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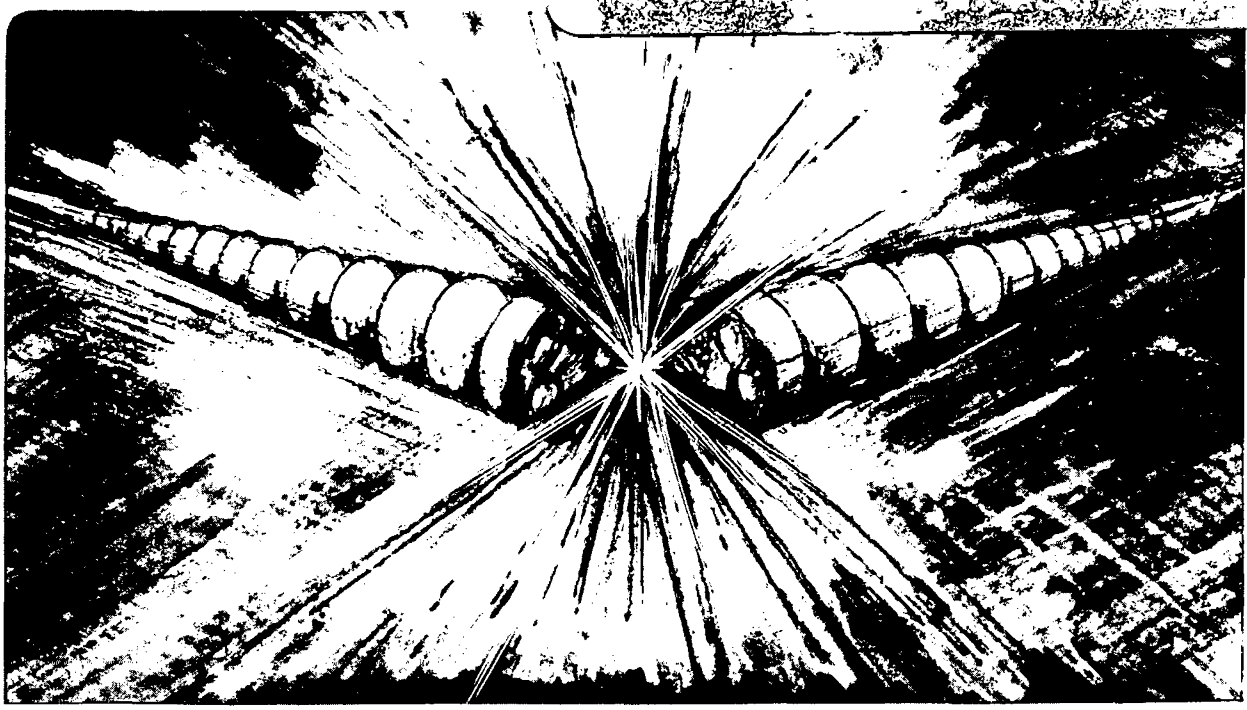

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ABSTRACT

The coupling between longitudinal and axially symmetric transverse perturbations is studied self-consistently by considering a beam of K-V distribution. The analysis is carried out within the context of linearized Vlasov-Maxwell equations and electrostatic approximation. It is shown that the coupling affects both the longitudinal and transverse modes significantly in the high density and low frequency region. In the medium and low density region, the separate treatment of longitudinal and transverse modes tends to be a good approximation. A new class of "coupling modes" is found which would not exist if the transverse motions of particles are neglected. The effect of resistive wall impedance on beam stability is also studied. It is found that the longitudinal impedance can cause a few transverse modes also to be weakly unstable.

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I. INTRODUCTION

In recent years, there has been a growing research activity in the study of using a heavy ion beam as a possible driver for inertial confinement fusion[1,2,3,4]. One of the main concerns in realizing the idea of heavy ion fusion is beam stability in transporting currents up to kilo-ampere range for distances of the order of kilometers. In spite of the fact that the stability of a charged particle beam has been studied over decades and major instabilities have been explored, one would still question the adequacy of existing theories when attempting to apply those theories to the beam in a proposed heavy ion fusion accelerator, which has a current several orders of magnitude higher than that in a conventional accelerator or storage ring.

For example, in a customary stability analysis of a continuous beam in an accelerator or storage ring, longitudinal and transverse effects are treated separately[5], a procedure which is valid because space charge forces are relatively weak and characteristic frequencies differ by orders of magnitude. In present day RF linacs, space charge effects are large and the frequencies in all degrees of freedom are comparable, but deterioration of beam quality is dominated by the effects of mismatches and nonlinear coupling in the first few drift tubes. A new situation arises when using an induction linac for heavy ion fusion and in the final transport lines to a reactor. Space charge forces are large and all frequencies are of the order of the plasma frequency, so that one might expect a significant coupling of longitudinal and transverse effects. The purpose of this paper is to present a first attempt to explore this regime.

The problem posed here is clearly a three or two dimensional stability analysis. Some three dimensional calculations have been carried out before[6,7], but the considerations were limited to laminar beams or rotating laminar beams or beams with very small radial motion of particles. These kind of beam models appear too simple for the present problem. In a strongly focused beam, as in the one which will be accelerated and transported in a heavy ion fusion driver, most particles have appreciable betatron oscillations in the radial direction. A previous analysis by Gluckstern has shown that the betatron oscillations of beam particles do support some transverse modes not found in a laminar or nearly laminar beam[8]. Our concern here will be concentrated on the coupling effect between those transverse modes and longitudinal modes.

We shall consider a simple model of an infinitely long, nonrelativistic beam of circular cross section under a constant linear external focusing force. The equilibrium configuration in phase space is assumed to have a K-V distribution[9] in the transverse direction and no velocity spread in the longitudinal direction. We also assume that the beam is surrounded by a circular pipe with arbitrary wall impedance. Our considerations will be limited to the linearized analysis of the stability of axially symmetric modes under electrostatic perturbation. In Section II, an integro-differential equation will be formulated from the Vlasov and Poisson equations. In Section III, the dispersion relation will be derived by expanding the perturbed electric potential into Legendre polynomials to solve the integro-differential equation. Numerical results will be presented in Section IV.

II. THEORETICAL MODEL

We consider an infinitely long, nonrelativistic beam of circular cross section with radius a and constant particle density n. Each individual particle has electric charge q and mass m. The beam is assumed to be surrounded by a conducting pipe of radius b and arbitrary wall impedance. The equilibrium state is maintained by a constant linear external transverse focusing force which can be represented as $m\nu_0^2 r$ in a cylindrical coordinate system of (r,φ,z). One can identify ν_0 as the betatron frequency of a particle in the absence of the beam's self-field. By taking both the external focusing force and the self-field of the beam into account, one finds the relation,

$$\nu^2 = \nu_0^2 - \frac{\omega_p^2}{2} , \qquad \text{(II-1)}$$

between the effective betatron frequency of particles, ν, and the plasma frequency, $\omega_p = \sqrt{4\pi n q^2/m}$. For simplicity, we assume that the equilibrium phase space configuration of particles is described by a distribution function of the form:

$$f_0(\vec{x}, \vec{v}) = \frac{n}{\pi} \delta\left[v_\perp^2 - \nu^2(a^2 - r^2)\right] \delta(v_z - v_0) , \qquad \text{(II-2)}$$

where $v_\perp = \sqrt{v_r^2 + v_\varphi^2}$ and v_z are particles' transverse and longitudinal speeds respectively, v_0 is a constant; $\delta(x)$ is the delta function. A straightforward calculation can show that the above distribution function satisfies the Vlasov equation, and the density profile obtained by integrating the above distribution function over velocity space is a constant equal to n for $r \le a$. Also indicated in Eq. (II-2) is that all the beam particles have the same transverse and longitudinal energies. In fact, the transverse distribution in Eq. (II-2) is just the axially symmetric case of the distribution function proposed by Kapchinskij and Vladimirskij[9] (abbr. K-V) years ago.

To make the mathematical calculation tractable, we shall consider electrostatic perturbations only and limit our analysis to the following linearized Vlasov and Poisson equations:

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} + \frac{d\vec{v}}{dt} \cdot \frac{\partial f_1}{\partial \vec{v}} = \frac{q}{m} \vec{\nabla} \phi_1 \cdot \frac{\partial f_0}{\partial \vec{v}} \quad , \tag{II-3}$$

and

$$\nabla^2 \phi_1 = - 4\pi q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\vec{x}, \vec{v}, t)\, d^3v \quad , \tag{II-4}$$

where ϕ is the electric potential and f is the distribution function; the subscripts 0 and 1 indicate the equilibrium and the perturbed quantities respectively. Also, to avoid complexity, we shall consider only azimuthally symmetric perturbations. Thus, only the coupling between radial and axial perturbations will be discussed.

Assuming that all perturbed quantities vary according to

$$f_1(\vec{x}, \vec{v}, t) = \tilde{f}(r, \vec{v}, k, \omega)e^{i(\omega t - kz)} \quad , \tag{II-5}$$

and

$$\phi_1(\vec{x}, t) = \tilde{\phi}(r)e^{i(\omega t - kz)} \quad , \tag{II-6}$$

then by substituting Eq. (II-2) into Eq. (II-3) and integrating both sides of Eq. (II-3) along the unperturbed particle orbit, we obtain that

$$\tilde{f}(r, \vec{v}, k, \omega) =$$

$$\frac{2nq}{\pi m}\int_0^{\infty} v_r' \; \frac{\partial\tilde{\phi}(r')}{\partial r'} \, \delta(v_z - v_o) \, e^{-i(\omega - kv_z)\tau} \frac{\partial}{\partial v_r'^2} \, \delta\left[v_\perp^2 - \nu^2(a^2 - r^2)\right] d\tau$$

$$- \frac{iknq}{\pi m}\int_0^{\infty} \tilde{\phi}(r') \, \delta\left[v_\perp{}^2 - \nu^2(a^2 - r^2)\right] e^{-i(\omega - kv_z)\tau} \; \frac{\partial\delta(v_z - v_o)}{\partial v_z} \, d\tau, \tag{II-7}$$

In the above equation, τ = t-t' is the duration between the present time t, and any past time, t'; r' and v_r' are the unperturbed particle orbit in the transverse phase space. In terms of τ , r' and v_r' are as follows:

$$r'(\tau) = \left[r^2\cos^2\nu\tau + \left(\frac{v_\perp}{\nu}\right)^2 \sin^2 \nu\tau - \frac{rv_\perp}{\nu} \sin 2\nu\tau \cos\theta\right]^{1/2} \tag{II-8}$$

and

$$v_r'(\tau) = \frac{dr'(\tau)}{d\tau} , \tag{II-9}$$

where

$$\theta = \cos^{-1}\left(\frac{v_r}{v_\perp}\right) . \tag{II-10}$$

We note that the conditions, $r'(0) = r$ and $v_r'(0) = v_r$, are implicitly contained in (II-8).

Substituting Eqs. (II-5), (II-6) and (II-7) into Eq. (II-4) and using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(v_r, v_\varphi) dv_r dv_\varphi = \int_0^{2\pi}\int_0^{\infty} h\left[v_r(v_\perp, \theta), v_\varphi(v_\perp, \theta)\right] v_\perp \, dv_\perp \, d\theta , \tag{II-11}$$

one then is able to obtain the following integro-differential equation for $\tilde{\phi}$ for $r \leq a$ by performing the integrations over $v_\perp$ and v_z:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\tilde{\phi}}{\partial r}\right) - k^2\tilde{\phi} =$$

$$\frac{k^2\omega_p^{\,2}}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} d\tau\tau e^{-i\Omega\tau}\ \tilde{\phi}(\rho)$$

$$- \frac{i\,\omega_p^{\,2}}{\pi\nu^2}\int_0^{2\pi} d\theta \int_0^{\infty} d\tau\Omega\ e^{-i\Omega\tau}\left[\sin^2 \nu\tau - \frac{r\sin(2\nu\tau)\cos\theta}{2\sqrt{a^2 - r^2}}\right]\frac{\partial\tilde{\phi}(\rho)}{\partial\rho^2}$$

$$+ \frac{\omega_p^{\,2} a}{\nu}\,\delta\,(r - a)\int_0^{\infty} d\tau \sin(2\nu\tau)e^{-i\Omega\tau}\ \frac{\partial\tilde{\phi}(R)}{\partial R^2} , \tag{II-12}$$

where $\Omega = \omega - kv_0$, (II-13)

$$R^2 = a^2 \cos^2 \nu\tau \ , \tag{II-14}$$

and
$$\rho^2 = \frac{a^2}{2}(1 - \cos 2\nu\tau) + r^2 \cos 2\nu\tau - r\sqrt{a^2 - r^2} \cos\theta \sin 2\nu\tau. \tag{II-15}$$

Equation (II-12), with $k = 0$, is the same as the one investigated by Gluckstern for stability of transverse modes.

To carry the analysis further, we need to match the solution of Eq. (II-12) to the electric potential exterior to the beam at the edge of the beam, $r = a$. Instead of pursuing the solution of Eq. (II-12) at this moment, we shall find the exterior electric potential here and leave the tedious procedure of solving Eq. (II-12) to the next section.

Assuming the space between the beam surface and pipe wall is vacuum and the electric potential in that region is $\tilde{\phi}_0(r)\, e^{i(\omega t - kz)}$, we find that $\tilde{\phi}_0(r)$ must satisfy the following Laplace equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\tilde{\phi}_0}{\partial r}\right) - k^2\tilde{\phi}_0 = 0 \ . \tag{II-16}$$

To include the wall impedance effects, we impose the boundary condition

$$\tilde{\phi}_0(b) = \frac{i\omega Z}{ck^2}\left(\frac{\partial\tilde{\phi}_0}{\partial r}\right)_{r=b} \tag{II-17}$$

on the solution of Eq. (II-16), where Z is the wall impedance in units of $Z_0 = 377$ ohms and c is the speed of light. One can easily show that the solution of Eq. (II-16), subject to the above boundary condition, is

$$\tilde{\phi}_0(r) = \Lambda \left\{ I_0(kr)K_0(kb) - I_0(kb)K_0(kr) - \left(\frac{i\omega Z}{ck}\right)\left[I_0(kr)K_0'(kb) - I_0'(kb)K_0(kr) \right] \right\} , \qquad \text{(II-18)}$$

where $I_0(x)$ and $K_0(x)$ are the zero order modified Bessel functions of first and second kind respectively, Λ is a constant to be determined by the boundary condition at $r = a$; primes denote the derivatives with respect to the arguments.

III. DISPERSION RELATION

Equation (II-12), in general, can not be solved for $\tilde{\phi}(r)$ with a finite series in r^2, except for the case of $k = 0$, for which Gluckstern obtained a solution in terms of finite Legendre polynomials as $P_\ell(1 - 2r^2/a^2) + P_{\ell+1}(1 - 2r^2/a^2)$ for transverse modes. The functional forms of these transverse modes suggest an expansion of $\tilde{\phi}(r)$ in Legendre polynomials of the same argument for the solution of Eq. (II-12) in order to study how these modes interact with longitudinal modes and with each other. We therefore assume that

$$\tilde{\phi}\,(r) = \sum_{\ell=0}^{\infty} G_\ell P_\ell\left(1 - \frac{2r^2}{a^2}\right) , \qquad \text{(III-1)}$$

for $r \leq a$. Substituting the above expansion into Eq. (II-12), with the aid of the recursion relations of Legendre polynomials and the identity

$$\ell\left[xP_\ell(x) - P_{\ell-1}(x) - (\ell + 1)(1 + x)P_\ell(x)\right] = (1 + x)\sum_{j=0}^{\ell-1} (-1)^{\ell+j-1}(j+1)^2\left[P_{j+1}(x) + P_j(x)\right] , \qquad \text{(III-2)}$$

one can show that the left hand side of Eq. (II-12) is equal to

$$-\frac{4}{a^2}\sum_{\ell=0}^{\infty}(\ell+1)\left[2(\ell+1)A_{\ell+1}+\left(\frac{ka}{2}\right)^2\left(\frac{G_\ell}{2\ell+1}+\frac{G_{\ell+1}}{2\ell+3}\right)\right]P_\ell^{(0,1)}\left(1-\frac{2r^2}{a^2}\right), \quad \text{(III-3)}$$

where

$$A_\ell=\sum_{j=\ell}^{\infty}(-1)^{\ell+j}\,G_j\,, \quad \text{(III-4)}$$

and $P_\ell^{(0,1)}(x)$ is the Jacobi polynomial which is related to Legendre polynomials by

$$P_\ell^{(0,1)}(x)=\left[P_\ell(x)+P_{\ell+1}(x)\right]\Big/(1+x)\;, \quad \text{(III-5)}$$

and

$$P_\ell(x)=\left[(\ell+1)P_\ell^{(0,1)}(x)+\ell P_{\ell-1}^{(0,1)}(x)\right]\Big/(2\ell+1)\;. \quad \text{(III-6)}$$

The quantities, A_ℓ, defined in Eq. (III-4) are the expansion coefficients of $\tilde{\phi}$ in Gluckstern's modes. Thus, in terms of A_ℓ, $\tilde{\phi}$ can be expressed as the following

$$\tilde{\phi}(r)=A_o+\sum_{j=1}^{\infty}A_j\left[P_j\left(1-\frac{2r^2}{a^2}\right)+P_{j-1}\left(1-\frac{2r^2}{a^2}\right)\right], \quad \text{(III-7)}$$

where

$$A_o=\sum_{\ell=0}^{\infty}(-1)^\ell\,G_\ell=\tilde{\phi}\,(a) \quad \text{(III-8)}$$

is the perturbed potential at the edge of the beam.

The algebra involved in manipulating the right hand side of Eq. (II-12) is tedious; we therefore summarized some intermediate steps in the Appendix and record the result as follows:

$$\text{RHS} = \left(\frac{\omega_p}{\nu}\right)^2 \sum_{\ell=0}^{\infty} (\ell + 1) \left\{ k^2 \left[\frac{G_\ell B_\ell'(\alpha)}{2\ell + 1} + \frac{G_{\ell+1}\, B_{\ell+1}'(\alpha)}{2\ell + 3}\right] - \frac{2A_{\ell+1}}{a^2}\, \alpha \left[B_{\ell+1}(\alpha) - B_\ell(\alpha)\right]\right\} P_\ell^{(0,1)}\left(1 - \frac{2r^2}{a^2}\right)$$

$$+ \frac{\omega_p^2}{a\nu^2}\, \delta(r - a) \sum_{\ell=0}^{\infty} (-1)^\ell\, G_\ell \left[1 - \alpha B_\ell(\alpha)\right] , \qquad \text{(III-9)}$$

where

$$B_\ell(\alpha) = i \int_0^\infty e^{-i\alpha\nu\tau}\, P_\ell(\cos 2\nu\tau)\, d(\nu\tau)$$

$$= \begin{cases} \dfrac{1}{\alpha} , & \text{for } \ell = 0 , \\[2ex] \dfrac{\left[\left(\frac{\alpha}{2}\right)^2 - 1\right]\left[\left(\frac{\alpha}{2}\right)^2 - 3^2\right] \cdots\cdots\cdots \left[\left(\frac{\alpha}{2}\right)^2 - (2m - 1)^2\right]}{\alpha\left[\left(\frac{\alpha}{2}\right)^2 - 2^2\right]\left[\left(\frac{\alpha}{2}\right)^2 - 4^2\right] \cdots\cdots\cdots \left[\left(\frac{\alpha}{2}\right)^2 - (2m)^2\right]} , & \text{for } \ell = 2m, \qquad \text{(III-10)} \\[2ex] \dfrac{\left(\frac{\alpha}{4}\right)\left[\left(\frac{\alpha}{2}\right)^2 - 2^2\right]\left[\left(\frac{\alpha}{2}\right)^2 - 4^2\right] \cdots\cdots\cdots \left[\left(\frac{\alpha}{2}\right)^2 - (2m - 2)^2\right]}{\left[\left(\frac{\alpha}{2}\right)^2 - 1\right]\left[\left(\frac{\alpha}{2}\right)^2 - 3^2\right] \cdots\cdots\cdots \left[\left(\frac{\alpha}{2}\right)^2 - (2m - 1)^2\right]} , & \text{for } \ell = 2m - 1, \\ & m = 1, 2, 3 \ldots, \end{cases}$$

$$B_\ell'(\alpha) = \frac{\partial B_\ell(\alpha)}{\partial \alpha} , \qquad \text{(III-11)}$$

and

$$\alpha = \frac{\omega - kv_0}{\nu} \qquad \text{(III-12)}$$

It is evident that the first term in expression (III-9) represents the perturbed volume charge density and the second term represents the perturbed surface charge density.

Equating the perturbed volume charge density to the expression in (III-3) and using the orthogonality relations of Jacobi polynomials, we find a recursion relation for A_ℓ:

$$W_\ell A_{\ell+1} + (U_\ell + W_\ell + W_{\ell-1})A_\ell + W_{\ell-1} A_{\ell-1} = 0, \qquad \text{(III-13)}$$

for $\ell \geq 1$,

where
$$U_\ell = 4\ell + \left(\frac{\omega_p}{\nu}\right)^2 \alpha \left[B_{\ell-1}(\alpha) - B_\ell(\alpha)\right] , \qquad \text{(III-14)}$$

and
$$W_\ell = \frac{k^2 a^2}{2(2\ell+1)} \left[1 + \left(\frac{\omega_p}{\nu}\right)^2 B_\ell'(\alpha)\right] . \qquad \text{(III-15)}$$

This recursion relation implicitly reduces the number of unknowns in the coefficients A_ℓ from infinity down to two, say, A_0 and A_1.

We now are ready to match the exterior and interior solutions to obtain the desired dispersion relation. The first matching condition requires that the potential be continuous at the edge of the beam, that is

$$\tilde{\phi}(a) = \tilde{\phi}_0(a) = A_0 \; . \tag{III-16}$$

The second condition requires that the jump of the derivatives of the potentials at the beam surface must be equal to the surface charge density; i.e.,

$$\left.\frac{\partial\tilde{\phi}_0}{\partial r}\right|_{r=a} - \left.\frac{\partial\tilde{\phi}}{\partial r}\right|_{r=a} = \frac{\omega_p^2}{a\nu^2} \sum_{\ell=0}^{\infty} (-1)^{\ell} \, G_{\ell}\left[1 - \alpha B_{\ell}(\alpha)\right] . \tag{III-17}$$

Substituting the expansion of Eq. (III-1) into the above condition and applying the recursion relation of A_ℓ in conjunction with the relations $\alpha B_0(\alpha) = 1$ and

$$G_\ell = A_\ell + A_{\ell+1} \quad , \tag{III-18}$$

Equation (III-17) is simplified to

$$a\left(\left.\frac{\partial\tilde{\phi}_0}{\partial r}\right|_{r=a}\right) = \left(\frac{\omega_p}{\nu}\right)^2 A_0\left[1 - \alpha \, B_0 \; (\alpha)\right] + W_0 G_0$$

$$= W_0(A_0 + A_1) \quad . \tag{III-19}$$

Dividing Eq. (III-19) by Eq. (III-16), yields

$$\frac{a}{\tilde{\phi}_0(a)}\left(\left.\frac{\partial\tilde{\phi}_0(r)}{\partial r}\right|_{r=a}\right) = W_0\left(1 + \frac{A_1}{A_0}\right) . \tag{III-20}$$

Applying the recursion relation (III-13) iteratively to the above equation, we obtain the dispersion relation in a form of infinite determinants or continuous fractions as the following:

$$\frac{a}{\tilde{\phi}_o(a)}\left[\frac{\partial\tilde{\phi}_o(r)}{\partial r}\right]_{r=a}$$

$$= W_o - \frac{W_o\begin{vmatrix} W_o & W_1 & 0 & 0 & \cdots \\ 0 & (W_1+W_2+U_2) & W_2 & 0 & \cdots \\ 0 & W_2 & (W_2+W_3+U_3) & W_3 & \cdots \\ 0 & 0 & W_3 & (W_3+W_4+U_4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}}{\begin{vmatrix} (W_o+W_1+U_1) & W_1 & 0 & 0 & \cdots \\ W_1 & (W_1+W_2+U_2) & W_2 & 0 & \cdots \\ 0 & W_2 & (W_2+W_3+U_3) & W_3 & \cdots \\ 0 & 0 & W_3 & (W_3+W_4+U_4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}}$$

$$= W_o - \frac{W_o^2|}{|(W_o+W_1+U_1)} - \frac{W_1^2|}{|(W_1+W_2+U_2)} - \frac{W_2^2|}{|(W_2+W_3+U_3)} - \cdots, \qquad \text{(III-21)}$$

where the frequency, ω, and the axial wave number, k, are contained in the quantities U_ℓ and W_ℓ defined in Eqs. (III-14) and (III-15).

Equation (III-21) is indeed a very complicated algebraic relation. In order to perform a practical numerical calculation, it is necessary to truncate the determinants at certain finite rank. Nevertheless, by taking the limits of some relevant parameters, a few special cases can still be studied analytically.

First, if $k = 0$ then Eq. (III-13) becomes $U_j = 0$ and the perturbed potential takes the simple form of $P_j(1 - 2r^2/a^2) + P_{j-1}(1 - 2r^2/a^2)$. Thus, Gluckstern's results are recovered as expected.

Next, we anticipate that Eq. (III-21) includes the dispersion relation for a cold beam. Recall that the temperature of the beam plasma is characterized by the effective betatron frequency, ν, and a value of zero for ν corresponds to a cold beam. One can easily prove that when ν approaches zero, the quantities U_ℓ and W_ℓ have the following limits:

$$\lim_{\nu\to 0} U_\ell(\alpha) = 4\ell\left(1 - \frac{\omega_p^2}{\Omega^2}\right), \tag{III-22}$$

and

$$\lim_{\nu\to 0} W_\ell(\alpha) = \frac{k^2a^2}{2(2\ell + 1)}\left(1 - \frac{\omega_p^2}{\Omega^2}\right) . \tag{III-23}$$

Substituting the above limits into Eq. (III-13), yields a recursion relation

$$\frac{(ka)^2 A_{\ell+1}}{4(2\ell + 1)} + \left[\frac{(ka)^2}{4(2\ell + 1)} + \frac{(ka)^2}{4(2\ell - 1)} + 2\ell\right] A_\ell + \frac{(ka)^2\ A_{\ell-1}}{4(2\ell - 1)} = 0 \quad , \tag{III-24}$$

for $\ell \geq 1$. Comparing Eq. (III-24) with the recursion relation of Bessel functions, we find that

$$\lim_{\nu\to 0} A_\ell = h(-1)^\ell\ I_{2\ell}(ka) \quad , \tag{III-25}$$

where h is an arbitrary constant independent of ℓ and ka. We then infer that when ν approaches zero, Eq. (III-20) becomes

$$\frac{1}{\tilde{\phi}_0(a)}\left[\frac{\partial\tilde{\phi}_0(r)}{\partial r}\right]_{r=a} = k\left(1 - \frac{\omega_p^2}{\Omega^2}\right)\frac{I_1(ka)}{I_0(ka)} \quad . \tag{III-26}$$

Upon substituting the exterior solution in Eq. (II-18) into the left hand side of the above equation, one immediately recognizes that Eq. (III-26) is just the cold beam dispersion relation derived by Hahn[10] and Ramo[11] years ago. In the long wavelength limit of $kb << 1$, Eq. (III-26) has the familiar solution

$$\Omega^2 = \frac{(ka)^2}{2}\,\omega_p^2\left[\ell n\left(\frac{b}{a}\right) - \frac{iv_0 Z}{bck}\right] . \tag{III-27}$$

Finally, by checking the regime of $0 < kb << 1$ and $\nu \neq 0$, we should perceive some effects of the betatron oscillations. For $ka < kb << 1$, a good approximation for Eq. (III-21) is

$$\frac{a}{\tilde{\phi}_0\,(a)}\left[\frac{\partial\tilde{\phi}_0(r)}{\partial r}\right]_{r=a} \approx W_0 - \frac{W_0^{\,2}}{W_0 + U_1} \quad . \tag{III-28}$$

Solving Eq. (III-28) for the Doppler shifted frequency Ω , we have

$$\Omega^2 \approx \frac{(ka)^2}{2}\,\omega_p^2\left[\frac{\nu^2}{4\nu^2 + \omega_p^2} + \ell n\left(\frac{b}{a}\right) - \frac{i\,v_0\,Z}{bck}\right] . \tag{III-29}$$

When ν is zero, the above equation is identical to Eq. (III-27). For non-zero ν, we see that due to the transverse motion of particles, the cold beam result is modified by the first term on the right hand side of Eq. (III-29). Also, in the long wavelength region, another class of modes owing to the betatron oscillation of particles can be found by the following observations:

$$\lim_{(\Omega/\nu)\to 0} \alpha B_\ell(\alpha) = \begin{cases} 0, & \text{if } \ell \text{ is odd,} \\ \overline{P}_\ell = \frac{1}{2\pi} \int_0^{2\pi} P_\ell(\cos x)dx \ , & \text{if } \ell \text{ is even.} \end{cases} \tag{III-30}$$

In this case, the dispersion relation, Eq. (III-21), admits a set of solutions

$$\Omega^2 \approx \frac{(ka\omega_p)^2\, \overline{P}_\ell}{8\ell(\ell+1)} \ , \quad \text{for } \left(\frac{\omega_p}{\nu}\right)^2 << 1 \ , \tag{III-31}$$

where ℓ is a positive even integer. For this kind of mode, the perturbed potential almost vanishes at the edge of the beam, i.e. $A_0 = \tilde{\phi}(a) = \tilde{\phi}_0(a) \approx 0$. We stress that this class of modes would not be found if we consider the longitudinal and transverse perturbations separately or if we neglect the betatron oscillations of particles.

IV. NUMERICAL RESULTS

As mentioned before, the infinite determinants in Eq. (III-21) have to be truncated at certain finite rank for a practical numerical computation. The rank of truncation will depend on the desired accuracy and the number of modes to be examined. With the aid of the limit considerations at the end of last section, one can estimate the number of modes that can be found at any particular truncation. However, the relation between the accuracy of the roots and the rank of truncation is much less apparent.

We know that when $ka = 0$, only the transverse modes described by $U_j = 0$ will exist. Therefore, if one wishes to study the interaction between the longitudinal modes and the transverse modes contained in $U_n = 0$, then the truncated determinants have to include U_n; that is one has to truncate the infinite determinants at rank of n at least. In principle, one could truncate the determinants at a rank much higher than n in order to achieve better accuracy in computing the roots. However, the larger the determinants used in the computation, the higher the number of roots that will be found in the dispersion relation and the smaller the intervals between roots will be, so that the identification of the many roots becomes more difficult.

When finite determinants are used in Eq. (III-21), the dispersion relation is then reduced to a finite order algebraic equation and hence the number of roots or modes can be definitely counted. In general, if $n \times n$ determinants were employed in Eq. (III-21), one can obtain $n(n+2)$ roots for even n or $(n+1)^2$ roots for odd n. Among these roots, $n(n+1)$ originate from the solutions of the equations $U_j = 0$ for $j = 1,2,\ldots,n$, one root of the longitudinal mode described in Eq. (III-26) and the remaining belong to the class described by Eq. (III-31). Since there is no strong necesssity to distinguish the roots among ℓ solutions of $U_\ell = 0$, we shall use the symbol T_ℓ to represent the whole family of solutions of $U_\ell = 0$. The ordinary longitudinal mode in Eq. (III-26) will be referred to as the L_1 mode and the "coupling modes" in Eq. (III-31) will be designated as L_j modes, for $j \geq 2$, in the order of their first appearance in the $(2j-1) \times (2j-1)$ determinants used in the dispersion relation.

In the present work, we limit our interest to examining how the longitudinal perturbations affect the T_1, T_2 and T_3 modes. We therefore have to truncate the determinants at the rank equal or higher than three, where the appropriate rank is chosen based on compromising the precision and the simplicity of computation. After comparing the results from different sizes of determinants used in the computation, we found that within the interesting ranges of relevant parameters, the 4x4 truncation can provide a reasonable accuracy without causing confusion. For this reason, the numerical results presented in the following are all computed according to the 4x4 truncations of the infinite determinants in Eq. (III-21). The use of 4x4 determinants in the computation also introduces four roots of T_4 modes. However, those roots always have poor accuracy; we therefore ignored them. If a more precise calculation of T_4 is needed, we then have to use 5x5 or 6x6 truncation.

There are four parameters involved in the dispersion relation: the ratio between the radius of the surrounding pipe and the radius of the beam, b/a, the axial wavelength parameter, ka, the tune depression, ν/ν_0, and the wall impedance parameter, $\mathcal{Z} = \omega Z/ck$. We only consider the case of $b/a = 1.5$ in order to narrow down the parameter space. For this relatively low value of b/a, the interesting range of ka would be from 0 up to 1; we therefore examined four different axial wavelength cases of $ka = 0.0$, 0.5, 1.0 and 1.5. The wall impedances considered are of perfect conductor and resistive types for $\mathcal{Z} = 0.0$, 0.1, 0.2 and 0.3, which are within the range corresponds to conditions anticipated in a linear induction accelerator. For each one of the sixteen combinations of the above mentioned axial wavelengths and wall impedances, we have calculated the ratio of the Doppler shifted frequency to the zero current betatron frequency, $\Omega/\nu_0 = (\omega - kv_0)/\nu_0$, for the full range of tune depressions; i.e. from $\nu/\nu_0 = 0$ to $\nu/\nu_0 = 1$.

Numerical results for $ka \neq 0$ are organized into the figures and tables. Since all the interesting results appear in the high tune depression region, we therefore only present the results in that region. Zero impedance cases are shown in Figs. 1 to 3 and examples of resistive impedance results are given in Tables 1 and 2.

The following is a summary of our numerical results:

A. Perfect Conducting Wall: ($\mathcal{Z} = 0$)

By comparing the Doppler shifted frequencies, $\Omega = \omega - kv_0$, obtained from different cases, we found that the transverse modes, T_1, T_2 and T_3, except for the lower T_2 and at the crossing of T_1 and T_3, are not significantly affected by the longitudinal perturbations. The growth rate of the well known instability in the confluent region of the two lower T_3 modes remains roughly the same as for $k = 0$, but the merging point is shifted from $\nu = .375\ \nu_0$ for $ka = 0$ to $\nu = .39\ \nu_0$ for $ka = 1.0$ and to $\nu = .4\ \nu_0$ for $ka = 1.5$ as can be seen in the figures.

The frequencies, Ω, of the ordinary longitudinal mode L_1 and the "coupling mode" L_2 all vanish at $\nu = \nu_0$. At $\nu = 0$, the L_1 mode approaches the cold beam limit in Eq. (III-26), while the L_2 mode joins with some of the T_2 and T_3 modes to fall to zero again. We notice that in the $|\mathrm{Re}(\Omega/\nu_0)|$ versus ν/ν_0 plots, the maximum of Ω for the L_1 mode occurs at $\nu > 0$ rather than at $\nu = 0$. This fact is due to the first term on the right hand side of Eq. (III-29) which indicates the effects of particles' transverse motion on the L_1 mode.

Pronounced longitudinal-transverse coupling appears in the high density and low frequency region, which is clearly shown in the figures. The first direct impact between longitudinal and transverse modes happens when the frequency of T_2 approaches the frequency of the L_1 mode, which occurs at $\nu = .44\ \nu_0$ for $ka = 1.0$ and $\nu = .51\ \nu_0$ for $ka = 1.5$. A more dramatic interaction appears when T_2 and L_2 modes merge together. In these confluent regions, the frequencies are complex conjugate pairs and so we expect that some modes are unstable. The general characteristics of the confluence are that both the span and the maximum growth rates increase with increasing ka.

In addition to the mixing of T modes and L modes, a very weak instability is also found when T_3 crosses T_1. When $k = 0$, T_1 and T_3 are independent modes and the frequencies are real at crossing.

B. <u>Resistive Wall Impedance</u>

It is found that the wall impedance affects modes L_1 and T_1 most. The familiar mode, L_1, is unstable in the familiar way and T_1, the envelope oscillation mode is weakly unstable. Effects of wall impedance on T_2 and T_3 modes are hardly perceptible. For complex values of $\mathfrak{z}$, the coefficients of the dispersion relation are not all real-valued, so that all the confluences are broken and there are no more complex conjugate pairs of frequencies. However, the confluence breaking seems to be a weak effect as can be seen in Table 1. Qualitatively, the confluence regions differ very little from those in the zero wall impedance results.

V. CONCLUSION

We have examined the stability properties of a monoenergetic, nonrelativistic heavy ion beam of axially symmetric K-V distribution. Our study is limited to the linear analysis of the azimuthally symmetric electrostatic perturbations. Although the model considered here is simple, the coupled Vlasov-Poisson equations have been solved rigorously and the results should give a qualitative indication of the coupling between longitudinal and transverse modes.

It is found that aside from the ordinary longitudinal mode and the previously found transverse modes, there also exists a class of "coupling modes" due to the interaction between the transversely oscillating particles and the longitudinal perturbed electric potential. In the high density region, both these "coupling modes" and the ordinary longitudinal mode interact with the lower transverse modes to create instabilities. However, for the situations with $ka \lesssim 1$, the dominant instabilities still originate from some previously known unstable transverse modes. In the medium and low density regions, there is no direct interaction between longitudinal and transverse modes and the influences of these two classes of modes on each other is insignificant. Thus, except for the very high density regime, the separate consideration of longitudinal and transverse modes tends to be a very good approximation. When considering the effect of a resistive wall impedance, we found that only the modes which involve an average axial field over the beam or surface motions are influenced.

The effects of various velocity spreads are still being pursued. A Lorentzian longitudinal velocity distribution of beam particles, i.e., $f(v_z) = \Delta/[\Delta^2 + (v_z-v_0)^2]$, introduces a damping rate of $\Delta|k|$, as it does for simpler problems. Then, in order to have damping of instabilities, we need

$$\frac{\Delta}{v_0} \geq \left| \frac{\nu_0}{k v_0} \, \mathrm{Im}\left(\frac{\Omega}{\nu_0}\right) \right| ; \tag{V-1}$$

where $\mathrm{Im}(\Omega/\nu_0)$ is taken from zero energy spread results. More realistic velocity spreads are being investigated.

APPENDIX

The calculations on the right hand side of Eq. (II-12) for the perturbed charge density will be carried out in this appendix. The right hand side of Eq. (II-12) can be written as

$$\mathrm{RHS} = I_1 + I_2 + I_3 , \tag{A-1}$$

where

$$I_1 = \frac{(k\omega_p)^2}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} d\tau\tau e^{-i\Omega\tau} \tilde{\phi}(\rho) , \tag{A-2}$$

$$I_2 = \frac{-i\omega_p^2}{\pi\nu^2} \int_0^{2\pi} d\theta \int_0^{\infty} d\tau\Omega\, e^{-i\Omega\tau} \left[\sin^2\nu\tau - \frac{r\sin(2\nu\tau)\cos\theta}{2\sqrt{a^2 - r^2}} \right] \frac{\partial\tilde{\phi}(\rho)}{\partial\rho^2} , \tag{A-3}$$

and

$$I_3 = \frac{a\omega_p^2}{\nu} \delta(r - a) \int_0^{\infty} d\tau \sin(2\nu\tau) e^{-i\Omega\tau} \frac{\partial\tilde{\phi}(R)}{\partial\rho^2} . \tag{A-4}$$

We consider I_2 first. Using the equality

$$\frac{dP_\ell(x)}{dx} = \frac{1}{2} \sum_{j=0}^{\ell} (2j + 1) \left[1 - (-1)^{\ell+j} \right] P_j(x) , \tag{A-5}$$

and the expansion of $\tilde{\phi}(r)$ in Eq. (III-1), we can derive from Eq. (A-3) the following

$$I_2 = \frac{-i\omega_p^2}{\pi\nu^2 a^2} \sum_{\ell=1}^{\infty} \sum_{j=0}^{\ell} G_\ell \left[(-1)^{\ell+j} - 1\right] (2j+1) \int_0^{\infty} d\tau\Omega\, e^{-i\Omega\tau}\ \Theta \ , \qquad \text{(A-6)}$$

where

$$\Theta = \int_0^{2\pi} d\theta \left[\sin^2\nu\tau - \frac{r\sin(2\nu\tau)\cos\theta}{2\sqrt{a^2 - r^2}}\right] P_j\left(1 - \frac{2\rho^2}{a^2}\right) , \qquad \text{(A-7)}$$

and ρ is a function of θ as shown in Eq. (II-15). For $r \leq a$, we can define a quantity η, such that

$$\frac{r}{a} = \sin\eta \ . \qquad \text{(A-8)}$$

The argument of the Legendre polynomial in Eq. (A-7) then can be expressed in terms of η as

$$1 - \frac{2\rho^2}{a^2} = \cos(2\eta)\cos(2\nu\tau) + \sin(2\eta)\sin(2\nu\tau)\cos\theta \ . \qquad \text{(A-9)}$$

With the aid of the above equality and the addition theorem of Legendre polynomials,we are able to perform the integration in Eq. (A-7) to obtain

$$\Theta = \pi\,(1 - \cos 2\nu\tau)P_j(s)P_j(T) - \frac{\pi}{j(j+1)} \tan\eta \sin(2\nu\tau)P_j^{\,1}(s)\ P_j^{\,1}(T) \qquad \text{(A-10)}$$

where
$$S = 1 - \frac{2r^2}{a^2} = \cos 2\eta \ , \tag{A-11}$$

$$T = \cos(2\nu\tau) \ , \tag{A-12}$$

and $P_j^1(x)$ is the Associated Legendre polynomial which is related to the Legendre Polynomials by

$$P_\ell^1(x) = \frac{(-1)^\ell}{\sqrt{1-x^2}}\left[\frac{\ell(\ell+1)}{2\ell+1}\right]\left[P_{\ell-1}(x) - P_{\ell+1}(x)\right] , \tag{A-13}$$

and

$$P_\ell^1(x) = \frac{\ell(-1)^\ell}{\sqrt{1-x^2}}\left[P_{\ell-1}(x) - xP_\ell(x)\right] . \tag{A-14}$$

Applying the above two relations together with the identity

$$xP_\ell(x) = \frac{1}{(2\ell+1)}\left[(\ell+1)P_{\ell+1}(x) + \ell P_{\ell-1}(x)\right] , \tag{A-15}$$

to Eq. (A-10), one obtains, after some algebra, that

$$\begin{aligned}\Theta &= \frac{\pi}{(1+s)}\left\{\left(\frac{j+1}{2j+1}\right)\left[P_j(s) + P_{j+1}(s)\right]\left[P_j(T) - P_{j+1}(T)\right]\right.\\ &\qquad \left. + \left(\frac{j}{2j+1}\right)\left[P_j(s) + P_{j-1}(s)\right]\left[P_j(T) - P_{j-1}(T)\right]\right\}\\ &= \frac{\pi}{(2j+1)}\left\{(j+1)P_j^{(0,1)}(s)\left[P_j(T) - P_{j+1}(T)\right] + jP_{j-1}^{(0,1)}(s)\left[P_j(T) - P_{j-1}(T)\right]\right\}\end{aligned} \tag{A-16}$$

where $P_\ell^{(0,1)}(x)$ is the Jacobi polynomial as introduced in Eq. (III-5). Substitution of Eq. (A-16) into Eq. (A-6) yields

$$I_2 = -\frac{\alpha\omega_p^2}{a^2\nu^2}\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell} G_\ell\left[(-1)^{\ell+j} - 1\right]\left\{(j+1)P_j^{(0,1)}(s)\left[B_j(\alpha) - B_{j+1}(\alpha)\right]\right.$$

$$\left. - j\,P_{j-1}^{(0,1)}(s)\left[B_{j-1}(\alpha) - B_j(\alpha)\right]\right\}$$

$$= -\frac{2\omega_p^2}{a^2\nu^2}\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1} G_\ell\,(-1)^{\ell+j}\,(j+1)\alpha\left[B_{j+1}(\alpha) - B_j(\alpha)\right]P_j^{(0,1)}(s)\ .$$

(A-17)

where α and $B_\ell(\alpha)$ have been defined in Eqs. (III-10) and (III-12). Using Eq. (III-4) and the transcription:

$$\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1} F_\ell\, H_j = \sum_{\ell=0}^{\infty}\sum_{j=\ell+1}^{\infty} F_j H_\ell\ , \tag{A-18}$$

for any F_ℓ and H_j, the equation (A-17) can be simplified to

$$I_2 = \frac{2\omega_p^{\ 2}}{a^2\nu^2}\sum_{\ell=0}^{\infty} A_{\ell+1}(\ell+1)\ \alpha\left[B_\ell(\alpha) - B_{\ell+1}(\alpha)\right]P_\ell^{(0,1)}\left(1 - \frac{2r^2}{a^2}\right).$$

(A-19)

To deal with the integrations in I_1, we can first use the relation (A-8) and the addition theorem of Legendre polynomials to obtain

$$\int_0^{2\pi} \tilde{\rho}(\rho)d\theta = 2\pi \sum_{\ell=0}^{\infty} G_\ell P_\ell (\cos 2\nu\tau) P_\ell(s) \ . \tag{A-20}$$

Then, by substituting the above result into Eq. (A-2), we have

$$I_1 = k^2\omega_p^2 \sum_{\ell=0}^{\infty} G_\ell P_\ell(s) \int_0^{\infty} \tau e^{-i\Omega\tau} P_\ell(\cos 2\nu\tau)d\tau$$

$$= \frac{k^2\omega_p^2}{\nu} \sum_{\ell=0}^{\infty} G_\ell P_\ell(s) \frac{\partial}{\partial\omega} B_\ell(\alpha)$$

$$= \frac{k^2\omega_p^2}{\nu^2} \sum_{\ell=0}^{\infty} (\ell + 1)\left[\frac{G_\ell B_\ell'(\alpha)}{2\ell + 1} + \frac{G_{\ell+1}B_{\ell+1}'(\alpha)}{2\ell + 3}\right] P_\ell^{(0,1)}\left(1 - \frac{2r^2}{a^2}\right), \tag{A-21}$$

where $B_\ell'(\alpha)$ has been defined in Eq. (III-11).

For the integration in I_3, we note that

$$dR^2 = d(a^2\cos^2\nu\tau) = -a^2\nu\sin(2\nu\tau)d\tau \ . \tag{A-22}$$

From this, it follows that

$$\int_0^{\infty} \sin(2\nu\tau) e^{-i\Omega\tau} \frac{\partial\tilde{\phi}(R)}{\partial R^2} d\tau$$

$$= \frac{1}{\nu a^2} \left\{ \tilde{\phi} \left[R(\tau = 0) \right] - i\alpha \int_0^{\infty} e^{-i\Omega\tau} \tilde{\phi}(R) d(\nu\tau) \right\}$$

$$= \frac{1}{\nu a^2} \sum_{\ell=0}^{\infty} G_\ell \left[P_\ell (-1)^\ell - i\alpha \int_0^{\infty} e^{-i\Omega\tau} P_\ell (1 - 2\cos^2 \nu\tau) d(\nu\tau) \right]$$

$$= \frac{1}{a^2 \nu} \sum_{\ell=0}^{\infty} (-1)^\ell G_\ell [1 - \alpha B_\ell(\alpha)] , \qquad \text{(A-23)}$$

and

$$I_3 = \frac{\omega_p^2}{a\nu^2} \delta(r - a) \sum_{\ell=0}^{\infty} (-1)^\ell G_\ell [1 - \alpha B_\ell(\alpha)] . \qquad \text{(A-24)}$$

Finally, by substituting Eqs. (A-19), (A-1) and (A-24) into (A-1), we obtain the expressions in (III-9).

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TABLE 1

Absolute values of real and imaginary parts of Ω/ν_0 (Re and Im) for ka = 1.0, b/a = 1.5, ν/ν_0 = 0.25 and various wall impedances. The confluence of L_2 and T_2 modes is broken by nonzero wall impedance.

	z = 0.0		z = 0.1		z = 0.2		z = 0.3	
mode	Re	Im	Re	Im	Re	Im	Re	Im
L_2	0.09884	0.10235	0.09884	0.10234	0.09884	0.10234	0.09884	0.10234
L_1	0.54502	0.00000	0.54763	0.02995	0.55516	0.05846	0.56683	0.08440
T_1	1.45589	0.00000	1.45589	0.00002	1.45589	0.00005	1.45590	0.00007
T_2	0.09884	0.10235	0.09884	0.10235	0.09884	0.10236	0.09884	0.10236
T_2	1.76675	0.00000	1.76675	0.00000	1.76675	0.00000	1.76675	0.00000
T_3	0.63343	0.20693	0.63343	0.20693	0.63343	0.20693	0.63343	0.20693
T_3	0.63343	0.20693	0.63343	0.20693	0.63343	0.20693	0.63343	0.20693
T_3	2.15894	0.00000	2.15894	0.00000	2.15894	0.00000	2.15894	0.00000

TABLE 2

Absolute values of real and imaginary parts of Ω/ν_0 (Re and Im) for ka = 1.0, b/a = 1.5, ν/ν_0 = 0.55 and various wall impedances.

	z = 0.0		z = 0.1		z = 0.2		z = 0.3	
mode	Re	Im	Re	Im	Re	Im	Re	Im
L_2	0.08742	0.00000	0.08742	0.00000	0.08742	0.00000	0.08742	0.00000
L_1	0.52093	0.00000	0.52279	0.02269	0.52818	0.04444	0.53662	0.06448
T_1	1.62119	0.00000	1.62120	0.00023	1.62123	0.00045	1.62129	0.00067
T_2	0.82364	0.00000	0.82364	0.00001	0.82364	0.00001	0.82364	0.00002
T_2	2.59381	0.00000	2.59381	0.00000	2.59382	0.00000	2.59382	0.00000
T_3	1.20489	0.00000	1.20489	0.00000	1.20849	0.00000	1.20489	0.00000
T_3	1.92841	0.00000	1.92841	0.00000	1.92841	0.00000	1.92841	0.00000
T_3	3.63098	0.00000	3.63098	0.00000	3.63098	0.00000	3.63098	0.00000

Figure Captions:

Fig. 1(a): Absolute value of the real part of Ω/ν_0 for various modes as a function of ν/ν_0 for ka = 0.5, b/a = 1.5 and zero wall impedance.

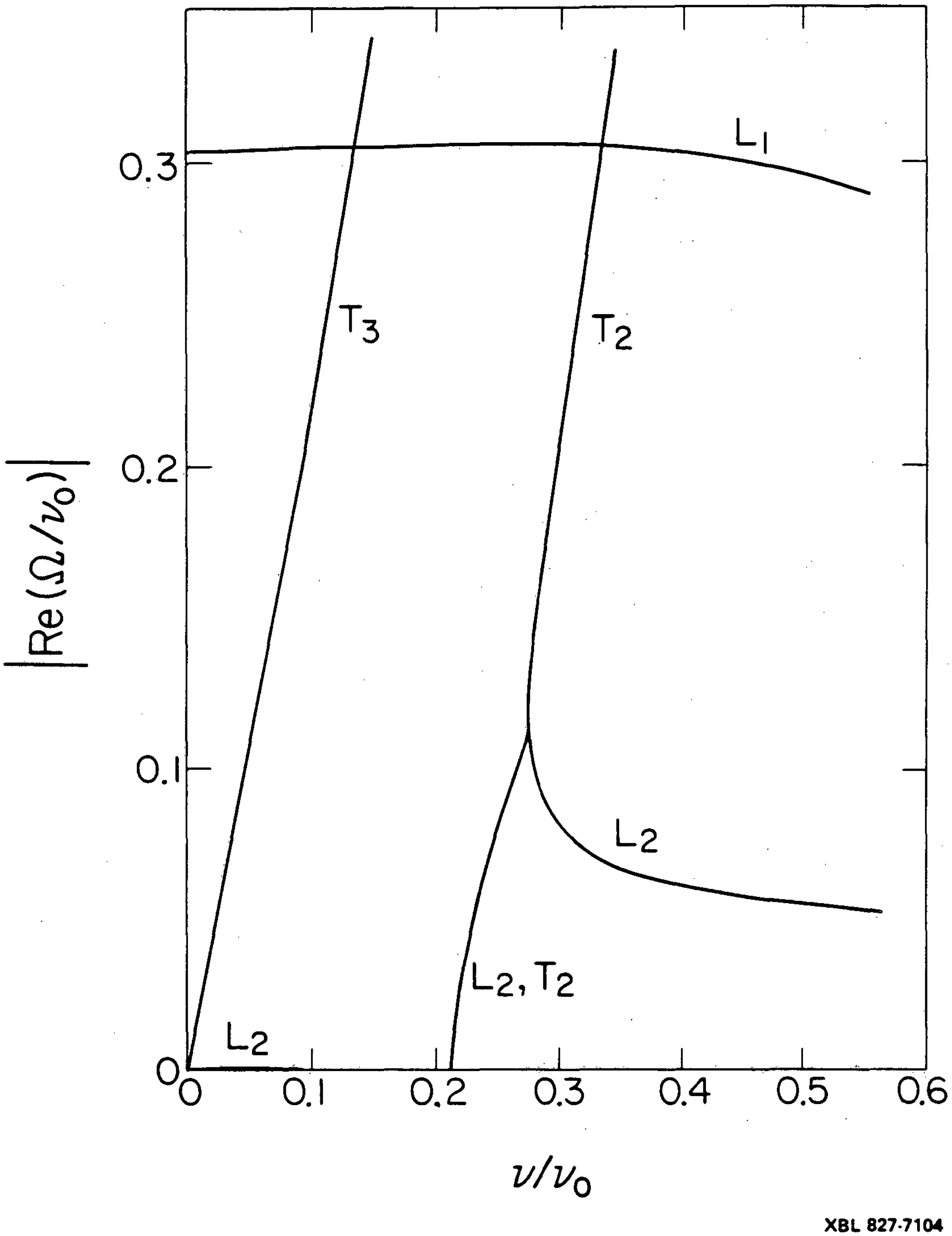
Fig. 1(b): Absolute value of the imaginary part of Ω/ν_0 for various modes as a function of ν/ν_0 for ka = 0.5, b/a = 1.5 and zero wall impedance.

Fig. 2(a): Absolute value of the real part of Ω/ν_0 for various modes as a function of ν/ν_0 for ka = 1.0, b/a = 1.5 and zero wall impedance.

Fig. 2(b): Absolute value of the imaginary part of Ω/ν_0 for various modes as a function of ν/ν_0 for ka = 1.0, b/a = 1.5 and zero wall impedance.

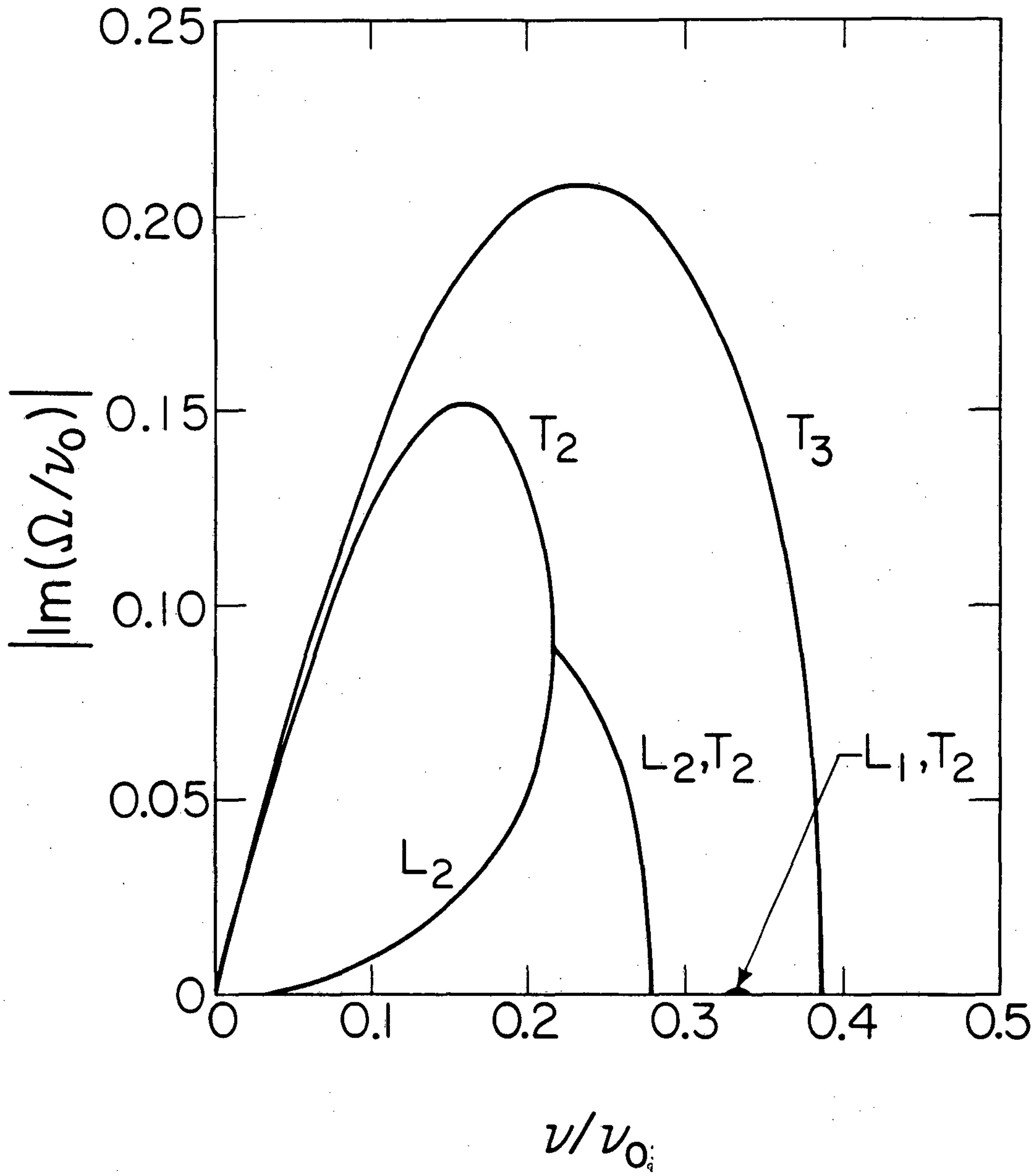
Fig. 3(a): Absolute value of the real part of Ω/ν_0 for various modes as a function of ν/ν_0 for ka = 1.5, b/a = 1.5 and zero wall impedance.

Fig. 3(b): Absolute value of the imaginary part of Ω/ν_0 for various mode as a function of ν/ν_0 for ka = 1.5, b/a = 1.5 and zero wall impedance. The kink in the L_2 mode at ν/ν_0 = 0.13 is due to a confluence of L_2 and T_4 modes.



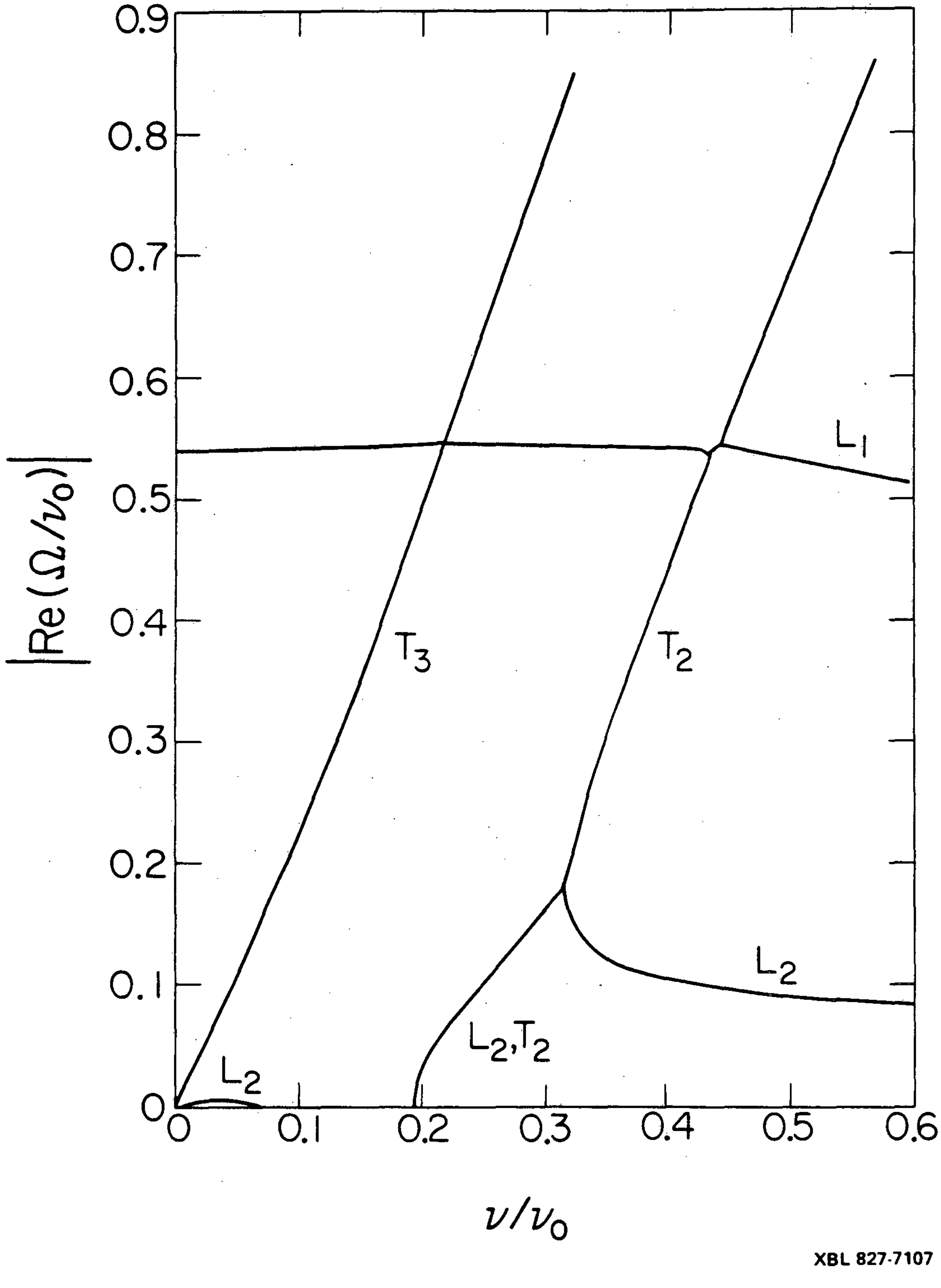
L1
T3
T2
L2
L2, T2
L2
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
|Re (Ω/ν0)|
ν/ν0
XBL 827-7104

Fig. 1(a)



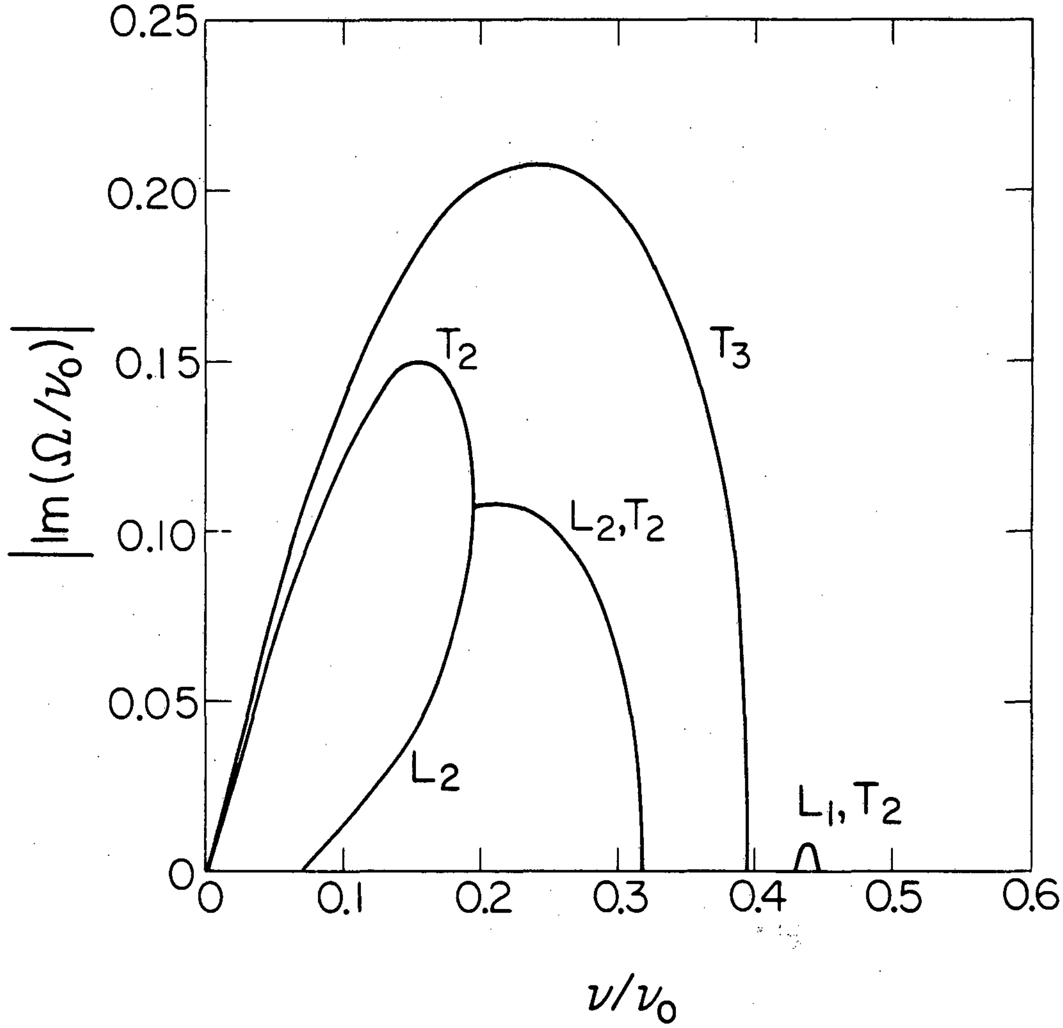
XBL 827-7105

Fig. 1(b)



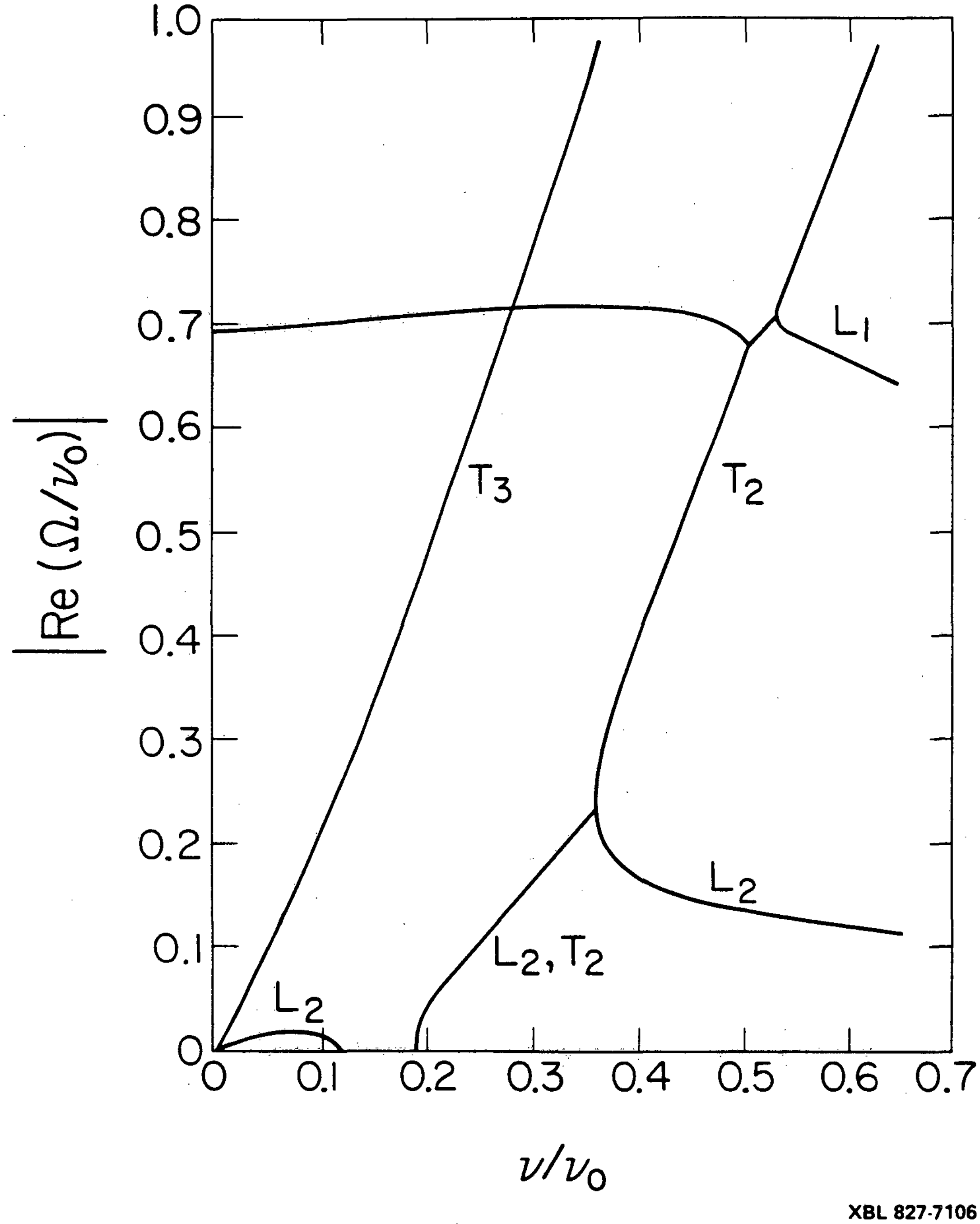
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
|Re(Ω/ν₀)|
T₃
T₂
L₁
L₂
L₂,T₂
L₂
0
0.1
0.2
0.3
0.4
0.5
0.6
ν/ν₀
XBL 827-7107

Fig. 2(a)



XBL 827-7102

Fig. 2(b)



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
|Re (Ω/ν₀)|
T₃
T₂
L₁
L₂
L₂, T₂
L₂
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
ν/ν₀
XBL 827-7106

Fig. 3(a)

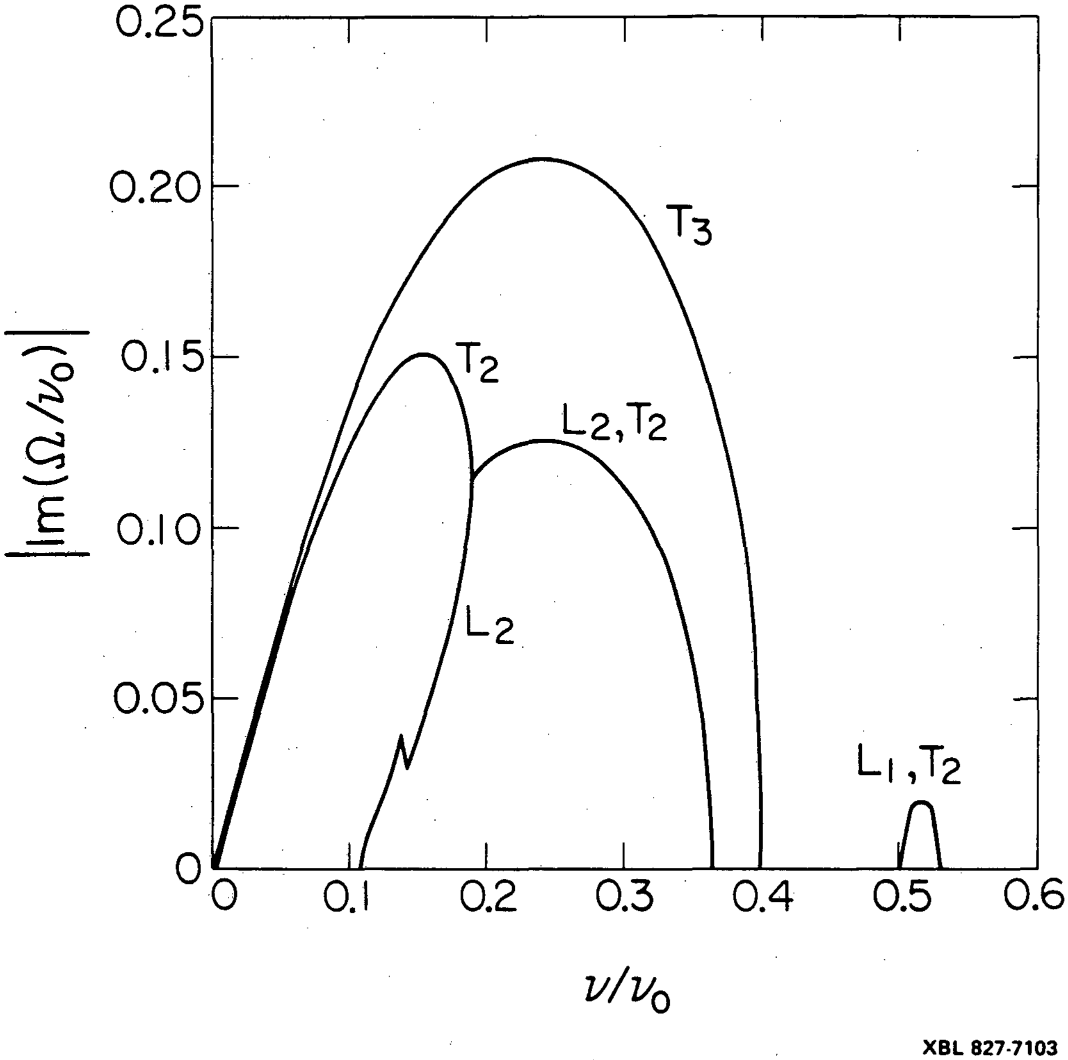

0.25
0.20
0.15
0.10
0.05
0
|Im(Ω/ν₀)|
T3
T2
L2,T2
L2
L1,T2
0
0.1
0.2
0.3
0.4
0.5
0.6
ν/ν₀
XBL 827-7103

Fig. 3(b)