

UC Berkeley

UC Berkeley Electronic Theses and Dissertations

Title

Well-posedness of the three-form field equation and the minimal surface equation in Minkowski space

Permalink

<https://escholarship.org/uc/item/8ft305r7>

Author

Ettinger, Boris

Publication Date

2013

Peer reviewed|Thesis/dissertation

**Well-posedness of the three-form field equation and the minimal surface
equation in Minkowski space**

by

Boris Ettinger

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Daniel Tataru, Chair
Professor Maciej Zworski
Professor Borivoje Nikolic

Spring 2013

**Well-posedness of the three-form field equation and the minimal surface
equation in Minkowski space**

Copyright 2013
by
Boris Ettinger

Abstract

Well-posedness of the three-form field equation and the minimal surface equation in Minkowski space

by

Boris Ettinger

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Daniel Tataru, Chair

This dissertation deals with the question of well-posedness for the three-form field equation in eleven dimensional supergravity and the minimal time-like hypersurface equation in Minkowski space. Both of these equations are nonlinear wave equations.

The three-form field equation arises in the classical field theory formulation that underlies string theory. After making simplifying assumptions, it becomes a system for a differential three-form on $\mathbb{R}^{3+1} \times K^7$, where K is a compact Riemannian manifold. A suitable gauge choice turns the the equation into a semilinear wave equation. We show that the three-form field equation is globally in time well posed for small, smooth, compactly supported initial data.

The minimal surface equation in Minkowski space $\mathbb{R}^{(n+1)+1}$ describes a submanifold, which has a vanishing mean curvature. If the submanifold is time-like, the equation is a quasilinear wave equation. We prove that for $n = 2, 3$ and under some further geometrical assumptions, the equation is locally well posed for initial data in $H^{\frac{n+3}{2}}(\mathbb{R}^n) \times H^{\frac{n+1}{2}}(\mathbb{R}^n)$.

The nonlinearities in the equations above belong to a restricted class of nonlinearities that are said to satisfy the null condition. The goal of the null condition is to select the nonlinearities that limit the amount of interactions between waves that have a small angle of incidence. The precise formulation of the null condition depends on the context of the Cauchy problem and it is not yet clear in all cases. In particular, our investigation of the minimal surface equation falls in the context of questions regarding large data quasilinear equations for which there is still only a conjectured formulation of the condition. We hope that our treatment of the particular example can shed some light on the general case.

To my wife, Bronislava.

Her love and support kept me going through this strange and wonderful journey of my
doctoral studies.

Contents

Contents	ii
1 Introduction	1
1.1 Nonlinear wave equations	2
1.2 The null condition	4
1.3 Null-form estimates	6
2 The Three-form field equation	10
2.1 Introduction	11
2.2 Background	12
2.3 Review of Hodge theory	19
2.4 The Linear Estimates	21
2.5 Analysis of the nonlinearity	27
2.6 Proof of Theorem 2.1.1	32
3 The Minimal Surface equation	38
3.1 Introduction	38
3.2 Preparations	41
3.3 The outline of the proof	46
3.4 The regularity of the characteristic surfaces	48
3.5 The wave-packet parametrix construction	71
3.6 Bilinear estimates	77
Bibliography	82

Acknowledgments

I would like to thank my advisor Daniel Tataru, for his wisdom and patience.

I would also like to thank Baoping Liu, my friend and chavruta¹. I'm indebted to Tobias Schottdorf for his friendship and countless hours of help.

I thank the members of my research circle, my "mathematical family": Ioan Bejenaru, Boaz Haberman, Benjamin Harrop-Griffiths, Sebastian Herr, Jeremy Louis Marzuola, Jason Metcalfe, Jacob Sterbenz, Mihai Tohaneanu and Tae Kwon (Danny) Wong for sharing their knowledge, mathematical and otherwise.

I benefited greatly from the atmosphere of comradery and mutual support that was fostered by the organizers and members (dwellers?) of HADES: Kiril Datchev, Semyon Dyatlov, Taryn Flock, Jeffrey Galkowski, Patrick LaVictoire, Diogo Olivera e Silva, Rene Quilodran, Betsy Stovall and Ivan Ventura and fellow students and analysts: Tonci Antunovic, Aron Chen, Subhroshekhar Ghosh, Alexandru Hening, Ryan Hynd, Paul Lee, Dmitri Pavlov, Trevor Potter, Claudiu Raicu, Sobhan Seyfaddini, Valentin Tonita, Hung Vinh Tran and Maxim Trokhimtchouk.

I will always cherish the companionship of my officemates (especially at the jolly 787): Jeff Doker, Erica Isaacson, Junjie Zhou, Benoit Zhubin.

I'm also grateful for the mentorship of the members of the Berkeley faculty, with whom I had many conversations: Maciej Zworski, Jon Wilkening, Konrad Waldorf, John Strain, Jan Reimann, Steve Evans, Craig Evans, Michael Christ and Alexander Chorin.

There are many others who I forget but deserve a credit in my successes. I offer to them my sincerest apologies for the omission.

Finally, I would like to thank University of California, Berkeley for allowing me to live in the San Francisco Bay area, where I met so many interesting people and made many great friends.

¹See <http://en.wikipedia.org/wiki/Chavruta>.

Chapter 1

Introduction

1.1 Nonlinear wave equations

A nonlinear wave equation is a partial differential equation for an unknown function $u : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ of the form

$$\sum_{\alpha, \beta=0}^n g^{\alpha\beta}(u, du) \partial_\alpha \partial_\beta u = F(u, du),$$

where g is a given symmetric quadratic form of signature $(-1, 1, \dots, 1)$ for every value of (u, du) and F is a given function, which is strictly nonlinear, i.e. $|F(x, y)| = o(|x|, |y|)$ for small values of $|x|, |y|$. This definition extends to systems and domains other than \mathbb{R}^n . We will say that the equation is semilinear if $g^{\alpha\beta} = m^{\alpha\beta} = \text{diag}(-1, 1, \dots, 1)$ and quasilinear otherwise. When the equation is augmented with the initial data

$$u|_{t=0} = u_0, \quad \frac{\partial}{\partial t} u|_{t=0} = u_1,$$

finding a solution u for each pair (u_0, u_1) is called the Cauchy problem.

Many nonlinear wave equations describe a variety of physical phenomena such as gravitation, electromagnetism, sound propagation and elasticity, but many others exhibit behavior which makes them unsuitable to serve as models of physical phenomena. Jacques Hadamard proposed a criterion to separate between these two cases. According to Hadamard, an equation can be a candidate to describe a physical model if the corresponding Cauchy problem is well posed, which means the following: there exists a space of initial data X_0 and the space of solutions X such that

- For every pair of initial data $(u_0, u_1) \in X_0$ there exists a solution $u \in X$.
- For every pair of initial data $(u_0, u_1) \in X_0$ the solution u is unique in X .
- The spaces X_0, X are endowed with topologies and the mapping $(u_0, u_1) \mapsto u$ is continuous as a function from X_0 to X .

Most often X, X_0 will be subsets of normed vector spaces of functions. If X is a set of functions on the space $[0, \infty) \times \mathbb{R}^n$ or $R \times \mathbb{R}^n$ (where the first coordinate represents time) we will say that the problem is globally (in time) well posed. If the domain of the function is of the form $[0, T] \times \mathbb{R}^n$ we will say that the problem is locally well-posed.

To motivate the statement of the theorems proven in this dissertation, let us take a further look at the examples of nonphysical behavior of nonlinear wave equations¹. Fritz John in [8] proved that solution to the equations such as²

$$\square u = (\partial_t u)^2$$

¹For the sake of concreteness we will concentrate on the three-dimensional case; which is the most physical, the two dimensional equations case exhibit even more singular behavior

²We will use the notation

$$\square u = m^{\alpha\beta} \partial_\alpha \partial_\beta u = -\partial_t^2 u + \sum_{i=1}^n \partial_{x_i}^2 u$$

or

$$\square u = (\partial_t^2 u)^2$$

will develop a singularity, in the sense that for some compactly supported smooth non-zero $F, G : \mathbb{R}^3 \rightarrow \mathbb{R}$ then for every solution u with initial conditions $(\varepsilon F, \varepsilon G)$, for every $\varepsilon > 0$ there exists a time T and point x such that $u(T, x) = \infty$. While the time of existence T can be quite large, indeed $T \gtrsim e^{\frac{C}{\varepsilon}}$, the blow up of small, smooth data is unacceptable as such a data is "experimentally" indistinguishable from the "rest state" or "vacuum state" $u \equiv 0$. Therefore, to avoid this situation, we would like to find ε, N, δ such that for the space of initial data

$$X_0 = \{(u_0, u_1) \mid \|u_0, u_1\|_{H^N} \leq \varepsilon, \text{supp}(u_0, u_1) \in B(0, 1)\}$$

and the space of solutions of the form

$$X = \{u \mid \|u(t, \cdot)\|_{H^N(\mathbb{R}^3)} \leq C\varepsilon(1+t)^\delta\}$$

the problem is well-posed³. This is the formulation that we prove for the three-form field equation in Chapter 2.

Another set of examples was found by Hans Lindblad in [20, 21], where he proves that for the equations of the form

$$\square u = (\partial_t u - \partial_{x_1} u)^2$$

there exist f, g with $\|f\|_{H^2} + \|g\|_{H^1}$ arbitrarily small such that a solution u will satisfy $\|u(t, \cdot)\|_{H^1(\mathbb{R}^3)} = \infty$ for any $t > 0$. Similarly, for the equation

$$\square u = u(\partial_t - \partial_{x_1})^2 u,$$

there are initial data arbitrarily small in $H^2 \times H^1$ such that $\|u(t, \cdot)\|_{H^2} = \infty$ for every $t > 0$. It is not straightforward to interpret the "experimental" meaning of rough (low Sobolev regularity) solutions, but since smooth compactly supported (and in a certain sense small) functions are dense in the Sobolev spaces, statements about blowup of low regularity solutions translate to "unpleasant" statements about smooth solutions. Moreover, the lower the regularity in which the problem well-posed, the more stability, less sensitivity to precision in formulation the model will have.

How low an exponent should one try reach? One of the thresholds is the scaling of the problem. Most non-linear wave equations have a scaling law, i.e. a transformation of the form $u(x, t) \mapsto \lambda^a u(\lambda t, \lambda x)$ which maps solutions to solutions. Then there is a critical exponent $s_c = \frac{n}{2} - a$, such that the \dot{H}^{s_c} Sobolev norm is invariant under scaling. These critical exponents indicates the balance between the linear and nonlinear parts of the equation. The range where the linear part is stronger $s > s_c$ is called subcritical, while the range $s < s_c$ is called supercritical. In a certain sense, the current methods are perturbative i.e. trying to solve the nonlinear equation as a perturbation of some linear one, therefore the best

³Ideally, we would like to have $\delta = 0$ but we will not be able to achieve this in all situations.

exponents one could hope to reach with such methods satisfy $s > s_c$ ⁴. However, Lindblad's counterexamples show that one cannot do better than $s > s_c + \frac{3}{4}$ in two dimensions and $s > s_c + \frac{1}{2}$ in three dimensions or higher for a generic nonlinear wave equation. In the case of the minimal surface equation, the critical exponent is $s_c = \frac{n+2}{2}$, while we will show well-posedness for $s = \frac{n+3}{2} = s_c + \frac{1}{2}$.

Thus to rule out instantaneous blowup, we will seek a time T which depends only on the norm of the initial data $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$, such that the equation is well posed for the space of initial data

$$X_0^K = \{(u_0, u_1) \mid \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \leq K\}$$

and the space of solutions

$$X^{T(K)} = \{u \mid u \in C([0, T(K)], H^s) \cap C^1([0, T(K)], H^{s-1})\}.$$

This is the formulation that we will seek to prove for the minimal surface equation in Chapter 3, up to some geometric considerations that will require us to limit some quantities that depend on $\|du\|_{L^\infty}$ as well.

1.2 The null condition

The examples in the previous section show that we cannot address the question of well-posedness of nonlinear wave equations with complete generality. We still would like to do it with a high level generality. To that end one can propose the following program: find a subclass of nonlinearities, which is can be effectively identified⁵, develop a set of tools, which will facilitate the investigation of well-posedness. One such subclass of nonlinearities is called equations that satisfy the null condition. The precise condition varies slightly depending on the context of the well-posedness question and it is not known in all the cases. We will provide several definitions in this section and survey some of the tools that were developed to investigate equations that satisfy the null condition in the next section.

Definition 1.2.1 (Null condition for small, smooth compactly supported data). *The equation*

$$g^{\alpha\beta}(du)\partial_\alpha\partial_\beta u = N(du),$$

satisfies a null condition for small smooth compactly supported data if g, N are smooth function and the bilinear form $B^{\alpha\beta} = \frac{\partial^2 N}{\partial(\partial_\alpha u)\partial(\partial_\beta u)} \Big|_{du=0}$ and the trilinear form $G^{\alpha\beta\gamma} = \frac{\partial g^{\alpha\beta}}{\partial(\partial_\gamma u)} \Big|_{du=0}$ satisfy

$$B^{\alpha\beta}\xi_\alpha\xi_\beta = 0, \quad G^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0$$

for every ξ of zero Minkowski length $m^{\alpha\beta}\xi_\alpha\xi_\beta = 0$.

⁴Also $s = s_c$ with small data is a regime where nonlinearity is weaker than the linear part.

⁵Saying "all the nonlinearities for which the problem is well-posed" is not an effective definition.

We will explain why only the quadratic term in N and linear term in g are important in Remark 1.3.1 in the next section after we introduce several other concepts. Incidentally, when considering local well-posedness for a semilinear wave equation $\square u = N(du)$, we can use a similar definition to ensure better results.

Definition 1.2.2 (Null condition for a semilinear wave equation). *Let $N(du) = B^{\alpha\beta}\partial_\alpha u\partial_\beta u + \dots$ be polynomial in du . Then the equation*

$$\square u = N(du)$$

satisfies the null condition for semilinear wave equation if

$$B^{\alpha\beta}\xi_\alpha\xi_\beta = 0,$$

for every ξ of zero Minkowski length $m^{\alpha\beta}\xi_\alpha\xi_\beta = 0$. A multiplication by a smooth, bounded function ϕ of u is harmless in this case, i.e. if $\square u = N(du)$ satisfies the null condition, so will $\square u = \phi(u)N(du)$.

Definition 1.2.1 and 1.2.1 are effective in improving the regularity in the corresponding Cauchy problem (see next section) but they are clearly stated for perturbations of the linear, constant coefficients wave equations since $g^{\alpha\beta}|_{du=0} = m^{\alpha\beta}$. The case of large data quasilinear wave equations cannot be considered a perturbation of the constant coefficient wave equation as changes in the metric shift the high frequency content compared to the unperturbed case and cause large changes in the Sobolev norms. Therefore the definitions above are insufficient. One suggestion to amend the definition was made in [32] and it is as follows:

Definition 1.2.3 (Conjectured definition for large data quasilinear wave equation,[32]). *The equation*

$$g^{\alpha\beta}(du)\partial_\alpha\partial_\beta u = 0$$

satisfies the null condition for large data quasilinear wave equation if

$$\frac{\partial g^{\alpha\beta}(u, du)}{\partial(\partial_\gamma u)}\xi_\alpha\xi_\beta\xi_\gamma = 0, \text{ for every } \xi \text{ such that } g^{\alpha\beta}(u, du)\xi_\alpha\xi_\beta = 0.$$

It is easy to check that the minimal surface equation satisfies this null condition, which we do in Corollary 3.2.2, but we do not know whether this is effective in lowering the regularity in every possible case.

We would like to add the following definition.

Definition 1.2.4 (Null-forms). *A bilinear quadratic form $Q(f, g) = B^{\alpha\beta}\partial_\alpha f\partial_\beta g$ is a null-form if*

$$B^{\alpha\beta}\xi_\alpha\xi_\beta = 0$$

for every ξ of zero Minkowski length $m^{\alpha\beta}\xi_\alpha\xi_\beta = 0$. Similarly, $Q(f, g)$ is a null-form with respect to the metric g if

$$B^{\alpha\beta}\xi_\alpha\xi_\beta = 0,$$

for every ξ of g -zero length $g^{\alpha\beta}\xi_\alpha\xi_\beta = 0$.

By decomposing the bilinear form to symmetric and anti-symmetric part, we can establish the following

Claim 1.2.5. *Q is a null-form with respect to metric g if and only if it is a linear combination of the following forms*

$$Q_0(f_1, f_2) = g^{\alpha\beta} \partial_\alpha f_1 \partial_\beta f_2,$$

$$Q_{ij}(f_1, f_2) = \partial_{x_i} f_1 \partial_{x_j} f_2 - \partial_{x_j} f_1 \partial_{x_i} f_2, \quad i, j = 0..n.$$

A way to rephrase this claim is that the only symmetric null-form is Q_0 .

1.3 Null-form estimates

To conclude this introduction, we would like to survey the line of research that proceeds as follows:

Step one Find spaces X_0, X, Y such that X_0 is the space of initial data, X is a space of solutions, Y is the "space of nonlinearity" or the space of the inhomogeneous term, which admit an energy estimate of the form

$$\|u\|_X \leq \|\square u\|_Y + \|(u_0, u_1)\|_{X_0}$$

with appropriate modifications for variable coefficient metrics.

Step two Prove a null-form estimate

$$\|Q(f, g)\|_Y \leq (\|f\|_X + \|(f_0, f_1)\|_{X_0})(\|g\|_X + \|(g_0, g_1)\|_{X_0}).$$

The following terminology was jokingly coined by Hart Smith:

Null-form estimates 1.0

This approach is called "The Klainerman vector fields method", which was formalized by Klainerman in [16]. The approach seeks to exploit symmetries of the constant coefficient wave equation. Index a basis of the set of conformal symmetries of the wave equation

$$\{Z_0, Z_1, \dots, Z_{10}\} = \{\partial_0, \dots, \partial_3, t\partial_t + r\partial_r, t\partial_j + x_j\partial_t, \dots, x_j\partial_k - x_k\partial_j\}, \quad i, j = 1..3, i \neq j.$$

Introduce a multi-index notation for compositions $Z^I = Z_{i_1} Z_{i_2} \dots Z_{i_{|I|}}$, choose a large M and define the following pair of norms

$$\|u\|_X = \sum_{|I| \leq M} \|\nabla Z^I u\|_{L_t^\infty L_x^2}$$

$$\|f\|_Y = \sum_{|I| \leq M} \|Z^I f\|_{L_t^1 L_x^2},$$

since $[\square, Z_I] = c_I \square$, the $X - Y$ energy estimate will follow from the conventional energy estimate. The null form estimates are a combination of two of the following estimates:

$$\|u(t, \cdot)\|_{L_x^\infty} \leq \frac{1}{1+t} \|u\|_X,$$

which are called Klainerman or Klainerman-Sobolev estimates and a pointwise estimate for a null-form

$$|Q(f, g)| \leq \frac{1}{1+t} (|\nabla f| |Zg| + |Zf| |\nabla g|).$$

The space of initial data will be

$$X_0 = \{(u_0, u_1) | \text{supp } u_i \in B(0, 1), \|u_0\|_{H^{N+1}} + \|u_1\|_{H^N} \leq \varepsilon\}.$$

Remark 1.3.1. *We will use the Klainerman vector field approach to explain why the cubic terms in the Taylor expansion of $N(du)$ and quadratic terms in $g(du)$ are "benign" i.e. for any such nonlinearity the problem is well-posed. We start with the energy estimate*

$$\|\nabla Z^I u(t)\|_{L_x^2} \leq e^{\|\nabla g\|_{L_t^1 L_x^\infty}} \left(\int_0^t \|Z^I N(du)(s)\| ds + \|dZ^I u(0)\|_{L_x^2} \right). \quad (1.3.1)$$

We can estimate

$$\sum_{|I| \leq M} \|Z^I N(du)\|_{L_x^2} \leq C \sum_{|J| \leq \frac{M}{2}} \|\nabla Z^J du\|_{L^\infty}^2 \sum_{I \leq M} \|\nabla Z^I N(du)\|_{L_x^2}.$$

Assume

$$\sum_{|I| \leq M} \|Z^I du\|_{L_t^\infty L_x^2} \leq \varepsilon,$$

we use Klainerman-Sobolev estimate to conclude

$$\|\nabla Z^J u(t)\|_{L_x^\infty}^2 \leq \frac{1}{(1+t)^2} \varepsilon^2, \quad |J| \leq \frac{M}{2}, \quad (1.3.2)$$

also

$$\|\nabla g\|_{L_t^1 L_x^\infty} \leq \int_0^\infty \frac{\varepsilon^2}{(1+t)^2} dt \leq C\varepsilon^2. \quad (1.3.3)$$

Using (1.3.2) and (1.3.3) in the energy estimate (1.3.1), we get

$$\|\nabla Z^I u(t)\|_{L_t^\infty L_x^2} \leq C e^{C\varepsilon^2} \left(\varepsilon^2 \int_0^\infty \frac{1}{(1+s)^2} ds + \|dZ^I u(0)\|_{L_x^2} \right).$$

We can easily sharpen this line of reasoning to set up a fixed point argument for semilinear equations or a bootstrap argument for quasilinear equations. The details can be found in textbooks [7, 28] and they are similar to the argument we will present in Chapter 2.

Remark. *By using M (the number of applications of the vector fields Z_i) high enough, one can prove the well-posedness for C^∞ solutions with C^∞ initial data.*

Null-form estimates 2.0

The step was undertaken by Klainerman and Machedon in [14], where they proved that for any null form

$$\|Q(f, g)\|_{H_{t,x}^{\frac{n-1}{2}}} \leq \|(f_0, f_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}} \|(g_0, g_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}},$$

for any solutions of the wave equation $\square f = 0 = \square g$. This estimate should be combined with Duhamel's principle to prove null-form estimate for solutions of inhomogeneous wave equation. Then one uses Sobolev space $H^{\frac{n+1}{2}}$ energy estimates to prove local well-posedness of various semilinear equations that satisfy the null condition for the spaces

$$X_0 = \{(u_0, u_1) \mid \|(u_0, u_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}} \leq K\},$$

$$X = L_t^\infty([0, T(K)), H^{\frac{n+1}{2}}), \quad Y = L^1([0, T(K)), H^{\frac{n+1}{2}}).$$

Null-form estimates 3.0

Klainerman and Machedon further refined their work in [13], where they considered spaces of the form

$$\|u\|_{\mathcal{X}^{s,b}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \hat{u}(\tau, \xi)\|_{L_{\tau,\xi}^2},$$

where $\hat{u}(\tau, \xi) = \int_{\mathbb{R}^{n+1}} e^{-i(t\tau + x\xi)} u(t, x) dt dx$ is the space-time Fourier transform. Then one defines

$$\|u\|_{X^{s,b}} = \|u\|_{\mathcal{X}^{s,b}} + \|\partial_t u\|_{\mathcal{X}^{s-1,b}}. \quad (1.3.4)$$

One also needs to apply a suitable time localization to get the space $X_T^{s,b}$. Then we can choose

$$X_0 = \{(u_0, u_1) \mid \|(u_0, u_1)\|_{H^{s+1} \times H^s} \leq K\},$$

$$X = X_T^{s,b}, \quad Y = \mathcal{X}_T^{s-1,b-1}.$$

As $X - Y$ energy estimates are not completely trivial but are fairly standard, then the appropriate null-form estimate is

$$\|Q(f, g)\|_{\mathcal{X}_T^{s-1,b-1}} \leq \|(f_0, f_1)\|_{H^{s+1} \times H^s} \|(g_0, g_1)\|_{H^{s+1} \times H^s} \quad (1.3.5)$$

for solutions of the linear wave equation $\square f = 0 = \square g$. And indeed such estimates were established for the following range of exponents: Q_0 form: $s > \frac{n}{2}, b > \frac{1}{2}$, and Q_{ij} forms $s > \max(\frac{n}{2}, \frac{n+3}{4}), b > \frac{1}{2}$. Finally, as $X_T^{s,b} \subseteq C([0, T], H^s)$ for $b > \frac{1}{2}$, the null form estimate leads to local well-posedness of semilinear equations that satisfy the null condition in Sobolev spaces of initial conditions for the exponents in which the null-form estimates are correct.

There are several extensions of these results: following [13], Foschi and Klainerman conjectured an extension of (1.3.5) for $L_t^p L_x^q$ type spaces, instead of the L^2 -based $X^{s,b}$ spaces. This conjecture has been mostly settled by Wolff [39], Tao [30] and Tataru [33].

Attempts to prove well-posedness for the critical regularity, i.e. $s = s_c$ has been successful for the wave-maps equation in two dimensions, see Tao [34] and Tataru [31] and the four dimensional Maxwell-Klein-Gordon equation by Krieger, Sterbenz and Tataru [17]. This is done by significantly refining the $X^{s,b}$ spaces.

For the metrics with variable coefficients, the version of (1.3.4) was obtained by Smith and Sogge in [26], using Fourier Integral Operators and the version of (1.3.5) by Tataru [33], using wave packets parametrices.

There is an alternative approach to the Klainerman vector fields technique, which is that of conformal compactification by Christodoulou [3]. The approach rewrites the non-linear wave equation on \mathbb{R}^{n+1} as a wave equation on a bounded subset of $\mathbb{R} \times \mathbb{S}^n$. The coefficients of the new equation will be analytic if and only if it satisfies the null condition, in which case the global-in-time problem on \mathbb{R}^{n+1} is reduced to a local problem on a bounded domain.

We will subscribe to the energy estimate combined with null-form estimates approach in this dissertation. The difficulty which we have to resolve is that the existing literature does not quite cover the conditions that are dictated by the circumstances of the theorems and we will have to adjust the proofs accordingly. One could say that our goal is to prove null-form estimates 1.1 for the three-form field equation in Chapter 2 and null-form estimate 2.1 for the minimal surface equation in Chapter 3.

Chapter 2

The three-form field equation in eleven dimensional supergravity

2.1 Introduction

Let K be a compact 7-dimensional Riemannian manifold. Then the product $\mathbb{R}^{3+1} \times K$ becomes an 11-dimensional Lorentzian manifold. For a 3-form u on $\mathbb{R}^{3+1} \times K$ we use the Hodge star $*$ and the de Rahm differential d of the product to formulate the following Cauchy problem:¹

$$\square_{\mathbb{R}^{3+1}} u + \Delta_K u = - * (du \wedge du), \quad (2.1.1a)$$

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1. \quad (2.1.1b)$$

In this chapter, we will prove the following statement.

Theorem 2.1.1. *There exist positive N, ϵ such that the Cauchy problem (2.1.1) is globally well-posed provided that initial data is localized in the ball of radius 1 in \mathbb{R}^3 for every point of K and obeys*

$$\|u_0\|_{H^N(\mathbb{R}^3 \times K)} + \|u_1\|_{H^{N-1}(\mathbb{R}^3 \times K)} \leq \epsilon.$$

Moreover, in such a case the solution u satisfies the following estimates

$$\sum_{|\alpha| \leq N} \|\nabla_{t,x,y} \Gamma^\alpha u(t)\|_{L^2(\mathbb{R}^3 \times K)} \leq C\epsilon(1+t)^{1/12},$$

$$\sum_{|\alpha| \leq N-10} \|\nabla_{t,x,y} \Gamma^\alpha u(t)\|_{L^2(\mathbb{R}^3 \times K)} \leq C\epsilon,$$

$$\begin{aligned} & (1+t) \sum_{|\alpha| \leq N-20} \|\nabla_{t,x,y} \Gamma^\alpha \mathcal{P}_0 u(t)\|_{L^\infty(\mathbb{R}^3 \times K)} \\ & + (1+t)^{3/2} \sum_{|\alpha| \leq N-20} \|\nabla_{t,x,y} \Gamma^\alpha \mathcal{P}_{>0} u(t)\|_{L^\infty(\mathbb{R}^3 \times K)} \leq C\epsilon, \end{aligned}$$

where $\mathcal{P}_0, \mathcal{P}_{>0}$ are the spectral projections of the operator Δ_K defined in Section 2.3, Γ^α are compositions of a subset of Klainerman vector fields together with the operator $(-\Delta_K)^{1/2}$, which are defined in Section 2.4, $\nabla_{t,x,y}$ is the gradient in all the derivatives of $\mathbb{R}^{3+1} \times K$ and C is a constant that depends only on N and the geometry of K .

The theorem is true if the Cauchy data is supported in a larger ball but then the constant ϵ has to decrease as a negative power of the size of the support.

The equations have a connection to the theory of supergravity, which we explain in the next section. The mathematical aspects of the supergravity theory have recently drawn the

¹ Our sign convention for the D'Alembertian is

$$\square_{\mathbb{R}^{3+1}} = -\frac{\partial^2}{\partial t^2} + \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2},$$

and the operator Δ_K is negative.

attention in the context of conformal geometry [6],[10], where the space-time was assumed to be a Riemannian manifold. The Lorentzian case was investigated earlier (see [2] and references therein).

Our methods are inspired by the work of Metcalfe, Sogge and Stewart [22] and Metcalfe and Stewart [23] who analyze the quasi-linear wave equation on $\mathbb{R}^{3+1} \times D$, where D is a bounded domain in \mathbb{R}^n with various non-linearities and boundary conditions. Their results do not cover the case in study but we employ some of their ideas in this work. See also [18] for the previous work on Klein-Gordon equations on $\mathbb{R}^{3+1} \times D$.

To explain the difference with respect to the null-form estimates research as surveyed in Chapter 1, we will need several definitions, therefore we defer that discussion to Remark 2.6.2 after Proposition 2.6.1.

The chapter is organized as follows. In section 2.2 we derive the equation from a gauge-invariant Lagrangian and explain how to fix the gauge. In section 2.3 we recall some necessary facts from Riemannian geometry. In section 2.4 we adapt the linear estimates for the wave equation on \mathbb{R}^{3+1} to the product $\mathbb{R}^{3+1} \times K$. In section 2.5 we perform a deeper analysis of the nonlinearity. In section 2.6, we provide the proof of Theorem 2.1.1.

We will denote by k a constant which depends only on N and the geometry of K , this constant may change from line to line but for each inequality below there is an a priori computable constant such that the inequality holds. We will also write $A \lesssim B$ to mean $A \leq kB$.

2.2 Background

Physical Motivation

The supergravity theory is a model of classical physics, which describes the low-energy, classical limit of the superstring theory. The model describes the interaction of the field of gravity with other fields. In one of the simplest setups, one considers an 11-dimensional Lorentzian manifold as a space-time, with gravity field g and a field, whose strength is described by a closed differential 4-form F . The Lagrangian is prescribed only locally by restricting the attention to an open, topologically trivial subset U of the space-time. Then one solves the equation for a potential of F on U :

$$dA = F.$$

With this, the Lagrangian can be written as

$$\mathcal{L} = \int_U R dv + \int_U F \wedge *F + \int_U A \wedge F \wedge F,$$

where R , dv and $*$ are the (scalar) Ricci curvature, the volume form and the Hodge $*$ corresponding to g , respectively. The reader should consult [38] and textbooks [37, Section 3.3], [25, Section 16.1.1] for physical aspects of this theory.

The Lagrangian and the equation

In this dissertation we wish to concentrate on the global aspects of dynamics of the field F , leaving the more difficult question of interaction of F with gravity for further research. Therefore we will simplify our setup to consider the product manifold $\mathbb{R}^{3+1} \times K$ as the fixed space-time, where K is a 7-dimensional compact Riemannian manifold without a boundary (thus dropping the term $\int R dv$ from the Lagrangian). The metric on the product space will be the product of the Minkowski metric and the metric on K . We will also assume that the field strength F is not only closed but also exact, namely there exists a global 3-form u , such that

$$du = F.$$

Then u will be the dynamical variable, for which we define a classical field theory Lagrangian

$$\mathcal{L}(u) = \int_{\mathbb{R}^{3+1} \times K} du \wedge *du + \int_{\mathbb{R}^{3+1} \times K} u \wedge du \wedge du.$$

The formal Euler-Lagrange equations are

$$d * du = -du \wedge du. \quad (2.2.1)$$

We will take the Hodge-dual on both sides of the equation and use the notation $\delta = - * d *$ to arrive to the following equation

$$\delta du = *(du \wedge du). \quad (2.2.2)$$

Since our space time is a product manifold then most of the operators which act on it can be decomposed in a natural way as operators acting on either on \mathbb{R}^{3+1} or K . We will denote by subscript \parallel the operators acting on \mathbb{R}^{3+1} and by subscript \perp the operators acting on K . For instance, we will have

$$d = d_{\mathbb{R}^{3+1} \times K} = d_{\mathbb{R}^{3+1}} \otimes \text{id}_K + \text{id}_{\mathbb{R}^{3+1}} \otimes d_K = d_{\parallel} \otimes \text{id}_K + \text{id}_{\mathbb{R}^{3+1}} \otimes d_{\perp}. \quad (2.2.3)$$

The tensor notation should be understood in terms of operations on differential forms $\Omega(\mathbb{R}^{3+1})$ and $\Omega(K)$. The equation (2.2.3) we will colloquially write

$$d = d_{\parallel} + d_{\perp}. \quad (2.2.4)$$

Hodge star and form Laplacian

Let us recall a few simple facts regarding the Hodge star operator, which is an operator that takes differential n -forms to differential $(11 - n)$ -forms. Let $x^i, i = 0..3$ be the coordinates on \mathbb{R}^{3+1} and $x^i, i = 4..10$ be a coordinate patch on K at a point where the metric tensor is

the identity and its derivative vanish, i.e. normal coordinates. Then the Hodge dual $*$ for an n -form $v = v_{i_0 i_1 \dots i_{n-1}} dx^{i_0} dx^{i_1} \dots dx^{i_{n-1}}$, $i_k = 0..10$ is defined as follows:

$$*v = \sum_{i_0, i_1, \dots, i_{10}=0}^{10} (-1)^{\alpha(\{i_0 \dots i_{n-1}\})} \varepsilon^{i_0 \dots i_{10}} v_{i_0 i_1 \dots i_{n-1}} dx^{i_n} dx^{i_{n+1}} \dots dx^{i_{10}}, \quad (2.2.5)$$

where

$$\varepsilon^{i_0 \dots i_{10}} = \begin{cases} 0, & i_k = i_l \text{ for some } k \neq l, \\ 1, & i_0 \dots i_{10} \text{ is an even permutation,} \\ -1, & i_0 \dots i_{10} \text{ is an odd permutation} \end{cases}$$

and

$$\alpha(\{i_0 \dots i_n\}) = \begin{cases} 1, & 0 \in \{i_0 \dots i_n\}, \\ 0, & 0 \notin \{i_0 \dots i_n\}. \end{cases}$$

Thus the $*$ operator exchanges the components of the forms, multiplying those containing the time x^0 coordinate by -1 . Next, we define δ which takes n -forms to $(n-1)$ -forms by

$$\delta u = (-1)^{\text{deg}u} * d(*u).$$

Lastly we define

$$\square_{\mathbb{R}^{3+1} \times K} = -d\delta - \delta d.$$

We have the following facts about $*$ and δ

- $**u = (-1)^{\text{deg}u} u$.
- $u \wedge *v = g(u, v) d\text{vol}$, where g is the Lorentzian metric on $\mathbb{R}^{3+1} \times K$ and $d\text{vol}$ is the volume form.
- The operator $*$ is an isometry and in particular $d*u = 0$ if and only if $\delta u = 0$.
- δ is the Lorentzian adjoint of d in the sense that

$$\int_{\mathbb{R}^{3+1} \times K} g(\delta u, v) d\text{vol} = \int_{\mathbb{R}^{3+1} \times K} g(u, dv) d\text{vol}.$$

- In normal coordinates and with Einstein summation convention we have

$$(\delta u)_{\alpha_1 \alpha_2 \dots} = -\partial^{\alpha_0} u_{\alpha_0 \alpha_1 \alpha_2 \dots}.$$

$$(\square_{\mathbb{R}^{3+1} \times K} u)_{\alpha_1 \alpha_2 \dots} = \partial^{\alpha_0} \partial_{\alpha_0} u_{\alpha_1 \alpha_2 \dots} + (f(R)u)_{\alpha_1 \alpha_2 \dots},$$

where $f(R)$ is a linear, zeroth-order tensor that depends on the Riemann curvature tensor.

- $\square_{\mathbb{R}^{3+1} \times K} * u = * \square_{\mathbb{R}^{3+1} \times K} u$.

- We have

$$\square_{\mathbb{R}^{3+1} \times K} du = d(\square_{\mathbb{R}^{3+1} \times K} u), \quad \square_{\mathbb{R}^{3+1} \times K} \delta u = \delta(\square_{\mathbb{R}^{3+1} \times K} u).$$

The first identity is the consequence of the fact that $d^2 = 0$. This fact also implies $\delta^2 = 0$ which leads to the second identity above.

Gauge fixing

There is an obvious gauge freedom in equation (2.2.2) - if u is a solution of the equation then for any two-form w , the three-form $u + dw$ is a solution as well. Therefore, we will fix the gauge by requiring that u satisfies

$$d * u = 0, \tag{2.2.6}$$

which is equivalent to

$$\delta u = 0.$$

This choice of gauge is similar to the Lorenz gauge of the Maxwell equations, where for a one-form \mathbf{a} one requires $d * \mathbf{a} = 0$. Since we work with 3-forms on a product manifold, the gauge is structurally more complicated. We will give a proof that equation (2.2.6) is a valid gauge choice in the end of this section but first we rewrite the equation (2.2.2) using the gauge. We defined the Laplace(d'Alembert)-Beltrami operator on forms as $\square_{\mathbb{R}^{3+1} \times K} u = -\delta d - d\delta$.² On a product manifold, the operator decomposes into $\square_{\mathbb{R}^{3+1} \times K} = \square_{\mathbb{R}^{3+1}} + \Delta_K$, where Δ_K is the Laplace-Beltrami operator on $\Omega(K)$ (the space of differential forms). Thus we can rewrite the main equation (2.2.2) as

$$\square_{\mathbb{R}^{3+1} \times K} u = -\delta du - d\delta u = - * (du \wedge du).$$

Therefore, the equation (2.2.2) with the gauge choice (2.2.6) becomes

$$\square_{\mathbb{R}^{3+1}} u + \Delta_K u = - * (du \wedge du). \tag{2.2.7}$$

Let us now address the validity of the gauge choice.

Proposition 2.2.1. *Let $F \in \Omega^4(\mathbb{R}^{3+1} \times K)$. Suppose there exists a solution $A \in \Omega^3(\mathbb{R}^{3+1} \times K)$ to the equation $dA = F$. Then there exists a solution to the system*

$$d\tilde{A} = F, \quad d * \tilde{A} = 0.$$

Moreover, we can choose \tilde{A} such that

$$\text{supp} \tilde{A} \subseteq \{(t, x) \times K \mid (s, y, z) \in \text{supp} A, |x - y| \leq |t - s|\}$$

for $t, s \in \mathbb{R}, x, y \in \mathbb{R}^3, z \in K$.

²The operator Δ_K is negative. See footnote on page 1.

Proof. Let A be as above. Denote $\delta A = e$. The two-form e is the error we wish to eliminate by finding a two-form b such that $\delta(A + db) = 0$. We solve the equation

$$-\square_{\mathbb{R}^{3+1} \times K} b = \delta db + d\delta b = -e \quad (2.2.8)$$

and define

$$\tilde{A} = A + db.$$

We have from equation (2.2.8)

$$\delta \tilde{A} = -d\delta b. \quad (2.2.9)$$

We have

$$\square \delta b = \delta \square b = -\delta e = -\delta^2 A = 0.$$

Thus δb solves the homogeneous wave equation. We will prove that $\delta b = 0$ by choosing suitable Cauchy data for b at the hypersurface $t = 0$. Our goal is to make the Cauchy data for δb be zero. The Cauchy data that we prescribe for b in the normal coordinates are as follows:

$$\begin{aligned} b_{\alpha_1 \alpha_2} &= 0, & \alpha_1, \alpha_2 &= 0..10. \\ \frac{\partial}{\partial t} b_{0\alpha} &= 0, & \alpha &= 1..10. \\ \frac{\partial}{\partial t} b_{\alpha_1 \alpha_2} &= A_{0\alpha_1 \alpha_2} = -A_{\alpha_1 0 \alpha_2}, & \alpha_1, \alpha_2 &= 0..10. \end{aligned} \quad (2.2.10)$$

We check the Cauchy data for δb at $t = 0$.

$$(\delta b)_\alpha|_{t=0} = \frac{\partial}{\partial t} b_{0\alpha} - \sum_{\beta \neq 0} \frac{\partial}{\partial x_\beta} b_{\beta\alpha} = 0.$$

This is because the Cauchy data for $b_{0\alpha}$ is zero and the function $b_{\beta\alpha} = 0$ for $\alpha \neq 0$ at $t = 0$ and so are the spatial derivatives. To see that the time derivative of δb at $t = 0$ is zero, we employ the equation for b

$$\begin{aligned} \frac{\partial}{\partial t} (\delta b)_\alpha|_{t=0} &= \frac{\partial^2 b_{0\alpha}}{\partial t^2} + \sum_{\mu \neq 0} \frac{\partial}{\partial t} \frac{\partial}{\partial x_\mu} b_{\mu\alpha} \\ &= (\Delta_{\mathbb{R}^3} + \Delta_K) b_{0\alpha} - (\delta A)_{0\alpha} + \sum_{\mu} \frac{\partial}{\partial t} \frac{\partial}{\partial x_\mu} b_{\mu\nu}. \end{aligned}$$

Observe that $(\Delta_{\mathbb{R}^3} + \Delta_K) b_{0\alpha} = 0$ since the function at $t = 0$ is zero and we take spatial derivatives and zero order term which are linear in b to compute the Laplacian. The second and the third terms cancel because since $A_{00\alpha} = 0$ by antisymmetry then

$$(\delta A)_{0\alpha} = - \sum_{\mu \neq 0} \frac{\partial}{\partial x_\mu} A_{\mu 0 \alpha} = \frac{\partial}{\partial t} \sum_{\mu \neq 0} \frac{\partial}{\partial x_\mu} b_{\mu\alpha},$$

by equation (2.2.10). Therefore

$$(\delta b)_\alpha|_{t=0} = 0, \quad \frac{\partial}{\partial t}(\delta b)_\alpha|_{t=0} = 0$$

and thus δb obeys a homogeneous wave equation with zero Cauchy data, which makes it identically zero. Therefore

$$\delta \tilde{A} = \delta(A + db) = d\delta b = 0.$$

Observe that the support of the Cauchy data for b is contained in the support for A and thus the statement on the support follows from finite speed of propagation. \square

We now wish to prove that the gauge condition $\delta u = 0$ persists for the equation (2.2.7). For that we need to discuss the initial conditions. Since the equation is of second order, the natural Cauchy data is $u|_{t=0}$ and $\frac{\partial}{\partial t}u|_{t=0}$. If we express (2.2.2) through the field strength $F = du$ we have

$$\delta F = *(F \wedge F). \quad (2.2.11)$$

The natural initial condition for this first order equation is $F|_{t=0}$ but we first need to observe that there is a certain compatibility condition in (2.2.11), which is not of the evolution form. For that we recall the notion of the interior product of a form by a vector field. Let α be an n -form and X be a vector field, then the interior product of α by X , denoted by $\alpha \rfloor_X$ is an $n - 1$ form defined by

$$\alpha \rfloor_X(X_1, X_2, \dots, X_{n-1}) = \alpha(X, X_1, X_2, \dots, X_{n-1}).$$

We will be interested in interior products by $\frac{\partial}{\partial t} = \frac{\partial}{\partial x_0}$. Such a construction can be simply described as freezing the first index of the form α to be the zero (i.e. time) index. Thus in coordinates

$$(\alpha \rfloor_{\frac{\partial}{\partial t}})_{a_1 a_2 \dots} = \alpha_{0 a_1 a_2 \dots}$$

With this notation we prove the following observation

Claim 2.2.2. *The form $(\delta F) \rfloor_{\frac{\partial}{\partial t}}$ does not contain the time derivatives of F .*

Proof. We will give the proof in normal coordinates. We have

$$[(\delta F) \rfloor_{\frac{\partial}{\partial t}}]_{a_1 a_2} = (\delta F)_{0 a_1 a_2} = \frac{\partial}{\partial x_0} F_{00 a_1 a_2} - \sum_{i=1}^{10} \frac{\partial}{\partial x_i} F_{i0 a_1 a_2}.$$

Only the first term contains the time derivative but $F_{00 a_1 a_2} = 0$ due to antisymmetry of F . \square

Thus applying $\rfloor_{\frac{\partial}{\partial t}}$ to (2.2.11) and restricting it to time $t = 0$ we see that both sides of the equality

$$(\delta F) \rfloor_{\frac{\partial}{\partial t}}|_{t=0} = [*(F \wedge F)] \rfloor_{\frac{\partial}{\partial t}}|_{t=0}$$

depend only on $F|_{t=0}$ and the spatial derivatives of $F|_{t=0}$ so they are functions of a gauge invariant part of the Cauchy data and express a compatibility condition, which must hold in both gauge-invariant and gauge-fixed versions of the equations. We thus make the following definition.

Definition 2.2.3. *The form $du|_{t=0}$ is compatible if*

$$(\delta du)|_{\frac{\partial}{\partial t}}|_{t=0} = *(du \wedge du)|_{\frac{\partial}{\partial t}}|_{t=0}.$$

We now can prove that the gauge condition $\delta u = 0$ persists in the equation for the compatible Cauchy data.

Proposition 2.2.4. *Let u solve*

$$\square_{\mathbb{R}^{3+1} \times K} u = - * (du \wedge du),$$

such that $\delta u|_{t=0} = 0$ and $du|_{t=0}$ is compatible. Then $\delta u = 0$ for all times.

Proof. We apply $\square_{\mathbb{R}^{3+1} \times K}$ to δu to see that

$$\begin{aligned} \square_{\mathbb{R}^{3+1} \times K}(\delta u) &= \delta(\square_{\mathbb{R}^{3+1} \times K} u) = \delta * (du \wedge du) = * d * *(du \wedge du) \\ &= * d(du \wedge du) = 2 * (d^2 u \wedge du) = 0. \end{aligned}$$

We check the Cauchy data: $\delta u|_{t=0} = 0$ by assumption. The term $\frac{\partial}{\partial t} \delta u|_{t=0}$ vanishes because of the equation and the compatibility condition. We prove that in normal coordinates. We have

$$[(d\delta u)]_{\frac{\partial}{\partial t}}]_{ab} = (d\delta u)_{0ab} = \frac{\partial}{\partial t}(\delta u)_{ab} - \frac{\partial}{\partial x_a}(\delta u)_{0b} + \frac{\partial}{\partial x_b}(\delta u)_{0a}.$$

If $a, b \neq 0$ then the last two terms above are spatial derivatives of δu which is zero when we compute at $t = 0$. Therefore, for $a, b \neq 0$

$$\begin{aligned} \frac{\partial}{\partial t}(\delta u)_{ab}|_{t=0} &= [(d\delta u)]_{\frac{\partial}{\partial t}}]_{ab}|_{t=0} = [(-\square u - \delta du)]_{\frac{\partial}{\partial t}}]_{ab}|_{t=0} = \\ &= \{[* (du \wedge du) - \delta du]\}_{\frac{\partial}{\partial t}}]_{ab}|_{t=0} = 0, \end{aligned}$$

where we applied the compatibility condition to conclude the last equality. Next we assume without loss of generality that $a = 0, b \neq 0$ then since $\delta^2 u = 0$ we have

$$\frac{\partial}{\partial t}(\delta u)_{0b} = \delta^2 u + \sum_{i \neq 0} \frac{\partial}{\partial x_i}(\delta u)_{ib} = \sum_{i \neq 0} \frac{\partial}{\partial x_i}(\delta u)_{ib}.$$

This vanishes because it is a sum of spatial derivatives of components of δu which vanish at $t = 0$. \square

Corollary 2.2.5. *Let u solve the equation*

$$\square_{\mathbb{R}^{3+1} \times K} u = - * (du \wedge du)$$

with $\delta u|_{t=0} = 0$ and compatible $du|_{t=0}$. Then u solves

$$\delta du = *(du \wedge du).$$

2.3 Review of Hodge theory

The objects of our study are 3-forms on $\mathbb{R}^{3+1} \times K$. The basic example of such a form would be $u_{\parallel} \wedge u_{\perp}$ where u_{\perp} is a k -form (for $k = 0, 1, 2, 3$) and u_{\parallel} is a $3-k$ form on \mathbb{R}^{3+1} . The action of the Hodge-Laplacian of K is clearly $\Delta_K(u_{\parallel} \wedge u_{\perp}) = u_{\parallel} \wedge (\Delta_K u_{\perp})$ and it extends through density on all the forms on $\mathbb{R}^{3+1} \times K$. Moreover, if we use the eigenvectors of Δ_K , $\Delta_K e_{\lambda} = -\lambda^2 e_{\lambda}$, we can further decompose any form on $\mathbb{R}^{3+1} \times K$ as a series $u(x, y) = \sum_{\lambda} u_{\lambda}(x) e_{\lambda}(y)$, where x is a variable on \mathbb{R}^{3+1} and y is the variable on K . Thus, we envision the equation being the system of equations on differential forms on \mathbb{R}^{3+1} which are indexed by λ , in which case the equation will become

$$\square u_{\lambda} - \lambda^2 u_{\lambda} = \sum_{\lambda', \lambda''} \mathcal{B}_{\lambda}^{\lambda', \lambda''}(u_{\lambda'}, u_{\lambda''}),$$

where \mathcal{B} 's are bilinear differential operators. Thus we see that u_{λ} for $\lambda = 0$ evolve under a non-linear wave equation, while u_{λ} for $\lambda \neq 0$ evolve under a non-linear Klein-Gordon equation. This analysis follows the ideas of Metcalfe, Sogge and Stewart [22] and Metcalfe and Stewart [23], who analyze the wave equation on $\mathbb{R}^{n+1} \times D$, where D is a bounded domain in \mathbb{R}^m with various boundary conditions. Their analysis splits the function to eigenfunctions of the Laplacian on D with appropriate boundary conditions.

In this section, we recall some properties of the eigenvectors of Δ_K which we need for the proof. The material is taken from textbooks, [9, section 2.1] and [35, Chapter 5, section 8]. For the rest of this section we will deal only with forms on K . We will continue to employ the subscript \perp to maintain consistency. We begin with the following facts.

Proposition 2.3.1. *1. The operator Δ_K is a differential operator acting on the space of forms $\bigoplus_{i=0}^7 \Omega^i(K)$ with the principal symbol $-g_{ij} \xi^i \xi^j Id$, where g is the Riemannian metric.*

2. The operator Δ_K has a self-adjoint nonpositive-definite extension to the space of L^2 -valued forms on K .

Denote $\mathcal{P}_0 = \chi_{\{0\}}(-\Delta_K)$, $\mathcal{P}_{>0} = \chi_{\{\lambda: \lambda > 0\}}(-\Delta_K)$. These are spectral projections on the zero-, non-zero subspace of the spectrum of $-\Delta_K$, respectively.

Hodge Theory

The range of \mathcal{P}_0 , i.e all the forms ω that satisfy $\Delta_K \omega = 0$, are called the harmonic forms. We have the following simple fact.

Claim 2.3.2.

$$d_{\perp} \mathcal{P}_0 = 0.$$

Proof. Let $\delta_K = d_\perp^*$ be the $L^2(K)$ adjoint of d_\perp . We have the following characterization of Δ_K (see [9, Definition 2.1.2]³)

$$-\Delta_K = \delta_K d_\perp + d_\perp \delta_K = d_\perp^* d_\perp + d_\perp d_\perp^*.$$

Therefore for $\omega = \mathcal{P}_0\omega$, we have $\Delta_K\omega = 0$. Thus

$$\begin{aligned} 0 &= -\langle \Delta_K\omega, \omega \rangle_{L^2(K)} = \langle d_\perp^* d_\perp \omega + d_\perp d_\perp^* \omega, \omega \rangle_{L^2(K)} \\ &= \|d_\perp \omega\|_{L^2(K)}^2 + \|d_\perp^* \omega\|_{L^2(K)}^2, \end{aligned}$$

where $L^2(K)$ is the space of L^2 valued differential forms on K . □

The full version of this claim can be found in [9, Proposition 2.1.5]. It is the basis of Hodge theory in algebraic topology. We will not require any of it in this dissertation but we will quote the following theorem for the sake of beauty.

Theorem 2.3.3. *Every non-empty de-Rham cohomology class of K contains precisely one harmonic form.*

See [9, Theorem 2.2.1] for the proof. Thus existence and properties of harmonic forms are connected to the topological properties of the manifold. For instance the sphere \mathbb{S}^7 will have only two harmonic forms - the constant 0-form and the volume 7-form. The torus \mathbb{T}^7 will have $\binom{7}{n}$ linearly independent harmonic n -forms. Observe that both of these statements are independent of the choice of the Riemannian metric.

Elliptic regularity for Δ_K

We recall some basic regularity results for the form Laplacian. We have the following estimate.

Claim 2.3.4. *Let ω be a form on K then*

$$\|\omega\|_{H^2(K)} \leq C(\|\Delta_K\omega\|_{L^2(K)} + \|\omega\|_{L^2(K)}).$$

Proof. Combine the rudimentary elliptic estimate [35, Chapter 5, Theorem 1.3]

$$\|\omega\|_{H^2(K)} \leq C(\|\Delta_K\omega\|_{L^2(K)} + \|\omega\|_{H^1(K)}),$$

with some basic interpolation theory [35, Chapter 4, Proposition 3.1]

$$\|\omega\|_{H^1(K)} \leq C(\|\omega\|_{H^2(K)}\|\omega\|_{L^2(K)})^{1/2} \leq \varepsilon\|\omega\|_{H^2(K)} + C_\varepsilon\|\omega\|_{L^2(K)}.$$

□

³Our definition is the negative of [9]

Corollary 2.3.5. 1. $\mathcal{P}_0 L^2(K)$ is finite dimensional.

2. For every N there exists a constant C_N such that for every $\omega \in L^2(K)$

$$\frac{1}{C_N} \|\mathcal{P}_0 \omega\|_{H^k(K)} \leq \|\mathcal{P}_0 \omega\|_{L^\infty(K)} \leq C_N \|\mathcal{P}_0 \omega\|_{H^k(K)}, k \leq N, x \in \mathbb{R}^3.$$

Corollary 2.3.6. For every N , there are constants A_N such that for every $\omega \in H^n(K)$

$$\frac{1}{A_N} \|\mathcal{P}_{>0} \omega\|_{H^n(K)} \leq \|(-\Delta_K)^{\frac{n}{2}} \omega\|_{L^2(K)} \leq A_N \|\mathcal{P}_{>0} \omega\|_{H^n(K)}, \quad \forall n \leq N.$$

See the discussion leading to [35, Equation (8.20)] for the proofs. The practical conclusion that we will draw from these two corollaries is that when measuring smoothness of the solution in K variables, we can ignore the question completely for $u = \mathcal{P}_0 u$ and use the $(-\Delta_K)^{1/2}$ operator for $u = \mathcal{P}_{>0} u$.

2.4 The Linear Estimates

In this subsection, we would like to obtain decay estimates for the linear inhomogeneous equation. We will leverage this decay by employing the following subset of Klainerman vector fields:

$$\tilde{\Gamma} = \{\partial_i, i = 0..3\} \cap \{\Omega_{ij}, i, j = 0..3\}, \quad (2.4.1)$$

where

$$\begin{aligned} \Omega_{ij} &= x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i, j \leq 3, \\ \Omega_{0j} &= x_0 \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_0}, \quad 1 \leq i \leq 3. \end{aligned}$$

We augment $\tilde{\Gamma}$ with the operator $(-\Delta_K)^{1/2}$

$$\Gamma = \tilde{\Gamma} \cup \{(-\Delta_K)^{1/2}\} = \{\partial_i, i = 0..3\} \cup \{\Omega_{ij}, i, j = 0..3\} \cup \{(-\Delta_K)^{1/2}\}.$$

We will index the set Γ by $i = 1..11$ and for a multi-index $I = (I_1, I_2, \dots, I_{|I|}) \in \{1, \dots, 11\}^{|I|}$ we define the composition

$$\Gamma^I = \Gamma_{I_1} \Gamma_{I_2} \dots \Gamma_{I_{|I|}}.$$

We will introduce some notation to simplify the presentation. We will denote for an integer N , an abstract vector valued function:

$$\mathbf{\Gamma}^{(N)} f = (\Gamma^\alpha f)_{|\alpha| \leq N}.$$

Accordingly we will interpret the following notations

$$|\mathbf{\Gamma}^{(N)} f| = \sum_{|\alpha| \leq N} |\Gamma^\alpha f|$$

and

$$\|\mathbf{\Gamma}^{(N)} f\|_p = \sum_{|\alpha| \leq N} \|\Gamma^\alpha f(t, x, y)\|_{L^p(\mathbb{R}^3 \times K)}.$$

We will also have a similar notation for the gradients

$$|\nabla \mathbf{\Gamma}^{(N)} f| = \sum_{|\alpha| \leq N} \sum_{i=0}^3 \left| \frac{\partial}{\partial x_i} \Gamma^\alpha f \right| + \sum_{|\alpha| \leq N} |(-\Delta_K)^{\frac{1}{2}} \Gamma^\alpha f|$$

and

$$\begin{aligned} \|\nabla \mathbf{\Gamma}^{(N)} f\|_p &= \sum_{|\alpha| \leq N} \sum_{i=0}^3 \left\| \frac{\partial}{\partial x_i} \Gamma^\alpha f \right\|_{L^p(\mathbb{R}^3 \times K)} \\ &\quad + \sum_{|\alpha| \leq N} \|(-\Delta_K)^{\frac{1}{2}} \Gamma^\alpha f\|_{L^p(\mathbb{R}^3 \times K)} \end{aligned}$$

All those norm will be taken at a certain time t , which we will drop from the notation when there is no ambiguity. We will fix coordinate patches on K , with the appropriate partition of unity. That will turn our objects into vector valued functions on $\mathbb{R}^{3+1} \times \mathbb{R}^7$, so that we will apply the vector fields $\tilde{\Gamma}$ simply by applying them on every component of u .

Linear estimates in \mathbb{R}^{3+1}

We recall the following estimates in \mathbb{R}^{3+1} which we seek to generalize to the product case $\mathbb{R}^{3+1} \times K$.

Proposition 2.4.1. *Let $w \in C^\infty(\mathbb{R}^{3+1})$ such that $\square_{\mathbb{R}^{3+1}} w(t, x) = 0$ for $|x| > t - B$ then for $t \geq 2B$ we have*

$$\begin{aligned} (1+t)|\nabla_{t,x} w(t, x)| &\lesssim \|\nabla_{t,x} \mathbf{\Gamma}^{(2)} w(2B, \cdot)\|_{L^2(\mathbb{R}^3)} \\ &\quad + \sum_k \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B, t]} 2^k \|\mathbf{\Gamma}^{(2)} \square w(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

This proposition is proved in [23, Proposition 3.1]. Although [23] proves it with zero Cauchy data, the estimate with non-zero Cauchy data is proven in the same manner.

Proposition 2.4.2. *Let $w \in C^\infty(\mathbb{R}^{3+1})$ such that $(\square_{\mathbb{R}^{3+1}} + 1)w(t, x) = 0$ for $|x| \geq t - B$ then*

$$\begin{aligned} (1+t)^{3/2} \sup_x |w(t, x)| &\lesssim \|\mathbf{\Gamma}^{(6)} w(2B, x)\|_{L^2(\mathbb{R}^3)} \\ &\quad + \sum_k \sup_{\substack{\tau \in \\ [2^{k-1}, 2^{k+1}] \cap [2B, t]}} 2^k \|\mathbf{\Gamma}^{(5)} F(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where $F = (\square_{\mathbb{R}^{3+1}} + 1)w$.

This proposition is proved in [7, Proposition 7.3.6], refining the previous work of Klainerman [15].

Linear estimates in $\mathbb{R}^{3+1} \times K$

We turn to obtaining estimates for the equation

$$\square u + \Delta_K u = F, \quad u(0) = u_0, \quad \frac{\partial}{\partial t} u(0) = u_1.$$

Since the spectral projections \mathcal{P}_A commute with this equation, we will split the equation into two equations

$$\square \mathcal{P}_0 u = \mathcal{P}_0 F$$

and

$$\square \mathcal{P}_{>0} u + \Delta_K \mathcal{P}_{>0} u = \mathcal{P}_{>0} F,$$

with the spectral projections applied to initial data as well. By elliptic regularity and the estimate for the wave equation, Proposition 2.4.1, we have the following estimate.

Proposition 2.4.3. *Let $\text{supp} F(\cdot, \cdot, y) \subseteq \{(t, x) : |t - |x|| \leq 1\}$ for every $y \in K$ then the solution of*

$$\square \mathcal{P}_0 u = \mathcal{P}_0 F$$

obeys the estimate

$$(1+t)|\nabla_{t,x}(\mathcal{P}_0 u)(t, x, y)| \lesssim \|\nabla_{t,x} \mathbf{\Gamma}^{(2)}(\mathcal{P}_0 u)(0, \cdot, \cdot)\|_2 + \sum_k \sup_{s \in [2^{k-1}, 2^{k+1}] \cap [0, t]} 2^k \|\mathbf{\Gamma}^{(2)} F(s, \cdot, \cdot)\|_2.$$

Proof. We wish to apply Proposition 2.4.1 with $B = 1$. For that we need to switch to a new coordinate $\tau = t + 2$ then the proposition applies with one reservation: the vector fields in τ, x, y coordinates are different from the vector fields in t, x, y coordinates but they are expressible in terms of sums of the old ones since $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t}$ and

$$\Omega_{0i}(\tau) = \tau \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial \tau} = (t+2) \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t} = \Omega_{0i}(t) + 2 \frac{\partial}{\partial t}.$$

Thus, the Proposition 2.4.1 applies with possibly a different constant and (t, x, y) vector fields to show that for every $y \in K$

$$(1+t)|\nabla_{t,x}(\mathcal{P}_0 u)(t, x, y)| \lesssim \|\nabla_{t,x} \mathbf{\Gamma}^{(2)}(\mathcal{P}_0 u)(0, x, y)\|_{L^2(\mathbb{R}^3)} + \sum_k \sup_{s \in I_k} 2^k \|\mathbf{\Gamma}^{(2)}(\mathcal{P}_0 F)(s, \cdot, y)\|_{L^2(\mathbb{R}^3)},$$

where $I_k = [2^{k-1}, 2^{k+1}] \cap [0, t]$. Apply elliptic regularity (Corollary 2.3.5) to dominate $(\mathcal{P}_0 F)(s, \cdot, y)$ by its $L^2(K)$ norm. \square

Theorem 2.4.4. *Let $u(t, x, y)$ solve*

$$(\square_{\mathbb{R}^{3+1}} + \Delta_K)\mathcal{P}_{>0}u = \mathcal{P}_{>0}F,$$

then

$$(1+t)^{3/2}|\mathcal{P}_{>0}u(t, x, y)| \lesssim \|\nabla\Gamma^{(9)}\mathcal{P}_{>0}u(0, \cdot, \cdot)\|_2 \\ + \sum_k \sup_{s \in [2^{k-1}, 2^{k+1}] \cap [0, t]} 2^k \|\Gamma^{(9)}\mathcal{P}_{>0}F(s, \cdot, \cdot)\|_2,$$

provided $\mathcal{P}_{>0}F(\cdot, \cdot, y)$ is supported in $\{(x, t) \mid |x| \leq 1 + |t|\}$ for every $y \in K$.

The proof of the theorem follows almost verbatim the proof of [7, Proposition 7.3.5] with the exception of the following modification of [7, Lemma 7.3.4]

Lemma 2.4.5. *Let K be a compact manifold. Let $u \in \Omega(K)$ solve the equation*

$$-\frac{\partial^2}{\partial t^2}\mathcal{P}_{>0}u + \Delta_K\mathcal{P}_{>0}u = \mathcal{P}_{>0}F, \quad u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = u_1,$$

then

$$\|\mathcal{P}_{>0}u\|_{L^\infty(K)} \leq \|\Delta_K^2\mathcal{P}_{>0}u_0\|_{L^2(K)} + \|(-\Delta_K)^{3/2}\frac{\partial}{\partial t}\mathcal{P}_{>0}u_0\|_{L^2(K)} \\ + \int_0^t \|(-\Delta_K)^{3/2}\mathcal{P}_{>0}F(s, \cdot)\|_{L^2(K)} ds.$$

Proof of Lemma 2.4.5. We combine the energy estimate for the equation

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta_K\right)(-\Delta_K)^{3/2}u = (-\Delta_K)^{3/2}F,$$

which is

$$\|\Delta_K^2 u(t)\|_{L^2(K)} \lesssim \|\Delta_K^2 u(0)\|_{L^2(K)} + \left\|\frac{\partial}{\partial t}(-\Delta_K)^{3/2}u(0)\right\|_{L^2(K)} \\ + \int_0^t \|(-\Delta_K)^{3/2}F(s)\|_{L^2(K)} ds,$$

with the Sobolev embedding for a 7-dimensional manifold:

$$\|\mathcal{P}_{>0}u(t)\|_{L^\infty(K)} \lesssim \|\Delta_K^2 u(t)\|_{L^2(K)}.$$

□

Proof of Theorem 2.4.4. We will denote $\tau = t + 2$. We will again employ operators Γ in τ variable, which are different from the operators in t variable but can be expressed as a linear combination with coefficients independent of u . See proof of Proposition 2.4.3 for details regarding this substitution. We will use the following statement

Proposition 2.4.6. *Let g be supported in $\{(\tau, x); T/2 \leq \tau \leq T, |x| \leq \tau\}$. Denote*

$$M(\rho) = \sup_{\tau^2 - |x|^2 = \rho^2} |g(\tau, x)|$$

then

$$T^2 \int M(\rho^2)^2 \rho d\rho \leq C \sum_{|I| \leq \frac{3}{2}} \int |\tilde{\Gamma}^I g(\tau, x)|^2 d\tau dx.$$

The proof of this statement is given in [7, Lemma 7.3.1]. Let u solve

$$\square u + \Delta_K u = -\frac{\partial^2 u}{\partial \tau^2} + \sum_{i=1}^3 3 \frac{\partial^2 u}{\partial x_i^2} + \Delta_K u = F.$$

Introduce the hyperbolic polar coordinates $(\tau, x) = \rho\omega$, $\rho = (\tau^2 - |x|^2)^{\frac{1}{2}}$, $\omega \in \mathbb{S}^2$. Then the equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial u}{\partial \rho} - \Delta_K u = \frac{1}{\rho^2} \Delta_H u + f,$$

where Δ_H is the Laplacian on the hyperbolic space. We have

$$\Delta_H = \sum_i (t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t})^2 - \sum_{k,j} (x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})^2$$

Thus $v = \rho^{\frac{3}{2}} u$ obeys

$$\frac{\partial^2 v}{\partial \rho^2} - \Delta_K v = \rho^{3/2} (\rho^{-2} (\Delta_H u + 3 \frac{u}{4}) + F).$$

We decompose the right-hand-side dyadically in time. Let χ be a smooth function, supported on $[\frac{1}{2}, 2]$ s.t. $\sum_{k=-\infty}^{\infty} \chi(\frac{\tau}{2^k}) = 1$. Denote $f_k = \chi(\frac{\tau}{2^k}) F$, $u_k = \chi(\frac{\tau}{2^k}) (\Delta_H u + n(n-2) \frac{u}{4})$. We apply the Lemma 2.4.5. We have

$$|v(\rho, \omega, y)| \leq \sum_k \int_0^\rho \rho^{3/2} (\rho^{-2} \|u_k(\rho, \omega, \cdot)\|_{H^4(K)} + \|f_k(\rho, \omega, \cdot)\|_{H^4(K)}) \quad (2.4.2)$$

We wish to estimate $u(\bar{\tau}, \bar{x})$, thus we need to estimate the sum of the integrals above. Denote $M_k(\rho^2) = \sup_\omega \|f_k(\rho, \omega, \cdot)\|_{H^4(K)}$ then we have

$$\int_0^{\bar{\rho}} \rho^{\frac{3}{2}} \|f_k\|_{H^4(K)} d\rho \leq \int_0^{\bar{\rho}} M_k(\rho^2) \rho d\rho \int_0^{\bar{\rho}} \rho^2 d\rho.$$

We have $2^{k-1} \leq t \leq 2^k$ therefore $\frac{\rho}{\bar{\tau}} \leq C\frac{\tau}{\bar{\tau}} \leq C\frac{2^k}{\bar{\tau}}$ in the support of u . Thus

$$\int \rho^2 d\rho \leq C\frac{\bar{\rho}^3 2^{3k}}{\bar{\tau}^3}$$

According to Proposition 2.4.6, we have

$$\begin{aligned} \int M_k(\rho^2)\rho d\rho &\leq 2^{-2k}C \sum_{|I| \leq \frac{5}{2}} \int |\tilde{\Gamma}^I \|f_k(\tau, x, \cdot)\|_{H^4(K)}|^2 d\tau dx \\ &\leq 2^{-2k}C \sum_{|I| \leq \frac{5}{2}} \int_{\tau \leq \bar{\tau}} \int_K |\tilde{\Gamma}^I \Delta_K^2 f_k(\tau, x, y)|^2 d\tau dx dy \end{aligned}$$

Therefore, we have

$$\begin{aligned} \int \rho^{3/2} \|f_k(\rho, \omega, \cdot)\|_{H^4(K)} &\leq \frac{2^{\frac{k}{2}} \rho^{3/2}}{\bar{\tau}^{3/2}} \sum_{|I| \leq \frac{5}{2}} \int_{2^{k-1} \leq \tau \leq 2^k} \int_K |\tilde{\Gamma}^I \Delta_K^2 f_k(\tau, x, y)|^2 d\tau dx dy \\ &\leq \rho^{3/2} 2^k \frac{1}{\bar{\tau}^{3/2}} \sum_{|I| \leq 7} \sup_{\tau \in [2^{k-1}, 2^{k+1}] \cap [2B, \bar{\tau}]} \|\Gamma^I f(\tau, \cdot, \cdot)\|. \end{aligned}$$

To deal with the first term on the right-hand side of equation (2.4.2), we repeat the argument in [7, Lemma 7.3.4] verbatim, to get the estimate

$$\int \rho^{-\frac{1}{2}} \|u_k\|_{H^4} \leq \frac{\bar{\rho}^{3/2}}{\bar{\tau}^{3/2}} \sum_{|I| \leq \frac{9}{2}} \sup_{2B \leq t} \|\tilde{\Gamma}^I u(t, \cdot)\|_{H^4(K)}.$$

We use the energy estimate to bound this expression by the estimate (2.4.3) and the initial energy to get after summation in k

$$\bar{\rho}^{\frac{3}{2}} |u| = |v| \leq \frac{\bar{\rho}^{\frac{3}{2}}}{\bar{\tau}^{\frac{3}{2}}} \sum_{|I| \leq 6} \left(\sum_i \|\Gamma^I \Gamma_i u_0\| + \sum_k 2^k \sup_{2^{k-1} \leq \tau \leq 2^{k+1}} \|\Gamma^I f(\tau)\| \right)$$

Dividing by $\frac{\bar{\rho}^{\frac{3}{2}}}{\bar{\tau}^{\frac{3}{2}}} = \frac{\bar{\rho}^{\frac{3}{2}}}{(t+1)^{\frac{3}{2}}}$ completes the proof. \square

Corollary 2.4.7. *Let M be an integer greater than 9. Let u solve*

$$(\square_{\mathbb{R}^{3+1}} + \Delta_K)u = F,$$

with initial data supported in the ball of radius 1 for every $y \in K$ and

$$\text{supp} F(\cdot, \cdot, y) \subseteq \{(t, x) : |x| \leq 1 + |t|\}$$

for every $y \in K$. Then the following estimate holds:

$$\begin{aligned} (1+t)\|\nabla\Gamma^{(M-10)}\mathcal{P}_0u(t)\|_\infty \\ + (1+t)^{3/2}\|\nabla\Gamma^{(M-10)}\mathcal{P}_{>0}u(t)\|_\infty \lesssim \|\nabla\Gamma^{(M)}u(0)\|_2 \\ + \sum_k \sup_{s \in I_k} 2^k \|\Gamma^{(M)}F(s, \cdot)\|_2, \end{aligned} \quad (2.4.3)$$

where $I_k = [2^{k-1}, 2^{k+1}] \cap [0, t]$.

Energy estimates

We combine the energy estimates for the solution of $\square_{\mathbb{R}^{3+1} \times K}u = F$ with the fact that the operators Γ are symmetries of the equation and use the notation introduced above.

Proposition 2.4.8. *Let u be the solution of $\square_{\mathbb{R}^{3+1} \times K}u = F$ then for any $M \geq 0$ we have*

$$\|\nabla\Gamma^{(M)}u(t)\|_2 \leq \|\nabla\Gamma^{(M)}u(0)\|_2 + \int_0^t \|\Gamma^{(M)}F(s)\|_{L^2} ds.$$

2.5 Analysis of the nonlinearity

In this section, we will treat the bilinear form $(u_1, u_2) \mapsto *(du_1 \wedge du_2)$. Recall from Subsection 2.2 the $*$ operator exchanges the components of the forms, multiplying those containing the time x^0 coordinate by -1 . The $*$ operator loses the simple form when the metric on K is no longer the identity, but because of tensoriality, it will be multiplied by a function depending only on x^4, \dots, x^{11} , which due to compactness will be bounded above and below. Therefore, when we take $L^2(\mathbb{R}^3 \times K)$ -norm at a certain time, we will consider $*v$ to be L^2 equivalent to v . Furthermore, we will be interested in the action of Γ operators on $*(du \wedge du)$. Clearly, the operators which act on \mathbb{R}^{3+1} componentwise will commute with $*$. The equation (2.2.5) shows that the Laplacian Δ_K commutes with $*$ simply because $\Delta_K = \sum_{i=4}^{10} \frac{\partial^2}{\partial x_i^2}$ at that point and the relation is tensorial. Thus any function of Δ_K will commute with $*$ and we have

$$(-\Delta_K)^{1/2} * v = *(-\Delta_K)^{1/2} v.$$

From this discussion we conclude that

$$\|\Gamma^\alpha(*v(t))\|_{L^2(\mathbb{R}^3 \times K)} \cong \|*\Gamma^\alpha v\|_{L^2(\mathbb{R}^3 \times K)} \cong \|\Gamma^\alpha v\|_{L^2(\mathbb{R}^3 \times K)},$$

for any multi-index α , time t , with constants which depend only on the manifold K .

The splitting of the nonlinearity

Recall that the operator d splits into $d = d_{\parallel} + d_{\perp}$. Also any form u on $\mathbb{R}^{3+1} \times K$ can be written as $u = \mathcal{P}_0 u + \mathcal{P}_{>0} u$. Therefore, we can write

$$\begin{aligned} *(du \wedge du) &= *[(d_{\parallel} + d_{\perp})(\mathcal{P}_0 u + \mathcal{P}_{>0} u) \wedge (d_{\parallel} + d_{\perp})(\mathcal{P}_0 u + \mathcal{P}_{>0} u)] \\ &= *(d_{\parallel} \mathcal{P}_0 u \wedge d_{\parallel} \mathcal{P}_0 u) \\ &\quad + 2*(d_{\parallel} \mathcal{P}_0 u \wedge d \mathcal{P}_{>0} u) + *(d \mathcal{P}_{>0} u \wedge d \mathcal{P}_{>0} u). \end{aligned}$$

where we used that $d_{\perp} \mathcal{P}_0 = 0$, which is the content of Claim 2.3.2. With this we proved the following splitting of the nonlinearity:

Claim 2.5.1. *Let u_1, u_2 be differential 3-forms on $\mathbb{R}^{3+1} \times K$. Denote*

$$B(\mathcal{P}_0 u_1, \mathcal{P}_0 u_2) = *(d_{\parallel} \mathcal{P}_0 u_1 \wedge d_{\parallel} \mathcal{P}_0 u_2),$$

$$C(\mathcal{P}_0 u_1, \mathcal{P}_{>0} u_2) = *(d_{\parallel} \mathcal{P}_0 u_1 \wedge d \mathcal{P}_{>0} u_2),$$

$$D(\mathcal{P}_{>0} u_1, \mathcal{P}_{>0} u_2) = *(d \mathcal{P}_{>0} u_1 \wedge d \mathcal{P}_{>0} u_2).$$

Then

$$*(du \wedge du) = B(\mathcal{P}_0 u, \mathcal{P}_0 u) + 2C(\mathcal{P}_0 u, \mathcal{P}_{>0} u) + D(\mathcal{P}_{>0} u, \mathcal{P}_{>0} u).$$

The basic estimate

Proposition 2.5.2. *Let F be any of the forms B, C, D defined in Claim 2.5.1 or the total nonlinearity which is $B + 2C + D$. Let N be a positive integer. Then there exists a constant k such that for any $M \leq N$ we have*

$$\begin{aligned} \|\Gamma^{(M)} F(v_1, v_2)\|_2 &\lesssim \|\nabla \Gamma^{(\frac{M}{2})} v_1\|_{p_1} \|\nabla \Gamma^{(M)} v_2\|_{q_1} \\ &\quad + \|\nabla \Gamma^{(M)} v_1\|_{q_2} \|\nabla \Gamma^{(\frac{M}{2})} v_2\|_{p_2}, \end{aligned}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}$, $2 \leq p_i, q_i \leq \infty$.

Proof. Choose a coordinate patch $x^i, i = 4..10$ for K . Then de Rham differentials $d, d_{\perp}, d_{\parallel}$ can be written as $a_i \frac{\partial}{\partial x_i}$ for a_i which depend only on $x^i, i \geq 4$. Thus we need to estimate an expression of the form

$$I = \sum_{i,j=0}^{10} \Gamma^{\alpha} (a_i a'_j \frac{\partial}{\partial x_i} v_1 \frac{\partial}{\partial x_j} v_2).$$

Observe that all the operators in Γ besides $(-\Delta_K)^{1/2}$ are vector fields and thus obey Leibniz's rule. So assume first that in the composite operator Γ^{α} there are no $(-\Delta_K)^{1/2}$ operators.

We treat the Hodge dual $*$ as a constant coefficient operator, which only permutes between different components. Next, we employ the Jacobi identity to write I as

$$\sum_{\alpha'+\alpha''=\alpha} C_{\alpha'\alpha''} \sum_{i,j=0}^{10} a_i a'_j \Gamma^{\alpha'} \left(\frac{\partial}{\partial x_i} v_1 \right) \Gamma^{\alpha''} \left(\frac{\partial}{\partial x_j} v_2 \right),$$

where $C_{\alpha'\alpha''}$ are constants. Observe that Γ_i commutes with $\frac{\partial}{\partial x_i}$, for $i \geq 4$ and for $i, j \leq 4$ we have

$$\left[\frac{\partial}{\partial x_k}, \Omega_{0j} \right] = \delta_{0k} \frac{\partial}{\partial x_j} + \delta_{jk} \frac{\partial}{\partial x_0}, \left[\frac{\partial}{\partial x_k}, \Omega_{ij} \right] = \delta_{jk} \frac{\partial}{\partial x_k} - \delta_{ik} \frac{\partial}{\partial x_j}.$$

Thus we commute $\frac{\partial}{\partial x_i}$ with Γ 's to get

$$I = \sum_{|\beta'|+|\beta''|\leq M} \sum_{i,j=0}^{10} C_{\beta'\beta''ij} \left(\frac{\partial}{\partial x_i} \Gamma^{\beta'} v_1 \right) \left(\frac{\partial}{\partial x_j} \Gamma^{\beta''} v_2 \right),$$

for some constants $C_{\beta'\beta''ij}$. In the expression above only one of the $|\beta'|, |\beta''|$ can be larger than $M/2$. We split the sum accordingly

$$\begin{aligned} |I| &\lesssim \sum_{|\beta'|\leq \frac{M}{2}, i} \left| \frac{\partial}{\partial x_i} \Gamma^{\beta'} v_1 \right| \sum_{|\beta''|\leq M, j} \left| \frac{\partial}{\partial x_j} \Gamma^{\beta''} v_2 \right| \\ &\quad + \sum_{|\beta'|\leq M, i} \left| \frac{\partial}{\partial x_i} \Gamma^{\beta'} v_1 \right| \sum_{|\beta''|\leq \frac{M}{2}, j} \left| \frac{\partial}{\partial x_j} \Gamma^{\beta''} v_2 \right|. \end{aligned}$$

We then obtain the required estimate by applying the L^2 norm to $|I|$ and using the appropriate Hölder inequalities.

In case Γ^α contains some m appearances of the operator $(-\Delta_K)^{1/2}$, we note that $(-\Delta_K)^{1/2}$ commutes with all the other Γ operators as the rest of Γ operate on \mathbb{R}^{3+1} only. By elliptic regularity

$$\|I\|_2 = \|(-\Delta_K)^{\frac{m}{2}} \Gamma^\alpha F\|_2 \lesssim \|\mathcal{P}_{>0} \Gamma^\alpha F\|_{H^m(K)} \leq \|\Gamma^\alpha F\|_{H^m(K)}.$$

Now $H^m(K)$ norm obeys a Jacobi's "inequality", which is a primitive form of the Kato-Ponce estimates, see [11],[36, Chapter II, Prop. 1.1] and thus the rest of the proof proceeds in a similar fashion. \square

Null form

Continuing with the notation of Claim 2.5.1, we need the observation that $B(\mathcal{P}_0 u_1, \mathcal{P}_0 u_2) = *(d_{\parallel} \mathcal{P}_0 u_1 \wedge d_{\parallel} \mathcal{P}_0 u_2)$ is a null form and the estimates that follow from it.

Proposition 2.5.3. *The bilinear form $B(\omega_1, \omega_2) = *(d_{\parallel} \omega_1 \wedge d_{\parallel} \omega_2)$ is a null-form*

We will give two proofs of the proposition.

Fourier. It is enough to compute $B(\omega_1, \omega_2)$ for $\omega_i = A_i e^{ik_i x}$ for two parallel null-vectors, with A_i being constant. If $B(\omega_1, \omega_2)$ vanishes in such a case then B is a null-form. But

$$B(\omega_1, \omega_2) = *(k_1 \wedge A_1 \wedge k_2 \wedge A_2) e^{i(k_1+k_2)x}.$$

When two vectors in the wedge product are parallel, the wedge product vanishes. Thus

$$B(\omega_1, \omega_2) = 0.$$

□

Coefficients. Let

$$A = A_{ijk} dx_i dx_j dx_k$$

and

$$B = B_{lmn} dx_l dx_m dx_n.$$

Then

$$d_{\parallel} A = \frac{\partial A_{ijk}}{\partial x_p} dx_p dx_i dx_j dx_k,$$

$$d_{\parallel} B = \frac{\partial B_{lmn}}{\partial x_s} dx_s dx_l dx_m dx_n.$$

Therefore

$$\begin{aligned} d_{\parallel} A \wedge d_{\parallel} B &= \frac{\partial A_{ijk}}{\partial x_p} \frac{\partial B_{lmn}}{\partial x_s} dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n \\ &\quad + \frac{\partial A_{ijk}}{\partial x_s} \frac{\partial B_{lmn}}{\partial x_p} dx_s dx_i dx_j dx_k dx_p dx_l dx_m dx_n \\ &= \left(\frac{\partial A_{ijk}}{\partial x_p} \frac{\partial B_{lmn}}{\partial x_s} - \frac{\partial A_{ijk}}{\partial x_s} \frac{\partial B_{lmn}}{\partial x_p} \right) dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n. \end{aligned}$$

Thus

$$\begin{aligned} *d_{\parallel} A \wedge d_{\parallel} B &= \left(\frac{\partial A_{ijk}}{\partial x_p} \frac{\partial B_{lmn}}{\partial x_s} - \frac{\partial A_{ijk}}{\partial x_s} \frac{\partial B_{lmn}}{\partial x_p} \right) \\ &\quad * (dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n). \end{aligned}$$

Since

$$\frac{\partial A_{ijk}}{\partial x_p} \frac{\partial B_{lmn}}{\partial x_s} - \frac{\partial A_{ijk}}{\partial x_s} \frac{\partial B_{lmn}}{\partial x_p}$$

is a null-form, $*d_{\parallel} A \wedge d_{\parallel} B$ is a null-form. Observe that we needed to assume that only dx_p and dx_s are co-vectors on \mathbb{R}^{3+1} ; all the other indices could have belonged to either K or

\mathbb{R}^{3+1} . In order to see that the sign in front of the second term is negative, we count the transpositions needed to transform

$$\omega_1 = dx_s dx_i dx_j dx_k dx_p dx_l dx_m dx_n$$

to

$$\omega_2 = dx_p dx_i dx_j dx_k dx_s dx_l dx_m dx_n$$

one needs four to bring dx_p to the front and then another three to bring dx_s behind $dx_i dx_j dx_k$ therefore there are 7 transpositions in total and the sign is minus, $\omega_1 = -\omega_2$. \square

Denote

$$Q_{ij}(f, g) = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i}.$$

As the proof of Proposition 2.5.3 shows the form $B(\omega_1, \omega_2) = *(d_{\parallel}\omega_1 \wedge d_{\parallel}\omega_2)$ is the sum of the forms Q_{ij} applied to different components. We have the following estimate.

Lemma 2.5.4.

$$|Q_{ij}(f, g)| \lesssim \frac{1}{1+t} (|\Gamma f| |\nabla_{x,t} g| + |\nabla_{x,t} f| |\Gamma g|).$$

Remark. *The proof is taken from [16, Lemma 1.1]. We reproduce it here to stress that we have the estimate involving only vector fields Γ and not the full set of Klainerman vector fields, which includes the radial scaling field $x_0 \frac{\partial}{\partial x_0} + \dots + x_3 \frac{\partial}{\partial x_3}$.*

Proof. We have

$$\frac{\partial}{\partial x_i} = -\frac{x_i}{t} \frac{\partial}{\partial x_0} + \frac{1}{t} \Omega_{0i}, \quad i = 1, 2, 3,$$

Thus we have for $i, j \geq 1$

$$Q_{ij}(f, g) = \frac{1}{t} \left[-\frac{\partial f}{\partial x_0} \Omega_{ij} g + (\Omega_{0i} f \frac{\partial g}{\partial x_j} - \Omega_{0j} f \frac{\partial g}{\partial x_i}) \right].$$

For $i = 0$, we have

$$Q_{0j} = \frac{1}{t} \left(\frac{\partial f}{\partial x_0} \Omega_{0j} g - \Omega_{0j} f \frac{\partial g}{\partial x_0} \right).$$

\square

We wish to prove a variant of the basic estimate specialized to the null form.

Proposition 2.5.5.

$$\begin{aligned} \|\Gamma^{(M)} B(v, v)\|_2 \lesssim & \frac{1}{1+t} \left(\|\Gamma^{(\frac{M}{2}+1)} v\|_{p_1} \|\nabla \Gamma^{(M)} v\|_{q_1} \right. \\ & \left. + \|\Gamma^{(M+1)} v\|_{q_2} \|\nabla \Gamma^{(\frac{M}{2})} v\|_{p_2} \right), \end{aligned}$$

Proof. The bilinear form B is a linear combination of the forms Q_{ij} . The forms Q_{ij} are preserved under applications of the Γ operators (with the exception of $(-\Delta_K)^{1/2}$) because of the following formula which appears in [16, Lemma 1.2] and which can be obtained by direct calculation

$$\Omega_{\alpha\beta}Q_{\gamma\delta}(f, g) = Q_{\gamma\delta}(\Omega_{\alpha\beta}f, g) + Q_{\gamma\delta}(f, \Omega_{\alpha\beta}g) + \tilde{Q}(f, g), \quad (2.5.1)$$

where

$$\tilde{Q}(f, g) = m_{\alpha\gamma}Q_{\beta\delta}(f, g) - m_{\beta\gamma}Q_{\alpha\delta}(f, g) + m_{\alpha\delta}Q_{\beta\gamma}(f, g) + m_{\beta\delta}Q_{\alpha\gamma}(f, g),$$

where $m_{\alpha\beta}$ are coefficients of the Minkowski metric. Thus we apply Lemma 2.5.4 and after grouping the multi-indices with order less than $\frac{M}{2}$ and applying the Hölder estimates we get the result like in Proposition 2.5.2. For $(-\Delta_K)^{1/2}$ we apply Kato-Ponce estimates. \square

2.6 Proof of Theorem 2.1.1

To prove the theorem we seek to establish the following apriori bounds for the solutions of (2.1.1):

$$(1+t)^{-\delta}\|\nabla\Gamma^{(N)}u(t)\|_2 \leq \epsilon, \quad (2.6.1a)$$

$$\|\nabla\Gamma^{(N-10)}u(t)\|_2 \leq \epsilon, \quad (2.6.1b)$$

$$(1+t)\|\nabla\Gamma^{(N-20)}\mathcal{P}_0u(t)\|_\infty + (1+t)^{3/2}\|\nabla\Gamma^{(N-20)}\mathcal{P}_{>0}u(t)\|_\infty \leq \epsilon, \quad (2.6.1c)$$

where δ is smaller than $\frac{1}{12}$. Any of the global in time estimates above implies uniqueness, existence and well-posedness for the semilinear wave equations by employing the local theory which is explained in [28, Chapter 2] or [7, section 6.2].

We will prove the estimates by bootstrapping. Namely, we will prove that (2.6.1) imply

$$(1+t)^{-\delta}\|\nabla\Gamma^{(N)}u(t)\|_2 \leq \frac{\epsilon}{2}, \quad (2.6.2a)$$

$$\|\nabla\Gamma^{(N-10)}u(t)\|_2 \leq \frac{\epsilon}{2}, \quad (2.6.2b)$$

$$(1+t)\|\nabla\Gamma^{(N-20)}\mathcal{P}_0u(t)\|_\infty + (1+t)^{3/2}\|\nabla\Gamma^{(N-20)}\mathcal{P}_{>0}u(t)\|_\infty \leq \frac{\epsilon}{2}, \quad (2.6.2c)$$

i.e. the right-hand side can be made $\frac{\epsilon}{2}$ instead of ϵ . Thus our goal is to establish the following statement.

Proposition 2.6.1. *Let N be an integer that satisfies*

$$\frac{N}{2} \leq N - 20.$$

Then there exists $\epsilon > 0$ small enough such that, for every $T > 0$, if a solution u to the Cauchy problem (2.1.1) with initial data that satisfies

$$\|\Gamma^{(N)}u_0\|_2 + \|\Gamma^{(N-1)}u_1\| \leq \frac{\epsilon}{4},^4$$

such that $u_0(\cdot, y), u_1(\cdot, y)$ are localized in a ball of radius 1 for every $y \in K$ and if the inequalities (2.6.1a), (2.6.1b), (2.6.1c) hold for every $0 < t \leq T$ then the inequalities (2.6.2a), (2.6.2b), (2.6.2c) hold for every $0 < t \leq T$, where δ is a positive exponent which depends on ϵ and is smaller than $\frac{1}{12}$

Remark. There are two considerations that affect the smallness of ϵ . One is that we will see that we can replace ϵ in the right-hand side of (2.6.1) by $\frac{\epsilon}{4} + k\epsilon^2$, where k is an a priori computable constant that depends only on N and the geometry of the manifold K . Thus we will need to decrease ϵ to achieve the inequality

$$\frac{\epsilon}{4} + k\epsilon^2 \leq \frac{\epsilon}{2}.$$

The second consideration is that the exponent δ depends linearly on ϵ , $\delta = C\epsilon$ with C depending on N and the geometry of K . We have to decrease ϵ so that $\delta = C\epsilon \leq \frac{1}{12}$.

Remark 2.6.2. We are now in position to explain the difference between our proof of the null-form estimates and the classical version in [16]. One of the main points is that the null-form estimate in Lemma 2.5.4 loses the gradient in front of the vector field. The energy estimates do not control the value of the solution, only its gradients. Therefore, we need to recover control over the value of the solution. The way it is done in [16] is by proving a delicate estimate, which involves the fundamental solution. More importantly, the estimate involves the powerful expansion vector field $t\partial_t + r\partial_r$ which is no longer a conformal symmetry in our problem and therefore not useful. Our technique is the use of interpolation and the basic Sobolev estimate $\|u\|_{L^6} \leq \|\nabla u\|_{L^2}$.

We prove the implication (2.6.2a) in the next lemma, while the implications (2.6.2b), (2.6.2c) are proved in Lemma 2.6.6.

Lemma 2.6.3. *Under conditions of Proposition 2.6.1, (2.6.1c) implies (2.6.2a).*

Proof. We use the energy estimate for the equation $\square\Gamma^{(N)}u = \Gamma^{(N)}(*du \wedge du)$.

$$\|\nabla\Gamma^{(N)}u(t)\|_2 \leq \|\nabla\Gamma^{(N)}u(0)\|_2 + \int_0^t \|\Gamma^{(N)} * du \wedge du(s)\|_2 ds. \quad (2.6.3)$$

We now use the Proposition 2.5.2 to estimate the nonlinearity. We have

$$\|\Gamma^{(N)} * du \wedge du(s)\|_2 \leq C\|\nabla\Gamma^{(\frac{N}{2})}u\|_\infty\|\nabla\Gamma^{(N)}u\|_2.$$

⁴Interpret high time derivatives of the initial data by using the equation.

Since $\frac{N}{2} \leq N - 20$ we employ the assumption (2.6.1c) in equation (2.6.3) to conclude

$$\|\nabla\Gamma^{(N)}u(t)\|_2 \leq \frac{\epsilon}{4} + \int_0^t \frac{C\epsilon}{1+s} \|\nabla\Gamma^{(N)}u(s)\|_2 ds.$$

After applying Gronwall inequality we conclude

$$\|\nabla\Gamma^{(N)}u(t)\|_2 \leq \frac{\epsilon}{4}(1+t)^{C\epsilon},$$

which is the required inequality. \square

We require the following interpolated intermediate result.

Claim 2.6.4. *Let $2 \leq p \leq \infty$ then:*

1. *Assumptions (2.6.1a) and (2.6.1c) imply*

$$(1+t)^{-\delta+1-\frac{2}{p}} \|\nabla\Gamma^{(N-10)}\mathcal{P}_0u\|_p + (1+t)^{-\delta+3/2(1-\frac{2}{p})} \|\nabla\Gamma^{(N-10)}\mathcal{P}_{>0}u\|_p \lesssim \epsilon. \quad (2.6.4)$$

2. *Assumptions (2.6.1b) and (2.6.1c) imply*

$$(1+t)^{1-\frac{2}{p}} \|\nabla\Gamma^{(N-20)}\mathcal{P}_0u\|_p + (1+t)^{3/2(1-\frac{2}{p})} \|\nabla\Gamma^{(N-20)}\mathcal{P}_{>0}u\|_p \lesssim \epsilon. \quad (2.6.5)$$

Proof. Obviously, the estimate for sum with more derivatives is true for the sum with less derivatives and thus we will interpolate between equation (2.6.1b) and equation (2.6.1c) to get the second conclusion. To prove the first point, we use the following intermediate result,

$$(1+t)^{-\delta+1} \|\nabla\Gamma^{(N-10)}\mathcal{P}_0u\|_\infty + (1+t)^{-\delta+3/2} \|\nabla\Gamma^{(N-10)}\mathcal{P}_{>0}u\|_\infty \lesssim \epsilon. \quad (2.6.6)$$

By interpolation, of equation (2.6.6) with equation (2.6.1a) we have

$$(1+t)^{-\delta+1-\frac{2}{p}} \|\nabla\Gamma^{(N-10)}\mathcal{P}_0u\|_p + (1+t)^{-\delta+3/2(1-\frac{2}{p})} \|\nabla\Gamma^{(N-10)}\mathcal{P}_{>0}u\|_p \lesssim \epsilon.$$

To establish (2.6.6) we use equation (2.4.3)

$$\begin{aligned} (1+t) \|\nabla\Gamma^{(N-10)}\mathcal{P}_0u\|_\infty \\ + (1+t)^{3/2} \|\nabla\Gamma^{(N-10)}\mathcal{P}_{>0}u\|_\infty \leq \|\nabla\Gamma^{(N)}u(0)\|_2 \\ + \sum_n \sup_{\tau \in I_n} 2^n \|\Gamma^{(N)}F(\tau)\|_2, \end{aligned}$$

where $I_n = [2^{n-1}, 2^{n+1}] \cap [0, t]$ and $F = *(du \wedge du)$. We use Proposition 2.5.2 and combine it with the assumptions to get

$$\begin{aligned} \|\Gamma^{(N)}F(\tau)\|_2 = \|\Gamma^{(N)} * du \wedge du(\tau)\|_2 &\lesssim C \|\nabla\Gamma^{(\frac{N}{2})}u\|_\infty \|\nabla\Gamma^{(N)}u\|_2 \\ &\lesssim \epsilon^2 (1+\tau)^{\delta-1}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 (1+t) \|\nabla \Gamma^{(N-10)} \mathcal{P}_0 u\|_\infty \\
 + (1+t)^{3/2} \|\nabla \Gamma^{(N-10)} \mathcal{P}_{>0} u\|_\infty &\lesssim \|\Gamma^{(N)} u(0)\|_2 \\
 &+ \sum_n \sup_{\tau \in [2^{n-1}, 2^{n+1}] \cap [0, t]} \epsilon 2^n \tau^{-1+\delta} \\
 &\leq \frac{\epsilon}{4} + k\epsilon^2(1+t)^\delta,
 \end{aligned}$$

which proves (2.6.6). \square

To complete the proof of Proposition 2.6.1, we analyze the equation

$$\square_{\mathbb{R}^{3+1} \times K} \Gamma^{(N-10)} u = \Gamma^{(N-10)} * du \wedge du.$$

We estimate the right-hand side of the equation in the following lemma.

Lemma 2.6.5. *Under assumptions (2.6.1a), (2.6.1b), (2.6.1c) we have*

$$\|\Gamma^{(N-10)} * du \wedge du(t)\|_2 \lesssim \epsilon^2(1+t)^{-1-\frac{1}{3}+\delta} \leq \epsilon^2(1+t)^{-1-\frac{1}{4}}.$$

Proof. Recall the splitting of the nonlinearity in Claim 2.5.1 and denote

$$B(t) = \|\Gamma^{(N-10)} B(\mathcal{P}_0 u(t), \mathcal{P}_0 u(t))\|_2, \quad (2.6.7a)$$

$$C(t) = \|\Gamma^{(N-10)} C(\mathcal{P}_0 u(t), \mathcal{P}_{>0} u(t))\|_2, \quad (2.6.7b)$$

$$D(t) = \|\Gamma^{(N-10)} D(\mathcal{P}_{>0} u(t), \mathcal{P}_{>0} u(t))\|_2. \quad (2.6.7c)$$

To prove the lemma, we will obtain the bound above for $B(t), C(t), D(t)$.

Estimate for $B(t)$ To estimate $B(t)$ we use Proposition 2.5.5 to get

$$\begin{aligned}
 \|\Gamma^{(N-10)} B(\mathcal{P}_0 u, \mathcal{P}_0 u)\|_2 &\leq \frac{k}{1+t} \left(\|\Gamma^{(\frac{N}{2}-4)} \mathcal{P}_0 u\|_6 \|\nabla \Gamma^{(N-10)} \mathcal{P}_0 u\|_3 \right. \\
 &\quad \left. + \|\Gamma^{(N-9)} \mathcal{P}_0 u\|_6 \|\nabla \Gamma^{(\frac{N}{2})} \mathcal{P}_0 u\|_3 \right).
 \end{aligned} \quad (2.6.8)$$

We wish to use the homogeneous Sobolev embedding in \mathbb{R}^3 . Thus, we have for every $y \in K$,

$$\|\Gamma^{(M)} \mathcal{P}_0 u(t, \cdot, y)\|_{L^6(\mathbb{R}^3)} \leq \|\nabla \Gamma^{(M)} \mathcal{P}_0 u(t, \cdot, y)\|_{L^2(\mathbb{R}^3)}.$$

Next we employ the compactness of K to see that the $L^6(K)$ norm of $\mathcal{P}_0 \Gamma^{(M)} u$ is dominated by the $L^\infty(K)$ norm. At the same time, elliptic regularity shows that the $L^\infty(K)$ norm of $\mathcal{P}_0 \nabla \Gamma^{(M)} u$ is dominated by its $L^2(K)$ norm. Therefore

$$\|\Gamma^{(\frac{N}{2}-4)} \mathcal{P}_0 u\|_6 \leq \|\Gamma^{(N-20)} \mathcal{P}_0 u\|_6 \leq \|\nabla \Gamma^{(N-20)} u\|_2 \lesssim \epsilon, \quad (2.6.9)$$

by (2.6.1b) and the fact that $\frac{N}{2} \leq N - 20$. Also

$$\|\mathbf{\Gamma}^{(N-9)}\mathcal{P}_0u\|_6 \leq \|\nabla\mathbf{\Gamma}^{(N-9)}u\|_2 \lesssim \epsilon(1+t)^\delta \quad (2.6.10)$$

by (2.6.1a). Employing (2.6.4) again we have

$$\|\nabla\mathbf{\Gamma}^{(N-10)}u\|_6 \lesssim \epsilon(1+t)^{\delta-\frac{1}{3}}. \quad (2.6.11)$$

Also equation (2.6.5) implies

$$\|\nabla\mathbf{\Gamma}^{(\frac{N}{2}-5)}\mathcal{P}_0u\|_3 \leq \|\nabla\mathbf{\Gamma}^{(N-20)}\mathcal{P}_0u\|_3 \lesssim \epsilon(1+t)^{-\frac{1}{3}}. \quad (2.6.12)$$

Thus, applying (2.6.9),(2.6.10),(2.6.11),(2.6.12) on (2.6.8) we have

$$B(t) = \|\mathbf{\Gamma}^{(N-10)}B(\mathcal{P}_0u, \mathcal{P}_0u)\|_2 \leq k\epsilon^2(1+t)^{\delta-\frac{1}{3}-1}. \quad (2.6.13)$$

Estimate for $C(t)$ For $C(t)$ we have the following estimate. We use Proposition 2.5.2 to see that

$$\begin{aligned} \|\mathbf{\Gamma}^{(N-10)}C(\mathcal{P}_0u, \mathcal{P}_{>0}u)\|_2 &\lesssim \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_0u\|_2 \|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_{>0}u\|_\infty \\ &\quad + \|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_0u\|_3 \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_6. \end{aligned}$$

By equation (2.6.5)

$$\|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_0u\|_3 \leq \epsilon(1+t)^{-\frac{1}{3}}$$

and by (2.6.4)

$$\|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_6 \leq \epsilon(1+t)^{\delta-1}.$$

For the first summand, we will use the bootstrap assumptions (2.6.1b) and (2.6.1c). Thus we get

$$C(t) = \|\mathbf{\Gamma}^{(N-10)}C(\mathcal{P}_0u, \mathcal{P}_{>0}u)\|_2 \leq k\epsilon^2((1+t)^{\delta-\frac{4}{3}} + (1+t)^{-\frac{3}{2}}). \quad (2.6.14)$$

Estimate for $D(t)$ Lastly, we estimate $D(t)$. We have

$$\|\mathbf{\Gamma}^{(N-10)}D(\mathcal{P}_{>0}u, \mathcal{P}_{>0}u)\|_2 \leq k(\|\nabla\mathbf{\Gamma}^{(\frac{N}{2})}\mathcal{P}_{>0}u\|_\infty \|\nabla\mathbf{\Gamma}^{(N-10)}\mathcal{P}_{>0}u\|_2),$$

which according to the bootstrap assumptions satisfies

$$D(t) = \|\mathbf{\Gamma}^{(N-10)}D(\mathcal{P}_{>0}u, \mathcal{P}_{>0}u)\|_2 \leq k\epsilon^2 \frac{1}{(1+t)^{\frac{3}{2}}}. \quad (2.6.15)$$

We combine (2.6.13),(2.6.14),(2.6.15) to obtain the required estimate. \square

We are now ready to complete the proof of the Proposition 2.6.1 in the following lemma.

Lemma 2.6.6. *Suppose the hypothesis of Proposition 2.6.1 is satisfied. Then the assumptions (2.6.1a), (2.6.1b), (2.6.1c) imply (2.6.2b), (2.6.2c).*

Proof. We analyze the equation

$$\square_{\mathbb{R}^{3+1} \times K} \Gamma^{(N-10)} u = \Gamma^{(N-10)} (*du \wedge du). \quad (2.6.16)$$

First apply the energy estimate to get

$$\begin{aligned} \|\nabla \Gamma^{(N-10)} u(t)\|_2 &\leq \|\nabla \Gamma^{(N-10)} u(0)\|_2 + \int_0^t \|\Gamma^{(N-10)} (*du \wedge du)(s)\|_2 ds \\ &\leq \frac{\epsilon}{4} + k\epsilon^2 \int_0^t \frac{1}{(1+s)^{1+1/4}} ds \leq \frac{\epsilon}{4} + k\epsilon^2 \leq \frac{\epsilon}{2}, \end{aligned}$$

where we used Lemma 2.6.5 and the assumptions. This establishes (2.6.2b). To address (2.6.2c), apply Corollary 2.4.7 to equation (2.6.16) to get

$$\begin{aligned} (1+t) \|\nabla \Gamma^{(N-20)} \mathcal{P}_0 u(t)\|_\infty \\ + (1+t)^{3/2} \|\nabla \Gamma^{(N-20)} \mathcal{P}_{>0} u(t)\|_\infty \\ \leq \|\nabla \Gamma^{(N-10)} \mathcal{P}_0 u(0)\|_2 \\ + \sum_n \sup_{s \in I_n} 2^n \|\Gamma^{(N-10)} (F(s))\|_2, \end{aligned}$$

where $I_n = [2^{n-1}, 2^{n+1}] \cap [0, t]$ and $F = *du \wedge du$. Since

$$\|\Gamma^{(N-10)} F(s)\|_2 \leq \epsilon^2 (1+s)^{-1-\frac{1}{4}}$$

by Lemma 2.6.5, we have

$$\begin{aligned} (1+t) \|\nabla \Gamma^{(N-20)} \mathcal{P}_0 u(t)\|_\infty \\ + (1+t)^{3/2} \|\nabla \Gamma^{(N-20)} \mathcal{P}_{>0} u(t)\|_\infty &\leq \frac{\epsilon}{4} + k \sum_{n \leq C \log t + 1} 2^n \epsilon^2 2^{-n(1+\frac{1}{4})} \\ &\leq \frac{\epsilon}{4} + k\epsilon^2, \end{aligned}$$

which establishes (2.6.2c) and completes the proof. \square

Chapter 3

The time-like minimal hypersurface equation in Minkowski space

3.1 Introduction

Let $u : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ solve the equation

$$-\frac{\partial}{\partial t} \left(\frac{u_t}{(1 - u_t^2 + |\nabla_x u|^2)^{\frac{1}{2}}} \right) + \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{u_{x_i}}{(1 - u_t^2 + |\nabla_x u|^2)^{\frac{1}{2}}} \right) = 0. \quad (3.1.1)$$

On an open set $V \subseteq \mathbb{R}^{1+n}$ where

$$|u_t|^2 - |\nabla_x u|^2 < 1, \quad (3.1.2)$$

the equation (3.1.1) is a quasi-linear wave equation. We will assume that the condition (3.1.2) occurs at time 0 for every $x \in \mathbb{R}^n$ and supply Cauchy data at time $t = 0$.

$$u(0) = u_0, \quad \frac{\partial}{\partial t} u(0) = u_1, \quad (3.1.3)$$

and consider the corresponding Cauchy problem. We will prove the following theorem:

Theorem 3.1.1. *Let $n = 2$ or 3 . For every $K, R > 0$, $\theta < 1$, there exists $T = T(K, R, \theta)$ such that for every pair $(u_0, u_1) \in H^{\frac{n+3}{2}} \times H^{\frac{n+1}{2}}$, satisfying*

$$\begin{aligned} \|u_0\|_{\dot{H}^{\frac{n+3}{2}}} + \|u_1\|_{\dot{H}^{\frac{n+1}{2}}} &\leq K, \\ \sup_{x \in \mathbb{R}^n} |\nabla_x u_0(x)| &\leq R \end{aligned}$$

and

$$\sup_{x \in \mathbb{R}^n} |u_1(x)|^2 - |\nabla_x u_0(x)|^2 \leq \theta,$$

there exists a unique solution $u \in C([0, T], H^{\frac{n+3}{2}}(\mathbb{R}^n))$ of (3.1.1) with (u_0, u_1) as the initial data.

The graph of the function $\{(t, x, u(t, x)) | x \in \mathbb{R}^n\}$ describes a submanifold of the Minkowski space $\mathbb{R}^{1+(n+1)}$ with vanishing mean curvature, i.e. a minimal hypersurface. To see this connection, apply the derivatives in (3.1.3) and multiply by $(1 - u_t^2 + |\nabla_x u|^2)^{\frac{1}{2}}$ to arrive to the equation

$$u_{tt}\left(-1 - \frac{u_t^2}{1 - u_t^2 + |\nabla_x u|^2}\right) + \sum_{i,j} u_{x_i t} \frac{2u_{x_i} u_t}{1 - u_t^2 + |\nabla_x u|^2} + u_{x_i x_j} \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1 - u_t^2 + |\nabla_x u|^2}\right) = 0.$$

Defining¹

$$g^{\alpha\beta} = m^{\alpha\beta} - \frac{m^{\alpha\gamma_1}(\partial_{\gamma_1} u)m^{\beta\gamma_2}(\partial_{\gamma_2} u)}{1 + m^{\gamma_3\gamma_4}(\partial_{\gamma_3} u)(\partial_{\gamma_4} u)}, \quad (3.1.4)$$

where $m^{\alpha\beta} = \text{diag}(-1, 1, 1, \dots, 1)$ is the Minkowski metric, the equation becomes

$$g^{\alpha\beta} \partial_\alpha \partial_\beta u = 0.$$

The metric $g^{\alpha\beta}$ is a metric on a cotangent bundle, the dual metric on the tangent bundle is

$$g_{\alpha\beta} = m_{\alpha\beta} + \partial_\alpha u \partial_\beta u,$$

which is the metric of the graph $\{(x_\alpha, u(x_\alpha)) | x_\alpha \in \mathbb{R}^{1+n}\} \subseteq \mathbb{R}^{1+(n+1)}$.

Remark. *The condition (3.1.2) says that the graph is a time-like hypersurface and it is necessary. The condition $|\nabla_x u| < \infty$ makes sure that the the hypersurface remains globally a graph and we assume it for the convenience of having global coordinates.*

Aside from the natural geometric settings, the equation arises also as Born-Infeld model of nonlinear electromagnetism [40, chapter 20] and related to the Born-Infeld electromagnetism, a model for evolution of branes in string theory [40, chapter 20],[5]. It is also been suggested that in a certain regime, some solutions to semilinear wave equations converge to minimal hypersurfaces[24].

Along with the potential applications of this model, our main interest is to attempt to understand the large data problem for quasilinear wave equations with null form nonlinearities. This question is completely understood in the case of small smooth rapidly decaying data after the works of Christodoulou [3] and Klainerman [16], it is also well-understood for semilinear wave equations following the work of Klainerman and Machedon [13] for the subcritical regularity and some examples of the critical regularity (for instance by Tao[31] and Tataru [34]).

¹A note regarding our conventions. We will rely on the geometric conventions, where we treat \mathbb{R}^{1+n} as (a coordinate patch of) a manifold. As such, we will have mostly greek letters as indices from 0 to n , denoting components of various tensors. We will try carefully denote the (tangent) vector components as superscripts and (cotangent) co-vector components as subscripts. We will assume a summation convention where the same index which appears as a subscript and a superscript is summed.

In both of these cases - the small smooth data and the semilinear equations, the null-form nonlinearity can be defined as an equation:

$$g^{\alpha\beta}(u, du)\partial_\alpha\partial_\beta u = N(du),$$

where $B^{\alpha\beta} = \frac{\partial^2 N}{\partial(\partial_\alpha u)\partial(\partial_\beta u)}\Big|_{du=0}$ and $G^{\alpha\beta\gamma} = \frac{\partial g^{\alpha\beta}}{\partial(\partial_\gamma u)}\Big|_{du=0}$ satisfy

$$B^{\alpha\beta}\xi_\alpha\xi_\beta = 0, \quad G^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0$$

for every ξ of zero Minkowski length². As one can see, this definition applies only to perturbations of the constant coefficient wave equation, which excludes the case of the large data quasilinear equations. There is still no clear picture for the latter case. One suggestion was made in [32] to define the null condition by

$$\frac{\partial g^{\alpha\beta}(u, du)}{\partial(\partial_\gamma u)}\xi_\alpha\xi_\beta\xi_\gamma = 0, \text{ for every } \xi \text{ such that } g^{\alpha\beta}(u, du)\xi_\alpha\xi_\beta.$$

It is easy to check that (3.1.1) satisfies this null-condition, which we do in Corollary 3.2.2 but we don't know whether this is effective in lowering the regularity in every possible case. We intend to check that in the near future.

The minimal hypersurface equation was studied by Lindblad [19], where a global in time regularity was established for small smooth rapidly decaying initial data in all dimensions, using the tools from both [16] and [3]. See also the work by Brendle [1] for a weaker result with more geometric proof. Another perturbative analysis was performed by Stefanov [29], where the global in time question for small data was addressed using Strichartz estimate for the constant coefficient wave equation.

For a general form of quasilinear wave equation, local well-posedness was obtained for $H^{\frac{11}{4}+\epsilon}$ for $n = 2$ and $H^{3+\epsilon}$ for $n = 3$ by Smith and Tataru in [27]³ This result is sharp in dimensions two and three in view of a counter-example by Lindblad [20]. We will modify the proof in [27] to lower the regularity in our specific case.

The technique employed in [27] is construction of a wave packet parametrized. The wave packets are built from thickened slices of special, light-like hypersurfaces $\Sigma_{\theta,r}$ ($\theta \in \mathbb{S}^{n-1}, r \in \mathbb{R}$) formed by flowing out a hyperplane $\theta \cdot x = r$ at time $t = 0$ in a uniform initial direction $\theta \cdot dx - dt$ by the geodesic flow. The required properties (namely energy and space-time estimates) of the parametrized are based on the regularity of these hypersurfaces. Specifically, denote by l the vector field of null-geodesics, which generates $\Sigma_{\theta,r}$. Augment l with a null frame $\{\underline{l}, \underline{l}, \mathbf{e}_a\}_{a=1..n-1}$ where \underline{l} is a null vector transverse to l and the rest of the vectors are normal to l (and thus tangent to $\Sigma_{\theta,r}$). To establish $H^{\frac{n+1}{2}}(\Sigma_{\theta,r})$ regularity of l , one then tries to establish $H^{\frac{n-1}{2}}(\Sigma_{\theta,r})$ of its derivatives. The most delicate part of those are the coefficients of the second fundamental form

$$\chi_{ab} = \langle \nabla_{\mathbf{e}_a} l, \mathbf{e}_b \rangle.$$

²A multiplication by a bounded smooth function can be harmless in some cases

³Note that their paper deals with the nonlinearity of the form $g(u)$, while in this paper the nonlinearity is of the form $g(\nabla u)$, requiring one more degree of regularity.

The observation that was made in [4] and [12] is that χ satisfies the Raychaudhuri equations of the form

$$l(\chi_{ab}) = \mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b) + \chi^2 + \dots,$$

where \mathcal{R} is the Riemann curvature tensor. Thus we would like to analyze the curvature term to integrate the Raychaudhuri equations. In the case of the graph $\{(x, u(x))\}$ the curvature can be expressed via products of the Hessian of u , which is a tensor made out of second derivatives of u . Therefore, the naive analysis fails since if we seek to prove that $\chi \in H^{\frac{n-1}{2}}$ then we would like to have roughly that $\mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b) \in L_t^1 H^{\frac{n-1}{2}}(\Sigma_{\theta,r}^t)$, where $\Sigma_{\theta,r}^t$ is the time-slice of $\Sigma_{\theta,r}$. But, even though $\nabla^2 u \in L_t^2 H^{\frac{n-1}{2}}(\Sigma_{\theta,r}^t)$ by characteristic energy estimates, the space variable regularity is critical since $\Sigma_{\theta,r}^t$ is $n-1$ dimensional, therefore the product of two such terms is not guaranteed to have $H^{\frac{n-1}{2}}$ regularity.

The null-form structure of the equation manifests itself in the absence of the term in the for $\underline{l}(u)$. Due to this structure, we will be able to observe that $\mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b) \in L_t^1 B_{2,1}^{\frac{n-1}{2}}(\Sigma_{\theta,r}^t) + l(L_t^\infty B_{2,1}^{\frac{n-1}{2}}(\Sigma_{\theta,r}^t))$, which can be integrated without further loss of regularity.

Another difference with [27] is the space-time estimates that are required to close the argument. Whereas [27] uses Strichartz estimates, we prove the following null-form estimate:

$$\|Q_0(f, f)\|_{H_{t,x}^{\frac{n-1}{2}}} \leq \|\nabla_{x,t} f(0, \cdot)\|_{H^{\frac{n-1}{2}}}^2,$$

for f - a free solution of the variable coefficient wave equation $\square_g f = 0$, where

$$Q_0(f, f) = g^{\alpha\beta} \partial_\alpha f \partial_\beta f.$$

Such estimates are a cornerstone of the work of [13] for the constant coefficient wave equation, which are then applied on the semilinear equations. Later, the null-form estimates were also established for rough variable coefficients in by Smith and Sogge[26]. We will not follow this approach since it uses Fourier Integral parametrix. Instead, we will adapt the method suggested in [33], which we have to strengthen since the argument in [33] works for H^s with $s > \frac{n+1}{2}$.

The rest of the paper is structured as follows. In section 2, we perform some standard reductions which will allow us to simplify the proof. In section 3, we outline the bootstrap argument which we intend to complete in the rest of the paper. In section 4, we deal with the regularity of the characteristic surfaces $\Sigma_{\theta,r}$, in section 5 construct the wave packet parametrix and in section 6 prove the required bilinear estimates.

3.2 Preparations

We observe that by a simple calculation for the metric in (3.1.4) we have

$$g^{\alpha\beta} \partial_\alpha \partial_\beta f = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\alpha\beta} \partial_\beta f) = \square_g f,$$

for any function f . We also define the null form

$$Q_0(f_1, f_2) = g^{\alpha\beta} \partial_\alpha f_1 \partial_\beta f_2,$$

to which we might occasionally add the metric as a subscript if ambiguity arises. We elucidate the notion of the solution to the equation.

Definition 3.2.1. *The Cauchy problem (3.1.1) and (3.1.3) is locally well posed in $H^{\frac{n+3}{2}} \times H^{\frac{n+1}{2}}$ if for each $K, R > 0$ and $\theta < 1$ there exist $T, M, C > 0$ so that the following properties are satisfied:*

1. For each initial data (u_0, u_1) satisfying

$$\|(u_0, u_1)\|_{\dot{H}^{\frac{n+3}{2}} \times \dot{H}^{\frac{n+1}{2}}} \leq K,$$

$$\|\nabla_x u_0\|_{L^\infty} + \|u_1\|_{L^\infty} \leq R$$

and

$$\|u_1^2 - |\nabla_x u_0|^2\|_{L^\infty} \leq \theta,$$

there exists a unique solution subject to a condition

$$du \in L_{t,x}^\infty([-T, T] \times \mathbb{R}^n)$$

.

2. The solution satisfies

$$du \in L_{t,x}^\infty([-T, T] \times \mathbb{R}^n) \leq M$$

and

$$\sup_{[-T, T] \times \mathbb{R}^n} |\partial_t u(t, x)|^2 - |\nabla_x u(t, x)|^2 \leq \theta + \frac{1 - \theta}{2}.$$

3. For each $t_0 \in [-T, T]$ the linear equation (with $g^{\alpha\beta}$ as in (3.1.4))

$$\begin{cases} g^{\alpha\beta} \partial_\alpha \partial_\beta v = 0, & (t, x) \in [-T, T] \times \mathbb{R}^n \\ v(t_0, \cdot) = v_0 \in H^{\frac{n+1}{2}}, & \partial_0 v(t_0, \cdot) = v_1 \in H^{\frac{n-1}{2}}, \end{cases}$$

admits a solution $v \in C([-T, T], H^{\frac{n+1}{2}}) \cap C^1([-T, T], H^{\frac{n-1}{2}})$ such that

$$\|v\|_{L^\infty H^{\frac{n+1}{2}}} + \|\partial_0 v\|_{L^\infty H^{\frac{n-1}{2}}} \leq C \|(v_0, v_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}}$$

and

$$\|Q_0(v, v)\|_{H_{t,x}^{\frac{n-1}{2}}} \leq C \|(v_0, v_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}}. \quad (3.2.1)$$

We will occasionally consider du as a parameter in the metric g . As such

$$g^{\alpha\beta}(p) = m^{\alpha\beta} - \frac{m^{\alpha\gamma_1} p_{\gamma_1} m^{\beta\gamma_2} p_{\gamma_2}}{1 + m^{\gamma_3\gamma_4} p_{\gamma_3} p_{\gamma_4}}.$$

By differentiating with respect to p we obtain the following useful formula:

$$\frac{\partial g^{\alpha\beta}}{\partial p_\gamma} = -\frac{m^{\alpha\delta_1} p_{\delta_1}}{1 + m^{\delta_2\delta_3} p_{\delta_2} p_{\delta_3}} g^{\beta\gamma} - \frac{m^{\beta\delta_1} p_{\delta_1}}{1 + m^{\delta_2\delta_3} p_{\delta_2} p_{\delta_3}} g^{\alpha\gamma}. \quad (3.2.2)$$

It leads immediately to the following

Corollary 3.2.2. *The metric $g^{\alpha\beta}$ satisfies the null condition of [32], i.e.*

$$\frac{\partial g^{\alpha\beta}}{\partial p_\gamma} \xi_\alpha \xi_\beta \xi_\gamma = 0, \text{ for every } \xi \text{ satisfying } g^{\alpha\beta} \xi_\alpha \xi_\beta = 0.$$

The formula (3.2.2) also allows to prove uniqueness and strong continuity in the lower Sobolev norms.

Lemma 3.2.3. *Let u, v satisfy $\square_{g(du)} u = 0, \square_{g(dv)} v = 0$ such that u satisfies Definition 3.2.1 and $dv \in L_{t,x}^\infty$ then*

$$\|u - v\|_{L^\infty H_x^{\frac{n+1}{2}}} \leq C(\|dv\|_{L^\infty}, \|du\|_{L^\infty}) \|(u_0 - v_0, u_1 - v_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}}.$$

Proof. We analyze the difference equation, since $\square_{g(du)} u = 0 = \square_{g(dv)} v$ then

$$\begin{aligned} \square_{g(du)}(v - u) &= (\square_{g(dv)} - \square_{g(du)})v \\ &= Q_{0,g(du)}(v - u, \partial_\alpha u) \frac{m^{\alpha\beta} \partial_\beta u}{1 + m^{\gamma_1\gamma_2} \partial_{\gamma_1} u \partial_{\gamma_2} u} \tilde{F}(du, dv) \\ &= Q_{0,g(du)}(v - u, \partial_\alpha u) F(dv, du), \end{aligned}$$

where we used (3.2.2) and the fact that $f(x) - f(y) = Df|_x(x - y)F(x, y)$ for some bounded function F . Now apply the Duhamel principle and the bilinear estimates (3.2.1) to imply the required estimates. \square

Reduction to small, smooth, compactly supported data

We would like to exploit the scaling symmetry of the equation and its Lorentz invariance in the ambient \mathbb{R}^{n+1+1} Minkowski space to simplify the setup.

Proposition 3.2.4. *There exist $\epsilon_3 \ll \epsilon_2 \ll 1$ such that for every smooth, supported in $B(0, 2)$ functions (u_0, u_1) that satisfy*

$$\|u_0\|_{H^{\frac{n+3}{2}}} + \|u_1\|_{H^{\frac{n+1}{2}}} \leq \epsilon_3.$$

there exists a smooth solution for the equations (3.1.1) and (3.1.3) on $[-1, 1]$ such that the following estimates hold:

Energy estimate

$$\|du\|_{L_t^\infty H_x^{\frac{n+1}{2}}} \leq \epsilon_2.$$

Bilinear estimate

$$\|Q_0(u, u)\|_{H^{\frac{n+1}{2}}} \leq \epsilon_2.$$

Estimates for the linear equation *Let v be a solution of the linear equation $\square_{g(du)}v = 0$ such that $(v_0, v_1) \in H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}$ then v satisfies*

$$\begin{aligned} \|v\|_{L^\infty H^{\frac{n+1}{2}}} &\lesssim \|(v_0, v_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}}, \\ \|Q_0(v, v)\|_{H^{\frac{n-1}{2}}} &\lesssim \|(v_0, v_1)\|_{H^{\frac{n+1}{2}} \times H^{\frac{n-1}{2}}}. \end{aligned}$$

We explain how to establish Theorem 3.1.1 from Proposition 3.2.4. As we will see momentarily there exists a function $F(\theta, R)$ such that by setting T to satisfy

$$MT^{\frac{1}{2}} \leq C\epsilon_3 F(\theta, R)$$

and using the scaling of our problem

$$\tilde{u}(t, x) = T^{-1}u(Tt, Tx),$$

we can achieve

$$\|d\tilde{u}(0, x)\|_{H^2} \leq F(\theta, R)\epsilon_3.$$

Next we truncate the data outside of a ball of $B(y, 2+)$. We translate the center of the ball to the origin. We observe that we can set $u(0) = 0$ as the equation only depends on derivatives. Next, apply a Lorentz boost $x' = Lx$ on \mathbb{R}^{n+1+1} in the direction

$$\omega = \text{sign}u_1(0) \left(\frac{\nabla_x u_0(0)}{(1 + |\nabla_x u(0)|)^{\frac{1}{2}}}, \frac{1}{(1 + |\nabla_x u(0)|)^{\frac{1}{2}}} \right) \in \mathbb{S}^n$$

with velocity

$$V = \frac{|u_1(0)|}{(1 + |\nabla_x u_0(0)|)^{\frac{1}{2}}}.$$

Next apply a rotation S in the $(\omega, e_{n+1} = (0, \dots, 1))$ plane to rotate ω to e_{n+1} . This defines a new solution $v(x')$ to the minimal surface equation such that the graphs $(x, u(x))$ and $(x', v(x'))$ are related by the composition SL and $v(0) = 0, dv(0) = 0$. Since the value and the derivatives at 0 are zero this implies that the inhomogeneous norms satisfy

$$\|v_0\|_{H^{\frac{n+3}{2}}} + \|v_1\|_{H^{\frac{n+1}{2}}} \leq \epsilon_3.$$

This allows us to apply Proposition 3.2.4 on a sequence of smooth approximation to (v_0, v_1) to obtain a solution to the equation in the limit. After applying $L^{-1}S^{-1}$ we will obtain a solution with the truncated, rescaled data.

Remark. *There are a few things that limit the time of existence beyond just scaling*

- *Due to the time dilation effect, time 1 in the boosted system will be smaller in the original system.*
- *We need to make sure that the equation remains hyperbolic $|\partial_t u|^2 - |\nabla_x u|^2 < 1$, which is a necessary condition and that it remains globally a graph $|\nabla_x u| < \infty$. Thus we will limit how much these parameters increase. In the boosted system, where we control the inhomogeneous norms, we can do that by controlling $\epsilon_3 + \epsilon_2$.*
- *Lastly, the Sobolev norms are not Lorentz invariant and they will grow when we apply a Lorentz transformation due to the length contraction. We need to have the norms smaller than ϵ_3 in the boosted coordinates, which further decreases the norm in the original coordinates.*

The considerations above require us to introduce a function $F(\theta, R)$ that limits the time of existence.

Let χ be a smooth function supported in $B(0, 3)$ and which equals to 1 in $B(0, 2)$. After rescaling the initial data, we truncate it around $y \in \mathbb{R}^n$ by defining

$$u_0^y(x) = \chi(x - y)(u_0(x) - u_0(y)).$$

Next, we translate y to zero and define the Lorentz transformation $S_y L_y$ as above and use it to obtain a solution v . We then define $u^y(x) = v(L^{-1}S^{-1}x - y) + u_0(y)$. We consider cones

$$K^y = \{t + |x - y| \leq 2, |t| < 1\}.$$

By finite speed of propagation, any two solutions that coincide on the common domain at the $t = 0$ of the basis of the cone must coincide in their common domain over the cone. Therefore, we define a partition of unity over centers y in a lattice $n^{-\frac{1}{2}}\mathbb{Z}^n$ such that

$$1 = \sum_{y \in n^{-\frac{1}{2}}\mathbb{Z}^n} \psi(x - y), \quad \forall x \in \mathbb{R}^n$$

and

$$\text{supp}\psi \subseteq K^0.$$

Then we define

$$u = \sum_{y \in n^{-\frac{1}{2}}\mathbb{Z}^n} \psi(x - y)u^y.$$

Similarly for a given initial values of (v_0, v_1) we similarly solve the truncated problems

$$\begin{cases} \square_{g(dx^y)} v^y = 0, \\ v^y(0) = \chi(x - y)v_0, \quad \partial_t v^y(0) = \chi(x - y)v_1. \end{cases}$$

and define

$$v = \sum_{y \in n^{-\frac{1}{2}}\mathbb{Z}^n} \psi(x - y)v^y.$$

The required well-posedness estimates follow from finiteness of overlaps and the Cauchy-Schwartz inequalities.

3.3 The outline of the proof

Frequency envelopes

We would like to encapsulate the notion of slowly varying sequence. Let $0 < \delta < 1/4$ be a small fixed parameter.

Definition 3.3.1. *A positive sequence $\{b_k\}_{k \in \mathbb{N}}$ is a (sloowly varying) frequency envelope if*

•

$$b_k \leq b_j 2^{\delta|k-j|}, \quad k, j \in \mathbb{N}.$$

•

$$b_k \geq C 2^{-\frac{k}{2}}.$$

Remark. • *Our definition is different from the usual definition of the frequency because of the second condition. It allows the usual embedding $H^{s+\frac{1}{2}} \hookrightarrow B_{2,1}^s$ on the level of frequencies since $\nu^{\frac{1}{2}}b_\nu \leq Cb_\nu^2$.*

• *For any positive sequence α_k , we can define an appropriate frequency envelope by*

$$\alpha'_k = \sup_{n \in \mathbb{N}} \alpha_n 2^{-\delta|n-k|} + C 2^{-\frac{k}{2}}.$$

• *Any frequency envelope $\{b_k\}$ satisfies*

$$\sum_{k < n} 2^{\frac{k}{4} - \frac{n}{4}} b_k, \sum_{k > n} 2^{\frac{n}{4} - \frac{k}{4}} b_k \leq C(\delta) b_n$$

and also

$$\sum_{k < n} 2^{\frac{k}{2} - \frac{n}{2}} b_k^2, \sum_{k > n} 2^{\frac{n}{2} - \frac{k}{2}} b_k^2 \leq C(\delta) b_n^2.$$

We choose a frequency envelope b_μ such that

$$\sum_{\mu \geq 1, \text{ dyadic}} b_\mu^2 \leq 1.$$

Let (u_0, u_1) be the initial data. Denote by \mathcal{H} the set of smooth solutions that satisfy

$$\begin{aligned} \mu^{\frac{n+3}{2}} \|P_\mu u_0\|_{L_x^2} + \mu^{\frac{n+1}{2}} \|P_\mu u_1\|_{L_x^2} &\leq \epsilon_3 b_\mu, \\ \mu^{\frac{n+3}{2}} \|S_\mu u\|_{L_t^\infty L_x^2} + \mu^{\frac{n+1}{2}} \|\partial_0 S_\mu u\|_{L_t^\infty L_x^2} &\leq 2\epsilon_2 b_\mu, \end{aligned}$$

where P_μ are the space variables Littlewood Paley projections and S_μ are the space-time Littlewood-Paley projections. Endow \mathcal{H} with C^∞ topology then we will seek to establish the following bootstrap argument:

Theorem 3.3.2. *There exists an \mathcal{H} -continuous functional $G : \mathcal{H} \rightarrow \mathbb{R}^+$, satisfying $G(0) = 0$, so that for each $u \in \mathcal{H}$ satisfying $G(u) \leq 2\epsilon_1$ the following holds*

1. *The function u satisfies $G(u) \leq \epsilon_1$.*
2. *The following estimate holds*

$$\mu^{\frac{n+3}{2}} \|S_\mu u\|_{L_t^\infty L_x^2} + \mu^{\frac{n+1}{2}} \|\partial_0 S_\mu u\|_{L_t^\infty L_x^2} \leq \epsilon_2 b_\mu.$$

3. *There exist $C, c, \delta > 0$ such that for every sequence of solutions v_μ of the inhomogeneous linear equation with frequency localized initial data $dv_\mu(0, \cdot) = P_\mu dv_\mu(0, \cdot)$ which satisfies*

$$\|dv_\mu(0, \cdot)\|_{L_x^2}, \|\square_{S_{\leq \mu} g} v\|_{L_t^1 L_x^2} \leq 1,$$

the following estimates hold:

$$\begin{aligned} \|S_\mu v_\mu\|_{L^\infty L^2} &\leq c, \\ \|Q_{0, S_{\leq \nu} g}(S_\mu v_\mu, S_\mu v_\mu)\|_{L_{t,x}^2} &\leq C \mu^{\frac{n-1}{2}}, \quad \forall \nu \leq \mu, \end{aligned}$$

$$\begin{aligned} &\int |Q_{0, S_{\leq \mu} g}(S_{\nu_1} v_{\nu_1}, S_\mu v_\mu) Q_{0, S_{\leq \mu} g}(S_{\nu_2} v_{\nu_2}, S_\mu v_\mu)| dt dx \\ &\leq C \nu_1^{\frac{n-1}{2}} \nu_2^{\frac{n-1}{2}} \left(\frac{\nu_2^{\frac{1}{2}}}{\nu_1^{\frac{1}{2}}} + \nu_1^{-\delta} \nu_2^{-\delta} \right), \quad \forall \nu_2 \leq \nu_1 \leq \mu. \end{aligned}$$

Partition the space-time into cones with base $B(x, 2)$, write the solution as a partition of unity over the cones multiplied by the local solution, prove that it satisfies Definition 3.2.1.

We will define the functional G and prove implication 1 in Section 3.4. We will construct a parametrix to the frequency localized equation in Section 3.5 and prove a bilinear estimate for solutions in Section 3.6, which is implication 3. This will allow us, using Duhamel's principle prove implication 2.

3.4 The regularity of the characteristic surfaces

Characteristic surfaces and null-frame coefficients, definition of $G(u)$

Let $\theta \in \mathbb{S}^{n-1}$ be a direction. Define a function $x_\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $x_\theta(x) = \theta \cdot x$. Let F_θ be the Hamiltonian flowout of x_θ with the initial null geodesic $V_\theta|_{t=0} = (sdt + dx_\theta)^*$ where $s(x)$ is chosen to make the initial direction null. Define $\Sigma_{\theta,r}$ to be the level set of F_θ at value r , i.e.

$$\Sigma_{\theta,r} = \{(t, x) | F_\theta(t, x) = r\}.$$

Let $V_\theta = \nabla F_\theta|_{\Sigma_{\theta,r}} = (dF_\theta)^*|_{\Sigma_{\theta,r}}$ (or in coordinates $V^\alpha = (g^{\alpha\beta} \partial_\beta F)|_{\Sigma_{\theta,r}}$) be the null geodesic tangent to $\Sigma_{\theta,r}$.

Claim 3.4.1. *Suppose $\|dF_\theta - (dt + \theta \cdot dx)\|_{L_{t,x}^\infty} \leq G(u) \leq 2\epsilon_1$ then for every θ, r there exists $\phi_{\theta,r} : \mathbb{R}_t \times \mathbb{R}_{x'}^{n-1} \rightarrow \mathbb{R}$ such that $\Sigma_{\theta,r}$ is given by*

$$\Sigma_{\theta,r} = \{x_\theta = \phi_{\theta,r}(t, x')\}.$$

Proof. The existence of this function is a local statement. Apply the Inverse Function Theorem to F . Since $\|dF - (dt + \theta \cdot dx)\|_{L^\infty} \leq 2\epsilon_1$ then $\phi_{\theta,r} = r + t + O(\epsilon_1)$. \square

Denote

$$\tau = dt(V).$$

Define

$$l = \tau^{-1}V. \tag{3.4.1}$$

Next, define a second vector field \underline{l} :

$$\underline{l} = 2(dt)^* - 2g^{tt}l,$$

where $(dt)^*$ is the g -dual vector field (on \mathbb{R}^{n+1}) to dt i.e. $\langle (dt)^*, X \rangle = dt(X)$ and $g^{tt} = \langle dt, dt \rangle_g$. This particular choice of \underline{l} achieves several things:

$$\begin{aligned} \langle \underline{l}, l \rangle &= 2, & \langle \underline{l}, \underline{l} \rangle &= 0, \\ dt(l) &= 1, & (dt)^* &\in \text{span}\{l, \underline{l}\}. \end{aligned}$$

Apply Gram-Schmidt orthogonalization procedure on $\{l, \underline{l}, \partial_{x'_a}\}_{a=1..n-1}$, where $\partial_{x'_a}$ is a set of constant vector fields which form a basis to $x \cdot \theta = 0$ at $t = 0$. The resulting frame $\{l, \underline{l}, \mathbf{e}_a\}_{a=1..n-1}$ will satisfy the following conditions:

$$\begin{aligned} \langle l, \underline{l} \rangle &= 2, & \langle \mathbf{e}_a, \mathbf{e}_b \rangle &= \delta_{ab}, \\ \langle l, l \rangle &= \langle \underline{l}, \underline{l} \rangle = \langle l, \mathbf{e}_a \rangle = \langle \underline{l}, \mathbf{e}_a \rangle = 0. \end{aligned}$$

Observe that since $dt(\mathbf{e}_a) = 0$, the vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ form a basis to the tangent space of the time slices of $\Sigma_{\theta,r}$. Form the second fundamental form of $\Sigma_{\theta,r}$ by defining

$$\chi_{ab} = \langle \nabla_{\mathbf{e}_a} l, \mathbf{e}_b \rangle.$$

Also let $c(s, y) : \mathbb{R} \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ be the flow map that maps initial conditions to solutions of $l(x) = 0$ and let $d(t, x) : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ be the inverse of c at time t , i.e.

$$c(s, \cdot) \circ d(s, \cdot) = \text{id}_{\mathbb{R}^{n-1}}, \quad d(s, \cdot) \circ c(s, \cdot) = \text{id}_{\mathbb{R}^{n-1}}.$$

Definition 3.4.2. *Let C be the smallest constant for which the following inequalities are true:*

$$\begin{aligned} \|P_\mu(\phi_{\theta,r} - r - t)\|_{L^2_{t,x'}} &\leq C\mu^{-\frac{n+3}{2}}b_\mu, & \forall \theta \in \mathbb{S}^{n-1}, r \in \mathbb{R}, \\ \|P_\mu\chi_{ab}\|_{L^\infty_t L^2_{x'}} &\leq C\mu^{-\frac{n-1}{2}}b_\mu^2, & \forall \theta \in \mathbb{S}^{n-1}, r \in \mathbb{R}, \\ \|P_\mu(c^i - x^i)\|_{L^\infty_t L^2_{x'}} &\leq C\mu^{-\frac{n+1}{2}}b_\mu^2, & i = 1..n-1, \\ \|P_\mu(d^i - x^i)\|_{L^\infty_t L^2_{x'}} &\leq C\mu^{-\frac{n+1}{2}}b_\mu^2, & i = 1..n-1, \end{aligned}$$

where for the rest of this section P_μ is the x' Littlewood-Paley projection. Then define

$$G(u) = C.$$

Remark. • *This definition implies*

$$\begin{aligned} \phi_{\theta,r} - r - t &\in L^2_t H^{\frac{n+3}{2}}_{x'}(\mathbb{R}^n), \\ \chi_{ab} &\in L^\infty_t B^{\frac{n-1}{2}}_{2,1,x'}(\mathbb{R}^{n-1}). \end{aligned}$$

The last space is an algebra in the space variables (in particular) unlike $L^2_t H^{\frac{n-1}{2}}_{x'}(\mathbb{R}^{n-1})$, which is the estimate that can be obtained by differentiating ϕ twice. Also, the flow map satisfies

$$\|c - \text{id}_{\mathbb{R}^{n-1}}\| \in L^\infty_t B^{\frac{n+1}{2}}_{2,1,x'}(\mathbb{R}^{n-1}).$$

- *This definition of G is different from [27] since it hardwires an extra regularity of the second fundamental form of the surface $\Sigma_{\theta,r}$ and of the flow map c .*
- *All the estimates are uniform in r , which is something that we will exploit in proving the bilinear estimates. The uniformity in the angle θ we will use in the construction of the wave packet parametrix.*

From now on, we will work under assumptions of Theorem 3.3.2 and work to prove it. We will split the proof into several sections. In this section we will tackle the first implication, which we split into two theorems.

Proposition 3.4.3. *Let $u \in \mathcal{H}$ satisfy $G(u) \leq 2\epsilon_1$. Let*

$$\chi_{ab} = \langle \nabla_{\mathbf{e}_a} l, \mathbf{e}_b \rangle,$$

then

$$\|P_\mu \chi_{ab}\|_{L_t^\infty L_x^2} \leq (c_2 \epsilon_2 + c_1 \epsilon_1^2) 2\mu^{-\frac{n-1}{2}} b_\mu^2,$$

where that do not depend on u .

Proposition 3.4.4. *Let $u \in \mathcal{H}$ satisfy $G(u) \leq 2\epsilon_1$ then*

$$\|P_\mu(\phi_{\theta,r} - r - t)\|_{L_{t,x'}^2} \leq (c_2 \epsilon_2 + c_1 \epsilon_1^2) \mu^{-\frac{n+3}{2}} b_\mu,$$

where c_i are constants that does not depend on u .

Proposition 3.4.5. *Let $u \in \mathcal{H}$ satisfy $G(u) \leq 2\epsilon_1$ then the flowmap c of solutions of l and it's instantaneous inverse d satisfy*

$$\|P_\mu(c - id_{\mathbb{R}^{n-1}})\|_{L_t^\infty L_x^2} \leq (c_2 \epsilon_2 + c_1 \epsilon_1^2) 2\mu^{-\frac{n+1}{2}} b_\mu^2,$$

$$\|P_\mu(d - id_{\mathbb{R}^{n-1}})\|_{L_t^\infty L_x^2} \leq (c_2 \epsilon_2 + c_1 \epsilon_1^2) 2\mu^{-\frac{n+1}{2}} b_\mu^2,$$

These three statements imply that $G(u) \leq \epsilon_1$ after choosing $c_2 \epsilon_2 + c_1 \epsilon_1^2 \leq \epsilon_1$.

Statement of auxiliary lemmas

In this subsection, we will state the necessary lemmas to prove Propositions 3.4.3 and 3.4.4. In the following subsections we will prove the theorems and then return to establishing the lemmas in the last three subsections of this Section.

Lemma 3.4.6 (Product estimates). *We will say that a function f is a multiplier on a set V if there exists $C > 0$ such that $Cfv \in V$ for every $v \in V$.*

1. Let f satisfy

$$\|P_\mu f\|_{L_{t,x'}^2} \leq \mu^{\frac{n+1}{2}} b_\mu.$$

Then f is a multiplier on the following sets

$$V_1 = \{v \mid \|P_\mu v\|_{L_{t,x'}^2} \leq \mu^{\frac{n+1}{2}} b_\mu\}, \quad V_2 = \{v \mid \|P_\mu v\|_{L_t^\infty L_x^2} \leq \mu^{\frac{n-1}{2}} b_\mu^2\},$$

$$V_3 = \{v \mid \|P_\mu v\|_{L_t^1 L_x^2} \leq \mu^{\frac{n-1}{2}} b_\mu^2\}.$$

2. Let g satisfy

$$\|P_\mu g\|_{L_t^\infty L_x^2} \leq \mu^{-\frac{n-1}{2}} b_\mu^2$$

then g is a multiplier on V_2 and V_3 .

Lemma 3.4.7 (Diffeomorphism lemma). *Let $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ be a diffeomorphism such that*

$$\|P_{\mu, x'}(\psi - id_{\mathbb{R}^{n-1}})\|_{L_{x'}^2} \lesssim \epsilon_1 \mu^{-\frac{n+1}{2}} b_\mu^2.$$

Then for any function $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ that satisfies $\|P_\mu f\|_{L^2} \leq \mu^{-\frac{n-1}{2}} b_\mu^2$, the function $g = f \circ \psi$ satisfies

$$\|P_\mu g\| \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2.$$

Lemma 3.4.8. *Let f be a solution of*

$$l(f) = g,$$

such that

$$\|P_\mu g\|_{L_t^1 L_{x'}^2}, \|P_\mu f|_{t=0}\|_{L_{x'}^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2,$$

then

$$\|P_\mu f\|_{L_t^\infty L_{x'}^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2.$$

Corollary 3.4.9. *Let f be a solution of*

$$l(f) = g + l(h),$$

such that $\|P_\mu g\|_{L_t^1 L_{x'}^2}, \|P_\mu h\|_{L_t^\infty L_{x'}^2}, \|P_\mu f|_{t=0}\|_{L_{x'}^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2$ then

$$\|P_\mu f\|_{L_t^\infty L_{x'}^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2.$$

Lemma 3.4.10. *Let $f = \sum_j h_{1j} l(h_{2j})$ such that*

$$\|P_\mu h_{ij}\|_{L_{t,x'}^2} \lesssim \mu^{-\frac{n+1}{2}} b_\mu,$$

then f can be written,

$$f = f_1 + l(f_2)$$

such that

$$\|P_\mu f_1\|_{L_{t,x'}^2} \lesssim \mu^{-\frac{n}{2}} b_\mu^2$$

and

$$\|P_\mu f_2\|_{L_{t,x'}^2} \lesssim \mu^{-\frac{n+1}{2}} b_\mu.$$

Lemma 3.4.11 (Minor). *Let h_1, h_2, h_3 satisfy*

$$\|P_\mu h_i\| \lesssim \mu^{-\frac{n+1}{2}} b_\mu,$$

then the expression $h_1 l(h_2) l(h_3)$ can be decomposed

$$h_1 l(h_2) l(h_3) = f_1 + l(f_2),$$

such that

$$\|P_\mu f_1\|_{L_t^1 L_{x'}^2}, \|P_\mu f_2\|_{L_t^\infty L_{x'}^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2.$$

Lemma 3.4.12 (Major). *Let $\mathcal{R}(\cdot, \cdot, \cdot, \cdot)$ be the Riemann curvature tensor of $g^{\alpha\beta}$. Let h satisfy*

$$\|P_\mu h\| \lesssim \mu^{-\frac{n+1}{2}} b_\mu,$$

then $h\mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b)$ has the following decomposition

$$h\mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b) = f_1 + l(f_2)$$

such that

$$\|P_\mu f_1\|_{L_t^1 L_{x'}^2}, \|P_\mu f_2\|_{L_t^\infty L_{x'}^2} \lesssim \epsilon_2^2 \mu^{-\frac{n-1}{2}} b_\mu^2.$$

In two dimensions the minor lemma implies the major lemma but in three dimensions some extra work is required.

Raychaudhuri equations, proof of Proposition 3.4.3

Decompose

$$\chi_{AB} = \langle \nabla_{\mathbf{e}_A} l, \mathbf{e}_B \rangle = \frac{1}{2} \theta \delta_{AB} + \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

Denote $\sigma_{\mathbb{C}} = \sigma_\times + i\sigma_+$,

$$\begin{aligned} \tilde{\Gamma} &= \langle \nabla_l \mathbf{e}_1, \mathbf{e}_2 \rangle, \\ l(\ln \kappa) &= \frac{1}{2} \langle \nabla_l \bar{l}, l \rangle. \end{aligned}$$

Claim 3.4.13. *The function $\tilde{\Gamma} = \langle \nabla_l \mathbf{e}_1, \mathbf{e}_2 \rangle$ can be written as*

$$\tilde{\Gamma} = \Gamma_1 + l(\Gamma_2),$$

with

$$\begin{aligned} \|P_\mu(\Gamma_1)\|_{L_{t,x'}^2} &\lesssim \epsilon_1 \mu^{-\frac{n}{2}} b_\mu^2, \\ \|P_\mu(\Gamma_2)\|_{L_{t,x'}^2} &\lesssim \epsilon_1 \mu^{-\frac{n+1}{2}} b_\mu. \end{aligned}$$

Proof. We will prove the claim by verifying that $\tilde{\Gamma}$ satisfies the assumptions of Lemma 3.4.10, i.e. there exists h_{ij} such that

$$\tilde{\Gamma} = \sum_i h_{1i} l(h_{2i}),$$

with

$$\begin{aligned} \|P_\mu h_{1j}\|_{L_{t,x'}^2} &\lesssim \mu^{-\frac{n+1}{2}} b_\mu, \\ \|P_\mu h_{2j}\|_{L_{t,x'}^2} &\lesssim \epsilon_1 \mu^{-\frac{n+1}{2}} b_\mu. \end{aligned}$$

The metric is

$$g_{\alpha\beta} = m_{\alpha\beta} + \partial_\alpha u \partial_\beta u.$$

Then the Christoffel symbols are:

$$\Gamma_{ij}^k = g^{km} \partial_m u \partial_i \partial_j u. \quad (3.4.2)$$

$$\langle \nabla_l \mathbf{e}_1, \mathbf{e}_2 \rangle = g_{\alpha\beta} l(\mathbf{e}_1^\alpha) \mathbf{e}_2^\beta + g_{\alpha\beta} \Gamma_{\alpha'\alpha''}^\alpha l^{\alpha'} \mathbf{e}_1^{\alpha''} \mathbf{e}_2^\beta.$$

Therefore

$$\langle \nabla_l \mathbf{e}_1, \mathbf{e}_2 \rangle = g_{\alpha\beta} \mathbf{e}_2^\beta l(\mathbf{e}_1^\alpha - \delta_{\alpha 1}) + \partial_\beta u \mathbf{e}_1^{\alpha''} \mathbf{e}_2^\beta l(\partial_{\alpha''} u), \quad (3.4.3)$$

since

$$\Gamma_{\alpha'\alpha''}^\alpha l^{\alpha'} = g^{\alpha\beta} \partial_\beta u \partial_{\alpha'\alpha''} u l^{\alpha'} = g^{\alpha\beta} \partial_\beta u l(\partial_{\alpha''} u).$$

$$\|P_\mu(\mathbf{e}_a^\alpha - \delta_{a\alpha})\|_{L^2_{t,x'}} \lesssim \epsilon_1 \mu^{-\frac{n+1}{2}} b_\mu,$$

by bootstraps and

$$\|P_\mu du\|_{L^2_{t,x'}} \lesssim \epsilon_2 \mu^{-\frac{n+1}{2}} b_\mu$$

by energy estimates. The same estimate applies to the products of these terms. Thus we arrive at the desired decomposition by (3.4.3). \square

Proof of Proposition 3.4.3. Denote $\mathcal{R}_{AB} = \mathcal{R}(l, \mathbf{e}_A, l, \mathbf{e}_B)$ and introduce

$$\mathcal{R}_{\mathbb{C}} = \frac{1}{2}(\mathcal{R}_{11} - \mathcal{R}_{22}) + i\mathcal{R}_{12}, \quad \text{tr}\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{22}.$$

The Raychaudhuri equations can be written in terms of functions θ , $\sigma_{\mathbb{C}}$, ω and they are

$$l(\theta) = \text{tr}\mathcal{R} - \left(\frac{1}{2}\theta^2 - 2|\sigma_{\mathbb{C}}^2| + 2\omega^2\right) + l(\ln \kappa)\theta.$$

$$l(\sigma_{\mathbb{C}}) = \mathcal{R}_{\mathbb{C}} + (\theta\sigma_{\mathbb{C}}) + (l(\ln \kappa) + 2i\tilde{\Gamma})\sigma_{\mathbb{C}}.$$

$$l(\omega) = \theta\omega + l(\ln \kappa)\omega.$$

Let $\tilde{\Gamma} = \tilde{\Gamma}_1 + l(\tilde{\Gamma}_2)$ be the decomposition of $\tilde{\Gamma}$ from Claim 3.4.13. Then we have

$$l(\kappa^{-1}\theta) = \kappa^{-1}\text{tr}\mathcal{R} - \kappa^{-1}\left(\frac{1}{2}\theta^2 - 2|\sigma_{\mathbb{C}}^2| + 2\omega^2\right).$$

$$l(\kappa^{-1}e^{-2i\tilde{\Gamma}_2}\sigma_{\mathbb{C}}) = \kappa^{-1}e^{-2i\tilde{\Gamma}_2}\mathcal{R}_{\mathbb{C}} + \kappa^{-1}e^{-2i\tilde{\Gamma}_2}(\theta + 2i\tilde{\Gamma}_1)\sigma_{\mathbb{C}}.$$

$$l(\kappa^{-1}\omega) = \kappa^{-1}\theta\omega.$$

We now apply the Major Lemma, Lemma 3.4.12 and Lemma 3.4.8 to arrive at the desired conclusion. \square

Jacobi fields equation, proof of Proposition 3.4.5

We start by constructing a new basis for the time slices, with the following lemma.

Lemma 3.4.14. *There exists a basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$ such that $\|P_\mu \tilde{\mathbf{e}}_A\|_{L^2_{t,x'}} \leq \mu^{-\frac{n+1}{2}} b_\mu$ and*

$$\|P_\mu(\langle \nabla_l \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \rangle)\|_{L^2_{t,x'}} \leq \epsilon_1 b_\mu^2 \mu^{-\frac{n}{2}}.$$

Proof. Let $\tilde{\Gamma} = \Gamma_1 + l(\Gamma_2)$ be the decomposition from Claim 3.4.13. Define

$$\begin{pmatrix} \tilde{\mathbf{e}}_1 \\ \tilde{\mathbf{e}}_2 \end{pmatrix} = e^{-\begin{pmatrix} 0 & \Gamma_2 \\ -\Gamma_2 & 0 \end{pmatrix}} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \cos \Gamma_2 & -\sin \Gamma_2 \\ \sin \Gamma_2 & \cos \Gamma_2 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Then

$$\langle \nabla_l \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \rangle = \Gamma_1,$$

which has the right regularity. \square

Proof of Proposition 3.4.5. Pick a direction y_j . We will analyze the derivative of the flow map in the direction y_j .

$$S = \frac{\partial c}{\partial y_j} \circ d.$$

Denote

$$S_A = \langle S, \tilde{\mathbf{e}}_A \rangle,$$

where $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$ is the basis from Lemma 3.4.14. To prove the statement for c , it is enough to prove

$$\|P_\mu S_A\|_{L_t^\infty L_{x'}^2} \leq \epsilon_1^2 \mu^{-\frac{n-1}{2}} b_\mu^2$$

and apply the diffeomorphism c which satisfies the condition of Lemma 3.4.7. Once we achieve the improved statement for $\frac{\partial c}{\partial y}$, we can immediately conclude the required improvement for $\frac{\partial d}{\partial x}$, since by the chain rule

$$\left(\frac{\partial c}{\partial y} \circ d\right) \frac{\partial d}{\partial x} = I.$$

Since $\frac{\partial c}{\partial y}$ satisfies the right bound and so $\frac{\partial c}{\partial y} \circ d$ by Lemma 3.4.7. We have

$$\begin{aligned} l &= \kappa^{-1} V, \quad \kappa = dt(V), \\ \nabla_l l &= -\ln(\kappa) l, \end{aligned}$$

where V is the geodesic. We compute

$$\begin{aligned}\nabla_l \nabla_l S &= \nabla_l \nabla_S l = \nabla_S \nabla_l l + \mathcal{R}(l, S, l) \\ &= -\nabla_S (\ln(\kappa)l) + \mathcal{R}(l, S, l) \\ &= -l(\ln \kappa) \nabla_l S + \mathcal{R}(l, S, l) - (Sl(\ln \kappa))l.\end{aligned}$$

Let $\tilde{\mathbf{e}}_A$ be a orthonormal basis from Lemma 3.4.14. We apply the Major lemma, Lemma 3.4.12 that says $\mathcal{R}_{AB} = \mathcal{R}(l, \tilde{\mathbf{e}}_A, l, \tilde{\mathbf{e}}_B)$ can be decomposed as

$$\kappa \mathcal{R}_{AB} = f_{AB} + l(\alpha_{AB}),$$

where $\|P_\mu f_{AB}\|_{L_t^1 L_x^2}, \|P_\mu \alpha_{AB}\|_{L_t^\infty L_x^2} \leq \epsilon_1^2 \mu^{-\frac{n-1}{2}} b_\mu^2$. We first estimate

$$Z_A = \langle \nabla_l S, \tilde{\mathbf{e}}_A \rangle.$$

$$l(\kappa Z_A) = \Gamma_1 \kappa Z_B + f_{AA} S_A + l(\alpha_{AA}) S_A + f_{AB} S_B + l(\alpha_{AB}) S_B,$$

where $\Gamma_1 = \langle \nabla_l \mathbf{e}_1, \mathbf{e}_2 \rangle$. We rewrite

$$l(\alpha_{AB}) S_B = l(\alpha_{AB} \langle S, \tilde{\mathbf{e}}_B \rangle) - \alpha_{AB} Z_B \pm \Gamma \alpha_{AC} S_C.$$

We conclude that

$$\|l(\kappa Z_A - \alpha_{AB} S_B - \alpha_{AA} S_A)\|_{L_t^1 L_x^2} \leq \epsilon_1^2 \mu^{-\frac{n-1}{2}} b_\mu^2.$$

Since $\|P_\mu \alpha_{AB}\|_{L_t^\infty L_x^2}, \|P_\mu S_B\|_{L_t^\infty L_x^2} \leq \epsilon_1 \mu^{-\frac{n-1}{2}} b_\mu^2$ by bootstraps this implies

$$\|P_\mu Z_A\|_{L_t^\infty L_x^2} \leq \epsilon_1^2 b_\mu^2 \mu^{-\frac{n-1}{2}}.$$

This then easily leads to $\|P_\mu S_A\|_{L_t^\infty L_x^2} \leq \epsilon_1^2 b_\mu^2 \mu^{-\frac{n-1}{2}}$, since

$$l(S_A) = l(\langle S, \tilde{\mathbf{e}}_A \rangle) = Z_A \pm \Gamma_1 S_B.$$

□

Proof of regularity, proof of Proposition 3.4.4

We will prove the theorem by analyzing $d\phi_{\theta,r}$ and proving that $d\phi_{\theta,r} \in H^{\frac{n+1}{2}}(\mathbb{R}^n)$. Observe that

$$dx_\theta - d\phi_{\theta,r} = \frac{1}{\tau} dF_\theta.$$

This implies that

$$l = (dx_\theta - d\phi_{\theta,r})^*.$$

Therefore, due to the properties of the metric and the product estimates, it is enough to prove the following statement.

Proposition 3.4.15. *Let l be the vector field defined in (3.4.1), then the components of l satisfy*

$$\mu^{\frac{n+1}{2}} \|P_\mu l^\alpha\|_{L^2_{t,x'}} \leq (c_1 \epsilon_1^2 + c_2 \epsilon_2) b_\mu.$$

Proof. To prove the Proposition, we will take the covariant derivatives of l in the null-frame. We need to establish

$$\begin{aligned} & \|P_\mu \langle \nabla_{\mathbf{e}_a} l, \mathbf{e}_a \rangle_g\|_{L^2_{t,x'}} + \|P_\mu \langle \nabla_{\mathbf{e}_a} l, \underline{l} \rangle_g\|_{L^2_{t,x'}} \\ & + \|P_\mu \langle \nabla_l l, \underline{l} \rangle_g\|_{L^2_{t,x'}} \leq (c_1 \epsilon_1^2 + c_2 \epsilon_2) b_\mu \mu^{-\frac{n-1}{2}}. \end{aligned}$$

The first term is covered by a much stronger statement of Proposition 3.4.3. For the second term, we have

$$\langle \nabla_{\mathbf{e}_a} l, \underline{l} \rangle_g = -\langle l, \nabla_{\mathbf{e}_a} \underline{l} \rangle_g = -\langle l, \nabla_{\mathbf{e}_a} (dt)^* \rangle_g.$$

This is a combination of Christoffel symbols (3.4.2) which are derivatives of u and therefore $\|P_\mu \Gamma_{ij}^0 \mathbf{e}_a^i l^j\| \lesssim \epsilon_2 \mu^{-\frac{n-1}{2}}$. \square

Proof of lemmas, Lemmas 3.4.6, 3.4.8-3.4.10

Proof of Lemma 3.4.6. Let $\|P_\mu f\|_{L^2_{t,x'}}, \|P_\mu g\|_{L^2_{t,x'}} \leq \mu^{-\frac{n+1}{2}} b_\mu$. We have

$$fg = \sum_{\mu, \nu} P_\mu f P_\nu g.$$

An application of P_μ splits the sum into two cases - balanced frequencies and unbalanced ones. For the balanced term, we have

$$\begin{aligned} \sum_{\lambda \geq \mu} \|P_\lambda f P_\lambda g\|_{L^2_{t,x'}} & \lesssim \sum_{\lambda \geq \mu} \lambda^{\frac{n}{2}} \|P_\lambda f\|_{L^2_{t,x'}} \|P_\lambda g\|_{L^2_{t,x'}} \lesssim \sum_{\lambda \geq \mu} \lambda^{-\frac{n+2}{2}} b_\lambda^2 \\ & \lesssim C \mu^{-\frac{n+1}{2}} b_\mu. \end{aligned}$$

For the unbalanced term, we have

$$\begin{aligned} \sum_{\nu \leq \mu} \|P_\mu f P_\nu g\|_{L^2_{t,x'}} & \lesssim \sum_{\nu \leq \mu} \nu^{\frac{n}{2}} \|P_\mu f\|_{L^2_{t,x'}} \|P_\nu g\|_{L^2_{t,x'}} \\ & \lesssim \mu^{-\frac{n+1}{2}} b_\mu \sum_{\nu \leq \mu} \nu^{-\frac{1}{2}} b_\nu \lesssim \mu^{-\frac{n+1}{2}} b_\mu \left(\sum_{\nu \leq \mu} \nu^{-1} \right)^{\frac{1}{2}} \left(\sum_{\nu \leq \mu} b_\nu^2 \right)^{\frac{1}{2}} \\ & \lesssim C \mu^{-\frac{n+1}{2}} b_\mu. \end{aligned}$$

Next we observe that if $\|P_\mu f\|_{L^2_{t,x'}} \leq \mu^{-\frac{n+1}{2}} b_\mu$ then by the properties of the frequency envelope, we have

$$b_1 \mu^{-\frac{1}{2}} \lesssim \sum_{\nu \leq \mu} b_\nu \nu^{\frac{1}{2}} \mu^{-\frac{1}{2}} \lesssim b_\mu.$$

Therefore, applying the Bernstein inequality on the time variable

$$\|P_\mu f\|_{L_t^\infty L_x^2} \lesssim \mu^{\frac{1}{2}} \|P_\mu f\|_{L_{t,x}^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu \mu^{-\frac{1}{2}} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2.$$

Let $\|P_\mu f\|_{L_t^\infty L_x^2}, \|P_\mu g\|_{L_t^1 L_x^2} \leq \mu^{-\frac{n-1}{2}} b_\mu^2$. Then for a balanced term of the product, we have

$$\begin{aligned} \sum_{\lambda \geq \mu} \|P_\lambda f P_\lambda g\|_{L_t^1 L_x^2} &\lesssim \sum_{\lambda \geq \mu} \|P_\lambda f\|_{L_t^\infty L_x^4} \|P_\lambda g\|_{L_t^1 L_x^4} \\ &\lesssim \sum_{\lambda \geq \mu} \lambda^{\frac{n-1}{2}} \|P_\lambda f\|_{L_t^\infty L_x^2} \|P_\lambda g\|_{L_t^1 L_x^2} \\ &\lesssim \sum_{\lambda \geq \mu} \lambda^{-\frac{n-1}{2}} b_\lambda^4 \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2 \sum_{\lambda \geq \mu} b_\lambda^2 \\ &\lesssim C \mu^{-\frac{n-1}{2}} b_\mu^2. \end{aligned}$$

For the unbalanced term

$$\begin{aligned} \sum_{\nu \leq \mu} \|P_\nu f P_\nu g\|_{L_t^1 L_x^2} &\lesssim \sum_{\nu \leq \mu} \|P_\nu f\|_{L_t^\infty L_x^2} \|P_\nu g\|_{L_t^1 L_x^\infty} \\ &\lesssim \sum_{\nu \leq \mu} \nu^{\frac{n-1}{2}} \|P_\nu f\|_{L_t^\infty L_x^2} \|P_\nu g\|_{L_t^1 L_x^2} \lesssim \mu^{-\frac{n-1}{2}} b_\mu^2 \sum_{\nu \leq \mu} b_\nu^2 \\ &\lesssim C \mu^{-\frac{n-1}{2}} b_\mu^2. \end{aligned}$$

The remaining estimate is for $\|P_\mu g\|_{L_t^\infty L_x^2} \leq \mu^{-\frac{n-1}{2}} b_\mu^2$, which is completely analogous to the case of $L_t^1 L_x^2$. \square

Proof of Lemma 3.4.8. If we apply c - the flow map of l to the equation $l(f) = h$ we get

$$\partial_t f_1 = h \circ c. \quad (3.4.4)$$

Since c satisfies the conditions of Lemma 3.4.7, we have

$$\|P_\mu h \circ c\|_{L_t^1 L_x^2} \leq \mu^{-\frac{n-1}{2}} b_\mu^2$$

then the solution to (3.4.4) satisfies

$$\|P_\mu f_1\|_{L_t^\infty L_x^2} \leq \mu^{-\frac{n-1}{2}} b_\mu^2.$$

The solution to the equation $l(f) = h$ is $f = f_1 \circ d$ which satisfies the same estimates since Lemma 3.4.7 applies to d as well. \square

Proof of Lemma 3.4.10. Clearly, it is enough to prove the lemma for just one term $g_1 l(g_2)$ and then sum the contributions to terms f_i . We will decompose the term

$$\sum_{\nu < \frac{\mu}{4}} (P_\nu g_1) l(P_\mu g_2) = l\left(\sum_{\nu < \frac{\mu}{4}} P_\nu g_1 P_\mu g_2\right) - \sum_{\nu < \frac{\mu}{4}} l(P_\nu g_1) P_\mu g_2.$$

Define

$$f_2 = \sum_{\nu < \frac{\mu}{4}} P_\nu g_1 P_\mu g_2.$$

Then

$$\|P_\nu f_2\|_{L^2} = \left\| \sum_{\nu < \frac{\mu}{4}} P_\nu g_1 P_\mu g_2 \right\|_{L^2} \leq \|P_\mu g_2\|_{L^2} \left\| \sum_{\nu < \frac{\mu}{4}} P_\nu g_1 \right\|_{L^\infty}.$$

We estimate

$$\begin{aligned} \left\| \sum_{\nu < \frac{\mu}{4}} P_\nu g_1 \right\|_{L^\infty} &\leq \sum_{\nu < \frac{\mu}{4}} \nu^{\frac{n}{2}} \|P_\nu g_1\|_{L^2} \\ &\leq \sum \nu^{-\frac{1}{2}} b_\nu \leq C \left(\sum b_\nu^2 \right)^{\frac{1}{2}} \leq C\epsilon. \end{aligned}$$

Thus f_2 satisfies the desired estimate. The rest of the terms will be f_1 . First, sum the balanced term where the high-frequency components of l do not contribute, we have

$$\begin{aligned} \|P_\mu f_{11}\|_{L^2} &\lesssim \sum_{\nu_1 \sim \nu_2} \mu \|P_{\nu_1} g_1 P_{\nu_2} g_2\|_{L^2} \leq \sum_{\nu_1 \sim \nu_2} \mu \nu_1^{\frac{n}{2}} \|P_{\nu_1} g_1\|_{L^2} \|P_{\nu_2} g_2\|_{L^2} \\ &\lesssim \mu \sum_{\nu \geq \mu} \nu^{\frac{n}{2}} \nu^{-n-1} b_\nu^2 \leq \mu^{-\frac{n}{2}} \sum_{\nu \geq \mu} \left(\frac{\mu}{\nu}\right)^{\frac{n+2}{2}} b_\nu^2 \leq \mu^{-\frac{n}{2}} b_\mu^2. \end{aligned}$$

For the unbalanced term, the contribution to the highest frequency will come from either coefficients l or the undifferentiated term, then we will have the low frequency differentiated term and the remaining term we can fuse via product estimates with one of these terms. We have

$$\begin{aligned} \|P_\mu f_{11}\|_{L^2} &\lesssim \sum_{\nu < \frac{\mu}{4}} \nu \|P_\mu g'_1 P_\nu g_2\|_{L^2} \leq \sum_{\nu < \frac{\mu}{4}} \nu^{\frac{n}{2}+1} \|P_\mu g'_1\|_{L^2} \|P_\nu g_2\|_{L^2} \\ &\leq \mu^{-\frac{n}{2}} b_\mu \sum_{\nu < \frac{\mu}{4}} \left(\frac{\nu}{\mu}\right)^{\frac{1}{2}} b_\nu \leq \mu^{-\frac{n}{2}} b_\mu^2. \end{aligned}$$

□

Proof of Diffeomorphism lemma, Lemma 3.4.7

In this subsection only, we will switch to dyadic index k , i.e. $\mu = 2^k$. Let $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ be diffeomorphism such that

$$\|P_k(\nabla\psi - I)\|_{L^2} \leq \epsilon b_k^2 2^{-\frac{n-1}{2}k},$$

where P_k is a Littlewood-Paley projection onto frequency 2^k and

$$\sum b_k^2 < 1.$$

Let $h \in L^2$ such that $\|h\|_{L^2} = 1$. Define

$$C_{I \rightarrow O}^K(h) = \|P_O[(P_I h) \circ \psi_{\leq K}]\|_{L^2}$$

and

$$C_{I \rightarrow O}^K = \sup_{\|h\|_{L^2}=1} C_{I \rightarrow O}^K(h).$$

The quantity $C_{I \rightarrow O}^K$ is the $L^2 \rightarrow L^2$ norm of the operator

$$T_{I \rightarrow O}^K h = P_O[(P_I h) \circ \psi_{\leq K}].$$

Similarly define

$$\begin{aligned} T_{I \rightarrow O} h &= P_O[(P_I h) \circ \psi], \\ C_{I \rightarrow O} &= \|T_{I \rightarrow O}\|_{L^2 \rightarrow L^2}. \end{aligned}$$

Our method is to bound $C_{I \rightarrow O}^K$ by bounding $C_{I \rightarrow O}^K(h)$ uniformly in h . We will observe uniformity of the bounds for large K , which leads to bounds for the full, untruncated diffeomorphism. As

$$\hat{\psi}_{\leq k}(\xi) = m_0\left(\frac{\xi}{2^k}\right)\hat{\psi}(\xi),$$

we can easily extend the operation of frequency truncation into the range of continuous k indices. Similarly, extend b_k by

$$b_k = \max\{b_{\lceil k \rceil}, b_{\lfloor k \rfloor}\}.$$

We observe that for every K , the map $h \mapsto P_O[(P_I h) \circ \psi_{\leq K}]$ is a map of smooth functions, which depends smoothly on K for $K > 0$. Therefore $C_{I \rightarrow O}^K(h)$ is Lipschitz in K and we will estimate it by estimating the derivative.

Claim 3.4.16. *For $k > \max\{I, O\}$ we have*

$$\frac{d}{dk} C_{I \rightarrow O}^k(h) \lesssim 2^I 2^{\frac{n-1}{2}O} 2^{-\frac{n+1}{2}k} \epsilon b_k^2$$

Proof. We have

$$\frac{d}{dk} P_I h(\psi_{\leq k}(x)) = \nabla [P_I h(\psi_{\leq k}(x))] \frac{d}{dk} \psi_{\leq k}(x).$$

Since

$$\hat{\psi}_{\leq k}(\xi) = m_0\left(\frac{\xi}{2^k}\right) \hat{\psi}(\xi),$$

then

$$\frac{d}{dk} \psi_{\leq k}(\xi) = \frac{C}{2^k} \xi m'_0\left(\frac{\xi}{2^k}\right) \hat{\psi}(\xi).$$

$m'_0\left(\frac{\xi}{2^k}\right)$ is a bounded multiplier supported in the frequency 2^k . We will treat it as P_k , this also makes the factor $\frac{\xi}{2^k}$ bounded and disposable. Therefore

$$\frac{d}{dk} \|P_O[(P_I h) \circ \psi_{\leq k}]\|_{L^2} \leq \|P_O[\nabla(P_I h(\psi_{\leq k}(x))) P_k \psi]\|_{L^2}.$$

Since $\nabla \psi$ is bounded by $1 + \epsilon$, we can multiply $\nabla(P_I h(\psi_{\leq k}(x)))$ by a bounded function to transform it to $(P_I \nabla h) \circ (\psi(x))$. We use Bernstein's inequality

$$\|P_O[P_I(\nabla h) \circ \psi_{\leq k} \nabla \psi_{\leq k} P_k \psi]\|_{L^2} \leq 2^{\frac{n-1}{2}O} \|P_I(\nabla h) \circ \psi_{\leq k} \nabla \psi_{\leq k} P_k \psi\|_{L^1}.$$

We estimate

$$\|P_I(\nabla h) \circ \psi_{\leq k} \nabla \psi_{\leq k} P_k \psi\|_{L^1} \leq 2^I \|h \circ \psi_{\leq k}\|_{L^2} \|\nabla \psi\|_{L^\infty} \|P_k \psi\|_{L^2}.$$

Since $\nabla \psi$ is bounded, we can estimate $\|h \circ \psi_{\leq k}\|_{L^2}$ by $(1 + \epsilon) \|h\|_{L^2}$. Also we have

$$\|P_k \psi\|_{L^2} \leq \epsilon b_k^2 2^{-\frac{n+1}{2}k}.$$

Combining all the estimates, we get the conclusion. \square

Corollary 3.4.17. *For $K > \max\{I, O\}$ we have*

$$C_{I \rightarrow O}^K(h) \leq C_{I \rightarrow O}^{\max\{I, O\}}(h) + C \epsilon b_{\max\{I, O\}}^2(h).$$

Proof.

$$C_{I \rightarrow O}^K(h) = C_{I \rightarrow O}^{\max\{I, O\}}(h) + \int_{\max\{I, O\}}^k \frac{d}{dk} C_{I \rightarrow O}^k(h) dk.$$

We use the previous claim to estimate

$$\frac{d}{dk} C_{I \rightarrow O}^k(h) \lesssim 2^{\frac{n+3}{2} \max\{I, O\}} 2^{-\frac{n+3}{2}k} \epsilon b_k^2.$$

Therefore, bounding the integral by the sum and using the frequency envelope condition, we get the desired conclusion. \square

One can state another version of the claim.

Claim 3.4.18. *For $k > O$ we have*

$$\frac{d}{dk} C_{I \rightarrow O}^k(h) \lesssim 2^{\frac{n-1}{2}} O 2^{-\frac{n-1}{2}k} \epsilon b_k^2 C_{I \rightarrow k}^k(h).$$

Proof. We again use

$$\frac{d}{dk} \|P_O[(P_I h) \circ \psi_{\leq k}]\|_{L^2} \leq \|P_O[\nabla(P_I h(\psi_{\leq k}(x)))P_k \psi]\|_{L^2}.$$

Since $P_k \psi$ is at frequency 2^k and 2^O is at lower frequency, we have to have $\nabla(P_I h(\psi_{\leq k}(x)))$ at comparable frequency (suppose it's exactly 2^k for simplicity). Then

$$\begin{aligned} \|P_O[\nabla(P_I h(\psi_{\leq k}(x)))P_k \psi]\|_{L^2} &= \|P_O[P_k(\nabla(P_I h(\psi_{\leq k}(x))))P_k \psi]\|_{L^2} \\ &= 2^k \|P_O[P_k(P_I h(\psi_{\leq k}))P_k \psi]\|_{L^2} \\ &= 2^k \|P_O[T_{I \rightarrow k}^k h P_k \psi]\|_{L^2}. \end{aligned}$$

Now use Bernstein and the fact that $\|T_{I \rightarrow k}^k h\|_{L^2} \leq C_{I \rightarrow k}^k \|h\|_{L^2}$ by definition to arrive at the conclusion of the lemma. \square

Claim 3.4.19. *Let $I < k$ then*

$$C_{I \rightarrow k}^k(h) \lesssim 2^{I-k}.$$

Proof. We have

$$\begin{aligned} \|P_k P_I h \circ \psi_{\leq k}\|_{L^2} &\leq 2^{-k} \|P_k \nabla(P_I h \circ \psi_{\leq k})\| \\ &\leq 2^{-k} \|(P_I \nabla h \circ \psi_{\leq k}) \nabla \psi_{\leq k}\|_{L^2} \\ &\leq 2^{I-k} \|(P_I h \circ \psi_{\leq k}) \nabla \psi_{\leq k}\|_{L^2}, \end{aligned}$$

where we used the chain rule. Since $\nabla \psi_{\leq k}$ is uniformly bounded by $1 + \epsilon$, we can change variables to estimate

$$\|(P_I h \circ \psi_{\leq k}) \nabla \psi_{\leq k}\|_{L^2} \lesssim \|h\|_{L^2}.$$

\square

Claim 3.4.20. *For $n = 3$ (two-dimensional maps) and $k \geq I$, the bound in Claim 3.4.19 can be improved to*

$$C_{I \rightarrow k}^k(h) \lesssim 2^{2(I-k)} + 2^{I-k} \epsilon b_k^2.$$

Proof. We iterate the argument of Claim 3.4.19

$$\|P_k P_I h \circ \psi_{\leq k}\|_{L^2} \leq 2^{-2k} \|P_k \nabla^2(P_I h \circ \psi_{\leq k})\|.$$

Using the chain rule twice, two applications of derivative lead to

$$\begin{aligned}\nabla^2(P_I h \circ \psi_{\leq k}) &= \nabla(\nabla(P_I \nabla h) \circ \psi_{\leq k} \nabla \psi_{\leq k}) \\ &= P_I(\nabla^2 h) \circ \psi_{\leq k} (\nabla \psi_{\leq k})^2 + P_I(\nabla h) \circ \psi_{\leq k} \nabla^2 \psi_{\leq k}.\end{aligned}$$

The L^2 norm of the first can be immediately estimated by $2^{2I} \|h\|_{L^2}$. We examine the application of P_k on the second term $P_k[P_I(\nabla h) \circ \psi_{\leq k} \nabla^2 \psi_{\leq k}]$. Since $\psi_{\leq k}$ has frequencies of only up to 2^k then one factors in the product has to have frequency 2^k . Therefore

$$\begin{aligned}P_k[P_I(\nabla h) \circ \psi_{\leq k} \nabla^2 \psi_{\leq k}] &= P_k[T_{I \rightarrow k}^k(\nabla h) \nabla^2 \psi_{\leq k}] \\ &\quad + P_k[P_I(\nabla h) \circ \psi_{\leq k} P_k \nabla^2 \psi].\end{aligned}$$

We estimate the contribution of each of these terms to $C_{I \rightarrow k}^k(h)$ after discarding the leading P_k . We have for the first term:

$$2^{-2k} 2^{I-k} 2^I \sum_{J < k} 2^J \epsilon b_J^2 \leq 2^{I-k} \epsilon b_k^2,$$

and for the second term, we estimate $P_I(\nabla h)$ in L^∞ using Bernstein's inequality:

$$2^{-2k} 2^{2I} \|P_k \nabla^2 \psi\|_{L^2} \leq 2^{-2k} 2^{2I} \epsilon b_k^2,$$

which completes the proof. □

Since the bounds are uniform in h , we have the following

Corollary 3.4.21. *For $I < O < k$, we have the bound*

$$C_{I \rightarrow O}^k \leq 2^{I-O},$$

and for $n = 3$

$$C_{I \rightarrow O}^k \leq 2^{2I-2O} + 2^{I-O} \epsilon b_O^2.$$

Since these bounds apply uniformly for large k , taking the supremum, we can apply them to the full diffeomorphism ψ . We summarize in the following statement.

Claim 3.4.22. *With the conditions and the notations in the beginning of the section, we have*

$$C_{I \rightarrow O} \lesssim \min\{1, 2^{I-O}\}$$

and

$$C_{I \rightarrow O} \lesssim \min\{1, 2^{2I-2O} + 2^{I-O} \epsilon b_O^2\}, \quad n = 3.$$

Proof of Lemma 3.4.7.

$$P_k g = \sum_j T_{j \rightarrow k} P_j f.$$

We apply Claim 3.4.22. For $n = 2$, we estimate

$$\|P_k g\|_{L^2} \lesssim 2^{-\frac{1}{2}k} \sum_{j \leq k} 2^{-\frac{1}{2}k} 2^{\frac{1}{2}j} b_j^2 + 2^{-\frac{1}{2}} \sum_{j \geq k} 2^{\frac{1}{2}k} 2^{-\frac{1}{2}j} b_j^2 \lesssim b_k^2 2^{-\frac{k}{2}}.$$

For $n = 3$, we estimate

$$\begin{aligned} \|P_k g\|_{L^2} &\lesssim 2^{-k} \sum_{j \geq k} 2^{k-j} b_j^2 + 2^{-k} \sum_{j \leq k} 2^{j-k} b_j^2 + \epsilon b_k^2 b_j^2 \\ &\lesssim 2^{-k} b_j^2. \end{aligned}$$

□

Proof of the Minor lemma, Lemma 3.4.11

Proof of the minor lemma, Lemma 3.4.11. In the term $P_\mu(h_1 l(h_2) l(h_2))$ there are in fact five different factors, as we need to worry about the high frequencies in the coefficients of l . But since these factors and their products satisfy the same estimate, therefore what matters is only the way different frequencies in differentiated and undifferentiated terms interact. First, we identify the $l(f_2)$ term which we will integrate by parts, this is the term:

$$I = \left[\sum_{\nu_i \leq \frac{\mu}{4}} P_{\nu_1} h_1 P_{\nu_4} (l^I) \frac{\partial}{\partial x_I} (P_{\nu_2} h_2) \right] l(P_\mu h_3).$$

Thus we define:

$$P_\mu f_2 = P_\mu h_3 \sum_{\nu_i \leq \frac{\mu}{4}} P_{\nu_1} h_1 P_{\nu_4} (l^I) \frac{\partial}{\partial x_I} (P_{\nu_2} h_2).$$

This leads to the formula

$$I = l(P_\mu f_2) - I_1 - I_2, \tag{3.4.5}$$

where

$$I_1 = P_\mu h_3 \sum_{\nu_i \leq \frac{\mu}{4}} P_{\nu_1} h_1 P_{\nu_4} (l^I) l\left(\frac{\partial}{\partial x_I} (P_{\nu_2} h_2)\right)$$

and I_2 is of the form

$$I_2 \sim P_\mu h_3 \sum_{\nu_i \leq \frac{\mu}{4}} l(P_{\nu_1} h_1) P_{\nu_4} (l^I) \frac{\partial}{\partial x_I} (P_{\nu_2} h_2),$$

along with a similar term where the l derivative falls on the low frequency coefficients of l . Before we turn to f_1 , we verify that f_2 satisfies the correct estimate.

$$\|P_\mu f_2\|_{L^2_{t,x'}} \lesssim \sum_{\nu_i \leq \frac{\mu}{4}, I} \nu_2 \|P_{\nu_1} h_1 P_{\nu_4}(l^I) P_{\nu_2} h_2 P_\mu h_3\|_{L^2_{t,x'}},$$

applying Sobolev embedding, we have

$$\lesssim \sum_{\nu_i \leq \frac{\mu}{4}, I} \nu_2^{\frac{n}{2}+1} \nu_1^{\frac{n}{2}} \nu_4^{\frac{n}{2}} \|P_{\nu_1} h_1\|_{L^2} \|P_{\nu_2} h_2\|_{L^2} \|P_{\nu_4} l^I\|_{L^2} \|P_\mu h_3\|_{L^2}.$$

For ν_1, ν_4 summations we have

$$\sum_{\nu_1} \nu_1^{\frac{n}{2}} \|P_{\nu_1} h_1\| \leq \sum_{\nu_1} \nu_1^{\frac{n}{2}} \nu_1^{-\frac{n+1}{2}} b_{\nu_1} \leq C \left(\sum b_{\nu_1}^2 \right)^{\frac{1}{2}} \lesssim 1. \quad (3.4.6)$$

Similarly,

$$\sum_{\nu_4} \nu_4^{\frac{n}{2}} \|P_{\nu_4} l^I\|_{L^2} \lesssim 1.$$

For the ν_2 summation, we use the frequency envelope property

$$\sum_{\nu_2 \leq \frac{\mu}{4}} \nu_2^{\frac{n}{2}+1} \|P_{\nu_2} h_2\| \leq \sum_{\nu_2 \leq \frac{\mu}{4}} \nu_2^{\frac{n}{2}+1} \nu_2^{-\frac{n+1}{2}} b_{\nu_2} \leq \mu^{\frac{1}{2}} \sum_{\nu_2 \leq \frac{\mu}{4}} \left(\frac{\nu_2}{\mu} \right)^{\frac{1}{2}} b_{\nu_2} \leq \mu^{\frac{1}{2}} b_\mu.$$

Combining with the estimate for $\|P_\mu h_3\|$ we obtain

$$\|P_\mu f_2\|_{L^2} \lesssim \mu^{\frac{n}{2}} b_\mu^2.$$

To proceed with the rest of the terms, we make a couple of observations to help us reduce the number of different types of terms from 31+1 to 4. First we observe that we can fuse the high frequency undifferentiated terms via product estimates. Secondly, following (3.4.6), we note that we can safely sum up the low undifferentiated terms. These two observations also imply that we can replace the variable coefficients l with just constant coefficients operator $|\nabla| = \sqrt{-\Delta}$. This leaves us with four types of terms.

Type I: Undifferentiated high frequency multiplied by two differentiated low frequencies:

$$I = \sum_{\nu_2 \leq \nu_1 \leq \frac{\mu}{4}} P_\mu g' |\nabla| (P_{\nu_1} g'_1) |\nabla| (P_{\nu_2} g'_2).$$

We estimate the norm:

$$\begin{aligned}
\|I\|_{L_t^1 L_{x'}^2} &\lesssim \sum_{\nu_2 \leq \nu_1 \leq \frac{\mu}{4}} \|P_\mu g' |\nabla|(P_{\nu_1} g'_1) |\nabla|(P_{\nu_2} g'_2)\|_{L_t^1 L_{x'}^2} \\
&\sim \sum_{\nu_2 \leq \nu_1 \leq \frac{\mu}{4}} \nu_1 \nu_2 \|P_\mu g' P_{\nu_1} g'_1 P_{\nu_2} g'_2\|_{L_t^1 L_{x'}^2} \\
&\lesssim \sum_{\nu_2 \leq \nu_1 \leq \frac{\mu}{4}} \nu_1 \nu_2 \|P_\mu g'\|_{L_{t,x'}^2} \|P_{\nu_1} g'_1\|_{L_t^2 L_{x'}^\infty} \|P_{\nu_2} g'_2\|_{L_{t,x}^\infty} \\
&\lesssim \sum_{\nu_2 \leq \nu_1 \leq \frac{\mu}{4}} \nu_1^{\frac{n+1}{2}} \nu_2^{\frac{n+2}{2}} \|P_\mu g'\|_{L_{t,x'}^2} \|P_{\nu_1} g'_1\|_{L_{t,x'}^2} \|P_{\nu_2} g'_2\|_{L_{t,x'}^2} \\
&\lesssim \sum_{\nu_2 \leq \nu_1 \leq \frac{\mu}{4}} \nu_1^{\frac{n+1}{2}} \nu_2^{\frac{n+2}{2}} \|P_\mu g'\|_{L_{t,x'}^2} \|P_{\nu_1} g'_1\|_{L_{t,x'}^2} \|P_{\nu_2} g'_2\|_{L_{t,x'}^2}.
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{\nu_2 \leq \nu_1} \nu_2^{\frac{n+2}{2}} \|P_{\nu_2} g'_2\|_{L_{t,x'}^2} &\leq \sum_{\nu_2 \leq \nu_1} \nu_2^{\frac{n+2}{2}} \nu_2^{-\frac{n+1}{2}} b_{\nu_2} \\
&\lesssim \nu_1^{\frac{1}{2}} \sum_{\nu_2 \leq \nu_1} \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{2}} b_{\nu_2} \lesssim \nu_1^{\frac{1}{2}},
\end{aligned}$$

after applying Cauchy-Schwartz inequality for $l_{\nu_2}^2$. Therefore

$$\begin{aligned}
\|I\|_{L_t^1 L_{x'}^2} &\lesssim \sum_{\nu_1 \leq \frac{\mu}{4}} \nu_1^{\frac{n+2}{2}} \|P_\mu g'\|_{L_{t,x'}^2} \|P_{\nu_1} g'_1\|_{L_{t,x'}^2} \leq \mu^{-\frac{n+1}{2}} b_\mu \sum_{\nu_1 \leq \frac{\mu}{4}} \nu_1^{\frac{1}{2}} b_{\nu_1} \\
&\leq \mu^{-\frac{n}{2}} b_\mu \sum_{\nu_1 \leq \frac{\mu}{4}} \left(\frac{\nu_1}{\mu}\right)^{\frac{1}{2}} b_{\nu_1} \leq \mu^{-\frac{n}{2}} b_\mu^2.
\end{aligned}$$

Type II: Undifferentiated high frequency with twice differentiated low frequency (the I_2 term in (3.4.5)):

$$II = \sum_{\nu \leq \frac{\mu}{4}} P_\mu g' P_\nu (|\nabla|^2 g'')$$

We estimate the norm:

$$\begin{aligned}
 \|II\|_{L_t^1 L_{x'}^2} &\lesssim \sum_{\nu \leq \frac{\mu}{4}} \|P_\mu g' P_\nu (|\nabla|^2 g'')\|_{L_t^1 L_{x'}^2} \sim \sum_{\nu \leq \frac{\mu}{4}} \nu^2 \|P_\mu g' P_\nu g''\|_{L_t^1 L_{x'}^2} \\
 &\lesssim \sum_{\nu \leq \frac{\mu}{4}} \nu^2 \|P_\mu g'\|_{L_{t,x'}^2} \|P_\nu g''\|_{L_t^2 L_{x'}^\infty} \\
 &\leq \sum_{\nu \leq \frac{\mu}{4}} \nu^{\frac{n+3}{2}} \|P_\mu g'\|_{L_{t,x'}^2} \|P_\nu g''\|_{L_{t,x'}^2} \\
 &\lesssim \mu^{-\frac{n+1}{2}} b_\mu \sum_{\nu \leq \frac{\mu}{4}} \nu^{\frac{n+3}{2}} \nu^{-\frac{n+1}{2}} b_\nu \\
 &\leq \mu^{-\frac{n-1}{2}} b_\mu \sum_{\nu \leq \frac{\mu}{4}} \left(\frac{\nu}{\mu}\right)^{\frac{1}{2}} b_\nu \\
 &\leq \mu^{-\frac{n+1}{2}} b_\mu^2.
 \end{aligned}$$

Type III: Undifferentiated high frequency factor with differentiated high and low frequency factors

$$III = \sum_{4\nu \leq \mu \leq \frac{\lambda}{4}} P_\lambda g'_1 |\nabla| (P_\lambda g'_2) |\nabla| (P_\nu g'_3)$$

We estimate the norm:

$$\begin{aligned}
 \|III\|_{L_t^1 L_{x'}^2} &\lesssim \sum_{4\nu \leq \mu \leq \frac{\lambda}{4}} \|P_\lambda g'_1 |\nabla| (P_\lambda g'_2) |\nabla| (P_\nu g'_3)\|_{L_t^1 L_{x'}^2} \\
 &\sim \sum_{4\nu \leq \mu \leq \frac{\lambda}{4}} \nu \lambda \|P_\lambda g'_1\|_{L_t^2 L_{x'}^4} \|P_\lambda g'_2\|_{L_t^2 L_{x'}^4} \|P_\nu g'_3\|_{L_{t,x'}^\infty} \\
 &\lesssim \sum_{4\nu \leq \mu \leq \frac{\lambda}{4}} \nu^{\frac{n+2}{2}} \lambda^{\frac{n+1}{2}} \|P_\lambda g'_1\|_{L_{t,x'}^2} \|P_\lambda g'_2\|_{L_{t,x'}^2} \|P_\nu g'_3\|_{L_{t,x'}^2} \\
 &\lesssim \sum_{\mu \leq \frac{\lambda}{4}} \lambda^{\frac{n+1}{2}} \lambda^{-n-1} b_\lambda^2 \sum_{4\nu \leq \mu} \nu^{\frac{n+2}{2}} \nu^{-\frac{n+1}{2}} b_\nu \\
 &\lesssim \mu^{-\frac{n}{2}} \sum_{\mu \leq \frac{\lambda}{4}} \left(\frac{\nu}{\lambda}\right)^{\frac{n+1}{2}} b_\lambda^2 \sum_{4\nu \leq \mu} \left(\frac{\nu}{\mu}\right)^{\frac{1}{2}} b_\nu \\
 &\lesssim \mu^{-\frac{n}{2}} b_\mu^2,
 \end{aligned}$$

after using the slow variation in the frequency envelope in λ sum and Cauchy-Schwartz in ν .

Type IV: Two differentiated high frequency factors

$$IV = \sum_{\mu \leq \frac{\lambda}{4}} |\nabla|(P_\lambda g'_1)|\nabla|(P_\lambda g'_2)$$

We have

$$\begin{aligned} \|IV\|_{L_t^1 L_{x'}^2} &\lesssim \sum_{\mu \leq \frac{\lambda}{4}} \| |\nabla|(P_\lambda g'_1)|\nabla|(P_\lambda g'_2) \|_{L_t^1 L_{x'}^2} \\ &\leq \sum_{\mu \leq \frac{\lambda}{4}} \lambda^2 \|P_\lambda g'_1\|_{L_t^2 L_{x'}^4} \|P_\lambda g'_2\|_{L_t^2 L_{x'}^4} \\ &\lesssim \sum_{\mu \leq \frac{\lambda}{4}} \lambda^{\frac{n+3}{2}} \|P_\lambda g'_1\|_{L_{t,x'}} \|P_\lambda g'_2\|_{L_{t,x'}} \\ &\lesssim \mu^{\frac{n-1}{2}} \sum_{\mu \leq \frac{\lambda}{4}} \left(\frac{\mu}{\lambda}\right)^{\frac{n-1}{2}} b_\lambda^2 \leq \mu^{\frac{n-1}{2}} b_\mu^2. \end{aligned}$$

□

Proof of the Major lemma, Lemma 3.4.12: structure of the curvature term

Proof of the Major Lemma, Lemma 3.4.12. We need to analyze the expression $\mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b)$. First we analyze the structure to see that, in two dimensions, the expression is covered by the Minor Lemma and, in three dimensions, provide the additional estimates for the term that doesn't fall under the Minor Lemma. We have

$$\begin{aligned} \mathcal{R}(l, \mathbf{e}_a, l, \mathbf{e}_b) &= \frac{1}{1 + |du|_m^2} \{ \text{Hess}_u(l, \mathbf{e}_a) \text{Hess}_u(l, \mathbf{e}_b) \\ &\quad - \text{Hess}_u(l, l) \text{Hess}_u(\mathbf{e}_a, \mathbf{e}_b) \}, \end{aligned}$$

where Hess_ψ is the Hessian of ψ . Observe that $\frac{1}{1+|du|^2}$ is of the form $F(du)$ then the first term is the easiest to deal with

$$\text{Hess}_u(l, \mathbf{e}_a) = \sum_{\alpha, \beta} \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} l^\beta \mathbf{e}_a^\alpha = \sum_{\alpha} \mathbf{e}_a^\alpha l \left(\frac{\partial u}{\partial x_\alpha} \right), \quad (3.4.7)$$

We have $\|P_\mu(\frac{1}{1+|du|^2}|_{\Sigma_{\theta,r}})\|_{L^2}, \|P_\mu \mathbf{e}_a^\alpha\|_{L^2} \leq \mu^{-\frac{n+1}{2}} b_\mu$, therefore the term

$$\frac{1}{1 + |du|_m^2} \text{Hess}_u(l, \mathbf{e}_a) \text{Hess}_u(l, \mathbf{e}_b)$$

has the structure to which the minor lemma applies. We turn to the second term. Since u satisfies the wave equation, we have

$$0 = \square_{g(du)}u = \text{TrHess}_u = -2\text{Hess}_u(l, \underline{l}) + \sum_{c=1}^{n-1} \text{Hess}_u(\mathbf{e}_c, \mathbf{e}_c). \quad (3.4.8)$$

Now we see that for $n = 2$, we have only one vector field \mathbf{e}_1 and therefore

$$\text{Hess}_u(\mathbf{e}_1, \mathbf{e}_1) = 2\text{Hess}_u(l, \underline{l}),$$

to which, along with $\text{Hess}_u(l, l)$, we apply a similar analysis as in (3.4.7) to see that the product falls under the Minor Lemma. For the case $n = 3$ we further write

$$\mathbf{e}_a = h_a l + \sum_b K_{ab} \frac{\partial}{\partial x_b}, \quad a, b = 1, 2.$$

Then

$$\begin{aligned} \text{Hess}_u(\mathbf{e}_a, \mathbf{e}_b) &= h_a h_b \text{Hess}(l, l) + \sum_c (h_a K_{cb} + h_b K_{ca}) \text{Hess}_u(l, \frac{\partial}{\partial x_a}) \\ &\quad + \sum_{c,d} K_{ca} K_{bd} \frac{\partial^2(u|_{\Sigma_{\theta,r}})}{\partial x_c \partial x_d}. \end{aligned}$$

Therefore, only the last term is not of the structure for the minor lemma. We rewrite (3.4.8) in the following form

$$\begin{aligned} \sum_{a,b,c} K_{ac} K_{bc} \frac{\partial^2(u|_{\Sigma_{\theta,r}})}{\partial x_a \partial x_b} &= - \sum_c h_c^2 \text{Hess}_u(l, l) - \sum h_c K_{ca} \text{Hess}_u(l, \frac{\partial}{\partial x_a}) \\ &\quad + 2\text{Hess}_u(l, \underline{l}). \end{aligned}$$

But since $g(du)$ is a small perturbation of the constant coefficient D'Ambertian, then the matrix $\sum_c K_{ac} K_{bc}$ is a small perturbation of δ_{ab} and therefore the left-hand side of the equation above defines an elliptic operator on \mathbb{R}^2 which has an $H_{x'}^2$ inverse with an $H_{x'}^1$ error, which is order -1 operator. Denote the inverse pseudo-differential operator V_t . We note that every term on the right-hand side can be written as a linear combination of the terms in the form $h_1 l(h_2)$ due to (3.4.7), where $\|P_\mu h_i\|_{L^2_{t,x'}} \leq \mu^{-\frac{n+1}{2}} b_\mu$. Thus we can write

$$\text{Hess}_u(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b}) = \sum_N R_{ab}^N l(h_{2N}) + E_N h_{2N},$$

where R_{ab}^N is an order 0 pseudodifferential operator which corresponds to the operator

$$f \mapsto \frac{\partial}{\partial x_a} \frac{\partial}{\partial x_b} V_t(h_{1N} f)$$

and E_N is the error, which is of order -1 , so we will neglect it, since we will apply only the low frequency parts of the symbol on the high frequency functions. Therefore, the analysis of the Minor Lemma, Lemma 3.4.11, needs to be modified to prove that a term of the form $h_1 l(h_2) R' l(h_3)$, where R' is an order zero pseudo-differential operator with $H_x^{\frac{n+1}{2}}$ symbol, satisfies the conclusions of the Major and Minor Lemmas, i.e.

$$h_1 l(h_2) R' l(h_3) = l(f_1) + f_2.$$

Let a' be the Kohn-Nirenberg (left quantization) symbol of R' , which we will denote as $R' = T_{a'}$. To prove the statement we write $R' = \sum_{\mu} R_{\mu}$ where $R_{\mu} = T_{P_{\mu} a'}$. We also split all the terms according to dyadic frequency. The proof proceeds in the same fashion as in the Minor Lemma, where we consider the balance of frequencies of different terms but we observe that only the estimate for first term depends on the precise structure, whereas for the rest of the terms, the specifics of the structure was not important, only the frequencies of differentiated terms. Therefore, we analyze the term

$$A = P_{<\mu} h_1 (P_{<\mu} l^i) \frac{\partial}{\partial x_i} (P_{<\mu} h_2) T_{a_{<\mu}} (P_{<\mu} l (P_{\mu} h_3)), \quad (3.4.9)$$

where

$$P_{<\mu} = \sum_{\nu \leq \frac{\mu}{4}} P_{\nu}, \quad a_{<\mu} = \sum_{\nu \leq \frac{\mu}{4}} a_{\nu}, \quad a_{\nu} = P_{\nu} a'.$$

We will use symbol calculus for the term in (3.4.9). Let $a_{\nu_1}(x, \xi)$ be the Kohn-Nirenberg symbol of R_{ν_1} and $L_{\nu_2}(x, \xi) = (P_{\nu_2} l^I) \xi_I$, the symbol of $P_{\nu_2} l^I \frac{\partial}{\partial x^I}$. The symbol of the composition is

$$c(x, \xi) = \int a_{\nu_1}(x, \xi + \eta) \hat{l}_{\nu_2}^I(\eta) \xi^I e^{2\pi i x \eta} d\eta.$$

Observe that for

$$c'(x, \xi) = \int l_{\nu_2}^I(x) \eta_I a_{\nu_1}(\eta, \xi) e^{2\pi i x \eta} d\eta,$$

we have

$$a L_{\nu_2} + c' = \int l^I(x)_{\nu_2} (\xi_I + \eta_I) a_{\nu_1}(\eta, \xi) e^{2\pi i x \eta} d\eta$$

is the symbol of the operator $P_{\nu_2} l^I \frac{\partial}{\partial x^I} \circ R_{\nu_1}$. Denote $c''_{\nu_1 \nu_2} = c - a_{\nu_1} L_{\nu_2}$. Denote

$$\begin{aligned} B' &= P_{<\mu} h_1 (P_{<\mu} l^i) \frac{\partial}{\partial x_i} (P_{<\mu} h_2) ((I - P_{<\mu}) l (R_{\mu} P_{\mu} h_3)), \\ B'' &= P_{<\mu} h_1 ((I - P_{<\mu}) l^i) \frac{\partial}{\partial x_i} (P_{<\mu} h_2) R_{\mu} (P_{<\mu} l (P_{\mu} h_3)). \end{aligned}$$

Then we have for A from (3.4.9)

$$A = P_{<\mu} h_1 l (P_{<\mu} h_2) l (R_{\mu} P_{\mu} h_3) - B' - B'' - C'' + C'', \quad (3.4.10)$$

where

$$C' = P_{<\mu} h_1 (P_{<\mu} l^i) \frac{\partial}{\partial x_i} (P_{<\mu} h_2) \sum_{\nu_i \leq \frac{\mu}{4}} T_{c'_{\nu_1 \nu_2}} (P_{\mu} h_3),$$

$$C'' = P_{<\mu} h_1 (P_{<\mu} l^i) \frac{\partial}{\partial x_i} (P_{<\mu} h_2) \sum_{\nu_i \leq \frac{\mu}{4}} T_{c''_{\nu_1 \nu_2}} (P_{\mu} h_3).$$

We observe that the first term in (3.4.10) is covered by the Minor lemma, Lemma 3.4.11. The terms B' and B'' are of type I or III from the proof of the Minor lemma. The precise type depends on whether the highest frequency terms appear once or twice. This means that B' and B'' , satisfy the correct estimate. So we are left with verifying that C', C'' satisfy the right estimate. To estimate the operators, we observe that after an application of Fourier transform, we have

$$\widehat{T_a h} = \int \widehat{a}(\eta - \xi, \xi) \widehat{h}(\xi) d\xi.$$

Therefore, we will estimate

$$\|T_a\|_{L^2 \rightarrow L^2} \leq \|\widehat{a}\|_{L^1_{\eta} L^{\infty}_{\xi}}.$$

For C' , we have that the symbol c' is merely the l derivative of the symbol a and it is still an order 0 pseudo-differential operator, thus

$$\begin{aligned} \|T_{c'_{\nu_1 \nu_2}}\|_{L^2 \rightarrow L^2} &\leq \|P_{\nu_2} l\|_{L^{\infty}_{t,x'}} \|\nu_1 \widehat{a}_{\nu_1}\|_{L^1_{\nu} L^{\infty}_{\xi}} \leq \nu_2^{\frac{n}{2}} \|P_{\nu_2} l\|_{L^2_x} \nu_1^{\frac{n}{2}+1} \|P_{\nu_1} a\|_{L^2_x} \\ &\leq \nu_2^{-\frac{1}{2}} b_{\nu_2} \nu_1^{\frac{1}{2}} b_{\nu_1}. \end{aligned}$$

Thus after summing the low frequency undifferentiated terms we have

$$\begin{aligned} \|C'\|_{L^1_t L^2_{x'}} &\lesssim \|P_{<\mu} h_1 (P_{<\mu} l^i) \frac{\partial}{\partial x_i} (P_{<\mu} h_2)\|_{L^2_t L^{\infty}_{x'}} \left\| \sum_{\nu_i \leq \frac{\mu}{4}} c'_{\nu_1 \nu_2} (P_{\mu} h_3) \right\|_{L^2_{t,x'}} \\ &\lesssim \sum_{\nu_i \leq \frac{\mu}{4}} \nu_1^{\frac{1}{2}} b_{\nu_1} \nu_2^{\frac{1}{2}} b_{\nu_2} \mu^{-\frac{n+1}{2}} b_{\mu} \\ &\lesssim \mu^{-\frac{n-1}{2}} b_{\mu} \sum_{\nu_i \leq \frac{\mu}{4}} \left(\frac{\nu_1}{\mu}\right)^{\frac{1}{2}} b_{\nu_1} \left(\frac{\nu_2}{\mu}\right)^{\frac{1}{2}} b_{\nu_2} \\ &\lesssim \mu^{-\frac{n-1}{2}} b_{\mu}^2. \end{aligned}$$

For the term C'' , we estimate the symbol $c''_{\nu_1 \nu_2}$ as follows

$$\begin{aligned} c''_{\nu_1 \nu_2}(x, \xi) &= c(x, \xi) - a_{\nu_1}(x, \xi) L_{\nu_2}(x, \xi) \\ &= \int [a_{\nu_1}(x, \xi + \rho) - a_{\nu_1}(x, \xi)] \widehat{l}_{\nu_2}^I(\rho) \xi^I e^{2\pi i x \rho} d\rho. \end{aligned}$$

Observe that due to frequency localization in l_{ν_2} , the integrand $|\rho| \sim \nu_2$ and we can estimate

$$|a_{\nu_1}(x, \xi + \rho) - a_{\nu_1}(x, \xi)| \leq \sup_{\rho} |\rho| |\partial_{\xi} a_{\nu_1}(x, \xi + \rho)|.$$

Since a is of order 0, we have $|\partial_{\xi} a_{\nu_1}(x, \xi + \rho)| \leq \frac{1}{\mu} \nu_1^{-\frac{1}{2}} b_{\nu_1}$, from which we estimate

$$\begin{aligned} \|c''_{\nu_1 \nu_2}\|_{L^1_{\eta} L^{\infty}_{\xi}} &\leq \nu_2 \|\hat{l}_{\nu_2}\|_{L^1_{\rho}} \|\partial_{\xi} a_{\nu_1}\|_{L^1_{\eta} L^{\infty}_{\xi}} \mu \\ &\leq \nu_2^{\frac{n+2}{2}} \|l_{\nu_2}\|_{L^2_x} \nu_1^{\frac{n}{2}} \|a_{\nu_2}(x, \xi)\|_{L^2_x L^{\infty}_{\xi}} \leq \nu_1^{-\frac{1}{2}} b_{\nu_1} \nu_2^{\frac{1}{2}} b_{\nu_2}, \end{aligned}$$

which is the same estimate as for c' with indices switched. The rest of the estimate proceeds similarly to the estimate of the C' term. This concludes the proof of the lemma. \square

3.5 The wave-packet parametrix construction

Construction of a wave packet

Sort of localized

We will fix a large number N and $c > 0$. Let $T \subseteq \mathbb{R}^m$ be a rectangle centered at the origin and \check{T} the dual rectangle with the dimensions inverse to dimensions of T . We will say that a function ψ is "sort of" localized in (T, \check{T}) if

$$\sum_{a \in L_{\check{T}}} |a|^N \|\psi|_{\check{T}+a}\|_{L^2} + \sum_{\hat{a} \in L_T} |\hat{a}|^N \|\hat{\psi}|_{T+\hat{a}}\|_{L^2} \leq 1,$$

where $L_T, L_{\check{T}}$ are the dual lattices for (T, \check{T}) and

$$\hat{\psi}(\xi) > c, \quad \forall \xi \in T,$$

Definition of a wave packet

Let $\theta \in \mathbb{S}^{n-1}$ be a direction. Let $x_{\theta} = x \cdot \theta$ and $x^{\perp} = x - x_{\theta} \theta$. Identify x^{\perp} coordinates with \mathbb{R}^{n-1} and consider the rectangle $T_{n-1} = [-\lambda^{\frac{1}{2}}, \lambda^{\frac{1}{2}}]^{n-1}$ in x^{\perp} variables. Fix a function a which is "sort of" localized in $(T_{n-1}, \check{T}_{n-1})$. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a function, which is sort of localized in the segment $[-1, 1]$, which is strictly localized in the segment $[-4, 4]$. Define an unnormalized mollifier on \mathbb{R}^n via

$$T_{\lambda} f = \check{\psi}_{\lambda} * f, \quad \check{\psi}_{\lambda}(y) = \check{\psi}(\lambda|y|), \quad y \in \mathbb{R}^n.$$

For a point $x \in \mathbb{R}^n$ and a direction $\theta \in \mathbb{S}^{n-1}$, let $r = x \cdot \theta$ and define

$$\psi_{\theta, r} = \lambda^{\frac{n-3}{4}} T_{\lambda}(zZ),$$

where

$$Z = \delta(F_\theta - r),$$

and

$$z(y) = z_0(\lambda^{\frac{1}{2}}(y' - \gamma(t))),$$

where γ be the null geodesic which passes through x in the null direction $sdt + \theta \cdot dx$, where s is an appropriate constant and z_0 is sort of localized in the rectangle $[-1, 1]^{n-1}$. The measure Z is a surface measure on the characteristic surface $\Sigma_{\theta, r}$ and we use the original optical function F_θ to make a particular choice of the area measure.

Proposition 3.5.1. *Let $\psi = \psi_{\lambda, \theta}$ be a normalized wave packet. We have*

$$\square_{g_\lambda} \psi = L(dg, dS_\lambda u) + \lambda^{\frac{n-3}{4}} S_\lambda T_\lambda \sum_{0,1,2} \zeta_m \delta^{(m)}(x_\theta - \phi_{\theta, r}),$$

where ζ_m satisfy

$$\|P_\mu \zeta_m\|_{L_t^\infty L_{x'}^2} \leq \epsilon_1^2 b_\mu^2 \lambda^{1-m} \mu^{-\frac{n-1}{2}}.$$

Proof. We have

$$\square_{g_\lambda} P_\lambda \psi_{\theta, r} = \lambda^{\frac{n-3}{4}} ((\square_{g_\lambda}, P_\lambda T_\lambda] + P_\lambda T_\lambda \square_{g_\lambda}) z \delta(F_\theta - r).$$

For the second term we have

$$\square_{g_\lambda} (z \delta(F_\theta - r)) = z \square_{g_\lambda} \delta(F_\theta - r) + (\square_{g_\lambda} z) \delta(F_\theta - r) + Q_0(z, \delta(F_\theta - r)).$$

Since $F = F_\theta$ be the defining function of the foliation, i.e. the Hamiltonian flowout of x_θ . Then $\nabla F = V$ is the light-like geodesic. Let $\{l, \underline{l}, \mathbf{e}_i\}_{i=1..n-1}$ be the orthonormal frame of $\Sigma_{\theta, r}$. Denote $\kappa = \langle V, \underline{l} \rangle$ and $l = \frac{1}{\kappa} V$. We have

$$\square \delta(F) = \operatorname{div} \nabla \delta(F) = \operatorname{div}(V \delta'(F)).$$

There are two terms in the divergence

$$\operatorname{div}(V \delta'(F)) = \langle V, \nabla \delta'(F) \rangle + \operatorname{div} \delta'(F).$$

The first term is

$$\langle V, \nabla \delta(F) \rangle = \langle V, V \rangle \delta''(F) = 0$$

because $\langle V, V \rangle$ vanishes to the second order. For the second term, we will use the frame

$$\operatorname{div}(V) = -\langle \nabla_l V, \underline{l} \rangle - \langle \nabla_{\underline{l}} V, l \rangle + \sum_i \langle \nabla_{\mathbf{e}_i} V, \mathbf{e}_i \rangle.$$

The first term vanishes since $\nabla_l V = \frac{1}{\kappa} \nabla_V V = 0$ since V is a geodesic. The second term also vanishes

$$\langle \nabla_{\underline{l}} V, l \rangle = \frac{1}{\kappa} \langle \nabla_{\underline{l}} V, V \rangle = \frac{1}{\kappa} \frac{1}{2} l(\langle V, V \rangle) = 0.$$

Thus we are left with

$$\langle \nabla_{\mathbf{e}_i} V, \mathbf{e}_i \rangle = \kappa \langle \nabla_{\mathbf{e}_i} l, \mathbf{e}_i \rangle = \kappa \chi_{ii}.$$

The sum over i gives the trace and the final result is

$$\square \delta(F) = \kappa \operatorname{tr} \chi \delta'(F).$$

□

Remark. As

$$|dr_\theta - (\theta \cdot dx - dt)| \leq \epsilon_1,$$

then

$$|\phi_{\theta,r} - \phi_{\theta,r'} - (r - r')| \lesssim \epsilon_1 |r - r'|,$$

therefore we can consider the support of the packet as being contained in a foliation of characteristic surfaces,

$$\operatorname{supp} \psi_{\lambda,\theta} \subseteq \bigcup_{r' \in C[-\frac{1}{\lambda}, \frac{1}{\lambda}]} \Sigma_{\theta,r'}.$$

Another conclusion that we can draw from this is that the change of coordinates $(r, t, x') \mapsto (t, x', \phi_{\theta,r}(t, x'))$ is Lipschitz continuous and thus the norms such as $L_{t,x}^p$ are equivalent to $L_r^p L^p(\Sigma_{\theta,r})$ norms.

The following lemma will be important in obtaining the null-form estimates in Section 3.6.

Lemma 3.5.2. *A wave packet $\psi = \psi_{\nu,\theta}$ satisfies*

1.

$$\|l(\psi)\|_{L_{t,x}^\infty} \leq \nu^{\frac{n+1}{4}} \|\nabla \psi\|_{L_{t,x}^2},$$

2.

$$\|l(\psi)\|_{L_{t,x}^\infty}, \|\mathbf{e}_a(\psi)\|_{L_{t,x}^\infty} \leq \nu^{\frac{n-1}{4}} \|\nabla \psi\|_{L_{t,x}^2}.$$

Proof. Since $\psi_{\nu,\theta}$ is an l^1 sum of functions which are frequency localized in a rectangle on volume $\nu^{\frac{n+1}{2}}$, applying Sobolev inequality to each rectangle and then summing gives item 1, after recalling that ψ is normalized to have norm 1 in H^1 .

For item 2, we would like to exploit the fact that in directions tangent to $\Sigma_{\theta,r}$ the wave packet changes on the scale of $\lambda^{\frac{1}{2}}$ as opposed to λ in the transversal directions. We observe the following as

$$\|dF_\theta - \theta \cdot dx - dt\|_{L_r^\infty H^{\frac{n+1}{2}}(\Sigma_{\theta,r})} \leq \epsilon_1,$$

we fix (t, x') and integrate this expression in θ direction to get

$$\|\phi_{\theta,r} - \phi_{\theta,r'} - (r - r')\|_{H^{\frac{n+1}{2}}(\Sigma_{\theta,r})} \leq \epsilon_1 (r - r'),$$

also since the bounds are uniform in r , we get

$$\|\phi_{\theta,r} - \phi_{\theta,r'}\|_{H^{\frac{n+3}{2}}(\Sigma_{\theta,r})} \leq \epsilon_1.$$

A simple interpolation implies

$$\|\nabla_{t,x'}\phi_{\theta,r} - \nabla_{t,x'}\phi_{\theta,r'}\|_{L^\infty(\Sigma_{\theta,r})} \leq (r - r')^{\frac{1}{2}}\epsilon.$$

This estimate, together with $H^{\frac{n+1}{2}}(\mathbb{R}^n) \subseteq C^{\frac{1}{2}}(\mathbb{R}^n)$ implies full $\frac{1}{2}$ -Hölder continuity for the tangential derivatives of ϕ . The fact that $d\phi - \theta \cdot dx$ is null can be used to recover the transversal component of $d\phi$ by using the metric, which is also $\frac{1}{2}$ -Hölder to conclude that $d\phi$ is $\frac{1}{2}$ -Hölder continuous. Since l is (metricly) adjoint to a multiple of $d\phi$ and the frame \mathbf{e}_a was constructed via Gramm-Schmidt process, we conclude that the vector fields l, \mathbf{e}_a have $\frac{1}{2}$ -Hölder continuous coefficients. Next we use $P_{\mu,t',x'}^\Sigma$ - the Littlewood-Paley projections in the t', x' coordinates of the (r, t', x') coordinate frame. We extend the operation on vector fields by

$$P_{\nu,t',x'}^\Sigma \mathbf{e} = P_{\nu,t',x'}^\Sigma \mathbf{e}^0 \frac{\partial}{\partial t'} + \sum_{i=1}^{n-1} P_{\mu,t',x'}^\Sigma \mathbf{e}^i \frac{\partial}{\partial x'_i}.$$

We split $\mathbf{e} = \mathbf{e}_{\leq \nu} + \mathbf{e}_{> \nu}$ by applying $P_{\leq \nu,t',x'}^\Sigma, P_{> \nu,t',x'}^\Sigma$ respectively. For $\mathbf{e}_{> \nu}$ we have

$$\|\mathbf{e}_{> \nu}^\alpha\|_{L^\infty} \leq \frac{1}{\nu^{\frac{1}{2}}} \|\mathbf{e}_{> \nu}^\alpha\|_{L_r^\infty H^{\frac{n+1}{2}}(\mathbb{R}^n)}$$

which means that $\mathbf{e}_{> \nu} \psi$ satisfies the right bound by application of item 1. For $\mathbf{e}_{\leq \nu}$, we observe that it is Hölder- $\frac{1}{2}$ continuous in (r, t, x') coordinates and since the coordinates are Lipschitz, it's also Hölder- $\frac{1}{2}$ continuous in the regular coordinates. Therefore since T_ν is a mollifier on the scale ν^{-1} we get:

$$\|[\mathbf{e}_{\leq \nu} T_\nu - T_\nu \mathbf{e}_{\leq \nu}](zZ)\|_{L^\infty} \leq \nu^{-\frac{1}{2}} \|\nabla T_\nu(zZ)\|_{L^\infty} \leq \nu^{\frac{n-1}{2}}.$$

We have

$$\mathbf{e}_{\leq \nu}(zZ) = \nu^{\frac{1}{2}} z_1 Z + \operatorname{div}_\Sigma \mathbf{e}_{\leq \nu} zZ,$$

where $z_1(\nu^{\frac{1}{2}}x) = z'(\nu^{\frac{1}{2}}x)$, which satisfy $\|z, z_1\|_{L^\infty} \leq \nu^{\frac{n}{2}}$ due to $\nu^{\frac{1}{2}}$ -scaling. We estimate

$$\|\operatorname{div}_\Sigma \mathbf{e}_{\leq \nu}\|_{L^\infty} \lesssim \|\nabla \mathbf{e}_{\leq \nu}^\alpha\|_{L^\infty} \leq \nu^{\frac{1}{2}} \|\mathbf{e}^\alpha\|_{L_r^\infty H_{t,x'}^{\frac{n-1}{2}}(\mathbb{R}^n)} \lesssim \nu^{\frac{1}{2}}.$$

Therefore, with the H^1 -normalization, we have

$$\|T_\nu \mathbf{e}_{\leq \nu}(zZ)\|_{L^\infty} \leq \nu^{\frac{n-1}{2}},$$

which completes the proof. \square

Decomposition of the initial data into wave packets

Proposition 3.5.3. *There exists $C > 0$ such that for every $(v_0, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ there exists a function of the form*

$$v = \sum a_{\lambda, \theta, k} \psi_{\lambda, \theta, k}$$

such that

$$v|_{t=0} = v_0, \quad \frac{\partial}{\partial t} v|_{t=0} = v_1$$

and

$$\sum_{\lambda, \theta, k} a_{\lambda, \theta, k}^2 \leq C(\|du_0\|_{L^2}^2 + \|u_1\|^2).$$

Proof. We will fix ψ a sort of localized function in the box $[-1, 1]^{n-1}$ and $K > 0$, which we specify momentarily. As $\psi_{\lambda, \theta, k}$ are frequency localized, we can assume without loss of generality that $v_0 = P_\lambda v_0, v_1 = P_\lambda v_1$ for some dyadic λ . Consider Ω a collection of maximally $\lambda^{-\frac{1}{2}}$ separated unit vectors. Use a partition of unity of \mathbb{S}^{n-1} subordinate to $\lambda^{-\frac{1}{2}}$ sectors around the vectors of Ω to write

$$\hat{v}_0 = \sum_{\omega \in \Omega} \hat{v}_0^\omega, \quad \hat{v}_1 = \sum_{\omega \in \Omega} \hat{v}_1^\omega.$$

For simplicity replace \hat{v}_1^ω with $\hat{v}_1^\omega \frac{1}{\omega \cdot \xi}$ and \hat{v}_0 with the sum of the new $\hat{v}_{i, \omega}$. We consider the support of \hat{v}_i^ω as contained in the box $B_\omega = [\lambda, 2\lambda]_\omega \times C$, where $C_\omega = [-K\lambda^{\frac{1}{2}}, K\lambda^{\frac{1}{2}}]_{\perp \omega}$. Next consider a rotation $R_\omega \in SO(n-2)$ such that $f_\omega = \hat{\psi}(\frac{1}{\lambda} R_\omega \xi)$ is supported in the box $[-\lambda^{\frac{1}{2}}, \lambda^{\frac{1}{2}}]_{\perp \omega}$. Then we take the function $\frac{\hat{v}_i^\omega}{f_\omega}$ as a periodic function in a box B_ω and with the corresponding Fourier series

$$b_{r, \alpha, i} = \int_{B_\omega} e^{-i(r\omega \cdot \xi + \theta \cdot \xi)} \frac{\hat{v}_i^\omega}{f_\omega} d\xi, \quad r \in \frac{1}{\lambda} \mathbb{Z}, \alpha \in L_C, i = 0, 1$$

where L_C is the dual lattice to the box C_ω . Define

$$\tilde{V}_{i, \omega}(x) = \sum_{r, \alpha} b_{r, \alpha, i} \check{\chi}_{[\lambda, 2\lambda]}(x \cdot \omega - r) \check{f}_\omega(x' - \alpha).$$

Our construction is geared toward achieving

$$\hat{V}_{i, \omega}|_{B_\omega} = \hat{v}_i^\omega, \quad i = 0, 1.$$

As the translates of $\check{\psi}$ are almost orthogonal, we have $\|\tilde{V}_{i, \omega}\|_{L^2} \leq c\|v_{i, \omega}\|_{\dot{H}^1}$. Therefore we define K such that $\|\psi - \psi|_{[-K, K]^{n-1}}\|_{\dot{H}^1} \leq \frac{1}{4c}\|\psi\|_{\dot{H}^1}$. This implies

$$\|v_i - \sum_{\omega \in \Omega} \tilde{V}_{i, \omega}\|_{\dot{H}^1} \leq \frac{1}{4}\|v_i\|_{\dot{H}^1},$$

This is enough, as we can apply the same construction on the difference $v_i - \sum_{\omega \in \Omega} \tilde{V}_{i,\omega}$ and subsequent errors to get a convergent sum such that

$$v_{i,\omega} = \sum_{r,\alpha} a_{r,\alpha,i} \tilde{\chi}_{[\lambda,2\lambda]}(x \cdot \omega - r) \check{f}_\omega(x' - \alpha).$$

Then assign the wave packet $\psi_{\omega,r,\alpha,\pm}$ be the wave packet that corresponds to the surface $\Sigma_{\omega,r,\pm}$ with initial data $\check{f}_\omega(x' - \alpha)$ on it. Then we can define

$$v = \sum_{\omega,r,\alpha} a_{\omega,r,\alpha,+} \psi_{\omega,r,\alpha,+} + a_{\omega,r,\alpha,-} \psi_{\omega,r,\alpha,-},$$

where

$$a_{\omega,r,\alpha,\pm} = a_{\omega,r,\alpha,0} + \frac{1}{s_\pm(r,\alpha)} a_{\omega,r,\alpha,1}, \quad \|s_\pm dt - \omega \cdot dx\|_g = 0.$$

□

Almost orthogonality

Proposition 3.5.4. *Let $v = \sum_{\theta,j} a_{\theta,j} \psi_{\theta,j}$ then*

$$\|dS_\lambda v\|_{L_t^\infty L_x^2} \lesssim \left(\sum a_{\theta,j}^2 \right)^{\frac{1}{2}}, \quad (3.5.1)$$

$$\|\square_{g_\lambda} dS_\lambda v\|_{L_t^\infty L_x^2} \lesssim \epsilon_1^2 \left(\sum a_{\theta,j}^2 \right)^{\frac{1}{2}}. \quad (3.5.2)$$

Proof. We will seek to prove fixed time energy estimates for expressions of the form

$$v = \sum_{\theta,j} a_{\theta,j} S_\lambda T_\lambda \tilde{z}(\lambda^{\frac{1}{2}} t, \lambda^{\frac{1}{2}} x_\theta^\perp) \zeta_{\theta,j} \delta_{x_\theta - \phi_{\theta,r}},$$

where $\zeta_{\theta,j}$ satisfy

$$\|P_\nu \zeta_{\theta,j}\|_{L_t^\infty L_x^2} \leq \epsilon_1^2 b_\nu^2 \nu^{-\frac{n-1}{2}}.$$

We will first combine packets with the same angle

$$v_\theta = \sum_j a_{\theta,j} S_\lambda T_\lambda \tilde{z}(\lambda^{\frac{1}{2}} t, \lambda^{\frac{1}{2}} x_\theta^\perp) \zeta_{\theta,j} \delta_{x_\theta - \phi_{\theta,r}}.$$

Next we apply the Littlewood Paley projection P_ν in the (t, x_θ^\perp) variables on every ζ :

$$v_{\theta,\nu} = \sum_j a_{\theta,j} S_\lambda T_\lambda \tilde{z}(\lambda^{\frac{1}{2}} t, \lambda^{\frac{1}{2}} x_\theta^\perp) P_\nu \zeta_{\theta,j} \delta_{x_\theta - \phi_{\theta,r}}.$$

We estimate

$$\left\| \sum_{\theta} v_{\theta, \nu} \right\|_{L_x^2}^2 = \sum_{\theta, \theta'} \langle \hat{v}_{\theta', \nu}, \hat{v}_{\theta, \nu} \rangle_{L_{\xi}^2(\mathbb{R}^n)}.$$

Consider two cases. First case: $\nu \geq \lambda^{\frac{1}{2}}$. Then the angular part Fourier support of $T_{\lambda} \tilde{z}_{\theta}$ in an angle $\lambda^{-\frac{1}{2}}$ around the direction θ . As the angles are $\lambda^{-\frac{1}{2}}$ separated, different wave-packets are almost orthogonal. Multiplication with an expression of the form $P_{\nu} \zeta_{\theta}$ is a convolution of \tilde{z} with a function supported in a rectangle of size $[-\nu, \nu]^{n-1}$ in the x_{θ}^{\perp} variables. Therefore, the angular support of the convolution will now be at an angle $\approx \frac{\nu}{\lambda}$ around θ , therefore v_{θ} will intersect $\approx \left(\frac{\nu}{\lambda^{\frac{1}{2}}}\right)^{n-1}$ different v'_{θ} . Next we estimate the L^2 norm of the product

$$\|\tilde{z} P_{\nu} \zeta\|_{L_{x'}^2} \leq \|\tilde{z}\|_{L^{\infty}} \|P_{\nu} \zeta\|_{L_{x'}^2} \leq \lambda^{\frac{n-1}{4}} \epsilon_1^2 b_{\nu}^2 \nu^{-\frac{n-1}{2}}.$$

Therefore we get

$$\left\| \sum_{\theta} v_{\theta, \nu} \right\|_{L_x^2}^2 \lesssim \nu^{n-1} \epsilon_1^4 b_{\nu}^4 \nu^{-(n-1)} \sum_{\theta} a_{\theta, j}^2.$$

The second case is $\nu \leq \lambda^{\frac{1}{2}}$ and we argue that the estimate remains the same since the angular support now does not grow significantly and the wave packets maintain their almost orthogonality. Then estimate $\|P_{\nu} \zeta\|_{L^{\infty}} \leq \nu^{\frac{n-1}{2}} \|P_{\nu} \zeta\|_{L^2}$ follows by Sobolev embedding which gives the same factor in the end.

To complete the estimate we simply sum in ν

$$\left\| \sum_{\theta} v_{\theta} \right\|_{L_x^2} \leq \sum_{\nu} \left\| \sum_{\theta} v_{\theta, \nu} \right\|_{L_x^2} \leq \left(\sum_{\theta} a_{\theta, j}^2 \right)^{\frac{1}{2}} \epsilon_1^2 \sum_{\nu} b_{\nu}^2.$$

□

3.6 Bilinear estimates

In this section, we intend to establish the following statement

Proposition 3.6.1. *Let v, w solve $\square_{g(du)} v = 0$, $\square_{g(du)} w = 0$ with initial data (v_0, v_1) and (w_0, w_1) , respectively. then*

$$\|Q_0(w, v)\|_{H_{t,x}^{\frac{n-1}{2}}} \lesssim \|(dw_0, w_1)\|_{H_x^{\frac{n-1}{2}}} \|(dv_0, v_1)\|_{H_x^{\frac{n-1}{2}}}.$$

We will write w in terms of wave packets by applying the Duhamel formula and presenting w as a time integral of wave packets which start at different times.

$$P_{\nu} w(t, x) = \int_0^t \sum_{\theta} a_{s, \nu, \theta} \psi_{\nu, \theta} ds.$$

Therefore for $\nu \leq \mu$ we can consider:

$$Q_0(P_\nu w, P_\mu v) = \sum_{\theta} a_{\nu, \theta} Q_0(\psi_{\nu, \theta}, P_\mu v).$$

For fixed ν, θ decompose Q_0 in terms of the $l_\theta, \underline{l}_\theta, \mathbf{e}_{a, \theta}$ of the packet, we get

$$\begin{aligned} Q_0(P_\nu w, P_\mu v) &= \sum_{\theta} a_{\nu, \theta} Q_0(\psi_{\nu, \theta}, P_\mu v) = \\ &= \sum_{\theta} a_{\nu, \theta} \left[l_\theta \psi_{\nu, \theta} \underline{l}_\theta(P_\mu v) + \sum_{b=1}^{n-1} \mathbf{e}_{b, \theta}(\psi_{\nu, \theta}) \mathbf{e}_{b, \theta}(P_\mu v) \right. \\ &\quad \left. + \underline{l}_\theta(\psi_{\nu, \theta}) l_\theta(P_\mu v) \right]. \end{aligned} \tag{3.6.1}$$

We analyze these three terms separately.

Claim 3.6.2. *There exist $\delta > 0, C$ such that*

$$\left\| \sum_{\theta} a_{\nu, \theta} l_\theta(\psi_{\nu, \theta}) \underline{l}_\theta(P_\mu v) \right\|_{L^2_{t,x}} \leq \nu^{\frac{n-1}{2} - \delta} \left(\sum_{\theta} a_{\nu, \theta}^2 \right)^{1/2} \|P_\mu v\|_{H^1_{t,x}}.$$

Proof. We will drop ν from the notation. We square the left hand side and write an integral of double sum

$$\int \sum_{\theta, \theta'} a_{\theta'} a_{\theta} l_{\theta'}(\psi_{\theta'}) l_{\theta}(\psi_{\theta}) \underline{l}_\theta(P_\mu v) \underline{l}_{\theta'}(P_\mu v) dt dx.$$

We have

$$\begin{aligned} \int l_{\theta'}(\psi_{\theta'}) l_{\theta}(\psi_{\theta}) \underline{l}_\theta(P_\mu v) \underline{l}_{\theta'}(P_\mu v) dt dx \leq \\ \|l_{\theta'}(\psi_{\theta'})\|_{L^\infty} \|l_{\theta}(\psi_{\theta})\|_{L^\infty} \int_I |\underline{l}_\theta(P_\mu v) \underline{l}_{\theta'}(P_\mu v)| dt dx, \end{aligned}$$

where $I = \text{supp}(\psi_{\nu, \theta}) \cap \text{supp}(\psi_{\nu, \theta'})$ is the intersection of the supports of each packet. We use item 2 in Lemma 3.5.2 to estimate the L^∞ norm of the packets. For the last term, we use energy estimates (combined with the fact that the coefficients of \underline{l} are bounded)

$$\begin{aligned} \int_I |\underline{l}_\theta(P_\mu v) \underline{l}_{\theta'}(P_\mu v)| dt dx &\lesssim \|\nabla(P_\mu v)\|_{L^2(I)}^2 \leq T_{\theta, \theta'} \|\nabla(P_\mu v)\|_{L^\infty_{t,x}}^2 \\ &\leq T_{\theta, \theta'} \|P_\mu v\|_{H^1_{t,x}}^2, \end{aligned}$$

where $T_{\theta, \theta'}$ is the maximal time dimension of the intersection of the support of the packets, which we call "time of interaction". Let α be the angle between θ and θ' and suppose α is small but not insignificant. Then we can estimate the time of the interaction projecting to (t, x_θ) plane, where we have a Lipschitz image of a rectangle at angle $\frac{\pi}{4}$ of length 1 and

width ν^{-1} and another Lipschitz image rectangle of length 1 and width $\lesssim \nu^{-\frac{1}{2}}$ at angle $\arccos(\frac{1}{2} + (1 + \epsilon)\frac{1}{2} \cos \alpha) \approx \alpha^2$ to the second rectangle. Therefore, we estimate the time of interaction by

$$T_{\theta, \theta'} \lesssim \min(1, \frac{\nu^{-\frac{1}{2}}}{\alpha^2}),$$

For angles smaller than α , we will estimate the time of interaction by 1 and for angles larger than α the time of interaction is smaller, therefore the estimate above is correct with α as a yet determined threshold. We can also sum in θ' as we observe that there are at most $\alpha^{\frac{n-1}{2}} \nu^{\frac{n-1}{2}}$ packets with angle less than α with respect to θ and less than $\nu^{\frac{n-1}{2}}$ packets overall. Therefore

$$\sum_{\theta'} T_{\theta, \theta'} \leq \alpha^{\frac{n-1}{2}} \nu^{\frac{n-1}{2}} + \frac{\nu^{-\frac{1}{2}}}{\alpha^2} \nu^{\frac{n-1}{2}} \leq \nu^{\frac{n-1}{2}} \nu^{-\frac{n-1}{2(n+3)}}.$$

after choosing $\alpha = \nu^{-\frac{1}{n+3}}$. Apply Schur's lemma to conclude

$$\sum_{\theta, \theta'} a_{\theta} a_{\theta'} T_{\theta, \theta'} \leq \nu^{\frac{n-1}{2}} \nu^{-\frac{n-1}{2(n+3)}} \sum_{\theta} a_{\theta}^2.$$

Combining the estimate, we conclude

$$\begin{aligned} \int \sum_{\theta, \theta'} a_{\theta'} a_{\theta} l_{\theta'}(\psi_{\theta'}) l_{\theta}(\psi_{\theta}) L_{\theta}(P_{\mu} v) L_{\theta'}(P_{\mu} v) dt dx \\ \lesssim \nu^{n-1-2\delta} \left(\sum_{\theta} a_{\theta}^2 \right) \|P_{\mu} v\|_{L_t^{\infty} H_x^1}, \end{aligned}$$

where $\delta = -\frac{n-1}{4(n+3)}$, which is the desired estimate. □

Claim 3.6.3. *There exist $\delta > 0, C$ such that*

$$\left\| \sum_{\theta} \sum_{b=1}^{n-1} a_{\nu, \theta} \mathbf{e}_{b, \theta}(\psi_{\nu, \theta}) \mathbf{e}_{b, \theta}(P_{\mu} v) \right\|_{L_{t,x}^2} \leq C \nu^{\frac{n-1}{2} - \delta} \left(\sum_{\theta} a_{\nu, \theta}^2 \right)^{1/2} \|P_{\mu} v\|_{L_t^{\infty} H_x^1}.$$

Proof. We can fix b by adding to multiplicative constant. Next, we estimate

$$\|\mathbf{e}_{b, \theta}(\psi_{\nu, \theta}) \mathbf{e}_{b, \theta}(P_{\mu} v)\|_{L_{t,x}^2} \lesssim \|\mathbf{e}_{b, \theta}(\psi_{\nu, \theta})\|_{L^{\infty}} \|\mathbf{e}_{b, \theta}(P_{\mu} v)\|_{L_{t,x}^2}.$$

For the first term, we use item 2 in Lemma 3.5.2 to get

$$\|\mathbf{e}_{b, \theta}(\psi_{\nu, \theta})\|_{L^{\infty}} \leq \nu^{\frac{n-1}{4}}.$$

For the second term, we use the characteristic energy estimate on the support of the wave packet.

$$\|\mathbf{e}_{b, \theta}(P_{\mu} v)\|_{L_{t,x}^2(\text{supp} \psi)} \lesssim \|\mathbf{e}_{b, \theta}(P_{\mu} v)\|_{L_r^2 L^2(\Sigma_{\theta, r} \cap \text{supp} \psi)}.$$

The support of ψ consists of l_1 combination of boxes of thickness ν^{-1} in the r variable. Therefore, we have

$$\begin{aligned} \|\mathbf{e}_{b,\theta}(P_\mu v)\|_{L_r^2 L^2(\Sigma_{\theta,r} \cap \text{supp}\psi)} &\lesssim \nu^{-\frac{1}{2}} \|\mathbf{e}_{b,\theta}(P_\mu v)\|_{L_r^\infty L^2(\Sigma_{\theta,r})} \\ &\lesssim \nu^{-\frac{1}{2}} \|P_\mu v\|_{L_t^\infty H_x^1}. \end{aligned}$$

Lastly, we recall that there are at most $\nu^{\frac{n-1}{2}}$ angles in the orthogonal decomposition, which we will use to convert the sum in θ into the sum of squares to get

$$\begin{aligned} \left\| \sum_{\theta} \sum_{b=1}^{n-1} a_{\nu,\theta} \mathbf{e}_{b,\theta}(\psi_{\nu,\theta}) \mathbf{e}_{b,\theta}(P_\mu v) \right\|_{L_{t,x}^2} &\lesssim \nu^{\frac{n-3}{4}} \sum_{\theta} \sum_{b=1}^{n-1} a_{\nu,\theta} \|P_\mu v\|_{L_t^\infty H_x^1} \\ &\lesssim C \nu^{\frac{n-2}{2}} \left(\sum_{\theta} a_{\nu,\theta}^2 \right)^{1/2} \|P_\mu v\|_{L_t^\infty H_x^1}. \end{aligned}$$

□

The presence of exponent δ allows to sum up the lower frequencies to produce the bilinear estimate. This leaves us with the task of estimating the last term in (3.6.1).

Claim 3.6.4. *Let $\nu_2 \leq \nu_1 \leq \mu$. Let $\psi_i = \psi_{\theta_i, \mu_i}$, $i = 1, 2$ be wave packets and $\square v = 0$ then*

$$\int l_{\theta_1}(\psi_1) l_{\theta_2}(\psi_2) l_{\theta_1}(P_\mu v) l_{\theta_2}(P_\mu v) \lesssim \left(\frac{\nu_2}{\nu_1} \right)^{\frac{1}{2}} \nu_1^{\frac{n-1}{4}} \nu_2^{\frac{n-1}{4}} \|P_\mu v\|_{L_t^\infty H_x^1}^2. \quad (3.6.2)$$

Proof. We decompose $l_{\theta_2} = c_1 l_{\theta_1} + c_2 l_{\theta_1} + c_3 \mathbf{e}_{\theta_1, a}$. After substituting this decomposition into (3.6.2) we see that the last two terms have the right form, since we can use the characteristic energy estimate like in the previous Claim for the support of the packet ψ_1 to obtain

$$\|l_{\theta_1}(P_\mu v) \mathbf{e}_{\theta_1}(P_\mu v)\|_{L^2} \leq \frac{1}{\nu_1} \|P_\mu v\|_{L_t^\infty H_x^1}^2.$$

Thus

$$\begin{aligned} &\int l_{\theta_1}(\psi_1) l_{\theta_2}(\psi_2) l_{\theta_1}(P_\mu v) \mathbf{e}_{\theta_1}(P_\mu v) \\ &\leq \|\nabla \psi_1\|_{L^\infty} \|\nabla \psi_2\|_{L^\infty} \|l_{\theta_1}(P_\mu v) \mathbf{e}_{\theta_1}(P_\mu v)\|_{L^2} \\ &\leq \nu_1^{\frac{n+1}{4}} \nu_2^{\frac{n+1}{4}} \nu_1^{-1} \|P_\mu v\|_{L_t^\infty H_x^1}^2 \\ &\leq \frac{\nu_2^{\frac{1}{2}}}{\nu_1^{\frac{1}{2}}} \nu_1^{\frac{n-1}{4}} \nu_2^{\frac{n-1}{4}} \|P_\mu v\|_{L_t^\infty H_x^1}^2. \end{aligned}$$

This has an off-diagonal decay in ν_1, ν_2 , which is what we look for. Therefore we need to estimate

$$\int l_{\theta_1}(\psi_1) l_{\theta_2}(\psi_2) l_{\theta_1}(P_\mu v) l_{\theta_1}(P_\mu v).$$

Let us drop the indices from the equation, as we are now only interested in the ν_1, θ_1 null-frame. We will integrate by parts and employ the equation

$$\begin{aligned} \underline{l}(l(P_\mu v)P_\mu v) &= \underline{l}(P_\mu v)l(P_\mu v) - \underline{l}l(P_\mu v)P_\mu v \\ &= \underline{l}(P_\mu v)l(P_\mu v) - \mathbf{e}_a \mathbf{e}_a(P_\mu v)P_\mu v. \end{aligned}$$

Therefore, we have

$$\underline{l}(P_\mu v)l(P_\mu v) = \underline{l}(l(P_\mu v)P_\mu v) + \mathbf{e}_a(\mathbf{e}_a(P_\mu v)P_\mu v) + \mathbf{e}_a(P_\mu v)\mathbf{e}_a(P_\mu v).$$

The last expression allows us to use the characteristic energy estimate of the lower frequency and thus has the off-diagonal decay. Integrating by parts in the first two expressions gives as a term of the form

$$\int \nabla \psi_1 \nabla^2 \psi_2 l(P_\mu v)P_\mu v.$$

We will use the characteristic energy estimate again

$$\|l(P_\mu v)P_\mu v\|_{L^2} \leq \frac{1}{\mu\nu_1} \|P_\mu v\|_{L_t^\infty H_x^1}^2.$$

Therefore, we have

$$\int \nabla \psi_1 \nabla^2 \psi_2 l(P_\mu v)P_\mu v \leq \nu_2^{\frac{n+1}{4}} \nu_1^{\frac{n+5}{4}} \frac{1}{\mu\nu_1} \|P_\mu v\|_{L_t^\infty H_x^1}^2,$$

which has the right form since $\nu_1 \leq \mu$. □

Combined with the fact that there are at most $\nu^{\frac{n-1}{2}}$ angles, we can now write

$$\begin{aligned} &\sum_{\nu \leq \nu' \leq \nu, \theta, \theta'} a_{\nu', \theta'} a_{\nu, \theta} \int \underline{l}_{\theta'}(\psi_{\theta'}) \underline{l}_\theta(\psi_\theta) l_\theta(P_\mu v) l_{\theta'}(P_\mu v) dt dx \\ &\leq \sum_{\nu \leq \nu'} \left(\frac{\nu}{\nu'}\right)^{\frac{1}{2}} \nu^{\frac{n-1}{2}} \nu'^{\frac{n-1}{2}} \left(\sum_\theta a_{\nu, \theta}^2\right)^{1/2} \left(\sum_\theta a_{\nu', \theta'}^2\right)^{1/2} \|P_\mu v\|_{L_t^\infty H_x^1}^2. \end{aligned}$$

Thus using the off-diagonal decay, we can sum ν, ν' variables to avoid the logarithmic divergence.

Bibliography

- [1] Simon Brendle. “Hypersurfaces in Minkowski space with vanishing mean curvature”. In: *Comm. Pure Appl. Math.* 55.10 (2002), pp. 1249–1279. ISSN: 0010-3640. DOI: 10.1002/cpa.10044. URL: <http://dx.doi.org/10.1002/cpa.10044>.
- [2] Yvonne Choquet-Bruhat. “The Cauchy problem in extended supergravity, $N = 1$, $d = 11$ ”. In: *Comm. Math. Phys.* 97.4 (1985), pp. 541–552. ISSN: 0010-3616. URL: <http://projecteuclid.org/getRecord?id=euclid.cmp/1103942186>.
- [3] Demetrios Christodoulou. “Global solutions of nonlinear hyperbolic equations for small initial data”. In: *Comm. Pure Appl. Math.* 39.2 (1986), pp. 267–282. ISSN: 0010-3640. DOI: 10.1002/cpa.3160390205. URL: <http://dx.doi.org/10.1002/cpa.3160390205>.
- [4] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*. Vol. 41. Princeton Mathematical Series. Princeton, NJ: Princeton University Press, 1993, pp. x+514. ISBN: 0-691-08777-6.
- [5] G. W. Gibbons. “Born-Infeld particles and Dirichlet p -branes”. In: *Nuclear Phys. B* 514.3 (1998), pp. 603–639. ISSN: 0550-3213. DOI: 10.1016/S0550-3213(97)00795-5. URL: [http://dx.doi.org/10.1016/S0550-3213\(97\)00795-5](http://dx.doi.org/10.1016/S0550-3213(97)00795-5).
- [6] Robin Graham. “Dirichlet-to-Neumann map for Poincaré-Einstein metrics”. In: *Oberwolfach Reports* ().
- [7] Lars Hörmander. *Lectures on nonlinear hyperbolic differential equations*. Vol. 26. Mathématiques & Applications (Berlin) [Mathematics & Applications]. Berlin: Springer-Verlag, 1997, pp. viii+289. ISBN: 3-540-62921-1.
- [8] Fritz John. “Blow-up for quasilinear wave equations in three space dimensions”. In: *Comm. Pure Appl. Math.* 34.1 (1981), pp. 29–51. ISSN: 0010-3640. DOI: 10.1002/cpa.3160340103. URL: <http://dx.doi.org/10.1002/cpa.3160340103>.
- [9] Jürgen Jost. *Riemannian geometry and geometric analysis*. Third. Universitext. Berlin: Springer-Verlag, 2002, pp. xiv+532. ISBN: 3-540-42627-2.
- [10] Joshua Kantor. “Eleven Dimensional Supergravity on Edge Manifolds”. PhD thesis. University of Washington, 2008.

- [11] Tosio Kato and Gustavo Ponce. “Commutator estimates and the Euler and Navier-Stokes equations”. In: *Comm. Pure Appl. Math.* 41.7 (1988), pp. 891–907. ISSN: 0010-3640. DOI: 10.1002/cpa.3160410704. URL: <http://dx.doi.org/10.1002/cpa.3160410704>.
- [12] S. Klainerman. “A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations”. In: *Internat. Math. Res. Notices* 5 (2001), pp. 221–274. ISSN: 1073-7928. DOI: 10.1155/S1073792801000137. URL: <http://dx.doi.org/10.1155/S1073792801000137>.
- [13] S. Klainerman and M. Machedon. “Smoothing estimates for null forms and applications”. In: *Duke Math. J.* 81.1 (1995). A celebration of John F. Nash, Jr., 99–133 (1996). ISSN: 0012-7094. DOI: 10.1215/S0012-7094-95-08109-5. URL: <http://dx.doi.org/10.1215/S0012-7094-95-08109-5>.
- [14] S. Klainerman and M. Machedon. “Space-time estimates for null forms and the local existence theorem”. In: *Comm. Pure Appl. Math.* 46.9 (1993), pp. 1221–1268. ISSN: 0010-3640. DOI: 10.1002/cpa.3160460902. URL: <http://dx.doi.org/10.1002/cpa.3160460902>.
- [15] Sergiu Klainerman. “Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions”. In: *Comm. Pure Appl. Math.* 38.5 (1985), pp. 631–641. ISSN: 0010-3640. DOI: 10.1002/cpa.3160380512. URL: <http://dx.doi.org/10.1002/cpa.3160380512>.
- [16] Sergiu Klainerman. “The null condition and global existence to nonlinear wave equations”. In: *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*. Vol. 23. Lectures in Appl. Math. Providence, RI: Amer. Math. Soc., 1986, pp. 293–326.
- [17] J. Krieger, J. Sterbenz, and D. Tataru. “Global well-posedness for the Maxwell-Klein Gordon equation in 4+1 dimensions. Small energy”. In: *ArXiv e-prints* (Nov. 2012). arXiv:1211.3527 [math.AP].
- [18] Peter H. Lesky and Reinhard Racke. “Nonlinear wave equations in infinite waveguides”. In: *Comm. Partial Differential Equations* 28.7-8 (2003), pp. 1265–1301. ISSN: 0360-5302. DOI: 10.1081/PDE-120024363. URL: <http://dx.doi.org/10.1081/PDE-120024363>.
- [19] Hans Lindblad. “A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time”. In: *Proc. Amer. Math. Soc.* 132.4 (2004), 1095–1102 (electronic). ISSN: 0002-9939. DOI: 10.1090/S0002-9939-03-07246-0. URL: <http://dx.doi.org/10.1090/S0002-9939-03-07246-0>.
- [20] Hans Lindblad. “Counterexamples to local existence for quasilinear wave equations”. In: *Math. Res. Lett.* 5.5 (1998), pp. 605–622. ISSN: 1073-2780.

- [21] Hans Lindblad. “Counterexamples to local existence for semi-linear wave equations”. In: *Amer. J. Math.* 118.1 (1996), pp. 1–16. ISSN: 0002-9327. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v118/118.1lindblad.pdf.
- [22] Jason Metcalfe, Christopher D. Sogge, and Ann Stewart. “Nonlinear hyperbolic equations in infinite homogeneous waveguides”. In: *Comm. Partial Differential Equations* 30.4-6 (2005), pp. 643–661. ISSN: 0360-5302. DOI: 10.1081/PDE-200059267. URL: <http://dx.doi.org/10.1081/PDE-200059267>.
- [23] Jason Metcalfe and Ann Stewart. “Almost global existence for quasilinear wave equations in waveguides with Neumann boundary conditions”. In: *Trans. Amer. Math. Soc.* 360.1 (2008), 171–188 (electronic). ISSN: 0002-9947. DOI: 10.1090/S0002-9947-07-04290-0. URL: <http://dx.doi.org/10.1090/S0002-9947-07-04290-0>.
- [24] John C. Neu. “Kinks and the minimal surface equation in Minkowski space”. In: *Phys. D* 43.2-3 (1990), pp. 421–434. ISSN: 0167-2789. DOI: 10.1016/0167-2789(90)90145-F. URL: [http://dx.doi.org/10.1016/0167-2789\(90\)90145-F](http://dx.doi.org/10.1016/0167-2789(90)90145-F).
- [25] Tomás Ortín. *Gravity and strings*. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge University Press, 2004, pp. xx+684. ISBN: 0-521-82475-3. DOI: 10.1017/CB09780511616563. URL: dx.doi.org/10.1017/CB09780511616563.
- [26] Hart F. Smith and Christopher D. Sogge. “Null form estimates for $(1/2, 1/2)$ symbols and local existence for a quasilinear Dirichlet-wave equation”. In: *Ann. Sci. École Norm. Sup. (4)* 33.4 (2000), pp. 485–506. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(00)00119-1. URL: [http://dx.doi.org/10.1016/S0012-9593\(00\)00119-1](http://dx.doi.org/10.1016/S0012-9593(00)00119-1).
- [27] Hart F. Smith and Daniel Tataru. “Sharp local well-posedness results for the nonlinear wave equation”. In: *Ann. of Math. (2)* 162.1 (2005), pp. 291–366. ISSN: 0003-486X. DOI: 10.4007/annals.2005.162.291. URL: <http://dx.doi.org/10.4007/annals.2005.162.291>.
- [28] Christopher D. Sogge. *Lectures on non-linear wave equations*. Second. Boston, MA: International Press, 2008, pp. x+205. ISBN: 978-1-57146-173-5.
- [29] Atanas Stefanov. “Global regularity for the minimal surface equation in Minkowskian geometry”. In: *Forum Math.* 23.4 (2011), pp. 757–789. ISSN: 0933-7741. DOI: 10.1515/FORM.2011.027. URL: <http://dx.doi.org/10.1515/FORM.2011.027>.
- [30] Terence Tao. “Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates”. In: *Math. Z.* 238.2 (2001), pp. 215–268. ISSN: 0025-5874. DOI: 10.1007/s002090100251. URL: <http://dx.doi.org/10.1007/s002090100251>.
- [31] Terence Tao. “Global regularity of wave maps. II. Small energy in two dimensions”. In: *Comm. Math. Phys.* 224.2 (2001), pp. 443–544. ISSN: 0010-3616. DOI: 10.1007/PL00005588. URL: <http://dx.doi.org/10.1007/PL00005588>.
- [32] Daniel Tataru. “Nonlinear wave equations”. In: *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*. Beijing: Higher Ed. Press, 2002, pp. 209–220.

- [33] Daniel Tataru. “Null form estimates for second order hyperbolic operators with rough coefficients”. In: *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*. Vol. 320. Contemp. Math. Providence, RI: Amer. Math. Soc., 2003, pp. 383–409. DOI: 10.1090/conm/320/05620. URL: <http://dx.doi.org/10.1090/conm/320/05620>.
- [34] Daniel Tataru. “On global existence and scattering for the wave maps equation”. In: *Amer. J. Math.* 123.1 (2001), pp. 37–77. ISSN: 0002-9327. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v123/123.1tataru.pdf.
- [35] Michael E. Taylor. *Partial differential equations I. Basic theory*. Second. Vol. 115. Applied Mathematical Sciences. New York: Springer, 2011, pp. xxii+654. ISBN: 978-1-4419-7054-1. DOI: 10.1007/978-1-4419-7055-8. URL: <http://dx.doi.org/10.1007/978-1-4419-7055-8>.
- [36] Michael E. Taylor. *Tools for PDE*. Vol. 81. Mathematical Surveys and Monographs. Pseudodifferential operators, paradifferential operators, and layer potentials. Providence, RI: American Mathematical Society, 2000, pp. x+257. ISBN: 0-8218-2633-6.
- [37] Bernard de Wit. “Supergravity”. In: *Unity from duality: gravity, gauge theory and strings (Les Houches, 2001)*. NATO Adv. Study Inst. EDP Sci., Les Ulis, 2003, pp. 1–135. DOI: 10.1007/3-540-36245-2_1. URL: http://dx.doi.org/10.1007/3-540-36245-2_1.
- [38] Edward Witten. “String theory dynamics in various dimensions”. In: *Nuclear Phys. B* 443.1-2 (1995), pp. 85–126. ISSN: 0550-3213. DOI: 10.1016/0550-3213(95)00158-0. URL: [http://dx.doi.org/10.1016/0550-3213\(95\)00158-0](http://dx.doi.org/10.1016/0550-3213(95)00158-0).
- [39] Thomas Wolff. “A sharp bilinear cone restriction estimate”. In: *Ann. of Math. (2)* 153.3 (2001), pp. 661–698. ISSN: 0003-486X. DOI: 10.2307/2661365. URL: <http://dx.doi.org/10.2307/2661365>.
- [40] Barton Zwiebach. *A first course in string theory*. Second. With a foreword by David Gross. Cambridge: Cambridge University Press, 2009, pp. xxii+673. ISBN: 978-0-521-88032-9.