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On tubular neighbourhoods of manifolds. I

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1. *Introduction.* Let X be a submanifold of Y , in either the topological, smooth, or piecewise linear (= PL) categories. A *normal cell bundle* on X in Y is a bundle $\xi = (p, E, X)$ in the category whose fibre is a closed cell, and such that E is a neighbourhood of X in Y and $p: E \rightarrow X$ is a retraction. The triple (Y, X, ξ) is a *tubular neighbourhood*, or briefly, a *tube*. For convenience we may refer to a tube by its cell bundle.

Two tubes (Y_i, X_i, ξ_i) ($i=0, 1$), are *isomorphic* if there is an isomorphism of the category $h: (Y_0, X_0) \rightarrow (Y_1, X_1)$ inducing a bundle map from ξ_0 to ξ_1 . That is, h takes fibres of ξ_0 onto fibres of ξ_1 . If $Y_0 = Y_1 = Y$, the two tubes are *I-cobordant* if there exists a tube $(Y \times I, Z, \xi)$ such that Z is isomorphic to $X_0 \times I$, and $Z \cap (Y \times i) = X_i \times i$, and $\xi|_{X_i \times i} = \xi_i \times i$. A tube is *trivial* if its cell bundle is a trivial bundle.

The purpose of this article is to prove the following theorem.

THEOREM A. *Let (Σ^{11}, Σ^7) be the standard pair of PL spheres. There are PL trivial normal cell bundles ξ_0 and ξ_1 on Σ^7 such that the tubes $(\Sigma^{11}, \Sigma^7, \xi_0)$ and $(\Sigma^{11}, \Sigma^7, \xi_1)$ are not PL I-cobordant, nor are they topologically isomorphic.*

A consequence of Theorem A and Hudson's isotopy extension theorem ((5)) is:

COROLLARY B. *There is a PL embedding $f: \Sigma^7 \times I^4 \rightarrow \Sigma^7 \times I^4$ which leaves $\Sigma^7 \times 0$ fixed, and which is not PL isotopic to any embedding taking each fibre $y \times I^4$ onto a fibre $z \times I^4$.*

2. *Terminology.* Three categories will be used:

- (1) the *topological category* $\mathcal{T}$ of spaces and continuous maps;
- (2) the *piecewise linear category* $\mathcal{P}$ of polyhedra and piecewise linear maps;
- (3) the *smooth category* $\mathcal{S}$ smooth manifolds and smooth maps.

Euclidean n -space R^n , the closed half space E^n and the closed interval $I = [0, 1]$ may be considered as objects in each of the three categories. An n -manifold in one of the categories $\mathcal{T}$ or $\mathcal{P}$ is a paracompact object that is locally isomorphic to E^n . The n -disc is the closed unit ball $D^n \subset R^n$, and is an object in $\mathcal{T}$ and $\mathcal{S}$. Its boundary is $\partial D^n = S^{n-1}$. The n -cube I^n is the Cartesian product of n copies of $[-1, 1]$; its boundary is Σ^{n-1} . A *cell* is an isomorph of I^n or D^n , depending on the category, for some appropriate n . Similarly, an n -sphere is an isomorph of S^n or Σ^n .

Embedding, isotopy, etc., have their usual interpretations in each category. We assume submanifolds to be closed as subsets.

Fix one of the categories; let Y be an object and $p: E \rightarrow B$ a map. The triple (p, E, B) is a Y -bundle if it is locally trivial in the category, with fibre Y . That is, B is covered by

open sets U for each of which there exists an isomorphism of the category $p^{-1}U \rightarrow U \times Y$ making commutative the usual diagram

$$\begin{array}{ccc} p^{-1}U & \longrightarrow & U \times Y \\ & \searrow & \swarrow \\ & U & \end{array}$$

For the theory of such bundles, see (4).

The notions of *tubular neighbourhood* and of *isomorphism* and *I-cobordism* of tubular neighbourhoods, defined in section 1, make sense in each of the three categories. Another equivalence relation that may hold between two tubes (Y, X_i, ξ_i) ($i = 0, 1$), is that of *h-cobordism*: there exists an *h-cobordism* $Z \subset Y \times I$ between the manifolds $X_0 \times 0$ and $X_1 \times 1$ and a normal cell bundle ξ on Z reducing to $\xi_i \times i$ on $X_i \times i$.

Let (Y, X, ξ) be a tube and $h: Y \rightarrow Y'$ an embedding. There is a unique normal cell bundle $h(\xi)$ on $h(X)$ in $h(Y)$ making $h: (Y, X, \xi) \rightarrow (h(Y), h(X), h(\xi))$ an isomorphism of tubes.

Let (Y, X, ξ) be a smooth tube; let $\xi = (p, E, X)$. A *triangulation* of ξ is a *PL* cell bundle $p: \hat{E} \rightarrow \hat{X}$, where $\hat{E}$ and $\hat{X}$ are smooth triangulations of E and X . The triangulation $\hat{E}$ always extends to a smooth triangulation $\hat{Y}$ of Y ; the resulting *PL* tube $(\hat{Y}, \hat{X}, \hat{\xi})$ is called a *triangulation* of (Y, X, ξ) . It follows from the triangulation theorems for smooth bundles proved in (4) that triangulations of smooth tubes always exist, and are unique up to *PL* isomorphism. In this paper all that is needed is the existence of triangulations for trivial tubes, and this is obvious.

3. *Geometrical facts.* We list in this section various theorems in differential and piecewise linear topology that are needed to prove Theorem A.

PROPOSITION 1 (Whitehead ((11))). *Let C be a collapsible polyhedron in a piecewise linear manifold M^m . Then C has a regular neighbourhood, and any regular neighbourhood of C is a *PL* m -cell.*

PROPOSITION 2 (Munkres ((8))). *A smooth manifold which can be smoothly triangulated by a *PL* cell is a smooth cell.*

PROPOSITION 3 (Hirsch ((1,2))). *Let M^m be a smooth manifold with a fixed compatible *PL* structure. Let $X \subset M$ be a *PL* $m-1$ dimensional submanifold lying on the boundary of a *PL* m -dimensional submanifold. Let $A \subset X$ be a closed subset having a neighbourhood in X which is a smooth submanifold of M . Then there exists a smooth submanifold $X' \subset M$ which can be smoothly triangulated by X , and such that $X \cap X'$ is a neighbourhood of A in X .*

Let $M \subset S^q$ be a smooth submanifold. Call M *depressible* if there exists a smooth embedding $F: M \times I \rightarrow S^q \times I$ such that $F(M \times 0) = M \times 0$, and $F(M \times 1) \subset S^{q-1} \times 1$.

PROPOSITION 4 (Levine ((7))). *There is a smoothly embedded 7-sphere in R^{11} which is not depressible, and which has a trivial smooth normal bundle.*

Proof. In (7) it is proved that $\tilde{\Sigma}^{11,7}$ has order 60, while $\tilde{\Sigma}^{10,7}$ has order 3 (where $\tilde{\Sigma}^{n,m}$ is the group of embedded m -spheres in S^n modulo the subgroup bounding framed manifolds). A non-zero $\alpha \in \Sigma^{11,7}$ of order 5 is represented by an indepressible embedded

$M^7 \subset R^{11}$. Moreover, M must have trivial normal bundle ν since ν is represented by the image of α under a homomorphism $\tilde{\Sigma}^{11,7} \rightarrow \Pi_6(SO_4) = Z_{12} \oplus Z_{12}$.

PROPOSITION 5. *A smooth submanifold $M \subset S^q$ is depressible provided there exists a smooth q -cell $D \subset S^q \times I$ with $M \times 0 \subset \partial D$.*

Proof. Deform D diffeotopically in $S^q \times I$, leaving ∂D fixed, until a small concentric q -cell D' coincides with a hemisphere of $S^q \times 1$ bounded by $S^{q-1} \times 1$. Since $D - \text{int } D'$ is diffeomorphic to $S^{q-1} \times I$, it is easy to construct the required embedding of $M \times I$.

PROPOSITION 6. *A smooth submanifold $M \subset S^q$ is depressible provided there exists a smooth triangulation of $S^q \times I$, a PL submanifold $X \subset S^q \times I$ of dimension q , and a PL q -cell $B \subset X$, such that:*

- (a) X lies on the boundary of a PL submanifold of dimension $q+1$;
- (b) $M \times 0 \subset \partial B$;
- (c) $M \times 0$ has neighbourhoods in B and X that are smooth submanifolds.

Proof. Apply Proposition 3 twice, first to get a smooth triangulation $f: X \rightarrow Y = a$ smooth submanifold of $S^q \times I$, and then to obtain a smooth triangulation $g: f(B) \rightarrow D = a$ smooth submanifold of Y , such that f and g leave fixed a neighbourhood of $M \times 0$ in X . By Proposition 2, D must be a smooth q -cell, and the proof is completed by applying Proposition 5.

PROPOSITION 7 (Hirsch–Zeeman; see also Irwin((6))). *Let W be a contractible PL m -manifold and $K \subset \partial W$ a compact polyhedron. If $\dim K \leq m-4$, there exists a PL m -cell C such that $K \subset C \subset W$.*

Proof. See the Engulfing theorem in (3).

PROPOSITION 8 (Zeeman((12))). *Let $A, B \subset \Sigma^n$ be PL embedded k -spheres. If $n \geq k+3$, there exists a PL homeomorphism of Σ^n taking A onto B .*

PROPOSITION 9 (Smale((10))). *A PL manifold of dimension ≥ 6 is a PL cell if it is compact and contractible and has a simply connected boundary.*

4. *Sections of tubes.* Let (Y, X, ξ) be a tube, with $\xi = (p, E, X)$. A section of ξ is a cross-section $f: X \rightarrow \partial E$ of the sphere bundle $(p, \partial E, X)$. The section f is *engulfable* provided there exists a contractible open set W of $Y - \text{int } E$ containing $f(X)$. If the tube is PL and f is a PL section, then f is *shrinkable* if there exists a collapsible polyhedron C such that $f(X) \subset C \subset Y - \text{int } E$.

PROPOSITION 10. *The existence of an engulfable section is an isomorphism invariant of tubes. The existence of a shrinkable section is a PL isomorphism invariant of PL tubes.*

Proof. Trivial.

PROPOSITION 11. *Let (Y, X, ξ) be a PL tube with X compact.*

- (a) *Every shrinkable section is engulfable.*
- (b) *If $\dim Y - \dim X \geq 4$, every PL engulfable section is shrinkable.*

Proof. Since a regular neighbourhood of a collapsible set is contractible, (a) is obvious. To prove (b), let $f: X \rightarrow \partial E$ be a PL engulfable section, and $W \subset Y - \text{int } E$ a contractible open set containing $f(X)$. The required collapsible polyhedron exists by virtue of Proposition 7.

PROPOSITION 12. Let $M \subset S^q$ be a smooth submanifold with a smooth normal cell bundle ν . Let $(\hat{S}^q, \hat{M}, \hat{\nu})$ be a triangulation of (S^q, M, ν) . If ν has an engulfable section, and

$$q - \dim M \geq 4,$$

then M is depressible.

Proof. Assume that ν has an engulfable section. By Proposition 10, $\hat{\nu}$ has a shrinkable section $f: \hat{M} \rightarrow \partial \hat{E}$, where $\nu = (p, E, M)$ and $\hat{\nu} = (p, \hat{E}, \hat{M})$. By Proposition 7 there is a PL q -cell $C \subset S^q - \text{int } E$ with $f(M) \subset C$. We may assume that $C \cap \partial E$ is a neighbourhood in ∂E of $f(M)$. Since E is a smooth submanifold of S^q , C is smooth in a neighbourhood of $f(M)$. Proposition 12 follows from Proposition 6 (or 3).

Remark. Using Proposition 3 it is easy to show that M is actually diffeotopic to a submanifold of S^{q-1} (i.e. M is compressible in the sense of (3)).

5. *Proof of Theorem A.* Let $M \subset S^{11}$ be the smoothly embedded 7-sphere of Proposition 4, and let ν be its smooth normal 4-cell bundle. Let $(\hat{S}^{11}, \hat{M}, \hat{\nu})$ be a triangulation (see (2)) of (S^{11}, M, ν) . By the smooth triangulation theorems of Whitehead ((11)) and Zeeman's unknotting theorem (Proposition 8), there is a PL normal cell bundle ξ_0 on $\Sigma^7 \subset \Sigma^{11}$ and a PL isomorphism $f: (\hat{S}^{11}, \hat{M}, \hat{\nu}) \rightarrow (\Sigma^{11}, \Sigma^7, \xi_0)$. By Proposition 12, $(\hat{S}^{11}, \hat{M}, \hat{\nu})$ does not have an engulfable section. Therefore neither has $(\Sigma^{11}, \Sigma^7, \xi_0)$, by Proposition 10. On the other hand Σ^7 obviously has another PL normal cell bundle in Σ^{11} , say ξ_1 , which does have an engulfable section, and even a shrinkable one. Therefore $(\Sigma^{11}, \Sigma^7, \xi_0)$ and $(\Sigma^{11}, \Sigma^7, \xi_1)$ are not topologically isomorphic. Moreover, both ξ_0 and ξ_1 are trivial bundles.

It remains to prove that ξ_0 is not PL I -cobordant to ξ_1 . This will follow from:

PROPOSITION 13. Let (S^q, M, ν) be a smooth tube with a triangulation $(\hat{S}^q, M_0, \xi_0)$ which is PL h -cobordant to a tube having a shrinkable section. If either (a) $\dim M \leq q - 3$, or (b) the h -cobordism is a PL I -cobordism, then M is depressible.

Proof. Let $(\hat{S}^q, M_1, \xi_1)$ be a PL tube with a shrinkable section $f: M_1 \rightarrow \partial E_1$, where $\xi_i = (p_i, E_i, M_i)$ for $i = 0, 1$. Let $Z \subset \hat{S}^q \times I$ be a PL h -cobordism between $M_0 \times 0$ and $M_1 \times 1$, and let ξ be a PL normal cell bundle on Z which extends $\xi_0 \times 0$ and $\xi_1 \times 1$. Put $\xi = (p, E, Z)$. Thus $Z \cap (S^q \times i) = M_i \times i$. Without loss of generality we assume that $E \cap (S^q \times [0, \frac{1}{2}]) = E_0 \times [0, \frac{1}{2}]$, so that E is smooth in a neighbourhood of $E_0 \times 0$. The shrinkable section $f: M_1 \times 1 \rightarrow \partial E_1 \times 1$ extends to a PL section $F: Z \rightarrow \partial E$, because $M_1 \times 1$ is assumed to be a deformation retract of Z , and the covering homotopy theorem is valid for PL bundles. Since f is shrinkable, there is a collapsible $C \subset S^q \times 1$ such that $C \cap \partial E = F(M_1 \times 1)$. Let $K = C \cup F(Z)$. It is clear that K is contractible, and that K is collapsible if Z is PL homeomorphic to $M_1 \times I$.

Let $T \subset E$ be the total space of the open cell bundle corresponding to ξ . That is, $T = E - \text{cl}(S^q \times I - E)$. Consider the submanifold $X \subset \hat{S}^q \times I$ defined by $X = (E \cup S^q \times 1) - T$. Observe that $K \subset X$. Let $B \subset X$ be a regular neighbourhood of K meeting $\partial X = \partial E_0 \times 0$ in a neighbourhood of $K \cap \partial E = F(M_0 \times 0)$. If K is collapsible, then B is a PL cell. If K is contractible, so is B , and if $\dim M \leq q - 3$, then $\dim K \leq \dim X - 3$ and hence ∂B is simply connected. In this case if $\dim X \geq 6$ then B is a PL cell by Proposition 9, while if $\dim X \leq 5$ the theorem is trivial using

standard embedding theorems. Thus we have a q -cell $B \subset X$; X clearly lies on the boundary of $(S^q \times I) - T$. Moreover, both B and X are smooth in a neighbourhood U of $F(M_0 \times 0)$ in B . Let $g: M_0 \times 0 \rightarrow \partial E_0 \times 0$ be a smooth section of ν_0 whose image lies in U . Then $g(M_0 \times 0)$ lies in the smooth part of B . Proposition 6 now implies that $g(M_0)$, and hence also M_0 , is depressible, proving Proposition 11.

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