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Geometry of Punctured Riemann Surfaces Moduli

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Jiayin Guo

2020

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ABSTRACT OF THE DISSERTATION

Geometry of Punctured Riemann Surfaces Moduli

by

Jiayin Guo
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2020
Professor Kefeng Liu, Chair

In this thesis, we study the geometry of Teichmüller space of punctured Riemann surfaces. We use L2 Hodge theory to describe the deformation theory for punctured Riemann surfaces, in which we defined Weil-Petersson metric, Hodge metric and Kodaira-Spencer map. We also give a new proof of Wolpert's curvature formula by computing the expansion of volume form and the Kodaira-Spencer map. We use Wolpert's formula to estimate upper bound for various curvature tensor. We construct an extension of pluricanonical form and compare it to the expansion of the Kodaira-Spencer map under Hodge metric.

The dissertation of Jiayin Guo is approved.

Hongquan Xu

Gang Liu

David Gieseker

Kefeng Liu, Committee Chair

University of California, Los Angeles

2020

To Mumu,

this dissertation is affectionately dedicated.

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CHAPTER 1

Introduction

Moduli spaces have been studied extensively by algebraic geometers. The moduli space of marked curves is a generalization to the moduli space of curves. Many theories on moduli space of curves can be established on marked one with some modification. The Weil-Petersson metric and Hodge metric are the main tools to investigate the geometry of such moduli space.

In this thesis, we use L2 Hodge theory to describe the deformation of punctured Riemann surfaces. We also give a new proof of Wolpert's curvature formula by computing the expansion of volume form and the Kodaira-Spencer Map. We use this formula to estimate upper bound for various curvature tensor. We construct an extension of pluricanonical form and compare it to the expansion of the Kodaira-Spencer map under Hodge metric. We believe this method can be applied to obtain more curvature formulas for Weil-Peterson metrics and Ricci metric over different moduli spaces.

In the rest of thesis, the punctured Riemann surfaces refers to closed Riemann surfaces with finite k points punctured. We always assume $2g - 2 + k > 0$ to ensure that the Riemann surface is hyperbolic. The convention for metric, curvatures is in consistent with [Bal06].

The Weil-Peterson metric is an Hermitian metric on the Teichmüller space of compact Riemann surfaces with $g > 1$. It is introduced by Weil based on Peterson's pairing for automatic forms [Wei58]. It is a Kählermetric [Ahl61b] and incomplete [Wol75] [Chu76]. It has negative scalar curvature, Ricci curvature,holomorphic sectional curvature and non-positive holomorphic bisectional curvature [Ahl61a], dual-Nakano-negative and semi Nakano-negative[LSY13]. The optimal upper bound for the holomorphic sectional curvature is conjectured by Royden[Roy74] and proved by Wolpert[Wol86] and Tromba[Tro86]. In Wolpert's proof, a formula is discovered to represent the curvature tensor. This formula is generalized to the case the moduli space of compact Kähler-Einstein manifolds with with $c_1 < 0$ [Siu86] and for the case with $c_1 \neq 0$ [Sch93]. For the moduli of compact Calabi-Yau manifolds with Weil-Peterson metric , a curvature formula is obtained for Calabi-Yau threefolds [Str90] and later for higher dimensions [Wan03]. The Weil-Peterson metric can also be defined over the direct image sheaf over the moduli of compact Kähler-Einstein manifolds with $c_1 \neq 0$ and a curvature formula is obtained [Sch12].

For the curvature formula for moduli of the compact or punctured Riemann surfaces. Wolpert's method using $SL(2)$ invariant first order differential operators considered in [Maa49]. Schumacher generalized Siu's method to treat the moduli space of punctured Riemann surface[ST08]. The proof in this paper is still base on Siu's method by constructing a harmonic lift. The Liu-Zhu's method [LZ18] and Sun's method [Sun12] leads to an expansion formula of pluricanonical forms under Hodge metric.

The classical deformation theory [MK71] treats holomorphic family of closed complex manifolds. Fixing one complex structure of the base smooth manifold M, a Beltrami differential $\varphi \in A^{0,1}(M, T^{1,0}M)$ is used to identify another complex structure if the deformation is small enough. In [Kur63], Kuranishi gauge for Beltrami differential is introduced to parametrize the complex structure. For the Teichmüller space of compact Riemann surface, it resembles the Bers coordinate in Teichmüller theory. Noticing that the L^2 Hodge theory for the punctured Riemann surface works similarly to the Hodge theory of closed Riemann surfaces.

Theorem 1.0.1 (Kähler package for punctured Riemann surfaces)**.** *Let (S, g, K, h) be the canonical bundle over punctured Riemann surface with the hyperbolic metric of constant curvature -1. The following holds for any $q \geq 0$.*

(a)The Hodge decomposition

$$L_2^{1,q}(S, K) = \mathrm{Im}(\bar{\partial}_m) \oplus \mathcal{H}^{1,q}(S, K) \oplus \mathrm{Im}(\delta_m).$$

(b)The Hodge isomorphism

$$\mathcal{H}^{1,q}(S,K) \cong H^{1,q}_{(2)}(S,K).$$

(c)Let H be the projection $\mathcal{H}^{1,q}(S,K)$ in Hodge decomposition. Then the Green operator $G = \Delta_{\bar{\partial}m}|_{(\mathcal{H}^{p,q}(S,K))^{\perp}}(I-H)$ is bounded, commuting with $\bar{\partial}_m$,δ_m and $\Delta_{\bar{\partial}m}$.

(d)L^2 Serre duality holds

$$H^{1,q}_{(2)}(S,K) \cong H^{0,1-q}_{(2)}(S,T^{1,0}S)^*.$$

We are able to gives analytic coordinates for Teichmüller space of punctured Riemann surfaces and give the definition for Hodge metric and Weil-Peterson metric.

Theorem 1.0.2. *Let S be a Riemann surface of finite type with hyperbolic metric and $\{\eta_a\}$ a basis of harmonic Beltrami differentials $\mathcal{H}^{0,1}(S,T^{1,0}S)$. There is a semi-universal deformation $(\mathscr{U},S_0,U,b,p)$ of S with a horizontally analytic trivialization F such that $\varphi(t) = \eta_a t^a$ with $t \in U \subset \mathbb{C}$ is the Beltrami differential corresponding to F.*

Moreover, U is open in $\mathcal{T}$ and $\{(t,U)\}$ for all S_0 gives a holomorphic coordinate cover of $\mathcal{T}$.

Definition 1.0.3 (Weil-Petersson metric)**.** Let $(\mathcal{S},B,\pi)$ be an analytic family of punctured Riemann surface , the Weil-Petersson h_{WP} is a a left conjugate linear Hermitian metric on $T_p\mathcal{T}_{g,n}$ defined as

$$h_{WP}(X,X') = \int_{S_p} (\mathcal{K}(X),\mathcal{K}(X')_{-1,1}V_p,$$

where $X,X' \in T_p\mathcal{T}_{g,n}$ $(\ ,\ )_{-1,1}$ is the left conjugate linear Hermitian metric induced by the left conjugate linear Hermitian metric h_p on $T^{1,0}S_p$, V_p is the volume form on S_p.

Definition 1.0.4 (Hodge metric)**.** Let $(\mathcal{S},B,\pi)$ be an analytic family of punctured Riemann surface . The Hodge bundle $E^m \to B$ is the vector bundle with fiber $H^{0,0}_{(2)}(S_p,K_p^m)$ at $p \in B$. The Hodge metric h^m is a right conjugate linear Hermitian metric on E_m defined as

$$h_H^m(s,s') = \int_{S_p} (s,s')_{V_p^{-m}} V_p,$$

where $(\ ,\ )_{V_p^{-m}}$ is the right conjugate linear Hermitian metric.

We apply Liu-Zhu's method to punctured Riemann surfaces case to give an expansion volume from V_t up to order (2,2) and claim this computation any order of expansion can be computed recursively.

Theorem 1.0.5. *Let $\varphi(t)$ and $\rho(t)$ be the setting as above,*

$$\rho(t) = D(\eta_i\eta_{\bar{j}})t^i\bar{t^{\bar{j}}} + O(|t|^3),$$

where $\eta_i\eta_{\bar{j}}$ is a globally defined continuous function and hence

$$V_t = (1 - \Delta_{\bar{\partial}} D(\eta_i\eta_{\bar{j}})t^i\bar{t^{\bar{j}}} + O(|t|^3))V_0.$$

Together with the expansion of Kodaira-Spencer map $\mathcal{K}_t$, we are able derive the expansion of Hodge metric $h_{\bar{i}j}(t)$ under a smooth frame $\{k_t\mathcal{K}_t(\partial_i)\}$, where k_t is the musical isomorphism.

Theorem 1.0.6. *Let $\varphi(t) = t^i\eta_i \in A^{0,1}_{(2)}(S, T^{1,0}S)$ be a harmonic Beltrami differential. The Hodge metric with respect to frame $\{k_t\mathcal{K}_t(\partial_i)\}$ coincides with the WP metric with respect to frame $\{\partial_i\}$ and have an explicit formula for any order. For order up to 2 is given as below.*

$$\begin{aligned}
\iota_0 h_{\bar{i}j}(t) &= \int_{S_0} \eta_{\bar{i}}\eta_j V_0,\\
\iota_0 \partial_{\bar{k}} h_{\bar{i}j}(t) &= 0,\\
\iota_0 \partial_l h_{\bar{i}j}(t) &= 0,\\
\iota_0 \partial_{\bar{k}l} h_{\bar{i}j}(t) &= \int_{S_0} \sigma_{ik}\eta_{\bar{i}}\eta_j D\eta_{\bar{k}}\eta_l V_0.
\end{aligned}$$

And hence we derive the curvature formula for WP metric.

$$R^W_{i\bar{j}k\bar{l}}(0) = \int_{S_0} \sigma_{ik}\eta_{\bar{j}}\eta_i D\eta_{\bar{l}}\eta_k V_0.$$

Based on the curvature formula, we proved estimated an upper bound for Ricci curvature, holomorphic curvature and scalar curvature, which is an analog for Wolpert's estimates for closed Riemann surfaces.

Theorem 1.0.7. *(1) The holomorphic sectional curvature and the Ricci curvature is bounded above by* $\frac{-1}{\pi(2g-2+k)}$.

(2)The scalar curvature is bounded by $\frac{-(3g-3+k)(3g-2+k)}{2\pi(2g-2+k)}$.

Using the techniques in [LZ18], we construct another set of smooth frame $\{E_t k_0 \mathcal{K}(\partial_i)\}$. Denote Hodge metric under a smooth frame $\{E_t k_0 \mathcal{K}(\partial_i)\}$ by $\tilde{h}_{\bar{i}j}(t)$.

Theorem 1.0.8. *Let* $\varphi(t) = t^i \eta_i \in A^{0,1}_{(2)}(S, T^{1,0}S)$ *be a harmonic Beltrami differential. Let* $s^a_0 = i\eta_{\bar{a}} g \mathrm{d}z^2, s^b_0 = i\eta_{\bar{b}} g \mathrm{d}z^2 \in A^{0,0}_{(2)}(S, K^m)$ *be holomorphic sections and* $E_t(s^a_0)$, $E_t(s^b_0)$ *be extension corresponding to* φ*. The Hodge metric with respect to frame* $\{E_t(s^a_0)\}$ *have an explicit formula for any order. For order up to 2 is given as below.*

$$\begin{aligned}
\iota_0 \tilde{h}^H_{\bar{i}j}(t) &= \int_{S_0} \eta_{\bar{i}} \eta_j V_0, \\
\iota_0 \partial_{\bar{k}} \tilde{h}^H_{\bar{i}j}(t) &= 0, \\
\iota_0 \partial_l \tilde{h}^H_{\bar{i}j}(t) &= 0, \\
\iota_0 \partial_{\bar{k}l} \tilde{h}^H_{\bar{i}j}(t) &= -\int_{S_0} \sigma_{ik} \eta_{\bar{i}} \eta_j D \eta_{\bar{k}} \eta_l V_0.
\end{aligned}$$

However, as $\{E_t k_0 \mathcal{K}(\partial_i)\}$ itself is not holomorphic nor related to any holomorphic sections. It still remains to be question that how one could utilize $\tilde{h}^H_{\bar{i}j}$ to get a curvature formula.

This thesis is organized as follows. In Chapter 2, we review some necessary concepts and results for the study of moduli space of punctured Riemann surface. In Section 2.1, we reviews some L^2 Hodge theories in order to show that the L^2 Hodge theory for holomorphic bundles over punctured Riemann surfaces behaves very similar to the compact cases. We first review the general L^2 Hodge theory over complete Kähler manifolds and gives a criteria, closedness of ∂_m range, for Kähler package to hold. Next we prove that the criteria is met for the case for holomorphic bundle over punctured Riemann surfaces. In Section 2.2, we introduce an analytic coordinate of Teichmüller space of punctured Riemann surfaces, which resembles the coordinate of Bers's. We introduce three equivalent ways to describe deformation near a fixed punctured Riemann surface, the definition introduced in Huber's book, K-S coordinates and Beltrami differentials. we defined Kodaira-Spencer Map, Weil-Peterson metric and Hodge metric at the end of the section. In Chapter 3, we generalized the Wolpert's formula to the case of punctured Riemann surfaces. To compute the curvature of WP metric, one way is to use K coordinate compute the expansion of a harmonic lift and the volume form with respect to a deformation $(\mathcal{S}, S_0, B, b, p)$. In Section 3.2, we computed the expansion of the volume form up to order 2 using a method suggested in [Sun12] which works for compact Kähler-Einstein cases.In Section 3.2, we recall the harmonic lift constructed by Siu[Siu86] and computed its expansion up to order 2. In Section 3.3, we ensemble the result from previous sections and derive the Wolpert's formula for punctured Riemann surfaces. We also use this formula to get a negative upper bound for the Curvature Tensor for the WP metric. In this chapter, we construct an extension of pluricanonical form and compare the Hoghe metric under the frame of extended pluricanonical forms. As suggested in [Sun12], one could derive Wolpert's formula using the expansion of Hodge metric, which can be derived from the expansion of volume form and the extension of pluricanonical form. However, this method could not to be directly related to the curvature formula for WP metric. This subtlety is due to the pluricanonical form we have constructed is not identical to the musical isomorphism of the harmonic lift defined in Section 3.2. In Section 4.1, we construct the extension of pluricanonical form for a deformation of punctured Riemann surfaces. In section 4.2, we discussed the relation between harmonic lift and the extension of pluricanonical form and gives an asymptotic formula.

CHAPTER 2

L^2 Hodge Theory and Deformation Theory

In this chapter, we review some necessary concepts and results for the study of moduli space of punctured Riemann surface. The main motivation is that each of these two theories should behave similar to those for closed Riemann surfaces. The major difficulties is the non-compactness of punctured Riemann surfaces, which is overcome by some choice of complete metric and technically refined definition for deformation.

2.1 L^2 Hodge Theory

This section reviews some L^2 Hodge theories in order to show that the L^2 Hodge theory for holomorphic bundles over punctured Riemann surfaces behaves very similar to the compact cases. We first review the general L^2 Hodge theory over complete Kähler manifolds and gives a criteria, closedness of ∂_m range, for Kähler package to hold. Next we prove that the criteria is met for the case for holomorphic bundle over punctured Riemann surfaces. We refer [MP90] for treatments for line bundles being trivial and [Ohs15] Chapter 2 for the general L^2 Hodge theory.

In this section, (M, g, E, h) means a holomorphic vector bundle E over a Kähler manifold M with the Chern connection, where h is an Hermitian metric on E and g is the Kählermetric on M.

Let $\bar{\partial}_{p,q}$ be the differential operator acting on smooth differential forms with compact support

$$\bar{\partial}_{p,q} : A_c^{p,q}(M, E) \longrightarrow A_c^{p,q+1}(M, E).$$

Its formal adjoint, denoted by $\delta_{p,q+1}$, still acts on smooth differential forms with compact support. We omit the subscript if it is not ambiguous.

Viewing $\bar{\partial}$ as a densely defined differential operator acting on $L_2^{p,q}(M,E) \to L_2^{p,q+1}(M,E)$, there is two canonical extensions of $\bar{\partial}$ as closed operators.

Definition 2.1.1. The maximal extension $\bar{\partial}_{max}$ of $\bar{\partial}$ is the adjoint operator of δ in Hilbert space $L_2^{p,q}(M,E)$. That is, $\alpha \in L_2^{p,q}(M,E)$ is in domain of definition $Dom(\bar{\partial}_{max})$ if there is $\alpha' \in L_2^{p,q+1}(M,E)$ such that $(\alpha, \delta\beta)_2 = (\alpha', \beta)_2$ holds for any $\beta \in A_c^{p,q+1}(M,E)$.

Definition 2.1.2. The minimal extension $\bar{\partial}_{min}$ of $\bar{\partial}$ is the closure of $\bar{\partial}$ in Hilbert space $L_2^{p,q}(M,E)$ under graph norm. That is, $\alpha \in L_2^{p,q}(M,E)$ is in domain of definition $Dom(\bar{\partial}_{min})$ if there is a sequence of $\alpha_n \in A_c^{p,q}(M,E)$ converging to α in L^2 norm such that $\bar{\partial}\alpha_n$ is convergent.

It is easy to check that $\delta_{min} = \bar{\partial}_{max}$ and $\delta_{max} = \bar{\partial}_{min}$. The following result of Gaffney [Gaf54] allow us to denote either of the operators by δ_m.

Proposition 2.1.3. *If M is complete as a Riemannian manifold, then $\bar{\partial}_{min} = \bar{\partial}_{max}$.*

Proposition 2.1.4 (Chernoff)**.** *Let D be an elliptic operator of order one with its symbol being uniformly bounded and let M be complete and connected. Then D^k is essentially self-adjoint for any $k \geq 1$. Moreover, $Dom(D^*D)_{max} \subseteq Dom(D)_{max}$.*

This result allows us to represent the closed extension of the formal Laplace operator.

Proposition 2.1.5. *Denote the formal Laplace operator by*

$$\Delta_{\bar{\partial}} : A_c^{p,q}(M,E) \longrightarrow A_c^{p,q}(M,E),$$

we have the unique closed extension $\Delta_{\bar{\partial}_m}$ being self-adjoint and

$$\Delta_{\bar{\partial}_m} = \delta_m \bar{\partial}_m + \bar{\partial}_m \delta_m.$$

Moreover, the domain of definition is

$$Dom(\Delta_{\bar{\partial}_m}) = \{\alpha \in Dom(\delta_m) \cap Dom(\bar{\partial}_m) | \delta_m \alpha \in Dom(\bar{\partial}_m), \bar{\partial}_m \alpha \in Dom(\delta_m)\}.$$

Proof. Applying Proposition 2.1.4 to $\bar{\partial} + \delta$, we have $\Delta_{\bar{\partial}}$ is essentially self-adjoint. Note $\delta_m \bar{\partial}_m + \bar{\partial}_m \delta_m$ is self-adjoint and with the same core as $\Delta_{\bar{\partial}}$. So $\Delta_{\bar{\partial}_m} = \delta_m \bar{\partial}_m + \bar{\partial}_m \delta_m$. $\square$

Denote $\operatorname{Ker} \Delta_{\bar{\partial}_m}$ by $\mathcal{H}^{p,q}(M, E)$. We obtain the following version of Kodaira decomposition.

Proposition 2.1.6 (Kodaira decomposition)**.**

$$L_2^{p,q}(M, E) = \overline{\operatorname{Im}(\bar{\partial}_m)} \oplus \mathcal{H}^{p,q}(M, E) \oplus \overline{\operatorname{Im}(\delta_m)}.$$

Proof. Since ∂_m and δ_m are adjoint to each other, we have

$$L_2^{p,q}(M, E) = \overline{\operatorname{Im}(\bar{\partial}_m)} \oplus \operatorname{Ker} \partial_m \cap \operatorname{Ker} \delta_m \oplus \overline{\operatorname{Im}(\delta_m)}.$$

On one hand, $\operatorname{Ker} \partial_m \cap \operatorname{Ker} \delta_m = \operatorname{Ker}(\partial_m + \delta_m) \subseteq \mathcal{H}^{p,q}(M, E)$. On the other hand, $\mathcal{H}^{p,q}(M, E) \subseteq Dom(\partial_m + \delta_m)$ by Proposition 2.1.4. And the identity $(\Delta_{\bar{\partial}_m} \alpha, \alpha)_2 = (\bar{\partial}_m \alpha, \bar{\partial}_m \alpha)_2 + (\delta\alpha, \delta\alpha)_2$ implies $\mathcal{H}^{p,q}(M, E) \subseteq \operatorname{Ker}(\partial_m + \delta_m)$. $\square$

In the L^2 cohomology theory, the following two cohomology groups are often considered.

Definition 2.1.7. Let (M, E) be a holomorphic vector bundle. The L^2 cohomology group of E is the following

$$H_{(2)}^{p,q}(M, E) = \frac{\operatorname{Ker} \bar{\partial}_{max}}{\operatorname{Im} \bar{\partial}_{max}}.$$

The L^2 reduced cohomology group is the following

$$H_{(2),red}^{p,q}(M, E) = \frac{\operatorname{Ker} \bar{\partial}_{max}}{\overline{\operatorname{Im} \bar{\partial}_{max}}},$$

where $\overline{\operatorname{Im} \bar{\partial}_{max}}$ means the closure under L^2 norm.

They coincide exactly when $\bar{\partial}_{max}$ has a closed range.

Proposition 2.1.8. *If* $\dim H_{(2)}^{p,q}(M, E) < +\infty$, *then* $\bar{\partial}_{max}^{p,q-1}$ *has a closed range.*

Proof. Note that $Dom(\bar{\partial}_{max})$ is the completion of normed space $A_c^{p,q}(M,E)$ equipped with the graph norm $\|\alpha\|_{\bar{\partial}}$. Thus $\bar{\partial}_{max}$ is a bounded operator from $Dom(\bar{\partial}_{max})$ to $\operatorname{Ker}\bar{\partial}_{max}$ with graph norm $\|\cdot\|_{\bar{\partial}}$ and L^2 norm $\|\cdot\|_2$ respectively. It has a closed image by Lemma 2.1.9 which is proved using the open mapping theorem. □

Lemma 2.1.9. *Let X,Y be Banach-Spaces and $T : X \to Y$ a linear and bounded operator. Then T has a closed range if TX is of finite codimension.*

Proof. We may assume T is injective. Let Z be a algebraic complimentary of TX in Y equipping with a norm $\|\|_Z$. Then the map

$$\tilde{T} : X \times Z \longrightarrow Y$$
$$(x, z) \mapsto T(x) + z$$

is a bijective continuous map. It is open and closed by the open mapping theorem and thus has a closed range. □

If we assume g is complete and $\bar{\partial}_m$ has closed range, the Hodge theory is very similar to the compact case. We state the Kähler package as the following theorem.

Theorem 2.1.10 (Kähler package)**.** *Let (M, g, E, h) be a holomorphic bundle over a complete Kähler manifold with the Chern connection. If $\bar{\partial}_m^{p,q}$ has a closed range for fixed p and all q, then following holds.*

(a)The Hodge decomposition

$$L_2^{p,q}(M,E) = \operatorname{Im}(\bar{\partial}_m) \oplus \mathcal{H}^{p,q}(M,E) \oplus \operatorname{Im}(\delta_m).$$

(b)The Hodge isomorphism $\mathcal{H}^{p,q}(M,E) \cong H_{(2)}^{p,q}(M,E)$.

(c)Let H be the projection $\mathcal{H}^{p,q}(M,E)$ in Hodge decomposition. Then the Green operator $G = \Delta_{\bar{\partial}m}|_{(\mathcal{H}^{p,q}(M,E))^{\perp}}(I - H)$ is bounded, commuting with $\bar{\partial}_m$,δ_m,$\Delta_{\bar{\partial}m}$

(d)The closed operator $\bar{\partial}_m$ for (M, E^*) also has closed range and L^2 Serre's duality holds

$$H_{(2)}^{p,q}(M,E) \cong H_{(2)}^{n-p,n-q}(M,E^*)^*.$$

Part (a), (b) and (c) by Proposition 2.1.6, Part(d) follows from [CS12] Theorem 2.

For the rest of the article we omit the subscript $\bar{\partial}_m$ if it is not ambiguous in the context.

Now consider the canonical bundle K over a punctured Riemann surface S of type (g,k), M is endowed with the hyperbolic metric g of constant curvature -1. K is endowed with the Hermitian metric h induced by g.

Proposition 2.1.11. *For (S,g,K,h) defined as above,we have*

$$H^{1,q}_{(2)}(S,K^m)=0$$

for $q>0$,$m>0$ and $\dim H^{1,0}_{(2)}(S,K)=3g-3+k$.

Proof. The $q>0$ case follows from K being positive and a vanishing theorems [Ohs15] Theorem 2.14.

For the case $q=0$, note that $\ker\bar{\partial}^{1,0}_m=\ker\bar{\partial}^{1,0}$. So the element s in $H^{1,0}_{(2)}(S,K)$ is a global holomorphic sections of $2K$ with finite L^2 norm.

Let $\bar{S}$ be the closed Riemann surface such that $S=\bar{S}-\{p_1,\ldots,p_k\}$. A small neighborhood U of a punctured point p is called a cusp neighborhood. We note from classical result from Fuchsian models of Riemann surfaces or from [TZ91]. There exists a coordinate z such that $z(p)=0$ and $g=\frac{|\mathrm{d}z|^2}{(|z|ln|z|)^2}$.

Let $s=f(z)\mathrm{d}z^2$, then

$$\int_U|f(z)|^2g^{-1}\mathrm{d}z\wedge\mathrm{d}\bar{z}<\infty.$$

Thus $f(z)=O(z^{-1})$ near 0. s is a meromorphic section of $\bar{S}$ with poles at most order 1 at punctured points. By Riemann-Roch theorem, $H^{1,0}_{(2)}(S,K)=3g-3+k$. This finishes the proof. $\square$

In particular, all the property mentioned in Proposition 2.1.10 can be applied for $H^{1,q}(S,K)$ as well as $H^{0,q}(S,T^{1,0}S)$.

Theorem 2.1.12 (Kähler package for punctured Riemann surfaces)**.** *Let (S,g,K,h) be the canonical bundle over punctured Riemann surface with the hyperbolic metric of constant curvature -1. The following holds for any $q\geq 0$.*

(a)The Hodge decomposition

$$L_2^{1,q}(S,K) = \mathrm{Im}(\bar{\partial}_m) \oplus \mathcal{H}^{1,q}(S,K) \oplus \mathrm{Im}(\delta_m).$$

(b)The Hodge isomorphism

$$\mathcal{H}^{1,q}(S,K) \cong H_{(2)}^{1,q}(S,K).$$

(c)Let H be the projection $\mathcal{H}^{1,q}(S,K)$ in Hodge decomposition. Then the Green operator $G = \Delta_{\bar{\partial}m}|_{(\mathcal{H}^{p,q}(S,K))^{\perp}}(I-H)$ is bounded, commuting with $\bar{\partial}_m$,δ_m and $\Delta_{\bar{\partial}m}$.

(d)L^2 Serre's duality holds

$$H_{(2)}^{1,q}(S,K) \cong H_{(2)}^{0,1-q}(S,T^{1,0}S)^*.$$

2.2 Deformation Theory for Punctured Riemann Surfaces

This section we introduce an analytic coordinate of Teichmüller space of punctured Riemann surfaces, which resembles the coordinate of Bers's [Ber58]. We first introduce three equivalent ways to describe deformation near a fixed punctured Riemann surface, the definition introduced in Huber's book, K-S coordinates and Beltrami differentials. , we define Kodaira-Spencer Map, Weil-Peterson metric and Hodge metric at the end of the section.

Recall the deformation family $(\mathcal{M}, B, \pi)$ for closed Riemann surfaces is a proper submersion p between complex manifolds with fibers being closed Riemann surfaces. For the punctured Riemann surfaces, the substitute for π being proper is the following definition [Hub16].

Definition 2.2.1. An analytic submersion $\pi : \mathcal{M} \longrightarrow B$ admits a locally horizontally analytic trivialization by a manifold M if there exists an open set $V \subset B$ and a diffeomorphism $F : M \times V \longrightarrow \pi^{-1}(V)$ commuting with π such that for any p in M and any $t \in V$, the map $F_p : t \mapsto F(p,t)$ is analytic.

Definition 2.2.2. An analytic family of Riemann surfaces is an analytic submersion

$p : \mathcal{S} \longrightarrow B$ of analytic manifolds such that the fibers S_t are 1-dimensional and π locally admits horizontally analytic trivialization.

Definition 2.2.3. a deformation $(\mathcal{S}, S_0, B, b, \pi)$ of Riemann surface S is an analytic family $(\mathcal{S}, B, \pi)$ of Riemann surfaces with $b \in B$, $S_0 = p^{-1}(b)$ and $S_0 \cong S$.

With these definitions and minor modification on the classical deformation theory [MK71] Chapter 4 we get another description of a deformation of Riemann surfaces.

Definition 2.2.4. A KS coordinate cover pair $(\omega_i, s_i, \mathscr{U}_i), (z_i, t_i, \mathscr{U}_i)$ with core over a marked analytical manifolds (B, b) of a Riemann surface S is a pair of two coordinates, the S-coordinate $(\omega_i, s_i, \mathscr{U}_i)$ and the K-coordinate $(z_i, t_i, \mathscr{U}_i)$ satisfying the following: :

1. There is an open set $V \subset B$ containing b such that both $(\omega_i, s_i, \mathscr{U}_i)$ and $(z_i, t_i, \mathscr{U}_i)$ are smooth coordinate covers of the smooth manifold $S \times V$.

2. K-coordinate $(z_i, t_i, \mathscr{U}_i)$ is an analytic coordinate cover induced by $S \times V$ as production of complex analytic manifolds. That is, z_i and t_i is the pull back of some analytic coordinate of S and V respectively.

3. For S-coordinate $(\omega_i, s_i, \mathscr{U}_i)$, $s_i = t_i$, $w_i(z_i, t_i)$ is holomorphic with respect to t_i and for any fixed $b' \in B$, the coordinate cover $(\omega_i, \mathscr{U}_i \cap S \times b')$ of $S \times b'$ has analytic transition function and thus endowed $S \times b'$ with a complex analytic structure.

4. Let $S_0 = S \times b$ and let $U_i = \mathscr{U}_i \cap S_0$. Then $t_i|_{U_i} = 0$, $w_i|_{U_i} = z_i|_{U_i}$.

The last requirement is just for the purpose of normalization and not necessary. For most of the times, we omit the subscript which indexing the cover and also the cover itself if they can be implied from context.

KS coordinate pairs is defined mainly for coordinate computation purposes which is extensively use in the following chapters. It is easy to verify that a deformation $(\mathcal{S}, S_0, B, b, \pi)$ with a specified horizontally analytic trivialization is equivalent to a pair of KS coordinates.

Definition 2.2.5. Let S be a Riemann surface, a smooth section $\phi \in A_2^{0,1}(S, T^{1,0}S)$ is called a Beltrami differential.

Proposition 2.2.6. *Let $(\mathcal{S}, S_0, B, b, \pi)$ be a deformation of a punctured Riemann surface S. Any horizontally analytic trivialization F to $p^{-1}(U)$, where (U, t) is a neighborhood of b with $t(b) = 0$, gives a family of Beltrami differential $\varphi(t)$ holomorphic with respect to t.*

Proof. By picking a horizontally analytic trivialization (F, V) of $(\mathcal{S}, S_0, B, b, \pi)$, we get KS coordinates pairs $(\omega, s), (z, t)$. one could define a global holomorphic tangent valued $(1, 0)$ form

$$\varphi(t) = (\frac{\partial \omega}{\partial z})^{-1} \bar{\partial} \omega \otimes \partial_z \tag{2.2.1}$$

such that $\varphi(0) = 0$, analytic in t. By choosing a sufficient small V, we can as assume $\|\varphi(t)\|_\infty < 1$ for any t in V and hence $\phi(t) \in \phi \in A_2^{0,1}(S, T^{1,0}S)$. Thus $\varphi(t)$ is a Beltrami differential for any t in V. □

For a complex manifold B, a fixed point $b \in B$ and Riemann surface S, a smooth family of Beltrami differential φ_t of S with $\varphi(0) = 0$ also gives a deformation $(\mathcal{S}, S, B, b, \pi)$, which is a consequence of the so called "complex" Frobenius theorem [NN57].

Theorem 2.2.7 (Newlander and Nirenberg)**.** *Let M be a complex manifold with holomorphic coordinate (z, U) and a Beltrami differential*

$$\varphi = \varphi_{\bar{i}}^j(z) \mathrm{d} z^{\bar{i}} \frac{\partial}{\partial z^j}.$$

Let

$$\begin{aligned} \bar{T}_{\bar{i}} &= \frac{\partial}{\partial z^{\bar{i}}} - \varphi_{\bar{i}}^j(z) \frac{\partial}{\partial z^j}, \\ T_i &= \frac{\partial}{\partial z^i} - \bar{\varphi}_i^{\bar{j}}(z) \frac{\partial}{\partial z^{\bar{j}}}. \end{aligned}$$

If T_i and $T_{\bar{i}}$ are complex linear independent and

$$\bar{\partial} \varphi - \frac{1}{2}[\varphi, \varphi] = 0,$$

then there is a smooth solution $(f_1(z), \ldots, f_n(z))$ such that

$$\bar{T}_{\bar{i}} f_j(z) = 0$$

for any j and $(f_1, \ldots, f_n, \bar{f}_1, \ldots, \bar{f}_n)$ forms a coordinate system on U.

Proposition 2.2.8. *Let S be a Riemann surface of finite type with hyperbolic metric and let $B \subseteq \mathbb{C}^n$ open with coordinate t. For any holomorphic family of Beltrami differential $\varphi(t)$, there is a deformation $(\mathscr{U}, S_0, U, b, p)$ of S with a horizontally analytic trivialization (F, U) with $U \subset B$, the Beltrami differential correspond to which coincides $\varphi(t)$ on U.*

Proof. For a coordinate (V, z) of S_0 and a cover $\{\mathscr{U}_j\}$ of $V \times U$. Let$T(z,t) = \partial_z - \bar{\varphi}(z,t)\partial_{\bar{z}}$, so T and $\bar{T}$ are linearly independent for $t \in U$. By the Theorem 2.2.7, equation

$$\bar{T} w_j(z,t) = 0,$$

has one smooth solution, for $t \in U \subseteq B$.

So $\{\mathscr{U}_j, w_j\}$ forms an analytic coordinate cover for $S \times U$ and the identity map gives one horizontally analytic trivialization of $S \times U$ as a complex manifold, denoted by $\mathscr{U}$. $\square$

The Teichmüller space has been extensively in the history. We use the following definition for general Riemann surfaces.

Definition 2.2.9. The Teichmüller space $\mathcal{T}$ for Riemann surface S is the space pairs (S, g) modulo a equivalence relation, where $g : S \to S$ a diffeomorphism. $(S, g), (S, h)$ are equivalent if $h \circ g^{-1} : S \to S$ is isotopic to identity. The pair (S, g) is called a marked Riemann surface of S.

For a marked Riemann surface (S, g), its associated Beltrami differential is

$$\varphi = (\frac{\partial w}{\partial z})^{-1} \bar{\partial} w \otimes \partial_z,$$

where $w = z \circ g^{-1}$ and z a holomorphic coordinate of S. We have a simple criteria for (S, g) being equivalent to (S, id).

Proposition 2.2.10. *Let ϕ be the associated Beltrami differential to marked Riemann surface (S, g). (S, g) equivalent to (S, id) if and only if $\varphi \in \mathrm{Im}(\bar{\partial}_m)$.*

Fixed a marked surface (S, id), to parametrize its neighborhood, one just need to construct a family of Beltrami differential which is not in the same equivalent class. The previous proposition suggest to construct a holomorphic family of Beltrami differential with $\operatorname{Im}\bar{\partial}_m$ part being zero.

With Kähler package proved in Section 2.1, we can just follow the proof line by line the proof, which is given by Kuranishi for the compact case. We just list the statements of propositions without proof to indicate which parts should be modified. Proof for compact case can be found in [MK71] Chapter 4.

Proposition 2.2.11. *Let S be a Riemann surface with a complete Kähler metric such that Kähler package 2.1.10 holds for $(S, T^{1,0}S)$. For any deformation of S, any Beltrami forms $\varphi(t)$ given by a horizontally analytic trivialization F , there exist uniquely a family of diffeomorphisms $f(t)$ of S_0 holomorphic in t such that Beltrami differential $\varphi_f(t)$ corresponding to $f \circ F$ is harmonic, that is*

$$\delta\varphi_f(t) = 0.$$

It is easy to verify that different choice of horizontally analytic trivialization leads to the same $\varphi_f(t)$. If the complete metric is the hyperbolic metric g, we can choose $\varphi(t)$ to be harmonic which can be identified as one element in $H^{0,1}_{(2)}(S, T^{1,0}S)$.

Definition 2.2.12. A deformation $(\mathcal{S}, S_0, B, b, p)$ of Riemann surface S is called semi-universal if for any deformation $(\mathcal{S}', S'_0, B', b', p')$ of S there is a holomorphic map $f : B' \longrightarrow B$ with $\operatorname{Im}(\mathrm{d}f)$ being unique such that $(\mathcal{S}', S'_0, B', b', p')$ is isomorphic to the pull back of f.

We summarize the discussion above to give a holomorphic coordinate of $\mathscr{T}$ by constructing a semi-universal deformation of S.

Theorem 2.2.13. *Let S be a Riemann surface of finite type with hyperbolic metric and $\{\eta_a\}$ a basis of harmonic Beltrami differentials $\mathcal{H}^{0,1}(S, T^{1,0}S)$. There is a semi-universal deformation $(\mathscr{U}, S_0, U, b, p)$ of S with a horizontally analytic trivialization F such that $\varphi(t) = \eta_a t^a$ with $t \in U \subset \mathbb{C}$ is the Beltrami differential corresponding to F.*

Moreover, U is open in $\mathcal{T}$ and $\{(t, U)\}$ for all S_0 gives a holomorphic coordinate cover of $\mathcal{T}$.

Proof. Let $\{\eta_a\}$ be a basis for $\mathcal{H}^{0,1}(S_0, T^{1,0}S_0)$ then $\varphi(t) = \eta_a t^a$ is a harmonic Beltrami differential for $t \in U$, where $U \subset \mathbb{C}$ is a small neighborhood of 0. Proposition 2.2.8 yields a deformation $\mathcal{H}^{0,1}(S, T^{1,0}S)$ with $\varphi(t) = \eta_a t^a$. The semi-university of $(\mathscr{U}, S_0, U, b, p)$ follows from proposition 2.2.11. The coordinate neighborhood (t, U) is a holomorphic one is a result of Ber's [Ber58]. $\square$

Definition 2.2.14 (Kodaira-Spencer map)**.** Let $(\mathcal{S}, B, \pi)$ be an analytic family of punctured Riemann surfaces with each fiber equipped with the unique hyperbolic metric of constant curvature -1. For any point $b \in B$, the Kodaira-Spencer map $\mathcal{K} : TB_b \to H^{0,1}_{(2)}(S_b, T^{1,0}S_b)$ is defined as

$$\mathcal{K} : TB_b \longrightarrow H^{0,1}_{(2)}(S_0, T^{1,0}S_0)$$
$$t^a \partial_a \mapsto t^a \frac{\partial \varphi(t)}{\partial t^a}|_{t=0}$$

where $\varphi(t)$ is harmonic and is given by a deformation $(\mathcal{S}, S_0, B, b, p)$ of S.

It can be shown that definition is independent of different choice of deformation.

Definition 2.2.15 (Weil-Petersson metric)**.** Let $(\mathcal{S}, B, \pi)$ be an analytic family of punctured Riemann surface , the Weil-Petersson h_{WP} is a a left conjugate linear Hermitian metric on $T_p\mathcal{T}_{g,n}$ defined as

$$h_{WP}(X, X') = \int_{S_p} (\mathcal{K}(X), \mathcal{K}(X')_{-1,1} V_p,$$

where $X, X' \in T_p\mathcal{T}_{g,n}$ $(\ ,\ )_{-1,1}$ is the left conjugate linear Hermitian metric induced by the left conjugate linear Hermitian metric h_p on $T^{1,0}S_p$, V_p is the volume form on S_p.

Definition 2.2.16 (Hodge metric)**.** Let $(\mathcal{S}, B, \pi)$ be an analytic family of punctured Riemann surface . The Hodge bundle $E^m \to B$ is the vector bundle with fiber $H^{0,0}_{(2)}(S_p, K_p^m)$ at $p \in B$. The Hodge metric h^m is a right conjugate linear Hermitian metric on E_m defined as

$$h^m_H(s, s') = \int_{S_p} (s, s')_{V_p^{-m}} V_p,$$

where $(\ ,\ )_{V_p^{-m}}$ is the right conjugate linear Hermitian metric.

Note that $\mathcal{T}$ is the base space of a semi-universal analytic family $(\mathcal{S}, \mathcal{T}, S)$ and $\mathcal{K}$ is isomorphic by semi-university. $\mathcal{T}$ thus endow with a WP metric on its holomorphic tangent bundle. If $m = 2$, by L^2 version of Serre Duality, $E^2 \to \mathcal{T}$ is isomorphic to the holomorphic cotangent bundle over $\mathcal{T}$ and thus the WP metric and the Hodge metric are co-metric to each other.

For a fixed point $b \in \mathcal{T}$, a basis $\{\eta_i\}$ of $H^{0,1}_{(2)}(S_b, T^{0,1}S_b)$, we pick a pair of KS coordinates (w, s) and (z, t) corresponding to $\varphi(t) = t^i \eta_i$. Denote $\pi^{-1}(t)$ by S_t, we have the following commuting diagram

$$\begin{tikzcd}
TB_t \arrow[r, "h_t^{WP}"] \arrow[d, "\mathcal{K}_t"] & T^*B_t \arrow[drr, "\mathcal{K}_t^*"] & & \\
H^{0,1}_{(2)}(S_t, T^{1,0}S_t) \arrow[r, "S_t"] & H^{2,0}_{(2)}(S_t, 2K_t)^* \arrow[rr, "h_t^{Hodge}"] & & H^{2,0}_{(2)}(S_t, 2K_t),
\end{tikzcd}$$

where h_t^{Hodge} and h_t^{WP} is the conjugate linear musical isomorphism with respect to metric and S_t is the complex linear Serre duality at t.

To compute the curvature of h_{WP}, one method is to find an explicit expansion up to order 2 of h_{WP} and to compute the curvature of h_{Rcci} which will be treated in Chapter 4, one need to compute find an explicit expansion up to order 4 of h_{WP}.

In K-coordinate (z, t), we use ∂_i as an abbreviation for ∂_{t^i}. Note that

$$h_{\bar{i}j}(t) = \int_{S_0} (\mathcal{K}_t(\partial_i), \mathcal{K}_t(\partial_j))_{V_t^{-1,1}} V_t$$

In Chapter 3, we compute the expansion of $\mathcal{K}_t(\partial_i)$ and V_t in K-coordinate and thus get a curvature formula for WP metric.

Denote $h_t^H \circ S_t$ by k_t, denote the dual basis of $\{\mathcal{K}_t(\partial_i)\}$ by ε^i. We have

$$k_t \mathcal{K}_t(\partial_i) = \varepsilon^i h_{i\bar{j}}.$$

Let

$$h^H_{\bar{i}j}(t) = h_H(k_t \mathcal{K}_t(\partial_i), k_t \mathcal{K}_t(\partial_j)) = h_{\bar{i}j}(t).$$

The curvature tensor of Hodge metric with respect to $\{k_t \mathcal{K}_t(\partial_i)\}$ is

$$R^{H}_{i\bar{j}\bar{l}}{}^{\bar{k}} = -\partial_{\bar{j}}(\partial_i h^H_{\bar{l}p} h_H^{p\bar{k}}). \tag{2.2.2}$$

Compared to the curvature tensor of WP metric with respect to $\{\partial_i\}$

$$R^{W\ \ l}_{i\bar{j}k} = -\partial_{\bar{j}}(\partial_i h_{\bar{q}k} h^{l\bar{q}}),$$

we have

$$R^W_{i\bar{j}k\bar{l}} = -R^H_{i\bar{j}k\bar{l}}.$$

In Chapter 4, for element in $H^{2,0}_{(2)}(S_0, 2K_0)$, we construct a extension operator E_t : $H^{2,0}_{(2)}(S_0, 2K_0) \to H^{2,0}_{(2)}(S_0, 2K_0)$, which is a smooth family of isomorphisms. Denote the following

$$\tilde{h}^H_{\bar{i}j}(t) = h^H(E_t k_0 \mathcal{K}_0(\partial_i), E_t k_0 \mathcal{K}_0(\partial_j)).$$

It may be tempted to use formula like (2.2.2) to compute the curvature tensor. However it will be proved in Chapter 4 that $\{(E_t k_0 \mathcal{K}_0(\partial_i)\}$ is a smooth et not a holomorphic frame over $B \subseteq \mathcal{T}$.

CHAPTER 3

Wolpert's Formula for Punctured Riemann Surfaces

In this chapter, we generalized the Wolpert's formula to the case of punctured Riemann surfaces. To compute the curvature of WP metric, one way is to use K coordinate compute the expansion of a harmonic lift and the volume form with respect to a deformation $(\mathcal{S}, S_0, B, b, p)$. In Section 3.2, we computed the expansion of the volume form up to order 2 using a method suggested in [Sun12] which works for compact Kähler-Einstein cases.In Section 3.2, we recall the harmonic lift constructed by Siu[Siu86] and computed its expansion up to order 2. In Section 3.3, we ensemble the result from previous sections and derive the Wolpert's formula for punctured Riemann surfaces. We also use this formula to get a negative upper bound for the Curvature Tensor for the WP metric.

3.1 Expansion of the Volume Form

For a punctured Riemann surface with hyperbolic metric, to expand the volume form, we follow the method in by [Sun12] section 3 treating compact Kähler manifold with non-flat Kähler-Einstein metric. More computation details are provided.

In K-coordinate ,we have the following notation.

Let $V_0 = ig(z)\mathrm{d}z \wedge \mathrm{d}\bar{z}$ be the volume form of S, $\varphi(t) = \varphi(z,t)\mathrm{d}\bar{z} \otimes \partial_z$ a harmonic Beltrami differential of a Riemann surface S, that is

$$\varphi_z = -\partial_z \log g\varphi.$$

in K-coordinate.

We are slightly abusing the notation of φ, it should be clearly from context if φ means a global section or a local function.

Let $e = \mathrm{d}z + \varphi \mathrm{d}\bar{z}$, $\tilde{V}_t = ige \wedge \bar{e}$. Let $\rho(z,t)$ be the real valued function such that $V_t = \tilde{V}_t e^{\rho}(z,t)$, where V_t is the volume form of S_t. It is easy check $\tilde{V}_t$ and ρ is defined globally.

In S-coordinate (w,s), we have the following notation.

The columns form $V_t = iG\mathrm{d}w \wedge \mathrm{d}\bar{w}$. By KE metric condition, we have Ricci form $-i\partial\bar{\partial}\log G$ is equal to the negative Kählerform $-V_t$, that is

$$(\log G)_{w\bar{w}} = G. \tag{3.1.1}$$

Since it works for any holomorphic coordinate with respect to S_t, we may expect to obtain an equation in K-coordinates, that is independent of terms like w_z. Indeed we have the following theorem.

Theorem 3.1.1. *Let φ and ρ be the setting as above, $T = \partial_z - \bar{\varphi}\partial_{\bar{z}}$, $\bar{T} = \partial_{\bar{z}} - \varphi\partial_z$ then*

$$\log(-\varphi_z \frac{1}{1-\bar{\varphi}\varphi} T\rho + \bar{T}(\frac{1}{1-\bar{\varphi}\varphi} T\rho) + g) = \rho + \log g + \log(1-\bar{\varphi}\varphi). \tag{3.1.2}$$

We provide a proof for how we get this equation, this proof can be generalized to higher dimension by keeping track of the simplification order in this proof.

Proof. The proof is mainly of two parts. The first is to show (3.1.1) with some modification is independent of choice of holomorphic coordinate w of S_t. This part works for any Beltrami differentials. The second part use harmonic Beltrami differential to largely simplify the equation.

Since w is a holomorphic coordinate with respect to φ, we have $\bar{T}w = 0$. Let $a = \frac{\partial w}{\partial z}$, and b by a^{-1} then

$$a_{\bar{z}} = \varphi_z a + \varphi a_z, \tag{3.1.3}$$

$$\bar{T}(b) = -b\varphi_z, \tag{3.1.4}$$

$$a_{z\bar{z}} = a_{zz}\varphi + 2a_z\varphi_z + \varphi_{zz}a, \tag{3.1.5}$$

$$T\log a = b(1-\bar{\varphi}\varphi)a_z - \bar{\varphi}\varphi_z \tag{3.1.6}$$

$$T\log \bar{a} = \bar{\varphi}_{\bar{z}}. \tag{3.1.7}$$

$$\bar{T}(b^2 a_z) = b\varphi_{zz}. \tag{3.1.8}$$

Denote L by $(1-\bar{\varphi}\varphi)^{-1}$ and A by $1-\bar{\varphi}$, we have

$$\partial_w = bLT, \tag{3.1.9}$$

Applying log to both side of (3.1.1), we have

$$\log(\bar{b}\bar{L}\bar{T}[bLT(\rho + \log g - \log a - \log \bar{a})]) = \rho + \log g - \log a - \log \bar{a} \tag{3.1.10}$$

Denote X by $\log(\bar{b}\bar{L}\bar{T}[bLT(\rho+\log g-\log a-\log\bar{a})])+\log a+\log\bar{a}$. We want to show that it is independent of a , the choice of holomorphic coordinate.

Using (3.1.6) and (3.1.7), we have

$$X = \log(\bar{b}\bar{L}\bar{T}[bLT(\rho + \log g) - b^2 a_z + bL\bar{\varphi}\varphi_z - bL\bar{\varphi}_{\bar{z}}]) + \log a + \log \bar{a},$$

Using (3.1.4) and (3.1.8) and let $\log a$ and $\log b$ cancel each other, we have

$$\begin{aligned}
X &= \log(\bar{L}\bar{T}[LT(\rho + \log g) + L\bar{\varphi}\varphi_z + L\bar{\varphi}_{\bar{z}}] - \varphi_z[LT(\rho + \log g) + B\bar{\varphi}\varphi_z - L\bar{\varphi}_{\bar{z}}] - \varphi_{zz}) \\
&= \log \bar{L} + \log(\bar{T}[LT\rho] - \varphi_z[LT\rho] + \bar{T}(LT\log g) + \bar{T}[L(\bar{\varphi}\varphi_z - \bar{\varphi}_{\bar{z}})] \\
&- L[\varphi_z T\log g + \bar{\varphi}\varphi_z^2 - \bar{\varphi}_{\bar{z}}\varphi_z] - \varphi_{zz}),
\end{aligned} \tag{3.1.11}$$

which in independent of w.

Denote $\bar{T}(LT\log g)+\bar{T}[L(\bar{\varphi}\varphi_z-\bar{\varphi}_{\bar{z}})]-L[\varphi_z T\log g+\bar{\varphi}\varphi_z^2-\bar{\varphi}_{\bar{z}}\varphi_z]-\varphi_{zz}$ by Y. Now using the harmonic condition $\varphi_z = -\varphi\log g_z$, we have

$$T\log g = \log g_z + \bar{\varphi}_{\bar{z}}, \tag{3.1.12}$$

$$\bar{T}L = -L^2\bar{T}A, \tag{3.1.13}$$

Y can be simplified as the following,

$$\begin{aligned}
Y &= \bar{T}[L(\log g_z + \bar{\varphi}\varphi_z)] - L[\varphi_z T\log g + \bar{\varphi}\varphi_z^2 - \bar{\varphi}_{\bar{z}}\varphi_z] - \varphi_{zz} \\
&= \bar{T}L(\log g_z + \bar{\varphi}\varphi_z) - L[\varphi_z \log g_z + \bar{\varphi}\varphi_z^2 - \bar{T}(\log g_z + \bar{\varphi}\varphi_z)] - \varphi_{zz} \\
&= \bar{T}L(\log g_z A) - L[\varphi_z \log g_z A - \bar{T}(\log g_z A)] - \varphi_{zz} \\
&= -L(\log g_z \bar{T}A + \varphi_z \log g_z A - \bar{T}(\log g_z A)) - \varphi_{zz} \\
&= \bar{T}\log g_z - \varphi_z \log g_z - \varphi_{zz} \\
&= g.
\end{aligned} \tag{3.1.14}$$

Putting X and Y back to (3.1.10), we get the desired equation.

$\square$

The following proof is modified from [Sun12] using the notation in this article.

Proposition 3.1.2. *Let $\varphi(t)$ is a family of harmonic Beltrami operator with expansion $\varphi(t) = t^i\eta_i$. Then*

$$\begin{aligned} \rho|_{t=0} &= 0, \\ \rho_i|_{t=0} &= 0, \\ \rho_{ik}|_{t=0} &= 0, \\ \rho_{i\bar{j}}|_{t=0} &= D(\eta_i\eta_{\bar{j}}), \end{aligned} \tag{3.1.15}$$

where $D = (\Delta_{\bar{\partial}} + 1)^{-1}$.

Proof. Denote $-\varphi_z\frac{1}{1-\varphi\bar{\varphi}}T\rho + \bar{T}(\frac{1}{1-\bar{\varphi}\varphi}T\rho) + g$ by $B(z,t)$. We have

$$B(z,0) = g, \rho(z,0) = 0, \tag{3.1.16}$$

Noting from Appendix ?? that in Riemann surface case the $\bar{\partial}$- Laplacian $\Delta_{\bar{\partial}}$ acting on function in local coordinate can be written as $\Delta_{\bar{\partial}}f = -g^{-1}f_{z\bar{z}}$ [Bal06] (5.51). Using (3.1.16), we have

$$B_t(0,z) = \rho_{z\bar{z}}(0,z), \tag{3.1.17}$$

Differentiate (3.1.2) and evaluate it at $t = 0$, we have

$$\Delta_{\bar{\partial}}\rho_i(0,z) = -\rho_i(0,z), \tag{3.1.18}$$

Since $\Delta_{\bar{\partial}}$ has no negative eigenvalue. we have $\rho_i(0,z) = 0$ and hence $\partial_i B(0,z) = 0$. Similarly we have $\rho_{\bar{j}}(0,z) = 0$, $\rho_{\bar{t}\bar{t}}(0,z) = 0$, and $\rho_{tt}(0,z) = 0$, and hence $B_{\bar{j}}(0,z) = 0$, $B_{\bar{j}\bar{l}}(0,z) = 0$, and $B_{ik}(0,z) = 0$.

For the $\rho_{z\bar{z}}(0,z)$ case, we still have

$$B_{i\bar{j}}(0,z) = \rho_{i\bar{j}z\bar{z}}(0,z).$$

But differentiate (3.1.2) with respect to t^i and $t^{\bar{j}}$, evaluating at $t=0$, we have

$$\Delta_{\bar{\partial}}\rho_{i\bar{j}}(z,0) = -\rho_{i\bar{j}}(z,0) + \eta_i(z)\eta_{\bar{j}}(z). \tag{3.1.19}$$

□

we may follow this idea further to get the higher order of expansion.

Theorem 3.1.3.

$$\begin{aligned} \rho_{ikm}|_{t=0} &= 0, \\ \rho_{ikmp}|_{t=0} &= 0, \\ \rho_{ik\bar{j}}|_{t=0} &= \sigma_{ik} DP_i D\eta_k \eta_{\bar{j}}, \\ \rho_{ikm\bar{j}}|_{t=0} &= \sigma_{ikm} DP_i DP_k D\eta_m \eta_{\bar{j}}, \\ \rho_{ik\bar{j}\bar{l}}|_{t=0} &= \sigma_{ik}\sigma_{\bar{j}\bar{l}}(DP_i D\bar{P}_{\bar{j}} D\eta_k \eta_{\bar{l}} + D\bar{P}_{\bar{j}} DP_i D\eta_k \eta_{\bar{l}} \\ &\quad + DQ_{i\bar{j}} D\eta_k \eta_{\bar{l}} + D\bar{Q}_{\bar{j}i} D\eta_k \eta_{\bar{l}}), \end{aligned} \tag{3.1.20}$$

where $Q_{i\bar{j}}(f) = -g^{-1}((\eta_i\eta_{\bar{j}})_{\bar{z}} f_z)$.

Proof.

$$\begin{aligned} B &= \rho_{z\bar{z}} - L(\varphi\rho_{zz} + \bar{\varphi}\rho_{\bar{z}\bar{z}}) \\ &\quad + L^2(-\varphi_z + \bar{\varphi}_{\bar{z}}\varphi + \bar{\varphi}\varphi_{\bar{z}} - \varphi^2\bar{\varphi}_z)\rho_z \\ &\quad + L^2(-\bar{\varphi}_{\bar{z}} + \varphi_z\bar{\varphi} + \varphi\bar{\varphi}_z - \bar{\varphi}^2\varphi_{\bar{z}})\rho_{\bar{z}} \end{aligned} \tag{3.1.21}$$

Note that

$$\rho(z,0) = 0, \rho_t(z,0) = 0.$$

Based on the inductive assumption that $\rho_{i_1,\ldots,i_{n-1}}(z,0) = 0$, we have

$$B_{i_1,\ldots,i_n}(z,0) = \rho_{z\bar{z}i_1,\ldots,i_n}(z,0)$$

Use (3.1.2), we have $(1+\Delta_{\bar{\partial}})\rho_{i_1,\ldots,i_n}(z,0) = 0$ and hence $\rho_{i_1,\ldots,i_n}(z,0) = 0$.

A direct computation shows that

$$B_{ik\bar{j}}(z,0) = \sigma_{ik}(-\eta_{iz}\rho_{zk\bar{j}}(z,0) - \eta_i\rho_{zzk\bar{j}}(z,0)) + \rho_{ik\bar{j}z\bar{z}}(z,0).$$

Denote a local operator $P_i(f) = -g^{-1}(\eta_i f_z)_z$, then we have

$$B_{ik\bar{j}} = g(\sigma_{ik} P_{\eta_i} \rho_{k\bar{j}}(z,0) - \Delta_{\bar{\partial}} \rho_{ik\bar{j}}(z,0)).$$

Use (3.1.2), we have

$$\rho_{ik\bar{j}}(z,0) = \sigma_{ik} D P_i D \eta_k \eta_{\bar{j}}$$

For expansion of order 4, we just need to consider the case $\rho_{ikm\bar{j}}(z,0)$ and $\rho_{ik\bar{j}\bar{l}}(z,0)$. As the process of taking the derivative become more and more complicated.

We denote operator of evaluating at $t = 0$ by ι_0. The following obvious formula is used repeatedly. For smooth function $A(z,t)$, let

$$\mathcal{N}(A) = \{\alpha \in \cup_{k=0}^{+\infty} \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}^k | \iota_0 A_\alpha = 0\}.$$

Then for smooth function $A(z,t), B(z,t)$, we have

$$\iota_0(AB)_\gamma = \sum_{\substack{(\alpha,\beta)\in\mathcal{P}(\gamma);\\ \alpha\notin\mathcal{N}(A),\beta\notin\mathcal{N}(B)}} \iota_0 A_\alpha \cdot \iota_0 B_\beta, \tag{3.1.22}$$

where $\mathcal{P}(\gamma)$ is the set of 2-partition of γ

For case of $\iota_0 \rho_{ikm\bar{j}}$, (3.1.2) yields

$$g^{-1} \iota_0 B_{ikm\bar{j}} = \iota_0 \rho_{ikm\bar{j}}. \tag{3.1.23}$$

From (3.1.21) and definition $\varphi(t) = \eta_i t^i$,

$$\begin{aligned}
\iota_0 B_{ikm\bar{j}} &= \iota_0 \rho_{z\bar{z}ikm\bar{j}} - \iota_0 L \iota_0 (\varphi \rho_{zz} + \bar{\varphi} \rho_{\bar{z}\bar{z}})_{ikm\bar{j}} \\
&\quad + \iota_0 L^2 \iota_0 ((-\varphi_z + \bar{\varphi}_{\bar{z}} \varphi + \bar{\varphi} \varphi_{\bar{z}} - \varphi^2 \bar{\varphi}_z) \rho_z)_{ikm\bar{j}} \\
&\quad + \iota_0 L^2 \iota_0 (-\bar{\varphi}_{\bar{z}} + \varphi_z \bar{\varphi} + \varphi \bar{\varphi}_z - \bar{\varphi}^2 \varphi_{\bar{z}}) \rho_{\bar{z}})_{ikm\bar{j}} \\
&= \iota_0 \rho_{z\bar{z}ikm\bar{j}} - \iota_0 (\varphi \rho_{zz})_{ikm\bar{j}} + \iota_0 (-\varphi_z \rho_z)_{ikm\bar{j}} \\
&= g(-\Delta_{\bar{\partial}} \iota_0 \rho_{ikm\bar{j}} + \iota_0 (P_\varphi(\rho))_{ikm\bar{j}}) \\
&= g(-\Delta_{\bar{\partial}} \iota_0 \rho_{ikm\bar{j}} + \sigma_{i,km} P_i \iota_0 \rho_{km\bar{j}}).
\end{aligned} \tag{3.1.24}$$

Combining (3.1.23) and (3.1.24) gives

$$\begin{aligned}
\iota_0 \rho_{ikm\bar{j}} &= \sigma_{i,km} D P_i (\iota_0 \rho_{km\bar{j}}) \\
&= \sigma_{ikm} D P_i D P_k D \eta_m \eta_{\bar{j}}
\end{aligned} \tag{3.1.25}$$

The computation above can be applied for α containing only one conjugate index. If $\alpha = (\beta, \bar\gamma)$ and $\bar\gamma \in \{\bar 1, \ldots, \bar n\}$ and $\beta \in \cup_{k=1}^{+\infty}\{1, \ldots, n\}^k$, then

$$\iota_0\rho_\alpha = \sigma_\alpha(\Pi_{k\in\alpha-i}DP_k)D\eta_i\eta_{\bar\gamma}.$$

For the case of $\iota_0\rho_{ik\bar j\bar l}$, (3.1.2) yields

$$g^{-1}\iota_0 B_{ik\bar j\bar l} = \iota_0\rho_{ik\bar j\bar l} - \eta_i\eta_{\bar j}\eta_k\eta_{\bar l}. \tag{3.1.26}$$

$$\begin{aligned}
\iota_0 B_{ik\bar j\bar l} &= \iota_0\rho_{z\bar z ik\bar j\bar l} \\
&\quad - \iota_0 L\iota_0(\varphi\rho_{zz} + \bar\varphi\rho_{\bar z\bar z})_{ik\bar j\bar l} - \sigma_{ik}\sigma_{\bar j\bar l}\iota_0(L)_{i\bar j}\iota_0(\varphi\rho_{zz} + \bar\varphi\rho_{\bar z\bar z})_{k\bar l} \\
&\quad + \iota_0 L^2\iota_0((-\varphi_z + \bar\varphi_{\bar z}\varphi + \bar\varphi\varphi_{\bar z} - \varphi^2\bar\varphi_z)\rho_z)_{ik\bar j\bar l} \\
&\quad + \iota_0 L^2\iota_0(-\bar\varphi_{\bar z} + \varphi_z\bar\varphi + \varphi\bar\varphi_z - \bar\varphi^2\varphi_{\bar z})\rho_{\bar z})_{ik\bar j\bar l} \\
&= \iota_0\rho_{z\bar z ik\bar j\bar l} - \iota_0((\varphi\rho_z)_z + (\bar\varphi\rho_{\bar z})_{\bar z})_{ik\bar j\bar l} \\
&\quad + \iota_0((\bar\varphi_{\bar z}\varphi + \bar\varphi\varphi_{\bar z})\rho_z)_{ik\bar j\bar l} + \iota_0((\varphi_z\bar\varphi + \varphi\bar\varphi_z)\rho_{\bar z})_{ik\bar j\bar l} \\
&= \iota_0\rho_{z\bar z ik\bar j\bar l} - \iota_0((\varphi\rho_z)_z + (\bar\varphi\rho_{\bar z})_{\bar z})_{ik\bar j\bar l} \\
&\quad + \sigma_{ik}\sigma_{\bar j\bar l}(\eta_i\eta_{\bar j})_{\bar z}\iota_0\rho_{zk\bar l} + \sigma_{ik}\sigma_{\bar j\bar l}(\eta_i\eta_{\bar j})_z\iota_0\rho_{\bar z k\bar l}
\end{aligned} \tag{3.1.27}$$

Introduce a new local operator $Q_{i\bar j}(f) = -g^{-1}((\eta_i\eta_{\bar j})_{\bar z} f_z)$

$$\iota_0 B_{ik\bar j\bar l} = g(-\Delta_{\bar\partial}\iota_0\rho_{ik\bar j\bar l} + \sigma_{ik}P_i\iota_0\rho_{k\bar j\bar l} + \sigma_{\bar j\bar l}\bar P_{\bar j}\iota_0\rho_{ik\bar l} + \sigma_{ik}\sigma_{\bar j\bar l}(Q_{i\bar j}\rho_{k\bar l} + \bar Q_{\bar j i})\rho_{k\bar l}). \tag{3.1.28}$$

Combining (3.1.27) and (3.1.28) gives

$$\iota_0\rho_{ik\bar j\bar l} = \sigma_{ik}\sigma_{\bar j\bar l}(DP_iD\bar P_{\bar j}D\eta_k\eta_{\bar l} + D\bar P_{\bar j}DP_iD\eta_k\eta_{\bar l} + DQ_{i\bar j}D\eta_k\eta_{\bar l} + D\bar Q_{\bar j i}D\eta_k\eta_{\bar l}). \tag{3.1.29}$$

□

We summarize the expansion up to order 2 for later use in this Chapter.

Theorem 3.1.4. *Let $\varphi(t)$ and $\rho(t)$ be the setting as above,*

$$\rho(t) = D(\eta_i\eta_{\bar j})t^i\bar t^{\bar j} + O(|t|^3),$$

where $\eta_i\eta_{\bar j}$ is a globally defined continuous function and hence

$$V_t = (1 - \Delta_{\bar\partial}D(\eta_i\eta_{\bar j})t^i\bar t^{\bar j} + O(|t|^3))V_0.$$

3.2 Expansion of the Harmonic Lift

In Definition (2.2.14), we define the KS map of a point $b \in \mathcal{T}$ by choosing a basis $\{\eta_i\}$ of $\mathcal{H}^{0,1}_{(2)}(S_b, T^{1,0}S_b)$ and finding the corresponding holomorphic normal coordinate (t, U) and we have by definition

$$\mathcal{K}(\partial_i)|_{t=0} = \eta_i.$$

However, for $t_0 \neq 0$, to compute $\mathcal{K}(\partial_i)|_{t=t_0}$ is not simple, as $(t - t_0, U)$ is not a normal holomorphic coordinate at point t_0.

In [Siu86], Siu proposed a method to construct a vector field over S-coordinate, which is also mentioned in [Sch93] and [LSY09].

Definition 3.2.1 (Siu)**.** Let $(\mathcal{M}, B, \pi)$ be an analytic family of Kähler manifolds, (w, s) a holomorphic coordinate of $\mathcal{M}$ and t a holomorphic coordinate of (B) with $s = \pi^* t$. For the local holomorphic vector fields $\partial_{t^1}, \ldots, \partial_{t^n}$,a set of smooth vector fields $v_1, \ldots, v_n$ on $\pi^{-1}(B)$ are called a harmonic lift if it satisfies the following.

(1) $\pi(v_i) = \partial_{t^i}$.

(2) $\bar\partial_F v_i \in \mathcal{H}^{0,1}_{(2)}(M_t, T^{1,0}M_t)$.

Siu proved the existence of harmonic lift for the case of closed Kähler manifolds with KE metric and negative Chen class of canonical bundle. Schumacher gives a construction which could be applied to the punctured Riemann surfaces, as being harmonic is a local condition.

Theorem 3.2.2. *Let $(\mathcal{S}, B, \pi)$ be an analytic family of punctured Riemann surfaces equipped with constant negative curvature -1. Let (w, s) be a local homomorphic coordinate of $\mathcal{S}$, with $s = \pi^* t$, holomorphic coordinate of B. Let Kähler metric be $iG\mathrm{d}w \wedge \mathrm{d}\bar w$, then*

$$v_i = \partial_{s_i} + (-G^{-1}\partial_{s^i \bar w} \log G)\partial_w$$

is a harmonic lift of ∂_{t^i}. Furthermore, its $\bar\partial_F$ image

$$\bar\partial_F(v_i) = -(G^{-1}\partial_{s^i}(\log G)_{\bar w})_{\bar w} \mathrm{d}\bar w \otimes \partial_w$$

is coincides with $\mathcal{K}(\partial_i)$ for any $t \in B$.

Our target is to find an expression of $\mathcal{K}(\partial_i)$ in K coordinate so that we could combine the result for the volume form to compute $h_{\bar{i}\bar{j}}$ in K coordinate. The expression above in S coordinate gives us a good staring point.

Theorem 3.2.3. *Denote*

$$A_{\bar{i}} = -G(G^{-1}\partial_{s^{\bar{i}}}(\log G)_w)_w c^2,$$

where $c(z,t) = \frac{\partial w}{\partial z}$*, we have* $A_{\bar{i}}$ *is independent the choice of S-coordinate* (w,s) *and*

$$\begin{aligned}
\iota_0 A_{\bar{i}} &= g\eta_{\bar{i}}, \\
\iota_0 \partial_{\bar{k}} A_{\bar{i}} &= 0, \\
\iota_0 \partial_l \log A_{\bar{i}} &= -g(g^{-1}\rho_{z\bar{i}l})_z, \\
\iota_0 \partial_{\bar{k}l} A_{\bar{i}} &= -g(g^{-1}\iota_0\rho_{z\bar{i}\bar{k}l})_z + \sigma_{ik} g\eta_{\bar{k}}\iota_0\rho_{\bar{i}j}.
\end{aligned}$$

The definition of $A_{\bar{i}}$ might seem confusing. If we use the notation to be define in Chapter 4, then

$$\bar{\partial}_F(v_i) = h_t \circ S_t \circ \sigma_{\varphi}^2(iA_{\bar{i}}\mathrm{d}z^2).$$

$A_{\bar{i}}$ is also a building block in calculating expansion of $h_{\bar{i}j}$. To illustrate this and also prepare for the proof of Theorem 3.2.3. We introduce and summarize some notation and relation about KS coordinates.

$$\begin{aligned}
T &= \partial_z - \varphi\partial_{\bar{z}}, \\
L &= (1-\varphi\bar{\varphi})^{-1}, \\
\partial_w &= c^{-1}LT, \\
\mathrm{d}w &= c(\mathrm{d}z + \bar{\varphi}\mathrm{d}\bar{z}) + w_i \mathrm{d}t^i, \\
\partial_{s^i} &= \partial_i - w_i\partial_w, \\
G &= |c|^{-2} g e^{\rho}, \\
V_t &= iG\mathrm{d}w\mathrm{d}\bar{w}.
\end{aligned}$$

And we have

$$h_{\bar{i}j}(t) = \int_{S_0} A_{\bar{i}} A_j |c|^{-4} G^{-2} V_t = \int_{S_0} A_{\bar{i}} A_j e^{-\rho}(1 - \varphi\bar{\varphi}) V_0$$

as well as some identities

$$G = (\log G)_{w\bar{w}}, \tag{3.2.1}$$

$$[\partial_{\bar{i}}, \partial_w] = -\bar{c}c^{-1} L\eta_{\bar{i}}\partial_{\bar{w}}, \tag{3.2.2}$$

$$[\partial_j, \partial_w] = (L\bar{\varphi}\eta_j - \partial_j \log c)\partial_w. \tag{3.2.3}$$

Applying (3.2.2), we have

$$\begin{aligned} A_{\bar{i}} &= G(-G^{-1}\partial_{\bar{i}}(\log G)_w + \bar{w}_{\bar{i}})_w c^2 \\ &= (\partial_w \log G \partial_{\bar{i}} \partial_w \log G - \partial_w \partial_{\bar{i}} \partial_w \log G)c^2 + g e^{\rho} L\eta_{\bar{i}}. \end{aligned} \tag{3.2.4}$$

In order to get expansion of $A_{\bar{i}}$ at $t = 0$, one just to get expansion of $\partial_w \log G$.

Let $f(z,t)$ be a smooth function in K coordinate. Denote the expansion of f of order (p,q) by $\iota_0\partial_\alpha\partial_{\bar{\beta}} f$, where $\alpha \in \{1,\dots,n\}^p$ and $\beta \in \{1,\dots,n\}^q$. We use the following scheme to simplify the commutation for function of form $\partial_w f$.

(1) If $p \neq 0$, use (3.2.3) to reduce recursively to the expansion of order $(0,q)$

$$\iota_0\partial_\alpha\partial_{\bar{\beta}} f = \iota_0\partial_{\alpha-j}\partial_{\bar{\beta}}\partial_w\partial_j f + \iota_0\partial_{\alpha-j}\partial_{\bar{\beta}}((L\bar{\varphi}\eta_j - \partial_j \log c)\partial_w f).$$

(2) If $p = 0$ and $q \neq 0$, use (3.2.2) to reduce recursively to the expansion of order $(0,0)$.

$$\iota_0\partial_\alpha\partial_{\bar{\beta}} f = \iota_0\partial_\alpha\partial_{\bar{\beta}-\bar{i}}\partial_w\partial_{\bar{i}} f - \iota_0\partial_\alpha\partial_{\bar{\beta}-\bar{i}}(\bar{c}c^{-1}L\eta_{\bar{i}}\partial_w f).$$

So to compute the expansion of $h_{\bar{i}j}$ up tp order (1,1), one just need to compute the expansion of $A_{\bar{i}}$ up tp order (1,1), and hence the expansion of $\partial_w \log G$ of order up to $(1,2)$.

Direct computation using the scheme gives the following

$$
\begin{aligned}
\iota_0\partial_{\bar{i}}\partial_w &= c^{-1}(\partial_{z\bar{i}} - \eta_{\bar{i}}\partial_{\bar{z}}),\\
\iota_0\partial_j\partial_w &= c^{-1}(\partial_{zj} - \partial_j \log c\partial_z),\\
\iota_0\partial_{jl}\partial_w &= c^{-1}(\partial_{zjl} - \sigma_{jl}\partial_l \log c\partial_{zj} - \partial_{jl}\log c\partial_z + \partial_j \log c\partial_l \log c\partial_z)\\
\iota_0\partial_{\bar{i}j}\partial_w &= \iota_0\partial_w\partial_{\bar{i}} - \partial_j \log c\iota_0\partial_{\bar{i}}\partial_w + \eta_{\bar{i}}\eta_j\iota_0\partial_w,\\
\iota_0\partial_{\bar{i}\bar{k}}\partial_w &= c^{-1}(\partial_{z\bar{i}\bar{k}} - \sigma_{ik}\eta_{\bar{i}}\partial_{\bar{z}\bar{k}} + \eta_{\bar{i}}\eta_{\bar{k}}\partial_z),\\
\iota_0\partial_{\bar{i}\bar{k}j}\partial_w &= \iota_0\partial_{\bar{i}\bar{k}}\partial_w\partial_j - \partial_j \log c\iota_0\partial_{\bar{i}\bar{k}}\partial_w + \sigma_{ik}\eta_{\bar{i}}\eta_j\iota_0\partial_{\bar{k}}\partial_w.
\end{aligned}
$$

Acting on $\partial_w(\log G)$, we have

$$
\begin{aligned}
\iota_0\partial_{\bar{i}}\partial_w \log G &= 0,\\
\iota_0\partial_j\partial_w \log G &= c^{-2}c_{jz} - c^{-2}c_j\partial_z \log g,\\
\iota_0\partial_{jl}\partial_w \log G &= 0!,\\
\iota_0\partial_{\bar{i}j}\partial_w \log G &= c^{-1}\iota_0\rho_{z\bar{i}j},\\
\iota_0\partial_{\bar{i}\bar{k}}\partial_w \log G &= 0,\\
\iota_0\partial_{\bar{i}\bar{k}j}\partial_w \log G &= c^{-1}\iota_0\rho_{z\bar{i}\bar{k}j}.
\end{aligned}
$$

Considering (3.2.4), we get

$$
\begin{aligned}
\iota_0 A_{\bar{i}} &= g\eta_{\bar{i}},\\
\iota_0\partial_{\bar{k}}A_{\bar{i}} &= 0,\\
\iota_0\partial_l A_{\bar{i}} &= -g(g^{-1}\rho_{z\bar{i}l})_z,\\
\iota_0\partial_{\bar{k}l}A_{\bar{i}} &= -g(g^{-1}\iota_0\rho_{z\bar{i}\bar{k}l})_z + \sigma_{ik}g\eta_{\bar{k}}\iota_0\rho_{\bar{i}l}.
\end{aligned}
$$

We further use the notation $f_{\bar{i}j}(t) = \int_{S_0} A_{\bar{i}}A_j g^{-2}V_0$ and get

$$
\begin{aligned}
\iota_0 f_{\bar{i}j}(t) &= \int_{S_0} \eta_{\bar{i}}\eta_j V_0,\\
\iota_0\partial_{\bar{k}} f_{\bar{i}j}(t) &= 0,\\
\iota_0\partial_l f_{\bar{i}j}(t) &= 0.
\end{aligned}
$$

For expansion of order (1,1), we have

$$\begin{aligned}\iota_0\partial_{\bar{k}l}f_{\bar{i}j}(t) &= \int_{S_0}(\iota_0\partial_{\bar{k}l}A_{\bar{i}}\iota_0A_j + \iota_0\partial_l A_{\bar{i}}\iota_0\partial_{\bar{k}}A_j + \iota_0A_{\bar{i}}\iota_0\partial_{\bar{k}l}A_j)g^{-2}V_0 \\ &= -\int_{S_0}(g^{-1}\iota_0\rho_{z\bar{i}\bar{k}l})_z\eta_jV_0 \\ &+\int_{S_0}\sigma_{ik}\eta_{\bar{k}}\iota_0\rho_{\bar{i}l}\eta_jV_0 \\ &+\int_{S_0}(g^{-1}\iota_0\rho_{z\bar{i}l})_z(g^{-1}\iota_0\rho_{\bar{z}\bar{k}j})_{\bar{z}}V_0 \\ &-\int_{S_0}(g^{-1}\iota_0\rho_{\bar{z}j\bar{k}l})_{\bar{z}}\eta_{\bar{i}}V_0 \\ &+\int_{S_0}\sigma_{jl}\eta_l\iota_0\rho_{j\bar{k}}\eta_{\bar{i}}V_0.\end{aligned} \tag{3.2.5}$$

The first and fourth integration is 0 since $(g\eta)_z = 0$. The third integration is simplified below.

$$\begin{aligned}&\int_{S_0}(g^{-1}\iota_0\rho_{z\bar{i}l})_z(g^{-1}\iota_0\rho_{\bar{z}\bar{k}j})_{\bar{z}}V_0 \\ =&\int_{S_0}(g^{-1}\iota_0\rho_{z\bar{i}l})_z(-\partial_{\bar{z}}\log g\iota_0\rho_{\bar{z}\bar{k}j}+\rho_{\bar{z}\bar{z}\bar{k}j})i\mathrm{d}z\mathrm{d}\bar{z} \\ =&\int_{S_0}(g^{-1}\iota_0\rho_{z\bar{i}l})_z(-\partial_{\bar{z}}\log g\iota_0\rho_{\bar{z}\bar{k}j}+\rho_{\bar{z}\bar{z}\bar{k}j})i\mathrm{d}z\mathrm{d}\bar{z} \\ =&\int_{S_0}\iota_0\rho_{z\bar{i}l}\iota_0\rho_{\bar{z}\bar{k}j}+g^{-1}\iota_0\rho_{z\bar{i}l}(\partial_{\bar{z}}\log g\iota_0\rho_{\bar{z}z\bar{k}j})+(g^{-1}\iota_0\rho_{z\bar{i}l})_{\bar{z}}\rho_{\bar{z}z\bar{k}j}i\mathrm{d}z\mathrm{d}\bar{z} \\ =&\int_{S_0}-\iota_0\rho_{\bar{i}l}\rho_{z\bar{z}\bar{k}j}+g^{-1}\iota_0\rho_{z\bar{z}\bar{i}l}\rho_{\bar{z}z\bar{k}j}i\mathrm{d}z\mathrm{d}\bar{z} \\ =&\int_{S_0}\iota_0(D^{-1}-1)\rho_{\bar{i}l}\rho_{\bar{k}j}+\iota_0(D^{-1}-1)\rho_{\bar{i}l}(D^{-1}-1)\rho_{\bar{k}j}V_0 \\ =&\int_{S_0}\eta_{\bar{i}}\eta_j\eta_{\bar{k}}\eta_l-\eta_{\bar{k}}\eta_jD\eta_{\bar{i}}\eta_lV_0.\end{aligned} \tag{3.2.6}$$

(3.2.5) reduced to

$$\iota_0\partial_{\bar{k}l}f_{\bar{i}j}(t) = \int_{S_0}\eta_{\bar{i}}\eta_l\eta_{\bar{k}}\eta_j - \eta_{\bar{k}}\eta_jD\eta_{\bar{i}}\eta_l + 2\sigma_{ik}\eta_{\bar{i}}\eta_jD\eta_{\bar{k}}\eta_lV_0. \tag{3.2.7}$$

Expansion of $h_{\bar{i}j}$ is obtained bt the expansion of $f_{\bar{i}j}$ and ρ.

Theorem 3.2.4. *Let $\varphi(t) = t^i\eta_i \in A^{0,1}_{(2)}(S, T^{1,0}S)$ be a harmonic Beltrami differential. The Hodge metric with respect to frame $\{k_t\mathcal{K}_t(\partial_i)\}$ coincides with the WP metric with* *respect to frame $\{\partial_i\}$ and have an explicit formula for any order. For order up to 2 is given as below.*

$$
\begin{aligned}
\iota_0 h_{\bar{i}j}(t) &= \int_{S_0} \eta_{\bar{i}}\eta_j V_0,\\
\iota_0 \partial_{\bar{k}} h_{\bar{i}j}(t) &= 0,\\
\iota_0 \partial_l h_{\bar{i}j}(t) &= 0,\\
\iota_0 \partial_{\bar{k}l} h_{\bar{i}j}(t) &= \int_{S_0} \sigma_{ik}\eta_{\bar{i}}\eta_j D\eta_{\bar{k}}\eta_l V_0.
\end{aligned}
$$

Using the convention in [Bal06], $R^W_{i\bar{j}k\bar{l}} = -R^W_{i\bar{j}k}\,h_{p\bar{l}}$ and

$$R^{WP}_{i\bar{j}k\bar{l}}(t) = \partial_{i\bar{j}}h_{k\bar{l}} - \partial_i h_{k\bar{p}}\partial_{\bar{j}}h_{q\bar{l}}h^{\bar{p}q}.$$

The curvature tensor formula for WP metric at $t = 0$ is

$$R^W_{i\bar{j}k\bar{l}}(0) = \int_{S_0} \sigma_{ik}\eta_{\bar{j}}\eta_i D\eta_{\bar{l}}\eta_k V_0.$$

This formula coincides in format with the Wolpert's one for moduli of closed Riemann surface.

3.3 Properties for Weil-Petersson Curvature Tensor

It is well known that the WP metric has multiple negative curvature properties. WP metric has negative scalar curvature, Ricci curvature,holomorphic sectional curvature and non-positive holomorphic bisectional curvature, dual-Nakano-negative and semi Nakano-negative. The optimal upper bound for the holomorphic sectional curvature is conjectured by Royden[Roy74] and proved by Wolpert and Tromba. The same things holds for the case of punctured Riemann surfaces. We provide the proof for the optimal upper bound which slightly differ from Wolpert's result since the number of punctured points matters.

Theorem 3.3.1. *For the WP metric of $\mathcal{T}(S)$ of punctured Riemann surfaces with genus g and k points punctured.*

(1) The holomorphic sectional curvature and the Ricci curvature is bounded above by

$$\frac{-1}{\pi(2g-2+k)}.$$

(2)The scalar curvature is bounded by

$$\frac{-(3g-3+k)(3g-2+k)}{2\pi(2g-2+k)}.$$

Proof. For any point $b \in \mathcal{T}$, choose a unitary basis $\{\eta_i\}$ of $\mathcal{H}^{0,1}_{(2)}(S_b, T^{1,0}S_b)$. The holomorphic sectional curvature of direction ∂_i is

$$-R^W_{i\bar{i}i\bar{i}} = -2\int_{S_b} |\eta_i|^2 D|\eta_i|^2 V_0.$$

As D is a self-adjoint compact operator, its has countable eigenvalues $\lambda_0, \lambda_1, \ldots 0$ with $\lambda_0 > \lambda_1 > \ldots$ and $\lim_{n\to\infty} \lambda_n = 0$. $\lambda_0 \leq 1$ as $D = (1+\Delta_{\bar{\partial}})^{-1}$. Let $|\eta_i|^2 = \sum_{k=0}^{\infty} \psi_{k,i}$, where $\psi_{k,i}$ is the eigenfunction for λ_k. By Lemma 3.3.2, all L^2 harmonic functions over S_b are constants. So $\psi_{0,i}$ is a constant function. We have

$$-2\int_{S_b} |\eta_i|^2 D|\eta_i|^2 V_0 = -2\sum_{k=0}^{\infty}\int_{S_b} |\psi_{k,i}|^2 \lambda_i V_0 \leq -2\int_{S_b} |\psi_{0,i}|^2 V_0.$$

Since the Poincaré dual of η_i is could be regard as a meromorphic section of $2K$ with at most k poles. η_i must have zero by Riemann-Roch theorem. Thus the inequality here is strict. Since $\int_{S_b} \psi_{0,i} V_0 = 1$, we have

$$\psi_{0,i} = \frac{1}{Area(S_b)} = \frac{1}{2\pi(2g+k-2)}.$$

For the Ricci curvature, we note that the Ricci curvature of direction ∂_i is

$$Ric(\partial_i, \partial_i) = -\sum_{j=1}^{3g-3+k} R^W_{i\bar{j}j\bar{i}}$$

. Since $R^W_{i\bar{j}j\bar{i}}$ is negative, we have $Ric(\partial_i, \partial_i) \leq R^W_{i\bar{i}i\bar{i}} < \frac{-1}{\pi(2g-2+k)}$.

For the scalar curvature, we have

$$\begin{aligned}
Scal &= \sum_{i=1}^{3g-3+k} Ric(\partial_i, \partial_i) \\
&= -\sum_{i=1}^{3g-3+k}\sum_{j=1}^{3g-3+k}\int_{S_b} \eta_{\bar{j}}\eta_i D\eta_j\eta_{\bar{i}} + |\eta_i|^2 D|\eta_j|^2 V_0 \\
&\leq -\sum_{i=1}^{3g-3+k}\sum_{j=1}^{3g-3+k}\int_{S_b} |\eta_i|^2 D|\eta_j|^2 V_0 - \sum_{i=1}^{3g-3+k} |\eta_i|^2 D|\eta_i|^2 V_0 \\
&\leq -\sum_{i=1}^{3g-3+k}\sum_{j=1}^{3g-3+k}\int_{S_b} \psi_{0,i}\psi_{0,j} V_0 - \sum_{i=1}^{3g-3+k} \psi_{0,i}^2 V_0 \\
&\leq \frac{-(3g-3+k)(3g-2+k)}{2\pi(2g-2+k)}.
\end{aligned}$$

For the same reason, the second last inequality is strict.

□

Lemma 3.3.2. *Let S be a Riemann surface wit k points punctured then*

$$\dim H^{0,0}_{(2)}(S, \mathbb{C}) = 1$$

and hence $\dim \mathcal{H}^{0,0}(S, \mathbb{C}) = 1$.

Proof. Note that $\ker \bar{\partial}^{0,0}_m = \ker \bar{\partial}^{0,0}$. So the element f in $H^{0,0}_{(2)}(S, \mathbb{C})$ is a global holomorphic function with finite L^2 norm.

Let $\bar{S}$ be the closed Riemann surface such that $S = \bar{S} - \{p_1, \ldots, p_k\}$. A small neighborhood U of a punctured point p is called a cusp neighborhood. We have a coordinate z such that $z(p) = 0$ and $g = \frac{|\mathrm{d}z|^2}{(|z|ln|z|)^2}$. Finite L^2 norm yields

$$\int_U |f(z)|^2 ig\mathrm{d}z \wedge \mathrm{d}\bar{z} < \infty.$$

Thus $f(z) = O(1)$ near 0. f extends to a holomorphic function on $\bar{S}$ and thus a constant. By Lemma 2.1.9, Kählerpackage holds. So $H^{0,0}_{(2)}(S, \mathbb{C}) \cong \mathcal{H}^{0,0}(S, \mathbb{C}) = 1$. □

CHAPTER 4

Extension Formula for Pluricanonical Form and a Curvature Formula for Ricci Metric

In this chapter, we construct an extension of pluricanonical form and compare the Hoghe metric under the frame of extended pluricanonical forms. As suggested in [Sun12], one could derive Wolpert's formula using the expansion of Hodge metric, which can be derived from the expansion of volume form and the extension of pluricanonical form. However, this method could not to be directly related to the curvature formula for WP metric. This subtlety is due to the pluricanonical form we have constructed is not identical to the musical isomorphism of the harmonic lift defined in Section 3.2.

In Section 4.1, we construct the extension of pluricanonical form for a deformation of punctured Riemann surfaces. In section 4.2, we discussed the relation between harmonic lift and the extension of pluricanonical form and gives an asymptotic formula.

4.1 Extension of Pluricanonical Form

Let S be a punctured Riemann surface, φ a Beltrami differential of S, S_φ the corresponding Riemann surface, then the map

$$\sigma_\varphi^m : A^0(S, K^m) \to A^0(S_\varphi, K_\varphi^m)$$

defined by

$$(\mathrm{d}z)^m \mapsto ((\mathrm{d}z + i_\varphi \mathrm{d}z)^m$$

is an isomorphism. The description of holomorphic pluri-canonical form can be stated as the following proposition.

Proposition 4.1.1. *Let S be a Riemann surface of finite type with the hyperbolic metric, φ a harmonic Beltrami differential of S , $s \in A^{0,0}(S, K^m)$ a smooth pluricanonical form. Then $\sigma_\varphi^m(s)$ is holomorphic on S_φ if and only if*

$$\bar\partial s = i_\varphi \nabla^{1,0} s. \tag{4.1.1}$$

Proof. Let $s = f\mathrm{d}z^m$, then

$$\nabla^{1,0} s = \mathrm{d}f - mf\partial_z \log g \mathrm{d}z \otimes \mathrm{d}z^m,$$
$$i_\varphi \nabla^{1,0} s = \varphi(f_z - mf\partial_z \log g)\mathrm{d}\bar z \otimes \mathrm{d}z^m,$$

Note $\sigma_\varphi^m(s) = fc^{-m}\mathrm{d}w^m$. $\sigma_\varphi^m(s)$ being holomorphic on S_φ is equivalent to $\partial_{\bar w}(fc^{-m}) = 0$,

$$\begin{aligned}\partial_{\bar w}(fc^{-m}) &= \bar c^{-1} c^{m+1} L(c\bar T f - mf\bar T c)\\ &= \bar c^{-1} c^{-m-1} L(c\bar T f - mf\bar T c)\\ &= \bar c^{-1} c^{-m} L(\partial_{\bar z} f - \varphi\partial_z f + mf\partial_z\varphi)\\ &= \bar c^{-1} c^{-m} L(\partial_{\bar z} f - \varphi\partial_z f - mf\varphi\partial_z \log g),\end{aligned} \tag{4.1.2}$$

which is equivalent to (4.1.1).

□

To solve (4.1.1) for $s \in A^{0,0}_{(2)}(S, K^m)$. First note that $\nabla^{1,0}s$ has finite L_2 norm. In fact, we have

$$\begin{aligned}(\nabla^{1,0}s, \nabla^{1,0}s) &= (\nabla^{1,0*}\nabla^{1,0}s, s)\\ &= (\Delta_{\bar\partial} s, s) + ([iR, \wedge]s, s)\\ &= (\bar\partial s, \bar\partial s) - m(s,s)\\ &\le C(s,s).\end{aligned}$$

The 1st equation follows from the Bochner-Kodaira-Nakano identity and the 2nd equation is because of the constant scalar curvature $[iR, \wedge]$ is $-m$. The 1st inequality is because of $\bar\partial$ has closed range.

We apply $\bar\partial^* G$ to the both side of (4.1.1). Using the condition $\bar\partial^*\varphi = 0$, we get

$$(1 - \bar\partial^* G i_\varphi \nabla^{1,0})s = Hs,$$

where Hs is the harmonic part of s. Denote $\bar{\partial}^* G i_\varphi \nabla^{1,0}$ by T_φ, since $\|T_\varphi\|_2 < 1$ when $\|\varphi\|_\infty$ small enough. we have

$$s = (1 - T_\varphi)^{-1} Hs,$$

the solution is determined by is harmonic part.

Conversely, let $s_0 \in A^{0,0}_{(2)}(M, K^m)$ be harmonic, then $s = (1 - T_\varphi)^{-1} s_0$ is a solution to (4.1.1). In fact, we have $\bar{\partial} s = H^\perp(i_\varphi \nabla^{1,0} s)$. For $m = 1$, $i_\varphi \nabla^{1,0} s = \partial(i_\varphi s)$. For $m > 2$, $i_\varphi \nabla^{1,0} s$ lives in $A^{0,1}_{(2)}(M, K^m)$. Since $H^{0,1}_{(2)}(S, K^m) = 0$ by Proposition 1.2, $i_\varphi \nabla^{1,0} s = H^\perp(i_\varphi \nabla^{1,0} s)$.This finishes the proof of the following proposition.

Theorem 4.1.2. *Let S be a punctured Riemann surface with the hyperbolic metric. Let φ be a harmonic Beltrami differential and $\|\varphi\|_\infty$ small enough. The solutions to* (4.1.1) *are exactly $s = (1 - T_\varphi)^{-1} s_0$, where $s_0 \in A^{0,0}_{(2)}(S, K^m)$ is a holomorphic section.*

Remark 4.1.3. To obtain this solution one only needs a complete metric with bounded scalar curvature.

Together with Proposition 4.1.1, we construct a smooth extension of a holomorphic section.

Theorem 4.1.4. *Let S be a punctured Riemann surface with the hyperbolic metric. For a Beltrami differential $\varphi(t)$, a harmonic Beltrami differential $\eta = \frac{\partial \varphi(t)}{\partial t}|_{t=0}$ with $\|\eta\|_\infty$ small enough and a holomorphic section $s_0 \in A^{0,0}_{(2)}(S, K^m)$, there is a smooth extension of s_0 with respect to Bers coordinate t*

$$\sigma^m_{\varphi(t)}(s(t)) = s_0 + T_{\eta_i} s_0 t^i + i_{\eta_i} s_0 t^i + O(|t^2|). \tag{4.1.3}$$

4.2 Comparison between Harmonic Lift and Extension of Pluricanonical Form

Since L^2 Serre duality holds for $H^{0,1}_{(2)}(S, T^{1,0}S)$, the Hodge metric h_H for $m = 2$ is the co-metric of WP metric h_W. As discussed in the end of Section 2.2. To compute the curvature of WP metric using $\tilde{h}^H_{\bar{i}j}(t)$ is not obvious , one could still compute for the Ricci tensor for the WP metric.

In this section, we first derive some useful lemmas illustrating the interplays of several operators and their local expression. Then we work on an expansion formula for $\tilde{h}^H_{\bar{i}j}(t)$ at the end of the section.

Let S be a punctured Riemann surface with hyperbolic metric. Let $\varphi(t) = t^i\eta_i \in A^{0,1}_{(2)}(S, T^{1,0}S)$ be a harmonic Beltrami differential. Let $s^a_0 = i\eta_{\bar{a}}g\mathrm{d}z^2, s^b_0 = i\eta_{\bar{b}}g\mathrm{d}z^2 \in A^{0,0}_{(2)}(S, K^m)$ be holomorphic sections and $E_t(s^a_0)$, $E_t(s^b_0)$ be the corresponding extension. We have

$$\begin{aligned}\tilde{h}^H_{\bar{a}b}(t) &= \int_{S_0} (\sigma^m_{\varphi(t)}(s^a(t), \sigma^m_{\varphi(t)}(s^b(t)))_{V_t^{-m}} V_t \\ &= \int_{S_0} (s^a(t), s^b(t))_{V_0^{-m}} (1 - \varphi\bar{\varphi}) e^{(1-m)\rho(t)} V_0.\end{aligned} \tag{4.2.1}$$

We further use the notation

$$\tilde{f}_{\bar{a}b}(t) = \int_{S_0} (s^a(t), s^b(t))_{V_0^{-m}} V_0. \tag{4.2.2}$$

Proposition 4.2.1. *The expansion of $\tilde{f}_{\bar{i}j}$ of order (p,q)at $t=0$ is the following.*

If $p = 0$ and $q = 0$

$$\iota_0 \tilde{f}_{\bar{a}b}(t) = \int_{S_0} \eta_{\bar{i}}\eta_j V_0.$$

If $p = 0$ or $q = 0$ but $(p,q) \neq (0,0)$

$$\iota_0 \partial_{\alpha\bar{\beta}} \tilde{f}_{\bar{b}}(t) = 0.$$

If $p \geq 1$ and $q \geq 1$

$$\iota_0 \partial_{\alpha\bar{\beta}} \tilde{f}_{\bar{b}}(t) = \sigma_\alpha \sigma_{\bar{\beta}} \int_{S_0} \eta_b \eta_{\bar{j}} (\Pi_{k\in\beta - j} \bar{P}_{\bar{j}} D)(1-D)(\Pi_{k\in\alpha - i} P_k D)(\eta_i \eta_{\bar{a}}),$$

where $\alpha \in \{1, \ldots, n\}^p$ and $\beta \in \{1, \ldots, n\}^q$, σ_α is the permutation of α.

Before the proof, we are going to deduce some lemmas.

Recall $T_\eta s = \bar{\partial}^* G i_\eta \nabla^{1,0} s$, the linear term vanishes. We have

$$h^m(T_\eta s^a_0, T_\eta s^b_0) = h^m(G i_\eta \nabla^{1,0} s^a_0), H^\perp i_\eta \nabla^{1,0} s^b_0) = h^m(G i_\eta \nabla^{1,0} s^a_0), i_\eta \nabla^{1,0} s^b_0), \tag{4.2.3}$$

since $\bar{\partial}(i_\eta \nabla^{1,0} s_0) = 0$ and $H^{0,1}_{(2)}(M, K^m) = 0$.

Definition 4.2.2. Let (S, L)be an Hermitian line bundle over Riemann surface S. Let z holomorphic coordinate of S, e holomorphic section of L. For $\varphi = \varphi^j_{\bar{i}} \mathrm{d}z^{\bar{i}} \partial_j \otimes e \in A^{0,1}(S, T^{1,0}S \otimes L)$, its divergence is defined to be $d \operatorname{div} \varphi = \operatorname{Tr} \nabla \varphi$. In local coordinate, it is

$$\operatorname{div} \varphi = (\varphi_z + \varphi \log gh) \mathrm{d}\bar{z} \otimes e,$$

where $2g$ is the Hermitian metric on S, h is the Hermitian metric on L.

The simplification of (4.2.3) follows from the following three lemmas [Sun12].

Lemma 4.2.3. *Let $\varphi = \varphi \mathrm{d}\bar{z} \otimes \partial_z \in A^{0,1}_{(2)}(S, T^{1,0}S)$ be a harmonic Beltrami differential, $s = f \mathrm{d}z^m \in A^{0,0}_{(2)}(S, K^m)$ be a smooth section. We have*

$$i_\varphi \nabla^{1,0} s = \operatorname{div} A_\varphi s,$$

where A_φ is a global operator defined as $A_i(s) = \eta_i \otimes s$ for $s \in A^{0,1}_{(2)}(S, K^m)$.

Proof. Note that

$$\begin{aligned}
\nabla^{1,0} s &= \mathrm{d}f - mf(\log g)_z \mathrm{d}z \otimes \mathrm{d}z^m, \\
i_\varphi \nabla^{1,0} s &= \varphi(f_z - mf(\log g)_z) \mathrm{d}\bar{z} \otimes \mathrm{d}z^m, \\
\operatorname{div}(\varphi \otimes) s &= (\varphi f)_z + \varphi f \log(g \cdot (2g)^{-m})_z \mathrm{d}\bar{z} \otimes \mathrm{d}z^m \\
&= \varphi(f_z - mf(\log g)_z) + f(\varphi_z + \varphi(\log g)_z) \mathrm{d}\bar{z} \otimes \mathrm{d}z^m.
\end{aligned}$$

Since $\operatorname{div} \varphi = 0$, we have $i_\varphi \nabla s = i_\varphi \nabla s$. □

For $\psi = \psi \mathrm{d}z \otimes e \in A^{0,1}(M, L)$, where e holomorphic section, S Riemann surface, we have

$$\operatorname{div}^* \psi = -(\psi g^{-1}) \mathrm{d}\bar{z} \otimes \partial_z \otimes e.$$

Lemma 4.2.4. *Let $\mu = \mu \mathrm{d}\bar{z} \otimes \partial_z \otimes \mathrm{d}z^m \in A^{0,1}_{(2)}(S, T^{1,0}S \otimes K^m)$, then*

$$div^* div \mu = \Delta_{\bar{\partial}} \mu.$$

Proof.

$$\begin{aligned}
\operatorname{div}\mu &= \mu_z + (1-m)\mu(\log g)_z \mathrm{d}\bar{z}\otimes \mathrm{d}z^m,\\
\operatorname{div}^*\operatorname{div}\mu &= -(g^{-1}(\mu_z + (1-m)\mu(\log g)_z))_{\bar{z}}\mathrm{d}\bar{z}\otimes\partial_z\otimes \mathrm{d}z^m,\\
\bar{*}h\mu &= i\bar{\mu}\mathrm{d}z\otimes \mathrm{d}z\otimes(\partial_z)^m(2g)^{1-m},\\
\bar{\partial}\bar{*}h\mu &= i(\bar{\mu}(2g)^{1-m})_{\bar{z}}\mathrm{d}\bar{z}\wedge \mathrm{d}z\otimes \mathrm{d}z\otimes(\partial_z)^m,\\
\bar{\partial}^*\mu &= \bar{*}h\bar{\partial}\bar{*}h\mu = -g^{-1}(\mu_z + (1-m)\mu(\log g)_z)\partial_z\otimes \mathrm{d}z^m,\\
\Delta_{\bar{\partial}} &= -(g^{-1}(\mu_z + (1-m)\mu(\log g)_z))_{\bar{z}}\mathrm{d}\bar{z}\otimes\partial_z\otimes \mathrm{d}z^m.
\end{aligned}$$

□

Lemma 4.2.5. *Let $\lambda = \lambda\mathrm{d}\bar{z}\otimes \mathrm{d}z^m \in A^{0,1}_{(2)}(S, K^m)$, S is a punctured Riemann surface with hyperbolic metric, then*

$$[\Delta_{\bar{\partial}}, \operatorname{div}^*]\lambda = (1-m)\operatorname{div}^*\lambda.$$

And hence for $m = 2$

$$\operatorname{div} D\lambda = G\operatorname{div}\lambda.$$

Proof.

$$
\begin{aligned}
\bar{*}h\lambda &= i\bar{\lambda}\mathrm{d}z \otimes (\partial_z)^m (2g)^{-m}, \\
\bar{\partial}\bar{*}h\lambda &= i(\bar{\lambda}(2g)^{-m})_{\bar{z}}\mathrm{d}\bar{z}\mathrm{d}z \otimes \partial_z^m, \\
\bar{\partial}^*\lambda &= \bar{*}h\bar{\partial}\bar{*}h\lambda = -g^{-1}(\lambda_z - m\lambda(\log g)_z)\mathrm{d}z^m, \\
\Delta_{\bar{\partial}}\lambda &= (g^{-1}(\lambda_z - m\lambda(\log g)_z))_{\bar{z}}\mathrm{d}\bar{z}\mathrm{d}z^m, \\
\operatorname{div}^* \Delta_{\bar{\partial}}\mu &= (g^{-1}(g^{-1}(\mu_z - m\mu(\log g)_z))_{\bar{z}})_{\bar{z}}\mathrm{d}\bar{z} \otimes \partial_z \otimes \mathrm{d}z^m \\
\Delta_{\bar{\partial}} \operatorname{div}^* \mu &= -(g^{-1}((\lambda g^{-1})_{z\bar{z}} + (1-m)(\lambda g^{-1})_{\bar{z}}(\log g)_z))_{\bar{z}}\mathrm{d}\bar{z} \otimes \partial_z \otimes \mathrm{d}z^m \\
[\Delta_{\bar{\partial}}, \operatorname{div}^*]\lambda &= (g^{-1}[(\lambda g^{-1})_{z\bar{z}} + (1-m)(\lambda g^{-1})_{\bar{z}}(\log g)_z - (g^{-1}(\lambda_z - m\lambda(logg)_z)_{\bar{z}})])_{\bar{z}} \\
&= (g^{-1}[(\lambda g^{-1})_{z\bar{z}} + (1-m)(\lambda g^{-1})_{\bar{z}}(\log g)_z - (g^{-1}\lambda_z)_{\bar{z}} \\
&+ m(g^{-1}\lambda)_{\bar{z}}(logg)_z + mg^{-1}\lambda(\log g)_{z\bar{z}})])_{\bar{z}} \\
&= (g^-1[(\lambda(g^-1)_z)_{\bar{z}} + (\lambda g^{-1})_{\bar{z}}(\log g)_z + mg^{-1}\lambda(\log g)_{z\bar{z}}])_{\bar{z}} \\
&= (g^{-1}[\lambda(g^-1)_{z\bar{z}} + \lambda(g^{-1})_{\bar{z}}(\log g)_z + mg^{-1}\lambda(\log g)_{z\bar{z}}])_{\bar{z}} \\
&= (g^{-1}(m-1)\lambda)_{\bar{z}} = (1-m)\operatorname{div}^* \lambda.
\end{aligned}
\tag{4.2.4}
$$

The condition of hyperbolic metric with constant curvature -1 is used in the second last row. If $m = 2$, then

$$(\Delta_{\bar{\partial}} + 1)\operatorname{div}^* \lambda = \operatorname{div}^* \Delta_{\bar{\partial}}\lambda.$$

By Proposition 2.1.11, $H^{0,1}_{(2)}(M, K^m) = 0$ and thus $\Delta_{\bar{\partial}} = G^{-1}$. By manipulating the terms, we have $\operatorname{div} D\lambda = G \operatorname{div} \lambda$. □

Lemma 4.2.6.

$$(1 + \Delta_{\bar{\partial}})^{-1}(h(\eta_k) \otimes \eta_i) = (1 + \Delta_{\bar{\partial}})^{-1}(\eta_i, \eta_k)e,$$

where $e = ig(\mathrm{d}\bar{z} \otimes \partial_z \otimes \mathrm{d}z^2)$.

Check [LSY13] Lemma 3.5 for the proof.

We ready for the proof for Proposition 4.2.1

Proof. We have

$$\partial_{\alpha\bar{\beta}}\tilde{f}_{\bar{a}b}(t) = \sigma_\alpha\sigma_{\bar{\beta}}h^H((\Pi_{i\in\alpha}T_i)s_0^a, (\Pi_{j\in\beta}T_j)s_0^b). \tag{4.2.5}$$

For the case $|\alpha| = |\beta| = 0$, it follows directly.

For the case $|\alpha| \geq 1$ and $|\beta| = 0$, $(\Pi_{i\in\alpha}T_i)s_0^a$ is in the range of $\bar{\partial}^*$ which is perpendicular to a holomorphic section s_0^b.

For the case $|\alpha| \geq 1$ and $|\beta| \geq 1$, For simplicity, we assume $\alpha = \{i, k\}$, $\beta = \{j\}$.

Use Lemma 4.2.3, Lemma 4.2.4 and Lemma 4.2.5 for the case $m = 2$, we have

$$\begin{aligned} h^H(T_kT_is_0^a, T_js_0^b) &= h^H(\bar{\partial}^* \operatorname{div} DA_k\bar{\partial}^* \operatorname{div} DA_is_0^a, \bar{\partial}^* \operatorname{div} DA_js_0^b) \\ &= h^H(\bar{\partial}^*G \operatorname{div} A_k\bar{\partial}^* \operatorname{div} DA_is_0^a, \operatorname{div} DA_js_0^b) \\ &= h^H(\Delta_{\bar{\partial}}G \operatorname{div} A_k\bar{\partial}^* \operatorname{div} DA_is_0^a, \operatorname{div} DA_js_0^b) \\ &= h^H(\operatorname{div}^* \operatorname{div} A_k\bar{\partial}^* \operatorname{div} DA_is_0^a, DA_js_0^b) \\ &= h^H((1-D)A_k\bar{\partial}^* \operatorname{div} DA_is_0^a, A_js_0^b), \end{aligned} \tag{4.2.6}$$

In a local K-coordinate, let $\mu = \mu e \in A^{0,1}_{(2)}(S, T^{1,0}S \otimes K^2)$, where $e = ig(\mathrm{d}\bar{z} \otimes \partial_z \otimes \mathrm{d}z^2)$. We have

$$A^*_{\bar{\partial}} \operatorname{div} \mu = P_i(\mu)e.$$

Combined with Lemma 4.2.6, we continue the simplification.

$$\begin{aligned} h^H(T_kT_is_0^a, T_js_0^b) &= h^H((1-D)A_k\bar{\partial}^* \operatorname{div} DA_is_0^a, A_js_0^b), \\ &= h^H((1-D)A_k\bar{\partial}^* \operatorname{div}(D(\eta_i\eta_{\bar{a}})e), \eta_j\eta_{\bar{b}}e), \\ &= h^H((1-D)(P_kD(\eta_i\eta_{\bar{a}})e), \eta_j\eta_{\bar{b}}e), \\ &= \int_{S_0} \eta_b\eta_{\bar{j}}(1-D)P_kD(\eta_i\eta_{\bar{a}})V_0 \end{aligned} \tag{4.2.7}$$

In general, we have by induction

$$\begin{aligned} \partial_{\alpha\bar{\beta}}\tilde{f}_{\bar{a}b}(t) &= \sigma_\alpha\sigma_{\bar{\beta}}h^H((\Pi_{i\in\alpha}T_i)s_0^a, (\Pi_{j\in\beta}T_j)s_0^b) \\ &= \sigma_\alpha\sigma_{\bar{\beta}} \int_{S_0} \eta_b\eta_{\bar{j}}(\Pi_{k\in\beta-j}\bar{P}_{\bar{j}}D)(1-D)(\Pi_{k\in\alpha-i}P_kD)(\eta_i\eta_{\bar{a}}). \end{aligned} \tag{4.2.8}$$

□

Combined with the expansion of ρ computed in Section (3.1.3) Let S be a punctured Riemann surface with hyperbolic metric.

Theorem 4.2.7. *Let $\varphi(t) = t^i\eta_i \in A^{0,1}_{(2)}(S, T^{1,0}S)$ be a harmonic Beltrami differential. Let $s_0^a = i\eta_{\bar{a}}g\mathrm{d}z^2, s_0^b = i\eta_{\bar{b}}g\mathrm{d}z^2 \in A^{0,0}_{(2)}(S, K^m)$ be holomorphic sections and $E_t(s_0^a)$, $E_t(s_0^b)$ be extension corresponding to φ. The Hodge metric with respect to frame $\{E_t(s_0^a)\}$ have an explicit formula for any order. For order up to 2 is given as below.*

$$\begin{aligned}
\iota_0\tilde{h}^H_{\bar{i}j}(t) &= \int_{S_0} \eta_{\bar{i}}\eta_j V_0, \\
\iota_0\partial_{\bar{k}}\tilde{h}^H_{\bar{i}j}(t) &= 0, \\
\iota_0\partial_l\tilde{h}^H_{\bar{i}j}(t) &= 0, \\
\iota_0\partial_{\bar{k}l}\tilde{h}^H_{\bar{i}j}(t) &= -\int_{S_0} \sigma_{ik}\eta_{\bar{i}}\eta_j D\eta_{\bar{k}}\eta_l V_0.
\end{aligned}$$

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