.2: The affine Dynkin diagram for A_{n-1}. There are n nodes in total, corresponding to n irreps. The number in each node suggests that each irrep is a 1-dimensional representation.

decomposition of $V_i \otimes V$. The above relationship implies that $2[V_i] = \sum_j [V_j]$ if V_j is connected to V_i, where $[V_i]$ denotes the dimension of the corresponding representation.

It turns out that the quiver diagram obtained in this fashion coincides with the affine Dynkin diagram of the corresponding ADE Lie algebra. The correspondence is

$$\mathbb{Z}_k \to A_{k-1}, \quad \mathrm{Dic}_n \to D_{n+2}, \quad \mathrm{T} \to E_6, \quad \mathrm{O} \to E_7, \quad \mathrm{I} \to E_8.$$

As an example, let $\Gamma = \mathbb{Z}_k$. The corresponding Lie algebra is A_{k-1}. Because Γ is abelian, all irreps are 1-dimensional. The i^{th} irrep has the generator ω^i, where $\omega = \exp(2\pi i/k)$ is the k^{th} root of unity. The affine Dynkin diagram for A_{k-1}, $k > 2$ reads where there are k nodes in total. The number in each node represents the dimension of the corresponding representation. Note that the sum of the neighboring nodes equals twice of each node, agreeing with the relation obtained earlier.

As a slightly more nontrivial example, the affine Dynkin diagram for E_6 reads where now there are three 1-dimensional irreps, three 2-dimensional irreps, and one 3-dimensional irrep.

2.4 Proposed Solution

Computing the action of S and T on the ground state Hilbert space of a non-abelian $N = 4$ SYM theory is hard. Our strategy is to use an AdS/CFT argument to find a dual system whose behavior under the corresponding S and T operations is understood. It turns out that this dual system is a certain Chern-Simons theory on T^2. In section 2.4, we give a brief review of the subject of string theory and D-branes on orbifold so as to put our holographic construction into a broader scope. or In section 2.4, we explain why the dual system is Chern-Simons on T^2 using holography. In section 2.4, we formulate the duality in another way using the language of Heisenberg-Weyl algebras.

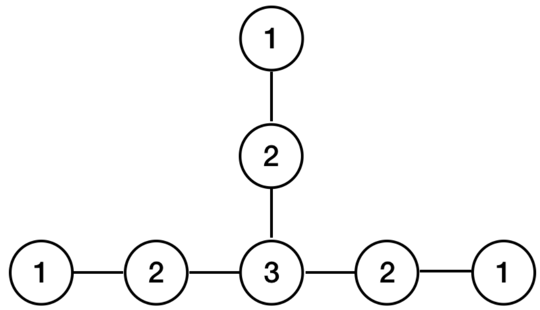

Figure 2.3: The affine Dynkin diagram for E_6. There are 7 nodes in total, corresponding to 7 irreps. In addition to the three 1-dimensional irreps, there are three 2-dimensional irreps and one 3-dimensional irrep.

String Theory and D-branes on Orbifold

There is a long history of the study of string theory and D-branes on orbifold singularity. Since the particular orbifold singularity that will concern us in this paper is the ADE orbifold singularity, we will in this section do a quick review of some past research on ADE singularity and string theory. We will discuss how this work is different from past research and motivate the Chern-Simons/SYM duality mentioned in the introductory section 3.1.

There are roughly speaking three uses of putting string theory or D-branes on ADE singularity. The first is to obtain a more realistic compactification of string theory down to lower dimensions. The second is to use string theory to probe the topology and geometry of the ADE singularity. The third is to use string theory construction to obtain new classes of quantum field theory.

One way to obtain ADE singulairty is the "blow-down" of K3 surface [39], the only nontrivial Calabi-Yau manifold in four (real) dimensions [40]. String theory on K3 surface leads to more realistic models of string compactification because the $SU(2)$ holonomy of K3 surface helps break half of the supersymmetry. It also leads to more examples of string dualities [40]. In fact, as we will see in section 2.4, the duality considered in this paper is derived using the duality between heterotic string on T^4 and IIA string on K3.

The second use partially overlaps with the first use because the low energy spectra of string theory on K3 sheds light on the topological invariants of the K3 surface [39, 40]. To probe the geometry and not just the topology of K3 surface, D-brane technologies must be used [41]. In [41], the metric of some ALE (asymptotically locally Euclidean) space was computed by probing the space using D1 branes *transverse* to the space. The fact that the moduli space of vacua for the low energy theory coincides with the ALE space suggests that the metric of the underlying space coincides with the metric that enters the kinetic term of the low energy theory.

The third use dates back to [22], where it was mentioned that M-theory on ADE singulairty Γ leads to a seven dimensional super Yang Mills theory transverse to the 4-dimensional ADE singularity with the gauge algebra $\mathfrak{g}(\Gamma)$ McKay-dual to the ADE singularity. In [42, 43], quiver gauge theories are constructed by putting D3 branes *transverse* to the orbifold singularity. In the language of AdS-CFT correspondence, in this construction, the singularity mods out the S^5 part of the bulk geometry and keeps the AdS_5 part untouched.

The difference between this work and the previous constructions is that here, the ADE singularity is longitudinal to the D3 brane world volume. In other words, the bulk geometry is $AdS_5/\Gamma \times S^5$. As will be derived in the next subsection, the long distance limit of this holographic system will involve a Chern-Simons theory in the AdS bulk direction. Constructions similar to this work can be found in [23, 15]. Differences between our construction and those in [23, 15] will be explained in the section.

Analysis via AdS/CFT

The holographic dual [12] of $N = 4$ SYM on $\mathbb{R}^4/\Gamma$ is type-IIB string theory on $(AdS_5/\Gamma) \times S^5$, with Γ acting as the obvious subgroup of the Lorentz group, leaving a codimension 4 singularity at the fixed points of Γ. The singularity meets the boundary at the origin (and at infinity), wraps the S^5, and carries a $(2, 0)$-theory associated with type-IIB string theory on $\mathbb{R}^4/\Gamma$ [22]. We claim that for $G = SU(q)$, after reducing on S^5, the singularity carries local low-energy degrees of freedom that are equivalent to a level q Chern-Simons theory with gauge algebra $G(\mathcal{D})$, compactified on T^2, and the S-duality group $SL_2(\mathbb{Z})$ of $N = 4$ SYM acts geometrically as the mapping class group of T^2.

To show this, consider in IIB string theory a stack of N D3 branes that in the Euclidean signature span the 0123 directions. The ADE singularity Γ acts on the world volume[4] of the D3 branes $\mathbb{C}^2$ to make it $\mathbb{C}^2/\Gamma$. Since M-theory on T^2 is dual to IIB string theory, we lift the D3 branes to M5 branes by first taking the T-duality in the 4 direction to turn them into D4 branes and then blowing up the M-theory circle in the # direction. The situation can be summarized in Table 2.1.

A stack of N M5 branes whose worldvolume is placed on an ADE singularity $\mathbb{C}^2/\Gamma$ has a near horizon geometry $AdS_7/\Gamma \times S^4$. Let the Lie algebra corresponding to the

[4]Note that this setup is different from the D-brane on orbifold setup considered in [42], where the orbifold action is along the transverse direction of the D-brane worldvolume rather than the longitudinal direction.

	0	1	2	3	4	5	6	7	8	9	#
D3	N	N	N	N	D	D	D	D	D	D	x
M5	N	N	N	N	N	D	D	D	D	D	N

Table 2.1: The string/M theory setup. Here, D denotes "Dirichlet" and N "Neumann". To avoid confusion, we use # to deonte the 10th direction. Since the D3 brane exists only in the 9+1 dimensional universe, an "x" is put under the 10th direction to indicate that the D3 brane does not exist in that direction.

ADE singularity be $\mathfrak{g}(\Gamma)$. The boundary theory lives on directions 012345. Let the bulk direction be 6. We claim that there exists a coupling

$$-\frac{1}{4\pi}\int \frac{C_3}{2\pi} \wedge \mathrm{tr}(F \wedge F) \tag{2.7}$$

where C_3 is the 3-form in M-theory and F is the 2-form field strength taking values in $\mathfrak{g}$. The integral is taken in the 456789# directions. The 5-plane transverse to the M5 branes can be decomposed into $\mathbb{R}^+ \times S^4$, where $\mathbb{R}^+$ is the direction into the bulk, and S^4 is the 4-sphere that surrounds the M5 branes. Integrating by parts, we obtain

$$\frac{1}{4\pi}\int_{T^2\times\mathbb{R}^+\times S^4} \frac{dC_3}{2\pi} \wedge \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) = \frac{N}{4\pi}\int_{T^2\times\mathbb{R}^+} \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \tag{2.8}$$

reproducing the Chern-Simons theory on T^2 at level N.

Before we derive this, we mention two similar constructions in the literature. Ref [23] reproduces a part of this duality by considering intersecting D4-D6 branes in type IIA string theory, so only the A-singularity is probed since D6 branes cannot create the D- or the E-singularity. Indeed, the leval-rank dual of this coupling is easy to see for $\Gamma = \mathbb{Z}_k$ where M-theory on $\mathbb{R}^4/\Gamma$ can be replaced with M-theory on a Kaluza-Klein monopole with k units of charge and further reduced to type-IIA with k D6-branes. Ref [15], on the other hand, considers Nakajima's instanton construction [44] on $\mathbb{R}^4/\Gamma$ using a certain twisted super Yang-Mills theory without explicitly mentioning the Chern-Simons theory. In [15], it is shown that a certain combination of the middle-dimensional cohomology of $U(q)$ n-instanton moduli space on $\mathbb{R}^4/\Gamma$ has the same structure as the highest weight representation of a level q state of algebra $\mathfrak{g}(\Gamma)$ (see equation 4.42 in [15]). The link to the Chern-Simons theory is implicit in their work because of the equivalence between WZW conformal blocks and Chern-Simons theory as shown in [25]. We derive the duality using a perspective different from these two constructions. Compared to [23], our construction has the virtue of incorporating the D- and the E-singularity. Compared to [15], our construction does not require twisting the super Yang-Mills theory and is easier to understand from a physicist's perspective.

The strategy we use to derive equation (2.7) is through the string-string duality. In the first step, we compactify the 4 direction and take the T-dual to end up in the Type IIA string theory. We now have N D4 branes along the 01234 directions. The ADE singularity acting on the 0123 directions can be locally thought of as a certain degeneration limit of a K3 surface. Although the latter is compact and the former is noncompact, the distinction will not be important in the following derivation. It is known that heterotic string on T^4 is dual to IIA string on K3, with the coupling constants being related by the equation

$$e^{\phi} = e^{-\phi'} \tag{2.9}$$

where unprimed quantities are for the heterotic theory and the primed quantities are for the IIA theory [45]. In addition, the NS-NS 2-forms are related in 6D as

$$dB = e^{-2\phi'} \star dB' \tag{2.10}$$

Now, both the T^4 and the K3 are along the Euclideanized 0123 directions. Recall that the NS-NS 2-form in IIA theory descends from the 3-form in M-theory as $dB' \wedge dx^{\#} = dC_3$. So if one blows up the M-theory circle and lift the above relation to 7D (i.e. adding the # direction), one gets

$$dB = e^{-2\phi'} \star dC_3 \tag{2.11}$$

We now have N M5 branes along the 01234# directions, where the 4 and the # directions are circles.

The heterotic theory has a modified action for its 3-form field strength (setting the curvature contribution to zero[5]):

$$\frac{1}{2}\int_{\mathbb{R}^6} e^{-2\phi}(dB - CS, dB - CS) \tag{2.12}$$

where CS denotes the Chern-Simons three form and where we considered the T^4 compactified action so the integration is along the directions 456789 in the IIA theory. Notice that $e^{-2\phi} = e^{2\phi'}$ is the heterotic dilaton coupling. If we consider blowing up the M-theory circle to lift the above action to 7D, and plug in the duality relation, the above term becomes

$$\frac{1}{2}\int_{\mathbb{R}^7 = T^2\times\mathbb{R}^5} e^{2\phi'}(e^{-2\phi'} \star dC_3 - CS, e^{-2\phi'} \star dC_3 - CS) \tag{2.13}$$

[5]It is amusing to note that if one were to include curvature, i.e. the gravitational Chern-Simons term, one would be able to give a string-theory derivation of the curvature counter term in the one-loop correction to the Chern-Simons action. In [25], this counter term arises out of the consideration for canceling the framing anomaly.

where the integral is taken along the directions 456789# and we split the directions along the torus (4#) and the $\mathbb{R}^5$ (56789) transverse to the N M5 branes (see Table 2). The $\mathbb{R}^5$ can be further decomposed into a radial direction times an S^4 that surrounds the N M5 branes. From this expression, we get a cross term

$$-\int (CS, \star dC_3) = \int CS \wedge \star\star\, dC_3 = \int CS \wedge dC_3 = N \int CS \tag{2.14}$$

where in the last step we integrated the 4-form fluxes along the S^4 to pick up the overall prefactor N, reproducing (2.7) and the Chern-Simons action we sought for.

In the IR limit, the Chern-Simons action is the unique action that has the lowest number of derivatives, and will characterize the ground state structure of the theory. Therefore, the ground state Hilbert space of $\mathcal{N} = 4$ $U(q)$ SYM on S^3/Γ is equivalent to that of the holographic dual which is the Hilbert space of level q Chern-Simons theory with gauge group $\mathfrak{g}(\Gamma)$. This Chern-Simons Hilbert space is given by the set (2.4). Let us now understand the corresponding duality for $\mathcal{N} = 4$ $SU(q)$ SYM on S^3/Γ. Changing the gauge group from $U(q)$ to $SU(q)$ reduces the number of ground states in the SYM ground state Hilbert space. What is the corresponding reduction on the Chern-Simons side? To see this, we use the example $\Gamma = \mathbb{Z}_k$. The $SU(q)$ SYM ground states Hilbert space admits a level-rank duality: the dimension of $SU(q)$ ground state Hilbert on $S^3/\mathbb{Z}_k$ is the same as that of the $SU(k)$ ground state Hilbert space on $S^3/\mathbb{Z}_q$[6]. The only nontrivial subspace of the Chern-Simons Hilbert space that is invariant under the level-rank duality is the one given by the set (3.2) [46].

Having established this mini AdS/CFT duality, we review the Hilbert space of Chern-Simons theory on T^2 in detail in the next section from the perspective of the Heisenberg-Weyl algebra. We will also make precise the dictionary that maps between the different theories.

Heisenberg-Weyl Algebra

It is well known that in the gauge $A_0 = 0$, the quantization of Chern-Simons theory on T^2 reduces to the representation theory of the Heisenberg-Weyl algebra [31] (briefly reviewed in section 2.6 below). The Hilbert space of our proposed solution is equivalent to a representation of this Heisenberg-Weyl algebra constructed from the weight lattice $\Lambda_w(\mathcal{D})$ associated with the Dynkin diagram. The weight lattice has an inner product $\langle \alpha, \beta \rangle$ (for $\alpha, \beta \in \Lambda_w$). The Heisenberg-Weyl algebra is specified by two pieces of data: the diagram $\mathcal{D}$ and a positive integer q. It is generated by the set of operators

$$\{\mathrm{U}(\alpha), \mathrm{V}(\alpha)\}_{\alpha \in \Lambda_w}$$

[6] A quick way to show this is to use the $SU(N)$ Ehrhart polynomials computed in (3.52) of chapter 3 and show that the qth term of $\mathrm{Ehr}_{\mathfrak{su}(k)}$ is the same as the kth term of $\mathrm{Ehr}_{\mathfrak{su}(q)}$.

with relations

$$\mathrm{U}(\alpha)\mathrm{U}(\beta) = \mathrm{U}(\beta)\mathrm{U}(\alpha), \qquad \mathrm{V}(\alpha)\mathrm{V}(\beta) = \mathrm{V}(\beta)\mathrm{V}(\alpha),$$

and

$$\mathrm{U}(\alpha)\mathrm{V}(\beta) = \mathrm{V}(\beta)\mathrm{U}(\alpha) \exp\left(\frac{2\pi i}{q}\langle\alpha, \beta\rangle\right).$$

Here q is the level of the Chern-Simons theory, which according to the holography argument is the q taken from $U(q)$, $SU(q)$ or $SU(q)/\mathbb{Z}_q$ on the SYM side. We will distinguish between these three cases later on. Note that if $\alpha \in q\Lambda_r$ (where Λ_r is the root lattice) then $\mathrm{U}(\alpha)$ is a central element, which we can identify with the identity operator.

Let $\mathrm{Hilb}_q(\mathcal{D})$ be the unique (up to isomorphism) irreducible representation of the Heisenberg-Weyl algebra at level q. The Weyl group $W(\mathcal{D})$ acts as an outer automorphism of the Heisenberg-Weyl algebra and can be extended to act on $\mathrm{Hilb}_q(\mathcal{D})$. (From here on we will omit the $\mathcal{D}$ from Hilb_q and W.) Let Hilb_q^W by the Weyl-invariant subspace of Hilb_q. We then propose:

$$\boxed{\begin{pmatrix} \text{Ground states of } U(q) \text{ SYM} \\ \text{on } S^3/\Gamma \end{pmatrix} \longleftrightarrow \mathrm{Hilb}_q^W} \tag{2.15}$$

Now let $\mathfrak{X}$ be the subalgebra of the Heisenberg algebra that is generated by

$$\{\mathrm{U}(q\alpha)\}_{\alpha\in\Lambda_w}$$

Let $\mathrm{Hilb}_q^{\mathfrak{X},W} \subset \mathrm{Hilb}_q$ be the subspace that is invariant under both $\mathfrak{X}$ and W. We propose

$$\boxed{\begin{pmatrix} \text{Ground states of } SU(q) \text{ SYM} \\ \text{on } S^3/\Gamma \end{pmatrix} \longleftrightarrow \mathrm{Hilb}_q^{\mathfrak{X},W}} \tag{2.16}$$

Let $\mathfrak{Y}$ be the subalgebra of the Heisenberg algebra that is generated by

$$\{\mathrm{V}(q\alpha)\}_{\alpha\in\Lambda_w}.$$

Let $\mathrm{Hilb}_q^{\mathfrak{Y},W} \subset \mathrm{Hilb}_q$ be the subspace that is invariant under both $\mathfrak{Y}$ and W. We propose

$$\boxed{\begin{pmatrix} \text{Ground states of } SU(q)/\mathbb{Z}_q \text{ SYM} \\ \text{on } S^3/\Gamma \end{pmatrix} \longleftrightarrow \mathrm{Hilb}_q^{\mathfrak{Y},W}} \tag{2.17}$$

As mentioned earlier, S-duality on the SYM side acts by exchanging the gauge groups $SU(q)$ and $SU(q)/\mathbb{Z}_q$. This is manifested in the dual Chern-Simons side by exchanging the operators U and V in the Heisenberg algebra. Since U and V do not commute, S-duality acts nontrivially on the ground states if the ground states were originally labeled by the U quantum numbers. By our proposal, this action should also appear naturally on the SYM side. In the next section, we will compute the S-duality action on the SYM side for a $U(1)$ gauge theory. In section 2.6, we will perform a detailed check of the first statement (2.15). Note that states in the second statement (2.16) are precisely those that also lie on the root lattice Λ_r. The motivation for the second statement (2.16) was discussed at the end of section 2.4.

2.5 The Action of S-Duality on $U(1)$ SYM

As a simple example, in this section, we compute the matrix elements of S for $G = U(1)$. The group is self-dual and we begin with $\Gamma = \mathbb{Z}_k$, corresponding to the A_{k-1} Dynkin diagram. Let γ be a generator of Γ, satisfying $\gamma^k = 1$. Set

$$\omega = e^{2\pi i/k}.$$

Then, a ground state corresponds to $\rho(\gamma) = \omega^p$ for $p = 0, \ldots, k-1$. We will denote the corresponding state by $|p\rangle \in \text{Hilb}$.

The metric on S^3/Γ can be described by a Hopf fibration

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + (d\chi - \tfrac{1}{2}k\cos\theta d\phi)^2, \qquad 0 \le \chi, \phi < 2\pi, \qquad 0 < \theta < \pi.$$

The good coordinates near $\theta = 0$ are

$$\sin\theta\sin\phi, \sin\theta\cos\phi, \begin{cases} \chi - \frac{1}{2}k\phi & \text{near } \theta = 0 \\ \chi + \frac{1}{2}k\phi & \text{near } \theta = \pi \end{cases}$$

A gauge field corresponding to $|p\rangle$ can be taken as $A_p = pA_1$, where

$$A_1 = \begin{cases} \frac{1}{k}d\chi - \frac{1}{2}d\phi & \text{for } 0 \le \theta \le \frac{1}{2}\pi \\ \frac{1}{k}d\chi + \frac{1}{2}d\phi & \text{for } \frac{1}{2}\pi \le \theta \le \pi \end{cases}$$

This clearly satisfies $F = 0$.

Let us start by computing the action of T, which is related to the Chern-Simons action by

$$T = \exp\left(\frac{i}{4\pi}\int A \wedge dA\right) \tag{2.18}$$

If A were globally defined, to compute the Chern-Simons action we would simply evaluate

$$\frac{1}{4\pi}\int A \wedge dA.$$

Since A is not globally defined, the correct procedure is to find a four-manifold W such that its boundary $\partial W = S^3/\Gamma$, extend A to the bulk of X_4, with $F = dA$ in the bulk, and evaluate

$$\frac{1}{4\pi}\int_W F\wedge F.$$

To define W, we can simply add a coordinate

$$0 \le r \le 1$$

and set

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + dr^2 + r^2(d\chi - \tfrac{1}{2}k\cos\theta d\phi)^2\,,$$

thus converting the circle fibers of the Hopf fibration to disks.

The boundary of W is at $r=1$. For fixed $0<\theta<\pi$ and $0\le\phi<2\pi$, let $D(\theta,\phi)\subset W$ be the disk corresponding to all values of $0\le r\le 1$ and $0\le\chi<2\pi$.

Instead of looking for A, we will look for $F = dA$ (locally). We would like to find F on W such that

$$\int_{D(\theta,\phi)} F = \frac{p}{k} \tag{2.19}$$

Note that $pdr\wedge A_1$ is not continuous. But we can take

$$F = \frac{p}{k}dr\wedge(d\chi - \frac{1}{2}k\cos\theta d\phi) + \frac{p}{2}r\sin\theta d\theta\wedge d\phi, \tag{2.20}$$

which is well-defined, because $(d\chi - \frac{1}{2}k\cos\theta d\phi)$ is globally defined. Note that $dF=0$. In fact, $(d\chi - \frac{1}{2}k\cos\theta d\phi)$ is the global angular form of the Hopf fibration, and F above can be identified as a Thom class [47]. Now we can evaluate

$$\frac{1}{4\pi}\int_W F\wedge F = \frac{\pi p^2}{k} \tag{2.21}$$

and therefore

$$T|p\rangle = e^{i\pi p^2/k}|p\rangle \tag{2.22}$$

For odd k there is an anomaly, because the sign of the action ot T is ambiguous, and only T^2 is well defined.

We can compute the S action on the ground states $|p\rangle$ using the explicit S-duality kernel quoted in [26]. We parameterize the general S action on the coupling constant τ as

$$S = \begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{pmatrix} \quad \tau' = S\tau = \frac{\mathbf{a}\tau+\mathbf{b}}{\mathbf{c}\tau+\mathbf{d}}$$

The S-duality kernel $S(\tilde{A}, A)$ helps map the wavefunction as $\Psi\{A\}$ a functional of the gauge field A from one duality framed (unprimed) to another (primed). This is represented by a path integral

$$\Psi\{\tilde{A}\} = \int [DA] S(\tilde{A}, A) \Psi\{A\} \tag{2.23}$$

The kernel $S(\tilde{A}, A)$ for $U(1)$ gauge group is [26]

$$S(\tilde{A}, A) = \exp\left\{\frac{i}{4\pi c}\int \mathbf{d}\tilde{A}\wedge d\tilde{A} - 2Ad\tilde{A} + \mathbf{a}A\wedge dA\right\} \tag{2.24}$$

In our problem, the gauge field A and the corresponding curvature F are divided into k topological sectors. The above equation shows that the kernel is topological, so the path integral reduces to a sum over the k topological sectors. Therefore, equation (2.24) reduces to the following form

$$\Psi\{\tilde{A}_{\tilde{p}}\} = \sum_{p=0}^{k-1} S(\tilde{A}_{\tilde{p}}, A_p)\Psi\{A_p\} \tag{2.25}$$

where A_p labels the connection in the pth topological sector. Restricting to the S action

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we evaluate the kernel (2.24) by extending A_p and $\tilde{A}_{\tilde{p}}$ into the bulk W:

$$S(\tilde{A}_{\tilde{p}}, A_p) = \exp\left\{i\frac{1}{2\pi}\int_W F_p \wedge \tilde{F}_{\tilde{p}}\right\} \tag{2.26}$$

We can use equation (2.20) for the explicit form of $\tilde{F}_{\tilde{p}}$ and F_p. Note that because of topological invariance, the only difference between these two quantities are the constants p and $\tilde{p}$. Using this, the S-duality kernel can be computed:

$$S(\tilde{A}_{\tilde{p}}, A_p) = \frac{1}{\sqrt{k}}\exp\left(\frac{2\pi i p\tilde{p}}{k}\right) \tag{2.27}$$

This is simply the Fourier matrix of rank k. Equations (2.22) and (2.27) define a (projective) k-dimensional representation of $SL_2(\mathbb{Z})$.

Does this representation agree with our duality proposal? According to our proposal, the ground states of this $U(1)$ SYM theory on $S^3/\mathbb{Z}_k$ are dual to the states of level 1 Chern-Simons theory with gauge group $SU(k)$. Our proposal implies that (2.22) and

(2.27) should be obtained from the T transformation ($\tau \to \tau + 1$) and S transformation ($\tau \to -1/\tau$) of $SU(k)$ Chern-Simons (on T^2) at level 1. In a basis of states of Chern-Simons theory on T^2 labeled by highest weights, the action of T is given by [46]

$$T_{\Lambda,\Lambda'} = \delta_{\Lambda,\Lambda'} e^{2\pi i m_\Lambda} \tag{2.28}$$

where m_Λ is the modular anomaly (at level q) defined as

$$m_\Lambda = \frac{|\lambda + \rho|^2}{2(q+h)} - \frac{|\rho|^2}{2h} \tag{2.29}$$

The action of S is given by the Verlinde matrix, which for simply laced Lie groups is [46]

$$S_{\Lambda,\Lambda'} = \frac{i^{|\Delta_+|}}{\sqrt{(\det C)(q+h)^r}} \sum_{w \in W} \epsilon(w) \exp\left[-2\pi i \frac{(w(\Lambda+\rho), \Lambda' + \rho)}{q+h}\right]. \tag{2.30}$$

In this expression, $|\Delta_+|$ is the number of positive roots, q is the level, h is the dual Coxeter number, r is the rank of the Lie algebra, W is the Weyl group, $\epsilon(w)$ is the determinant of the Weyl group element w, and ρ is the half-sum of positive roots. In addition, Λ and Λ' are highest weights of the integrable representation of the corresponding affine Lie algebra at level k. The Weyl group has order $k!$ for $SU(k)$, so the sum contains $k!$ terms. For level $q = 1$ there are k choices for the weights Λ, Λ' and the Verlinde matrix (2.30) is a $k \times k$ matrix.

It is important to note that in comparing (2.28) to (2.22), and (2.30) to (2.27), we should in principle also include an integral of a Berry connection. Starting with a given τ, and a path in τ-space from τ to $-1/\tau$, we must multiply the Verlinde matrix by the path-ordered integral $P\exp\left(\int A_{\text{Berry}}\right)$ of the Berry connection A_{Berry} before comparing to (2.27). However, we can work at $\tau = i$ which is invariant under $\tau \to -1/\tau$, and for which the Berry connection factor is trivial. Another way of obtaining this is to note that (2.30) and (2.27) describe the action of S on characters of an affine Lie algebra. The characters are functions of the modular parameter τ and additional data that can be interpreted as parameterizing holonomies of the Chern-Simons gauge fields along one of the cycles of T^2. The characters can thus be understood as wavefunctions of the states of Chern-Simons theory on T^2, and (2.30) and (2.27) correspond to different bases of the Hilbert space of states on T^2. However, the base-change matrix depends on τ and is not generally invariant under S. The difference between (2.30) and (2.27) therefore also includes the nontrivial transformation of the base change matrix elements under S.

For $\tau = i = -1/\tau$ the Berry phase is trivial, and our conjecture requires the Verlinde matrix (2.30) to be equivalent to the Fourier matrix (2.27), but it is not obvious from the expression that, after summing $k!$ terms by restricting to level $q = 1$, one ends up with a rank-k Fourier matrix. In fact, for $k = 1, 2, 3, 4$, one can numerically check that (2.30) reduces to the Fourier matrix, but this is no longer the case when $k > 4$. However, the eigenvalues of the Verlinde S-matrix computed using (2.30) agree with the eigenvalues of the Fourier matrix of the corresponding rank, which suggests that a change of basis has happened: the basis states in the SYM side are in general different from the basis states used to compute the Verlinde S-matrix on the Chern-Simons side. Indeed, it was shown in [48] that, in a suitable basis of affine characters, the Verlinde S matrix for $SU(k)$ at level $q = 1$ is equivalent to the rank-k Fourier matrix, which is a manifestation of level-rank duality, and is consistent with our proposal.

We will now show this explicitly in a way that will also demonstrate the equivalence of the $SL_2(\mathbb{Z})$ representations.

To see this, Denote by Hilb$[U(1)_k]$ the Hilbert space of $U(1)$ Chern-Simons theory on T^2 at level k (for a fixed τ). Denote by Hilb$[SU(n)_k]$ the Hilbert space of $SU(n)$ Chern-Simons theory on the same T^2, at level k, and denote by Hilb$[U(n)_{k,nk}]$ the Hilbert space of $U(n)$ Chern-Simons theory at level (k, nk) (where the first factor refers to the $SU(n)$ level, and the second to the level of the $U(1)$ center). Then, we have an equivalence [31]

$$\text{Hilb}[U(n)_{k,nk}] \simeq \left(\text{Hilb}[U(1)_k]\right)^{\otimes n}/S_n$$

where the RHS is the symmetric part of the tensor product of n factors of Hilb$[U(1)_k]$.

Now set $k = 1$. Then, Hilb$[U(1)_k]$ is one-dimensional and the RHS is therefore also one-dimensional. It follows that Hilb$[U(n)_{1,n}]$ is one dimensional. But since $U(n) = [SU(n) \times U(1)]/\mathbb{Z}_n$ we can construct any state of Hilb$[U(n)_{1,n}]$ as a state in the tensor product Hilb$[SU(n)_1] \otimes$ Hilb$[U(1)_n]$. We can find a basis in which the state of this one-dimensional space takes the form

$$\sum_{p=0}^{n-1} |p\rangle_{SU(n)} \otimes |p\rangle_{U(1)}$$

where $|p\rangle_{U(1)}$ are the states of the Fourier basis that appear in, say, (2.22). Also, as a projective representation of $SL_2(\mathbb{Z})$ the state of Hilb$[U(n)_{k,nk}]$ transforms trivially. It follows that the states $|p\rangle_{SU(n)}$ transform in a dual way to the states $|p\rangle_{U(1)}$.

2.6 Matching of the Dimension of the Hilbert Space for $U(q)$ Gauge Group

We now show that the SYM theory and the dual Chern-Simons theory have the same dimension of the ground state Hilbert space. In the first subsection, we compute the ground state Hilbert space dimension of the SYM theory with gauge group $U(q)$ on S^3/Γ. According to our proposal, the dual theory is the Chern-Simons theory with gauge group $G(\Gamma)$ at level n. The dimension of the Chern-Simons Hilbert space will be reviewed in section 2.6.

Dimension of the Ground State Hilbert Space for $U(q)$ SYM Theory

Let us calculate the number of ground states for $U(q)$, for general n. We need to find the number of inequivalent homomorphisms $\Gamma \to U(q)$, i.e., the number of inequivalent q-dimensional unitary representations of Γ. Since all irreducible representations of a finite group are equivalent to unitary representations [38], we can get unitary representations if we combine the unitary irreducible representations.

Let C_q be the number of ground states for gauge group $U(q)$, and let $\sum_{q=0}^{\infty} C_q t^q$ be the generating function (with $C_0 = 1$ by definition), where t is an auxiliary variable that we introduced to keep track of q. A finite group Γ has a finite number of irreducible representations. According to the McKay correspondence reviewed earlier, the number of irreducible representations is the number of nodes in the corresponding Dynkin diagram $\mathcal{D}(\Gamma)$. If the rank of the corresponding simple Lie algebra is r, then this number is simply $r + 1$. (This is also the number of conjugacy classes in the group [38].) Let $m_0, \ldots, m_r$ be the dimensions of the irreducible representations, with $m_0 = 1$ corresponding to the trivial representation. Since finite dimensional representations are built out of copies of irreducible representations, we can write the generating function as:

$$\Phi(t) = \sum_{n=0}^{\infty} C_n t^n = \prod_{i=0}^{r} \frac{1}{1 - t^{m_i}} \tag{2.31}$$

This formula expresses the number of ways C_n in which n can be partitioned into dimensions of irreducible representations.

Using the McKay correspondence, we can read off m_i from the numbers in the corresponding affine Dynkin diagram. We have the following list for $m_1, \ldots, m_r$:

$$\begin{aligned}
A_n &: \quad \underbrace{1, 1, 1, \ldots, 1}_{n}\,; \\
D_n &: \quad 1, 1, 1, \underbrace{2, 2, 2, \ldots, 2}_{n-3}\,; \\
E_6 &: \quad 1, 1, 2, 2, 2, 3\,; \\
E_7 &: \quad 1, 2, 2, 2, 3, 3, 4\,; \\
E_8 &: \quad 2, 2, 3, 3, 4, 4, 5, 6\,.
\end{aligned}$$

So, for example, for $n = 2$ and E_6 we have $C_n = 9$ since there are three 1-dimensional representations and therefore 6 ways to decompose $2 = 1 + 1$, and there are in addition three 2-dimensional irreducible representations. In chapter 3 we produce the generating functions for $SU(q)$ representations; they can be built from the generating functions for the $U(q)$ representation.

Dimension of the Hilbert Space for Chern-Simons Theory with Gauge Group G at Level q

Canonical quantization of Chern-Simons theory on T^2 was discussed in [31], and we briefly review the result. In temporal ($A_0 = 0$) gauge, the constraint $F_{ij} = 0$ ($i, j = 1, 2$ along T^2) restricts the gauge field to be flat already at the level of the path integral, not just the equations of motion, thus recasting the problem as quantization of the moduli space of flat connections [25]. For T^2, the holonomies of the gauge field along the 1-cycles of T^2 commute, and with a gauge transformation if necessary, the components A_i of the gauge field can be conjugated to take values in the Cartan subalgebra corresponding to $\mathcal{D}(\Gamma)$. We denote those time-dependent elements of the Cartan subalgebra by $\theta_i(t)$. We choose a basis of the Cartan subalgebra and denote the components of θ_i in that basis by θ_i^I [$I = 1, \ldots, \text{rank}\mathcal{D}(\Gamma)$]. Let C_{IJ} be the Cartan matrix corresponding to $\mathcal{D}(\Gamma)$, in the chosen basis. The action in terms of $\theta(t)$ becomes

$$S = \frac{q}{2\pi} \int dt C_{IJ} \theta_1^I \dot{\theta}_2^J,$$

and its quantization leads to a basis of quantum states that can be put in one-to-one correspondence with the elements of a quotient

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r} \tag{2.32}$$

where Λ_w and Λ_r are the weight lattice and the root lattice for $\mathcal{D}(\Gamma)$, respectively, and W is the Weyl group. This is the same Hilbert space as the one that appeared in section 2.4 as a representation of a Heisenberg-Weyl algebra. Recall that there, we identified states that differ by the action of the Weyl group, and set states that are q times the root to be trivial.

We also recall the connection with Affine Lie algebras [25]. The affine Weyl group at level q is given by [46]

$$W \ltimes q\Lambda_{cr} \tag{2.33}$$

where Λ_{cr} is the coroot lattice. For ADE gauge groups which are simply laced, the coroot lattice coincides with the root lattice $\Lambda_{cr} = \Lambda_r$. Therefore, (2.32) means that the number of independent states is the same as the number of highest weight representations of the affine Lie algebra of G at level q. At first sight, it is not obvious that the number of states in (2.32) is the same as the one obtained in section 2.6 by counting flat connections of $U(q)$ on S^3/Γ. We will show in section 2.6 that they are in fact the same.

Matching at Large q

Before we show that the dimensions of Hilbert spaces of the two theories (Chern-Simons and ground states of SYM on S^3/Γ) match for all q, it is illuminating to look at the large q case in which there exists a semiclassical formula for the number of states in the Chern-Simons theory. On the SYM side, we will see that the large q counting reproduces a remarkable formula in the theory of Lie algebra.

For large q, the number of states described by the theory

$$S = \frac{q}{2\pi} \int dt C_{ij} \theta^i \dot{\theta}^j \tag{2.34}$$

can be computed by going to the phase space and demanding that each 2π cell (we set $\hbar = 1$) in the phase space contains one degree of freedom. In this fashion, one obtains the number of states for the Chern-Simons theory

$$N_{CS} = \frac{q^r \det C}{|W|} \tag{2.35}$$

where r is the rank of the Lie algebra and $|W|$ the size of the Weyl group. For related work see [49]. Let us now look at the SYM theory.

For the SYM theory, recall that the number of states is encoded in the C_n coefficients in the generating function which we reproduce here for the convenience of the reader (m_i represents the numbers in the affine Dynkin diagram):

$$\Phi(t) = \sum_{q=0}^{\infty} C_q t^q = \prod_{i=0}^{r} \frac{1}{1 - t^{m_i}} \tag{2.36}$$

To compute the coefficient C_q when q is large, we want to isolate the term with the highest power (in the denominator) in t in the polynomial.

Using

$$(1 - t^{m_i}) = (1 - t)(1 + t + t^2 + ... + t^{m_i - 1}) \tag{2.37}$$

we can put the generating function in the form

$$\Phi(t) = \frac{1}{(1 - t)^{r+1}} \prod_{i=0}^{r} \frac{1}{1 + t + t^2 + ... + t^{m_i - 1}} \tag{2.38}$$

The generating function can be broken into a sum with a leading term and some subleading terms

$$\Phi(t) = \frac{a_1}{(1-t)^{r+1}} + O\left(\frac{1}{(1-t)^r}\right) \tag{2.39}$$

where a_1 can be found by multiplying $\Phi(t)$ by $(1-t)^{r+1}$ and setting $t = 1$. From (2.38) we see that

$$a_1 = \prod_{i=0}^{r} \frac{1}{m_i} = \prod \frac{1}{\text{numbers on the affine Dynkin diagram}} \tag{2.40}$$

Next, extracting the coefficient multiplying t^q in the leading term using simple combinatoric formula, we obtain

$$C_q = \frac{(n+r)!}{r!n!} \prod \frac{1}{m_i} \approx \frac{n^r}{r!} \prod \frac{1}{\text{numbers on the affine Dynkin diagram}} \tag{2.41}$$

In the theory of Lie algebra, there exists a remarkable formula that computes the dimension of the Weyl group of any Lie algebra. It is given by [50]

$$|W| = r! \times \prod(\text{numbers on the affine Dynkin diagram}) \times \det C \tag{2.42}$$

where C is the Cartan matrix. Using this in our formula, we obtain

$$C_q = \frac{n^r \det C}{|W|} \tag{2.43}$$

which matches the large q Chern-Simons counting obtained in (2.35).

Matching for all q

Having discussed the case for large q, we now give a proof of the match between the number of ground states for all q. The argument is simple. On the Chern-Simons side, we saw in the previous section that the number of states is the number of highest weight representation of G at level q. On the SYM side, we note that the highest root θ can be constructed from a linear combination of simple roots α_i with the coefficient m_i read off from the ith node in the Dynkin diagram [46]:

$$\theta = \sum_{i=1}^{r} m_i \alpha_i \tag{2.44}$$

Since the q dimensional representations are built out of the irreducible representations, one of which is the 1-dimensional trivial representation $m_0 = 1$, the number C_q

of q dimensional representations are given by the possible ways of choosing a set of nonnegative integers λ_i such that

$$\sum_i \lambda_i m_i \leq q \tag{2.45}$$

The number λ_i means the number of times the ith irreducible representation appears in constructing the q dimensional representation. The above formula makes sense because the way we construct the q dimensional representation here is to first ignore the trivial irreducible representation $m_0 = 1$, fill up the q dimensional representation using the remaining nontrivial r irreducible representations (hence the $\leq$ sign), and then fill up the rest (if any) by inserting some number(s) of trivial representation.

However, this formula can be "lifted" to an affine Lie algebra interpretation by introducing another integer λ_0 to complete the "weight" vector $\lambda = (\lambda_0, \lambda_1, ..., \lambda_r)$. The level q and λ_0 is related [46]:

$$\lambda_0 = q - (\lambda, \theta)$$

Highest weight representation at level q means that $\lambda_0 \leq 0$, which reproduces the previous inequality. Therefore, the possible choices of λ_i that satisfies the inequality gives the number of highest weight representations of the affine Lie algebra, which is the same as the number of states in the Chern-Simons theory. The argument here shows that this number is also the number of ground states in the SYM theory. This completes the matching for all q.

2.7 Matching of the Dimension of the Hilbert Space for $SU(q)$ Gauge Group

In the previous section, we showed that the Hilbert space of $U(q)$ SYM on S^3/Γ matches that of the level q Chern-Simons theory with gauge group G on T^2. In this section, we investigate the matching when the SYM theory has gauge group $SU(q)$. In particular, we will prove formula (2.16), copied here for convenience:

$$\begin{pmatrix} \text{Ground states of } SU(q) \text{ SYM} \\ \text{on } S^3/\Gamma \end{pmatrix} \longleftrightarrow \mathrm{Hilb}_q^{\mathfrak{X},W}$$

where we recall that $\mathrm{Hilb}_q^{\mathfrak{X},W}$ refers to states in the Heisenberg algebra Hilb_q that are invariant under both the Weyl transformation and the transformation generated by

$$\mathfrak{X} = \{\mathrm{U}(q\alpha)\}_{\alpha\in\Lambda_w}$$

We also recall that the identity element in the Heisenberg algebra is

$$\{\mathrm{U}(q\alpha)\}_{\alpha\in\Lambda_r}$$

and that the states in the Heisenberg algebra can be given a geometric interpretation [31] (see (2.32)):

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r}$$

The approach we take in proving the formula is to show that the states on both sides satisfy the same constraints. We first focus on the constraints for the Heisenberg algebra system.

Because of the identification by the Weyl group action, the states are weights restricted to lie inside the fundamental Weyl chamber of the Lie algebra. Because of the identification of q times the root lattice, weights that are also on $q\Lambda_r$ are identified as the identity element. Therefore, the states invariant under $\mathfrak{X}$ are precisely states that lie on

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r} \cap \Lambda_r$$

The generating function for counting such states is computed in chapter 3, and is shown to agree with the generating function for counting the corresponding states in the $SU(q)$ SYM theory. We here give a sketch of the proof. Let x be a state such that

$$x \in \frac{\Lambda_w}{W \ltimes q\Lambda_r} \text{ and } x \in \Lambda_r$$

Let the Dynkin label of a state x be $x = [x_1, x_2, ..., x_r]$, where each entry is a nonnegative integer. The first condition implies that

$$\sum_{i=1}^{r} a_i x_i \leq q \tag{2.46}$$

where a_i is expansion coefficient of the longest root in terms of the simple roots.

The second condition means that x can be written as a linear combination of simple roots with nonnegative coefficients. To find the expansion coefficients in terms of the simple roots, we simply need to multiply the Dynkin label as a vector by the inverse of the Cartan matrix. Therefore, we need to impose the constraint

$$\sum_{j=1}^{r} C^{-1}_{ij} x_j \in \mathbb{Z} \tag{2.47}$$

The key claim in chapter 3 is that the constraints satisfied by the states on the SYM side are exactly (2.46) and (2.47), thereby establishing the proof.

It is easy to see that (2.46) is one of the constraints on the SYM side. Let x_i $i = 0, 1, ..., r$ be the number of times the irreducible representation i of Γ appears in the homomorphism $\Gamma \to SU(q)$. Note that x_0 is the trivial 1-dimensional representation. By the McKay correspondence, the dimension of the ith irreducible representations is a_i, where we define $a_0 = 1$. Therefore, the constraint that the dimension of the representation is q gives

$$\sum_{i=0}^{r} a_i x_i = q \tag{2.48}$$

Treating x_0 as a slack variable, the above constraint is the same as equation (2.46). This was the same constraint we obtained for the $U(q)$ case encountered in §2.6.

Now, the new ingredient here is to impose a further constraint such that the representation has unit determinant. The general case is proved in 3, so here we will look at a specific case where $\Gamma = \mathbb{Z}_k$. The jth irreducible representation has determinant

$$\omega^j \tag{2.49}$$

where $\omega = \exp(2\pi i/k)$ and $j = 0, 1, ..., k-1$. The unit determinant constraint is

$$\prod_i \omega^{i x_i} = 1 \tag{2.50}$$

or, equivalently

$$\sum_{i=1}^{k-1} \frac{i x_i}{k} \in \mathbb{Z} \tag{2.51}$$

At first sight, claiming this constraint to be equal to the constraint derived in (2.47) seems impossible, since (2.47) actually contains $r = k-1$ constraints for the Lie algebra $\mathfrak{su}_k$. In fact, we shall see that only one of the $r = k-1$ constraints in (2.47) is independent. To show this, we give the formula for the inverse of the $\mathfrak{su}_k$ Cartan matrix:

$$C^{-1}_{ij} = \frac{1}{k}\left[\min(i,j) \times k - ij\right] \tag{2.52}$$

In particular, modulo 1, constraint (2.47) reads

$$\sum_{j=1}^{k-1} \frac{i j x_j}{k} \in \mathbb{Z}$$

for each $i = 1, 2, ..., k-1$. We see that if the constraint is satisfied for $i = 1$, the rest of the cases $i = 2, 3, ..., k-1$ are automatically satisfied. Therefore, the only independent constraint is

$$\sum_{j=1}^{k-1} \frac{j x_j}{k} \in \mathbb{Z} \tag{2.53}$$

We see that the constraints for counting states are the same for both systems when $\Gamma = \mathbb{Z}_k$. The idea used in this proof can be generalized to prove the case for the rest of the ADE Γ. In chapter 3, we derive the generating function for counting $SU(q)$ states for the SYM theory on S^3/Γ for $\Gamma = \mathbb{Z}_k, \mathrm{Dic}_k, E_6, E_7, E_8$[7].

2.8 More Examples of S Matrices

In section 2.5, we computed the S matrix for $U(1)$ SYM on $S^3/\mathbb{Z}_k$, and found that in a certain basis it matches with the level 1 Verlinde S matrix for gauge group $SU(k)$. In this section, we give some examples of S matrices for non-abelian gauge theory on S^3/Γ computed from the Chern-Simons theory side.

Example 1: S Matrix for $U(k)$ SYM on $S^3/\mathbb{Z}_2$

The first non-abelian example we focus on is $U(k)$ SYM on $S^3/\mathbb{Z}_2$. There are $k+1$ ground states in total. According to our proposal, the S matrix of this theory is the same as the level k Verlinde S matrix for gauge group $SU(2)$. This is given by [46]

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin \frac{(i+1)(j+1)\pi}{k+2} \tag{2.54}$$

where $i, j = 0, 1, ..., k$.

Example 2: S Matrix for $U(2)$ SYM on $S^3/\mathbf{Dic}_2$

For this example, we consider the ground states of nonabelian $U(2)$ SYM theory on S^3/Dic_2, where Dic_2 is the dicyclic (binary dihedral) group of order 2. According to our proposal, the ground states of this theory are dual to the ground states of the level 2 Chern-Simons theory with gauge group $SO(8)$ on T^2. There are 11 such ground states, labeled by the Dynkin label $[x_1, x_2, x_3, x_4]$ that satisfies the level 2 constraint

$$x_1 + 2x_2 + x_3 + x_4 \leq 2$$

where the coefficients multiplying x_i are the expansion coefficients of the highest root in terms of the simple roots. We use the explicit formula for the Verlinde S-matrix to compute the S-matrix. Note that the Weyl group for $\mathfrak{so}(8)$ is $(\mathbb{Z}_2)^3 \ltimes S_4$, so to compute each matrix element using the formula one needs to sum over $4! \times 2^3 = 192$ terms. We wrote a Mathematica code to compute the 11 by 11 S-matrix. The result is

[7] The case for E_8 is trivial. All irreducible representations have unit determinant, so the $U(q)$ representation coincides with the $SU(q)$ representation.

$$\frac{1}{4\sqrt{2}}\begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2\sqrt{2} & 0 & 0 & 2 & -2 & -2 & 0 & 0 & -2\sqrt{2} & 0 \\ 2 & 0 & 2\sqrt{2} & 0 & -2 & 2 & -2 & 0 & -2\sqrt{2} & 0 & 0 \\ 2 & 0 & 0 & 2\sqrt{2} & -2 & -2 & 2 & -2\sqrt{2} & 0 & 0 & 0 \\ 1 & 2 & -2 & -2 & 1 & 1 & 1 & -2 & -2 & 2 & 2 \\ 1 & -2 & 2 & -2 & 1 & 1 & 1 & -2 & 2 & -2 & 2 \\ 1 & -2 & -2 & 2 & 1 & 1 & 1 & 2 & -2 & -2 & 2 \\ 2 & 0 & 0 & -2\sqrt{2} & -2 & -2 & 2 & 2\sqrt{2} & 0 & 0 & 0 \\ 2 & 0 & -2\sqrt{2} & 0 & -2 & 2 & -2 & 0 & 2\sqrt{2} & 0 & 0 \\ 2 & -2\sqrt{2} & 0 & 0 & 2 & -2 & -2 & 0 & 0 & 2\sqrt{2} & 0 \\ 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & -4 \end{pmatrix}$$

The eigenvalues of the S-matrix are -1 and $+1$, with multiplicities 4 and 7, respectively.

2.9 Discussion

By putting the $\mathcal{N} = 4$ SYM on a singular space S^3/Γ, we are able to isolate the degenerate subspace of ground states and study how S-duality acts on this subspace. Remarkably, this ground state Hilbert space has a dual formulation in terms of the dual Chern-Simons theory. On this dual theory, the action of S and T operators are well-known, and it is natural to expect that the SYM theory has the same matrix elements for S in a suitable basis given that the Berry phase vanishes at $\tau = i$ as discussed in section 2.1.

In the next two chapters, we will perform more tests on this duality.

Chapter 3

Chern-Simons Theory, Ehrhart Polynomials, and Representation Theory

3.1 Introduction

As discussed in chapter 2, the states of Chern-Simons theory with simply-laced gauge algebra $\mathfrak{g}$ quantized on T^2 can be identified with points in the set (2.4), reproduced here for convenience

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r} \tag{3.1}$$

where Λ_w, Λ_r are the weight and the root lattice of $\mathfrak{g}$, and W is the Weyl group. In this paper, we study in detail the counting of a special class of states that lie in the set

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r} \cap \Lambda_r \tag{3.2}$$

where the intersection with Λ_r picks out the states in the set (2.4) that are also roots. The motivation for studying this special class of states is because they correspond to the states mentioned in the statement (2.16).

We will show that the counting of such states leads to a curious connection to Ehrhart polynomials, McKay correspondence, and representation theory. Ehrhart polynomials were first constructed to count lattice points in rational polytopes [34], a problem that is in general NP-hard [51] to solve by computer. Ehrhart polynomials have been found to connect different areas of mathematics such as number theory, geometry, and topology[1]. Since closed-form Ehrhart polynomials are rare and are only for very

[1]For an extensive introduction, see the book [52]

special geometries, our computation in this paper will add more examples to the known collection of Ehrhart polynomials.

We will also show that the Ehrhart polynomials we obtain from the geometric point of view can also be obtained from a representation theory point of view. In fact, the latter comes from a dual formulation of the problem. The two approaches are connected by the McKay correspondence [21], which gives a ADE Dynkin diagram classification of discrete subgroups of $SU(2)$ and their irreducible representations. The physics behind this dual formulation of the problem comes from a holographic system of D-branes on ADE singularity as disucssed in section 2.4 in chapter 2. To repeat the statement, there is a duality between the ground state Hilbert space $SU(q)$ $\mathcal{N}=4$ Supersymmetric Yang-Mills theory on S^3/Γ and a certain subspace (3.2) of the Hilbert space of level q Chern-Simons theory on T^2 with gauge algebra $\mathfrak{g}(\Gamma)$ given by the McKay correspondence.

This chapter is organized as follows. In section 3.2, we review the quantization of Chern-Simons theory on T^2 and formulate the Hilbert space geometrically in terms of certain points on the Lie algebra lattice. This partially overlaps but expands on the discussion in introductory section in chapter 2. To illustrate the rather abstract notation, we give an example of states of $\mathfrak{g}=\mathfrak{su}(3)$. In section 3.3, we pose the problem of counting the number of the special class of states in the set defined in (3.2). We shall find that the problem is the same as counting lattice points in rational polytopes, and that the generating function for counting the states is exactly the corresponding Ehrhart polynomial. In section 3.4, we compute the explicit form of the Ehrhart polynomial for a specific case by using the Ω operator introduced by MacMahon [53]. The general case is solved by reverse-engineering some representation theory formulas in section 3.5. We will see that the Ehrhart polynomial that counts the special states at level q with gauge algebra $\mathfrak{g}$ is the same as the generating function for the $SU(q)$ representation of the ADE subgroup given by the McKay correspondence. In section 3.6, we extend our result to the D-series where the gauge algebra of Chern-Simons theory is $\mathfrak{so}(2(N+2))$, $N\geq 1$. In section 3.7, we discuss some curious representation theory properties implied by the inverse of Cartan matrices modulo 1 for ADE Lie algebras by focusing on the exceptional series $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$. In section 3.8, we compute the Ehrhart polynomials for the exceptional Lie algebras. This chapter is based on the work [54].

3.2 Quantization of Chern-Simons Theory on T^2 and the States

The Chern-Simons theory with gauge algebra $\mathfrak{g}$ on T^2 can be quantized using the standard quantization procedure by choosing a gauge $A_0=0$ and imposing the constraint

$\delta S/\delta A_0 = 0$, where S is defined in equation (2.3). The constraint gives the condition that the connections are flat $F = dA + A \wedge A = 0$. Imposing the constraint, using the remaining freedom to gauge transform the connection A into the maximal torus of $\mathfrak{g}$, and imposing the canonical commutation relation, one obtains that the states are in one-to-one correspondence with points in the set [31] described in equation (2.4).

In words, states are points on the weight lattice of $\mathfrak{g}$ with two states being identified if they differ by some combinations of the Weyl transformation and q times the root lattice translation. We shall henceforth call this set the state set. Notice that because of the identification by $q\Lambda_r$, the number of states is finite, a fact that can also be seen from the compactness of the Chern-Simons phase space on T^2. In the large q limit, the number of states is given by

$$\frac{q^r \det C}{W} \tag{3.3}$$

where r is the rank of $\mathfrak{g}$, C is the Cartan matrix, and W is the size of the Weyl group. This formula can either be derived from equation (2.4) by noting that $\det C = \Lambda_w/\Lambda_r$ or going back to canonical quantization and demanding that the phase space contains one state per 2π cell (we set $\hbar = 1$).

To give an example of the state set of $\mathfrak{su}(3)$ Chern-Simons theory at level $q = 1$ and $q = 2$, we present the figure (see Fig. 3.1) from [55][2].

Fig. 3.1 shows the unique states in the Hilbert space for the level 1 and the level 2 theory, respectively. Notice that the pattern continues to all levels: all states lie in the fundamental Weyl chamber, and that as the level increases, the number of state increases quadratically (roughly as the area of the shaded region). In fact, the number of states for the $\mathfrak{su}(3)$ theory at level q is

$$\frac{q(q+1)}{2} \tag{3.4}$$

This formula can be derived as follows. For simply laced algebra (which is our only focus in this paper), the root lattice Λ_r is the same as the coroot lattice Λ_{cr}. A theory in affine Lie algebra shows that the state set (2.4) is simply the highest weight representation of the corresponding affine Lie algebra $\tilde{\mathfrak{g}}$ at level q [46]. Let a state be labeled by the Dynkin label $(a_1, a_2, ..., a_r)$, where each a_i is some nonnegative integer. The highest weight states satisfy the inequality

$$\sum_{i=1}^{r} c_i a_i \leq q \tag{3.5}$$

[2] In [55], a more pedestrian way of quantizing the Chern-Simons theory on T^2 is used.

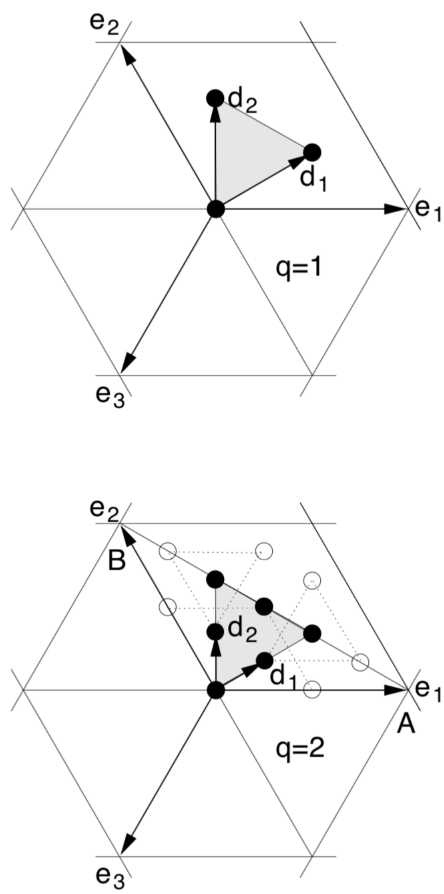

Figure 3.1: In both pictures, e_1 and e_2 are the simple roots, and d_1 and d_2 are the fundamental weights. In the top picture, three black dots represent the three unique states of the level $q = 1$ theory. In the bottom picture, the six black dots are the unique states of the level $q = 2$ theory. It is easy to convince oneself that one can reach the white dots or other weight lattice points through a combination of Weyl reflections and q times the root lattice translation on the black dots. Figure retrieved from [55].

where $c_i \in \mathbb{Z}_{>0}$ is the coefficient multiplying the ith simple root α_i in the expression for the highest root θ:

$$\theta \equiv \sum_{i=1}^{r} c_i \alpha_i \tag{3.6}$$

For the case of $\mathfrak{su}(3)$, the numbers are $c_1 = c_2 = 1$, so the number of Chern-Simons states at level q is the same as the number of solutions to the inequality

$$a_1 + a_2 \leq q$$

and solving for this gives exactly the quadratic formula (3.4).

Although the example is given for $\mathfrak{su}(3)$, the reader should keep in mind the generalization of the picture to other ADE gauge algebras. In the next section, we look at a special class of states on the state lattice.

3.3 A Special Class of States

The Geometry and the Counting Problem

The special states we want to focus on are those that belong to the set

$$Q_q = \frac{\Lambda_w}{W \ltimes q\Lambda_r} \cap \Lambda_r \tag{3.7}$$

namely states that also roots. As discussed earlier, the physical motivation for considering such states is mentioned in (2.16). Following the $\mathfrak{su}(3)$ example in Fig. 3.1, at level 1 there is one state that belongs to Q_q, the state at the origin. At level 2 there are two states, the additional one being at the position $d_1 + d_2$.

The question we pose is: given $\mathfrak{g}$ and level q, how many states are in Q_q? There are two equivalent formulations of this counting problem. The first formulation uses the Cartan matrix and it naturally leads to the concept of Ehrhart polynomials. The second (dual) formulation uses the inverse of the Cartan matrix, and it leads to representation theory and connections to string theory. The rest of the section focuses on the first formulation of the problem.

Let the Cartan matrix of $\mathfrak{g}$ be C. In the basis of Dynkin labels, the rows of C give the Dynkin coefficients of the simple roots. Since we are dealing with simply laced Lie algebras, $C^T = C$, so that the columns of C also give the representation of the simple roots. If $y \in Q_q$, y being the Dynkin label (as a column vector) of some state in the theory, then $y \in \Lambda_r$ by definition, so that it can be represented as a linear combination of the simple roots:

$$y = Cx$$

where $x \in \mathbb{Z}^r_{\geq 0}$ is a column of nonnegative integers. The nonnegativity of x is because states in Q_q are all dominant weights. What constraints does y have to satisfy? The first constraint on y is that the entries are nonnegative (note that $x \geq 0$ does not imply $y \geq 0$). The second constraint is the level q constraint in equation (3.5). In terms of x, the constraints read

$$\sum_{j=1}^{r} C_{ij} x_j \geq 0 \tag{3.8}$$

$$\sum_{i,j=1}^{r} c_i C_{ij} x_j \leq q \tag{3.9}$$

Geometrically, the above constraints define a rational polytope, a polygon whose vertices have rational coordinates. The solutions to the above constraints are simply integer points contained inside the rational polytope. By adding slack variables, any rational polytope can be represented as a system of linear equations [52]

$$Ax' = b \tag{3.10}$$

for some matrix A, vector b, and unknowns x'.

For our problem, we need $r+1$ slack variables $k_1, k_2, ..., k_{r+1} \in \mathbb{Z}_{\geq 0}$, so that the system of inequalities reduces to the system of equalities:

$$\sum_{j=1}^{r} C_{ij} x_j - k_i = 0 \tag{3.11}$$

$$\sum_{i,j=1}^{r} c_i C_{ij} x_j + k_{r+1} = q \tag{3.12}$$

Stacking the vector x and k into $x' = (x, k)$, we have the system of linear equation $Ax' = b$ where A is $r+1$ by $2r+1$ and b is $r+1$ by 1:

$$A = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1r} & -1 & 0 & 0 & \dots & 0 \\ C_{21} & C_{22} & \dots & C_{2r} & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{r1} & C_{r2} & \dots & C_{rr} & 0 & 0 & 0 & -1 & 0 \\ \sum_i c_i C_{i1} & \sum_i c_i C_{i2} & \dots & \sum_i c_i C_{ir} & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \tag{3.13}$$

$$b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q \end{pmatrix} \tag{3.14}$$

The problem of counting states in Q_q is then transformed into counting the solutions to this system of linear equations.

Ehrhart Polynomials

What we did in the previous section is under the guise of Ehrhart polynomials [34]. They are generating functions for counting lattice points contained in polytopes. Let us formulate the theory following the notation of [52]. Let a rational polytope $\mathcal{P}$ be specified by a system of linear equations with slack variables added:

$$\mathcal{P} = \{x \in \mathbb{R}^d_{\geq 0} : Ax = b\} \tag{3.15}$$

for some integer valued matrix A and integer valued vector b. One considers the tth dilate of $\mathcal{P}$, defined as

$$t\mathcal{P} = \{x \in \mathbb{R}^d_{\geq 0} : Ax = tb\} \tag{3.16}$$

where $t \in \mathbb{Z}_{>0}$. Let $L_{\mathcal{P}}(t)$ denote the number of lattice points contained in the $t\mathcal{P}$:

$$L_{\mathcal{P}}(t) = \#\{x \in \mathbb{Z}^d_{\geq 0} : Ax = tb\} \tag{3.17}$$

The Ehrhart polynomial associated to the polytope $\mathcal{P}$ is defined as

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t \tag{3.18}$$

In our problem, the polytope is given as the qth dilate of

$$Ax' = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

with A and x' defined in the previous section (see equation (3.13) and equation (3.14)). We call this base polytope $\mathcal{Q}_{\mathfrak{g}}$, where the dependence on the Lie algebra $\mathfrak{g}$ is made explicit as each simply laced Lie algebra has its unique polytope. Since we are interested in computing the number of special states for each level $q > 0$, we want to find the number of lattice points contained in the qth dilate of $\mathcal{Q}_{\mathfrak{g}}$ for each positive q. Therefore, the question posed in the previous section can be now phrased as finding the Ehrhart polynomial of $\mathcal{Q}_{\mathfrak{g}}$:

$$\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z) = ?$$

This concludes the first formulation of our problem. The dual formulation of the problem in terms of the inverse of Cartan matrices will be introduced in section 3.5. In the next section, we develop a formal method to compute $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$ in the most general way possible.

3.4 Computation of $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$

In this section and the next, we compute $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$ for all simply laced $\mathfrak{g}$. We give two approaches to this computation. The first approach uses MacMahon's Ω operator method [53] but quickly becomes tedious when the level q becomes large. However, the merit of this approach is that it is general: it can be applied to solving for all Ehrhart polynomials given the constraints, and it will be amiss if we do not discuss the general solution. Due to the computational difficulty, we shall only use this approach to give the explicit formula for $\mathcal{Q}_{\mathfrak{g}}$ for the case $\mathfrak{g} = \mathfrak{su}(2)$. MacMahon's method is reviewed in section 3.4 and the computation for $\mathfrak{su}(2)$ is done in section 3.4.

The second formulation of the problem uses a hint from representation theory, and leads to the expression of $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$ in one full sweep. The second approach is inspired from the duality relation constructed from string theory (see section 2.4 in chapter 2), without which it is not obvious how one can make a connection of $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$ to being solved by representation theory.

The Ω Operator

In computing the number of ways of partitioning some integer u into a sum of n non-negative integers $a_1 + a_2 + ... + a_n = u$, the order of the integers in the sum does not matter. Therefore, the problem of counting n-partitions of u is quite different from the problem of counting solutions to the the equation $a_1 + a_2 + ... + a_n = u$, where the number of solutions is the coefficient of the x^u term in the generating function

$$\frac{1}{(1-x)^n} \tag{3.19}$$

To introduce the Ω operator, we focus on the number partition problem, and therefore we can assume an ordering of a_i to be $a_1 \geq a_2 \geq ... \geq a_n$ without loss of generality. One way to impose this ordering constraint is to consider the expression [53]

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda_1 x)(1-\frac{\lambda_2}{\lambda_1}x)(1-\frac{\lambda_3}{\lambda_2}x)...(1-\frac{\lambda_n}{\lambda_{n-1}}x)} \tag{3.20}$$

where the notation $\underset{\geq}{\Omega}$ means restricting terms that have only nonnegative powers of each λ and setting each λ to be one in the end. This is easily verified by expanding each fraction in power series.

The $\underset{\geq}{\Omega}$ leads to many identities. For example, one can again expand in power series and verify that

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x^{p_1})(1-\frac{x^{p_2}}{\lambda})} = \frac{1}{(1-x^{p_1})(1-x^{p_1+p_2})} \tag{3.21}$$

Repeated use of this identity shows that equation (3.20) is equal to

$$\frac{1}{(1-x)(1-x^2)(1-x^3)...(1-x^n)} \tag{3.22}$$

which is exactly the generating function to count the n-partition of some integer.

One can also compose the $\underset{\geq}{\Omega}$ operation. We modify the notation accordingly if there is any ambiguity with extra variables:

$$\begin{aligned}\underset{\lambda\geq}{\Omega}\,\underset{\mu\geq}{\Omega}\frac{1}{(1-\lambda\mu x)(1-\frac{y}{\lambda^2\mu})} &= \underset{\lambda\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\frac{xy}{\lambda})}\\ &= \frac{1}{(1-x)(1-x^2y)}\end{aligned}$$

This expression counts the number of partition into two nonnegative integers a_1 and a_2 such that the constraint $a_1 \geq 2a_2$ is satisfied[3].

In fact, one can also have two other operators $\underset{\leq}{\Omega}$ and $\underset{=}{\Omega}$, defined in a self-explanatory way. As we shall see, we will be interested in identities involving $\underset{=}{\Omega}$. The three operators satisfy some useful algebraic relations listed in [53], and can be used to compute $\underset{=}{\Omega}$. For example, let $F(\lambda)$ be some polynomial depending on λ as the variable used in the Ω operators, the following expression

$$\underset{=}{\Omega}F(\lambda) = \underset{\geq}{\Omega}F(\lambda) + \underset{\geq}{\Omega}F(\lambda^{-1}) - F(1) \tag{3.23}$$

can be used to derive identities involving $\underset{=}{\Omega}$ in terms of the identities involving $\underset{\geq}{\Omega}$. In the next subsection we will use an identity $\underset{=}{\Omega}$ to compute $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$ for $\mathfrak{g} = \mathfrak{su}(2)$. We will also sketch the idea of computing $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$ for a general simply laced $\mathfrak{g}$.

Applying the Ω Operator

To illustrate the use of the Ω operator, we use it to write down a formal expression for the Ehrhart polynomial $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$. Recall that the base polytope is given by

$$Ax' = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3.24}$$

[3]Reason: $\underset{\mu\geq}{\Omega}$ implements the condition that ordering does not matter, and $\underset{\lambda\geq}{\Omega}$ implements the constraint $a_1 \geq 2a_2$.

where A ($r+1$ by $2r+1$) was given in equation (3.13). Let $\mathbf{z} \equiv (z_1, z_2, ..., z_r, z)$. The Ehrhart polynomial is computed by the formal expression

$$\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z) = \underset{z_1=}{\Omega}\,\underset{z_2=}{\Omega} \ldots \underset{z_r=}{\Omega} \left(\prod_{i=1}^{2r+1} \frac{1}{1 - z_1^{A_{1i}} z_2^{A_{2i}} \ldots z_r^{A_{ri}} z^{A_{r+1,i}}} \right) \tag{3.25}$$

where a composition of r $\underset{=}{\Omega}$ is applied, each time restricting the polynomial to the 0th order term of some z_i. This formula looks intimidating, but can be easily derived as follows. The term involving the product is simply the generating function for counting the combinations of Ax'. Imposing the constraint that the right hand side of the equation has r vanishing entries means that one must restrict to the 0th order term of $z_1, ...z_r$. Since the last entry of the column on the right hand side is 1, the number of solutions to the qth dilate of the polytope is then the the coefficient of the z^q term. This completes the argument that the formal expression in (3.25) computes the Ehrhart polynomial.

It is nice to have a formal expression like eqn. (3.25) for the Ehrhart polynomial. If one wants to compute the first few terms, a computer can easily do the job. However, we want to take a step further and obtain a closed form solution, which as we shall see exists for all simply laced $\mathfrak{g}$.

To illustrate how one might obtain a closed form solution using the current formalism, we first focus on the case of $\mathfrak{su}(2)$. The Cartan matrix for $\mathfrak{su}(2)$ is simply 2, the coefficient for the highest root is $c_1 = 1$. Therefore the A matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \tag{3.26}$$

Using expression (3.25), the Ehrhart polynomial is given by

$$\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{su}(2)}}(z) = \underset{z_1=}{\Omega} \left(\frac{1}{1 - z_1^2 z^2} \frac{1}{1 - z_1^{-1}} \frac{1}{1 - z} \right) \tag{3.27}$$

Using the Ω operator identity [53]

$$= \underset{\lambda=}{\Omega} \frac{1}{(1 - \lambda^2 x)(1 - y\lambda^{-1})} = \frac{1}{1 - xy^2} \tag{3.28}$$

we find that the Ehrhart polynomial for $\mathfrak{g} = \mathfrak{su}(2)$ is

$$\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{su}(2)}}(z) = \frac{1}{(1 - z)(1 - z^2)} \tag{3.29}$$

Now let us generalize to arbitrary $\mathfrak{g}$. The first thing to note is that one needs more general Ω identities in addition to equation (3.28), since a variable Z_i can appear more than two times in the product in equation (3.25). However, the Ω identities for a general polynomial are not documented and do not have clean solutions. This problem has been solved by [56], where the authors developed the Omega Package in Mathematica to compute Ω identities for a general polynomial

$$\frac{P(x_1, ..., x_n; \lambda_1, ...\lambda_r)}{\prod_{i=1}^{n}(1 - x_i\lambda_1^{v_1(i)}...\lambda_r^{v_r(i)})} \tag{3.30}$$

For our problem, we can apply the program r times to eliminate $z_1, ...z_r$ to find $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$ of rank r. In the next section, we will use a trick from representation theory to compute all $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$ by hand, bypassing the need for the computer program computation.

3.5 $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$ From Representation Theory

To connect $\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z)$ to representation theory, we briefly summarize the discussion of the McKay correspondence [21] in section 2.3. The McKay correspondence associates any simply laced $\mathfrak{g}$ to some discrete subgroup $\Gamma(\mathfrak{g})$ of $SU(2)$. More specifically,

$$\Gamma(\mathfrak{su}(N)) = \mathbb{Z}_N \tag{3.31}$$

$$\Gamma(\mathfrak{so}(2(N+2))) = \mathrm{Dic}_N \tag{3.32}$$

$$\Gamma(\mathfrak{e}_i) = E_i \qquad \text{i=6,7,8} \tag{3.33}$$

where $\mathbb{Z}_N$ is the cyclic group of order N, Dic_N is the dicyclic (binary dihedral) group of order $4N$, and E_6, E_7, E_8 are the binary tetrahedral group, binary octahedral group, and the binary icosahedral group, respectively (also called $2T$, $2O$, and $2I$).

For some simply laced $\mathfrak{g}$, consider the homomorphism $\Gamma(\mathfrak{g}) \to SU(q)$, or, in other words, $SU(q)$ representations of the group $\Gamma(\mathfrak{g})$. Let the number of Weyl-inequivalent[4] representations $\Gamma(\mathfrak{g}) \to SU(q)$ (inequivalent also under $SU(q)$ conjugation) be b_q, so that one forms the generating function to count the number of $SU(q)$ representations

$$\Phi_\mathfrak{g}(z) \equiv 1 + \sum_{i=1} b_i z^i \tag{3.34}$$

We claim that this generating function is exactly the same as the Ehrhart polynomial corresponding to the same Lie algebra:

$$\boxed{\mathrm{Ehr}_{\mathcal{Q}_\mathfrak{g}}(z) = \Phi_\mathfrak{g}(z)} \tag{3.35}$$

[4]This means that two diagonal $SU(q)$ matrices are identified if the diagonal elements differ by some permutation.

This beautiful equation connects geometry (left hand side) with representation theory (right hand side) via the McKay correspondence and string theory (the equality sign). A string-theory proof of this equation was given in section 2.4 using the mini AdS/CFT argument. The rest of this section is devoted to proving this equality mathematically and using this equality to compute the closed form $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$ for $\mathfrak{g} = \mathfrak{su}(N)$ and $\mathfrak{g} = \mathfrak{so}(2(N+2))$. The Ehrhart polynomials for the exceptional Lie algebras are computed in section 3.8.

The strategy we use here is to look at the dual formulation by starting with weights, expressing them in terms of the simple roots, and imposing the constraints. This reverse process is carried out using the inverse the Cartan matrix. Since the inverse of Cartan matrices in general has fractional entries, this dual formulation is less suited for geometric arguments we had in the previous sections. Instead, we will use purely algebraic arguments to prove equation (3.35).

Let $\mathfrak{g} = \mathfrak{su}(r+1)$ and the corresponding Cartan matrix be C. Start with some weight x in the state set (2.4) which we reproduce here for convenience

$$\frac{\Lambda_w}{W \ltimes q\Lambda_r} \tag{3.36}$$

We can represent x as a vector in $\mathbb{Z}^r_{\geq 0}$ by using its Dynkin label $(x_1, ..., x_r)$. It satisfies the constraint

$$\sum_{i=1}^{r} x_i \leq q \tag{3.37}$$

as argued in section 3.2.

Since the simple roots span $\mathbb{R}^r$, the weight x has a unique expansion in terms of the simple roots $\alpha_1, ..., \alpha_r$:

$$x = \sum_{i=1}^{r} l_i \alpha_i \tag{3.38}$$

The expansion coefficients l_j are simply given by multiplying x (as a vector) by the inverse of the Cartan matrix[46]

$$l_i = \sum_{j=1}^{r} C^{-1}_{ij} x_j \tag{3.39}$$

Since we want x to lie on the root lattice Λ_r as well, the only other constraint we need to impose is that each l_j is an integer

$$l_i = \sum_{j=1}^{r} C^{-1}_{ij} x_j \in \mathbb{Z} \tag{3.40}$$

The inverse of the Cartan matrix for $\mathfrak{su}(N)$ algebra has an interesting modular structure. Here, we display C^{-1} for $\mathfrak{su}(4)$ and $\mathfrak{su}(7)$:

$$C^{-1}_{\mathfrak{su}(4)} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \tag{3.41}$$

$$C^{-1}_{\mathfrak{su}(7)} = \frac{1}{7}\begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 10 & 8 & 6 & 4 & 2 \\ 4 & 8 & 12 & 9 & 6 & 3 \\ 3 & 6 & 9 & 12 & 8 & 4 \\ 2 & 4 & 6 & 8 & 10 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \tag{3.42}$$

In general, the formula for $C^{-1}_{\mathfrak{su}(N)}$ is [57]

$$C^{-1}_{\mathfrak{su}(N),ij} = \frac{1}{N}\left[\min(i,j) \times N - ij\right] \tag{3.43}$$

Even though we have r number of constraints from equation (3.40), we shall see that because of the peculiar property of $C^{-1}_{\mathfrak{su}(N)}$, there is effectively only one constraint, the one imposed by the last row of the matrix:

$$\frac{1}{N}\sum_{i=1}^{N-1} i x_i \in \mathbb{Z} \tag{3.44}$$

A quick proof that constraint (3.44) implies the rest of the constraints is as follows. The matrix elements $C^{-1}_{\mathfrak{su}(N),ki}$ of the kth row are

$$C^{-1}_{\mathfrak{su}(N),ki} = \begin{cases} \frac{(N-k)i}{N} & k > i \\ \frac{(N-i)k}{N} & k \leq i \end{cases}$$

Using this, we obtain the constraint imposed by the kth row:

$$\begin{aligned} l_k &= \sum_{j=1}^{N-1} C^{-1}_{\mathfrak{su}(N),kj} x_j \\ &= \sum_{i=1}^{k-1} \frac{(N-k)i}{N} x_i + \sum_{i=k}^{N-1} \frac{(N-i)k}{N} x_i \\ &= -\frac{k}{N}\sum_{i=1}^{N-1} i x_i \mod 1 \end{aligned}$$

Up to integers, the constraint imposed by the kth row is simply k times that of the constraint imposed by the last row. Therefore, the only unique constraint in (3.40) is the one given by the last row. In summary, there are two constraints on x:

$$\sum_{i=1}^{N-1} x_i \leq q \tag{3.45}$$

$$\frac{1}{N}\sum_{i=1}^{N-1} i x_i \in \mathbb{Z} \tag{3.46}$$

We would now like to argue that this is exactly the same constraints satisfied by $SU(q)$ representations of $\Gamma(\mathfrak{su}(N)) = \mathbb{Z}_N$.

$SU(q)$ representations of $\Gamma(\mathfrak{su}(N)) = \mathbb{Z}_N$

Since finite dimensional representations are built up from irreducible representations, we first look at the irreducible representations of $\mathbb{Z}_N$. There are in total N 1-dimensional irreducible representations. The kth irreducible representation is given by the kth power of the Nth root of unity

$$\omega_k = \exp\left(\frac{2\pi i k}{N}\right) \tag{3.47}$$

Here, k runs from 0 to $N-1$, with $k=0$ being the trivial representation. The $SU(q)$ representations are constructed by inserting x_k copies of the kth irreducible representation. By definition, the determinant of any $SU(q)$ matrix must be 1, so we have

$$\prod_{k=0}^{N-1} \omega_k^{x_k} = \exp\left(\frac{2\pi i}{N}\sum_{k=0}^{N-1} k x_k\right) = 1 \tag{3.48}$$

But this is equivalent to the constraint 3.46:

$$\frac{1}{N}\sum_{i=1}^{N-1} i x_i \in \mathbb{Z}$$

The constraint that the dimensions of the irreducible representations add up to q is

$$\sum_{i=0}^{N-1} x_i = q \tag{3.49}$$

However, treating x_0 as a slack variable, this constraint is equivalent to (3.45)

$$\sum_{i=1}^{N-1} x_i \leq q$$

Therefore, we have shown that the two counting problems satisfy the same constraints and are secretly one and the same. This concludes the proof of equation (3.35) for the case $\mathfrak{g} = \mathfrak{su}(N)$:

$$\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z) = \Phi_{\mathfrak{g}}(z)$$

In the next subsection, we give explicit formulae for $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{g}}}(z)$ by computing $\Phi_{\mathfrak{g}}(z)$.

Computation of $\Phi_{\mathfrak{g}}(z)$

We want to find the generating function (3.34)

$$\Phi_{\mathfrak{g}}(z) = 1 + \sum_i b_i z^i$$

in which b_q counts the number of inequivalent $SU(q)$ representations of $\mathbb{Z}_N$. We saw in the previous subsection that the N irreducible representations are given by the Nth roots of unity. Consider the function[5]

$$\frac{1}{(1-z)(1-wz)(1-w^2z)...(1-w^{N-1}z)} \tag{3.50}$$

The coefficient of z^q is in general a sum of n terms (n is some integer), representing n ways of building a q-dimensional representation. Each of the n terms has some coefficient w^m for some integer $m \in \mathbb{Z}$, which represents the determinant of that representation. We want to retain terms of determinant 1 only. To project out terms of non-unit determinant, we simply have to sum over w in w^m and divide by N, since we know from Fourier analysis that

$$\sum_w w^m = \begin{cases} N & m = 0 \mod N \\ 0 & m \neq 0 \mod N \end{cases}$$

where the sum is taken over the Nth root of unity. Therefore, the generating function is

$$\Phi_{\mathfrak{su}(N)}(z) = 1 + \sum_i b_i z^i = \frac{1}{N}\sum_{i=0}^{N-1} \frac{1}{(1-z)(1-w^iz)(1-w^{2i}z)...(1-w^{(N-1)i}z)} \tag{3.51}$$

One can check that b_i is a quasi-polynomial in i, a property shared by Ehrhart polynomial. Finally, we use the theorem proved in the last subsection to give the explicit expression for the Ehrhart polynomial

$$\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{su}(N)}}(z) = \frac{1}{N}\sum_{i=0}^{N-1} \frac{1}{(1-z)(1-w^iz)(1-w^{2i}z)...(1-w^{(N-1)i}z)} \tag{3.52}$$

[5] We thank O. Ganor for pointing out this trick.

For example, let $\mathfrak{g} = \mathfrak{su}(2)$. Then

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{su}(2)}}(z) &= \frac{1}{2}\left(\frac{1}{(1-z)(1-z)} + \frac{1}{(1-z)(1+z)}\right) \\ &= \frac{1}{(1-z)(1-z^2)}\end{aligned}$$

which agrees with what we obtained in equation (3.29) using the Ω operator calculus.

3.6 Ehrhart Polynomial for $\mathfrak{so}(2(N+2))$

In the previous subsections, we obtained the Ehrhart polynomial for $\mathfrak{su}(N)$ by counting the $SU(q)$ representations of $\mathbb{Z}_N$. We now use the same method to compute the Ehrhart polynomial for $\mathfrak{so}(2(N+2))$, $N \geq 1$. Here, N is shifted by 2 because of convenience. The Dynkin diagram associated to $\mathfrak{so}(2(N+2))$ is D_{N+2}. According to the McKay correspondence, we should be looking for the representation of the discrete group Dic_N, the dicyclic (or binary dihedral) group of order N. Since Dic_N is less well-known than $\mathbb{Z}_N$, we analyze the group structure in detail and derive the irreducible representations in the next subsection. After that, we will use the irreducible representations to construct the $SU(q)$ representation of Dic_N and obtain the generating function $\Phi(z)$. We prove that, in a spirit similar to what we did for the $\mathfrak{su}(N)$ case, the generating function $\Phi(z)$ coincides with the Ehrhart polynomial for $\mathfrak{so}(2(N+2))$.

$SU(q)$ representation of Dic_N

The group Dic_N is defined by the following multiplication rules:

$$\begin{aligned} r^{2N} &= e \\ s^2 &= r^N \\ s^{-1}rs &= r^{-1} \end{aligned}$$

where e is the identity element. The reader may have noticed a similarity to the dihedral group of order $2N$, identifying r with the fundamental rotation and s with the rotation. The difference here is that the reflection s does not square to the identity. Instead, it squares to a central element of the group.

To analyze the irreducible representations of this group, we need to understand the conjugacy classes. For Dic_N, there are $N+3$ conjugacy classes:

$$\{e\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}, ..., \{r^{N-1}, r^{-N+1}\}, \{r^N\}, \{sr^{2k}\}, \{sr^{2k+1}\} \tag{3.53}$$

where for the last two conjugacy classes, k is an integer running from 0 to $2N-1$. Therefore, there are $N+3$ irreducible representations. The affine Dynkin diagram of D_{N+2} is shown in Fig. 3.2 [46].

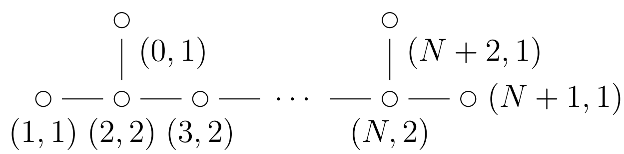

Figure 3.2: The affine Dynkin diagram for D_{N+2}. The tuple (x, y) indicates the xth simple root with (co)mark y. The mark is the same as the comark here because the algebra is simply-laced. By the McKay correspondence, the mark also indicates the dimension of the corresponding irreducible representation. Therefore, there are four 1-dimensional irreducible representations and $N-1$ 2-dimensional irreducible representations.

By the McKay correspondence [21] or by abelianizing the group, one sees that there are four 1-dimensional irreducible representations and $N-1$ 2-dimensional irreducible representations. The 1-dimensional irreducible representations for Dic_N behave differently for N even and N odd. In the following, we restrict N to be an even integer, since the case for N odd can be treated analogously. We present the character table:

$\{e\}$	$\{r, r^{-1}\}$	...	$\{r^{N-1}, r^{-N+1}\}$	$\{r^N\}$	$\{sr^{2k}\}$	$\{sr^{2k+1}\}$
1	1	...	1	1	1	1
1	1	...	1	1	-1	-1
1	-1	...	-1	1	1	-1
1	-1	...	-1	1	-1	1
2	$w + w^{-1}$	...	$w^{N-1} + w^{-(N-1)}$	-2	0	0
2	$w^2 + w^{-2}$	...	$w^{2(N-1)} + w^{-2(N-1)}$	2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	$w^{N-1} + w^{-N+1}$	...	$w^{(N-1)^2} + w^{-(N-1)^2}$	-2	0	0

Table 3.1: Character table for the Dic_N group. The first four lines are the characters for the four 1-dimensional irreducible representations. Note that the first line is the trivial 1-dimensional representation. The rest of the irreducible representations are 2-dimensional. w represents the $2N$th root of unity $\exp(\pi i/N)$. One can check that the character orthogonality relation holds.

For the $N-1$ 2-dimensional irreducible representations, we note that r and s take on the following form:

$$r = \begin{pmatrix} \exp(m\pi i/N) & 0 \\ 0 & \exp(-m\pi i/N) \end{pmatrix} \quad m = 1, ..., N-1$$

$$s = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, m \text{ odd} \\ \begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix}, m \text{ even} \end{cases}$$

In particular, for the 2-dimensional irreducible representations, $\det r = 1$, and a non-unit determinant can only come from s. The formula for s shows that the determinant of the 2-dimensional representations[6] alternates between 1 and -1. For example, Dic_2 has one 2-dimensional irreducible representation with determinant 1, whereas Dic_4 has three 2-dimensional irreducible representations, with determinant 1, -1, 1, respectively.

We now use the above information to compute the generating function for the $SU(q)$ representation of Dic_N. The $SU(q)$ representation is constructed out of the $N+3$ irreducible representations so that the dimensions add up to q. Let the number of the four 1-dimensional representations be $x_0, x_1, x_{N+1}, x_{N+2}$, respectively[7]. Let the number of 2-dimensional representations be labeled by $x_2, x_3, ..., x_N$. The constraint on the size of the representation is

$$x_0 + x_1 + 2x_2 + 2x_3 + ... + 2x_N + x_{N+1} + x_{N+2} = q \tag{3.54}$$

In addition to the size constraint, we also have the unit-determinant constraint. From the character table, we see that a -1 determinant can only come from three columns[8]: $\{r, r^{-1}\}$, $\{sr^{2k}\}$, and $\{sr^{2k+1}\}$. The unit-determinant constraints coming

[6] In our case, we define the determinant of a representation as follows. If all conjugacy classes in the representation have determinant 1, we say that the representation has determinant 1. If some conjugacy classes have non-unit determinant, we pick the determinant D that, in the polar decomposition, has the smallest angle $\theta \in [0, 2\pi)$ and call D the determinant of the representation. For example, if ω is the 3rd root of unity, $\omega = \exp(2\pi i/3)$. Suppose there are two conjugacy classes with non-unit determinant, one with determinant ω and the other ω^2. Because ω has a smaller polar angle than ω^2, we say that the representation has determinant ω.

[7] The numbering here is to make connection with the numbering of the nodes in the affine Dynkin diagram.

[8] Note that the entries in the character table contain the trace of the conjugacy classes. The determinant coincides with the trace for 1-dimensional representations. For 2-dimensional representations the determinant in this case was computed earlier by looking at the explicit matrix representations. There are other columns that have -1 determinant, but one can show that there are only these three independent columns to consider.

from the three columns are

$$\frac{x_{N+1}+x_{N+2}}{2} \in \mathbb{Z} \tag{3.55}$$

$$\frac{x_1+x_3+x_5+...+x_{N-1}+x_{N+2}}{2} \in \mathbb{Z} \tag{3.56}$$

$$\frac{x_1+x_3+x_5+...+x_{N-1}+x_{N+1}}{2} \in \mathbb{Z} \tag{3.57}$$

Note that the constraints are not independent, since (3.56) and (3.57) imply (3.55). Therefore, we effectively only have two determinant constraints.

To write down the generating function $\Phi_{\mathfrak{so}(2(N+2))}$, we can use the independent constraints (3.56) and (3.57). The idea is similar to the $\mathfrak{su}(N)$ case. Consider the function

$$\frac{1}{(1-z)(1-w_1z)(1-w_2z)(1-w_1w_2z)(1-z^2)^{N/2}(1-w_1w_2z^2)^{N/2-1}} \tag{3.58}$$

Here, $w_1, w_2 \in \{1, -1\}$. The first four terms represent the contributions of inserting the four 1-dimensional representations, and the last two terms represent the contributions of inserting the $N-1$ copies of the 2-dimensional representations. If we expand the fractions into a power series, a generic term would look like

$$cw_1^{k_1}w_2^{k_2}z^n \tag{3.59}$$

where $c, k_1, k_2 \in \mathbb{Z}$. The $w_2^{k_2}$ term represents the determinant contribution from the two 1-dimensional representations labeled by $N+1$ and $N+2$. We want to retain the term that satisfies $w_2^{k_2} = 1$. This can be done by a projection similar to what we did for the $\mathfrak{su}(N)$ case, except now we need to sum over $\omega_2 \in \{1, -1\}$ and divide by 2. We also need to do a similar projection on w_1. In fact, the two projections help us retain the terms that satisfy the constraints (3.56) and (3.57). Therefore, we need to compute

$$\Phi_{\mathfrak{so}(2(N+2))}(z) = \frac{1}{4}\sum_{w_1,w_2} \frac{1}{(1-z)(1-w_1z)(1-w_2z)(1-w_1w_2z)(1-z^2)^{N/2}(1-w_1w_2z^2)^{N/2-1}}$$

e which gives us the answer for $\Phi_{\mathfrak{so}(2(N+2))}(z)$:

$$\frac{1}{4}\left(\frac{1}{(1-z)^4(1-z^2)^{N-1}} + \frac{2}{(1-z^2)^2(1-z^2)^{N/2}(1+z^2)^{N/2-1}} + \frac{1}{(1-z^2)^2(1-z^2)^{N-1}}\right) \tag{3.60}$$

As an example, we set $N=2$, so that we are looking at the generating function for $SU(q)$ representations of Dic_2. The first few terms of (3.60) are

$$\Phi_{\mathfrak{so}(8)}(z) = 1 + z + 5z^2 + ...$$

which suggests that there are one $SU(1)$ representation and five $SU(2)$ representations. The one $SU(1)$ representation is just the trivial representation. Even though there are four unitary 1-dimensional representations for Dic_2, only the trivial representation has unit determinant as can be seen from the character table. The reader can easily work out the five $SU(2)$ representations, one of which comes from the unique *irreducible* 2-dimensional representation of Dic_2. The rest come from the *reducible* 2-dimensional representations by combining the 1-dimensional irreducible representations such that the determinant constraint is satisfied.

We will show in the next subsection that this is exactly the Ehrhart polynomial $\mathrm{Ehr}_{\mathcal{Q}_{\mathfrak{so}(2(N+2))}}(z)$ for the D_{N+2} polytope defined in a similar fashion as in section 3.3.

Equivalence to the Ehrhart polynomial

We are looking at the level q highest weight representations of $\mathfrak{so}(2(N+2))$ which also lie on the root lattice. To show that the Ehrhart polynomial coincides with the generating function obtained in the previous subsection, we simply show that the two systems have the same constraints. Let the Dynkin label of some highest weight representation be $\lambda = (x_1, x_2, ..., x_{N+2})$, each term some nonnegative integer. The level q constraint yields

$$(\lambda, \theta) = x_1 + 2x_2 + 2x_3 + ... + 2x_N + x_{N+1} + x_{N+2} \leq q \tag{3.61}$$

where θ is the highest root whose expansion coefficients in terms of the simple roots can be read off from the affine Dynkin diagram. By adding a slack variable to turn the inequality into an equality, we reproduce the first constraint (3.55). Next, we need to demand that the weights are expressed as integer combinations of simple roots:

$$\sum_{j=1}^{N+2} C_{ij}^{-1} x_j \in \mathbb{Z} \tag{3.62}$$

The inverse of the Cartan matrix for the D-series has an interesting form [58]. Since the matrix is symmetric, we only give the values for the upper half of the matrix. Let C^{-1} be the inverse of the Cartan matrix for D_{N+2}. We have

$$C_{ij}^{-1} = \begin{cases} i, & 1 \leq i \leq j \leq N \\ i/2, & i \leq N, j = N+1 \text{ or } N \\ \frac{N}{4}, & i = N+1, j = N+2 \\ \frac{N+2}{4}, & i = j = N+1 \text{ or } N+2 \end{cases}$$

We give an example of C_{ij}^{-1} for D_6 where $N = 4$:

$$C^{-1}_{\mathfrak{so}(12)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1/2 & 1/2 \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3/2 & 3/2 \\ 1 & 2 & 3 & 4 & 2 & 2 \\ 1/2 & 1 & 3/2 & 2 & 3/2 & 1 \\ 1/2 & 1 & 3/2 & 2 & 1 & 3/2 \end{pmatrix}$$

Note that the N by N block is integer-valued, so it does not enter into the constraint (3.62). We only have to worry about the last two columns and the last two rows (which are the same as the last two columns by symmetry). For our case, N is an even number, so the only fraction that enter into the constraints modulo 1 is 1/2. Restricting our attention to the last two columns, we see that as the row number increases from 1 to N, the values of the last two columns alternate between being half-integer valued and integer valued. Therefore, from the first N rows of C^{-1}, we effectively get only one constraint:

$$\frac{x_{N+1} + x_{N+2}}{2} \in \mathbb{Z} \tag{3.63}$$

The constraints coming from the last two rows can be deduced similarly. Restricting ourselves to the last two rows, as the column number j increases from 1 to N, the values of $C^{-1}_{N+1,j}$ and $C^{-1}_{N+2,j}$ alternate between being half-integer valued and integer valued. Taking into account of the last two columns, the constraints are

$$\frac{x_1 + x_3 + x_5 + ... + x_{N-1} + x_{N+2}}{2} \in \mathbb{Z} \tag{3.64}$$

$$\frac{x_1 + x_3 + x_5 + ... + x_{N-1} + x_{N+1}}{2} \in \mathbb{Z} \tag{3.65}$$

We see that (3.61), (3.64), and (3.64) are exactly the same constraints we obtained in the last subsection. This establishes the equivalence of the generating function for $SU(q)$ representation of Dic_N group and the Ehrhart polynomial for the $\mathfrak{so}(2(N+2))$ polytope. In fact, one can repeat the same argument to show that the equivalence holds for the exceptional Lie algebras as well. This concludes the proof of remarkable formula (3.35).

3.7 A New Perspective on the McKay Correspondence

Let C be some Cartan matrix of an ADE type Lie algebra of rank r. Let G_C be the discrete $SU(2)$ subgroup corresponding to C according to the McKay correspondence.

Define $[C^{-1}]$ as C^{-1} modulo 1, where modulo 1 is done element-wise. Our computation in the previous two sections shows that a great deal of information is hidden in this object. In particular, $[C^{-1}]$ can tell us about the determinant[9] of the irreducible representations of the group G_C. To make it more precise, we define the following $\vee$ operator acting on the rational numbers in the *congruence class* of 1 as

$$a \vee b = \begin{cases} a+b, & \text{if } a=0 \text{ or } b=0 \\ a, & b \text{ is a nonzero integer multiple of } a \\ b, & a \text{ is a nonzero integer multiple of } b \end{cases}$$

Note that this definition comes with a priority structure: there could be cases where condition 2 and 3 are both satisfied. In that case, we stick with condition 2 and demand that $a \vee b = a$. For example, bearing in mind that we are working within the congruence class of 1, the above rules imply

$$\begin{aligned} 2/3 \vee 1/3 &= 2/3 \quad \text{because } 1/3 \text{ is 2 times } 2/3 \\ 5/7 \vee 6/7 &= 5/7 \quad \text{because } 6/7 \text{ is 4 times } 5/7 \\ 1/4 \vee 0 &= 1/4 \\ 1/2 \vee 1/2 &= 1/2 \end{aligned}$$

As we shall see, we never have to worry about the case when a and b do not satisfy the three cases. To use the $\vee$ operator, let us define X_i to be the ith row of $[C^{-1}]$. Let the operator $\vee$ act element-wise on X_i. Define the row vector X as

$$X \equiv X_1 \vee X_2 \vee ... \vee X_r \tag{3.66}$$

The dual group G_C has $r+1$ irreducible representations, among which 1 of them is the trivial representation. We claim that the determinant of the rest of the r non-trivial representations, encoded in the row vector $D = (d_1, ...d_r)$, can be found by

$$D = \exp 2\pi i X \tag{3.67}$$

where the exponential is taken element-wise on X, producing another row vector.

This claim can be proved by checking for all ADE Lie algebras. The information presented in the previous sections is enough for the readers to check the claim for the A and the D series. Here we will focus on the exceptional series $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$. The determinants for the irreducible representations of the exceptional groups are worked out in section 3.8. We give a brief summary.

[9]The determinant of a representation is defined in footnote 6.

- $\mathfrak{e}_6$. The dual group has six nontrivial irreducible representations, of which two have determinant $\exp(2\pi i/3)$ and two have determinant $\exp(4\pi i/3)$. The other two have determinant 1.

- $\mathfrak{e}_7$. The dual group has seven nontrivial irreducible representations, of which three have determinant -1 while the rest has determinant 1.

- $\mathfrak{e}_8$. All representations of the dual group have unit determinant.

Let us now compare the prediction made by equation (3.67) to the facts cited above.

The most trivial case to check is $\mathfrak{e}_8$, whose inverse Cartan matrix modulo 1 vanishes. In this case, $X_1 = X_2 = ... = X_8 = 0$, so according to the formula above, all irreducible representations of the binary icosahedral group have unit determinant.

For $\mathfrak{e}_7$, $[C^{-1}]$ is

$$[C^{-1}_{\mathfrak{e}_7}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 1/2 \end{pmatrix}$$

The $\vee$ sum of all the rows gives $(0, 0, 0, 1/2, 0, 1/2, 1/2)$. Equation (3.67) shows that the determinants are $(1, 1, 1, -1, 1, -1, -1)$, agreeing with the facts above.

For $\mathfrak{e}_6$, $[C^{-1}]$ is

$$[C^{-1}_{\mathfrak{e}_6}] = \begin{pmatrix} 1/3 & 2/3 & 0 & 1/3 & 2/3 & 0 \\ 2/3 & 1/3 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 1/3 & 2/3 & 0 \\ 2/3 & 1/3 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The $\vee$ sum of all the rows gives $(1/3, 2/3, 0, 1/3, 2/3, 0)$. Equation (3.67) shows that the determinants are $(w, w^2, 1, w, w^2, 1)$, where w is the third root of unity. This also agrees with the facts cited above, and completes the proof of equation (3.67).

3.8 Generating Functions for $SU(q)$ Representations for Exceptional Groups

In this section, we derive the Ehrhart polynomials for the exceptional groups, i.e. generating function that computes the number of $SU(q)$ representation of $2T$ (binary tetrahedral group) and $2O$ (binary octahedral group), the exceptional groups corresponding to E_6 and E_7 singularity, respectively. Because all finite dimensional representations of $2I$ (binary icosahedral group) have determinant 1 and can therefore be treated using the technique introduced earlier in this paper, we will not be discussing the E_8 group here.

$SU(q) \to E_6$

The group $E_6 = 2T$ is defined by the following presentation:

$$r^2 = s^3 = t^3 = rst$$

where each term is the central element of order 2.

The affine Dynkin diagram of E_6 is given in Fig. 2.3, from which we can read off the number of irreducible representations and their dimensions for the $2T$ group. There are three 1-dimensional irreducible representations, three 2-dimensional irreducible representations, and one 3-dimensional irreducible representation.

Let R_1 be one of the two nontrivial 1-dimensional irreducible representations of $2T$, where $\gamma \in 2T$ is represented by a phase $e^{i\phi(\gamma)} \in U(1)$. Since R_1 is nontrivial, some of the phases are nontrivial, and $R_1 \otimes R_1$ is a different representation. Since there are two nontrivial 1-dimensional irreducible representations, one of $R_1 \otimes R_1$ or $R_1 \otimes R_1 \otimes R_1$ must be trivial. We claim that it is $R_1^{\otimes 3}$. Suppose to the contrary that $R_1^{\otimes 2}$ is trivial, which means that all the phases $e^{i\phi(\gamma)}$ are (± 1). But then the other 1-dimensional nontrivial representation R_2 also has phases that are all ± 1 (otherwise $\overline{R}_2$ is a different nontrivial representation that is neither R_1 nor R_2), and then R_1, R_2, $R_1 \otimes R_2$ are three nontrivial inequivalent 1-dimensional representations, contradicting the statement that there are only two nontrivial inequivalent 1-dimensional representations.

Thus, $R_1^{\otimes 3}$ is the trivial representations and all the phases of R_1 are $e^{\pm\frac{2\pi i}{3}}$ or 1. The second nontrivial 1-dimensional representation is $R_2 = \overline{R}_1$ where all the phases are conjugated.

Since $2T$ has only one 3-dimensional representation, R_6, all its matrices must have determinant 1, otherwise the determinants would be phases $e^{\pm\frac{2\pi i}{3}}$ (since the determinant must be one of the representations R_1 or R_2) and the complex conjugate representation $\overline{R}_6$ of this 3-dimensional representation would have different determinants, and so can't be equivalent to R_6. Note that we must have

$$R_6 = \overline{R}_6 = R_6 \otimes R_1 = R_6 \otimes R_2,$$

since there is only one inequivalent 3-dimensional irreducible representation.

Note that tensoring a 3-dimensional representation with R_1 or R_2 doesn't change the determinant, since the phases of R_1 and R_2 are all $e^{\pm\frac{2\pi i}{3}}$ or 1.

Let R_3, R_4, R_5 be the 2-dimensional irreducible representations. Then $R_1 \otimes R_3$ and $R_2 \otimes R_3$ have different determinants for at least some matrices, and so must be different representations. They must therefore be R_4 and R_5. One of the determinants $\det R_3$, $\det R_4$, or $\det R_5$, must be the trivial representation, so without loss of generality we can assume that it is $\det R_3$ and then that $\det R_4 = R_1$ and $\det R_5 = R_2$.

Now we can write the generating function counting $SU(n)$ representations of $2T$.

First, the generating function for $U(n)$ representations, is

$$\Phi_1(z) \equiv \frac{1}{(1-z)^3(1-z^2)^3(1-z^3)}\,.$$

Now, set

$$\omega = e^{\frac{2\pi i}{3}}$$

and let us look at the expression

$$\Phi_2(z) \equiv \frac{1}{(1-z)(1-\omega z)(1-\omega^2 z)(1-z^2)(1-\omega z^2)(1-\omega^2 z^2)(1-z^3)}\,.$$

when expanding Φ_2 we will get sums of terms of the form $\omega^k t^n$ that correspond to particular ways to decompose n into a sum of dimensions of irreducible representations, and ω^k represents the determinant of the corresponding n-dimensional representation in the sense that if $\omega^k = 1$ we have determinant 1, and if $\omega^k \neq 1$ we have determinant that is not 1 for some group elements.

Noting that $1 + \omega^k + \omega^{-k} = 0$ if $\omega^k \neq 1$ and $1 + \omega^k + \omega^{-k} = 3$ of $\omega^k = 1$, we see that if we add to Φ_2 two similar expressions, one in which ω is replaced by 1, and another in which ω is replaced by ω^{-1}, we should get the generating function that we need, up to a factor of 3. Thus,

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1}{3}\left[\Phi_1(z) + 2\Phi_2(z)\right] \tag{3.68}$$

$$= \frac{1}{3}\left[\frac{1}{(1-z)^3(1-z^2)^3(1-z^3)} + \frac{2}{(1-z^6)(1-z^3)^2}\right] \tag{3.69}$$

$$= 1 + z + 3z^2 + 8z^3 + 14z^4 + 26z^5 + 49z^6 + \cdots \tag{3.70}$$

$SU(q) \to E_7$

The affine Dynkin diagram associated with the group $E_7 = 2O$ is the affine Dynkin diagram of E_7 (see Fig. 3.3), from which we see that there is only one nontrivial 1-dimensional representation R_1.

The phases of R_1 must be ± 1, since $R_1^{\otimes 2}$ must be the trivial representation. The two 3-dimensional representations, R_5, R_6, must be related by $R_6 = R_5 \otimes R_1$, and we

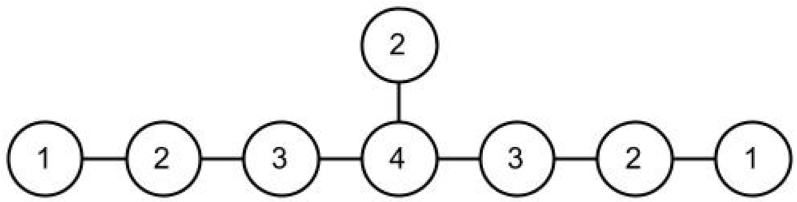

Figure 3.3: The affine Dynkin diagram for E_7.

Class	1	z	sz	t^2	r	s	t	tz
Size	1	1	8	6	12	8	6	6
Order	1	2	3	4	4	6	8	8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	1	-1	-1
χ_3	2	2	-1	2	0	-1	0	0
χ_4	2	-2	-1	0	0	1	$\sqrt{2}$	$-\sqrt{2}$
χ_5	2	-2	-1	0	0	1	$-\sqrt{2}$	$\sqrt{2}$
χ_6	3	3	0	-1	1	0	-1	-1
χ_7	3	3	0	-1	-1	0	1	1
χ_8	4	-4	1	0	0	-1	0	0

Table 3.2: Character table for the binary octahedral group $2O$.

can assume $\det R_5 = 1$ and $\det R_6 = R_1$. That leaves the question of what are the determinants of the even dimensional representations R_2, R_3, R_4 and R_7. Tensoring any of these with R_1 doesn't change their determinants.

The group is defined in terms of generators r, s, t as

$$r^2 = s^3 = t^4 = rst$$

where each term is the central element of order two.

The character table is

For 2×2 matrices

$$\det M = \frac{1}{2}\left[\mathrm{tr}(M)^2 - \mathrm{tr}(M^2)\right]$$

and for 4×4 matrices

$$\det M = \frac{1}{24}(\operatorname{tr} M)^4 + \frac{1}{3}(\operatorname{tr} M)\operatorname{tr}(M^3) - \frac{1}{4}(\operatorname{tr} M)^2 \operatorname{tr}(M^2) + \frac{1}{8}[\operatorname{tr}(M^2)]^2 - \frac{1}{4}\operatorname{tr}(M^4)$$

Using this we can check that the determinant of the matrix representing r in the 3^{rd} representation is -1, while in the 4^{th}, 5^{th}, and 8^{th} it is $+1$.

Now the generating function for $U(n)$ is

$$\Psi_1(z) = \frac{1}{(1-z)^2(1-z^2)^3(1-z^3)^2(1-z^4)}$$

and also define

$$\Psi_2(z) = \frac{1}{(1-z)(1+z)(1+z^2)(1-z^2)^2(1-z^3)(1+z^3)(1-z^4)}$$

where we inserted (-1) for every representation with determinant R_2. Then, the generating function for $SU(n)$ is

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1}{2}(\Psi_1 + \Psi_2) \tag{3.71}$$

$$= \frac{1}{2(1-z)^2(1-z^2)^3(1-z^3)^2(1-z^4)} + \frac{1}{2(1-z^2)^2(1-z^4)^2(1-z^6)} \tag{3.72}$$

$$= 1 + z + 4z^2 + 6z^3 + 15z^4 + 22z^5 + 44z^6 + \cdots \tag{3.73}$$

$SU(q) \to E_8$

From the affine Dynkin diagram of E_8 in Fig. 3.4, we see that there is only one 1-dimensional irreducible representation which corresponds to the trivial representation. Therefore, all other irreducible representations must have unit determinant. The generating function for $SU(q)$ representation of $E_8 = 2I$ is therefore the same as that for the $U(q)$ representation. It is given by

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1}{(1-z)(1-z^2)^2(1-z^3)^2(1-z^4)^2(1-z^5)(1-z^6)} \tag{3.74}$$

$$= 1 + z + 3z^2 + 5z^3 + 10z^4 + 15z^5 + 27z^6 + \cdots \tag{3.75}$$

3.9 Discussion

We have shown that the counting of root lattice states of level q Chern-Simons theory Hilbert space on T^2 can be solved by computing the exact generating function using

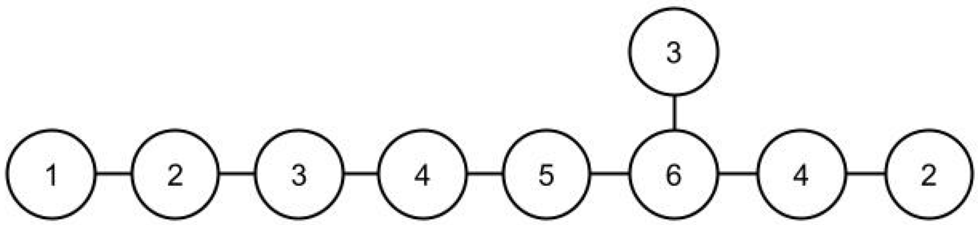

Figure 3.4: The affine Dynkin diagram for E_8.

either the Ω operator calculus or using the duality proposed in this dissertation. The reader might wonder why we framed the problem in terms of the Ehrhart polynomials. First, the number of root lattice states grows as some quasi-periodic polynomial, a property shared by Ehrhart polynomials. As we saw in section 3.3, the geometric formulation of the counting problem leads very naturally to the idea of Ehrhart polynomials. Second, and most importantly, recent development of mathematics and physics shows that it is always fruitful to find connections between different subfields of mathematics and physics. Since Ehrhart polynomials connect various branches of mathematics such as number theory and topology, formulating the problem using Ehrhart polynomials is an attempt at achieving more unification.

Despite being a decades-old subject, Chern-Simons theory is at the heart of the inter-connectedness explored in this paper. In fact, as shown in section 2.4, the Chern-Simons theory arise naturally at the long distance limit of some holographic system in string theory. Another key ingredient is the McKay correspondence [21]. In fact, we saw in section 3.7 that the simple Lie algebras know a lot more than the content of the McKay correspondence through the inverse of the Cartan matrices. The simple Lie algebras secretly know the determinant of the irreducible representations of the corresponding ADE subgroup.

Chapter 4

One Loop Correction and S-Duality

4.1 Introduction

The previous two chapters give a detailed proof of the duality by using a string/M theory construction and by matching the dimension of the Hilbert space of the two theories. In this chapter, we show that the classically flat Wilson lines of $\mathcal{N} = 4$ SYM on S^3/Γ with gauge group $SU(2)$ is also flat quantum mechanically. We expect that quantum correction does not lift the ground state degeneracy, because the dual of the SYM flat Wilson lines are states of some Chern-Simons theory which is one-loop exact. The way we compute the one-loop quantum correction to the ground state energy is by using the supersymmetric index [59].

Recent years saw a proliferation of index technology in understanding the strongly-coupled regime of supersymmetric Yang-Mills theories and dualities [60, 61, 62], although the use of index dates back to [59]. For a review, see [63] and [64] and the references therein. Using supersymmetric index, one can obtain more information on the theory if the underlying geometry has nontrivial first fundamental group and preserves some number of supersymmetry, since nontrivial flat Wilson lines from the vector multiplet (or from background flavor gauge fields) can wind around such geometry and divides the theory into different holonomy sectors. A salient example is the lens space supersymmetric index [65, 66, 67]. The particular lens space considered in these works is $L(r,1)$, or $S^3/\mathbb{Z}_r$, defined by identifying the point (z_1, z_2) on S^3 with the point $(wz_1, w^{-1}z_2)$ where w is the rth root of unity. There are two notable properties of this identification. First, the identification acts only on the holomorphic part of the coordinates, suggesting that one can choose some supersymmetry to be preserved. Second, the identification forms an abelian discrete subgroup of $SU(2)$ and acts separately on z_1 and z_2. There exist other discrete subgroups of $SU(2)$ that satisfy the first property but break the second (i.e. they are non-abelian). In fact, as mentioned in the previous chapters, all discrete subgroups of $SU(2)$ have been classified and are found to correspond to simply-laced Dynkin diagrams [21]. The A-series corresponds to the abelian $\mathbb{Z}_k$ subgroups, and produces the lens space $L(k, 1)$. The D-series corresponds to the non-abelian Dic_k (binary dihedral) subgroups. The E-series corresponds to the double-cover of symmetry groups for the tetrahedron, the octahedron, and the icosahedron (respectively for E6, E7, and E8). Although modding out S^3 by the D and the E subgroups leads to fundamental groups that are non-abelian, it is nevertheless sensible to consider the supersymmetric index on these more complicated geometries because they can preserve the same number of supercharge as the lens space. In this paper, we will compute the supersymmetric index on these nontrivial geometries.

Supersymmetry is critical in maintaining the degeneracy of the ground state Wilson lines. When supersymmetry is not present, the ground state degeneracy is in general lifted. Although in this dissertation we deal with flat Wilson lines taking discrete values according to the ADE group Γ, we give some examples using Wilson lines taking continuous values to support this claim. Without supersymmetry, toroidal compactification in general will dynamically generate a potential for the dilaton field and localizes it [2], the dilaton being analogous to the classically flat Wilson line. Another example where quantum corrections lift the classical degeneracy in flat Wilson lines is 2D Yang-Mills on a circle. The quantum mechanical partition function of this theory is [68]

$$Z = \sum_{R} e^{-TLc_2(R)} \tag{4.1}$$

where T and L are the lengths of the time and the spatial circle, the sum is over the irreducible representations, and $c_2(R)$ is the quadratic Casimir of irreducible representation R. From the expression, one sees that classically flat Wilson lines now gain different amount of energy depending on the representation R.

Even with the help of supersymmetry, Wilson lines (dilatons) exhibit different behaviors depending on the amount of supersymmetry. Let us start with the example of n coincident D4 branes in type IIA string theory with world volume in the 01234 directions. This system breaks half of the 32 supersymmetries and leads to the 5D $U(n)$ supersymmetry Yang-Mills theory on the worldvolume. Now, we compactify the 4th directions of the D4 branes. We can adjust the value of the Wilson line by tuning the fourth component of the gauge field A_4. Doing so does not cost any energy classically, and we can conjugate A_4 to take value in the Cartan subalgebra of $\mathfrak{u}(n)$. That this does not cost any energy quantum mechanically can be seen by going to the T-dual picture, where the system now contains n D3 branes separated along the 4th direction given by the value of A_4. The distance X^4 is given by the T-dual relation $X^4 = 2\pi\alpha' A_4$ where α' is the string length [2]. This system also breaks half of the 32 supersymmetries, and no potential is generated for the classically flat Wilson line [2]. In contrast, consider the system of a D4 brane with world volume in the 01234 directions and a D2 brane in the

034 directions in type IIA string theory. We further suppose that there is no separation between the D-branes in the 56789 directions. This system breaks one quarter of the 32 supersymmetries as opposed to one half, and it leads to $\mathcal{N} = 2$ supersymmetry as measured in 4D. In one picture, the D2 brane dissolves in the D4 brane, leaving behind some RR flux. In another picture, we compactify the 4th direction, and tune the Wilson line as before such that it is classically flat. In the T-dual picture, this separates the D3 brane and the D1 brane in the 4 direction, and it is further U-dual to the F1-D1 system. It is known that the F1-D1 system is unstable and forms a bound state [69]. There is therefore an attractive force between the D3 brane and the D1 brane, causing the dilaton to gain a potential.

The lesson in the previous paragraph is that the number of supersymmetry can affect whether or not classically flat Wilson lines can gain a potential. In this work, we deal with discrete flat Wilson lines, and a priori there is no reason to expect such statement to hold for discrete flat Wilson lines. Nevertheless, as we will see in section 4.2, the discrete flat Wilson lines can become nonflat when the number of supersymmetry is reduced from 4 to 2.

We define the one-loop correction to the jth ground state energy as the supersymmetric Casimir energy E_j mentioned in [66]. It is the nonsingular part of

$$E_j = -\frac{1}{2} \lim_{\beta \to 0} \frac{\partial}{\partial \beta} \hat{I}_j \tag{4.2}$$

where $\hat{I}_j$ is the single letter supersymmetric index of the theory for the jth ground state Wilson line and β is some fugacity parameter coupled to some suitable Hamiltonian (to be discussed later) that commutes with the supersymmetry subalgebra used in computing the index. In this work, we will show that

$$E_1 = E_2 = ... = E_q \tag{4.3}$$

where q is the dimension of the ground state Hilbert space and 1 corresponds to the trivial Wilson line (i.e. the identity). We show this for $\mathcal{N} = 4$ SYM with $SU(2)$ gauge group on different ADE singularities Γ, and conjecture that this relation holds for all $SU(N)$ or $U(N)$ gauge groups. The reason we focus on the $SU(2)$ gauge group in this work is twofold. First, it is the simplest non-abelian gauge group and it makes the index computation tractable. Second, we want to compare our result with the conformal $\mathcal{N} = 2$ four-flavor theory which also has gauge group $SU(2)$. A novelty of this work is that the single-letter supersymmetric index on the D-singularity is computed for the first time, complementing the A-singularity (lens space) result.

One reason we choose to measure the Casimir energy using a Hamiltonian that commutes with the supersymmetry subalgebra is that we want to have an energy measure we can trust at strong coupling. At strong coupling, we can use S-duality to go to a weakly coupled theory and compute the supersymmetric Casimir energy of the flat t' Hooft lines. There will be two nontrivial tests of S-duality. First, the number of flat t' Hooft lines must match the number of the number of flat Wilson lines. Second, let us denote the supersymmetric Casimir energy of the ith flat t' Hooft lines by $\tilde{E}_i$. S-duality predicts

$$\tilde{E}_1 = \tilde{E}_2 = ... = \tilde{E}_q = E_1 = E_2 = ... = E_q \tag{4.4}$$

namely the flat t' Hooft lines must also be degenerate in supersymmetric Casimir energy as the flat Wilson lines. These two facts will be checked in section 4.7.

The chapter is organized as follows. In section 4.2, we analyze the $\mathcal{N} = 4$ superconformal algebra and set the convention we use in defining the supersymmetric index. In section 4.3 and section 4.4, we compute the single letter supersymmetric index for $N = 4$ $SU(2)$ SYM on A- and D-singularities, respectively. We set up the calculation for the E-singularity in section 4.5. In section 4.6, we compare our result to theories with less supersymmetry, namely $\mathcal{N} = 2$ supersymmetry. We find that generically, only $\mathcal{N} = 4$ theory has an exact degeneracy of supersymmetric Casimir energy. This makes sense from the holographic duality perspective, since the duality considered in this dissertation has a D3(M5) brane realization where the number of supercharges is 16 (see section 2.4). No such duality exists for supersymmetry less than $\mathcal{N} = 4$ except for some particular $\mathcal{N} = 2$ theories (class S theories). In particular, we will find a surprising result that there is no ground state degeneracy for the conformal $\mathcal{N} = 2$ four-flavor theory on S^3/Γ: not all classically flat Wilson lines are created equal for this theory. Finally, in section 4.7, we compute the supersymmetry Casimir energy for the t' Hooft lines to give yet another test of S-duality. This chapter is based on the work [70].

4.2 Analysis of Supersymmetry

As discussed earlier, the approach we take to compute the supersymmetric Casimir energy is through the supersymmetric index calculation. Since the index calculation hinges on understanding the $N = 4$ superconformal algebra, we take a moment to review the algebra in this section. We shall mostly follow the convention of [61].

Since we are dealing with $\mathcal{N} = 4$ supersymmetry, the superalgebra of interest has an $SU(4)$ R-symmetry. Without the conformal part of the algebra, the superalgebra is closed under the fermionic symmetry generators $Q^{\alpha i}$, $\bar{Q}_i^{\dot{\alpha}}$, the Lorentz symmetry generator $J^{\alpha}_{2\beta}$, $J^{\dot{\alpha}}_{2\beta}$, the $SU(4)$ R-symmetry generator R^i_j, and the translation generator $P^{\alpha\dot{\beta}}$. Here, $\alpha, \dot{\alpha}$ are $SU(2)_L$ and $SU(2)_R$ indices coming from the Lorentz group, and $i = \{1, 2, 3, 4\}$ is the $SU(4)$ R-symmetry index. Because the theory is conformal even at the quantum level, one can enlarge the superalgebra by adding in the conformal algebra. The enhancement produces the fermionic counterpart of Q and $\bar{Q}$: $S_{\alpha i}$, $\bar{S}^{i}_{\dot{\alpha}}$, the bosonic counterpart of P: $K^{\alpha\dot{\beta}}$, and the dilitation operator D. These form the superalgebra $SU(2,2|4)$.

In radial quantization, the S generators and the Q generators are Hermitian conjugate of each other, so that we have the positive definite anticommutator [61]

$$\{Q^{\dagger}_{\alpha i}, Q^{\beta j}\} = \delta^j_i J^{\beta}_{1\alpha} + \delta^{\beta}_{\delta} R^j_i + \delta^j_i \delta^{\beta}_{\alpha} \frac{D}{2} \tag{4.5}$$

A similar relation holds if we replace Q by $\bar{Q}$, the undotted indices by dotted indices, and J_1 by J_2. A comprehensive list of other (anti)commutators can be found in the appendix of [61]. The above anticommutator will be enough for our purposes.

To define the index, we need to pick out a particular $Q, Q^{\dagger}$ pair and look for states annihilated by $\{Q^{\dagger}, Q\}$. Let us pick $Q = Q^{-1/2,1}$, which according to equation (4.5) gives the anticommutator

$$2\{Q^{\dagger}, Q\} = D - 2J_1 - \left(\frac{3}{2}R_1 + R_2 + \frac{1}{2}R_3\right) \tag{4.6}$$

where R_1, R_2, R_3 are the maximally commuting $SU(4)$ charges and J_1 is the $SU(2)_L$ charge. States that are annihilated by the above anticommutator form short BPS multiplets whose contribution to the index

$$Z = \mathrm{tr}(-1)^F \tag{4.7}$$

does not change as the coupling constant is varied, providing a reliable probe of the strongly-coupled regime of the theory [59, 61]. Nevertheless, equation (4.7) is not the most general quantity that is protected when one restricts the partition function in the Hilbert space annihilated by the anticommutator (4.53). One can also add in other quantum numbers that commute with the superalgebra generated by Q and $Q^{\dagger}$. The commuting subalgebra is $SU(2,1|3)$, whose bosonic Cartan elements are

$$D + J_1, J_2, R_2, R_3 \tag{4.8}$$

Therefore, if one wants to obtain maximal information on the protected spectrum of the theory, one can compute

$$Z = \mathrm{tr}((-1)^F e^{-\beta(D+J_1+\Omega_2 J_2+tR_2+uR_3)}) \tag{4.9}$$

where β, $\beta\Omega_2$, βt, βJ are the bookkeeping parameters (fugacities) that help us distinguish states with different quantum numbers. They also help regulate[1] the infinite sum

[1]Since the $\mathcal{N} = 4$ theory is a UV finite theory, we will see that there will be no UV divergence in the index calculation.

in (4.7). Since we are ultimately interested in the supersymmetric Casimir energy, we set Ω_2, t, u to be zero and focus only on the $D + J_1$ part. Another reason for setting Ω_2 to zero is that nonabelian singularities such as the D- and the E-singularity do not commute with J_2, so J_2 is a bad quantum number[2].

For this work, it is sensible to define the "supersymmetric energy" as $D + J_1$ for four reasons. First, D generates time translation in the radially quantized picture, so to define a Hamiltonian we need a quantity that contains D. Second, $D + J_1$ commutes with $\{Q^\dagger, Q\}$, so an index calculation involving this quantity gives a well-defined answer, independent of the value of the coupling constant. This property is crucial for our S-duality calculation in section 4.7, since we need a quantity that can be meaningfully compared in two different strong-weak duality frames. Had we chosen a Hamiltonian that does not commute with $\{Q^\dagger, Q\}$, we would not have been able to have a good test of S-duality. Third, as we will check later, $D + J_1$ is positive definite on the BPS states annihilated by $\{Q^\dagger, Q\}$. Finally, $D + J_1$ is what matters when we take the so-called MacDonald limit [71] of the index, setting $t = 0$. In this limit, we are counting states that are annihilated by one more pair of supersymmetry charges. If the theory contains states that are annihilated by at least two pairs of supersymmetry charges, the index must have a well-defined MacDonald limit $t = 0$.

Having discussed why we chose to define the supersymmetric energy as $D + J_1$, we now analyze how to compute the Casimir part of it. Because the index is independent of the coupling constant, the computation of

$$Z = \mathrm{tr}(-1)^F e^{-\beta(D+J_1)} \tag{4.10}$$

can be carried out by doing a twisted path integral on the free theory, the result being some determinant factors [66]. The result is trivially one-loop exact (since the coupling constant can be safely set to 0 without affecting the result), and the supersymmetric Casimir energy corresponding to $D + J_1$ is given by [66] as in equation (4.2). A quick way to see why (4.2) yields the supersymmetric Casimir energy without going through the path integral derivation is to recall that this is how the $-1/24$ normal ordering constant arises in bosonic string theory [2]. There, the single letter "index" is simply a sum over $e^{-\beta L_0}$ on the vacuum and its Virasoro descendent where L_0 is the Cartan element of the Virasoro generators.

There is one more ingredient we need to discuss before computing the index. In our setting, the ground states are labeled by different flat Wilson lines g_j taking values in

[2]As an example, the defining two-dimensional representation of the reflection element in the Dic_k group is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose action turns $\partial_{++}^{n_1}\partial_{+-}^{n_2}$ into $(-1)^{n_2}\partial_{++}^{n_2}\partial_{+-}^{n_1}$, thus changing the J_2 value of this operator from $n_1 - n_2$ to $n_2 - n_1$. This situation does not occur for the A-singularity (lens space) which is abelian.

$SU(2)$ and furnishing a representation of $\pi_1(S^3/\Gamma) = \Gamma$. To count the contribution to the single letter index, we gauge away the flat Wilson lines at the cost of introducing a twist to other fields when transported around a nontrivial loop. To see this, let the nontrivial loop starting from a and ending at b be l and let the gauge field A in the jth flat Wilson line sector be denoted by $A^{(j)}$. Note that point a and point b are identified under some group element $\gamma \in \Gamma$: $b = \gamma a$. The Wilson line wrapping l is

$$g_j(\gamma) = P \exp i \oint_l A^{(j)} \quad \in SU(2) \tag{4.11}$$

The notation $g_j(\gamma)$ is suggestive, as the Wilson lines are representations of the group Γ. Here, g_j is the jth representation of the specific group element $\gamma \in \Gamma$. This Wilson line can be gauged away by using a non-periodic gauge transformation $U \in SU(2)$ such that

$$\begin{aligned} U(a) &= 1 \\ U(\gamma a) &= g_j(\gamma) \end{aligned}$$

The effect of this nonperiodic gauge transformation on a field ϕ that transforms in the fundamental of $SU(2)$ is such that, when ϕ is transported from a to γa, it is multiplied by $U(\gamma a) = g_j(\gamma)$. For a field Φ that transforms in the adjoint of $SU(2)$, the field becomes $g_j(\gamma)^{-1}\Phi g_j$ as it is transported around the loop. In the $N = 4$ SUSY theory, we only have adjoint fields, so we write down the transformation rule for a general adjoint field Φ around a loop l with flat Wilson line $g_j(\gamma)$:

$$\Phi(\gamma a) = g_j(\gamma)^{-1}\Phi(a) g_j(\gamma) \tag{4.12}$$

The group Γ acts geometrically on the underlying space. This geometric action induces an action on the fields Φ such that the field values at a and γa are related. Define this induced action as

$$\Phi(\gamma a) = \gamma \Phi(a) \tag{4.13}$$

Using this, we see from the previous equation that the adjoint field Φ must satisfy the constraint

$$\gamma \Phi = g_j^{-1}(\gamma) \Phi g_j(\gamma) \tag{4.14}$$

In the Wilson line sector j, for a field Φ to enter into the single letter index counting, it must satisfy the above constraint for all $\gamma \in \Gamma$. If the above equation holds for all the k generators $\gamma_1, ...\gamma_k$ of Γ, it holds for all $\gamma \in \Gamma$. Therefore, we effectively have only k constraints for each j. As we will see, the k value for the A, D, E groups are 1, 2, and 2, respectively.

The strategy for computing the single letter index is now clear. By the state-operator correspondence, We start with a field Φ that is annihilated by $\{Q^\dagger, Q\}$ and act on it using the derivatives $\partial_{+\pm}$ which are also annihilated by $\{Q^\dagger, Q\}$, keeping the BPS condition. We project out all the operators that do not satisfy the constraint (4.14). For those that do, we add to the single letter index their Boltzmann weight

$$(-1)^F e^{-\beta(D+J_1)} = (-1)^F t^{2(D+J_1)} \tag{4.15}$$

in which, following the convention of [66], we define

$$t = e^{-\beta/2} \tag{4.16}$$

It is useful to have a list of fields that satisfy the BPS condition so that we could view their quantum numbers D, j_1, j_2. Such a list of fields and their quantum numbers is provided by [61]. Although the j_2 quantum number does not explicitly appear in the constraint (4.14), it enters implicitly on the left hand side (the geometric part of the action). The reason is that we chose to embed Γ in $SU(2)_R$ which breaks half of the supersymmetry. Therefore, Γ can act nontrivially on fields with a nonzero j_2 value. We will write down the action explicitly for each of the ADE groups we encounter later.

Letter	$(-1)^F[E, j_1, j_2]$
X, Y, Z	$[1, 0, 0]$
$\psi_{+,0;-++}, \psi_{+,0;+-+}, \psi_{+,0;++-}$	$-[3/2, 1/2, 0]$
F_{++}	$[2, 1, 0]$
$\partial_{++}\psi_{0,-;+++} + \partial_{+-}\psi_{0,+;+++} = 0$	$[5/2, 1/2, 0]$
$\psi_{0,\pm;+++}$	$-[3/2, 0, \pm 1/2]$
$\partial_{+\pm}$	$[1, 1/2, \pm 1/2]$

Table 4.1: A list of operators that satisfy the BPS condition and their E, j_1, j_2 quantum numbers. The R_1, R_2, R_3 quantum numbers are omitted and can be found in [61].

The operators listed in this table are all BPS. In the fourth operator, the minus sign for fermion is canceled out by the Dirac equation since we want to subtract its contribution from the index to avoid overcounting. With the exception of the last two operators, everything else has zero j_2 quantum number. We are now ready to compute the single letter index and the supersymmetric Casimir energy on each ADE geometry.

4.3 A-Singularity

The index calculation for the A-series (lens space) has been done before [67, 66, 65] for $\mathcal{N} = 2$ theories. We here redo the calculation for the $\mathcal{N} = 4$ theory in a way that can be generalized to the D- and the E-singularity. In this section, we will discuss the logic of our computation and relegate the details in the later sections.

The A-series are the groups $\Gamma = \mathbb{Z}_k$ whose group presentation is

$$r^k = e \tag{4.17}$$

We first discuss the geometric action and then the Wilson line action. The geometric action, as discussed in the previous section, has to do with the generators of the group. For $\mathbb{Z}_k$ there is a single generator which we call r. According to the McKay correspondence [21], the defining geometric action of r on $SU(2)$ doublet is [40]

$$r = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} \tag{4.18}$$

where $w = \exp(2\pi i/k)$. We would now like to understand how r acts geometrically on various BPS operators. From Table 4.1, we see that $\partial_{+\pm}$ transforms as a doublet under $SU(2)$, since their j_2 quantum numbers are $\pm 1/2$. Thus, under r, they transform as

$$\begin{pmatrix} \partial_{++} \\ \partial_{+-} \end{pmatrix} \to \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} \partial_{++} \\ \partial_{+-} \end{pmatrix} \tag{4.19}$$

This suggests that, if we start with a field Φ that has $j_2 = 0$ and acts on it by $\partial_{++}^{n_1}\partial_{+-}^{n_2}$, it will transform geometrically as

$$r(\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi) = w^{n_1-n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi \tag{4.20}$$

The only fields in the theory that has $j_2 \neq 0$ are $\psi_{0,\pm;+++}$, which have $j_2 = \pm 1/2$. This suggests that, just as $\partial_{+\pm}$, $\psi_{0,\pm;+++}$ also transform as a doublet under r. Denote such a doublet field by Φ_μ, where $\mu = \pm 1$. We have

$$r(\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_\mu) = w^{n_1-n_2+\mu}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi \tag{4.21}$$

This concludes the discussion of the geometric action of r on various operators. We now discuss the action of the Wilson lines which take value in $SU(2)$ representations of $\mathbb{Z}_k$. There are k irreducible representations for the group $\mathbb{Z}_k$. From the k irreducible representations we can build up $\lfloor k/2 \rfloor$ $SU(2)$ representations of the form

$$g_n(r) = \begin{pmatrix} w^n & 0 \\ 0 & w^{-n} \end{pmatrix} \quad j = 0, 1, ..., \lfloor k/2 \rfloor \tag{4.22}$$

A truncation at $n = \lfloor k/2 \rfloor$ happens because of the Weyl group symmetry, which exchanges the diagonal elements of the $SU(2)$ matrix. The nth Wilson line g_n acts on the $SU(2)$ adjoint fields by conjugation. Denote a general adjoint field by Φ_l,

$l = -1, 0, 1$, so that $\Phi_0 \in \mathbb{R}$ and $\Phi_1 = \Phi_{-1}^* \in \mathbb{C}$. Since the Wilson line does not act on the geometric part of the field, we suppress the j_2 doublet index μ if there is any. Assembling the three components of the adjoint fields into a matrix:

$$\Phi = \begin{pmatrix} \Phi_0 & \Phi_1 \\ \Phi_{-1} & -\Phi_0 \end{pmatrix} \tag{4.23}$$

We see that under $\Phi \to g_n(r)\Phi g_n^{-1}(r)$ the components transform as

$$g_n(r)(\Phi_0, \Phi_1, \Phi_{-1})g_n^{-1}(r) = (\Phi_0, w^{2n}\Phi_1, w^{-2n}\Phi_{-1})$$

as is familiar from the theory of angular momentum.

Consider the nth ground state Wilson line. We would like to compute the supersymmetric Casimir energy from the single letter index, which, by the state operator correspondence, has contribution from BPS derivatives $\partial_{++}^{n_1}\partial_{+-}^{n_2}$ on BPS fields. Without the $\mathbb{Z}_k$ action, all nonnegative n_1 and n_2 lead to valid single letter operator that can contribute to the index. The $\mathbb{Z}_k$ Wilson line will, however, project out many states. The goal is to figure out the constraint on n_1 and n_2. We first discuss the case where the BPS field Φ_l has $j_2 = 0$. Here, $l = -1, 0, 1$ is the $SU(2)$ adjoint index. The most general linear combination of single letter operators is

$$\Psi = \sum_{n_1 n_2 l} C_{n_1 n_2 l}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_l \tag{4.24}$$

where $n_1, n_2 \geq 0$. We would like to understand for what values of n_1, n_2, l is $C_{n_1 n_2 l}$ nonvanishing, because that would indicate a contribution to the single letter index. Applying the constraint equation (4.14), we need to have

$$r\Psi = g_n(r)\Psi g_n(r)^{-1} \tag{4.25}$$

where the left hand side is the geometric action as in equation (4.21). Substituting in the expression for Ψ, we can turn the above equation into

$$\sum_{n_1 n_2 l} w^{n_1 - n_2} C_{n_1 n_2 l}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_l = \sum_{n_1 n_2 l} w^{2nl} C_{n_1 n_2 l}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_l \tag{4.26}$$

Assuming the states corresponding to the operators are orthogonal as in 2D CFT, we obtain

$$\exp\left(\frac{2\pi i(n_1 - n_2)}{k}\right) C_{n_1 n_2 l} = \exp\left(\frac{2\pi i n l}{k}\right) C_{n_1 n_2 l} \tag{4.27}$$

which suggests that C_{mnl} is nonvanishing only when

$$n_1 - n_2 - 2nl = 0 \mod k \tag{4.28}$$

There are three cases to consider $l = 0$, $l = 1$, and $l = -1$, although the last two should lead to the same contribution to the index by symmetry. When $l = 0$, we need to sum over all $n_1, n_2 \geq 0$ such that

$$n_1 - n_2 = 0 \mod k \tag{4.29}$$

The contribution of $\partial_{++}^{n_1}\partial_{+-}^{n_2}$ to the Boltzmann weight (4.15) is $t^{3n_1+3n_2}$, so we need to compute the sum

$$F^n_{\mathbb{Z}_k, j_2=0} \equiv \sum_{n_1 n_2} t^{3n_1+3n_2} \tag{4.30}$$

in which n_1 and n_2 obey the constraint (4.29). Assume that k is an even number, it is easy to see that the constrained sum is

$$F^n_{\mathbb{Z}_k, j_2=0} = \frac{3(1+t^{3k})}{(1-t^6)(1-t^{3k})}, \quad n = 0, k/2 \tag{4.31}$$

$$F^n_{\mathbb{Z}_k, j_2=0} = \frac{1+t^{3k}+2(t^{6n}+t^{3k-6n})}{(1-t^6)(1-t^{3k})}, \quad n \neq 0, k/2 \tag{4.32}$$

In section 4.9, the constrained sum $F'^n_{\mathbb{Z}_k, j_2=\pm 1/2}$ for $j_2 = \pm 1/2$ fields are computed as in equation (4.126) and (4.127). The final step is to add in the contribution of the BPS fields. According to Table 4.1, the single letter index for the nth ground state Wilson line is thus

$$\hat{I}^n_{\mathbb{Z}_k} = F^n_{\mathbb{Z}_k, j_2=0}(3t^2 - 3t^4 + 2t^6) - F^n_{\mathbb{Z}_k, j_2=\pm 1/2} t^3, \quad n = 0, k/2 \tag{4.33}$$

$$\hat{I}^n_{\mathbb{Z}_k} = F'^n_{\mathbb{Z}_k, j_2=0}(3t^2 - 3t^4 + 2t^6) - F'^n_{\mathbb{Z}_k, j_2=\pm 1/2} t^3, \quad n \neq 0, k/2 \tag{4.34}$$

A similar result can be obtained for the case k is odd, as is done in section **??**.

To obtain the supersymmetric Casimir energy, we can expand equations (4.33) and (4.34) to first order in β:

$$\hat{I}^n_{\mathbb{Z}_k} = 3 - \frac{4\beta}{3k} + O(\beta^2), \quad n = 0, k/2 \tag{4.35}$$

$$\hat{I}^n_{\mathbb{Z}_k} = 1 - \frac{4\beta}{3k} + O(\beta^2), \quad n \neq 0, k/2 \tag{4.36}$$

There are two notable features. First, there are no terms divergent in β as we take $\beta \to 0$. This is expected as $N = 4$ is a UV finite theory. This feature does not occur when the number of supersymmetry is less than four, as we will see later. Second, applying the supersymmetric Casimir energy formula (4.2), we find that the ground states are all degenerate:

$$E^n_{\mathbb{Z}_k} = \frac{2}{3k} \quad n = 0, 1, ..., k/2 \tag{4.37}$$

The exact value of the Casimir energy does not concern us, because in the path integral derivation [66] there is an overall constant one is free to shift. The important point is that the Casimir energy is the same for all ground states. Just as there are no terms divergent as $\beta \to 0$, the ground state degeneracy is also a special feature for $\mathcal{N}=4$ supersymmetry and does not generically hold for less supersymmetric theories. Before we compare with theories with less supersymmetry, in the next section we shall compute the Casimir energy for the Dic_k group where we have to confront with nonabelian geometric actions and nonabelian Wilson lines. We will see that the same degeneracy holds for Dic_k group as well. This lends strong support to the duality we proposed.

4.4 D-Singularity

The presentation for the Dic_k (binary dihedral) group is

$$r^{2k} = e, \quad s^2 = r^k, \quad s^{-1} r s = r^{-1} \tag{4.38}$$

which is reminiscent of the dihedral group of the symmetry of a $2k$-gon. The difference here is that the reflection s does not square to the identity. Instead, it squares to a central element of order two. The group Dic_k contains $4k$ elements generated by r and s. The geometric action of r and s on $SU(2)$ doublets is [40]

$$r = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$$
$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\omega = \exp(\pi i/k)$. The novelty here compared with the lens space case is that there is a generator which acts geometrically in a nonabelian way.

The ground state Wilson lines are $SU(2)$ representations of Dic_k. We would like to know how many ground state Wilson lines there are and their explicit forms. The group representation of Dic_k behaves differently depending on whether k is even or odd. Here we focus on the case when $k \in 2\mathbb{N}$, since the odd case can be worked out similarly. In general, let the number of $SU(q)$ ground state Wilson lines for Dic_k be a_q. The generating function $\Phi(z)$ for computing a_q is worked out in section 3.6 in chapter 3 and is given by equation (3.60):

$$\begin{aligned} \Phi_{\mathrm{Dic}_k} &\equiv 1 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \ldots \qquad (4.39)\\ &= \frac{1}{4}\left(\frac{1}{(1-z)^4(1-z^2)^{k-1}} + \frac{2}{(1-z^2)^2(1-z^2)^{k/2}(1+z^2)^{k/2-1}} + \frac{1}{(1-z^2)^2(1-z^2)^{k-1}}\right) \end{aligned}$$

Since we are interested in the $SU(2)$ representation, we expand the generating function to second order in z and find

$$a_2 = 4 + \frac{k}{2}, \quad k \in 2\mathbb{N} \tag{4.40}$$

We can confirm this formula by working out explicitly the $SU(2)$ representations. Since we only care about the representation of the two generators r and s, we write out the $SU(2)$ representations for r and s only. See section 3.6. First, we have four abelian $SU(2)$ representations:

$$g_1(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.41}$$

$$g_2(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_2(s) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.42}$$

$$g_3(r) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_3(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.43}$$

$$g_4(r) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_4(s) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.44}$$

In addition to the four abelian Wilson lines, we also have $k/2$ nonabelian Wilson lines (labeled by a prime on g in the following) given by

$$g'_n(r) = \begin{pmatrix} \exp(n\pi i/k) & 0 \\ 0 & \exp(-n\pi i/k) \end{pmatrix} \quad n = 1, 3, 5, ..., k-1 \tag{4.45}$$

$$g'_n(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad n = 1, 3, 5, ..., k-1$$

This shows that the number of $SU(2)$ representation of Dic_k is indeed $4 + k/2$ for k even.

We can work out the single letter index in the same way we did in the previous section for the lens space, except for each Wilson line we have two constraints given by the two generators instead of one. The single letter indices for the four abelian Wilson lines must be the same, because being ± 1 they all act in the same way in the adjoint. The details are worked out in section 4.10, and we quote the result here:

$$\begin{aligned} \hat{I}^{\text{abelian}}_{\mathrm{Dic}_k} &= 3\left(\frac{t^{6k}}{(1-t^6)(1-t^{6k})} + \frac{1}{1-t^{12}}\right)(3t^2 - 3t^4 + 2t^6) - 3\frac{t^3 + t^{6k-3}}{(1-t^6)(1-t^{6k})}t^3 \\ &= 3 - \frac{\beta}{3k} + O(\beta^2) \end{aligned}$$

from which we can use (4.2) to read off the supersymmetric Casimir energy:

$$E^{\text{abelian}}_{\text{Dic}_k} = \frac{1}{6k} \tag{4.46}$$

The $k/2$ nonabelian ground states have different single letter index labeled by n. From section 5.2, they are given by

$$\begin{aligned}\hat{I}^n_{\text{Dic}_k} &= \left(\frac{t^{6k}+t^{6n}+t^{6k-6n}}{(1-t^6)(1-t^{6k})}+\frac{t^6}{1-t^{12}}\right)(3t^2-3t^4+2t^6)\\ &\quad -\frac{t^{6n+3}+t^{6n-3}+t^{6k-3-6n}+t^{6k+3-6n}+t^3+t^{6k-3}}{(1-t^6)(1-t^{6k})}t^3\\ &= -\frac{\beta}{3k}+O(\beta^2)\end{aligned}$$

From the above expression, we see that the single letter index is different for each nonabelian Wilson line labeled by n, but the expansion in β shows that they all agree to first order. This suggests that the $k/2$ nonabelian ground states all have the same Casimir energy

$$E^n_{\text{Dic}_k} = \frac{1}{6k}, \quad n = 1, 3, 5, ..., k-1 \tag{4.47}$$

Comparing this with the supersymmetric Casimir energy of the abelian Wilson lines in equation (4.46), we come to the conclusion that all $4 + k/2$ ground states are degenerate, just like the case for lens space.

4.5 E-Singularity

In this section we comment on the supersymmetric Casimir energy for ground states on S^3/E_k ($k = 6, 7, 8$) where E_k is any of the three symmetry groups for the platonic solids, also known as 2T (binary tetrahedral group), 2O (binary octahedral group), and 2I (binary icosahedral group). We will not be able to prove, like we did for the A- and the D-singularity, the exact degeneracy of supersymmetric Casimir energy for the E-singularity. Instead, we will start with a general discussion and use the specific example of E_6 to conjecture that degeneracy indeed happens for the E-singularity.

Like Dic_k, all three E_k groups have two generators. However, it is convenient to write down the group presentation for E_k using three dependent generators r, s, t:

$$r^2 = s^3 = t^{k-3} = rst = -1, \quad k = 6, 7, 8 \tag{4.48}$$

where -1 denotes the central element of the group of order 2. It is easy to see that one of the generators can be expressed using the other two independent ones.

In the following, we focus on the simplest of the three groups, E_6, the binary tetrahedral group of order 24. By the McKay correspondence [21], there are three 1-dimension irreducible representations, three 2-dimensional irreducible representations, and one 3-dimensional irreducible representation. By a trick proposed in..., we can quickly compute the determinant of the irreps using the inverse of the Cartan matrix. The result is as follows. The three 1-dimensional irreps $\rho_1, \rho_1', \rho_1''$ have determinant, $1, \omega, \omega^2$, where ω is the third root of unity. The three 2-dimensional irreps $\rho_2, \rho_2', \rho_2''$ have determinant $1, \omega, \omega^2$. The 3-dimensional irrep ρ_3 has determinant 1. There are seven conjugacy classes labled by $I, -I, \beta, \gamma, \gamma^2, \gamma^4, \gamma^5$, and the character table for E_6 is as follows [72]. The number in the parenthesis in the first line represents the order of any element in the conjugacy class.

	$I(1)$	$-I(1)$	$\beta(4)$	$\gamma(6)$	$\gamma^2(3)$	$\gamma^4(3)$	$\gamma^5(6)$
ρ_1	1	1	1	1	1	1	1
ρ_2	2	-2	0	1	-1	-1	1
ρ_3	3	3	-1	0	0	0	0
ρ_2'	2	-2	0	ω	$-\omega$	$-\omega^2$	ω^2
ρ_2''	2	-2	0	ω^2	$-\omega^2$	$-\omega$	ω
ρ_1'	1	1	1	ω	ω	ω^2	ω^2
ρ_1''	1	1	1	ω^2	ω^2	ω	ω

Table 4.2: Character table for the group E_6, retrieved from [72]. Here, ω is the third root of unity. The number in the parenthesis represents the order of the conjugacy class.

We pick the two generators to sit in the β and the γ conjugacy classes and without any confusion we use the same letters to denote the generators. The defining two-dimensional geometric actions by these two generators are

$$\beta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix} \tag{4.49}$$

where ϵ is the eighth root of unity. To see that these two generators indeed generate the whole group, we can look at the quarternionic representation of E_6. The 24 group elements can be represented by the following unit quarternions.

- 1 element of order 1: 1.
- 1 element of order 2: -1.
- 6 elements of order 4: $\pm i, \pm j, \pm k$.

- 8 elements of order 6: $(1 \pm i \pm j \pm k)/2$.
- 8 elements of order 6: $(-1 \pm i \pm j \pm k)/2$.

To make contact with Table 4.2, we note that the first three lines each forms a conjugacy class, and each of the last two lines (elements of order 6) splits into two conjugacy classes. We rewrite the above using the language of conjugacy classes:

- I: 1 element of order 1: 1.
- $-I$: 1 element of order 2: -1.
- β: 6 elements of order 4: $\pm i, \pm j, \pm k$.
- γ: 4 elements of order 6: $(1 + i + j + k)/2$, with even numbers of sign flips for i, j, k.
- γ^5: 4 elements of order 6: $(1 - i + j + k)/2$, with odd numbers of sign flips for i, j, k.
- γ^2: 4 elements of order 3: $(-1 + i + j + k)/2$, with even numbers of sign flips for i, j, k.
- γ^4: 4 elements of order 3: $(-1 - i + j + k)/2$, with odd numbers of sign flips for i, j, k.

One can check that, for example, i and $(1 + i + j + k)/2$ can generate all elements in the group. In the following, we shall therefore take β, γ in equation (4.49) as the generators, and we would like to find the $SU(2)$ Wilson lines representing these two elements.

How many ground state Wilson lines are on S^3/E_6? The generating function for computing the number of $SU(2)$ representations of E_6 is computed in equation (3.68):

$$\begin{aligned}\Phi_{E_6} &= \frac{1}{3}\left(\frac{1}{(1-z)^3(1-z^2)^3(1-z^3)} + \frac{2}{(1-z^6)(1-z^3)^2}\right)\\ &= 1 + z + 3z^2 + ...\end{aligned}$$

which suggests that there are 3 $SU(2)$ ground state Wilson lines. From the character table and from our previous discussion on the determinant of the representations, it is easy to see that the ground state Wilson lines are

$$\rho_1 \oplus \rho_1, \quad \rho_2, \quad \rho_1' \oplus \rho_1'' \tag{4.50}$$

where $\rho_1 \oplus \rho_1$ is the trivial Wilson line. In the following, we discuss the trivial Wilson line only. The only relevant action for the trivial Wilson line is the geometric action. Start with a $j_2 = 0$ field Φ. The descendant $\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi$ transforms under the two generator β as (see equation (4.49))

$$\beta(\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi) = i^{n_1+n_2}\partial_{++}^{n_2}\partial_{+-}^{n_1}\Phi \tag{4.51}$$

What makes the index on S^3/E_6 so hard to compute compared with the A- and the D-singularity case is the action of γ:

$$\begin{aligned}\gamma(\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi) &= \frac{1}{2^{(n_1+n_2)/2}}(\epsilon\partial_{++}+\epsilon^3\partial_{+-})^{n_1}(\epsilon\partial_{++}-\epsilon^3\partial_{+-})^{n_2}\Phi \\ &= \frac{1}{2^{(n_1+n_2)/2}}\sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\binom{n_1}{l}\binom{n_2}{m}(-1)^{n_2-m}\epsilon^{3n_1+3n_2-2m-2l}\partial_{++}^{m+l}\partial_{+-}^{n_1+n_2-l-m}\Phi\end{aligned}$$

The equation above says that, if we start with an operator $\partial_{++}^{n}\Phi$, then the action of γ would generate a sum of terms

$$\partial_{++}^{l}\partial_{+-}^{m}\Phi$$

such that $l+m=n$. Therefore, a good ansats for an operator so that it is potentially invariant under γ is

$$\sum_{l+m=n} C_{lm}^{n}\partial_{++}^{l}\partial_{+-}^{m}\Phi \tag{4.52}$$

for a given n. Using this ansatz, one can impose the constraints from β and γ to determine what kind of C_{lm}^{n} is nonzero. We leave this for a future work and encourage the reader to work on this as well.

4.6 Comparison with $\mathcal{N}=2$ Supersymmetry

We mentioned earlier that the exact degeneracy in the ground state supersymmetric Casimir energy is a special feature for the $N=4$ theory and does not hold for less supersymmetric theories. In this section, we compute the supersymmetric Casimir energy for the $N=2$ $SU(2)$ super Yang-Mills theory on S^3/Dic_k to give support to this claim. The R-symmetry for $\mathcal{N}=2$ supersymmetry is $U(2)$, which can be decomposed into an $SU(2)$ part with quantum number R and a $U(1)$ part with quantum number r. We pick the same Q as in the $\mathcal{N}=4$ theory to define the index. The anticommutator in the $\mathcal{N}=2$ theory is

$$2\{Q^\dagger, Q\} = D - 2J_1 - R - \frac{r}{2} \tag{4.53}$$

The BPS operators annihilated by the above anticommutator are listed in Table 4.3.

In the $\mathcal{N}=2$ $SU(2)$ SYM theory, one can have hypermultiplets, vector multiplets, or a combination of both so that the "quarks" are charged under some flavor

Letter	$(-1)^F[E, j_1, j_2]$
X	$[1, 0, 0]$
ψ_{1+}	$-[3/2, 1/2, 0]$
F_{++}	$[2, 1, 0]$
$\partial_{++}\psi^2_- + \partial_{+-}\psi^2_+ = 0$	$[5/2, 1/2, 0]$
$\bar{\psi}_{1\pm}$	$-[3/2, 0, \pm 1/2]$
$\bar{q}$	$[1, 0, 0]$
ψ_+	$-[3/2, 1/2, 0]$
$\partial_{+\pm}$	$[1, 1/2, \pm 1/2]$

Table 4.3: A list of operators in $\mathcal{N} = 2$ supersymmetric theories that satisfy the BPS condition and their E, j_1, j_2 quantum numbers. The first operators along with the derivative operators are relevant for the vector multiplet, whereas $\bar{q}$ and ψ_+ along with the derivative operators are relavant for the hypermultiplet. The R, r quantum numbers are omitted and can be found in the appendix of [66].

symmetry. One can also consider the class-S theory [10] in which the topology of the Riemann surface determines the matter and gauge content of the theory. In the following subsections, we compute the supersymmetric Casimir energy for 1) the $SU(2)$ vector multiplet, 2) the $SU(2)$ hypermultiplet which transforms in the fundamental, and 3) the conformal $N_f = 4$ theory. We also comment on the general class-S theory.

One thing to note is that the $\mathcal{N} = 2$ vector multiplet (or the hypermultiplet) by itself does not lead to a conformal field theory, and so we cannot interpret the calculation as computing the conformal dimension of the ground state Wilson line operators. Also, the state-operator correspondence will be lost and we cannot interpret each term in the Boltzmann sum as corresponding to a particular state of the theory. However, it is nevertheless okay to compute the index away from the conformal point by using the BPS spectrum at a conformal point.

$\mathcal{N} = 2$ Vector Multiplet

The $\mathcal{N} = 2$ vector multiplet transforms in the adjoint of $SU(2)$, so in computing the contribution of the constrained sum $t^{3n_1 3+n_2}$ to the single letter index, we can simply re-use the constrained sum we obtained for the $\mathcal{N} = 4$ theory. The only difference is the BPS field content[3].

[3]For example, in the $\mathcal{N} = 4$ theory there are six real scalar field which leads to three holomorphic combinations, explaining the letters X, Y, Z in Table 4.1. In the $\mathcal{N} = 2$ theory there are two real scalar fields in the vector multiplet, which leads to only one holomorphic combination. This is why in Table 4.3 there is only one BPS scalar field as indicated by the letter "X".

To compute the single letter index for the four trivial abelian Wilson lines, we can use the Boltzmann sum $F^{\text{abelian}}_{\text{Dic}_k,j_2=0}$ for $j_2 = 0$ BPS fields from equation (4.133) and $F^{\text{abelian}}_{\text{Dic}_k,j_2=\pm 1/2}$ from equation (4.139) for $j_2 = \pm 1/2$ BPS fields as before:

$$\hat{I}^{\text{vec,abelian}}_{\text{Dic}_k} = F^{\text{abelian}}_{\text{Dic}_k,j_2=0}(t^2 - t^4 + 2t^6) - F^{\text{abelian}}_{\text{Dic}_k,j_2=\pm 1/2}t^3 \tag{4.54}$$

where $t^2 - t^4 + 2t^6$ is the Boltzmann weight for the $j_2 = 0$ BPS base fields and $-t^3$ is the Boltzmann weight for the $j_2 = \pm 1/2$ base fields as in Table 4.3. Expanding to first order in β, on obtains

$$\hat{I}^{\text{vec,abelian}}_{\text{Dic}_k} = -\frac{2}{3k\beta} + 3 + \frac{-2 - 27k - 9k^2}{18k}\beta + O(\beta) \tag{4.55}$$

For the nonabelian Wilson line ground states labeled by $n = 1, 3, ..., k-1$, we can use $F^n_{\text{Dic}_k,j_2=0}$ from (4.151) and $F^n_{\text{Dic}_k,j_2=\pm 1/2}$ from (4.157) to compute the single letter index

$$\hat{I}^{\text{vec},n}_{\text{Dic}_k} = F^n_{\text{Dic}_k,j_2=0}(t^2 - t^4 + 2t^6) - F^n_{\text{Dic}_k,j_2=\pm 1/2}t^3 \tag{4.56}$$

Expanding to first order in β, we have

$$\hat{I}^{\text{vec},n}_{\text{Dic}_k} = -\frac{2}{3k\beta} + \frac{-2 + 9k - 9k^2 + 36nk - 36n^2}{18k}\beta + O(\beta^2) \tag{4.57}$$

As anticipated at the end of section 4.3 and as can be seen from equation (4.55) and equation (4.57), the single letter index here (for both the abelian ground states and the nonabelian ground states) has a divergent term proportional to $1/\beta$, which is not present in the $\mathcal{N} = 4$ SUSY case. We can read off the supersymmetric Casimir energy of the abelian ground states and for the nonabelian ground states labled by n:

$$E^{\text{vec,abelian}}_{\text{Dic}_k} = \frac{2 + 27k + 9k^2}{36k} \tag{4.58}$$

$$E^{\text{vec},n}_{\text{Dic}_k} = \frac{2 - 9k + 9k^2 - 36nk + 36n^2}{36k} \tag{4.59}$$

which shows that the degeneracy is partly lifted as the supersymmetric Casimir energy depends on n for the nonabelian ground states.

$\mathcal{N} = 2$ Hypermultiplet

The $\mathcal{N} = 2$ hypermultiplet transforms in the fundamental of $SU(2)$. The hypermultiplet fields transform differently under the $SU(2)$ Wilson line from the vector multiplet fields, so we need to recompute the Boltzmann sum $\sum t^{3n_1+3n_2}$ for the derivative operators. One simplification here is that neither of the hypermultiplet base fields, $\bar{q}$ and ψ_+, has

$j_2 = \pm 1/2$, so we only need to consider one Boltzmann sum for the $j_2 = 0$ fields.For the $k/2$ nonabelian Wilson line ground states, a generic BPS operator is given by the linear combination

$$\Psi = \sum C_{n_1 n_2 \mu} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu \tag{4.60}$$

where $\mu = \pm 1$ is the $SU(2)$ fundamental index. We again demand two constraints for each Wilson line labeled by $n = 1, 3, 5..., k-1$ (see equation (4.45)):

$$\sum C_{n_1 n_2 \mu} \exp(\pi i(n_1 - n_2)/k) \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu = \sum C_{n_1 n_2 \mu} \exp(\pi i n \mu/k) \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu \tag{4.61}$$

$$\sum C_{n_1 n_2 \mu} \exp(\pi i n_2) \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_\mu = \sum C_{n_1 n_2 \mu} (-1)^{(\mu+1)/2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu} \tag{4.62}$$

They imply

$$n_1 - n_2 - n\mu = 0 \mod 2k \tag{4.63}$$

$$C_{n_2 n_1, -\mu} \exp(\pi i n_1) = (-1)^{(\mu+1)/2} C_{n_1 n_2 \mu} \tag{4.64}$$

The second equation tells us that we can just set $\mu = 1$ and sum over all (n_1, n_2) pairs such that the first equation is satisfied. We do not have to worry about the $\mu = -1$ case. The constrained Boltzmann sum therefore gives

$$\sum t^{3n_1 + 3n_2} = \frac{t^{3n} + t^{6k-3n}}{(1-t^6)(1-t^{6k})} \tag{4.65}$$

From Table 4.3, the Boltzamnn weight for the scalar BPS field $\bar{q}$ is t^2 and that for the fermion BPS field ψ_+ is $-t^4$. So overall the single letter index for the $SU(2)$ hypermultiplet on S^3/Dic_k for the nth nonabelian ground state is

$$\hat{I}_{\text{Dic}_k}^{\text{hyper},n} = \frac{t^{3n} + t^{6k-3n}}{(1-t^6)(1-t^{6k})} (t^2 - t^4) \tag{4.66}$$

Expanding to first order in β, we obtain

$$\hat{I}_{\text{Dic}_k}^{\text{hyper},n} = \frac{2}{9k\beta} + \frac{-8 + 18k^2 - 54nk + 27n^2}{108k} \beta + O(\beta^2) \tag{4.67}$$

which implies a supersymmetric Casimir energy of

$$E_{\text{Dic}_k}^{\text{hyper},n} = \frac{8 - 18k^2 + 54nk - 27n^2}{216k} \tag{4.68}$$

Similarly, one can compute the single letter index and the supersymmetric Casimir energy for the four abelian ground states. The results are

$$E_{\mathrm{Dic}_k}^{\text{hyper, abelian,1}} = \frac{4-27k-9k^2}{36k} \tag{4.69}$$

$$E_{\mathrm{Dic}_k}^{\text{hyper, abelian,2}} = \frac{4+27k-9k^2}{36k} \tag{4.70}$$

$$E_{\mathrm{Dic}_k}^{\text{hyper, abelian,3}} = \frac{8+9k^2}{216k} \tag{4.71}$$

$$E_{\mathrm{Dic}_k}^{\text{hyper, abelian,4}} = \frac{8+9k^2}{216k} \tag{4.72}$$

From the above equations, we see that just as the vector multiplet case (4.59), for the hypermultiplet the degeneracy is also partially lifted and a discrete quadratic potential generated for the nonabelian ground states.

$\mathcal{N}=2$ $N_f=4$ Theory

In this section we consider the $\mathcal{N}=2$ $N_f=4$ $SU(2)$ theory where N_f indicates the number of flavors. This means that the theory contains one $SU(2)$ vector multiplets and four $SU(2)$ hypermultiplets. This theory is special because it is conformal. One way to see this is to compute the β function explicitly. Another way to see this is to use the D-brane construction [73] where we put two NS5 branes separeted in the 6 direction and two D4 branes in between the NS5 branes. The worldvolume of the NS5 brane is along the 012345 direction, while that of the D4 brane is along the 01236 direction. The D4 branes create dimples at where they intersect the NS5 branes, bending the NS5 branes toward the D4 branes. This bending is interpreted as the running coupling constant. To make the NS5 branes straight (no running coupling constant), one can attach two semi-infinite D4 branes to the left of the first NS5 brane and to the right of the second NS5 brane. These four semi-infinite D4 branes create the four hypermultiplets in the $N_f=4$ theory.

To find the supersymmetric Casimir energy of this theory, we simply have to add four times the hypermultiplet Casimir energy as in (4.68) to the vector multiplet Casimir energy as in (4.59). Therefore, the supersymmetric Casimir energy for the nonabelian Wilson line ground states for this $N_f=4$ theory is

$$E_{\mathrm{Dic}_k}^{N_f=4,n} = \frac{11}{54k} - \frac{1}{4} - \frac{1}{12}k + \frac{n^2}{2k}, \quad n=1,3,5...,k-1 \tag{4.73}$$

For the four abelian Wilson line ground states, we have (the trivial Wilson line ground state is labeled by 1)

$$E_{\mathrm{Dic}_k}^{N_f=4,\text{abelian},1} = \frac{11}{54k} - \frac{1}{4} - \frac{1}{12}k \tag{4.74}$$

$$E_{\mathrm{Dic}_k}^{N_f=4,\text{abelian},2} = \frac{11}{54k} + \frac{7}{4} - \frac{1}{12}k \tag{4.75}$$

$$E_{\mathrm{Dic}_k}^{N_f=4,\text{abelian},3} = \frac{11}{54k} + \frac{3}{4} + \frac{7}{24}k \tag{4.76}$$

$$E_{\mathrm{Dic}_k}^{N_f=4,\text{abelian},4} = \frac{11}{54k} + \frac{3}{4} + \frac{7}{24}k \tag{4.77}$$

From the five equations above, we see that degeneracy of the $N_f = 4$ theory ground states is also partly lifted. The most important and surprising feature of the above equations is that the superconformal Casimir energy is minimized for the unique ground state: the trivial Wilson line ground state as in equation (4.74). This is important for the following reason. In 2D CFT, the conformal dimension for operators is defined on $\mathbb{C}$. The Casimir energy for 2D operators is computed on a cylinder. The Casimir energy and the conformal dimension are related through the anomalous transformation property of the energy momentum tensor T (which is not primary)

$$z^2 T_{zz} = T_{ww} + \frac{c}{24} \tag{4.78}$$

where c is the central charge of the 2D CFT, z is the coordinate of $\mathbb{C}$ and w is the coordinate on the cylinder. This equation implies that there is a universal shift constant $c/24$ that must be added to the Casimir energy in order to obtain the conformal dimension of an operator.

The computation of the supersymmetric Casimir energy that we are doing in this paper is analogous to the Casimir energy on the 2D cylinder. To obtain the conformal dimensions of the ground state Wilson lines, one should add a constant to each Casimir energy value. This constant is characterized by the 4D conformal anomaly. Since the conformal anomaly depends on the geometry only, each ground state Casimir energy must be shifted by the same constant to obtain the conformal dimension. Here, for the $N_f = 4$ theory, we assume that the trivial Wilson line corresponds to the unit operator, which has a conformal dimension of 0. According to equation (4.74), this shift constant is simply

$$\frac{11}{54k} - \frac{1}{4} - \frac{1}{12}k \tag{4.79}$$

Applying the shift to all ground state Wilson lines, one finds their confomral dimensions D to be:

$$D^{N_f=4,\text{abelian},1}_{\text{Dic}_k} = 0 \tag{4.80}$$
$$D^{N_f=4,\text{abelian},2}_{\text{Dic}_k} = 2 \tag{4.81}$$
$$D^{N_f=4,\text{abelian},3}_{\text{Dic}_k} = 1 + \frac{3}{8}k \tag{4.82}$$
$$D^{N_f=4,\text{abelian},4}_{\text{Dic}_k} = 1 + \frac{3}{8}k \tag{4.83}$$
$$D^{N_f=4,n}_{\text{Dic}_k} = \frac{n^2}{2k}, \quad n = 1, 3, 5, ..., k-1 \tag{4.84}$$

In particular, the trivial Wilson line ground state, having conformal dimension 0, is the true ground state of this theory. All other classical ground states have positive conformal dimension. This result is significant because the $N_f = 4$ conformal theory, being superconformal, has a unitarity bound, so all operators should have nonnegative conformal dimensions. It is impossible for nontrivial ground state Wilson lines to obtain a supersymmetric Casimir energy smaller than that of the trivial ground state Wilson line. On the other hand, this result is also surprising. The uninitiated might be tempted to conclude that the ground states for the conformal $SU(2)$ $N_f = 4$ theory on S^3/Γ ($\Gamma = \text{Dic}_k$) are simply Weyl-inequivalent homomorphisms from Γ to $SU(2)$ as he would do for the case of $\mathcal{N} = 4$ SUSY. Our detailed calculation in this section shows that for the $N_f = 4$ theory there is only one unique ground state corresponding to the trivial Wilson line, unlike the case of $\mathcal{N} = 4$ SUSY where all classically flat Wilson lines have the same conformal dimension 0 (after shifting by some anomaly constant).

How should we make of this surprising result? We postpone this discussion to the end of the next subsection on class-S theory.

Class-S Theory

The way the duality was derived in section 2.4 was to consider a stack of q M5 branes compactified on a torus T^2. The ADE singularity Γ acts on $\mathbb{R}^4$, the rest of the world volume of the M5 branes. It was found there that the dual Chern-Simons theory with level q and gauge group given by $G(\Gamma)$ (recall that $G(\Gamma)$ is the gauge group McKay dual to Γ) lives on the torus T^2. The number of ground states on the Chern-Simons theory side is simply the number of level q WZW conformal blocks [25].

A variant of this construction is to replace T^2 by a genus g Riemann surface. It is natural to expect that the ground state duality still holds, except that the theories on both sides of the duality are modified.

- The SYM side: The supersymmetry is reduced from $\mathcal{N} = 4$ to $\mathcal{N} = 2$. The theory becomes the $\mathcal{N} = 2$ class-S theory [10]. Let the number of M5 branes be two. The fundamental building block of the class-S theory in this case is the trinion theory. One way to visualize it is to imagine a vertex with three legs (or, in the blow-up limit, a pair of pants). The legs (either form the same vertex or different vertices) can be contracted to obtain a graph which corresponds to some degeneration limit of some Riemann surface. Each contracted leg yields a $\mathcal{N} = 2$ vector multiplet, and hence a gauge field to create a Wilson line that wraps around the ADE singularity.

- The Chern-Simons side: The Chern-Simons theory is still the same as before with the same level q and the same gauge group $G(\Gamma)$, except that it is quantized on the genus g Riemann surface instead of on T^2. The fundamental building blocks of the ground states of this theory are the fusion rule coefficients N_{ijk}. Any genus-two Riemann surfaces can be constructed by splicing together pair-of-pants topologies. The three holes of the pants correspond to the indices i, j, k. The pants are contracted to form the Riemann surface, and the number of Chern-Simons ground states is simply a contraction of a series of fusion rule coefficients N_{ijk}.

Naively, if some degeneration limit of the Riemann surface of the $SU(2)$ class-S theory has p internal lines, and if each vector field can create h flat Wilson lines by wrapping around the ADE singularity, one would expect (without knowing the duality argument) the number of ground states to be p^h. This would have been true had we still been dealing with $\mathcal{N} = 4$ theories; we showed earlier that exact degeneracy of supersymmetric Casimir energy does not occur for $\mathcal{N} = 2$ theories. There has to be a truncation of ground states, and in the language of the duality this truncation comes about because not all N_{ijk} are nonzero. A zero N_{ijk} would imply an impossible combination of Wilson lines at a particular trinion vertex. Such a combination of Wilson lines will lead to a supersymmetric Casimir energy different from the one for the state involving only the trivial Wilson lines ($N_{111} = 1$ where the subscript 1 corresponds to the identity element). The detail of this computation using the example of Dic_2 singularity is fleshed out in chapter 5.

4.7 Implications For S-Duality

S-duality maps the weak coupling regime of the $\mathcal{N} = 4$ SYM theory with gauge group G to the strong coupling regime of the same theory with the Langlands dual gauge group $\tilde{G}$ [74]. It exchanges the electric degrees of freedom with the magnetic degrees of freedom and elementary particles with solitons. In particular, the Wilson lines are exchanged with the t' Hooft lines [75, 76]. In our setting, S-duality turns the ground state Wilson lines wrapping the Γ singularity into the ground state t' Hooft lines. Because the

Wilson lines take values in $SU(2)$, the ground state t' Hooft lines will take values in the Langlands dual group $SU(2)/\mathbb{Z}_2 \approx SO(3)$. S-duality makes two predictions here. First, the number of ground state t' Hooft lines must be equal to the number of ground state Wilson lines. In other words, the number of $SU(2)/\mathbb{Z}_2$ representations of Γ (up to identification by conjugation and Weyl group) must be equal to the number of $SU(2)$ representations of Γ. Second, for a given Γ, the ground state t' Hooft lines must be exactly degenerate in the supersymmetric Casimir energy, just like their Wilson line counterparts. In addition to this degeneracy, the supersymmetric Casimir energy of the t' Hooft lines must be the same as their Wilson line counterparts. This is because S-duality maps the weakly-coupled electric ground state Hilbert space to the strongly-coupled magnetic ground state Hilbert space.

In the following subsections, we check these two predictions explicitly for the A- and the D-singularity using the gauge group $SU(2)$. Since the supersymmetry Casimir energy $D + J_1$ is protected by supersymmetry, we can imagine doing an index calculation using the dual theory t' Hooft lines and dual theory fields to compute the supersymmetry Casimir energy of the dual theory.

S-Duality on $S^3/\mathbb{Z}_k$

As mentioned earlier, the dual t' Hooft line will take values in $SU(2)/\mathbb{Z}_2$, furnishing some representation of $SU(2)/\mathbb{Z}_2 \to \mathbb{Z}_k$. In other words, let a t' Hooft line be $\tilde{g}(r) \in SU(2)/\mathbb{Z}_2$. It must satisfy the relation

$$\tilde{g}(r)^k = \pm \mathbf{1} \tag{4.85}$$

and we identify two t' Hooft lines $\tilde{g}(r)$ and $\tilde{g}'(r)$ if they differ by $-\mathbf{1}$ (and by conjugation and Weyl group transformation). We would like to show that the number of $SU(2)/\mathbb{Z}_2 \to \mathbb{Z}_k$ representations is the same as that of $SU(2) \to \mathbb{Z}_k$ representations. Before we prove this, we give a specific example for $k = 2$. In this case, there are two Wilson lines

$$g_0(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$g_1(r) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

However, in the t' Hooft line picture, these two solutions correspond to the same t' Hooft line because they differ by $-\mathbf{1}$. A moment's thought reveals that there are indeed two t' Hooft lines:

$$\tilde{g}_0(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\tilde{g}_1(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Although $\tilde{g}_1(r)$ does not square to $\mathbf{1}$, it squares to $-\mathbf{1}$ which is allowed by (4.85). We now prove the general case. First, consider the case where k is odd. In this case, the claim is that the t'Hooft lines take the same values as the Wilson lines

$$\tilde{g}_j(r) = \begin{pmatrix} e^{2\pi i j/k} & 0 \\ 0 & e^{-2\pi i j/k} \end{pmatrix}, \quad j = 0, 1, ..., \lfloor k/2 \rfloor \tag{4.86}$$

It is easy to see that no two solutions are identified under conjugation or under the Weyl group. To see that no two solutions are identified by multiplication by $-\mathbf{1}$, we assume the contrary and suppose that for some $p, q \in \mathbb{Z}$

$$\tilde{g}_p(r) = -\tilde{g}_q(r) \tag{4.87}$$

or

$$e^{2\pi i(p-q)/k} = -1 \tag{4.88}$$

But this is impossible since, k being an odd integer by assumption, $(p-q)/k$ can never be an odd mulitple of $1/2$. In addition to the solutions (4.91), one can also have solutions of the form

$$\tilde{g}'_j(r) = \begin{pmatrix} e^{\pi i j/k} & 0 \\ 0 & e^{-\pi i j/k} \end{pmatrix}, \quad j = 1, 3, ..., k \tag{4.89}$$

since their kth power is $-\mathbf{1}$, satisfying equation (4.85). However, these solutions are redundant: each of these solutions is identified with an old solution in equation (4.91). To see this, consider -1 times $\tilde{g}'_j(r)$, which gives

$$-\tilde{g}'_j(r) = \begin{pmatrix} e^{\pi i (j+k)/k} & 0 \\ 0 & e^{-\pi i (j+k)/k} \end{pmatrix}, \quad j = 1, 3, ..., k \tag{4.90}$$

By assumption, k is odd, and so $j+k$ must be even. Solutions of this form are just the old solutions (4.91). This concludes the proof that for odd k, the number of flat Wilson lines is the same as that of the t' Hooft lines. When k is even, the solutions to the t' Hooft lines are of the form

$$\tilde{g}_j(r) = \begin{pmatrix} e^{\pi i j/k} & 0 \\ 0 & e^{-\pi i j/k} \end{pmatrix}, \quad j = 0, 1, ..., k/2 \tag{4.91}$$

There are $k/2+1$ solutions, which agree with the number of Wilson line solutions. The proof is similar to the above, so we omit it.

The next step is to compute the supersymmetric Casimir energy of the t' Hooft lines, which is expected to be the same as that of the Wilson lines. To do this, we imagine starting with the electric theory and dialing up the coupling constant g_{YM} to a very large value. The $D+J_1$ value does not change in this process, since it is a protected quantity. Next, we use S-duality to go to the weakly coupled magnetic theory in which the elementary fields create/annihialte monopoles (as perceived in the electric frame). We assume that the spectrum of the BPS operators does not change, i.e. in the dual magnetic theory we have a similar BPS operator spectrum the same as Table 4.1. We also assume that the elementary fields transform in the adjoint of the magnetic gauge group $SU(2)/\mathbb{Z}_2$. The last two assumptions imply that we can carry out the same index computation we did before for the $\mathbb{Z}_k$ case, except now using the t' Hooft line solutions.

The case where k is odd is easy because we proved earlier that the t' Hooft line solutions for k odd are the same as the Wilson line solutions. Therefore, for odd k the supersymmetric Casimir energy of the t' Hooft line ground states is the same as that of the Wilson line ground states given by equation (4.37). The novelty here is the k even case. The ingredients that go into the single letter index for k even are worked out in section 4.11. The single letter index $\tilde{I}^n$ for the nth t' Hooft line is

$$\tilde{I}_k^0 = T^0_{\mathbb{Z}_k, j_2=0}(3t^2 - 3t^3 + 2t^6) - T^0_{\mathbb{Z}_k, j_2=\pm 1/2}t^3 \tag{4.92}$$

$$\tilde{I}_k^n = T^n_{\mathbb{Z}_k, j_2=0}(3t^2 - 3t^3 + 2t^6) - T^n_{\mathbb{Z}_k, j_2=\pm 1/2}t^3, \quad n \neq 0 \tag{4.93}$$

where the T functions are defined in equations (4.161) (4.162) (4.163) (4.164). Expanding to first order in β, we have

$$\tilde{I}_k^0 = 3 - \frac{4}{3k}\beta + O(\beta^2) \tag{4.94}$$

$$\tilde{I}_k^n = 1 - \frac{4}{3k}\beta + O(\beta^2), \quad n \neq 0 \tag{4.95}$$

which suggests that all $k/2+1$ t' Hooft line ground states have supersymmetric Casimir energy of $2/3k$, in agreement with the Wilson line ground state energy (4.37). This is a nontrivial statement of S-duality on the ground states for the exactly marginal $\mathcal{N}=4$ SYM theory. It does not in general hold for less supersymmetric theory.

S-Duality on S^3/Dic_k

Although the group Dic_k is nonabelian, we expect that ground state S-duality holds on S^3/Dic_k just as well since Dic_k can be embedded in the same $SU(2)_L$ as $\mathbb{Z}_k$. To compare with the Wilson line results, we take k to be even throughout this subsection.

To find the t' Hooft lines, we need to solve for $r, s \in SU(2)/\mathbb{Z}_2$ so that the following equations are satisfied

$$r^{2k} = \pm e, \quad s^2 = \pm r^k, \quad s^{-1} r s = \pm r^{-1} \tag{4.96}$$

for any of the eight choices of the $\pm$ sign combination. Because there are two generators and because the group is nonabelian, the t' Hooft line solutions for the Dic_k case are not as easy to find as the abelian $\mathbb{Z}_k$ case. For example, we found in section 4.4 that there are $4 + k/2$ Wilson line ground states. Under the -1 identification, the four abelian solutions (4.41) (4.42) (4.43) (4.44) are actually one and the same t' Hooft line solution. In addition, some of the nonabelian solutions are also identified under -1. As shown in section 4.12, there are indeed $4 + k/2$ t' Hooft line solutions, in agreement with the Wilson line result. However, the structure of the t' Hooft line solutions is very different from that of the Wilson line solutions. For the Wilson line case, there are four universal solutions as in equations (4.41) (4.42) (4.43) (4.44) and $k/2$ solutions having k dependence as in equation (4.45). For the t' Hooft line solutions, there are five universal solutions and $k/2 - 1$ solutions that have a k dependence[4]. The former will be labeled by $\tilde{g}_j, j = 1, ..., 5$, and the latter by $\tilde{g}'_n$, $n = 2, 4, ..., k-2$:

$$\tilde{g}_1(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tilde{g}_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.97}$$

$$\tilde{g}_2(r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tilde{g}_2(s) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.98}$$

$$\tilde{g}_3(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \tilde{g}_3(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.99}$$

$$\tilde{g}_4(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \tilde{g}_4(s) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.100}$$

$$\tilde{g}_5(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \tilde{g}_4(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.101}$$

$$\tilde{g}'_n(r) = \begin{pmatrix} \exp(n\pi i/2k) & 0 \\ 0 & \exp(-n\pi i/2k) \end{pmatrix} \quad n = 2, 4, 6, ..., k-2 \tag{4.102}$$

$$\tilde{g}'_n(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad n = 2, 4, 6, ..., k-2$$

Having shown that the number of ground state t' Hooft lines is the same as that of the ground state Wilson lines, we now show that they also have the same supersymmetric Casimir energy as predicted by S-duality. Just as the $\mathbb{Z}_k$ case, the Dic_k indices

[4] For the base case Dic_2, the only solutions are the five universal solutions.

all have the same structure:

$$\tilde{I}^n_{\mathrm{Dic}_k} = T^n_{\mathrm{Dic}_k, j_2=0}(3t^2 - 3t^3 + 2t^6) - T^n_{\mathrm{Dic}_k, j_2=\pm 1/2} t^3 \tag{4.103}$$

The T functions can be calculated using the approach from section 4.10 and are tabulated in section 4.12. Using the result from section 4.12 , we can compute the single letter indices and expand to first order in β. The result is

$$\tilde{I}^1_{\mathrm{Dic}_k} = 3 - \frac{\beta}{3k} \tag{4.104}$$
$$\tilde{I}^2_{\mathrm{Dic}_k} = 1 - \frac{\beta}{3k} \tag{4.105}$$
$$\tilde{I}^3_{\mathrm{Dic}_k} = 1 - \frac{\beta}{3k} \tag{4.106}$$
$$\tilde{I}^4_{\mathrm{Dic}_k} = 1 - \frac{\beta}{3k} \tag{4.107}$$
$$\tilde{I}^5_{\mathrm{Dic}_k} = -\frac{\beta}{3k} \tag{4.108}$$
$$\tilde{I}'^n_{\mathrm{Dic}_k} = -\frac{\beta}{3k} \tag{4.109}$$

where the first five lines $\tilde{I}^j_{\mathrm{Dic}_k}$ are the single letter indices for the five universal solutions and the last line $\tilde{I}'^n_{\mathrm{Dic}_k}$ is for the $k/2-1$ solutions, with the superscript $n = 2, 4, ..., k-2$. Since the above indices all have the same first order term, all t' Hooft line ground states have the same supersymmetric Casimir energy $1/6k$ for a given k. Comparing this result with the Wilson line result (4.46), we find that both the Wilson line ground states and the t' Hooft line ground states have the same supersymmetric Casimir energy $1/6k$, agreeing with the prediction of S-duality.

4.8 Discussion

In this chapter, We find that, for the $N = 4$ $SU(2)$ SYM theory, the Wilson line ground states on S^3/Γ where $\Gamma = \mathbb{Z}_k$ (lens space) or $\Gamma = \mathrm{Dic}_k$ have the same supersymmetric Casimir energy. This result can be viewed as a one-loop test of the duality that relates the ground states of $N = 4$ $U(q)$ (or $SU(q)$) SYM on S^3/Γ to the ground states (or a subspace of the ground states) of the level q Chern-Simons theory with the McKay dual gauge group $G(\Gamma)$. Such degeneracy in the ground state supersymmetric Casimir energy is not found in theories with fewer than four supercharges. We showed this using the example of $\mathcal{N} = 2$ supersymmetry and briefly mentioned the role of supersymmetry Casimir energy in class-S theory. In fact, for the conformal $N_f = 4$ $SU(2)$ theory, our result shows that there is only one true ground state: the one involving the trivial Wilson line. This is surprising at first, but can be potentially explained by looking at the Hilbert space dimension of the corresponding Chern-Simons theory on a four-punctured sphere with all the in and the out states in the identity representation.

We also find that the number of t' Hooft line ground states equals that of the Wilson line ground states, and that the supersymmetry Casimir energy of the t' Hooft line ground states is the same as that of the Wilson line ground states. This provides yet another check of the prediction of S-duality.

Although the numerics done in this paper assumes that the SYM theory has $SU(2)$ gauge group, we have good faith (based on our earlier derivation in section 2.4) to conjecture that the degeneracy in supersymmetry Casimir energy in both the Wilson line sector and the t' Hooft line sector holds for all $SU(q)$ and $U(q)$ gauge group on S^3/Γ where Γ can be any of the ADE singularities.

In the next few sections, we work out some computational details cited in the previous sections of this chapter. We also compute the number of ground state t' Hooft lines for the exceptional singularities E_6, E_7, E_8, and check that the number of ground state t' Hooft lines equals that of the ground state Wilson lines.

4.9 $\mathbb{Z}_k$ Index Calculation

In this section we work out the contribution of the $j_2 = \pm 1/2$ fields to the index for $\mathbb{Z}_k$ to complement the $j_2 = 0$ result in section 4.3. Let $l = -1, 0, 1$ be the $SU(2)$ adjoint index, and $\mu = -1, 1$ the doublet index. The latter responds under the geometric action r as

$$r\Phi_{\mu l} = \exp(2\pi i\mu/k)\Phi_{\mu l} \tag{4.110}$$

The nth Wilson line is $\exp(2\pi ni/k)$, where n ranges from 0 to $*k/2$ in integer steps. Under the adjoint action of the nth holonomy $g_n(r)$, the field transform as

$$g_n(r)(\Phi_{\mu 0}, \Phi_{\mu 1}, \Phi_{\mu,-1})g_n^{-1}(r) = (\Phi_{\mu 0}, \exp(4\pi in/k)\Phi_{\mu 1}, \exp(-4\pi in/k)\Phi_{\mu,-1}) \tag{4.111}$$

as explained in section 4.3.

A general single letter operator can be written as

$$\Psi' = \sum C_{n_1 n_2 \mu l}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{\mu l} \tag{4.112}$$

Because there is only one generator r for the group $\mathbb{Z}_k$, there is only one constraint the operator Ψ' needs to satisfy. It is given by equation (4.14) and reads

$$\sum \exp(2\pi i(n_1 - n_2 + \mu)/k)C_{n_1 n_2 \mu l}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{\mu l} = \sum \exp(4\pi inl/k)C_{n_1 n_2 \mu l}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{\mu l} \tag{4.113}$$

For $C_{n_1 n_2 \mu l}$ to be nonvanishing, one must have

$$n_1 - n_2 + \mu - 2nl = 0 \mod k \tag{4.114}$$

Since μ can take values in $\{-1, 1\}$ and l in $\{-1, 0, 1\}$, there are six cases we need to consider. However, by the symmetry of equation (4.114), we only need to consider the case $\{l = 0, \mu = -1\}$, $\{l = 1, \mu = -1\}$, and $\{l = 1, \mu = 1\}$, since the rest of cases, $\{l = 0, \mu = 1\}$, $\{l = -1, \mu = 1\}$, and $\{l = -1, \mu = -1\}$ give the same contribution to the single letter index as the previous three cases, respectively. Let us first assume that k is even.

Case $\{l = 0, \mu = -1\}$. In this case we need to sum over all (n_1, n_2) pairs in

$$\sum t^{3n_1+3n_2} \tag{4.115}$$

such that equation (4.114) is satisfied for $l = 0, \mu = 1$, or

$$n_1 - n_2 - 1 = 0 \mod k \tag{4.116}$$

The constrained sum gives

$$\sum t^{3n_1+3n_2} = \frac{t^3 + t^{3k-3}}{(1 - t^6)(1 - t^{3k})} \tag{4.117}$$

Case $\{l = 1, \mu = -1\}$. In this case we need to sum over all (n_1, n_2) pairs in

$$\sum t^{3n_1+3n_2} \tag{4.118}$$

such that they satisfy the equation

$$n_1 - n_2 - 1 - 2n = 0 \mod k \tag{4.119}$$

If $n = 0$ or $n = k/2$ (since we assumed k even), the constrained sum is

$$\sum t^{3n_1+3n_2} = \frac{t^3 + t^{3k-3}}{(1 - t^6)(1 - t^{3k})} \tag{4.120}$$

Otherwise, we have

$$\sum t^{3n_1+3n_2} = \frac{t^{6n+3} + t^{3k-6n-3}}{(1 - t^6)(1 - t^{3k})} \tag{4.121}$$

Case $\{l = -1, \mu = -1\}$. In this case we need to sum over all (n_1, n_2) pairs in

$$\sum t^{3n_1+3n_2} \tag{4.122}$$

such that they satisfy the equation

$$n_1 - n_2 - 1 + 2n = 0 \mod k \tag{4.123}$$

If $n = 0$ or $n = k/2$ (since we assumed k even), the constrained sum is

$$\sum t^{3n_1+3n_2} = \frac{t^3 + t^{3k-3}}{(1-t^6)(1-t^{3k})} \tag{4.124}$$

Otherwise, we have

$$\sum t^{3n_1+3n_2} = \frac{t^{6n-3} + t^{3k-6n+3}}{(1-t^6)(1-t^{3k})} \tag{4.125}$$

In summary, the constrained sum of $t^{3n_1+3n_2}$ for fields having $j_2 = \pm 1/2$ is

$$F'^n_{\mathbb{Z}_k, j_2=\pm 1/2} = \frac{6(t^3 + t^{3k-3})}{(1-t^6)(1-t^{3k})} \quad n = 0, k/2 \tag{4.126}$$

$$F'^n_{\mathbb{Z}_k, j_2=\pm 1/2} = \frac{2(t^3 + t^{3k-3} + t^{6n+3} + t^{3k-6n-3} + t^{6n-3} + t^{3k-6n+3})}{(1-t^6)(1-t^{3k})} \quad n \neq 0, k/2 \tag{4.127}$$

4.10 Dic$_k$ Index Calculation

In this section we work out the Dic$_k$ single letter index in detail. First, we deal with the abelian holonomies. As discussed in section 4.4, there are four $SU(2)$ abelian holonomies where $g_j(r)$ and $g_j(s)$ ($j = 1, 2, 3, 4$) take values in ± 1. Since all of the fields transform in the ajdoint of $SU(2)$, the four abelian holonomies act as identity on the fields. We first compute the contribution to the index by the $j_2 = 0$ fields. A general single letter operator can be written as

$$\Psi = \sum_{n_1 n_2 l} C_{n_1 n_2 l} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_l \tag{4.128}$$

where $n_1, n_2 \geq 0$ and $l = -1, 0, 1$ is the $SU(2)$ adjoint index. The operator has to satisfy two constraints (4.14) given by the generators r and s. The two constraints are

$$\sum_{n_1 n_2 l} \exp((n_1 - n_2)\pi i/k) C_{n_1 n_2 l} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_l = \sum_{n_1 n_2 l} C_{n_1 n_2 l} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_l \tag{4.129}$$

$$\sum_{n_1 n_2 l} \exp(n_2 \pi i) C_{n_1 n_2 l} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_l = \sum_{n_1 n_2 l} C_{n_1 n_2 l} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_l \tag{4.130}$$

The first equation implies that for $C_{n_1 n_2 l}$ to be nonvanishing, we need

$$n_1 - n_2 = 0 \mod 2k \tag{4.131}$$

Exchanging the label n_1 with n_2, we see that the second equation implies

$$\exp(n_1\pi i)C_{n_2n_1l}\exp(n_1\pi i) = C_{n_1n_2l} \tag{4.132}$$

Equation (4.132) shows that we only need to sum over $n_1 \geq n_2$ subject to the condition (4.131), since $C_{n_1n_2l}$ is a function of $C_{n_2n_1l}$. For $n_1 = n_2$, equation (4.132) shows that we need to sum over $n_1 = n_2 = 2\mathbb{N}$ only, since an odd value of n_1 would lead to a vanishing $C_{n_1n_2l}$. Therefore, the constrained sum on $t^{3n_1+3n_2}$ becomes

$$F^{\text{abelian}}_{\text{Dic}_k, j_2=0} \equiv 3\sum t^{3n_1+3n_2} = \frac{3t^{6k}}{(1-t^6)(1-t^{6k})} + \frac{3}{1-t^{12}} \tag{4.133}$$

where the prefactor comes from the fact that l can take three values.

Now we discuss the case for $j_2 = \pm 1/2$ fields. The most general operator for such fields $\Phi_{\mu l}$ is

$$\Psi' = \sum C_{n_1n_2\mu l}\partial^{n_1}_{++}\partial^{n_2}_{+-}\Phi_{\mu l} \tag{4.134}$$

where $\mu = \pm 1$ is the $SU(2)$ doublet index since the fields, having $j_2 = \pm 1/2$, now form a doublet. The difference between this case and the previous $j_2 = 0$ case is that now, the geometric action of r and s will be affected by the μ index. The r and s constraints on Ψ' are

$$\sum \exp((n_1 - n_2 + \mu)\pi i/k)C_{n_1n_2\mu l}\partial^{n_1}_{++}\partial^{n_2}_{+-}\Phi_{\mu l} = \sum C_{n_1n_2\mu l}\partial^{n_1}_{++}\partial^{n_2}_{+-}\Phi_{\mu l} \tag{4.135}$$

$$\sum \exp(n_2\pi i)(-1)^{(\mu-1)/2}C_{n_1n_2\mu l}\partial^{n_2}_{++}\partial^{n_1}_{+-}\Phi_{-\mu l} = \sum C_{n_1n_2\mu l}\partial^{n_1}_{++}\partial^{n_2}_{+-}\Phi_{\mu l} \tag{4.136}$$

They imply

$$n_1 - n_2 + \mu = 0 \mod 2k \tag{4.137}$$

$$\exp(n_1\pi i)(-1)^{-(\mu+1)/2}C_{n_2n_1,-\mu,l} = C_{n_1n_2\mu l} \tag{4.138}$$

Equation (4.138) suggests that we need to sum over all n_1, n_2 pairs such that equation (4.137) holds for $\mu = 1$, since the coefficient for $\mu = -1$ is related to that for $\mu = 1$ via equation (4.138). Therefore, the constrained sum on $t^{3n_1+3n_2}$ becomes

$$F^{\text{abelian}}_{\text{Dic}_k, j_2=\pm 1/2} \equiv 3\sum t^{3n_1+3n_2} = \frac{3t^3 + 3t^{6k-3}}{(1-t^6)(1-t^{6k})} \tag{4.139}$$

At this point, we have computed the contribution of $\partial_{+\pm}$ to the index. We also need to add the contribution to the index by the base fields (i.e. those acted on by the derivative operators). Looking up Table 4.1, we can read off the $t^{2(D+j_1)}$ value for each field, and the total single letter index is

$$\hat{I}^{\text{abelian}}_{\text{Dic}_k} = F^{\text{abelian}}_{\text{Dic}_k, j_2=0}(3t^2 - 3t^4 + 2t^6) - F^{\text{abelian}}_{\text{Dic}_k, j_2=\pm 1/2}t^3 \tag{4.140}$$

and its small β expansion up to first order is

$$\hat{I}^{\text{abelian}}_{\text{Dic}_k} = 3 - \frac{\beta}{3k} + O(\beta^2) \tag{4.141}$$

Next, we work out the single letter index for the nonabelian holonomies g'_n, $n = 1, 3, 5, ..., k-1$, as discussed in section 4.4. In this case, the holonomies will act non-trivially on the fields. Recall from section 4.4 that

$$\begin{aligned} g'_n(r) &= \begin{pmatrix} \exp(n\pi i/k) & 0 \\ 0 & \exp(-n\pi i/k) \end{pmatrix} \quad n = 1, 3, 5, ..., k-1 \\ g'_n(s) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad n = 1, 3, 5, ..., k-1 \end{aligned}$$

The adjoint action of the nth holonomy on the fields is

$$g'_n(r)(\Phi_0, \Phi_1, \Phi_{-1})g'^{-1}_n(r) = (\Phi_0, e^{2n\pi i/k}\Phi_1, e^{-2n\pi i/k}\Phi_{-1}) \tag{4.142}$$
$$g'_n(s)(\Phi_0, \Phi_1, \Phi_{-1})g'^{-1}_n(s) = -(\Phi_0, \Phi_{-1}, \Phi_1) \tag{4.143}$$

In particular, $g'_n(s)$ exchanges the 1 and the -1 components of the field.

As we did before, we first discuss the contribution to the index by fields having $j_2 = 0$. The most general single letter operator one can form is as before

$$\Psi = \sum_{n_1 n_2 l} C_{n_1 n_2 l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_l \tag{4.144}$$

Using equations (4.142) and (4.143) in equation (4.14), we find the constraints

$$\sum_{n_1 n_2 l} \exp((n_1 - n_2)\pi i/k) C_{n_1 n_2 l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_l = \sum_{n_1 n_2 l} \exp(2\pi n l i/k) C_{n_1 n_2 l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_l \tag{4.145}$$
$$\sum_{n_1 n_2 l} \exp(n_2 \pi i) C_{n_1 n_2 l} \partial^{n_2}_{++} \partial^{n_1}_{+-} \Phi_l = -\sum_{n_1 n_2 l} C_{n_1 n_2 l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_{-l} \tag{4.146}$$

They imply

$$n_1 - n_2 - 2nl = 0 \mod 2k \tag{4.147}$$
$$-e^{n_1 \pi i} C_{n_2 n_1, -l} = C_{n_1 n_2 l} \tag{4.148}$$

For the case $l = 0$, equation (4.148) tells us to sum over all $n_1 > n_2$ pairs satisfying equation (4.147), and for $n_1 = n_2$ we need to sum over odd n_1. The constrained $t^{3n_1}t^{3n_2}$ sum is thus

$$\sum t^{3n_1} t^{3n_2} = \frac{t^{6k}}{(1-t^6)(1-t^{6k})} + \frac{t^6}{1-t^{12}} \tag{4.149}$$

For the case $l = 1$, we need to sum over all (n_1, n_2) pairs that satisfy equation (4.147), and according to equation (4.148) this will take care of the $l = -1$ case automatically. The constrained $t^{3n_1} t^{3n_2}$ sum in this case is

$$\sum t^{3n_1} t^{3n_2} = \frac{t^{6n} + t^{6k-6n}}{(1-t^6)(1-t^{6k})} \tag{4.150}$$

Overall, the constrained sum is

$$F^n_{\mathrm{Dic}_k, j_2=0} = \frac{t^{6k} + t^{6n} + t^{6k-6n}}{(1-t^6)(1-t^{6k})} + \frac{t^6}{1-t^{12}} \tag{4.151}$$

Now consider the fields that have $j_2 = \pm 1/2$. The most general operator built out of such fields is

$$\Psi' = \sum C_{n_1 n_2 \mu l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_{\mu l} \tag{4.152}$$

where now we have a doublet index $\mu = \pm 1$. The constraints are

$$\sum \exp((n_1 - n_2 + \mu)\pi i/k) C_{n_1 n_2 \mu l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_{\mu l} = \sum C_{n_1 n_2 \mu l} \exp(2\pi n l i/k) \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_{\mu l} \tag{4.153}$$

$$\sum \exp(n_2 \pi i)(-1)^{(\mu-1)/2} C_{n_1 n_2 \mu l} \partial^{n_2}_{++} \partial^{n_1}_{+-} \Phi_{-\mu l} = -\sum C_{n_1 n_2 \mu l} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi_{\mu, -l} \tag{4.154}$$

They imply

$$n_1 - n_2 + \mu - 2nl = 0 \mod 2k \tag{4.155}$$

$$-e^{n_1 \pi i}(-1)^{-(\mu+1)/2} C_{n_2 n_1, -\mu, -l} = C_{n_1 n_2 \mu l} \tag{4.156}$$

Equation (4.156) suggests that we only need to sum over all (n_1, n_2) pairs satisfying equation (4.155) for each of the three independent cases: $(\mu = -1, l = 0)$, $(\mu = -1, l = 1)$, $(\mu = -1, l = -1)$. Adding up the contributions to $t^{3n_1+3n_2}$ from all three cases, one obtains

$$F^n_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{t^3 + t^{6k-3} + t^{6n+3} + t^{6k-6n-3} + t^{6n-3} + t^{6k-6n+3}}{(1-t^6)(1-t^{6k})} \tag{4.157}$$

Therefore, the total single letter index for the nth ground state Wilson line is

$$\hat{I}^n_{\mathrm{Dic}_k} = F^n_{\mathrm{Dic}_k, j_2=0}(3t^2 - 3t^4 + 2t^6) - F^n_{\mathrm{Dic}_k, j_2=\pm 1/2} t^3 \tag{4.158}$$

Expanding in β up to first order, we get

$$\hat{I}^n_{\mathrm{Dic}_k} = -\frac{\beta}{3k} + O(\beta^2) \tag{4.159}$$

4.11 Ground State t' Hooft Lines on $S^3/\mathbb{Z}_k$

As mentioned in section 4.7, for k odd the t' Hooft line solutions are the same as the Wilson line solutions. For k even, the t' Hooft line solutions are

$$\tilde{g}_n(r) = \begin{pmatrix} e^{\pi in/k} & 0 \\ 0 & e^{-\pi in/k} \end{pmatrix}, \quad n = 0, 1, ..., k/2 \tag{4.160}$$

Using the method from the previous appendices, we compute the T functions that go into the computation of single letter indices as in equation (4.92). It turns out that the $n = 0$ case is different from the rest, so we list the T functions separately for both cases.

$$T^0_{\mathbb{Z}_k, j_2=0} = 3\frac{1+t^{3k}}{(1-t^6)(1-t^{3k})} \tag{4.161}$$

$$T^n_{\mathbb{Z}_k, j_2=0} = \frac{1+t^{3k}+2(t^{3n}+t^{3k-3n})}{(1-t^6)(1-t^{3k})}, \quad n \neq 0 \tag{4.162}$$

$$T^0_{\mathbb{Z}_k, j_2=\pm 1/2} = 6\frac{t^3+t^{3k-3}}{(1-t^6)(1-t^{3k})} \tag{4.163}$$

$$T^n_{\mathbb{Z}_k, j_2=\pm 1/2} = 2\frac{t^3+t^{3k-3}+t^{3n-3}+t^{3k+3-3n}+t^{3n+3}+t^{3k-3-3n}}{(1-t^6)(1-t^{3k})}, \quad n \neq 0 \tag{4.164}$$

4.12 Ground State t' Hooft Lines on $S^3/\mathbf{Dic}_k$

In this section, we first show that the number of $SU(2)/\mathbb{Z}_2$ t' Hooft line ground states is the same as that of the $SU(2)$ Wilson line ground states on S^3/Dic_k, and then tabulate the T functions used in the computation of single letter index. Let $\tilde{g}(r)$ and $\tilde{(g)}(s)$ denote a t' Hooft line representation for the generators r and s. As mentioned in section 4.7 they must satisfy

$$\tilde{g}(r)^{2k} = \pm\mathbf{1}, \quad \tilde{g}(r)^k = \pm\tilde{g}(s)^2, \quad \tilde{g}^{-1}(s)\tilde{g}(r)\tilde{g}(s) = \pm\tilde{g}^{-1}(r) \tag{4.165}$$

We choose to diagonalize $\tilde{g}(r)$. It is easy to see that

$$\tilde{g}_n(r) = \begin{pmatrix} e^{\pi in/2k} & 0 \\ 0 & e^{-\pi in/2k} \end{pmatrix}, \quad n = 0, ..., k \tag{4.166}$$

satisfies the first equation in (4.165). We do not need $n > k$, since, in the same spirit as the $\mathbb{Z}_k$ case, these solutions are identified with the $n \leq k$ ones under multiplication by -1 and the Weyl group exchanging the diagonal elements in the matrix. Let us parameterize $\tilde{g}(s)$ by $\tilde{g}(s) = \exp(i\theta\hat{n}\cdot\vec{\sigma})$ where $\vec{\sigma}$ is a vector of the three Pauli matrices. The second and the third equation in (4.165) implies

$$\cos(\pi n/2)\mathbf{1} + i\sin(\pi n/2)\sigma_3 = \pm(\cos(2\theta)\mathbf{1} + i\sin(2\theta)\hat{n}\cdot\vec{\sigma}) \tag{4.167}$$

$$\cos(\pi n/2k)\mathbf{1} + i\sin(\pi n/2k)\tilde{g}(s)^{-1}\sigma_3\tilde{g}(s) = \pm\cos(\pi n/2k)\mathbf{1} - i\sin(\pi n/2k)\sigma_3 \tag{4.168}$$

There are three cases to consider.

- Case $n = 0$. In this case there are two solutions to $\tilde{g}(s)$:

$$\tilde{g}(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.169}$$

  It might seem that there can be another solution for $\tilde{g}(s)$, namely

$$\tilde{g}(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.170}$$

  But this $\tilde{g}(r), \tilde{g}(s)$ pair can be obtained from the previous one by an $SU(2)$ conjugation.

- Case $n = k$. Recall that k is assumed even, so in this case there are three solutions to $\tilde{g}(s)$:

$$\tilde{g}(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.171}$$

  Although the last of the above solutions by itself can be obtained from conjugating the previous solution, the conjugation would change $\tilde{g}(r)$ in this case, making the last solution unique.

- Case $0 < n < k$. In this case it is easy to see that there is no solution for odd n. The only solutions are

$$\tilde{g}_n(r) = \begin{pmatrix} e^{\pi i n/2k} & 0 \\ 0 & e^{-\pi i n/2k} \end{pmatrix}, \quad n = 2, 4, ..., k-2 \tag{4.172}$$

$$\tilde{g}_s(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad n = 2, 4, ..., k-2 \tag{4.173}$$

In total, there are $4 + k/2$ solutions, in agreement with the number of Wilson line ground states. Note that the five solutions from the first two cases are universal. They are valid for all k.

Next, we tabulate the T functions used in equation (4.103), first for the five universal solutions and then for the $k/2-1$ solutions dependent on k. We denote the T functions as $T^j, j = 1, ..., 5$ for the five universal solutions and as $T'^n, n = 2, 4, ..., k-2$ for the $k/2-1$ solutions.

$$T^1_{\mathrm{Dic}_k, j_2=0} = \frac{3t^{6k}}{(1-t^6)(1-t^{6k})} + \frac{3}{1-t^{12}} \tag{4.174}$$

$$T^1_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{3t^3 + 3t^{6k-3}}{(1-t^6)(1-t^{6k})} \tag{4.175}$$

$$T^2_{\mathrm{Dic}_k, j_2=0} = \frac{3t^{6k}}{(1-t^6)(1-t^{6k})} + \frac{1+2t^6}{1-t^{12}} \tag{4.176}$$

$$T^2_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{3t^3 + 3t^{6k-3}}{(1-t^6)(1-t^{6k})} \tag{4.177}$$

$$T^3_{\mathrm{Dic}_k, j_2=0} = \frac{t^{6k} + 2t^{3k}}{(1-t^6)(1-t^{6k})} + \frac{1}{1-t^{12}} \tag{4.178}$$

$$T^3_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{t^3 + t^{6k-3} + 2(t^{3k-3} + t^{3k+3})}{(1-t^6)(1-t^{6k})} \tag{4.179}$$

$$T^4_{\mathrm{Dic}_k, j_2=0} = \frac{t^{6k} + 2t^{3k}}{(1-t^6)(1-t^{6k})} + \frac{1}{1-t^{12}} \tag{4.180}$$

$$T^4_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{t^3 + t^{6k-3} + 2(t^{3k-3} + t^{3k+3})}{(1-t^6)(1-t^{6k})} \tag{4.181}$$

$$T^5_{\mathrm{Dic}_k, j_2=0} = \frac{t^{6k} + 2t^{3k}}{(1-t^6)(1-t^{6k})} + \frac{t^6}{1-t^{12}} \tag{4.182}$$

$$T^5_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{t^3 + t^{6k-3} + 2(t^{3k-3} + t^{3k+3})}{(1-t^6)(1-t^{6k})} \tag{4.183}$$

$$T'^n_{\mathrm{Dic}_k, j_2=0} = \frac{t^{6k} + t^{3n} + t^{6k-3n}}{(1-t^6)(1-t^{6k})} + \frac{t^6}{1-t^{12}}, \quad n = 2, 4, ..., k-2 \tag{4.184}$$

$$T'^n_{\mathrm{Dic}_k, j_2=\pm 1/2} = \frac{t^3 + t^{6k-3} + t^{3n-3} + t^{6k+3-3n} + t^{3n+3} + t^{6k-3-3n}}{(1-t^6)(1-t^{6k})}, \quad n = 2, 4, ..., k-2 \tag{4.185}$$

4.13 Ground State t' Hooft lines on S^3/E_6

In this section, we show that the number of ground state t' Hooft lines on S^3/E_6 equals that of the ground state $SU(2)$ Wilson lines on S^3/E_6.

As discussed in section 4.5, the binary tetrahedral group E_6 has two generators satisfying the relations

$$(st)^2 = (ts)^2 = s^3 = t^3$$

Without loss of generality we can make t diagonal:

$$t = \exp(i\theta\sigma_3)$$

Let s be

$$s = \exp(i\phi\hat{n} \cdot \vec{\sigma})$$

The condition $s^3 = t^3$ leads to

$$\cos 3\phi I + i \sin 3\phi\hat{n} \cdot \vec{\sigma} = \cos 3\theta I + i \sin 3\theta\sigma_3$$

But now, for t' Hooft lines, $s^3 = \pm t^3$ gives

$$\cos 3\phi I + i \sin 3\phi\hat{n} \cdot \vec{\sigma} = \pm(\cos 3\theta I + i \sin 3\theta\sigma_3)$$

We first discuss when $\sin 3\theta = 0$, or $t^3 = \pm I$.

Case $t^3 = \pm I$

In this case $3\theta = n\pi$ and $3\phi = n'\pi$, $n, n \in \mathbb{Z}$. Or

$$\theta = \frac{n\pi}{3}$$
$$\phi = \frac{n'\pi}{3}$$

As before, the $sts = \pm t^2$ equation is equivalent to four conditions.

$$\begin{aligned}
\cos\theta\cos 2\phi - n_3 \sin 2\phi \sin\theta &= \pm\cos 2\theta \\
n_1(\sin 2\phi\cos\theta - 2n_3\sin^2\phi\sin\theta) &= 0 \\
n_2(\sin 2\phi\cos\theta - 2n_3\sin^2\phi\sin\theta) &= 0 \\
\sin 2\phi\cos\theta n_3 + \cos^2\phi\sin\theta + \sin^2\phi\sin\theta(n_1^2 + n_2^2 - n_3^2) &= \pm\sin 2\theta
\end{aligned}$$

We can enumerate the values of θ and ϕ to see if it leads to any solutions for $\hat{n}$. Note that ϕ is no longer constrained to take on only three values based on the value of θ. Now, ϕ can take on any of the six values.

- $\theta = 0, \phi = 0$. One solution $t = s = I$.

- $\theta = 0, \phi = \pi$. One solution $t = I, s = -I$.
- $\theta = 0, \phi = \pm\pi/3$. No solution.
- $\theta = 0, \phi = \pm 2\pi/3$. No solution.
- $\theta = \pi, \phi = 0$. One solution $t = -I, s = I$.
- $\theta = \pi, \phi = \pi$. One solution $t = s = -I$.
- $\theta = \pi, \phi = \pm\pi/3$. No solution.
- $\theta = \pi, \phi = \pm 2\pi/3$. No solution.
- $\theta = \pi/3, \phi = 0$. No solution.
- $\theta = \pi/3, \phi = \pi$. No solution.
- $\theta = \pi/3, \phi = \pm\pi/3$. One solution with $t = \exp i\sigma_3\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = 1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp i\sigma_3\pi/3$, $s = \exp(-i\sigma_3\pi/3)$.
- $\theta = \pi/3, \phi = \pm 2\pi/3$. One solution with $t = \exp i\sigma_3\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}2\pi/3$, $n_3 = -1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp i\sigma_3\pi/3$, $s = \exp(i\sigma_3 2\pi/3)$.
- $\theta = -\pi/3, \phi = 0$. No solution.
- $\theta = -\pi/3, \phi = \pi$. No solution.
- $\theta = -\pi/3, \phi = \pm\pi/3$. One solution with $t = \exp -i\sigma_3\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = -1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp -i\sigma_3\pi/3$, $s = \exp(i\sigma_3\pi/3)$.
- $\theta = -\pi/3, \phi = \pm 2\pi/3$. One solution with $t = \exp -i\sigma_3\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}2\pi/3$, $n_3 = 1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp -i\sigma_3\pi/3$, $s = \exp(-i\sigma_3 2\pi/3)$.
- $\theta = 2\pi/3, \phi = 0$. No solution.
- $\theta = 2\pi/3, \phi = \pi$. No solution.
- $\theta = 2\pi/3, \phi = \pm\pi/3$. One solution with $t = \exp i\sigma_3 2\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = -1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp i\sigma_3 2\pi/3$, $s = \exp i\sigma_3\pi/3$.
- $\theta = 2\pi/3, \phi = \pm 2\pi/3$. One solution with $t = \exp i\sigma_3 2\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}2\pi/3$, $n_3 = 1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp i\sigma_3 2\pi/3$, $s = \exp -i\sigma_3 2\pi/3$
- $\theta = -2\pi/3, \phi = 0$. No solution.

- $\theta = -2\pi/3, \phi = \pi$. No solution.
- $\theta = -2\pi/3, \phi = \pm\pi/3$. One solution with $t = \exp -i\sigma_3 2\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = 1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp -i\sigma_3 2\pi/3$, $s = \exp -i\sigma_3\pi/3$.
- $\theta = -2\pi/3, \phi = \pm 2\pi/3$. One solution with $t = \exp -i\sigma_3 2\pi/3$, $s = \exp i\hat{n}\cdot\vec{\sigma}2\pi/3$, $n_3 = -1/3$, $n_1^2 + n_2^2 = 8/9$. Another solution with $t = \exp -i\sigma_3 2\pi/3$, $s = \exp i\sigma_3 2\pi/3$.

In summary, for the case $t^4 = \pm I$, we have found 3 inequivalent solutions. They are (listing only one member of each equivalent class)

$$
\begin{aligned}
&t = s = I \\
&t = \exp i\sigma_3\pi/3, s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3, n_3 = 1/3, n_1^2 + n_2^2 = 8/9. \\
&t = \exp i\sigma_3 2\pi/3, s = \exp -i\sigma_3 2\pi/3
\end{aligned}
$$

Case $t^3 \neq \pm I$

In this case, consider the first constraint $s^3 = \pm t^3$ again

$$\cos 3\phi I + i\sin 3\phi\hat{n}\cdot\vec{\sigma} = \pm(\cos 3\theta I + i\sin 3\theta\sigma_3)$$

We must have $\hat{n} = (0, 0, 1)$, so that s commutes with t. There are two cases.

- "+" sign. Then $3\phi = 3\theta + 2\pi k, k \in \mathbb{Z}$. Imposing $s^2 = \pm t$, we have

  $$\cos 2\phi + i\sin 2\phi\sigma_3 = \pm(\cos\theta + i\sin\theta\sigma_3)$$

  Taking the "+" sign of this equation gives $2\phi = \theta + 2\pi k', k' \in \mathbb{Z}$. Combined with the previous equation this gives us $\phi = 2/3\pi k, k \in \mathbb{Z}$, which leads to the following solutions

  $$
  \begin{aligned}
  &\theta = \phi = 0 \to t = s = I \\
  &\theta = 4\pi/3, \phi = 2\pi/3 \to t = \exp(\sigma_3 i4\pi/3), s = \exp(\sigma_3 i2\pi/3) \\
  &\theta = 2\pi/3, \phi = 4\pi/3 \to t = \exp(\sigma_3 i2\pi/3), s = \exp(\sigma_3 i4\pi/3)
  \end{aligned}
  $$

  but the last two solutions are conjugate.

  Taking the "-" sign gives $2\phi = \theta + \pi + 2k'\pi$, which leads to solutions

  $$
  \begin{aligned}
  &\theta = \pi, \phi = k\pi \to t = -I, s = \pm I \\
  &\theta = 5\pi/3, \phi = 4\pi/3 + k\pi \to t = \exp(\sigma_3 i5\pi/3), s = \pm\exp(\sigma_3 i4\pi/3) \\
  &\theta = 1\pi/3, \phi = 5\pi/3 + k\pi \to t = \exp(\sigma_3 i\pi/3), s = \pm\exp(\sigma_3 i5\pi/3)
  \end{aligned}
  $$

  where the last two solutions are conjugate.

- "$-$" sign. Then $3\phi = 3\theta + \pi + 2\pi k$. Imposing $s^2 = \pm t$, we have

$$\cos 2\phi + i \sin 2\phi\sigma_3 = \pm(\cos\theta + i\sin\theta\sigma_3)$$

Taking the "+" sign of this equation gives $2\phi = \theta + 2\pi k', k' \in \mathbb{Z}$. Combined with the previous equation this gives us $\phi = \pi/3 + 2/3\pi k, k \in \mathbb{Z}$, which leads to the following solutions

$$\begin{aligned}
&\theta = 2\pi/3, \phi = \pi/3 \to t = \exp(\sigma_3 i 2\pi/3), s = \exp(\sigma_3 i\pi/3)\\
&\theta = 0, \phi = \pi \to t = I, s = -I)\\
&\theta = 4\pi/3, \phi = 5\pi/3 \to t = \exp(\sigma_3 i 4\pi/3), s = \exp(\sigma_3 i 5\pi/3)
\end{aligned}$$

where the first and the third solutions are conjugate.

Taking the "$-$" sign gives $2\phi = \theta + \pi + k'\pi, k' \in \mathbb{Z}$, which leads to

$$\begin{aligned}
&\theta = 5\pi/3, \phi = 4\pi/3 + k\pi \to t = \exp(\sigma_3 i 5\pi/3), s = \pm\exp(\sigma_3 i 4\pi/3)\\
&\theta = \pi, \phi = k\pi \to t = -I, s = \pm I)\\
&\theta = \pi/3, \phi = 5\pi/3 + k\pi \to t = \exp(\sigma_3 i\pi/3), s = \pm\exp(\sigma_3 i 5\pi/3)
\end{aligned}$$

where the first and the last solutions are conjugate.

Compared with the previous analysis, we see that none of the solutions here are new.
Summary: For the $E_6 \to SU(2)/\mathbb{Z}_2$ case, we see that there are in fact two solutions:

$$\begin{aligned}
&t = s = I\\
&t = \exp i\sigma_3\pi/3, s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3, n_3 = 1/3, n_1^2 + n_2^2 = 8/9.\\
&t = \exp i\sigma_3 2\pi/3, s = \exp -i\sigma_3 2\pi/3
\end{aligned}$$

so the number of ground state t' Hooft lines is 3, agreeing with the number of Wilson lines as computed in section 4.5.

4.14 Ground State t' Hooft Lines on S^3/E_7

In this section, we show that the number of ground state t' Hooft lines on S^3/E_7 is the same as that of the $SU(2)$ ground state Wilson lines on S^3/E_7. The generating function for computing the number of ground state $SU(q)$ Wilson lines on S^3/E_7 is computed in (3.71), from which we see that there are 4 Wilson line ground states.

The generator relations for the octahedral group are

$$(st)^2 = s^3 = t^4$$

We again parametrize s and t as in the E_6 case. But now for the t' Hooft lines $s^3 = \pm t^4$ gives

$$\cos 3\phi I + i \sin 3\phi \hat{n} \cdot \vec{\sigma} = \pm(\cos 4\theta I + i \sin 4\theta \sigma_3)$$

As before, we first discuss when $\sin 4\theta = 0$, or $t^3 = \pm I$.

Case $t^4 = \pm I$

In this case $4\theta = n\pi$ and $3\phi = n'\pi$, $n, n' \in \mathbb{Z}$. Or

$$\theta = \frac{n\pi}{4}$$
$$\phi = \frac{n'\pi}{3}$$

Imposing $sts = \pm t^3$ we have LHS$= \pm$RHS:

$$\begin{aligned}
\text{LHS} &= (\cos^2\phi\cos\theta - \sin^2\phi\cos\theta) + 2i\sin\phi\cos\phi\cos\theta\hat{n}\cdot\vec{\sigma} - \sin\phi\cos\phi\sin\theta\{\hat{n}\cdot\vec{\sigma}, \sigma_3\} \\
&\quad + i\cos^2\phi\sin\theta\sigma_3 - i\sin^2\phi\sin\theta\hat{n}\cdot\vec{\sigma}\sigma_3\hat{n}\cdot\vec{\sigma} \\
\text{RHS} &= \cos(3\theta) + i\sin(3\theta)\sigma_3 \\
\text{LHS} &= \pm\text{RHS}
\end{aligned}$$

As before the LHS can be cleaned up:

$$\begin{aligned}
\text{LHS} = &(\cos\theta\cos 2\phi - n_3\sin 2\phi\sin\theta) \\
&+ in_1(\sin 2\phi\cos\theta - 2n_3\sin^2\phi\sin\theta)\sigma_1 \\
&+ in_2(\sin 2\phi\cos\theta - 2n_3\sin^2\phi\sin\theta)\sigma_2 \\
&+ i(\sin 2\phi\cos\theta n_3 + \cos^2\phi\sin\theta + \sin^2\phi\sin\theta(n_1^2 + n_2^2 - n_3^3))\sigma_3
\end{aligned}$$

Setting LHS equal to RHS we obtain four equations

$$\begin{aligned}
\cos\theta\cos 2\phi - n_3\sin 2\phi\sin\theta &= \pm\cos 3\theta \\
n_1(\sin 2\phi\cos\theta - 2n_3\sin^2\phi\sin\theta) &= 0 \\
n_2(\sin 2\phi\cos\theta - 2n_3\sin^2\phi\sin\theta) &= 0 \\
\sin 2\phi\cos\theta n_3 + \cos^2\phi\sin\theta + \sin^2\phi\sin\theta(n_1^2 + n_2^2 - n_3^3) &= \pm\sin 3\theta
\end{aligned}$$

We can enumerate the values of θ and ϕ to see if it leads to any solutions for $\hat{n}$. Note that, if we choose one θ, then by our previous relations there are only 3 options for ϕ: $\phi = 4\theta/3 + 2m\pi/3$.

- $\theta = 0, \phi = 0$. one solution $t = s = I$.
- $\theta = 0, \phi = \pi$. one solution $t = I, s = -I$.
- $\theta = 0, \phi = \pm\pi/3$. No solution.
- $\theta = 0, \phi = \pm 2\pi/3$. No solution.
- $\theta = \pi/4, \phi = 0$. No solution.
- $\theta = \pi/4, \phi = \pi$. No solution.
- $\theta = \pi/4, \phi = \pm\pi/3$. One solution $t = \exp i\sigma_3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = 1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = \pi/4, \phi = \pm 2\pi/3$. One solution $t = \exp i\sigma_3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}2\pi/3$, $n_3 = -1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = \pi/2, \phi = 0$. One solution $t = \exp i\sigma_3\pi/2$, $s = I$.
- $\theta = \pi/2, \phi = \pi$. One solution $t = \exp i\sigma_3\pi/2$, $s = -I$.
- $\theta = \pi/2, \phi = \pm\pi/3$. One solution $t = \exp i\sigma_3\pi/2$, $s = \exp(\pm i\hat{n}\cdot\vec{\sigma}\pi/3)$, $\hat{n} = (n_1, n_2, 0)$.
- $\theta = \pi/2, \phi = \pm 2\pi/3$. One solution $t = \exp i\sigma_3\pi/2$, $s = \exp(\pm i\hat{n}\cdot\vec{\sigma}2\pi/3)$, $\hat{n} = (n_1, n_2, 0)$.
- $\theta = 3\pi/4, \phi = 0$. No solution.
- $\theta = 3\pi/4, \phi = \pi$. No solution.
- $\theta = 3\pi/4, \phi = \pm\pi/3$. One solution $t = \exp i\sigma_3 3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = -1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = 3\pi/4, \phi = \pm 2\pi/3$. One solution $t = \exp i\sigma_3 3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}2\pi/3$, $n_3 = 1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = \pi, \phi = 0$. One solution $t = -I$, $s = I$.
- $\theta = \pi, \phi = \pi$. One solution $t = -I$, $s = -I$.
- $\theta = \pi, \phi = \pm\pi/3$. No solution.
- $\theta = \pi, \phi = \pm 2\pi/3$. No solution.
- $\theta = -3\pi/4, \phi = 0$. No solution.

- $\theta = -3\pi/4, \phi = \pi$. No solution.
- $\theta = -3\pi/4, \phi = \pm\pi/3$. One solution $t = \exp -i\sigma_3 3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = 1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = -3\pi/4, \phi = \pm\pi/3$. One solution $t = \exp -i\sigma_3 3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = -1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = -\pi/2, \phi = 0$. One solution $t = \exp -i\sigma_3\pi/2$, $s = I$.
- $\theta = -\pi/2, \phi = \pi$. One solution $t = \exp -i\sigma_3\pi/2$, $s = -I$.
- $\theta = -\pi/2, \phi = \pm\pi/3$. One solution $t = \exp -i\sigma_3\pi/2$, $s = \exp(\pm i\hat{n}\cdot\vec{\sigma}\pi/3)$, $\hat{n} = (n_1, n_2, 0)$.
- $\theta = -\pi/2, \phi = \pm 2\pi/3$. One solution $t = \exp -i\sigma_3\pi/2$, $s = \exp(\pm i\hat{n}\cdot\vec{\sigma}2\pi/3)$, $\hat{n} = (n_1, n_2, 0)$.
- $\theta = -\pi/4, \phi = 0$. No solution.
- $\theta = -\pi/4, \phi = \pi$. No solution.
- $\theta = -\pi/4, \phi = \pm\pi/3$. One solution $t = \exp -i\sigma_3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = -1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.
- $\theta = -\pi/4, \phi = \pm 2\pi/3$. One solution $t = \exp -i\sigma_3\pi/4$, $s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3$, $n_3 = 1/\sqrt{3}$, $n_1^2 + n_2^2 = 2/3$.

In summary, for the case $t^4 = \pm I$, we have found 4 inequivalent solutions. They are (writing only one of the solutions in each equivalence class)

$$\begin{aligned}
&t = s = I \\
&t = \exp i\sigma_3\pi/4, s = \exp i\hat{n}\cdot\vec{\sigma}\pi/4, n_3 = 1/\sqrt{3}, n_1^2 + n_2^2 = 2/3. \\
&t = \exp i\sigma_3\pi/2, s = I \\
&t = \exp i\sigma_3\pi/2, s = \exp i\hat{n}\cdot\vec{\sigma}\pi/3, \hat{n} = (n_1, n_2, 0).
\end{aligned}$$

Case $t^4 \neq \pm I$

In this case the solution to

$$\cos 3\phi + i\sin 3\phi\hat{n}\cdot\vec{\sigma} = \pm(\cos 4\theta + i\sin 4\theta\sigma_3)$$

must have $\hat{n} = (0, 0, 1)$. The "+" sign implies $3\phi = 4\theta + 2\pi k$, and the "−" sign $3\phi = 4\theta + \pi + 2\pi k$.

Since s now commutes with t, the condition $sts = \pm t^3$ is the same as $s^2 = \pm t^2$, or

$$\cos 2\phi + i \sin 2\phi \hat{n} \cdot \vec{\sigma} = \pm(\cos 2\theta + i \sin 2\theta \sigma_3)$$

The "+" sign implies $\phi = \theta + k'\pi$, and the "−" sign $\phi = \theta + \pi/2 + k'\pi$. In total there are 4 cases to consider.

- "++". This leads to

$$\theta = 3\pi k' - 2\pi k$$
$$\phi = 4\pi k' - 2\pi k$$

  or $\theta = \pi, \phi = 0$, $\theta = 0, \pi = 0$, corresponding to the single solution $t = s = I$.

- "+−". This leads to

$$\theta = 3\pi/2 + 3\pi k' - 2\pi k$$
$$\phi = 2\pi + 4\pi k' - 2\pi k$$

  or $\theta = \pm\pi/2$, $\phi = 0$, corresponding to the single solution $t = \exp(i\pi/2\sigma_3), s = I$.

- "−+". This leads to

$$\theta = 3\pi k' - \pi - 2\pi k$$
$$\phi = 4\pi k' - \pi - 2\pi k$$

  or $\theta = 0, \phi = \pi$, $\theta = \phi = \pi$, corresponding to the single solution $t = s = I$.

- "−−". This leads to

$$\theta = 3\pi k' + \pi/2 - 2\pi k$$
$$\phi = 4\pi k' + \pi - 2\pi k$$

  or $\theta = \pm\pi/2, \phi = \pi$, $\theta = \phi = \pi$, corresponding to the single solution $t = \exp(i\pi/2\sigma_3), s = I$.

We see that no new solution is generated. Therefore, the number of ground state t' Hooft lines is 4, agreeing with that of the ground state Wilson lines.

4.15 Ground State t' Hooft Lines on S^3/E_8

In this section, we show that the number of ground state t' Hooft Lines on S^3/E_8 is the same as that of the ground state $SU(2)$ Wilson lines on S^3/E_8 (which is 3 according to the generating function (3.74)).

The binary icosahedral group E_8 has two generators satisfying the relations

$$(st)^2 = (ts)^2 = s^3 = t^5$$

We again parametrize s and t as in the E_6 case. But now for the t' Hooft lines $s^3 = \pm t^5$ gives

$$\cos 3\phi I + i \sin 3\phi \hat{n} \cdot \vec{\sigma} = \pm(\cos 5\theta I + i \sin 5\theta \sigma_3)$$

As before, we first discuss when $\sin 5\theta = 0$, or $t^5 = \pm I$.

Case $t^3 = \pm I$

In this case $5\theta = n\pi$ and $3\phi = n'\pi$, $n, n \in \mathbb{Z}$. Or

$$\theta = \frac{n\pi}{5}$$
$$\phi = \frac{n'\pi}{3}$$

Let us now impose the final constraint

$$sts = \pm t^4 \rightarrow \exp(i\phi\hat{n} \cdot \vec{\sigma}) \exp(i\theta\sigma_3) \exp(i\phi\hat{n} \cdot \vec{\sigma}) = \pm \exp(4i\theta\sigma_3)$$

As before this implies four equations.

$$\begin{aligned} \cos\theta \cos 2\phi - n_3 \sin 2\phi \sin\theta &= \pm \cos 4\theta \\ n_1(\sin 2\phi \cos\theta - 2n_3 \sin^2\phi \sin\theta) &= 0 \\ n_2(\sin 2\phi \cos\theta - 2n_3 \sin^2\phi \sin\theta) &= 0 \\ \sin 2\phi \cos\theta n_3 + \cos^2\phi \sin\theta + \sin^2\phi \sin\theta(n_1^2 + n_2^2 - n_3^3) &= \pm \sin 4\theta \end{aligned}$$

and we can now enumerate the solutions:

- $\theta = 0$, $\phi = 0$. One solution with $t = s = I$.
- $\theta = 0$, $\phi = \pi$. One solution with $t = I, s = -I$.
- $\theta = 0$, $\phi = \pi/3$. No solution.
- $\theta = 0$, $\phi = 2\pi/3$. No solution.

- $\theta = \pi/5$, $\phi = 0$. No solution.
- $\theta = \pi/5$, $\phi = \pi$. No solution.
- $\theta = \pi/5$, $\phi = \pi/3$. One solution with $t = \exp(i\sigma_3\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}\pi/3)$, $n_3 = p_1/(\sqrt{3}q_1)$.
- $\theta = \pi/5$, $\phi = 2\pi/3$. One solution with $t = \exp(i\sigma_3\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}2\pi/3)$, $n_3 = -p_1/(\sqrt{3}q_1)$.
- $\theta = 2\pi/5$, $\phi = 0$. No solution.
- $\theta = 2\pi/5$, $\phi = \pi$. No solution.
- $\theta = 2\pi/5$, $\phi = \pi/3$. One solution with $t = \exp(i\sigma_3 2\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}\pi/3)$, $n_3 = p_2/(\sqrt{3}q_2)$.
- $\theta = 2\pi/5$, $\phi = 2\pi/3$. One solution with $t = \exp(i\sigma_3 2\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}2\pi/3)$, $n_3 = -p_2/(\sqrt{3}q_2)$.
- $\theta = 3\pi/5$, $\phi = 0$. No solution.
- $\theta = 3\pi/5$, $\phi = \pi$. No solution.
- $\theta = 3\pi/5$, $\phi = \pi/3$. One solution with $t = \exp(i\sigma_3 3\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}\pi/3)$, $n_3 = -p_2/(\sqrt{3}q_2)$.
- $\theta = 3\pi/5$, $\phi = 2\pi/3$. One solution with $t = \exp(i\sigma_3 3\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}2\pi/3)$, $n_3 = p_2/(\sqrt{3}q_2)$.
- $\theta = 4\pi/5$, $\phi = 0$. No solution.
- $\theta = 4\pi/5$, $\phi = \pi$. No solution.
- $\theta = 4\pi/5$, $\phi = \pi/3$. One solution with $t = \exp(i\sigma_3 4\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}\pi/3)$, $n_3 = -p_1/(\sqrt{3}q_1)$.
- $\theta = 4\pi/5$, $\phi = 2\pi/3$. One solution with $t = \exp(i\sigma_3 4\pi/5)$, $s = \exp(i\hat{n}\cdot\vec{\sigma}2\pi/3)$, $n_3 = p_1/(\sqrt{3}q_1)$.
- $\theta = \pi$, $\phi = 0$. One solution with $t = -I, s = I$.
- $\theta = \pi$, $\phi = \pi$. One solution with $t = s = -I$.
- $\theta = \pi$, $\phi = \pi/3$. No solution.

- $\theta = \pi$, $\phi = 2\pi/3$. No solution.

In summary, we have 3 solutions (we pick a representative from each conjugacy class):

$$
\begin{aligned}
&t = s = I \\
&t = \exp(i\sigma_3\pi/5), s = \exp(i\hat{n}\cdot\vec{\sigma}\pi/3), n_3 = p_1/(\sqrt{3}q_1). \\
&t = \exp(i\sigma_3 3\pi/5), s = \exp(i\hat{n}\cdot\vec{\sigma}\pi/3), n_3 = -p_2/(\sqrt{3}q_2)
\end{aligned}
$$

Case $t^3 \neq \pm I$

In this case, consider the first constraint $s^3 = \pm t^5$ again

$$\cos 3\phi I + i\sin 3\phi\hat{n}\cdot\vec{\sigma} = \pm(\cos 5\theta I + i\sin 5\theta\sigma_3)$$

The "+" sign implies $3\phi = 5\theta + 2\pi n$, and the "−" sign $3\phi = 5\theta + \pi + 2\pi n$. The next constraint $s^2 = \pm t^3$ similarly has two cases. The "+" sign corresponds to $2\phi = 3\theta + 2\pi k$, and the "−" sign $2\phi = 3\theta + \pi + 2\pi k$. There are in total four cases to consider:

- "++". This case was solved before, leading to the trivial result $t = s = I$.
- "+−". In this case we can solve the two equations to obtain

$$
\begin{aligned}
\theta &= 3\pi + 6\pi k - 4\pi n \\
\phi &= 5\pi + 10\pi k - 6\pi n
\end{aligned}
$$

  or $\theta = \phi = \pi$, which corresponds to $s = t = -I$.

- "−+". In this case we can solve the two equations to obtain

$$
\begin{aligned}
\theta &= 6\pi k - 2\pi - 4\pi n \\
\phi &= 10\pi k - 3\pi - 6\pi n
\end{aligned}
$$

  or $\theta = 0, \phi = \pi$, which corresponds to $t = I, s = -I$.

- "−−". In this case we can solve the two equations to obtain

$$
\begin{aligned}
\theta &= 3\pi + 6\pi k - 2\pi - 4\pi n \\
\phi &= 5\pi + 10\pi k - 3\pi - 6\pi n
\end{aligned}
$$

  or $\theta = \phi = \pi$, which corresponds to $t = s = -I$.

All of these cases correspond to the same solution $t = s = \pm I$, so in total the number of ground state t' Hooft lines is 3, agreeing with the number of ground state Wilson lines.

Part II

Singular M5 Branes on Higher Genus Riemann Surfaces

Chapter 5

Class S Theory on ADE Singularity

5.1 Introduction

In this chapter, we explore a natural generalization of the brane construction proposed in section 2.4. Instead of on T^2, we shall put the M5 branes on higher genus Riemann surfaces. This is the class S construction proposed and explored in [10, 77]. The T^2 in the brane construction is where the dual Chern-Simons theory lives. Therefore, we expect to see that the dual Chern-Simons theory lives on the same Riemann surface the M5 branes are compactified on. Moreover, the gauge group and the level of the dual Chern-Simons theory should not change as we change T^2 into some higher genus Riemann surface. We expect to see a one-to-one match between the SYM ground states and the Chern-Simons states on the higher genus Riemann surface. The latter are built out of the fusion rule N_{ijk} [37] defined on the pair-of-pants topology. The simplest higher-genus Riemann surface built out of the pair-of-pants topology is the genus two Riemann surface. Therefore, for the class S theory side we shall focus on the theory which comes from compactifying two coincident M5 branes on a genus two Riemann surface where the classical ground states are now Wilson line triplets corresponding to the three handles on the genus two Riemann surface.

The outline of this chapter is as follows. In section 5.2 we review the duality argument proposed earlier and use an example of a genus two Riemann surface to understand the constraints on the ground state Wilson line triplets coming from the dual Chern-Simons theory. In section 5.2, we outline the computation of the superconformal index (and hence supersymmetric Casimir energy) for the $SU(2)$ trinion theory on S^3/Dic_2 and use the computation in section 5.3 to explicitly show that the duality works for the Dic_2 singularity. The way we show this explicitly is by computing the supersymmetric Casimir energy of the classically flat Wilson line triplets and comparing the number of true ground states to the number of dual Chern-Simons states. In section 5.4 we apply the duality to a non-Lagrangian class S theory to find its ground states. This chapter is based on the work done with Emil Albrychiewicz, Andres Franco Valiente, and Ori Ganor.

5.2 The Duality and the Statement of the Problem

In section 2.4, we used a brane construction to derive the duality between SYM ground states on S^3/Γ and states of some Chern-Simons theory. After lifting the D3 branes which lie along the 0123 directions to M-theory, the corresponding M5 branes are compactified along the 4# directions as in Table 2.1.

A natural question to ask is, if one replaces the 4# directions by any higher genus Riemann surfaces while keeping the ADE singularity along the 0123 directions, what are the ground states of the 4D theory? Since the compactification of q M5 branes on Riemann surfaces is the class S construction [10], the question is equivalently phrased as finding the ground states of class S theory on ADE singularities.

We illustrate the problem using the example of a genus 2 Riemann surface $\mathcal{R}$. We take $q = 2$ and let the gauge group to be $SU(2)$ and the ADE singularity is $\mathbb{Z}_2$. This means that we have two coincident M5 branes whose world volume is $\mathbb{R}^4/\mathbb{Z}_2 \times \mathcal{R}$. We can think about the resulting 4D theory by first compactifying the 6D theory to 5D SYM along the handle direction and then reducing it further to 4D analogous to the construction in [66]. Figure 5.1 illustrates this idea. The left figure is the genus-2 Riemann surface $\mathcal{R}$, and the right figure is obtained after reducing the Riemann surface along the handle direction. This can be viewed as a quiver diagram of the trifundamental theory.

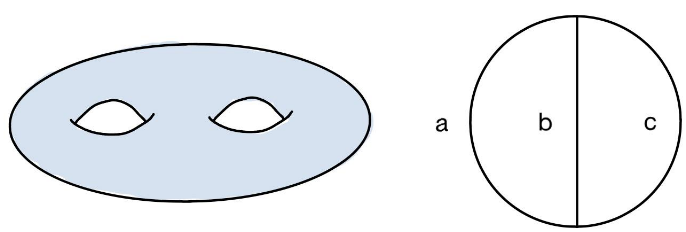

Figure 5.1: 6D to 5D to 4D. In the right figure, the intersections of the lines represent the 4D SYM theory. There are three $SU(2)$ gauge fields labeled by a, b, c, respectively. These gauge fields are expected to satisfy certain constraints to be discussed later.

In figure 5.1, each internal line represents an $SU(2)$ gauge field, labeled by a, b, and c. The classical ground states of this theory can therefore be written as a tensor product

$$g_a \otimes g_b \otimes g_c \tag{5.1}$$

where g_i represents the $SU(2)$ flat Wilson line of gauge field i. It is easy to see that there are only two solutions: g_i can be either $\mathrm{diag}(1,1)$ or $\mathrm{diag}(-1,-1)$. Therefore, naively the number of ground states is $2^3 = 8$. However, the fact that the internal lines end on the same points suggests that there should be some nontrivial constraints on the gauge fields.

The constraint can be most easily derived from the duality, according to which the flat connection g_i is dual to a state i on the Chern-Simons theory. To determine whether a specific ground state $g_a \otimes g_b \otimes g_c$ is allowed, one needs to check if the $SU(2)_2$ fusion coefficient N_{abc} is nonzero[1]. This idea was alluded to earlier in section 4.6 of chapter 4. For our case, the number of Chern-Simons states is given by

$$\sum_{a,b,c} N_{abc} N^{abc}, \tag{5.2}$$

where the sum is only over the restricted states that lie in the set (3.2).

For us, the dual Chern-Simons theory has gauge group $SU(2)$ and level 2, so i, j, k can take the value of either 0 or 2. Note that a value of 1 corresponds to a weight rather than a root. The nonzero N_{ijk} are

$$\begin{aligned} N_{000} &= 1 \\ N_{022} &= 1 \\ N_{220} &= 1 \\ N_{202} &= 1 \end{aligned}$$

Therefore, we can see that the counting from the Chern-Simons side tells us that there are 4 ground states in total, different from the naive answer $2^3 = 8$. It is natural to identify the flat Wilson lines of the $SU(2)$ class S theory on $\mathcal{R}$ with the two Chern-Simons states as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow 0, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Longleftrightarrow 2 \tag{5.3}$$

The corresponding four ground states on the SYM side can be represented as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[1]For $SU(2)$, the fusion coefficients can only be either 1 or 0 [46].

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$
$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$
$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

In this section, we used the Chern-Simons ground states to conjecture the ground states of the dual trinion theory. To establish this duality more closely for the pair-of-pants topology, in the next section we will explicitly compute the ground state degeneray for the trinion theory on S^3/Dic_2. Similar works with a focus on the A-singularity have been done in the past. See [65, 66, 67]. The novelty here is that we will, for the first time, compute the superconformal index for the $SU(2)$ trinion theory on a nonabelian singularity S^3/Dic_2.

Ground States of Class-S Theory on S^3/Dic_2

The group Dic_k was discussed earlier in section 3.6 and in section 4.4. We recall that the group is generated by two elements (r, s) satisfying the following properties:

$$r^{2k} = e, \quad s^2 = r^k, \quad s^{-1} r s = r^{-1}. \tag{5.4}$$

In the rest of this section, we compute the single letter index for Dic_2 for the trinion theory. Therefore, we still take the gauge group to be $SU(2)$, so the flat Wilson lines are $SU(2)$ representations of Dic_2 up to identifications by $SU(2)$ conjugation and by the Weyl group. The inequivalent $SU(2)$ representations of Dic_k for the case $k = 2\mathbb{N}$ was worked out in section 3.6. For the specific case $k = 2$, the representations for the two generators r and s are listed in the Table 5.1.

The Wilson line triplet abc is built out of the combination of the individual Wilson lines listed in the table. Below, we outline the idea of the computation of the index in a spirit similar to what we did in section 4.3, 4.4, and 4.5. The essential difference is that there, we were dealing with a single Wilson line at a time, whereas here the ground state has a triplet of Wilson lines. This will complicate the computation and is the reason why we chose to focus on the simplest nonabelian singularity Dic_2.

The contribution to the index can be broken down to two parts, one from the vector multiplet and one from the hypermultiplet. We first focus on the index contribution from one vector multiplet, which transforms under the *corresponding* Wilson line in the triplet abc. For example, the second vector multiplet only responds to the second Wilson line b in the Wilson line triplet abc, and to compute the total single letter index from the vector multiplet we simply have to add up the individual contributions.

j	$g_j(r)$	$g_j(s)$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
3	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
4	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
5	$\begin{pmatrix} e^{\pi i/2} & 0 \\ 0 & e^{-\pi i/2} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Table 5.1: Ground state Wilson lines on S^3/Dic_2.

Such individual single letter indices from the vector multiplet were already computed in section 4.6. Here, we change the notation a little bit. We denote the single letter index of the abelian Wilson line (labeled 1 through 4 in Table 5.1) as I_1^{vec} (recall each abelian Wilson line leads to the same single letter index since the abelian Wilson lines here, being either $\mathbf{1}$ or $-\mathbf{1}$, act in the same way in the adjoint).

Having discussed the vector multiplet, we now turn to the hypermultiplet, which transforms under all Wilson lines in the Wilson line triplet $m = (abc)$, where $a, b, c \in \{1, 2, 3, 4, 5\}$. This is because of the covariant derivative on a hypermultiplet field Φ takes the form

$$D\Phi = (d + A^1 + A^2 + A^3)\Phi$$

in the Lagrangian, where A^j is the jth gauge field.

All hypermultiplet BPS fields have $j_2 = 0$, so a generic single letter operator constructed out of a BPS field Φ_{ijk} (where $i, j, k \in \{-1, 1\}$ denote the indices for the $SU(2)$ fundamental representation) is

$$\Psi \equiv \sum_{n_1, n_2, i, j, k} C_{n_1 n_2 ijk} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{ijk}. \tag{5.5}$$

It satisfies two constraints, one from the r generator and one from the s generator. The Wilson lines will now act in the fundamental representation. For the Wilson line triplet abc, the constraints are

$$r\Psi = g_a(r)g_b(r)g_c(r)\Psi, \tag{5.6}$$

$$s\Psi = g_a(s)g_b(s)g_c(s)\Psi. \tag{5.7}$$

We need to define the right hand side. The Wilson lines act only on the base field Φ_{ijk} in Ψ and do not act on the derivatives $\partial_{\pm +}$. The base field Φ_{ijk} transforms under the Wilson line triplet abc as

$$g_a(r)g_b(r)g_c(r)\Phi_{ijk} = \sum_{i',j',k'} (g_a(r))_{ii'}(g_b(r))_{jj'}(g_c(r))_{kk'}\Phi_{i'j'k'} \tag{5.8}$$

and similarly for s.

This concludes the setup of the problem of computing the single letter index on S^3/Dic_2. In the following section (section 5.3), we will compute the single letter index and thus the Casimir energy for each classical ground state (or Wilson line triplet, as we use the language interchangeably). The result is as follows.

The Wilson line triplets break into four groups, each having a particular supersymmetric Casimir energy. First, there are 28 states (including the triplet 111) that have the same supersymmetric Casimir energy as the trivial Wilson line triplet 111 (the triplet that contains only the trivial Wilson lines). Those are: 122, 133, 144, 234, 155, 255, 355, 455. In total there are 28 of them, since we need to include the permutations. These states have the lowest positive supersymmetric Casimir energy, of the value $E_0 = 35/108$.

The Wilson line triplets of the form $ab5$, $a, b \in \{1, 2, 3, 4\}$ and permutations have the same supersymmetric Casimir energy $E_1 = 143/108$. Note that E_1 is one unit above E_0.

The Wilson line triplets of the form 112, 113, 114, 233, 244, 222, 134, 224, 334, 223, 344, 333, 444, 123, 124 and permutations have the same supersymmetric Casimir energy $E_2 = 251/108$ that is two units above E_0.

Finally, the Wilson line triplet 555 has a negative supersymmetric Casimir energy, $E_5 = -19/108$, that is 1/2 units below E_0. If we shift Casimir energy of all states by a number representing the conformal anomaly (as discussed in section 4.6) so that the ground state corresponding to the trivial Wilson line triplet 111 has zero conformal dimension, the state 555 will lead to a negative conformal dimension of $-1/2$. This seems to violate the unitarity bound of this superconformal theory, so 555 is an impossible state.

The significance of the this calculation is as follows. From the duality, we expect that the ground states of the $SU(2)$ trinion theory on S^3/Dic_2 are in one-to-one correspondence with the $SO(8)$ level-2 Chern-Simons theory states on the genus two Riemann surface the M5 branes are compactified on. The number of such Chern-Simons states is given by

$$\sum_{i,j,k \in \Lambda_r} N_{ijk}N^{ijk} = 28$$

where, by the duality, i, j, k correspond to the level-2 $SO(8)$ Chern-Simons states that also lie on the $SO(8)$ root lattice (3.2). On the other hand, as our supersymmetric

Casimir energy shows, there are precisely 28 ground states in the super Yang-Mills theory, agreeing with the prediction of the duality. This calculation gives strong support to the generalization of the duality to higher genus Riemann surfaces, and we conjecture that this result generalizes to other D- and E-singularities and for all $SU(N)$ gauge groups. Namely, if we were to consider a potentially non-Lagrangian, non-conformal trifundamental theory with gauge group $SU(N)$ on S^3/Γ, the ground state Wilson line triplets are in one-to-one correspondence with the level-N $G(\Gamma)$ Chern-Simons theory fusion rule N_{ijk} where the indices (i, j, k) are restricted to correspond to states on the $G(\Gamma)$ root lattice.

5.3 Details of S^3/Dic_2 Index Calculation

In this section, we give a detailed computation of the superconformal single letter index on S^3/Dic_2. The idea behind the computation is discussed in section 5.2. A quick summary is that, for a BPS operator to contribute to the single letter index, it must satisfy two constraints given by the two generators r and s. The contribution of an operator to the index is given by the Boltzamnn weight $t^{2(E+j_2)}$. The goal of this section is to solve for the constraints and sum over the Boltzmann weights to find the index. In total, there are 125 Wilson line tripets, given by the following decomposition:

- 111: 1 triplet involving only the trivial Wilson line.
- 115: 3 triplets involving two trivial Wilson lines and one nonabelian Wilson line. There are 3 of Wilson line triplets of this type due to permutation.
- $11a$: 9 triplets involving two trivial Wilson lines and one nontrivial abelian Wilson line. Here, $a \in \{2, 3, 4\}$ represents any of the three nontrivial abelian Wilson lines Wilson lines.
- $1ab$: 18 triplets involving one trivial Wilson line and two distinct nontrivial abelian Wilson lines.
- $1aa$: 9 triplets involving one trivial Wilson line and two identical nontrivial abelian Wilson lines.
- $j55$: 12 triplets involving one abelian Wilson line ($j \in \{1, 2, 3, 4\}$) and two identical nontrivial nonabelian Wilson lines.
- $1a5$: 18 triplets involving one trivial Wilson line, one nontrivial abelian Wilson line, and one nonabelian Wilson line.
- abc: 6 triplets involving three distinct nontrivial abelian Wilson lines.

- *aab*: 18 triplets involving two nontrivial abelian Wilson lines that are identical and one that is distinct from the two.
- *aaa*: 3 triplets involving three nontrivial abelian Wilson lines that are identical.
- *ab*5: 18 triplets involving two nontrivial abelian Wilson lines that are distinct and one nonabelian Wilson line.
- *aa*5: 9 triplets involving two nontrivial abelian Wilson lines that are identical and one nonabelian Wilson line.
- 555: 1 triplet involving three nonabelian Wilson lines.

Although the number of Wilson line triplets is huge, as we will see in the rest of this section, many of these triplets lead to the same single letter index.

We denote a generic BPS field with $j_2 = 0$ as Φ, and a generic BPS field with $j_2 = \pm 1/2$ as $\lambda_\pm$. The geometric action of r and s on BPS descendants of fields with $j_2 = 0$ is discussed in section 4.10:

$$\begin{aligned} r\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi &= e^{i(n_1-n_2)/2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi \\ s\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi &= e^{\pi i n_2}\partial_{++}^{n_2}\partial_{+-}^{n_1}\Phi \end{aligned}$$

For the latter case, the trick is to note that $\lambda_\pm$ themselves carry angular momentum and transform into each other as an $SU(2)$ doublet under s:

$$\begin{aligned} r\partial_{++}^{n_1}\partial_{+-}^{n_2}\lambda_\mu &= e^{i(n_1-n_2+\mu)/2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\lambda_\mu \\ s\partial_{++}^{n_1}\partial_{+-}^{n_2}\lambda_\mu &= (-1)^{(\mu-1)/2}e^{\pi i n_2}\partial_{++}^{n_2}\partial_{+-}^{n_1}\lambda_{-\mu} \end{aligned}$$

The action of the Wilson lines on the fields have already been discussed in section 5.2. Therefore, we now have all the ingredients we need to compute the single letter indices. Since the three vector multiplets do not couple to each other, we can compute the vector multiplet contributions to the index for each Wilson line in the Wilson line triplet. As a preliminary, we compute the vector multiplet contribution to Wilson line 1-5.

I^{vec}

First, we note that the vector multiplet contribution for Wilson line 1-4 is the same.

$$I_1^{vec} = I_2^{vec} = I_3^{vec} = I_4^{vec}$$

This is because these Wilson lines lead to the same adjoint actions. We first commpute the contribution from BPS base fields Φ that have spin $j_2 = 0$. As discussed in section 5.2, we need a linear combination of the descendants to satisfy the constraints:

$$\sum c^{n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi$$

The r and s constraints leads to the equations

$$\begin{aligned} \sum e^{i\pi/2(n_1-n_2)} c^{n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi &= \sum c^{n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi \\ \sum e^{i\pi n_2} c^{n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi &= \sum c^{n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi \end{aligned}$$

The first equation suggests that, for $c^{n_1 n_2}$ to be nonvanishing, we need

$$n_1 - n_2 = 0 \mod 4$$

and the second constraint shows that we only need to sum over $n_1 \geq n_2$. However, for $n_1 = n_2$, we need to sum over $n_1 = 0 \mod 2$ only, since the coefficient vanishes otherwise. Therefore, the constrained sum (where the prefactor 3 below comes from the 3 $SU(2)$ adjoint degrees of freedom)

$$F = 3 \sum t^{3n_1} t^{3n_2}$$

becomes

$$F = \frac{3t^{12}}{(1-t^6)(1-t^{12})} + \frac{3}{1-t^{12}} = 3\frac{1-t^6+t^{12}}{(1-t^6)(1-t^{12})}$$

Therefore, the contribution of the spin 0 fields and EoM in the vector multiplet to the single letter index for Wilson line 1-4 is

$$F(t^2 - t^4 + 2t^6) \tag{5.9}$$

Now let us deal with the spin ± 1 fields λ_μ. Coupled to the spin, a generic operator has the form

$$\sum c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu$$

Imposing the constraints we obtain

$$\begin{aligned} \sum e^{\pi i/2(n_1-n_2+\mu)} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu &= \sum c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu \\ \sum (-1)^{(\mu-1)/2} e^{\pi i n_2} c^{\mu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \lambda_{-\mu} &= \sum c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu \end{aligned}$$

The first constraint shows that when $\mu = 1$, we have

$$n_1 - n_2 = 3 \mod 4$$

and the second constraint shows that we need to sum over all n_1 and n_2 subject to the above equation. We don't have to do this again for the $\mu = -1$ case, since c^{-1,n_1n_2} is completely determined from c^{1,n_1n_2} from the second constraint. Therefore, the sum becomes

$$F' = 3\sum t^{3n_1}t^{3n_2} = 3\frac{t^3 + t^9}{(1-t^6)(1-t^{12})}$$

and the contribution of the $\lambda_\pm$ fields to the index is

$$-F't^3 \tag{5.10}$$

Adding (5.9) and (5.10), we find that the vector multiplet contribution to the single letter index for Wilson line 1-4 is

$$\begin{aligned} I_1^{vec} &= F(t^2 - t^4 + 2t^6) - F't^3 \\ &= \frac{3(1-t^6+t^{12})(t^2-t^4+2t^6) - 3(t^3+t^9)t^3}{(1-t^6)(1-t^{12})} \end{aligned}$$

Next, we compute I_5^{vec}, the vector multiplet single letter index for the 5th Wilson line. This case is a bit harder because the 5th Wilson line is non-abelian. As discussed in section 5.2, under $g_5(r)$ and $g_5(s)$, the adjoint fields Φ^p, $p = 0, -1, 1$ transform as

$$\begin{aligned} g_5(r)(\Phi^0, \Phi^1, \Phi^{-1})g_5^{-1}(r) &= (\Phi^0, -\Phi^1, -\Phi^{-1}) \\ g_5(s)(\Phi^0, \Phi^1, \Phi^{-1})g_5^{-1}(s) &= (-\Phi^0, -\Phi^{-1}, -\Phi^1) \end{aligned}$$

As before, we do the commputation for spin 0 and spin ± 1 fields separately. For spin 0 fields Φ, a generic operator that descends from it is

$$\sum c^{pn_1n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi^p$$

where the adjoint $SU(2)$ index p takes values in $0, \pm 1$. The r and s constraints are

$$\begin{aligned} \sum e^{i\pi/2(n_1-n_2)}c^{pn_1n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi^p &= \sum(-1)^p c^{pn_1n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi^p \\ \sum e^{i\pi n_2)}c^{pn_1n_2}\partial_{++}^{n_2}\partial_{+-}^{n_1}\Phi^p &= -\sum c^{pn_1n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi^{-p} \end{aligned}$$

They imply

$$\begin{aligned} e^{i\pi/2(n_1-n_2)}c^{pn_1n_2} &= (-1)^p c^{pn_1n_2} \\ e^{i\pi n_1}c^{-p,n_2n_1} &= -c^{pn_1n_2} \end{aligned}$$

There are two cases to consider.
Case 1: $p = 0$. For this case, the first constraint shows that

$$n_1 - n_2 = 0 \mod 4$$

and the second constraint instructs us to sum over only $n_1 \geq n_2$, but for $n_1 = n_2$ we need only sum over $n_1 = 1 \mod 2$. Therefore, the sum becomes

$$F^0 = \frac{t^{12}}{(1-t^6)(1-t^{12})} + \frac{t^6}{1-t^{12}} = \frac{t^6}{(1-t^6)(1-t^{12})} \tag{5.11}$$

Case 2: $p = 1$. For this case, the first constraint shows that

$$n_1 - n_2 = 2 \mod 4$$

and the second constraint tells us to sum over all such n_1 and n_2 obeying the above equation and that we do not have to consider the $p = -1$ case. The sum is therefore

$$F^+ = \frac{2t^6}{(1-t^6)(1-t^{12})} = 2F^0 \tag{5.12}$$

So overall the contribution to the single letter vector multiplet index for Wilson line 5 from fields that have $j_2 = 0$ is

$$(F^0 + F^+)(t^2 - t^4 + 2t^6) = 3F^0(t^2 - t^4 + 2t^6) \tag{5.13}$$

Now let us consider the $j_2 = \pm 1/2$ contribution. A generic operator that descends from the field is

$$\sum c^{p\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu^p$$

The constraints are

$$\sum e^{\pi i/2(n_1-n_2+\mu)} c^{p\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu^p = \sum (-1)^p c^{p\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu^p$$
$$\sum (-1)^{(\mu-1)/2} e^{\pi i n_2} c^{p\mu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \lambda_{-\mu}^p = -\sum c^{p\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \lambda_\mu^{-p}$$

They imply

$$e^{\pi i/2(n_1-n_2+\mu)} c^{p\mu n_1 n_2} = (-1)^p c^{p\mu n_1 n_2}$$
$$(-1)^{(-\mu-1)/2} e^{\pi i n_1} c^{-p,-\mu,n_2 n_1} = -c^{p\mu n_1 n_2}$$

There are three cases to consider.
Case 1: $p = 0, \mu = 1$. For this case, the first constraint implies

$$n_1 - n_2 = 3 \mod 4$$

and the second constraint shows that the case $p = 0, \mu = -1$ is taken care of if we sum over all such n_1 n_2 pairs. The sum is

$$F^{01} = \frac{t^3 + t^9}{(1 - t^6)(1 - t^{12})} \tag{5.14}$$

Case 2: $p = 1, \mu = 1$. For this case, the first constraint implies

$$n_1 - n_2 = 1 \mod 4$$

and the second constraint shows that the case $p = -1, \mu = -1$ is taken care of if we sum over all such n_1 n_2 pairs. The sum is

$$F^{11} = \frac{t^3 + t^9}{(1 - t^6)(1 - t^{12})} = F^{01} \tag{5.15}$$

Case 3: $p = 1, \mu = -1$. For this case, the first constraint implies

$$n_1 - n_2 = 3 \mod 4$$

and the second constraint shows that the case $p = -1, \mu = 1$ is taken care of if we sum over all such n_1 n_2 pairs. The sum is

$$F^{1,-1} = \frac{t^3 + t^9}{(1 - t^6)(1 - t^{12})} = F^{01} \tag{5.16}$$

Adding this up, we find that the $j_2 = \pm 1/2$ contribution to the single letter vector multiplet index is

$$-(F^{01} + F^{11} + F^{1,-1})t^3 = -3F^{01}t^3 \tag{5.17}$$

Adding the $j_2 = 0$ contribution from (5.13) and the $j_2 = \pm 1/2$ contribution from (5.17), we get the vector multiplet contribution to the index for Wilson line 5:

$$\begin{aligned} I_5^{vec} &= 3F^0(t^2 - t^4 + 2t^6) - 3F^{01}t^3 \\ &= 3\frac{-t^6 + t^8 - t^{10} + t^{12}}{(1 - t^6)(1 - t^{12})} \end{aligned}$$

We are now done with the vector multiplet computation for the single letter index. Here's a summary:

$$I_1^{vec} = \frac{3(1 - t^6 + t^{12})(t^2 - t^4 + 2t^6) - 3(t^3 + t^9)t^3}{(1 - t^6)(1 - t^{12})} \tag{5.18}$$

$$I_5^{vec} = 3\frac{-t^6 + t^8 - t^{10} + t^{12}}{(1 - t^6)(1 - t^{12})} \tag{5.19}$$

Let us now move on to compute the trifundamental contribution to the single letter index.

$I^{trifund}$

First, let us deal with the abelian Wilson lines. We are interested in computing $I^{trifund}_{abc}$ where $a, b, c \in \{1, 2, 3, 4\}$. According to equation (5.8), there are only 4 independent combinations that give unique constraints: 111, 112, 113, 114 (for example, 122 would be equivalent to 111 in terms of constraints on the trifundamental fields), so we only have four such indices to compute. We do them one by one.

Note that all base fields in the hypermultiplet has $j_2 = 0$, so we need not worry about the $j_2 = \pm 1/2$ contribution.

Case $I^{trifund}_{111}$. A generic operator that descends from the field has the form

$$\sum c^{n_1 n_2} \partial^{n_1}_{++} \partial^{n_2}_{+-} \Phi$$

where Φ denotes three fields in the triplet. For clarity, we do not show the $SU(2)$ trifundamental indices on the field Φ as in equation (5.8). The r and the s constraints give

$$\begin{aligned} e^{i\pi/2(n_1 - n_2)} c^{n_1 n_2} &= c^{n_1 n_2} \\ e^{i\pi n_1} c^{n_2 n_1} &= c^{n_1 n_2} \end{aligned}$$

The first constraint suggests that

$$n_1 - n_2 = 0 \mod 4$$

and the second constraint instructs us to sum over $n_1 \geq n_2$ only, but for $n_1 = n_2$ we need $n_1 = 0 \mod 2$. Overall, summing over the Boltzamnn weights, we have

$$F_{111} = 8\frac{t^{12}}{(1-t^6)(1-t^{12})} + 8\frac{1}{1-t^{12}} = 8\frac{1 - t^6 + t^{12}}{(1-t^6)(1-t^{12})}$$

where the prefactor 8 comes from 2^3 choices of the $SU(2)$ trifundmanetal indices.

Therefore,

$$I^{trifund}_{111} = F_{111}(t^2 - t^4) = 8\frac{(1 - t^6 + t^{12})(t^2 - t^4)}{(1-t^6)(1-t^{12})} \tag{5.20}$$

Case $I^{trifund}_{112}$. The constraints are

$$\begin{aligned} e^{i\pi/2(n_1 - n_2)} c^{n_1 n_2} &= c^{n_1 n_2} \\ e^{i\pi n_1} c^{n_2 n_1} &= -c^{n_1 n_2} \end{aligned}$$

which suggests

$$F_{112} = 8\frac{t^{12}}{(1-t^6)(1-t^{12})} + 8\frac{t^6}{1-t^{12}} = 8\frac{t^6}{(1-t^6)(1-t^{12})}$$

and so

$$I_{112}^{trifund} = F_{112}(t^2 - t^4) = 8\frac{t^6(t^2 - t^4)}{(1 - t^6)(1 - t^{12})} \tag{5.21}$$

Case $I_{113}^{trifund}$. The constraints are

$$\begin{aligned} e^{i\pi/2(n_1 - n_2)} c^{n_1 n_2} &= -c^{n_1 n_2} \\ e^{i\pi n_1} c^{n_2 n_1} &= c^{n_1 n_2} \end{aligned}$$

The first constraint implies

$$n_1 - n_2 = 2 \mod 4$$

and the second one tells us to sum over all n_1, n_2 pairs. This gives

$$F_{113} = 8\frac{t^6}{(1 - t^6)(1 - t^{12})} = F_{112}$$

and

$$I_{113}^{trifund} = 8F_{113}(t^2 - t^4) = 8\frac{t^6(t^2 - t^4)}{(1 - t^6)(1 - t^{12})} \tag{5.22}$$

Case $I_{114}^{trifund}$. The constraints are

$$\begin{aligned} e^{i\pi/2(n_1 - n_2)} c^{n_1 n_2} &= -c^{n_1 n_2} \\ e^{i\pi n_1)} c^{n_2 n_1} &= -c^{n_1 n_2} \end{aligned}$$

Comparing the 114 constraints with the 113 constraints, we see that they have the same solutions, so

$$I_{114}^{trifund} = I_{113}^{trifund} = 8\frac{t^6(t^2 - t^4)}{(1 - t^6)(1 - t^{12})} \tag{5.23}$$

This completes the computation for when all three Wilson lines are abelian. Now, we consider the case when one/two/three of the Wilson lines in the triplet are the 5th Wilson line. First, consider the case where there is only one non-abelian Wilson line. Similar to the argument we had previously, there are only 4 independent cases to consider: 115, 125, 135, 145.

Case $I_{115}^{trifund}$. Now, the $SU(2)$ index of the third component of the trifundamental field matters, because the nonabelian Wilson line does not act diagonally anymore. This consideration leads to the most general operator as in

$$\sum c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu$$

where $\mu = \pm 1$ is the third component of the trifundamental index. We do not show the first two components for clarity. The constraints are

$$\sum e^{i\pi/2(n_1-n_2)} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu = \sum e^{\mu\pi i/2} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu$$
$$\sum e^{i\pi n_2} c^{\mu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_\mu = \sum (-1)^{(\mu-1)/2} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu}$$

which leads to the equations

$$e^{i\pi/2(n_1-n_2)} c^{\mu n_1 n_2} = e^{\mu\pi i/2} c^{\mu n_1 n_2}$$
$$e^{i\pi n_1} c^{-\mu, n_2 n_1} = (-1)^{(\mu-1)/2} c^{\mu n_1 n_2}$$

For $\mu = 1$, The first constraint implies

$$n_1 - n_2 = 1 \mod 4$$

and the second constraint tells us to sum over all n_1, n_2 pairs, which would take care of the case $\mu = -1$. Therefore, summing over the Boltzmann weights, we get

$$F_{115} = 4 \frac{t^3 + t^9}{(1-t^6)(1-t^{12})}$$

and

$$I_{115}^{trifund} = F_{115}(t^2 - t^4) = 4 \frac{(t^3+t^9)(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.24}$$

where the 4 comes from 2^2 choices of the first two $SU(2)$ trifundamental indices. The factor $t^2 - t^4$ comes from the contribution of the BPS base fields.

Case $I_{125}^{trifund}$. The constraints are

$$\sum e^{i\pi/2(n_1-n_2)} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu = \sum e^{\mu\pi i/2} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu$$
$$\sum e^{i\pi n_2} c^{\mu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_\mu = -\sum (-1)^{(\mu-1)/2} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu}$$

The only difference from the previous 115 case is the minus sign for the s condition, so

$$I_{125}^{trifund} = I_{115}^{trifund} = 4 \frac{(t^3+t^9)(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.25}$$

Case $I_{135}^{trifund}$. The constraints are

$$\sum e^{i\pi/2(n_1-n_2)} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu = -\sum e^{\mu\pi i/2} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_\mu$$
$$\sum e^{i\pi n_2} c^{\mu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_\mu = \sum (-1)^{(\mu-1)/2} c^{\mu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu}$$

Now, the $\mu = 1$ case leads to

$$n_1 - n_2 = 3 \mod 4$$

and it is easy to see that

$$I_{135}^{trifund} = I_{115}^{trifund} = 4\frac{(t^3+t^9)(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.26}$$

By the same logic, we have

$$I_{145}^{trifund} = I_{115}^{trifund} = 4\frac{(t^3+t^9)(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.27}$$

This completes the computation for when only one of the three Wilson lines is nonabelian. We now consider the four cases where two of the Wilson lines are nonabelian. The four cases are: 155, 255, 355, and 455.

Case $I_{155}^{trifund}$. Now, the second and the third $SU(2)$ trifundamental indices matter, since the Wilson lines will act nondiagonally on them separately. So the general operator combination is

$$\sum c^{\mu\nu n_1 n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{\mu\nu}$$

where we do not show the first $SU(2)$ trifundamental index.

The constraints are

$$\begin{aligned}\sum e^{\pi i/2(n_1-n_2)}c^{\mu\nu n_1 n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{\mu\nu} &= \sum e^{\pi(\mu+\nu)i/2}c^{\mu\nu n_1 n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{\mu\nu}\\ \sum e^{\pi i n_2}c^{\mu\nu n_1 n_2}\partial_{++}^{n_2}\partial_{+-}^{n_1}\Phi_{\mu\nu} &= \sum(-1)^{(\mu+\nu)/2-1}c^{\mu\nu n_1 n_2}\partial_{++}^{n_1}\partial_{+-}^{n_2}\Phi_{-\mu,-\nu}\end{aligned}$$

from which we have

$$\begin{aligned}e^{\pi i/2(n_1-n_2)}c^{\mu\nu n_1 n_2} &= e^{\pi(\mu+\nu)i/2}c^{\mu\nu n_1 n_2}\\ e^{\pi i n_1}c^{-\mu,-\nu,n_2 n_1} &= (-1)^{(\mu+\nu)/2-1}c^{\mu\nu n_1 n_2}\end{aligned}$$

Case 1: $\mu = \nu = 1$. The first constraint implies

$$n_1 - n_2 = 2 \mod 4$$

and the second constraint says that summing over all such pairs will help us take care of the case $\mu = \nu = -1$ case. The Boltzmann weight sum is

$$F_{11} = 2\frac{2t^6}{(1-t^6)(1-t^{12})}$$

where the prefactor 2 comes from 2 choices of the first $SU(2)$ trifundamental index.

Case 2: $\mu = 1, \nu = -1$. The first constraint implies

$$n_1 - n_2 = 0 \mod 4$$

and the second constraint says that summing over all such pairs will help us take care of the case $\mu = -1, \nu = 1$ case. The Boltzmann weight sum is

$$F_{1,-1} = 2\frac{1+t^{12}}{(1-t^6)(1-t^{12})}$$

Adding these up, we have

$$I_{155}^{trifund} = (F_{11} + F_{1,-1})(t^2 - t^4) = \frac{2(1+2t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.28}$$

Case $I_{255}^{trifund}$. The constraints are

$$\begin{aligned}\sum e^{\pi i/2(n_1-n_2)} c^{\mu\nu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu} &= \sum e^{\pi(\mu+\nu)i/2} c^{\mu\nu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu} \\ \sum e^{\pi i n_2} c^{\mu\nu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_{\mu\nu} &= -\sum (-1)^{(\mu+\nu)/2-1} c^{\mu\nu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu,-\nu}\end{aligned}$$

The only difference from the previous 155 case is the minus sign in the second equation, but this does not affect the number of independent solutions. This suggests that

$$I_{255}^{trifund} = I_{155}^{trifund} = \frac{2(1+2t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.29}$$

Case $I_{355}^{trifund}$. The constraints are

$$\begin{aligned}\sum e^{\pi i/2(n_1-n_2)} c^{\mu\nu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu} &= -\sum e^{\pi(\mu+\nu)i/2} c^{\mu\nu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu} \\ \sum e^{\pi i n_2} c^{\mu\nu n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_{\mu\nu} &= \sum (-1)^{(\mu+\nu)/2-1} c^{\mu\nu n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu,-\nu}\end{aligned}$$

from which we have

$$\begin{aligned}e^{\pi i/2(n_1-n_2)} c^{\mu\nu n_1 n_2} &= -e^{\pi(\mu+\nu)i/2} c^{\mu\nu n_1 n_2} \\ e^{\pi i n_1} c^{-\mu,-\nu,n_2 n_1} &= (-1)^{(\mu+\nu)/2-1} c^{\mu\nu n_1 n_2}\end{aligned}$$

Case 1: $\mu = \nu = 1$. The first constraint implies

$$n_1 - n_2 = 0 \mod 4$$

Case 2: $\mu = 1, \nu = -1$. The first constraint implies

$$n_1 - n_2 = 2 \mod 4$$

But these are the same constraints as the 155 case, except the order is different. Therefore,

$$I_{355}^{trifund} = I_{155}^{trifund} = \frac{2(1+2t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.30}$$

By the same logic,

$$I_{455}^{trifund} = I_{155}^{trifund} = \frac{2(1+2t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.31}$$

In summary, we have the result that, for nonabelian Wilson lines that appear twice in the triplet,

$$I_{155}^{trifund} = I_{255}^{trifund} = I_{355}^{trifund} = I_{455}^{trifund} = \frac{2(1+2t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} \tag{5.32}$$

Finally, we need to attack the last case, Wilson line 555. A general operator is

$$\sum c^{\mu\nu\rho n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu\rho}$$

where now all three trifundamental $SU(2)$ indices ($\mu, \nu, \rho = \pm 1$) matter. The constraints are

$$\begin{aligned}\sum e^{\pi i/2(n_1-n_2)} c^{\mu\nu\rho n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu\rho} &= \sum e^{\pi i/2(\mu+\nu+\rho)} c^{\mu\nu\rho n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{\mu\nu\rho} \\ \sum e^{\pi i n_2} c^{\mu\nu\rho n_1 n_2} \partial_{++}^{n_2} \partial_{+-}^{n_1} \Phi_{\mu\nu\rho} &= \sum (-1)^{(\mu+\nu+\rho-3)/2} c^{\mu\nu\rho n_1 n_2} \partial_{++}^{n_1} \partial_{+-}^{n_2} \Phi_{-\mu,-\nu,-\rho}\end{aligned}$$

from which we have

$$\begin{aligned}e^{\pi i/2(n_1-n_2)} c^{\mu\nu\rho n_1 n_2} &= e^{\pi i/2(\mu+\nu+\rho)} c^{\mu\nu\rho n_1 n_2} \\ e^{\pi i n_1} c^{-\mu,-\nu,-\rho n_2 n_1} &= (-1)^{(\mu+\nu+\rho-3)/2} c^{\mu\nu\rho n_1 n_2}\end{aligned}$$

Case 1: $\mu = 1, \nu = 1, \rho = 1$. In this case we have

$$n_1 - n_2 = 3 \mod 4$$

and the sum yields the Boltzamnn weight sum

$$F_{+++} = \frac{t^3+t^9}{(1-t^6)(1-t^{12})}$$

Case 2: $\mu = 1, \nu = 1, \rho = -1$. In this case we have

$$n_1 - n_2 = 1 \mod 4$$

and the sum yields

$$F_{++-} = \frac{t^3 + t^9}{(1-t^6)(1-t^{12})} = F_{+++}$$

Case 3: $\mu = 1, \nu = -1, \rho = 1$. In this case we have

$$n_1 - n_2 = 1 \mod 4$$

and the sum yields

$$F_{+-+} = \frac{t^3 + t^9}{(1-t^6)(1-t^{12})} = F_{+++}$$

Case 4: $\mu = 1, \nu = -1, \rho = -1$. In this case we have

$$n_1 - n_2 = 3 \mod 4$$

and the sum yields

$$F_{+--} = \frac{t^3 + t^9}{(1-t^6)(1-t^{12})} = F_{+++}$$

Adding these up, and including the $t^2 - t^4$ contribution from the BPS base fields, we have

$$I_{555}^{trifund} = 4\frac{(t^3 + t^9)(t^2 - t^4)}{(1-t^6)(1-t^{12})} \tag{5.33}$$

We now have all the ingredients we need to compute the supersymmetric Casimir energy.

Supersymmetric Casimir Energy Computation

When we sum over the index to compute the Casimir energy, we need to multiply the vector multiplet contribution by 1/2, since each vector multiplet is shared between two trinions. For example, for the triplet 111, we have

$$\begin{aligned} I_{111} &= I_{111}^{trifund} + \frac{3}{2} I_1^{vec} \\ &= 8\frac{(1-t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} + \frac{3}{2}\frac{3(1-t^6+t^{12})(t^2-t^4+2t^6) - 3(t^3+t^9)t^3}{(1-t^6)(1-t^{12})} \\ &= -\frac{1}{18\beta} + \frac{9}{2} - \frac{35\beta}{54} + O(\beta^2) \end{aligned}$$

which suggests that the supersymmetric Casimir energy is

$$E_{111} = \frac{35}{108} \tag{5.34}$$

It is easy to see that

$$E_{111} = E_{122} = E_{133} = E_{144} = E_{234} = \frac{35}{108} \tag{5.35}$$

since these Wilson lines have the same index contributions as we computed in the previous subsection.

For other Wilson lines that do not involve Wilson line 5:

$$\begin{aligned} I_{112} &= I_{112}^{trifund} + \frac{3}{2} I_1^{vec} \\ &= 8\frac{t^6(t^2-t^4)}{(1-t^6)(1-t^{12})} + \frac{3}{2}\frac{3(1-t^6+t^{12})(t^2-t^4+2t^6)-3(t^3+t^9)t^3}{(1-t^6)(1-t^{12})} \\ &= -\frac{1}{18\beta} + \frac{9}{2} - \frac{251\beta}{54} + O(\beta^2) \end{aligned} \tag{5.36}$$

$$\begin{aligned} I_{114} = I_{113} &= I_{113}^{trifund} + \frac{3}{2} I_1^{vec} \\ &= 8\frac{t^6(t^2-t^4)}{(1-t^6)(1-t^{12})} + \frac{3}{2}\frac{3(1-t^6+t^{12})(t^2-t^4+2t^6)-3(t^3+t^9)t^3}{(1-t^6)(1-t^{12})} \\ &= -\frac{1}{18\beta} + \frac{9}{2} - \frac{251\beta}{54} + O(\beta^2) \end{aligned} \tag{5.37}$$

These show that

$$E_{112} = E_{113} = E_{114} = E_{222} = E_{332} = E_{442} = E_{134} = \frac{251}{108} \tag{5.38}$$

$$E_{224} = E_{334} = E_{444} = E_{223} = E_{333} = E_{443} = E_{123} = E_{124} = \frac{251}{108} \tag{5.39}$$

For Wilson lines that involve a single nonabelian Wilson line, we have

$$\begin{aligned} I_{115} &= I_{115}^{trifund} + I_1^{vec} + \frac{1}{2} I_5^{vec} \\ &= 4\frac{(t^3+t^9)(t^2-t^4)}{(1-t^6)(1-t^{12})} + \frac{3(1-t^6+t^{12})(t^2-t^4+2t^6)-3(t^3+t^9)t^3}{(1-t^6)(1-t^{12})} + \frac{1}{2}3\frac{-t^6+t^8-t^{10}+t^{12}}{(1-t^6)(1-t^{12})} \\ &= -\frac{1}{18\beta} + 3 - \frac{143\beta}{54} + O(\beta^2) \end{aligned} \tag{5.40}$$

which shows that

$$E_{115} = E_{225} = E_{335} = E_{445} = \frac{143}{108} \tag{5.41}$$

Similarly, one can show that $I_{125} = I_{135} = I_{145} = I_{115}$, so

$$E_{ab5} = E_{115} = \frac{143}{108} \tag{5.42}$$

where $a, b \in \{1, 2, 3, 4\}$. Wilson line triplets that involve only one nonabelian Wilson line have the same Casimir energy.

Now, consider Wilson line triplets that contain two nonabelian Wilson lines.

$$\begin{aligned} I_{155} &= I_{155}^{trifund} + \frac{1}{2} I_1^{vec} + I_5^{vec} \\ &= \frac{2(1+2t^6+t^{12})(t^2-t^4)}{(1-t^6)(1-t^{12})} + \frac{1}{2}\frac{3(1-t^6+t^{12})(t^2-t^4+2t^6)-3(t^3+t^9)t^3}{(1-t^6)(1-t^{12})} + \\ &+ 3\frac{-t^6+t^8-t^{10}+t^{12}}{(1-t^6)(1-t^{12})} \\ &= -\frac{1}{18\beta} + \frac{3}{2} - \frac{35\beta}{54} + O(\beta^2) \end{aligned} \tag{5.43}$$

What is surprising (and also in a sense expected) is that

$$E_{155} = \frac{35}{108} = E_{111} \tag{5.44}$$

From the previous section, we see that

$$E_{255} = E_{355} = E_{455} = E_{155} = \frac{35}{108} \tag{5.45}$$

The result we have so far, ranked from lowest energy to highest, is

- $E_0 = 35/108$: 111, 122, 133, 144, 234, 155, 255, 355, 455. These lead to 28 states, and they correspond to the states in the dual $SO(8)$ Chern-Simons theory at level 2.

- $E_1 = 143/108 = E_0 + 1$: $ab5$, $a, b \in \{1, 2, 3, 4\}$. These lead to 48 states.

- $E_2 = 251/108 = E_0{+}2$: 112, 113, 114, 332, 442, 222, 134,224,334,223,443,333,444,123, 124 These lead to 48 states.

There is a single state, 555, that is left to consider.

$$\begin{aligned} I_{555} &= I_{555}^{trifund} + \frac{3}{2} I_5^{vec} \\ &= 4\frac{(t^3+t^9)(t^2-t^4)}{(1-t^6)(1-t^{12})} + \frac{9}{2}\frac{-t^6+t^8-t^{10}+t^{12}}{(1-t^6)(1-t^{12})} \\ &= -\frac{1}{18\beta} + \frac{19}{54}\beta + O(\beta^2) \end{aligned} \tag{5.46}$$

The supersymmetric Casimir energy is negative compared to the 111 trivial ground state:

$$E_{555} = -\frac{19}{108} = E_0 - \frac{1}{2} \tag{5.47}$$

The treatment for this statement is discussed in the previous section.

5.4 A Non-Lagrangian Example

At the end of section 5.2, we gave the example of counting the ground states of the $SU(2)$ theory coming from compactifying the two M5 branes on a genus-2 Riemann surface. The resulting four-dimensional theory was put on $S^3\mathbb{Z}_2$. This theory has an explicit Lagrangian description. In the previous two sections, we also focused on computing the ground states of the Lagrangian $SU(2)$ theory, albeit on a different geometry S^3/Dic_2. In this section, we explicitly list the ground states of a non-Lagrangian $SU(N)$ theory on the same genus-2 Riemann surface (see figure 5.1)) and with the same ADE geometry $S^3/\mathbb{Z}_2$. The classically flat Wilson lines are simply N by N matrices with 1 and -1 on the diagonal. The number of -1s must be even in order to satisfy the determinant constraint. Therefore, up to Weyl group, there are in total $\lfloor N/2 \rfloor + 1$ classically flat Wilson lines. Let $a, b, c \in SU(N)$ be the classically flat Wilson lines in the trifundamental theory as in figure 5.1. We claim that the true ground states of this theory are of the form

$$a \otimes b \otimes c \tag{5.48}$$

where a, b, c satisfies the equation

$$abc = \mathbf{1} \tag{5.49}$$

up to Weyl permutation. The right hand side is the N by N identity matrix and the left hand side is matrix multiplication. Why is this true? According to the duality and the conjecture, the ground states of this theory must satisfy the $SU(N)_2$ fusion rule, which, by the level-rank duality, is the $SU(2)_N$ fusion rule.

We claim that equation (5.49) encodes exactly this information. Here's a quick proof. We denote a classically flat Wilson line by j_k where k is the number of -1s along the diagonal. By assumption, $0 \leq k \leq N$ and k is even. We ask the question: if we multiply two such matrices together, say j_l and j_m, what can the solution be? Remember that we are free to permute the diagonal because of the Weyl symmetry. To answer this question, we first notice that the least number of -1s we can get in the resulting product is by overlapping as many -1s from j_l as possible with -1s from j_m. Without loss of generality, let us assume $l > m$. It is not hard to see that the least number of -1s is

$$l - m$$

which means that j_{l-m} is a solution of the product $j_l \cdot j_m$. For example, setting $N = 5$, $l = 4$, and $m = 2$, we have, for the least number of -1s:

$$j_4 \cdot j_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = j_2$$

What is the maximum number of -1s we can get from multiplying j_l and j_m? Starting from the configurations that yield the least number of -1s in the product, we can move one -1 from j_m to a position so that the corresponding position in j_n is a $+1$. This changes the number of -1 in the result by 2, yielding j_{l-m+2}. In our example, this action is illustrated by shifting the second -1 in j_2 to the last slot on the diagonal:

$$j_4 {\cdot} j_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = j_4$$

We can keep doing this, until we reach the result that has the most number of -1s in it. What is this number? There are two cases to consider. In the first matrix, there are $N - l$ slots that contain $+1$. If we can manage to shift all -1s in the second matrix to those $N - l$ slots, the resulting product will contain $l + m$ -1s. Therefore, if

$$N - l \geq m, \quad \text{or} \quad m + l \leq N$$

the last state in the sequence will be j_{m+l}. The second case is where this condition doesn't hold, namely

$$m + l > N$$

In this case, there will be some overlaps of -1s for the two matrices. The number of overlaps is $l + m - n$, which suggests that the resulting product has $N - (l + m - N) = 2N - (l + m)$ -1s, giving

$$j_{2N-(l+m)}$$

as the final states. The two cases can be summarized in one line: the last state contains

$$\min(m+l, 2N-m-l)$$

-1s. Therefore, the multiplication rule that we have is

$$j_l \otimes j_m = j_{l-m} \oplus j_{l-m+2} \oplus \ldots \oplus j_{\min(m+l,2N-m-l)}$$

which is exactly the $SU(2)_N$ fusion rule [78]. This shows that the ground states of the non-Lagrangian $SU(N)$ theory on Σ and $\mathbb{Z}_2$ singularity can be listed exactly following the rule in equation 5.49.

Aside from the reason from the fusion rule, we can give another reason for why the rule 5.49 gives the true ground states. First, we know that the trivial Wilson line triplet 111 must be a true ground state. The question is what other states share the same supersymmetry Casimir energy as the trivial ground state 111. This can happen when the single letter superconformal index of a state matches that of the trivial ground state. The superconformal index comes from two parts, the vector multiplet and the vector multiplet. However, due to the special form of the classical ground states in this problem (all diagonal elements of the Wilson lines are ± 1), the vector multiplet contributes to the same index for each Wilson line triplet. To compute the hypermultiplet contribution to the index, we again use the familiar method of imposing the constraint as in section 4.9. In the constraint equation, the only thing that might be different for different Wilson line triplets is the action from the Wilson lines. For the trivial ground state 111, the action of the Wilson lines is trivial, so if a Wilson line triplet abc leads to the trivial Wilson line action, it must have the same superconformal index as the trivial ground state 111, and hence the same supersymmetric Casimir energy[2]. For single letters, the Wilson line action is simply given by multiplying abc up to Weyl permutation, so if $abc = \mathbf{1}$ up to Weyl permutation, then abc must share the same index as the trivial ground state and hence the same supersymmetric Casimir energy. Therefore, these abc are the true ground states of the theory.

5.5 Conclusion

In this dissertation, we proposed a way to find the action of S-duality on ground states of $\mathcal{N} = 4$ super Yang-Mills theory on S^3/Γ where Γ is a discrete subgroup of $SU(2)$.

[2]There are two loopholes of this statement. One is that, it could happen that two Wilson line triplets, having different Wilson line actions, lead to the same supersymmetric index. Another is that, two different supersymmetric indices may lead to the same supersymmetric Casimir energy, since the only requirement that they lead to the same Casimir energy is a match in the $O(\beta)$ term in the expansion.

The solution is, through a mini AdS/CFT argument, to identify the SYM ground states with the states of a certain Chern-Simons theory on T^2. We gave support to this duality by using a top-down brane construction, explicit state counting, and a one-loop computation using the superconformal index. We also generalized our duality argument to $\mathcal{N} = 2$ theories and uncovered some surprising result regarding the ground state structure of $\mathcal{N} = 2$ theories. In particular, the ground state of the conformal $\mathcal{N} = 2$ four-flavor $SU(2)$ theory on S^3/Γ is nondegenerate, contrary to conventional wisdom that the ground states can be any classically flat Wilson lines furnishing a representation of Γ. We also looked at other $\mathcal{N} = 2$ theories by compactifying the M5 branes on a genus two Riemann surface. We found, by explicit calculations, that the allowed Wilson line triplets are in one-to-one correspondence with nonvanishing fusion rule coefficient N_{ijk}.

Finally, we say a few words on where this work might go in the future. First, it is interesting to generalize the SYM gauge group from $SU(q)$ and $U(q)$ to other classical gauge groups such as $SO(q)$ and $Sp(2q)$. This will presumably involve putting an orientifold along the D3 branes in the brane construction in section 2.4. Second, we might be able to give a physics derivation of the current algebra fusion rules. For example, we could compute the superconformal index for the Lagrangian $SU(2)$ trinion theory on S^3/Dic_k for all k. By counting what Wilson line triplets shares the same supersymmetric Casimir energy as the trivial ground state, we could presumably obtain the level 2 fusion rule of $SO(2(k+2))$ for states that also lie on the $SO(2(k+2))$ root lattice. Last but not least, it will be interesting to compute explicitly the S-duality matrices for nonabelian SYM gauge theories using the SYM theory only as we did in 2.5 for the $U(1)$ theory and compare with the Chern-Simons result.

Bibliography

[1] E. T. Jaynes. *Probability Theory: The Logic of Science*. Ed. by G. LarryEditor Bretthorst. Cambridge University Press, 2003. DOI: 10.1017/CBO9780511790423.

[2] Joseph Polchinski. *String Theory*. Vol. 1. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998. DOI: 10.1017/CBO9780511816079.

[3] John Cardy. *Scaling and Renormalization in Statistical Physics*. Cambridge Lecture Notes in Physics. Cambridge University Press, 1996. DOI: 10.1017/CBO9781316036440.

[4] Steven Weinberg. *The quantum theory of fields. Vol. 3: Supersymmetry*. Cambridge University Press, June 2013. ISBN: 978-0-521-67055-5, 978-1-139-63263-8, 978-0-521-67055-5.

[5] Michael Dine, Willy Fischler, and Mark Srednicki. "Supersymmetric Technicolor". In: *Nucl. Phys. B* 189 (1981). Ed. by J. P. Leveille, L. R. Sulak, and D. G. Unger, pp. 575–593. DOI: 10.1016/0550-3213(81)90582-4.

[6] Ian Affleck, Michael Dine, and Nathan Seiberg. "Dynamical Supersymmetry Breaking in Supersymmetric QCD". In: *Nucl. Phys. B* 241 (1984), pp. 493–534. DOI: 10.1016/0550-3213(84)90058-0.

[7] Edward Witten. "Dynamical Breaking of Supersymmetry". In: *Nucl. Phys. B* 188 (1981), p. 513. DOI: 10.1016/0550-3213(81)90006-7.

[8] N. Seiberg. "Electric - magnetic duality in supersymmetric nonAbelian gauge theories". In: *Nucl. Phys. B* 435 (1995), pp. 129–146. DOI: 10.1016/0550-3213(94)00023-8. arXiv: hep-th/9411149.

[9] N. Seiberg and Edward Witten. "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory". In: *Nucl. Phys. B* 426 (1994). [Erratum: Nucl.Phys.B 430, 485–486 (1994)], pp. 19–52. DOI: 10.1016/0550-3213(94)90124-4. arXiv: hep-th/9407087.

[10] Davide Gaiotto. "N=2 dualities". In: *JHEP* 08 (2012), p. 034. DOI: 10.1007/JHEP08(2012)034. arXiv: 0904.2715 [hep-th].

[11] Yuji Tachikawa. *N=2 supersymmetric dynamics for pedestrians*. Dec. 2013. DOI: 10.1007/978-3-319-08822-8. arXiv: 1312.2684 [hep-th].

[12] Juan Martin Maldacena. “The Large N limit of superconformal field theories and supergravity”. In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 231–252. DOI: 10.4310/ATMP.1998.v2.n2.a1. arXiv: hep-th/9711200.

[13] C. Montonen and David I. Olive. “Magnetic Monopoles as Gauge Particles?” In: *Phys. Lett. B* 72 (1977), pp. 117–120. DOI: 10.1016/0370-2693(77)90076-4.

[14] Ashoke Sen. “Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and SL(2,Z) invariance in string theory”. In: *Phys. Lett. B* 329 (1994), pp. 217–221. DOI: 10.1016/0370-2693(94)90763-3. arXiv: hep-th/9402032.

[15] Cumrun Vafa and Edward Witten. “A Strong coupling test of S duality”. In: *Nucl. Phys. B* 431 (1994), pp. 3–77. DOI: 10.1016/0550-3213(94)90097-3. arXiv: hep-th/9408074.

[16] M. Bershadsky et al. “Topological reduction of 4-d SYM to 2-d sigma models”. In: *Nucl. Phys. B* 448 (1995), pp. 166–186. DOI: 10.1016/0550-3213(95)00242-K. arXiv: hep-th/9501096.

[17] Jeffrey A. Harvey, Gregory W. Moore, and Andrew Strominger. “Reducing S duality to T duality”. In: *Phys. Rev. D* 52 (1995), pp. 7161–7167. DOI: 10.1103/PhysRevD.52.7161. arXiv: hep-th/9501022.

[18] Anton Kapustin. “Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality”. In: *Phys. Rev. D* 74 (2006), p. 025005. DOI: 10.1103/PhysRevD.74.025005. arXiv: hep-th/0501015.

[19] Anton Kapustin and Edward Witten. “Electric-Magnetic Duality And The Geometric Langlands Program”. In: *Commun. Num. Theor. Phys.* 1 (2007), pp. 1–236. DOI: 10.4310/CNTP.2007.v1.n1.a1. arXiv: hep-th/0604151.

[20] Davide Gaiotto and Edward Witten. “S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory”. In: *Adv. Theor. Math. Phys.* 13.3 (2009), pp. 721–896. DOI: 10.4310/ATMP.2009.v13.n3.a5. arXiv: 0807.3720 [hep-th].

[21] John McKay. “Graphs, Singularities, and Finite Groups”. In: *Proc. Symp. Pure Math.* 37 (1980).

[22] Edward Witten. “Some comments on string dynamics”. In: *STRINGS 95: Future Perspectives in String Theory*. July 1995, pp. 501–523. arXiv: hep-th/9507121.

[23] Robbert Dijkgraaf et al. “Supersymmetric gauge theories, intersecting branes and free fermions”. In: *JHEP* 02 (2008), p. 106. DOI: 10.1088/1126-6708/2008/02/106. arXiv: 0709.4446 [hep-th].

[24] Edward Witten. "Geometric Langlands From Six Dimensions". In: (May 2009). arXiv: 0905.2720 [hep-th].

[25] Edward Witten. "Quantum Field Theory and the Jones Polynomial". In: *Commun. Math. Phys.* 121 (1989). Ed. by Asoke N. Mitra, pp. 351–399. DOI: 10.1007/BF01217730.

[26] Ori J. Ganor and Yoon Pyo Hong. "Selfduality and Chern-Simons Theory". In: (Dec. 2008). arXiv: 0812.1213 [hep-th].

[27] Frank Wilczek. "Magnetic Flux, Angular Momentum, and Statistics". In: *Phys. Rev. Lett.* 48 (1982), pp. 1144–1146. DOI: 10.1103/PhysRevLett.48.1144.

[28] Frank Wilczek and A. Zee. "Linking Numbers, Spin, and Statistics of Solitons". In: *Phys. Rev. Lett.* 51 (1983), pp. 2250–2252. DOI: 10.1103/PhysRevLett.51.2250.

[29] A. Zee. "Quantum Hall fluids". In: *Lect. Notes Phys.* 456 (1995). Ed. by H. B. Geyer, pp. 99–153. DOI: 10.1007/BFb0113369. arXiv: cond-mat/9501022.

[30] R. B. Laughlin. "Anomalous quantum Hall effect: An Incompressible quantum fluid with fractionallycharged excitations". In: *Phys. Rev. Lett.* 50 (1983), p. 1395. DOI: 10.1103/PhysRevLett.50.1395.

[31] Shmuel Elitzur et al. "Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory". In: *Nucl. Phys. B* 326 (1989), pp. 108–134. DOI: 10.1016/0550-3213(89)90436-7.

[32] Edward Witten. "AdS / CFT correspondence and topological field theory". In: *JHEP* 12 (1998), p. 012. DOI: 10.1088/1126-6708/1998/12/012. arXiv: hep-th/9812012.

[33] Ori J. Ganor, Hao-Yu Sun, and Nesty R. Torres-Chicon. "Double-Janus Linear Sigma Models and Generalized Reciprocity for Gauss Sums". In: *JHEP* 05 (2021), p. 227. DOI: 10.1007/JHEP05(2021)227. arXiv: 1912.11471 [hep-th].

[34] E. Ehrhart. "Sur les polyèdres rationnels homothétiques à n dimensions". In: *Comptes rendus de l'Académie des Sciences* 254 (1962), pp. 616–618.

[35] Po-Shen Hsin, Anton Kapustin, and Ryan Thorngren. "Berry Phase in Quantum Field Theory: Diabolical Points and Boundary Phenomena". In: *Phys. Rev. B* 102 (2020), p. 245113. DOI: 10.1103/PhysRevB.102.245113. arXiv: 2004.10758 [cond-mat.str-el].

[36] Anton Kapustin and Lev Spodyneiko. "Higher-dimensional generalizations of Berry curvature". In: *Phys. Rev. B* 101.23 (2020), p. 235130. DOI: 10.1103/PhysRevB.101.235130. arXiv: 2001.03454 [cond-mat.str-el].

[37] Erik P. Verlinde. “Fusion Rules and Modular Transformations in 2D Conformal Field Theory”. In: *Nucl. Phys. B* 300 (1988), pp. 360–376. DOI: 10.1016/0550-3213(88)90603-7.

[38] William Fulton and Joe Harris. *Representation Theory: A First Course*. Springer-Verlag, Jan. 1991. ISBN: 978-0-387-97495-8, 978-1-4612-0979-9, 978-3-54-000539-1, 978-0-387-97527-6.

[39] Lance J. Dixon et al. “Strings on Orbifolds”. In: *Nucl. Phys. B* 261 (1985). Ed. by B. Schellekens, pp. 678–686. DOI: 10.1016/0550-3213(85)90593-0.

[40] Paul S. Aspinwall. “K3 surfaces and string duality”. In: *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*. Nov. 1996, pp. 421–540. arXiv: hep-th/9611137.

[41] Clifford V. Johnson and Robert C. Myers. “Aspects of type IIB theory on ALE spaces”. In: *Phys. Rev. D* 55 (1997), pp. 6382–6393. DOI: 10.1103/PhysRevD.55.6382. arXiv: hep-th/9610140.

[42] Michael R. Douglas and Gregory W. Moore. “D-branes, quivers, and ALE instantons”. In: (Mar. 1996). arXiv: hep-th/9603167.

[43] Duiliu-Emanuel Diaconescu, Michael R. Douglas, and Jaume Gomis. “Fractional branes and wrapped branes”. In: *JHEP* 02 (1998), p. 013. DOI: 10.1088/1126-6708/1998/02/013. arXiv: hep-th/9712230.

[44] Hiraku Nakajima. “Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras”. In: *Duke Math. J.* 76.2 (1994), pp. 365–416. DOI: 10.1215/S0012-7094-94-07613-8.

[45] Edward Witten. “String theory dynamics in various dimensions”. In: *Nucl. Phys. B* 443 (1995), pp. 85–126. DOI: 10.1016/0550-3213(95)00158-O. arXiv: hep-th/9503124.

[46] Victor G. Kac. *Infinite-Dimensional Lie Algebras*. 3rd ed. Cambridge University Press, 1990. DOI: 10.1017/CBO9780511626234.

[47] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Springer-Verlag, Jan. 1982. ISBN: 978-0-387-90613-3, 978-1-4757-3951-0, 978-1-4419-2815-3.

[48] Claude Itzykson. “Level one Kac-Moody characters and modular invariance”. In: *Nuclear Physics B - Proceedings Supplements* 5.2 (1988), pp. 150–165. ISSN: 0920-5632. DOI: https://doi.org/10.1016/0920-5632(88)90378-7. URL: https://www.sciencedirect.com/science/article/pii/0920563288903787.

[49] Jonathan J. Heckman and Luigi Tizzano. “6D Fractional Quantum Hall Effect”. In: *JHEP* 05 (2018), p. 120. DOI: 10.1007/JHEP05(2018)120. arXiv: 1708.02250 [hep-th].

[50] R. Borcherds et al. "Lie Groups". In: (2016).

[51] J. De Loera. " The many aspects of counting lattice points in polytopes". In: *Math. Semesterber.* 52 (Sept. 2005), pp. 175–195. DOI: 10.1007/s00591-005-0094-9.

[52] M. Beck and S. Robins. *Computing the Continuous Discretely*. Springer, Jan. 2015. ISBN: 978-1-4939-2969-6, 978-1-4939-2968-9, 978-1-4939-3858-2.

[53] P. A. MacMahon. *Combinatory Analysis*. American Mathematical Society, May 2001. ISBN: 978-0821828328.

[54] Chao Ju. "Chern-Simons Theory, Ehrhart Polynomials, and Representation Theory". In: (Apr. 2023). arXiv: 2304.11830 [math-ph].

[55] X. G. Wen and A. Zee. "Topological degeneracy of quantum Hall fluids". In: *Phys. Rev. B* 58 (1998), p. 15717. DOI: 10.1103/PhysRevB.58.15717. arXiv: cond-mat/9711223.

[56] George E. Andrews, Peter Paule, and Axel Riese. "MacMahon's Partition Analysis: The Omega Package". In: *European Journal of Combinatorics* 22.7 (2001), pp. 887–904. ISSN: 0195-6698. DOI: https://doi.org/10.1006/eujc.2001.0527. URL: https://www.sciencedirect.com/science/article/pii/S019566980190527X.

[57] E. B. Dynkin. "Semisimple Subalgebras of Semisimple Lie Algebras". In: *American Mathematical Society Translations* 30 (1952), pp. 349–462.

[58] Yangjiang Wei and Yi Ming Zou. "Inverses of Cartan matrices of Lie algebras and Lie superalgebras". In: *Linear Algebra and its Applications* 521 (May 2017), pp. 283–298. ISSN: 0024-3795. DOI: 10.1016/j.laa.2017.01.036. URL: http://dx.doi.org/10.1016/j.laa.2017.01.036.

[59] Edward Witten. "Constraints on Supersymmetry Breaking". In: *Nucl. Phys. B* 202 (1982), p. 253. DOI: 10.1016/0550-3213(82)90071-2.

[60] Christian Romelsberger. "Counting chiral primaries in N = 1, d=4 superconformal field theories". In: *Nucl. Phys. B* 747 (2006), pp. 329–353. DOI: 10.1016/j.nuclphysb.2006.03.037. arXiv: hep-th/0510060.

[61] Justin Kinney et al. "An Index for 4 dimensional super conformal theories". In: *Commun. Math. Phys.* 275 (2007), pp. 209–254. DOI: 10.1007/s00220-007-0258-7. arXiv: hep-th/0510251.

[62] Ofer Aharony et al. "3d dualities from 4d dualities". In: *JHEP* 07 (2013), p. 149. DOI: 10.1007/JHEP07(2013)149. arXiv: 1305.3924 [hep-th].

[63] Leonardo Rastelli and Shlomo S. Razamat. "The supersymmetric index in four dimensions". In: *J. Phys. A* 50.44 (2017), p. 443013. DOI: 10.1088/1751-8121/aa76a6. arXiv: 1608.02965 [hep-th].

[64] Abhijit Gadde. “Lectures on the Superconformal Index”. In: *J. Phys. A* 55.6 (2022), p. 063001. DOI: 10.1088/1751-8121/ac42ac. arXiv: 2006.13630 [hep-th].

[65] Shlomo S. Razamat and Brian Willett. “Global Properties of Supersymmetric Theories and the Lens Space”. In: *Commun. Math. Phys.* 334.2 (2015), pp. 661–696. DOI: 10.1007/s00220-014-2111-0. arXiv: 1307.4381 [hep-th].

[66] Francesco Benini, Tatsuma Nishioka, and Masahito Yamazaki. “4d Index to 3d Index and 2d TQFT”. In: *Phys. Rev. D* 86 (2012), p. 065015. DOI: 10.1103/PhysRevD.86.065015. arXiv: 1109.0283 [hep-th].

[67] Shlomo S. Razamat and Masahito Yamazaki. “S-duality and the N=2 Lens Space Index”. In: *JHEP* 10 (2013), p. 048. DOI: 10.1007/JHEP10(2013)048. arXiv: 1306.1543 [hep-th].

[68] Yuji Tachikawa. “A brief review of the 2d/4d correspondences”. In: *J. Phys. A* 50.44 (2017), p. 443012. DOI: 10.1088/1751-8121/aa5df8. arXiv: 1608.02964 [hep-th].

[69] Edward Witten. “Bound states of strings and p-branes”. In: *Nucl. Phys. B* 460 (1996), pp. 335–350. DOI: 10.1016/0550-3213(95)00610-9. arXiv: hep-th/9510135.

[70] Chao Ju. “Exact Degeneracy of Casimir Energy for $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory on ADE Singularities and S-Duality”. In: (Nov. 2023). arXiv: 2311.18223 [hep-th].

[71] Abhijit Gadde et al. “Gauge Theories and Macdonald Polynomials”. In: *Commun. Math. Phys.* 319 (2013), pp. 147–193. DOI: 10.1007/s00220-012-1607-8. arXiv: 1110.3740 [hep-th].

[72] A. Craw. “The McKay Correspondence and Birational Geometry”. In: *Sympletic Singularities in Geometry and Representation Theory* (2022).

[73] Edward Witten. “Solutions of four-dimensional field theories via M theory”. In: *Nucl. Phys. B* 500 (1997), pp. 3–42. DOI: 10.1016/S0550-3213(97)00416-1. arXiv: hep-th/9703166.

[74] P. Goddard, J. Nuyts, and David I. Olive. “Gauge Theories and Magnetic Charge”. In: *Nucl. Phys. B* 125 (1977), pp. 1–28. DOI: 10.1016/0550-3213(77)90221-8.

[75] Gerard ’t Hooft. “On the Phase Transition Towards Permanent Quark Confinement”. In: *Nucl. Phys. B* 138 (1978), pp. 1–25. DOI: 10.1016/0550-3213(78)90153-0.

[76] Gerard 't Hooft. "A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories". In: *Nucl. Phys. B* 153 (1979), pp. 141–160. DOI: 10.1016/0550-3213(79)90595-9.

[77] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. "Wall-crossing, Hitchin systems, and the WKB approximation". In: *Adv. Math.* 234 (2013), pp. 239–403. DOI: 10.1016/j.aim.2012.09.027. arXiv: 0907.3987 [hep-th].

[78] Philippe Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Springer Science & Business Media, 2012.