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Monoidal Categories of Line Operators and 3d N = 4 Mirror Symmetry

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Monoidal Categories of Line Operators and 3d $\mathcal{N} = 4$ Mirror Symmetry

By

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Abstract

We study the simplest example of 3d mirror symmetry for $\mathcal{N} = 4$ supersymmetric gauge theories: the A-twist of a free hypermultiplet and the B-twist of $U(1)$ gauge theory coupled to a single hypermultiplet (SQED[1]). Our interest is primarily directed towards the category of line operators in each theory as well as the tensor structure they each possess. After reviewing these topics in a general setting, we return to our main example and identify the categories of line operators therein as appropriate module subcategories of certain vertex operator algebras; our approach is analogous to that used by Witten and uses results of Costello-Gaiotto. One vertex operator algebra that appears is the familiar $\beta\gamma$ system, the second one is related to the affine superalgebra $\widehat{\mathfrak{gl}(1|1)}$. We provide an explicit description of these module categories, which are strictly larger than those previously studied by Ridout-Wood and Allen-Wood. We additionally prove that these module categories are equivalent as braided tensor categories, culminating in an equivalence between the categories of line operators of original interest. This result completes a nontrivial check of the 3d mirror symmetry conjecture. We compute the tensor structure induced on the category of $\beta\gamma$ modules from this equivalence, extending the work of Allen-Wood. We finally comment on work in progress generalizing this equivalence to theories with arbitrary abelian gauge groups.

Contents

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1 Introduction

Dualities are powerful tools that allow physicists to probe a theory where traditional perturbative approaches may fail. The theory of quantum chromodynamics (QCD), which describes the interactions of quarks and gluons, is known to be strongly coupled in the infrared [GW73, Pol73]. In this low-energy regime, the usual perturbative approach to computing scattering amplitudes and correlation functions does not yield reliable results due to the magnitude of the coupling constants. Various creative workarounds have been used to perform trustworthy QCD computations in the IR, such as lattice QCD. Another approach often used to circumvent these types of difficulties in general contexts is that of a duality: a pair of seemingly distinct theories that are equivalent in some high- or low-energy regime. Their utility is that a computation which proves practically impossible in one theory is equivalent to a feasible calculation in the dual theory. Seiberg duality is a famous duality for a *supersymmetric extension* of QCD conjectured in [Sei95]; it states that two SQCD theories with different gauge groups and matter content are equivalent in the IR where one theory is weakly coupled and the other is strongly coupled. In this thesis, we focus solely on the duality approach for understanding supersymmetric gauge theories.

A theory is *supersymmetric* if its global symmetry algebra contains an extension of the usual Poincaré algebra by fermionic supercharges Q_i that anti-commute back into the Poincaré algebra. Theories with supersymmetry have been of great interest to physicists for a multitude of reasons: they allow certain quantities to be computed exactly (e.g. via supersymmetric localization) [KWY10a, Sei94, APS96, CK19], they get rid of the pesky tachyon and anomalies in string theory [GS84, Pol07b], and they present a candidate solution to the hierarchy problem in the standard model [DR81, DFS81]. Our interest in supersymmetric theories primarily arises from the rich mathematical structures they contain, which we review in this thesis.

It is an unfortunate truth that dualities are extremely difficult to prove. Much progress in this area comes in the form of checks that simply *support* the conjecture of a duality; i.e.

they ensure that certain computations are indeed consistent with the duality. In this thesis, we contribute rigorous, nontrivial evidence that supports a particular duality conjecture.

In a bit more detail, given a choice of Lie group G and a representation V, one can construct a 3d $\mathcal{N} = 4$ supersymmetric gauge theory with gauge group G and hypermultiplets transforming in the representation $V \oplus V^*$ of G. For each nilpotent element in the SUSY algebra, one can twist the theory by choosing to instead work in the *cohomology* of the operators with respect to this element. We focus on the simplest examples of such theories: the A-twist of a free hypermultiplet and the B-twist of SQED. We rigorously demonstrate that the category of bulk line operators in each of these theories match with each other. The proposal that these twisted theories are dual falls under a set of conjectures known as 3d mirror symmetry [IS96, BEHT15, BHOO97, BHOOY97].

Three dimensional theories with $\mathcal{N} = 4$ supersymmetry possess rich mathematical structures rendering them especially amenable to study. A particular example that plays a central role in this thesis are the boundary conditions, and subsequently the twists, that these types of theories support. The presence of a boundary breaks the supersymmetry algebra down to a 2d subalgebra (at least) simply due to the partially broken Poincaré invariance. The 3d $\mathcal{N} = 4$ supersymmetry algebra contains two important 2d subalgebras: the $\mathcal{N} = (2, 2)$ subalgebra and the $\mathcal{N} = (0, 4)$ subalgebra [CO17]. The class of boundary conditions that preserve the former subalgebra support topological twists, whereas boundary conditions that preserve the latter support holomorphic twists which give rise to vertex operator algebras. We discuss both of these boundary conditions and twists in detail in this thesis; they are paramount to our work since they help us analyze the category of line operators in a manner analogous to [Wit89]. Along the way we will encounter connections to various mathematical structures such as braided tensor categories and supersymmetric indices.

The structure of this thesis is as follows: Section 2 begins with a definition of the 3d $\mathcal{N} = 4$ supersymmetry algebra and a plethora of its relevant supersymmetric subalgebras. This is followed by a discussion of the common representations of these algebras and about how the representations decompose when restricting to a subalgebra. We then review some general features and structures that 3d $\mathcal{N} = 4$ theories possess.

Section 3 provides a brief introduction to 3d mirror symmetry wherein we discuss how mirror symmetry acts on the features and structures of 3d $\mathcal{N} = 4$ theories, describe some simple examples, and introduce the main conjecture that we tackle in this thesis.

We begin introducing the technical background pertinent to our work in Section 4. We define and discuss the category of line operators and the mathematical structures it possesses, providing intuition for their physical counterparts along the way. We review the story of [Wit89] to provide the unfamiliar reader with a foundational understanding of their technique of artificially cutting with a boundary since it is important to our analysis in later section. Vertex operator (super)algebras make their first appearance here, albeit perhaps rather covertly, so we define them and discuss their categorical structures at great length, working through multiple examples from both a mathematical and physics perspective.

The various boundary conditions one can impose on the artificial boundary, as well as their physical implications, are dealt with in Section 5. Many simple examples are provided for the reader's benefit. We also introduce the boundary conditions relevant for our work and will see how they give rise to the $\beta\gamma$ vertex operator algebra $V_{\beta\gamma}$ and the vertex operator algebra $V(\widehat{\mathfrak{gl}(1|1)})$ associated to the affine central extension of $\mathfrak{gl}(1|1)$. Lastly we present the notion of the half-index in these theories with boundary, compute them in the examples provided earlier, and verify that they match in accordance with 3d mirror symmetry.

Section 6 contains the bulk of the main results of this thesis and is adapted from [BN22]. Therein we begin by defining the $V_{\beta\gamma}$ representation category of physical relevance $\mathcal{C}_{\beta\gamma}$ and performing the nontrivial task of classifying its objects and extension structure. Next we propose an equivalent category of $V(\widehat{\mathfrak{gl}(1|1)})$ representations, classify its objects, and most importantly compute its tensor structure. This is followed by establishing the existence of a tensor functor between these two categories, allowing us to translate these results to understand the monoidal structure of $\mathcal{C}_{\beta\gamma}$. We conclude with a brief discussion about connections between $\mathcal{C}_{\beta\gamma}$ and representations of a certain quiver algebra and of the quantum group $\overline{U}_q^H(\mathfrak{sl}_2)$, the latter of which can be used to help compute the braiding structure of $\mathcal{C}_{\beta\gamma}$.

This thesis culminates with Section 7 which summarizes some work in progress [BCDN22] generalizing our check of 3d mirror symmetry to pairs of theories with *arbitrary* abelian gauge group (the gauge group in the Section 6 was $U(1)$). Of note is a detailed calculation of the theory's half-index utilizing a free field realization.

2 3d $\mathcal{N} = 4$ preliminaries and conventions

We begin by reviewing some basic objects and features of 3d $\mathcal{N} = 4$ SUSY gauge theories. In the process, we establish conventions and notation used throughout this thesis.

2.1 The 3d $\mathcal{N} = 4$ SUSY algebra and relevant subalgebras

We work in a flat, 3-dimensional, Euclidean spacetime $\mathbb{R}^3$ with coordinates (x^1, x^2, x^3). In this thesis we will consider theories in the presence of line operators and boundary conditions which, in general, only preserve a subalgebra of the 3d $\mathcal{N} = 4$ SUSY algebra. We now define and discuss a few such algebras that will be relevant for us.

As a side note, the calligraphic $\mathcal{N}$ does not directly correspond to the number of independent supercharges in these algebras, but rather it is equal to the number of supercharges divided by the dimension of the smallest irreducible spinorial representation of the Lorentz group $SO(d)$ [Car14].

2.1.1 The 3d $\mathcal{N} = 4$ SUSY algebra

The fermionic part of the 3d $\mathcal{N} = 4$ SUSY algebra is generated by 8 supercharges $Q_\alpha^{a\dot{a}}$ where $\alpha, a, \dot{a} \in \{+, -\}$ are $SU(2)$ indices. It can be obtained by dimensionally reducing the 4d $\mathcal{N} = 2$ SUSY algebra (e.g. [WB92]) to 3 dimensions wherein the 4^{th} translation generator becomes central. An upper index denotes that the object transforms in the fundamental representation of the corresponding $SU(2)$ group; lower indices denote that the object transforms in the anti-fundamental $SU(2)$ representation. There is an isomorphism of representations between the fundamental and anti-fundamental representations of $SU(2)$ given by

$$A^a = \epsilon^{ab} A_b \tag{2.1}$$

where

$$\epsilon^{+-} = \epsilon_{-+} = 1. \tag{2.2}$$

The $(\alpha, a, \dot{a})$ indices on $Q_\alpha^{a\dot{a}}$ indicate that it lives in a representation of $SU(2)_E \times SU(2)_H \times SU(2)_C$, where $SU(2)_E$ is the (Euclidianized) spin group and $SU(2)_H \times SU(2)_C$ is the R-symmetry automorphism group of the SUSY algebra.

The (super-)Lie brackets of the supercharges are

$$\{Q_\alpha^{a\dot{a}}, Q_\beta^{b\dot{b}}\} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^\mu_{\alpha\beta}P_\mu - i\epsilon_{\alpha\beta}(\epsilon^{ab}m^{\dot{a}\dot{b}} + \epsilon^{\dot{a}\dot{b}}t^{ab}) \tag{2.3}$$

where

$$(\sigma^1)^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\sigma^2)^\alpha{}_\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (\sigma^3)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

are the usual Pauli matrices and the $m^{\dot{a}\dot{b}}$ and t^{ab} are mass and Fayet-Iliopoulos terms, respectively. For future reference, we include

$$(\sigma^1)_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\sigma^2)_{\alpha\beta} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \qquad (\sigma^3)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

and

$$(\sigma^1)^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\sigma^2)^{\alpha\beta} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \qquad (\sigma^3)^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

With an eye towards considering holomorphic boundary conditions, we decompose $\mathbb{R}^3$ into $\mathbb{C} \times \mathbb{R}$ by

$$(x^1, x^2, x^3) \mapsto (z, t) := (x^1 + ix^2, x^3). \tag{2.7}$$

When considering functions on this space, we are really expanding the space so that x^1 and x^2 are independent *complex* coordinates and will eventually restrict back to the original space by taking the real slice defined by $x^1, x^2 \in \mathbb{R}$ [DMS97]. Hence in terms of

$$P_z = \frac{1}{2}(P_1 - iP_2) \qquad P_{\bar{z}} = \frac{1}{2}(P_1 + iP_2) \qquad P_t = P_3 \tag{2.8}$$

the Lie brackets become

$$\{Q_+^{a\dot{a}}, Q_+^{b\dot{b}}\} = -2\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}P_{\bar{z}} \qquad \{Q_-^{a\dot{a}}, Q_-^{b\dot{b}}\} = 2\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}P_z \tag{2.9}$$

$$\{Q_+^{a\dot{a}}, Q_-^{b\dot{b}}\} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}P_t + i(\epsilon^{ab}m^{\dot{a}\dot{b}} + \epsilon^{\dot{a}\dot{b}}t^{ab}) \tag{2.10}$$

2.1.2 The 3d $\mathcal{N} = 2$ SUSY algebra

The 3d $\mathcal{N} = 2$ SUSY algebra is generated by 4 supercharges $Q_\alpha, \overline{Q}_\alpha$ with non-zero Lie brackets

$$\{Q_\alpha, \overline{Q}_\beta\} = \sigma^\mu_{\alpha\beta}P_\mu + 2i\epsilon_{\alpha\beta}Z \tag{2.11}$$

where Z is a central charge. We can identify a 3d $\mathcal{N} = 2$ subalgebra with generators defined by

$$Q_\alpha^{\text{3d }\mathcal{N}=2} := Q_\alpha^{+\dot{+}} \qquad \overline{Q}_\alpha^{\text{3d }\mathcal{N}=2} := Q_\alpha^{-\dot{-}}. \tag{2.12}$$

The only non-zero Lie brackets are

$$\begin{aligned}
\{Q_+, \overline{Q}_+\} &= -2P_{\bar{z}} \\
\{Q_+, \overline{Q}_-\} &= P_t + i(m^{\dot{+}\dot{-}} + t^{+-}) \\
\{Q_-, \overline{Q}_+\} &= P_t - i(m^{\dot{+}\dot{-}} + t^{+-}) \\
\{Q_-, \overline{Q}_-\} &= 2P_z
\end{aligned} \tag{2.13}$$

Looking at (2.9), one might have thought the subalgebras that include only $Q_+^{a\dot{a}}$ or $Q_-^{b\dot{b}}$ would be 3d $\mathcal{N} = 2$ SUSY algebras. However, it would be incorrect to call these *3d* SUSY algebras since the supercharges can only generate translations in a single direction. More importantly though, the Lie bracket of these with the 3d Lorentz generators would not close, hence they are not actually subalgebras.

Only a $U(1)$ subgroup of the original 3d $\mathcal{N}=4$ R-symmetry group $SU(2)_H \times SU(2)_C$, given by the standard diagonal embedding

$$U(1) \hookrightarrow SU(2) \overset{\Delta}{\hookrightarrow} SU(2)_H \times SU(2)_C, \tag{2.14}$$

preserves this 3d $\mathcal{N}=2$ subalgebra.

2.1.3 The 2d $\mathcal{N}=(0,4)$ SUSY algebra

When working on theories with boundary (e.g. $\mathbb{C} \times \mathbb{R}_{\geq 0}$), the generator of translations perpendicular to the boundary is no longer a global symmetry of the theory. Since this generator is in the image of the adjoint action of some of the supercharges on 3d SUSY algebras, we conclude that the boundary explicitly breaks the 3d SUSY algebra to a subgroup preserving the boundary. Such an object is a 2d SUSY algebra that only contains translation generators in directions tangent to the boundary. We now discuss a few of their flavors.

The 2d $\mathcal{N}=(0,4)$ SUSY algebra can be obtained from the 3d $\mathcal{N}=4$ SUSY algebra as follows. It is generated by

$$Q^{a\dot{a}, \text{2d } \mathcal{N}=(0,4)} := Q_+^{a\dot{a}} \tag{2.15}$$

which has Lie brackets

$$\{Q^{a\dot{a}}, Q^{b\dot{b}}\} = -2\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} P_{\bar{z}} \tag{2.16}$$

This algebra has R-symmetry $SU(2) \times SU(2)$. When bulk hypermultiplets are present, theories preserving this boundary algebra require that either all of the scalars are set to a constant on the boundary, or that all of the fermions in the hypermultiplet are set to zero on the boundary.

2.1.4 The 2d $\mathcal{N} = (2,2)$ SUSY algebra

This superalgebra can be obtained as a subalgebra of the 3d $\mathcal{N} = 4$ SUSY algebra via

$$Q_+^{\text{2d}\,\mathcal{N}\,=\,(2,2)} := Q_+^{+\dot{-}} \qquad Q_-^{\text{2d}\,\mathcal{N}\,=\,(2,2)} := Q_-^{+\dot{+}} \tag{2.17}$$

$$\overline{Q}_+^{\text{2d}\,\mathcal{N}\,=\,(2,2)} := Q_+^{-\dot{+}} \qquad \overline{Q}_-^{\text{2d}\,\mathcal{N}\,=\,(2,2)} := Q_-^{-\dot{-}}. \tag{2.18}$$

Its non-zero Lie brackets are

$$\{Q_+, \overline{Q}_+\} = 2P_{\bar{z}} \qquad \{Q_+, Q_-\} = -it^{++} \qquad \{Q_+, \overline{Q}_-\} = im^{\dot{-}\dot{-}} \tag{2.19}$$

$$\{Q_-, \overline{Q}_-\} = 2P_z \qquad \{Q_-, \overline{Q}_+\} = -im^{\dot{+}\dot{+}} \qquad \{\overline{Q}_-, \overline{Q}_+\} = it^{--}. \tag{2.20}$$

Boundary conditions that preserve this subalgebra will typically set “one half” of each bulk hypermultiplet to zero. This SUSY algebra possesses a $U(1)_V \times U(1)_A$ R-symmetry described in [HV00, Section 2.1].

2.1.5 The 2d $\mathcal{N} = (0,2)$ SUSY algebra

This superalgebra can be obtained as a subalgebra of the 2d $\mathcal{N} = (2,2)$ SUSY algebra by taking the “anti-holomorphic piece”

$$Q^{\text{2d}\,\mathcal{N}\,=\,(0,2)} = Q_+^{\text{2d}\,\mathcal{N}\,=\,(2,2)} \qquad \overline{Q}^{\text{2d}\,\mathcal{N}\,=\,(0,2)} = \overline{Q}_+^{\text{2d}\,\mathcal{N}\,=\,(2,2)}. \tag{2.21}$$

The Lie brackets and superspace representations are easily read off from the previous section. This SUSY algebra has a $U(1)$ R-symmetry where Q carries charge 1 and $\overline{Q}$ carries charge -1.

The 2d $\mathcal{N} = (0,2)$ SUSY algebra also is a subalgebra of the 3d $\mathcal{N} = 2$ SUSY algebra, as follows

$$Q^{\text{2d}\,\mathcal{N}\,=\,(0,2)} = Q_+^{\text{3d}\,\mathcal{N}\,=\,2} \qquad \overline{Q}^{\text{2d}\,\mathcal{N}\,=\,(0,2)} = -\overline{Q}_+^{\text{3d}\,\mathcal{N}\,=\,2}. \tag{2.22}$$

Boundary conditions that preserve this subalgebra will typically set "one half" of each 3d chiral or 3d Fermi multiplet to zero.

2.1.6 The 1d $\mathcal{N} = 4$ SUSY algebra

When infinite straight line operators are present in a theory, only the generator of translations tangent to the line can remain a global symmetry of the theory; the other generators are explicitly broken. Therefore only subalgebras of the 3d $\mathcal{N} = 4$ SUSY algebra that don't have these broken generators in the image of the Lie bracket can be global symmetries of these theories. We now discuss a few such superalgebras.

The 1d $\mathcal{N} = 4$ SUSY algebra is generated by 4 supercharges Q^a and $\overline{Q}^a$ where $a \in \{+, -\}$. The Lie brackets are

$$\{Q^a, \overline{Q}^b\} = 2\epsilon^{ab}(P + Z_1) \qquad \{Q^a, Q^b\} = \{\overline{Q}^a, \overline{Q}^b\} = Z_2^{ab} \tag{2.23}$$

where Z_1 and Z_2^{ab} are symmetric central charges. This theory has a $U(1) \times SU(2)$ R-symmetry when the central charges vanish.

There are essentially two classes of 1d $\mathcal{N} = 4$ SUSY algebras sitting in the 3d $\mathcal{N} = 4$ SUSY algebra [AG15] and we refer to them as SQM_A and SQM_B. The former is given by

$$Q_A^{\dot{a}} := Q_+^{+\dot{a}} + Q_-^{-\dot{a}} \qquad \overline{Q}_A^{\dot{a}} := Q_+^{+\dot{a}} - Q_-^{-\dot{a}} \tag{2.24}$$

and the latter is given by

$$Q_B^{a} := Q_+^{a\dot{+}} + Q_-^{a\dot{-}} \qquad \overline{Q}_B^{a} := Q_+^{a\dot{+}} - Q_-^{a\dot{-}}. \tag{2.25}$$

The class of half-BPS line operators is characterized by which of these 1d $\mathcal{N} = 4$ subalgebras they preserve. The central charges will not play a significant role in the rest of this thesis, so from here on out, we set them to zero.

2.2 Multiplets

We review common multiplets that will play a role throughout the rest of this thesis. This is by no means an exhaustive list of representations for each of the SUSY algebras discussed in the previous section. For more exotic multiplets, see [LIR94] for example. The multiplets we discuss will decompose into a direct sum of multiplets when only requiring invariance under a SUSY subalgebra. Therefore it makes sense to first describe multiplets for the smallest SUSY algebras and then successively describe multiplets for larger SUSY algebras.

2.2.1 1d $\mathcal{N} = 4$ multiplets

We can represent the action of this SUSY algebra as derivations on superspace in the following manner

$$Q^+ = \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{d}{dt} \qquad \overline{Q}^- = \frac{\partial}{\partial \bar{\theta}^-} + \theta^+ \frac{d}{dt} \tag{2.26}$$

$$Q^- = \frac{\partial}{\partial \theta^-} - \bar{\theta}^+ \frac{d}{dt} \qquad \overline{Q}^+ = \frac{\partial}{\partial \bar{\theta}^+} - \theta^- \frac{d}{dt}. \tag{2.27}$$

To construct certain multiplets, it is convenient to introduce the following superderivatives that commute with the SUSY charges

$$D^+ = \frac{\partial}{\partial \theta^+} - \bar{\theta}^- \frac{d}{dt} \qquad \overline{D}^- = \frac{\partial}{\partial \bar{\theta}^-} - \theta^+ \frac{d}{dt} \tag{2.28}$$

$$D^- = \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{d}{dt} \qquad \overline{D}^+ = \frac{\partial}{\partial \bar{\theta}^+} + \theta^- \frac{d}{dt}. \tag{2.29}$$

Hypermultiplet:

The hypermultiplet $\Phi_{1\mathrm{d}\ \mathcal{N}=4}$ is a map on the superspace $\mathbb{R}^{1|4}$ defined by the condition

$$\overline{D}\Phi_{1\mathrm{d}\ \mathcal{N}=4} = 0. \tag{2.30}$$

Solving this equation, we find its superspace component expansion is

$$\Phi_{1\mathrm{d}\,\mathcal{N}=4}(t,\theta,\bar{\theta}) = \phi(t) + \theta^{+}\psi_{+}(t) + \theta^{-}\psi_{-}(t) + \theta^{+}\theta^{-}F(t) + \theta^{-}\bar{\theta}^{+}\dot{\phi}(t) - \theta^{+}\bar{\theta}^{-}\dot{\phi}(t) + \theta^{+}\theta^{-}\bar{\theta}^{-}\dot{\psi}_{-}(t) + \theta^{+}\theta^{-}\bar{\theta}^{+}\dot{\psi}_{+}(t) - \theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\ddot{\phi}(t) \quad (2.31)$$

where $\phi(t)$ is a complex scalar.

Vector multiplet:

The vector superfield $V_{1\mathrm{d}\,\mathcal{N}=4}$ is defined by the reality condition

$$V_{1\mathrm{d}\,\mathcal{N}=4} = V^{\dagger}_{1\mathrm{d}\,\mathcal{N}=4}. \quad (2.32)$$

Solving this equation, we find its superspace component expansion is

$$V_{1\mathrm{d}\,\mathcal{N}=4}(t,\theta,\bar{\theta}) = \sigma(x) + \theta^{+}\bar{\lambda}_{-}(x) + \theta^{-}\eta_{+}(x) - \bar{\theta}^{+}\lambda_{-}(x) - \bar{\theta}^{-}\bar{\eta}_{+}(x) + \theta^{+}\theta^{-}F(x) + \theta^{+}\bar{\theta}^{-}\varphi(x) + \theta^{+}\bar{\theta}^{+}A_{+}(x) + \theta^{-}\bar{\theta}^{-}A_{-}(x) + \theta^{-}\bar{\theta}^{+}\bar{\varphi}(x) - \bar{\theta}^{+}\bar{\theta}^{-}\bar{F}(x) + \theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\lambda_{+}(x) + \theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}\bar{\eta}_{-}(x) + \theta^{+}\theta^{-}\bar{\theta}^{-}\bar{\lambda}_{+}(x) + \theta^{+}\theta^{-}\bar{\theta}^{+}\eta_{-}(x) + \theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}D(x) \quad (2.33)$$

where $\{\sigma, A_{\pm}, D\}$ are real.

2.2.2 2d $\mathcal{N}=(0,2)$ multiplets

We can represent the action of this SUSY algebra as derivations on superspace in the following manner [HK+03]

$$Q = \frac{\partial}{\partial\theta^{+}} + i\bar{\theta}^{+}\frac{\partial}{\partial\bar{z}} \qquad \overline{Q} = -\frac{\partial}{\partial\bar{\theta}^{+}} - i\theta^{+}\frac{\partial}{\partial\bar{z}}. \quad (2.34)$$

The + superscripts are unnecessary, but we include them to illustrate how everything fits together with our realization of this algebra as a subalgebra of the 2d $\mathcal{N} = (2,2)$ SUSY algebra. To construct certain multiplets, it is convenient to introduce the following superderivatives that commute with the SUSY charges:

$$D = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\frac{\partial}{\partial\bar{z}} \qquad\qquad \overline{D} = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\frac{\partial}{\partial\bar{z}}. \tag{2.35}$$

Chiral multiplet:

The chiral multiplet $\Phi_{2\text{d }\mathcal{N}=(0,2)}$ is a map on the superspace $\mathbb{R}^{3|1,1}$ defined by the condition

$$\overline{D}\Phi_{2\text{d }\mathcal{N}=(0,2)} = 0. \tag{2.36}$$

Solving this equation, we find its superspace component expansion

$$\Phi_{2\text{d }\mathcal{N}=(0,2)}(x,\theta,\bar{\theta}) = \phi(x) + \theta^+\psi_+(x) - i\theta^+\bar{\theta}^+\partial_{\bar{z}}\phi(x) \tag{2.37}$$

where $\phi(x)$ is a complex scalar.

Antichiral multiplet:

The antichiral multiplet $\overline{\Phi}_{2\text{d }\mathcal{N}=(0,2)}$ is a map on the superspace $\mathbb{R}^{3|1,1}$ defined by the condition

$$D\Phi_{2\text{d }\mathcal{N}=(0,2)} = 0. \tag{2.38}$$

Solving this equation, we find its superspace component expansion

$$\overline{\Phi}_{2\text{d }\mathcal{N}=(0,2)}(x,\theta,\bar{\theta}) = \bar{\phi}(x) + \bar{\theta}^+\bar{\psi}_+(x) + i\theta^+\bar{\theta}^+\partial_{\bar{z}}\bar{\phi}(x). \tag{2.39}$$

Note that $\phi(x)$ and $\bar{\phi}(x)$ are independent functions in Euclidean signature.

Fermi multiplet:

The Fermi multiplet is defined by the same equation as the chiral multiplet but the lowest component is fermionic. It has the superspace component expansion

$$\Psi_{\text{2d } \mathcal{N} = (0,2)} = \psi_-(x) + \theta^+ f(x) - i\theta^+\bar{\theta}^+\partial_{\bar{z}}\psi_-(x). \tag{2.40}$$

Conjugate Fermi multiplet:

The conjugate Fermi multiplet is defined by the same equation as the antichiral multiplet but the lowest component is fermionic. It has the superspace component expansion

$$\overline{\Psi}_{\text{2d } \mathcal{N} = (0,2)} = \bar{\psi}_-(x) + \bar{\theta}^+ \bar{f}(x) + i\theta^+\bar{\theta}^+\partial_{\bar{z}}\bar{\psi}_-(x). \tag{2.41}$$

Vector multiplet:

The vector multiplet is comprised of two separate superfields

$$A_{\text{2d } \mathcal{N} = (0,2)} = \theta^+\bar{\theta}^+ A_+ \qquad V_{\text{2d } \mathcal{N} = (0,2)} = A_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D \tag{2.42}$$

in Wess-Zumino gauge [DGP18].

2.2.3 2d $\mathcal{N} = (2,2)$ multiplets

We can represent the action of this SUSY algebra as derivations on superspace in the following manner [HV00]

$$Q_+ = \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+\frac{\partial}{\partial\bar{z}} \qquad \overline{Q}_+ = -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\frac{\partial}{\partial\bar{z}} \tag{2.43}$$

$$Q_- = \frac{\partial}{\partial\theta^-} + i\bar{\theta}^-\frac{\partial}{\partial z} \qquad \overline{Q}_- = -\frac{\partial}{\partial\bar{\theta}^-} - i\theta^-\frac{\partial}{\partial z}. \tag{2.44}$$

To construct certain multiplets, it is convenient to introduce the following superderivatives that commute with the SUSY charges:

$$D_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\frac{\partial}{\partial\bar{z}} \qquad \overline{D}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\frac{\partial}{\partial\bar{z}} \tag{2.45}$$

$$D_- = \frac{\partial}{\partial\theta^-} - i\bar{\theta}^-\frac{\partial}{\partial z} \qquad \overline{D}_- = -\frac{\partial}{\partial\bar{\theta}^-} + i\theta^-\frac{\partial}{\partial z}. \tag{2.46}$$

Chiral multiplet:

The chiral multiplet $\Phi_{\text{2d }\mathcal{N}\,=\,(2,\,2)}$ is defined by the conditions

$$\overline{D}_\pm\Phi_{\text{2d }\mathcal{N}\,=\,(2,\,2)} = 0. \tag{2.47}$$

Solving these equations, we find its superspace component expansion

$$\begin{aligned}\Phi_{\text{2d }\mathcal{N}\,=\,(2,\,2)}(x,\theta,\bar{\theta}) = \phi(x) + \theta^\alpha\psi_\alpha(x) + \theta^+\theta^-F(x) - i\theta^+\bar{\theta}^+\partial_{\bar{z}}\phi(x) - i\theta^-\bar{\theta}^-\partial_z\phi(x)\\ - i\theta^+\theta^-\bar{\theta}^-\partial_z\psi_+(x) + i\theta^+\theta^-\bar{\theta}^+\partial_{\bar{z}}\psi_-(x) + \theta^+\theta^-\bar{\theta}^+\bar{\theta}^-\partial_z\partial_{\bar{z}}\phi(x)\end{aligned} \tag{2.48}$$

where $\alpha \in \{+,-\}$. Note that, under the embedding described in (2.21), $\Phi_{\text{2d }\mathcal{N}\,=\,(2,\,2)}$ decomposes into a 2d $\mathcal{N} = (0,2)$ chiral multiplet plus a 2d $\mathcal{N} = (0,2)$ Fermi multiplet:

$$\Phi_{\text{2d }\mathcal{N}\,=\,(0,\,2)} = \Phi_{\text{2d }\mathcal{N}\,=\,(2,\,2)}\Big|_{\theta^-=\bar{\theta}^-=0} \tag{2.49}$$

$$\Psi_{\text{2d }\mathcal{N}\,=\,(0,\,2)} = D_-\Phi_{\text{2d }\mathcal{N}\,=\,(2,\,2)}\Big|_{\theta^-=\bar{\theta}^-=0}. \tag{2.50}$$

Since the generators of the 2d $\mathcal{N} = (0,2)$ subalgebra $Q_+, \overline{Q}_+$ properly commute with D_- and with setting θ^- and $\bar{\theta}^-$ to zero, we verify that these two superfields are indeed closed representations of the $\mathcal{N} = (0,2)$ SUSY algebra.

Antichiral multiplet:

The antichiral multiplet $\overline{\Phi}_{\text{2d }\mathcal{N}=(2,2)}$ is defined by the conditions

$$D_{\pm}\overline{\Phi}_{\text{2d }\mathcal{N}=(2,2)} = 0. \tag{2.51}$$

Solving these equations, we find its superspace component expansion

$$\begin{aligned}\overline{\Phi}_{\text{2d }\mathcal{N}=(2,2)}(x,\theta,\bar{\theta}) &= \bar{\phi}(x) + \bar{\theta}^{\alpha}\bar{\psi}_{\alpha}(x) + \bar{\theta}^{+}\bar{\theta}^{-}\bar{F}(x) + i\theta^{+}\bar{\theta}^{+}\partial_{\bar{z}}\bar{\phi}(x) + i\theta^{-}\bar{\theta}^{-}\partial_{z}\bar{\phi}(x) \\ &+ i\theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{\bar{z}}\bar{\psi}_{-}(x) - i\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{z}\bar{\psi}_{+}(x) + \theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{z}\partial_{\bar{z}}\bar{\phi}(x).\end{aligned} \tag{2.52}$$

Under the embedding described in (2.21), $\overline{\Phi}_{\text{2d }\mathcal{N}=(2,2)}$ decomposes into a 2d $\mathcal{N}=(0,2)$ antichiral multiplet plus a 2d $\mathcal{N}=(0,2)$ conjugate Fermi multiplet:

$$\overline{\Phi}_{\text{2d }\mathcal{N}=(0,2)} = \overline{\Phi}_{\text{2d }\mathcal{N}=(2,2)}\Big|_{\theta^{-}=\bar{\theta}^{-}=0} \tag{2.53}$$

$$\overline{\Psi}_{\text{2d }\mathcal{N}=(0,2)} = \overline{D}_{-}\overline{\Phi}_{\text{2d }\mathcal{N}=(2,2)}\Big|_{\theta^{-}=\bar{\theta}^{-}=0} \tag{2.54}$$

Vector multiplet:

In Wess-Zumino gauge, the vector multiplet is

$$\begin{aligned}V_{\text{2d }\mathcal{N}=(2,2)}(x,\theta,\bar{\theta}) &= \theta^{+}\bar{\theta}^{+}A_{+}(x) + \theta^{-}\bar{\theta}^{-}A_{-}(x) + \theta^{-}\bar{\theta}^{+}\varphi(x) + \theta^{+}\bar{\theta}^{-}\bar{\varphi}(x) + 2\theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}D(x) \\ &+ 2i(\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\lambda_{-}(x) + \theta^{+}\theta^{-}\bar{\theta}^{+}\eta_{+}(x) + \theta^{+}\theta^{-}\bar{\theta}^{-}\bar{\lambda}_{-}(x) + \theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}\bar{\eta}_{+}(x)).\end{aligned} \tag{2.55}$$

Under the embedding described in (2.21), $V_{\text{2d }\mathcal{N}=(2,2)}$ decomposes into a 2d $\mathcal{N}=(0,2)$ vector multiplet plus a 2d $\mathcal{N}=(0,2)$ chiral multiplet and a 2d $\mathcal{N}=(0,2)$ antichiral:

$$V_{\text{2d }\mathcal{N}=(0,2)} = \overline{D}_- D_- V_{\text{2d }\mathcal{N}=(2,2)}\Big|_{\theta^-=\bar{\theta}^-=0} \tag{2.56}$$

$$A_{\text{2d }\mathcal{N}=(0,2)} = V_{\text{2d }\mathcal{N}=(2,2)}\Big|_{\theta^-=\bar{\theta}^-=0} \tag{2.57}$$

$$\Phi_{\text{2d }\mathcal{N}=(0,2)} = \frac{\partial}{\partial\bar{\theta}^+} D_- V_{\text{2d }\mathcal{N}=(2,2)}\Big|_{\theta^-=\bar{\theta}^-=0} \tag{2.58}$$

$$\overline{\Phi}_{\text{2d }\mathcal{N}=(0,2)} = \frac{\partial}{\partial\theta^+} \overline{D}_- V_{\text{2d }\mathcal{N}=(2,2)}\Big|_{\theta^-=\bar{\theta}^-=0}. \tag{2.59}$$

2.2.4 2d $\mathcal{N}=(0,4)$ multiplets

There unfortunately is not a nice superspace representation of this SUSY algebra [PSY16], so we describe the 2d $\mathcal{N}=(0,4)$ multiplets in terms of their decomposition into 2d $\mathcal{N}=(0,2)$ multiplets.

Hypermultiplet:

A 2d $\mathcal{N}=(0,4)$ hypermultiplet Φ consists of two 2d $\mathcal{N}=(0,2)$ chiral multiplets Φ_1 and Φ_2 [Ton14]. To specify the action of the full 2d $\mathcal{N}=(0,4)$ SUSY algebra on Φ, it remains to describe how Φ_1 and Φ_2 transform under the $SU(2)_1 \times SU(2)_2$ R-symmetry. The scalar components of Φ_1 and Φ_2 transform as a doublet under $SU(2)_1$ and are invariant under the action of $SU(2)_2$, whereas the fermions are invariant under $SU(2)_1$ but transform in a doublet under $SU(2)_2$.

Vector multiplet:

The 2d $\mathcal{N}=(0,4)$ vector multiplet V decomposes into a 2d $\mathcal{N}=(0,2)$ vector multiplet V' and a 2d $\mathcal{N}=(0,2)$ Fermi multiplet Ψ that takes values in the adjoint representation of the gauge group [Ton14]. The gauge field in V' is invariant under $SU(2)_1 \times SU(2)_2$, and the fermions in Ψ transform as a doublet under $SU(2)_1$ and $SU(2)_2$.

2.2.5 3d $\mathcal{N}=2$ multiplets

We can represent the action of this SUSY algebra as derivations on superfields in the following manner [CO17]

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \qquad \overline{Q}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - \frac{i}{2}(\sigma^\mu\theta)_\alpha\partial_\mu \tag{2.60}$$

To construct certain multiplets, it is convenient to introduce the following superderivatives that commute with the SUSY charges:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \frac{i}{2}(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \qquad \overline{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{i}{2}(\sigma^\mu\theta)_\alpha\partial_\mu. \tag{2.61}$$

Chiral multiplet:

The chiral multiplet $\Phi_{3\text{d }\mathcal{N}=2}$ is defined by the conditions

$$\overline{D}_\pm\Phi_{3\text{d }\mathcal{N}=2} = 0. \tag{2.62}$$

Solving these equations, we find its superspace component expansion

$$\begin{aligned}\Phi_{3\text{d }\mathcal{N}=2}(x,\theta,\bar{\theta}) &= \phi(x) + \theta^+\psi_+(x) + \theta^-\psi_-(x) + \theta^+\theta^-F(x)\\ &+ i\theta^+\bar{\theta}^+\partial_{\bar{z}}\phi(x) - i\theta^-\bar{\theta}^-\partial_z\phi(x) - \frac{i}{2}\theta^+\bar{\theta}^-\partial_t\phi(x) - \frac{i}{2}\theta^-\bar{\theta}^+\partial_t\phi(x)\\ &+ \theta^+\theta^-\bar{\theta}^-\left(\frac{i}{2}\partial_t\psi_-(x) - i\partial_z\psi_+(x)\right) - \theta^+\theta^-\bar{\theta}^+\left(\frac{i}{2}\partial_t\psi_+(x) + \partial_{\bar{z}}\psi_-(x)\right)\\ &- \theta^+\theta^-\bar{\theta}^+\bar{\theta}^-\left(\frac{1}{4}\partial_t^2\phi(x) + \partial_z\partial_{\bar{z}}\phi(x)\right).\end{aligned} \tag{2.63}$$

Under the embedding described in (2.22), $\Phi_{3\text{d }\mathcal{N}=2}$ decomposes into a 2d $\mathcal{N}=(0,2)$ chiral multiplet plus a 2d $\mathcal{N} = (0,2)$ Fermi multiplet:

$$\Phi_{\text{2d } \mathcal{N} = (0,2)} = \Phi_{\text{3d } \mathcal{N} = 2}\Big|_{\theta^- = \bar{\theta}^- = 0} \tag{2.64}$$

$$\Psi_{\text{2d } \mathcal{N} = (0,2)} = D_- \Phi_{\text{3d } \mathcal{N} = 2}\Big|_{\theta^- = \bar{\theta}^- = 0}. \tag{2.65}$$

Antichiral multiplet:

The antichiral multiplet $\overline{\Phi}_{\text{3d } \mathcal{N} = 2}$ is defined by the conditions

$$D_{\pm} \overline{\Phi}_{\text{3d } \mathcal{N} = 2} = 0. \tag{2.66}$$

Solving these equations, we find its superspace component expansion

$$\begin{aligned}\overline{\Phi}_{\text{3d } \mathcal{N} = 2}(x, \theta, \bar{\theta}) = {} & \bar{\phi}(x) + \bar{\theta}^+ \bar{\psi}_+(x) + \bar{\theta}^- \bar{\psi}_-(x) + \bar{\theta}^+ \bar{\theta}^- \bar{F}(x) \\ & - i\theta^+ \bar{\theta}^+ \partial_{\bar{z}} \bar{\phi}(x) + i\theta^- \bar{\theta}^- \partial_z \bar{\phi}(x) + \frac{i}{2}\theta^+ \bar{\theta}^- \partial_t \bar{\phi}(x) + \frac{i}{2}\theta^- \bar{\theta}^+ \partial_t \bar{\phi}(x) \\ & + \theta^- \bar{\theta}^+ \bar{\theta}^- \left(\frac{i}{2}\partial_t \bar{\psi}_-(x) - i\partial_z \bar{\psi}_+(x)\right) - \theta^+ \bar{\theta}^+ \bar{\theta}^- \left(i\partial_{\bar{z}} \bar{\psi}_-(x) + \frac{i}{2}\partial_t \bar{\psi}_+(x)\right) \\ & - \theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- \left(\frac{1}{4}\partial_t^2 \bar{\phi}(x) + \partial_z \partial_{\bar{z}} \bar{\phi}(x)\right).\end{aligned} \tag{2.67}$$

Vector multiplet:

In Wess-Zumino gauge, the vector multiplet is [CO17]

$$V_{\text{3d } \mathcal{N} = 2} = -\sigma^{\mu}_{\alpha\beta} \theta^{\alpha} \bar{\theta}^{\beta} A_{\mu}(x) + i\epsilon_{\alpha\beta} \theta^{\alpha} \bar{\theta}^{\beta} \sigma(x) - i\epsilon_{\alpha\beta} \theta^{\alpha} \bar{\theta}^{\beta} \theta^{\gamma} \bar{\lambda}_{\gamma}(x) + i\epsilon_{\alpha\beta} \theta^{\alpha} \bar{\theta}_{\beta} \bar{\theta}^{\gamma} \lambda_{\gamma}(x) - \frac{1}{2}\theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- D(x). \tag{2.68}$$

2.2.6 3d $\mathcal{N} = 4$ multiplets

A nice description of 3d $\mathcal{N} = 4$ hypermultiplets as a superfield is not currently known (one needs a superfield containing infinitely many auxiliary fields) [HKLR87, GIKOS84]. Thus we describe 3d $\mathcal{N} = 4$ multiplets in terms of the 3d $\mathcal{N} = 2$ multiplets they decompose into under the 3d $\mathcal{N} = 2$ subalgebra.

Hypermultiplet:

A 3d $\mathcal{N} = 4$ hypermultiplet $\Phi_{3\text{d}\,\mathcal{N}=4}$ decomposes into two 3d $\mathcal{N} = 2$ chiral multiplets X and Y. To specify the full 3d $\mathcal{N} = 4$ action, we need to describe how the R-symmetry $SU(2)_H \times SU(2)_C$ acts on X and Y. The scalars of X and $\overline{Y}$ form an $SU(2)_H$ doublet and transform trivially under $SU(2)_C$. The complex fermions ψ_a^X and ψ_a^Y form an $SU(2)_C$ doublet and transform trivially under $SU(2)_H$.

In terms of the 2d $\mathcal{N} = (2, 2)$ subalgebra, a 3d $\mathcal{N} = 4$ hypermultiplet decomposes into two 2d $\mathcal{N} = (2, 2)$ chiral multiplets which we can express in terms of the 3d $\mathcal{N} = 2$ decomposition as [BDGH16]

$$\Phi_X = \{X, \psi_+^X, \bar{\psi}_-^Y, F_Y\} \qquad \Phi_Y = \{Y, \psi_+^Y, \bar{\psi}_-^X, F_X\}. \tag{2.69}$$

Vector multiplet:

The off shell degrees of freedom for the 3d $\mathcal{N} = 4$ vector multiplet are a 3d gauge field A_μ, a real plus a complex scalar field σ, φ that together transform in the adjoint representation of $SU(2)_C$, a complex fermion λ_α transforming in the fundamental of $SU(2)_C$ and another η_α transforming in the fundamental of $SU(2)_H$, as well as auxiliary fields D and F [BDGH16, DGGH20].

In terms of the 2d $\mathcal{N} = (2, 2)$ subalgebra, a 3d $\mathcal{N} = 4$ vector multiplet decomposes into a 2d $\mathcal{N} = (2, 2)$ a chiral superfield S and a twisted-chiral superfield Σ

$$S = \{A_\perp, \sigma, \bar{\lambda}_+, \eta_-, F\} \qquad \Sigma = \{\varphi, \eta_+, \lambda_-, D, A_1, A_2\}. \tag{2.70}$$

2.3 General features of 3d $\mathcal{N} = 4$ gauge theories

2.3.1 Higgs and Coulomb branches

We discuss generic features of the moduli of vacua in 3d $\mathcal{N} = 4$ supersymmetric gauge theories. The topics here are not crucial for understanding the rest of the thesis, but we felt we should at least briefly mention them for some sense of completeness and general discussion. We summarize the discussion found in [AHISS97], which focuses on 3d $\mathcal{N} = 2$, and comment about the enhanced properties of these moduli spaces when we have $\mathcal{N} = 4$ supersymmetry.

Higgs branch:

The Higgs branch is defined to be the space of vacua that minimize the potential for the matter multiplets. This requires that the vector multiplets be set to zero, hence is parametrized by gauge-invariant combinations of the VEVs of the scalar component of the matter fields. The classical description of the Higgs branch turns out to describe the Higgs branch in the quantum theory as well; there are no quantum corrections to the Higgs branch [APS96]. In 3d $\mathcal{N} = 4$ gauge theories, the Higgs branch is a hyperkähler manifold obtained by taking the hyperkähler quotient by the gauge group [HKLR87].

Coulomb branch:

The (classical) Coulomb branch is defined to be the space of vacua which minimize the vector multiplet contributions to the action. This requires that the matter multiplets are set to zero, hence is parametrized by the VEV of the scalars in the vector multiplet and is a hyperkähler manifold in 3d $\mathcal{N} = 4$ theories [HKLR87]. The Coulomb branch receives quantum corrections from loops diagrams and from instantons. In general, the Coulomb branch is a difficult object to compute due to these correction. We refer the reader to papers such as [CHZ14, BDG17, BFN19] for some advanced techniques on computing them. Mirror symmetry states that the Higgs branch for one theory should match the Coulomb branch for the mirror theory, and vice versa. By “match”, we mean they are isomorphic as hyperkähler manifolds when computed in the infrared.

2.3.2 Operator content

In this thesis, we will primarily be interested in studying the category of line operators in 3d $\mathcal{N} = 4$ gauge theories. We discuss line operators in much greater detail in section 4, but for now, it suffices to say that they are operators supported on 1-dimensional submanifolds (e.g. Wilson lines). Towards the end, we will also reveal a secondary interest in the local operators in these theories: they contribute to half-indices of theories with boundary. These are essentially traces of SUSY and flavor symmetry grading operators over the vector space of boundary local operators. These objects arise physically as the partition function of the theory on $HS^2 \times S^1$. Our interest stems from the desire to perform nontrivial checks of mirror boundary conditions; half-indices must match since they capture information about the boundary local operators. Section 5.6 describes in greater detail how one computes these objects with many examples.

2.3.3 Twists

Given a nilpotent supercharge Q in the SUSY algebra acting on some theory, one can construct the Q-*twisted* theory by taking cohomology with Q playing the role of the differential. In physics terminology, this equates to restricting one's attention to Q-invariant operators and considering those that differ by a supersymmetry transformation as equivalent [Wit88]. One desires our cohomology theory to retain Lorentz symmetry, and since Q generally transforms non-trivially under the Lorentz group, a modification must be made. One redefines the spacetime Lorentz group as an appropriate subgroup of the product of the Lorentz group with the R-symmetry group such that Q is invariant under these combined transformations. This point is a bit subtle and won't play any significant role in our local computations.

We will see that twisting a theory tends to enhance its structure; the details of exactly what structure it gains depends on the choice of nilpotent supercharge [ESW20]. In fact, one of main reason we consider $\mathcal{N} = 4$ theories is that 3d theories with fewer supercharges do not possess twists that yield the nice mathematical structures we are interested in [CG19]. When we consider theories with boundary later on, we will see that only certain twists are possible depending on the details of the boundary condition. In other words, the nilpotent supercharge must preserve the boundary condition in order for the twisting procedure to make sense.

Topological A-twist:

If we define

$$Q_A := Q_+^{+\dot{+}} + Q_-^{-\dot{+}}, \tag{2.71}$$

note that

$$P_z = \left\{ Q_A, -\frac{1}{2} Q_-^{+\dot{-}} \right\} \tag{2.72}$$

$$P_{\bar{z}} = \left\{ Q_A, -\frac{1}{2} Q_+^{-\dot{-}} \right\} \tag{2.73}$$

$$P_t = \{ Q_A, Q_-^{-\dot{-}} \}. \tag{2.74}$$

This means that all of the translation generators are Q_A-exact, hence translation acts (locally) trivially in the A-twist. We call such a theory "topological" since correlation functions are invariant under moving operator insertions around; the *local* translation invariance does not allow us to move operators through each other, hence the correlation function still depends on the topological type of the configuration. The A-twist in a 3d $\mathcal{N} = 4$ theory can be obtained from a dimensional reduction of the 4d Donaldson-Witten twist [Wit88].

Topological B-twist:

If we define

$$Q_B := Q_+^{+\dot{+}} + Q_-^{+\dot{-}}, \tag{2.75}$$

note that

$$P_z = \left\{ Q_B, -\frac{1}{2} Q_-^{-\dot{+}} \right\} \tag{2.76}$$

$$P_{\bar{z}} = \left\{ Q_B, -\frac{1}{2} Q_+^{-\dot{-}} \right\} \tag{2.77}$$

$$P_t = \{ Q_B, Q_-^{-\dot{-}} \}. \tag{2.78}$$

By the same reasoning, twisting by Q_B also yields a topological theory. The B-twist cannot be obtained via dimensional reduction of a higher dimensional theory. It was first discovered in various 3d $\mathcal{N} = 4$ gauge theories in [BT97] and [RW97]. The A- and B-twists can more directly be distinguished by which factor of the $SU(2) \times SU(2)$ R-symmetry group they preserve.

Holomorphic-topological twist:

If we define

$$Q_{HT} := Q_+^{+\dot{+}}, \tag{2.79}$$

note that

$$P_{\bar{z}} = \left\{ Q_B, -\frac{1}{2} Q_+^{-\dot{-}} \right\} \tag{2.80}$$

$$P_t = \{ Q_B, Q_-^{-\dot{-}} \} \tag{2.81}$$

but P_z is not in the image of $\{Q_{HT}, -\}$. This means that in the HT-twist, the t-direction is topological and $P_{\bar{z}} \propto \partial_{\bar{z}}$ acts as zero, hence correlation functions are holomorphic.

3 3d mirror symmetry

In this section we review various aspects of 3d mirror symmetry of $\mathcal{N} = 4$ gauge theories and set the stage for the main result of this thesis. Throughout, by 'mirror symmetry', we specifically mean 3d mirror symmetry; one otherwise would usually understand this phrase as a reference to the class of dualities of *two*-dimensional theories established in [HV00].

Mirror symmetry is an infrared duality of theories. The theories need not be supersymmetric, but the action of mirror symmetry becomes much richer if they are. Additionally if the theory supports a topological twist, then the duality will hold at all energy scales since the renormalization group flow acts trivially. Let us briefly review a simple non-supersymmetric example of mirror symmetry to begin.

3.1 Particle-vortex duality

Particle-Vortex duality is a duality between 3d theories where the local operators constructed from the fundamental fields (i.e. particles) in one theory map to monopole operators (i.e. vortices) in the dual theory. Some of the first examples dealt with abelian bosons on a lattice [Pes78, DH81, FL89].

3.2 Action on main structures

Recall from section 2.3.1 the definition of the Higgs and Coulomb branches of a supersymmetric theory. Given a mirror pair of theories $(\mathcal{T}, \mathcal{T}')$, the Higgs (resp. Coulomb) branch of $\mathcal{T}$ is the same as the Coulomb (resp. Higgs) branch of $\mathcal{T}'$ as hyperkähler manifolds [IS96]. In other words, the action of mirror symmetry *swaps* the Higgs branch of $\mathcal{T}$ with the Coulomb branch of $\mathcal{T}'$, etc. Along similar lines, these theories also possess a flavor symmetry group $G_C \times G_H$. The Coulomb branch symmetry G_C originates from the topological symmetry acting on the components of the gauge fields corresponding to $U(1)$ factors of the gauge group; essentially

$$G_C = U(1)^{\#\, U(1) \text{ factors in } G}. \tag{3.1}$$

The Higgs branch symmetry group G_H is defined to be the normalizer of the gauge group G in $USp(N)$ where N is the number of hypermultiplets. One typically requires that the hypermultiplets transform under a quaternionic representation of G, hence when we take the normalizer, we mean the image of G in $USp(N)$. Mirror symmetry swaps G_C and G_H, meaning that the Coulomb symmetry in $\mathcal{T}$ is the Higgs symmetry in $\mathcal{T}'$, etc.

Given a pair $(\mathcal{T}, \mathcal{T}')$ of mirror theories, one can consider their A- and B-twists. Mirror symmetry conjectures that the A-twist (B-twist, resp.) of $\mathcal{T}$ is mirror to the B-twist (A-twist, resp.) of $\mathcal{T}'$. As mentioned in Section 4.1, when considering line operators we restrict to the category of line operators that preserve the nilpotent supercharge we twist by. Thus mirror symmetry swaps the category of A-type line operators in $\mathcal{T}$ with the category of B-type line operators in $\mathcal{T}'$, etc.

Lastly, if the pair of bulk mirror theories each possess a boundary, then imposing Neumann boundary conditions in one theory is expected to be mirror dual to the mirror theory with generic Dirichlet boundary conditions [BDGH16]. If one instead imposes exceptional Dirichlet boundary conditions, one obtains so called 'enriched Neumann' boundary conditions which are essentially standard Neumann boundary conditions coupled to additional 2d matter living on the boundary [BZ21]. We expand our discussion of mirror symmetry and boundary conditions in Section 5.

3.3 Simple examples

Mirror symmetry of 3d theories *with* supersymmetry was first studied in [IS96] wherein the authors conjectured many new dualities between 3d $\mathcal{N} = 4$ SUSY gauge theories for various families of gauge groups. Shortly after, [BHOO97] enlarged these conjectures to include supersymmetric quiver gauge theories. These papers checked the proposed dualities and mirror maps by computing the dimensions of the Higgs and Coulomb branches and the metrics on the respective spaces, as well as counting the number of mass and FI parameters.

The action of mirror symmetry on the line operators in a theory (see Section 4) was uncovered in [AG15]. The authors conjectured that half-BPS Wilson loops are exchanged with vortex loops, and this was checked by computing supersymmetric partition functions (i.e. indices) in the presence of each of these loops.

3.4 Our example

Let us introduce the main pair of theories we study in this thesis. We start off with SQED[1]: $U(1)$ gauge theory with a single hypermultiplet. Its mirror dual is a single free twisted 3d $\mathcal{N} = 4$ hypermultiplet. It was checked in [KS99, KWY10b, KWY20] that the partition functions of these theories match, and it was further shown in [BKW02] that the monopole operators indeed satisfy other properties required by 3d mirror symmetry (e.g. they carry the expected conformal weights).

We let T_A be the A-twist of a free twisted hypermultiplet. According to section 3.2, T_A should be dual to the B-twist of SQED[1] T_B. The pair (T_A, T_B) constitute the simplest example of 3d abelian mirror symmetry of twisted SUSY gauge theories and will be the primary focus of this thesis.

As explained in Section 2.3.3, T_A and T_B are topological theories. While mirror symmetry is an *infrared* duality, topological invariance implies that T_A and T_B are actually equivalent at *all* energy scales. This allows us to perform computations using the ultraviolet degrees of freedom, which is more straightforward.

In particular, we study the categories of half-BPS line operators $\mathcal{L}_A$ ($\mathcal{L}_B$, resp.) that preserve the 1d $\mathcal{N} = 4$ SUSY subalgebras containing Q_A (Q_B, resp.) in T_A (T_B, resp.). These are the line operators that survive when we take the A- or B-twist; i.e. they are compatible with taking $Q_{A/B}$-cohomology. Mirror symmetry suggests that $\mathcal{L}_A$ is equivalent $\mathcal{L}_B$; we perform the nontrivial check that this is indeed the case in Section 6. We furthermore exploit this equivalence to study the structures of $\mathcal{L}_A$ by way of analyzing the corresponding structures on $\mathcal{L}_B$ where they are much easier to compute. We now transition to a presentation of the requisite background material before describing these main results.

4 Line operators

In contrast to local operators, which are operators that depend on a single point (or rather, an arbitrarily small neighborhood of a point), line operators are operators that depend on a 1-dimensional submanifold. A well known example are the Wilson loops constructable in any gauge theory

$$W_\gamma = \mathcal{P}e^{\int_\gamma \gamma^* \rho(A)} \tag{4.1}$$

where γ is a closed path and ρ is a representation of the Lie algebra. As stated in Section 3.4, local operators were studied and shown to match between certain conjectural mirror pairs of theories. A main focus/outcome of this thesis is to describe non-trivial progress we have made supporting one of these conjectures by proving that the line operators of these theories match. Some pioneering work in this direction can be found in [AG15]. Before we get there, we review some basic facts about line operators and review some simple examples.

4.1 Categorical structures

We will mainly be interested in the local structure of line operators, so we will restrict ourselves to studying line operators that are (semi)-infinite, straight, and parallel. Moreover, we would like these 1d defects to preserve as much of the 3d $\mathcal{N} = 4$ SUSY algebra as possible, which is one of the 1d $\mathcal{N} = 4$ subalgebras described in 2.1.6. As described in section 2.3.3, each of these subalgebras contain supercharges that yield topological theories when twisted by them. We further restrict focus to the set of line operators preserved by either of these supercharges, one at a time.

4.1.1 Morphisms

Let L and L' be two semi-infinite line operators that share an endpoint. The vector space of local operators $\mathrm{Ops}(L, L')$ insertable[1] at this endpoint is of particular interest since it is

[1] In general, the set of operators that can be inserted at a junction of lines L and L' is restricted. For example, the stacking of L and L' might not be gauge-invariant, thus by 'insertable operators', we mean the

a natural candidate of a notion of a morphism $L \to L'$. One can try to define a notion of composition of these operators

$$\circ\colon \mathrm{Ops}(L', L'') \otimes \mathrm{Ops}(L, L') \to \mathrm{Ops}(L, L'') \tag{4.2}$$

by a limiting procedure in which the local operators are brought close together by shrinking L' to a point, depicted in Figure 1. Unfortunately this map will not be associative in general

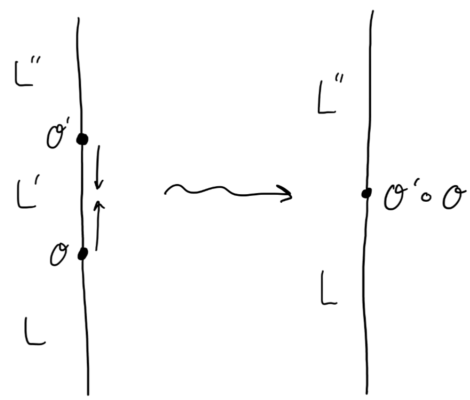


Figure 1: Depiction of composition map of local operators bound to line operators.

and heavily depends on the limiting procedure. We can remedy these issues by restricting attention to the set of A-type line operators or of B-type line operators. If we instead define morphism spaces as the Q_A- or Q_B-cohomology of $\mathrm{Ops}(L, L')$

$$\mathrm{Hom}_{A/B}(L, L') := H^\bullet(\mathrm{Ops}(L, L'), Q_{A/B}), \tag{4.3}$$

the composition map defined above will then be associative since translation of the operators along the line and the shrinking of L' are $Q_{A/B}$-exact operations. In conclusion, sets of line operators preserving a supercharge that is topological along the direction of the line can be given the structure of a category in a natural manner.

set of local operators whose gauge transformation properties exactly cancel the gauge covariance of L and L'.

4.1.2 Monoidal structure

Similar to the collision of local operators described in the previous section, one can also consider colliding parallel line operators, such as that depicted in Figure 2. By the principle of locality, we expect the result of such a collision in any correlation function is equivalent to an insertion of some other line operator in the theory. In general, this will not define a

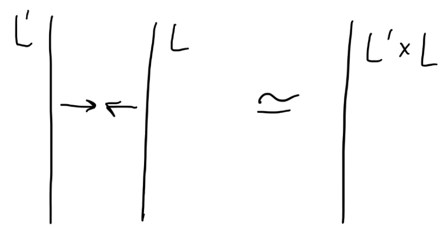


Figure 2: The collision of line operators, giving rise to a monoidal structure.

product with any of the usual/nice properties one expects and desires. For example, the product may be heavily dependent on the exact collision prescription. See [BBBDN20] for a more in-depth discussion.

In theories that preserve a fully topological supercharge, for example, colliding line operators in the corresponding twisted theory will not depend on the details of the collision and will define an associative product on the category of line operators. Theories that preserve an HT-twist with the holomorphic plane perpendicular to the line operators also possess a nice product given by the operator product expansion (OPE). This product is not strictly associative but it still satisfies an appropriate notion of associativity that we describe in Section 4.3.1.

Categories with a notion of multiplication on its objects that is associative and contain an identity object are called *monoidal* categories. In detail, a category $\mathcal{C}$ is said to be monoidal if it is equipped with

- a bifunctor

$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{4.4}$$

- functorial associativity isomorphisms

$$\alpha_{UVW}\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W) \tag{4.5}$$

  for all $U, V, W \in \mathrm{Ob}(\mathcal{C})$

- an identity object $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ with functorial isomorphisms

$$\begin{aligned} \lambda_V &\colon \mathbb{1} \otimes V \to V \\ \rho_V &\colon V \otimes \mathbb{1} \to V \end{aligned} \tag{4.6}$$

satisfying various properties that are intuitively understood but technical to state; see [BK01, Definition 1.1.7] for full details.

4.1.3 Braiding

One may encounter two line operators that have been twisted around each other, such as in Figure 3. We usually cannot just untwist them and compute as if they were disentangled.

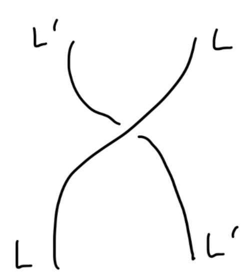


Figure 3: Braided line operators.

However, depending on the structures present in the theory, we can often relate these "braided" line operators to pairs of unbraided line operators if they are sufficiently close together. In topological theories for example, the relation is trivial since we are free to homotopically untwist the line operators (modulo subtleties involving asymptotic boundary conditions). In the following sections, we will describe theories that possess non-trivial braiding.

Mathematically speaking, a monoidal category $\mathcal{C}$ is a *braided tensor category* if, for every $V, W \in \text{Ob}(\mathcal{C})$, there exist a functorial isomorphism

$$\sigma_{V,W} : V \otimes W \to W \otimes V \tag{4.7}$$

such that the set of all such *braiding maps* are compatible with the monoidal structure and the relations of the braid group (see [BK01, Definition 1.2.3] for full details).

4.2 Warm up story, Act I

Some of the categorical structure of the previous section has been explicitly computed in the work of [DGGH20] where the authors studied Wilson lines in the B-twist of 3d $\mathcal{N} = 4$ gauge theories, vortex lines in the A-twist, and the categories they generate. In particular, the authors computed the morphism spaces, defined in Section 4.1.1, within these classes of line operators. Morphisms between half-BPS Wilson lines L and L' carrying representations R and R', for example, were found to be the space of insertable local operators living in the representation $R^* \otimes R'$ subject to an equivalence relation that sets the complex moment map to zero.

We pause to review the famous work of [Wit89] about line operators in non-abelian 3d Chern-Simons theory and highlight how the categorical structures introduced in the previous section appear naturally. This theory is topological, hence the set of line operators will have the rich structure described in section 4.1, although this modern language was not explicitly stated in the paper. We point out that this theory is not constructed from a topological twist of a supersymmetric theory, in contrast to many of the theories we will be interested in.

We fix an oriented 3-manifold M, a simple compact Lie group G, and a G-bundle E with connection 1-form A. We work in the theory described by the Chern-Simons Lagrangian

$$\mathcal{L} = \frac{k}{4\pi} \int_M \text{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big) \tag{4.8}$$

where $k \in \mathbb{Z}$. Wilson loops $W_R(\gamma)$ in this theory are labeled by an oriented closed curve $\gamma \in M$ and an irreducible representation R of G.

Once quantized, monopole operators yield isomorphisms between Wilson loops carrying different representations, hence the quantized theory only possesses a finite set of distinct Wilson loops corresponding to simple representations of G [EMSS89]. As a consequence, the category of line operators in quantized 3d Chern-Simons theory forms a finite semisimple category. This category was shown to additionally possess the structure of a braided tensor category by [Wit89, EMSS89].

Our goal is to be able to compute correlation functions involving arbitrary, mutually non-intersecting, Wilson loops, such as in Figure 4. The topology of M and the arrangement

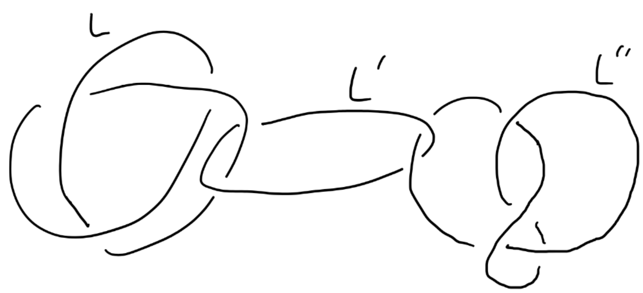


Figure 4: Example of an arbitrary collection of line operators.

of the Wilson loops can be quite complicated, so we would like to be able slice/decompose M into pieces that are easier to study. Locally, the region around each slice of M by an oriented Riemannian submanifold Σ looks like $\mathbb{R} \times \Sigma$. If a Wilson line $W_{R_i}(\gamma_i)$ intersects Σ at a point p_i, then it is natural to consider the p_i as marked points of Σ and to associate each point with the representation R_i or R_i^* depending on the relative orientation. This decomposition procedure yields a connection to vertex operator algebras (defined in Section 4.3) where the braided tensor structure will make its appearance. The authors explain how this structure can be used to express the expectation value of any link of Wilson loops in arbitrary representations in terms of expectation values of unknotted Wilson loops, which themselves are determined by the braided tensor structure. Let us summarize this procedure.

Suppose we start out with the Wilson loop arrangement in Figure 4. If the Wilson lines could pass through themselves, then everything could be unlinked and unknotted and the expectation value would simply be the product of the expectation value of the resulting unknots. Unfortunately the Wilson loops are not specters that can pass freely through each other without changing the expectation value. However, the braiding structure effectively lets us do just this by keeping track of how the expectation value changes as we pass lines past each other. Let us cut the space M along the hypersurface Σ. If we focus on the part of M on one side of Σ, something special happens on the manifold's putative boundary Σ: there is a 2-dimensional vertex operator algebra on Σ, and the points p_i where the Wilson lines end on Σ correspond to modules of the VOA. We justify these statements in Section 4.4, but let us describe the upshot before delving into the details. The idea is that we would like to be able to drag around the p_i so that, when we glue back the two regions of M, the links becomes untangled. The VOA's braiding structure tells us how to do this. We now take an intermission to introduce the unfamiliar reader to the relevant definitions and examples of VOAs and explain their relation to CFTs; they will play a central role in the rest of this thesis.

4.3 Vertex operator algebras

Vertex operator algebras (VOAs) are the mathematically rigorous formulation of the holomorphic sector of 2d conformal field theories (CFTs). CFTs are quantum field theories (QFTs) that are invariant under spacetime diffeomorphisms that rescale the metric by a positive function. In 2 spacetime dimensions, the group of such conformal transformations is infinite dimensional, imposing strong relations that the CFT's correlation functions must satisfy. The implications of these constraints were first explored in the pioneering paper [BPZ84]. Some standard references for CFT basics are [DMS97, Gin88, Sim17].

While VOAs are inherently two-dimensional objects, they can appear naturally in theories living in dimensions strictly larger than two that greatly aid the study of the original theory.

For example, we saw in the previous section how VOAs can be used to study a certain 3d theory by inserting an artificial codimension one surface. Similar techniques have been used to study 4d theories using codimension two hypersurfaces (see, e.g. [LL16, BL+15, BL20]).

4.3.1 Definitions and explanation

A good introductory book on vertex operator algebras is [FB04]. We will follow the development of the theory of VOAs contained in [HLZ10-11].

Definition 4.1 (Definition 2.2 in [HLZ10-11]). *A vertex operator algebra is a $\mathbb{Z}$-graded vector space*

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)} \tag{4.9}$$

together with two distinguished vectors:

- vacuum vector: $|0\rangle \in V_{(0)}$
- conformal vector: $\omega \in V_{(2)}$

and a linear map

$$\begin{aligned} Y(-,z)\colon V &\to End(V)[\![z,z^{-1}]\!] \\ Y(v,z) &= \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \end{aligned} \tag{4.10}$$

called the state-operator correspondence, satisfying the following condition:

1. lower truncation: $v_n w = 0$ *for any* $v, w \in V$ *and all* n *sufficiently large (i.e.* $Y(v,z)w \in V(\!(z)\!)$*)*
2. vacuum property: $Y(|0\rangle, z) = \mathbb{1}_V$*, the identity operator on* V
3. creation property: $Y(v,z)|0\rangle \in V[\![z]\!]$ *and* $\lim_{z\to 0} Y(v,z)|0\rangle = v$

4. Jacobi identity:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(v,z_1)Y(w,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(w,z_2)Y(v,z_1)$$
$$=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(v,z_0)w,z_2) \quad (4.11)$$

where

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right):=z_0^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{z_1-z_2}{z_0}\right)^n=\sum_{m\in\mathbb{N},n\in\mathbb{Z}}(-1)^m\binom{n}{m}z_0^{-n-1}z_1^{n-m}z_2^m \quad (4.12)$$

is an element of $\mathbb{C}[z_0,z_0^{-1}][\![z_1,z_1^{-1},z_2]\!]$

5. internal Virasoro algebra: *with* $L(n):=\omega_{n+1}$*, i.e.*

$$Y(\omega,z)=\sum_{n\in\mathbb{Z}}L(n)z^{-n-2}, \quad (4.13)$$

there exists some $c\in\mathbb{C}$*, called the theory's central charge, such that*

$$[L(m),L(n)]=(m-n)L(m+n)+\frac{1}{12}m(m+1)(m-1)\delta_{m+n,0}\,c\mathbb{1}_V \quad (4.14)$$

6. $L(-1)$-derivative property:

$$\frac{d}{dz}Y(v,z)=Y(L(-1)v,z) \quad (4.15)$$

7. weight condition: *for each* $n\in\mathbb{Z}$ *and* $v\in V_{(n)}$*,*

$$L(0)v=nv. \quad (4.16)$$

It follows that if $v\in V_{(n)}$, then v_m maps $V_{(k)}$ to $V_{(k+n+m-1)}$; i.e. the $\mathbb{Z}$-grading is compatible with the internal action of the VOA on itself.

Let us elaborate upon this behemoth of a definition and explain how it connects to familiar notions in CFT. The vector space V is the Hilbert space of states of the CFT, which is regarded as living at the origin of the complex plane formally parametrized by the variable z. While our notation does not directly coincide with [HLZ10-11], the reader may find it helpful to take note of Remark 1.3 therein to keep in mind when variables are formal versus complex. We require z be formal to ensure certain equations are well-defined by forcing an algebraic interpretation.

Given a state $v \in V$, one can construct an associated field $v(z)$ using the state operator map: $v(z) := Y(v, z)$. In physics, this map is achieved by a path integral with appropriate boundary conditions reflecting the insertion of the state v and point z (see [Sim17, Section 6] and [Pol07a, Chapter 2]). The fact that this map is an isomorphism results from the conformal symmetry: Goddard's uniqueness theorem states that every field is determined by a *unique* state given by the creation property [FB04, Section 3.1]. This correspondence manifests itself in the existence of the state-operator correspondence map $Y(v, z)$ (a.k.a. operator-field correspondence, or the state-field correspondence) satisfying the conditions in Definition 4.1.

There is a division between the mathematical and physics literature regarding the indexing of the *modes* (i.e. coefficients) of fields $Y(v, z)$. We follow the mathematical conventions wherein all modes, aside from ones in the stress-energy tensor $Y(\omega, z)$, are indexed so that

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}. \tag{4.17}$$

If v had conformal weight h_v, a physicist would have indexed the modes in the following manner

$$Y(v, z) = \sum_{n \in \mathbb{Z} - h_v} v_n z^{-n-h_v}. \tag{4.18}$$

For a discussion comparing the advantages of each convention, see [KR18, Section 2].

The implications of the horrendous Jacobi identity can be briefly summarized in that it leads to the existence of the operator product expansion (OPE)

$$Y(a,z)Y(b,w)c = Y(Y(a,z-w)b,w)c \tag{4.19}$$
$$= \sum_{n\in\mathbb{Z}} \frac{Y(a_n \cdot b, w)}{(z-w)^{n+1}} c \tag{4.20}$$

where $a, b, c \in V$. This can be considered as a version of an associativity condition in the realm of VOAs. The lower truncation condition ensures that the OPE only contains singular terms up to a finite order.

In physics, one encounters this equation in the guise

$$Y(a,z) \cdot Y(b,w) \sim \sum_{n\geq 0} \frac{\mathcal{O}_n(w)}{(z-w)^{n+1}} \tag{4.21}$$

where $\mathcal{O}_n(w)$ are fields. Note that only singular terms are written in equation (4.21); the non-singular terms are determined by the easily computable normal-ordered product of $Y(a,z)$ and $Y(b,z)$, hence are omitted from the right-hand side for convenience. The precise meaning of the OPE (4.19) as an algebraic equation is a bit technical to spell out (see [FB04, Section 3.3]), but it essentially states that if field insertion points z and w are sufficiently close together, then the product $Y(a,z)Y(b,w)$ is equal to an expansion in the small parameter $z - w$ when $|z| > |w|$.

The quantized conformal symmetry algebra in 2d is the Virasoro Lie algebra defined by (4.14), hence the fields in our theory carry an action of this algebra dictating how the fields transform under local conformal transformations.

We would like the Virasoro action to be generated by a state in the Hilbert space V; this is the meaning of the *internal* Virasoro algebra condition in Definition 4.1. The field $Y(\omega, z)$ is the stress-energy tensor $T(z)$ in CFT and it's components generate local conformal transformations by taking residues against various powers of z. In particular, it's $(-1)^{\text{th}}$ component generates holomorphic translations; this requirement is encapsulated by the $L(-1)$-derivative property. Dilations are generated by the 0^{th} component, which is the meaning of the weight condition. The weight condition requires that the grading on V coincides with the usual grading by conformal weight. Since one typically considers conformally invariant vacua in physics, we now understand why the vacuum vector must live in $V_{(0)}$.

In the following section, we will present multiple examples of vertex operator algebras to familiarize the reader, but also since some will play central roles in the remainder of this thesis. We will also encounter vertex operator *super*algebras (VOSAs) in this thesis. Our definition of a vertex operator superalgebra (VOSA) closely follows that of a VOA but there is an additional $\mathbb{Z}_2$ grading present and all VOA structures and conditions are rephrased to make them compatible (e.g. commutativity conditions are replaced by *graded* commutativity conditions); see remark 1 of [FB04, Section 1.3.2] and [CKM17, Section 1.4] for further details.

The final definition we give before proceeding to examples is that of a *module* for a VOA. These appear in physical theories whose space of states is not simply generated by the action of the mode algebra on the VOA's vacuum vector (i.e. Fock module). Canonical examples are modules for the free boson generated by vertex operators (see Section 4.3.2).

Definition 4.2 (Definitions 2.9 and 2.12 of [HLZ10-11])**.** *A generalized module for a VOA V is a $\mathbb{C}$-graded vector space*

$$W = \coprod_{\alpha \in \mathbb{C}} W_{[\alpha]} \tag{4.22}$$

together with a linear map

$$Y_W(-, z) \colon V \to End(W)[\![z, z^{-1}]\!] \tag{4.23}$$

with mode expansion

$$Y_W(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \tag{4.24}$$

satisfying the following conditions:

1. lower truncation: *for each* $v \in V$ *and* $w \in W$, $v_n w = 0$ *for all* n *sufficiently large*

2. vacuum property: $Y_W(|0\rangle, z) = \mathbb{1}_W$, *where* $\mathbb{1}_W$ *is the identity operator on* W

3. Jacobi identity for Y_W:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_W(u,z_1)Y_W(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y_W(v,z_2)Y_W(u,z_1) \\ = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y_W(Y(u,z_0)v,z_2) \tag{4.25}$$

4. Virasoro action: *defining* $L_W(n)$ *by*

$$Y_W(\omega,z) = \sum_{n\in\mathbb{Z}} L_W(n) z^{-n-2}, \tag{4.26}$$

   we have

$$[L_W(m), L_W(n)] = (m-n)L_W(m+n) + \frac{1}{12}m(m+1)(m-1)\delta_{m+n,0}\, c\mathbb{1}_V \tag{4.27}$$

   where the central charge takes the same value as for the VOA V *itself*

5. $L(-1)$-derivative property:

$$\frac{d}{dz}Y_W(v,z) = Y_W(L(-1)v,z) \tag{4.28}$$

6. generalized weight condition: *for each* $\alpha \in \mathbb{C}$ *and* $w \in W_{[\alpha]}$, *there exists some* $m > 0$ *such that*

$$(L(0)-\alpha)^m w = 0. \tag{4.29}$$

The field $Y_W(v, z)$ is called the vertex operator on W associated to $v \in V$ and describes how the VOA V acts on W. These conditions are direct analogues to the ones found in Definition 4.1, hence the physical motivation is similar. We only comment/emphasize that we do *not* require $L(0)$ to act semisimply: in many physical theories of interest, such as the ones we consider in Section 6, one naturally encounters modules that are reducible but indecomposable. Such theories are called *logarithmic* VOAs/CFTs due to the appearance of logarithms within the OPEs, correlation functions, and intertwiners of the theory; see [CR13a] for a good, physically motivated introduction on this topic and [HLZ10-11] for a comprehensive mathematical discussion. A *rational* VOA is one in which the representation category *is* semisimple, although we warn the reader that the meaning of this term varies slightly in the literature.

4.3.2 Example 1: free boson

The free boson is perhaps the simplest VOA one encounters in a first course on VOAs and CFTs. From a physics perspective, it is described by a field $X(z, \bar{z})$ subject to the action

$$S = \frac{1}{2\pi} \int_{\mathbb{C}} \partial X \bar{\partial} X. \tag{4.30}$$

The equations of motion imply that the field is of the form

$$X(z, \bar{z}) = \frac{1}{2}(x(z) + \bar{x}(\bar{z})). \tag{4.31}$$

Once quantized, the modes in the Laurant expansion of x

$$x(z) = \tilde{x} + x_0 \log z - \sum_{n \neq 0} \frac{1}{n} x_n z^{-n}. \tag{4.32}$$

are found to satisfy

$$[x_n, \tilde{x}] = \delta_{n,0} \qquad\qquad [x_n, x_m] = n\delta_{n,-m}. \tag{4.33}$$

The term involving $\log z$ indicates that $x(z)$ is not a field directly associated to a state in the VOA. However, its derivative

$$\partial x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1} \tag{4.34}$$

is a perfectly valid field associated to the state $x_{-1}|0\rangle$ and has OPE

$$\partial x(z) \cdot \partial x(w) \sim \frac{-1}{(z-w)^2}. \tag{4.35}$$

We halt our physics description of the free boson CFT and restart it from the mathematical point of view, describing how one constructs the free boson *VOA*. We hope this helps bridge the two perspectives together. For further exposition from the physics point of view, we refer the reader to [Gin88, Section 2.3] and [DMS97, Section 6.3].

To construct the free boson VOA, more commonly known as the Heisenberg VOA or the Weyl VOA, we start by constructing the Heisenberg *Lie* algebra as the central extension

$$0 \longrightarrow \mathbb{C} \cdot \mathbb{1} \longrightarrow \mathcal{H} \longrightarrow \mathbb{C}((t)) \longrightarrow 0 \tag{4.36}$$

with Lie bracket

$$[f(t) + a\mathbb{1}, g(t) + b\mathbb{1}] := -(\mathrm{Res}_{t=0} f(t) g'(t))\mathbb{1} \tag{4.37}$$

where the residue operator returns the coefficient of t^{-1}. Concretely, if we define $b_n := t^n$, then

$$[b_n, b_m] = n\delta_{m,-n}\mathbb{1}. \tag{4.38}$$

The vector space upon which the VOA structure is built is obtained as a representation space of the Heisenberg Lie algebra in the following manner. Note that $\mathcal{H}$ can essentially be split into a space of "negative modes" $\langle b_k \rangle_{k<0}$ of "non-negative modes" $\langle b_k \rangle_{k \geq 0}$. Intuitively, we would like to construct a vector space V generated by a vector $|0\rangle$ upon which the negative modes act freely and that is killed by the non-negative modes. This is a familiar construction of the Fock space one encounters in a course on Quantum Mechanics. Such a space is realized mathematically as the polynomial ring in infinitely many commuting variables

$$V := \mathbb{C}[b_k]_{k<0}. \tag{4.39}$$

The action of the negative modes is the usual multiplication in the polynomial ring, and the non-negative modes act as derivations $b_n = n\frac{\partial}{\partial b_{-n}}$. The vertex operator algebra structure on V is defined as follows. The state-operator map is defined by

$$Y(b_{k_1}b_{k_2}\cdots b_{k_n}, z) = \frac{1}{\prod_{j=1}^{n}(-k_j-1)!}:\prod_{j=1}^{n}\partial_z^{-k_j-1}b(z): \tag{4.40}$$

where

$$b(z) := Y(b_{-1}, z) = \sum_{k\in\mathbb{Z}} b_k z^{-k-1}. \tag{4.41}$$

The normally ordered product of fields $:A_1(z)\cdots A_n(z):$ is a non-associative multilinear map that rearranges the coefficients at each order in z so that all of the positive modes are placed to the right of the non-positive modes [FB04, Definition 2.2.2]. Mathematically, this seemingly bizarre operation is actually necessary for the VOA structure to be well-defined. From a physics point of view, normal ordering is seen as a regularization technique that gets rid of pesky infinities. The vacuum vector is $|0\rangle = 1 \in V$ and there is a 1-parameter family of conformal vectors given by

$$\omega_\lambda = \frac{1}{2}b_{-1}^2 + \lambda b_{-2} \qquad \lambda \in \mathbb{C} \tag{4.42}$$

which determines the grading on V and is independent of λ. If we compute the OPE, we find

$$b(z)\cdot b(w) \sim \frac{1}{(z-w)^2} \tag{4.43}$$

which matches (4.35). Thus we see the connection between our physical and mathematical expositions: the CFT and VOA are identified via $b(z) \longleftrightarrow i\partial X(z)$! The mode $\tilde{x}$ of $X(z)$

(4.32), which is absent in $\partial X(z)$, allows us to easily construct a family of modules for V in the following manner. This mode commutes with all other modes aside from x_0

$$[\tilde{x}, x_n] = -\delta_{n,0}. \tag{4.44}$$

Therefore if we define the vector space

$$F_\lambda := e^{\lambda \tilde{x}} \cdot V, \tag{4.45}$$

we find that F_λ carries the same action of $\mathcal{H}$ as V except now b_0 acts as $\lambda \mathbb{1}_{F_\lambda}$. In physics one usually denotes $e^{\lambda\tilde{x}}|0\rangle$ as $|\lambda\rangle$, in which case it becomes natural to realize F_λ as simply another Fock module for the free boson VOA! We remark that one can also use $\tilde{x}$ to construct reducible but indecomposable modules. For example, the module F_0^2 generated by $\tilde{x}|0\rangle$ has unique submodule F_0 generated by $|0\rangle$, and the quotient F_0^2/F_0 is isomorphic to F_0, but $F_0^2 \ncong F_0 \oplus F_0$.

4.3.3 Example 2: lattice VOA

Some of the Fock modules F_λ can be used to construct a VOA extension of V. By this we mean a larger VOA that contains the original one as a sub-VOA. For any integer $n \in 2\mathbb{Z}\setminus\{0\}$[2], the lattice $\sqrt{n}\mathbb{Z}$ corresponds to a VOA built as

$$V_{\sqrt{n}\mathbb{Z}} := \bigoplus_{m\in\mathbb{Z}} F_{m\sqrt{n}}. \tag{4.46}$$

and with the same conformal element from F_0. The state-operator map is extended from V by

$$Y(|m\sqrt{n}\rangle, z) := e^{m\sqrt{n}\tilde{x}} z^{m\sqrt{n}b_0} e^{-m\sqrt{n}\sum_{n<0}\frac{b_n}{n}z^{-n}} e^{-m\sqrt{n}\sum_{n>0}\frac{b_n}{n}z^{-n}}. \tag{4.47}$$

[2]When $n \in 2\mathbb{Z}+1$, one obtains a VOSA.

Note that $z^{m\sqrt{n}b_0}$ acts as an *integral* power of z on any element of $V_{\sqrt{n}\mathbb{Z}}$ (as it must to be a state-operator map) and that $e^{m\sqrt{n}\tilde{x}}$ maps $F_{a\sqrt{n}} \to F_{(m+a)\sqrt{n}}$. One should compare equation (4.47) to [DMS97, Equation 6.59] to witness the match to the physics perspective, however they consider a more general setting where one allows for arbitrary complex power of z. There are mathematical frameworks that generalize the notion of a VOA to allow for rational powers of z in the state-operator map, such as [DL93]. In general though, the RHS of equation (4.47) does not define a valid field for arbitrary $\alpha \in \mathbb{C}$, but instead defines an intertwining operator; we describe these objects in Section 4.3.5.

To construct modules for $V_{\sqrt{n}\mathbb{Z}}$, we need to ensure that the action of $z^{m\sqrt{n}b_0}$ in (4.47) results in integral powers of z. The natural object to look at is the dual lattice $\frac{1}{\sqrt{n}}\mathbb{Z}$. Since the action of modes in $V_{\sqrt{n}\mathbb{Z}}$ can map a vector of b_0-weight λ to any weight in $\lambda + \sqrt{n}\mathbb{Z}$, any module corresponding to a point in the dual lattice p must involve all points in the dual lattice of the form $p + m\sqrt{n}$ to ensure the action is closed. Thus we are naturally led to consider modules labeled by the quotient $\frac{1}{\sqrt{n}}\mathbb{Z}/\sqrt{n}\mathbb{Z}$, which has n elements. In conclusion, $\Lambda \in \frac{1}{\sqrt{n}}\mathbb{Z}/\sqrt{n}\mathbb{Z}$ defines a $V_{\sqrt{n}\mathbb{Z}}$-module

$$\mathcal{F}_\Lambda := \bigoplus_{\lambda \in \Lambda} F_\lambda. \tag{4.48}$$

One can generalize this notion of a lattice VOA corresponding to lattices of higher rank and with non-definite bilinear form. In Section 6.3.3 we make use of such a VOA to realize and study a VOA of interest as a sub-VOA, so we briefly review the relevant construction here. The construction is very similar to the construction of the rank-1 lattice that we just treated.

Let $L_\mathbb{Z}$ be a rank r ($r < \infty$) lattice with symmetric bilinear form $(-,-)\colon L_\mathbb{Z} \times L_\mathbb{Z} \to \mathbb{Z}$. Note we do not require this form to be positive-definite. Let $L := \mathbb{C} \otimes_\mathbb{Z} L_\mathbb{Z}$ be a minimal complex vector space containing $L_\mathbb{Z}$. The rank r Heisenberg Lie algebra associated to L is the central extension

$$0 \longrightarrow \mathbb{C}\cdot \mathbb{1} \longrightarrow \hat{L} \longrightarrow L(\!(t)\!) \longrightarrow 0 \tag{4.49}$$

with Lie bracket

$$[v \otimes f(t), w \otimes g(t)] := -(v, w)\mathrm{Res}_{t=0} f(t) g'(t) \mathbb{1} \tag{4.50}$$

where the bilinear form on $L_{\mathbb{Z}}$ has been extended to a bilinear form on L in the natural way. The construction of the Weyl algebra $\mathcal{H}_L$ associated to L is similar to the rank 1 case; the modes satisfy

$$[v_n, w_m] = n(v, w)\delta_{n,-m} \tag{4.51}$$

for $v, w \in L$. The final step to obtaining a VOA is to construct the state-operator map on a particular Fock module of $\mathcal{H}_L$. We have the usual (vector space) decomposition of $\mathcal{H}_L$ into non-negative and negative parts, which given a vector $v \in V$, defines a Fock module

$$F_v := \mathrm{Ind}_{\mathcal{H}_{L \geq 0}}^{\mathcal{H}_L} \mathbb{C}|v\rangle. \tag{4.52}$$

The notation $\mathrm{Ind}_B^A M$ means the following: suppose we have a Lie algebra A with sub-Lie algebra B and M is a representation of B. Then

$$\mathrm{Ind}_B^A M := \mathcal{U}(A) \otimes_{\mathcal{U}(B)} M \tag{4.53}$$

where $\mathcal{U}(A)$ is the universal enveloping algebra of A. This space naturally carries the structure of an A-module.

In our current setting, the positive modes of $\mathcal{H}_{L\geq 0}$ annihilate $|v\rangle$ and the zero-modes act as

$$w_0|v\rangle = (w, v)|v\rangle. \tag{4.54}$$

The module F_0 can be given the structure of a VOA similar to the rank-1 case.

4.3.4 Affine Kac-Moody VO(S)As / WZW models

When a CFT possesses a flavor symmetry described (infinitesimally) by a Lie algebra $\mathfrak{g}$, the corresponding conserved current $J(z)$ in the quantized theory generates a sub-VOA of the theory. This is called an affine Kac-Moody VOA. In general, one obtains a VOSA if the symmetry contains a fermionic part; such an operator algebra will be central to a main result of this thesis. We describe the non-super story here, but will explicitly describe the VOSA of relevance when it becomes appropriate in Section 6.2.

Let us describe how they are constructed mathematically. Let $\mathfrak{g}$ be the Lie algebra of G and let κ be an invariant bilinear form on $\mathfrak{g}$. For now we assume $\mathfrak{g}$ to be finite-dimensional and simple. Let

$$L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}(\!(t)\!) \tag{4.55}$$

be the formal loop algebra of $\mathfrak{g}$. This inherits a Lie algebra structure from $\mathfrak{g}$:

$$[A \otimes f(t), B \otimes g(t)]_{L\mathfrak{g}} := [A, B]_{\mathfrak{g}} \otimes f(t)g(t). \tag{4.56}$$

The Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ is the central extension of $L\mathfrak{g}$ by a 1-dimensional Lie algebra with generator K

$$0 \longrightarrow \mathbb{C} \cdot K \longrightarrow \hat{\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0 \tag{4.57}$$

with Lie bracket

$$[A \otimes f(t), B \otimes g(t)]_{\hat{\mathfrak{g}}} := [A \otimes f(t), B \otimes g(t)]_{L\mathfrak{g}} - (\mathrm{Res}_{t=0} f(t)g'(t))\kappa(A, B) \cdot K. \tag{4.58}$$

The construction of the Kac-Moody VOA associated to $\hat{\mathfrak{g}}$ is a straightforward generalization of the construction of the free boson VOA: it will be a $\hat{\mathfrak{g}}$-module induced relative to a splitting of $\hat{\mathfrak{g}}$ into negative and non-negative modes. Note that $\mathfrak{g}[\![t]\!] \oplus \mathbb{C}K$ is a commutative Lie subalgebra of $\hat{\mathfrak{g}}$. We define the vacuum representation of $\hat{\mathfrak{g}}$ at level $k \in \mathbb{C}$ to be

$$V_k := \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[\![t]\!] \oplus \mathbb{C}\cdot K} \mathbb{C}_k, \tag{4.59}$$

wherein K acts on $\mathbb{C}_k$ by $k\mathbb{1}_{\mathbb{C}_k}$ and $\mathfrak{g}[\![t]\!]$ acts as zero. Defining the vacuum vector to be

$$v_k := 1 \otimes 1 \in \hat{\mathfrak{g}} \tag{4.60}$$

and letting

$$J_n := J \otimes t^n, \tag{4.61}$$

we see that $J_n \cdot v_k = 0$ for $n \geq 0$. The state-operator map is defined by

$$Y(J^{a_1}_{n_1} \cdots J^{a_m}_{n_m} v_k, z) := \, :\prod_{\ell=1}^{m} \frac{1}{(-n_\ell - 1)!} \partial_z^{-n_\ell - 1} J^{a_\ell}(z): \tag{4.62}$$

where

$$J^{a_\ell}(z) := Y(J^{a_\ell}_{-1} v_k, z) = \sum_{k \in \mathbb{Z}} J^{a_\ell}_k z^{-k-1}. \tag{4.63}$$

This vertex algebra has a conformal vector when $k \neq -h^\vee$ given by the Segal-Sugawara vector: if $\{J^a\}$ is a basis for $\mathfrak{g}$, then letting $\{J_a\}$ be the dual basis with respect to κ, the Segal-Sugawara vector is given by

$$S := \frac{1}{2} \sum_{k=1}^{\dim \mathfrak{g}} J_{a,-1} J^a_{-1} v_k. \tag{4.64}$$

When the level is a positive integer, V_k is reducible as a $\hat{g}$-module, in which case one often quotients by the unique maximal ideal to obtain a simple VOA L_k. The simple quotient of a Kac-Moody VOA for $\mathfrak{g}$ simple is rational when k is a positive integer [FZ92]; Wess-Zumino-Witten (WZW) models are the physical CFT counterparts to a precisely this subclass of Kac-Moody VOAs (see [DMS97, Chapter 15] for an overview of WZW models). We've actually already encountered Kac-Moody VOAs in section 4.4!

4.3.5 Braided tensor structure

We have given the mathematical definitions of a braided tensor category in sections 4.1.2 and 4.1.3 and described how a category of line operators may possess these structures in certain theories. In this section, we describe the monoidal and braiding structures that certain VOA module categories enjoy. Along the way, we illustrate how these structures naturally arise from a physics perspective in 2d CFTs.

For the rest of this section, we fix some $z \in \mathbb{C}^\times$. The choice of z is immaterial as there are isomorphisms to all of the following objects and notions if we had instead picked some other z'.

To discuss the tensor product, we must first introduce the notion of a $P(z)$-intertwining map.

Definition 4.3 (Definition 4.2 of [HLZ10-11]). *Given three generalized modules W_1, W_2, W_3 for a VOA V, a $P(z)$-intertwining map of type $\begin{pmatrix} W_3 \\ W_1\, W_2 \end{pmatrix}$ is a linear map*

$$I\colon W_1 \otimes_{\mathbb{C}} W_2 \to \overline{W}_3, \tag{4.65}$$

where $\otimes_{\mathbb{C}}$ is the vector space tensor product and $\overline{W}$ denotes the completion of W with respect to the L_0 grading, satisfying a grading compatibility condition, a lower truncation condition, a Jacobi identity:

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right) Y_3(v, z_1) I(w_1 \otimes w_2) \\ &\qquad = z^{-1}\delta\left(\frac{z_1 - z_0}{z}\right) I(Y_1(v, z_0) w_1 \otimes w_2) + z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right) I(w_1 \otimes Y_2(v, z_1) w_2), \end{aligned} \tag{4.66}$$

and a compatibility with the Virasoro action (see the citation for full details). The Jacobi identity here encodes that the map I appropriately intertwines the action of V.

After fixing a branch, these $P(z)$-intertwining maps are in 1-1 correspondence with logarithmic intertwining operators [HLZ10-11, Proposition 4.8]. In the free boson VOA, the vertex operators $:e^{\alpha X(z)}:$ are examples of $P(z)$-intertwining maps.

Definition 4.4 (Definition 4.13 of [HLZ10-11])**.** *Given two modules W_1, W_2 of a VOA V, a $P(z)$-product of W_1 and W_2 is another V-module W_3 (with state operator map $Y_3(-,z)$) together with a $P(z)$-intertwining map I of type* $\begin{pmatrix} W_3 \\ W_1\ W_2 \end{pmatrix}$.

Notice the use of the article 'a'; in general there may be many (non-isomorphic) $P(z)$-products of two modules. The tensor product of two objects is an initial object in an appropriate category of $P(z)$-products.

Definition 4.5 (Definition 4.15 of [HLZ10-11])**.** *Given two modules W_1, W_2 of a VOA V, a $P(z)$-tensor product of W_1 and W_2 is a $P(z)$-product $(W, Y; I)$ of W_1 and W_2 such that, for any other $P(z)$-product $(W', Y'; I')$ of W_1 and W_2, there exists a unique module map $\eta \colon W \to W'$ such that the following diagram commutes*

$$\begin{array}{ccc} & W_1 \otimes_{\mathbb{C}} W_2 & \\ {\scriptstyle I} \swarrow & & \searrow {\scriptstyle I'} \\ \overline{W} & \xrightarrow{\quad \overline{\eta} \quad} & \overline{W'} \end{array} \tag{4.67}$$

where $\overline{\eta}$ is the extension of η to the completions of W and W'.

We denote the $P(z)$-tensor product as

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)}). \tag{4.68}$$

This definition may feel very alien and abstract for a physicist. Let us briefly review the tensor product from the physics perspective, point out it's naïvity, and explain how one fixes the issue.

In physics, the fusion product often plays the role of the tensor product bifunctor. One envisions that we have two modules W_1 and W_2 inserted at points, say, 0 and z. Given some operator $\mathcal{O}(\zeta)$ from our VOA V inserted at some point $|\zeta| > |z|$, using principles of locality, we attempt to define an action of each mode $\mathcal{O}_n$ on $W_1 \otimes_{\mathbb{C}} W_2$ by taking residues of $\mathcal{O}(\zeta)$ around a loop containing 0 and z with appropriate powers of ζ inserted [MS89]. This is an approach to define a coproduct $\Delta_{z,0}\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ on the chiral algebra (i.e. the algebra generated by the End(V)-valued coefficients that appear in the image of the state-operator map) defined by the following equation

$$\Delta_{z,0}(\mathcal{O}_n) := \oint_{|\zeta - z|=1} d\zeta\, \zeta^{n+\Delta-1} \left(\sum_{m\in\mathbb{Z}} (\zeta - z)^{-m-\Delta} \mathcal{O}_m \right) \otimes_{\mathbb{C}} \mathbb{1} + \mathbb{1} \otimes_{\mathbb{C}} \mathcal{O}_n \tag{4.69}$$

$$= \sum_{k=0}^{\infty} \binom{n+\Delta-1}{k} z^{n+\Delta-1-k} \mathcal{O}_{1+k-\Delta} \otimes_{\mathbb{C}} \mathbb{1} + \mathbb{1} \otimes_{\mathbb{C}} \mathcal{O}_n. \tag{4.70}$$

Here Δ is the conformal weight of $\mathcal{O}(\zeta)$.

One should note that, were the insertion point of W_1 at any point other than zero, the simple $\mathbb{1} \otimes_{\mathbb{C}} \mathcal{O}_n$ term would become equally as messy as the $\mathcal{O}_n \otimes_{\mathbb{C}} \mathbb{1}$ terms. This intuitive notion leads to an attempt at a definition of fusion of modules known as the Nahm-Gaberdiel-Kausch fusion algorithm [Nah94, Gab94b, Gab94a, GK96]. We do not copy the result here, but suffice it to say that the fusion product is defined by a quotient of the vector space $W_1 \otimes_{\mathbb{C}} W_2$ by a certain relation: a choice was implicitly made in equation (4.69) to take the OPE of $\mathcal{O}(\zeta)$ with the field in W_1. One obtains a different notion of the coproduct $\tilde{\Delta}_{z,0}$ if one instead choose to take the OPE of $\mathcal{O}(\zeta)$ with the field from W_2. Physically it should not matter which field we take the OPE of $\mathcal{O}(\zeta)$ with, hence the relations are generated by $\Delta_{z,0} - \tilde{\Delta}_{z,0}$. Kanade and Ridout [KR18] cleanly demonstrated the unfortunate mathematical failure of this intuitive attempt: these relations are not well-defined. A successful construction of the $P(z)$-tensor product for many VOAs is given in full detail in [HLZ10-11, Section 5.2] and summarized in [KR18, Section 6].

In a CFT, one obtains nontrivial braiding when correlation functions involve terms with branch cuts (e.g. logarithms). In such a case, there is a nontrivial monodromy when transporting a field around the other. The mathematical structure that captures this notion of braiding are the commutativity isomorphism defined in [HLZ10-11, Section 12.2]. The definition is fairly technical and we don't need to concern ourselves with the details, but the intuition is essentially the same: it is defined in terms of maps that transport a module half-way around another. Braiding is thus obtained by essentially squaring the commutativity isomorphism. The braiding structures in the VOA aid our computations of the corresponding structures in the category of line operators, as we now demonstrate.

4.4 Warm up story, Act II

Let us resume the story we started in Section 4.2. The first loose end we tie up is explaining how the VOA arises on the superficial cut $\partial M = \Sigma$. One must specify which boundary conditions we impose on Σ; a common choice in 3d Chern-Simons are holomorphic boundary conditions that set

$$A_{\bar{z}}\Big|_{\Sigma} = 0. \tag{4.71}$$

This ultimately ensures that all boundary local operators depend holomorphically on Σ and will form a VOA.

Now if a Wilson line $W_{R_i}(\gamma)$ ends on Σ at point p_i, then the local operator sitting at the endpoint of γ can be acted upon by sufficiently close boundary local operators. This gives a map from line operators in the theory to modules of the boundary VOA; in 3d Chern-Simons, this map is actually an isomorphism [EMSS89].

Here we see the utility of slicing M with an imaginary surface Σ: after imposing suitable boundary conditions, the category of line operators in 3d Chern-Simons is equivalent to a certain category of representations for the affine Kac-Moody VOA living on the boundary. Due to the monopole isomorphisms discussed in Section 4.2, the set of line operators is finite and matches the set of integrable representations of the loop group of G[3], hence the category is semisimple. Next we describe how to keep track of braiding Wilson lines around each other along Σ.

Braiding Wilson lines in the bulk is equivalent to moving their endpoints around each other on Σ. Since these points on Σ correspond to VOA modules, we can compute this braiding action using the definitions discussed in 4.3.5. This was performed in [MS88].

[3]Line operators corresponding to non-integrable representations were shown to decouple by [GW86].

5 Boundary conditions

Boundary conditions (BCs) arise naturally when studying a theory on a manifold with boundary; they are necessary to ensure that the action is minimized when imposing the bulk equations of motion. As we saw in section 4.4, BCs can add interesting structures to a theory.

The presence of the boundary itself breaks translation symmetry in the direction perpendicular to the boundary. Since the translation generators can be in the image of the SUSY algebra Lie bracket, this will restrict which SUSY subalgebras the boundary can preserve and what cohomology theories we can form; some of the differentials (i.e. supercharges) are not compatible with the boundary. Let us go through a few examples in detail; we work in the half space $M := \mathbb{C} \times \mathbb{R}_{\geq 0}$ throughout.

5.1 Example 1: free 3d $\mathcal{N} = 4$ hypermultiplet

Recall the action describing a bulk free 3d $\mathcal{N} = 4$ hypermultiplet [CO17]

$$S = \int_M d^3x \left[-\partial_\mu \bar{\phi}_1 \partial^\mu \phi_1 - i\bar{\psi}_1 \sigma^\mu \partial_\mu \psi_1 + \bar{F}_1 F_1 - \partial_\mu \bar{\phi}_2 \partial^\mu \phi_2 - i\bar{\psi}_2 \sigma^\mu \partial_\mu \psi_2 + \bar{F}_2 F_2 \right]. \tag{5.1}$$

When varying the action with respect to $\psi_{1,\alpha}$ and $\psi_{2,\alpha}$, an integration by parts yields boundary terms:

$$\delta_{\psi_{1,+}} S \supseteq \int_{\partial M} \bar{\psi}_{1,-}\, \delta\psi_{1,+} \qquad \delta_{\psi_{1,-}} S \supseteq \int_{\partial M} \bar{\psi}_{1,+}\, \delta\psi_{1,-} \tag{5.2}$$

$$\delta_{\psi_{2,+}} S \supseteq \int_{\partial M} \bar{\psi}_{2,-}\, \delta\psi_{2,+} \qquad \delta_{\psi_{2,-}} S \supseteq \int_{\partial M} \bar{\psi}_{2,+}\, \delta\psi_{2,-}. \tag{5.3}$$

All fermions must be set to zero at the boundary in order for these to vanish. Varying with respect to ϕ_1 and ϕ_2 yield boundary terms proportional to the normal derivative of $\bar{\phi}$. We have derived Neumann[4] boundary conditions which preserve a 2d $\mathcal{N} = (0, 4)$ subalgebra.

Let us consider the effect of adding a boundary superpotential describing a coupling of the

[4]Even though Dirichlet BCs are imposed on the fermions, the BC is usually named by the BC given to the lowest component of the multiplet.

bulk fields with an extra boundary multiplet Γ:

$$S_\partial = \int_{\partial M} d^2x \, (F_1 \Gamma_\phi - \psi_{1,-}\Gamma_\psi + \text{ c.c.}) \tag{5.4}$$

Minimizing the action still requires $\psi_{1,-}\big|_{\partial M} = \psi_{2,-}\big|_{\partial M} = 0$ but we now find that $\psi_{1,+}$ and $\psi_{2,+}$ must satisfy Neumann BCs. These BCs preserve 2d $\mathcal{N} = (2,2)$ SUSY.

5.2 Example 2: SQED[1]

Let us now consider a 3d $\mathcal{N} = 4$ hypermultiplet coupled to a $U(1)$ vector multiplet. Recall the decomposition into 2d $\mathcal{N} = (2,2)$ multiplets contained in equation (2.70). The action describing this system is long and can be found in [CO17] or [BDGH16, Appendix A.1]. Through a similar procedure, one finds the same choices of BCs for the hypermultiplet as were found in the previous section. For the vector multiplet, one has the following additional choices that preserve a 2d $\mathcal{N} = (2,2)$ subalgebra

$$\text{Neumann:} \quad S\Big|_{\partial M} = 0 \qquad\qquad \text{Dirichlet:} \quad \Sigma\Big|_{\partial M} = 0. \tag{5.5}$$

5.3 Enhanced boundary structure in twisted theories

Recall from Section 2.3.3 that twisting a theory by a nilpotent supercharge Q (i.e. passing to Q-cohomology) can enhance the theory's structure since operators are now considered invariant under Q-exact transformations. In a theory with boundary, one can only twist by Q if it is preserved by the boundary. Moreover, since the boundary generically breaks the global SUSY algebra to a subalgebra, the structure present on the boundary in the twisted theory may differ from the (local) structure present in the bulk; such a situation arises in the main examples of interest in this thesis where the twisted theory is topological in the bulk but holomorphic on the boundary. Obtaining a twisted theory of this type isn't quite as straightforward as one may hope, rather one must take a suitable *deformation* of a 2d

$\mathcal{N} = (0,4)$ BC to preserve a topological supercharge in the bulk [CG19]. Recall that 2d $\mathcal{N} = (0,4)$ BCs preserve a holormorphic supercharge (equation (2.79)); we can deform this to a topological supercharge (e.g. in equations (2.71) or (2.75)).

One way to deform a BC is to add a term to the Lagrangian with δ-function support on the boundary (i.e. adding to the action a term integrated only over ∂M). For example, in the case of the free 3d $\mathcal{N} = 4$ hypermultiplet discussed in Section 5.1, the addition of the term

$$\int_{\partial M} c\partial_\perp \phi_1 \tag{5.6}$$

together with its SUSY completion modifies δS so that an entire 2d $\mathcal{N} = (2,2)$ multiplet is required to equal the constant c, instead of 0, at the boundary, deforming the exceptional Dirichlet boundary condition to a generic Dirichlet boundary condition [DGP18].

In some cases, the deformed BC further breaks the supersymmetry preserved by the original BC, perhaps including the chosen nilpotent supercharge Q. To remedy this twisting obstruction, one can deform Q by adding another supercharge to it so that the action is invariant under the linear combination.

5.4 Example 3: $\beta\gamma$ VOA from twisted BCs

We would like to impose BCs on a free 3d $\mathcal{N} = 4$ hypermultiplet living on M that preserves a 2d $\mathcal{N} = (0,4)$ subalgebra. One choice would be to impose Neumann boundary conditions on the scalars in the hypermultiplet and Dirichlet BCs on the fermions. This is the choice we initially make, but they must be deformed to obtain a BC that supports a holomorphic twist; the VOA we obtain living on ∂M after taking the A-twist is the $\beta\gamma$ VOA $V_{\beta\gamma}$ [GR19, Appendix E]. Let us describe this VOA and fix notation.

The VOA $V_{\beta\gamma}$ is strongly generated by two bosonic fields

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-1} \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n} \tag{5.7}$$

satisfying the operator product expansions

$$\beta(z)\beta(w) \sim 0 \qquad \gamma(z)\gamma(w) \sim 0 \qquad \beta(z)\gamma(w) \sim \frac{-1}{z-w}. \tag{5.8}$$

It admits the structure of a $\mathbb{Z}_{\geq 0}$-graded VOA when equipped with the following choice of conformal element/stress-energy tensor [RW15]

$$\omega = -\beta_{-1}\gamma_{-1} \qquad T(z) = -:\beta(z)\partial\gamma(z):= \sum_{\ell\in\mathbb{Z}} z^{-\ell-2}\left[\sum_{k\in\mathbb{Z}} k :\beta_{\ell-k}\gamma_k:\right]. \tag{5.9}$$

The OPEs imply that the coefficients of $\beta(z)$ and $\gamma(z)$ possess the following commutation relation

$$[\beta_m, \beta_n] = 0 \qquad [\gamma_m, \gamma_n] = 0 \qquad [\beta_m, \gamma_n] = -\delta_{m,-n}\mathbb{1}. \tag{5.10}$$

For each $n \in \mathbb{Z}$, $\beta\gamma_n := \mathbb{C}[\beta_n, \gamma_{-n}]$ is a 1-dimensional Weyl algebra under the identification $\beta_n \leftrightarrow x_{-n}$ and $\gamma_n \leftrightarrow \partial_n$. Our analysis of the category of VOA modules later on in the thesis strongly relies on this simple observation. We denote the universal enveloping algebra generated by $\{\beta_n, \gamma_n\}_{n\in\mathbb{Z}}$, which can be thought of as an infinite-dimensional Weyl algebra, by $\beta\gamma$.

The $\beta\gamma$ VOA possesses an additional global $U(1)$ symmetry (i.e. contains a $U(1)$ Kac-Moody VOA) whose associated current is

$$J(z) =:\beta(z)\gamma(z):= \sum_{\ell\in\mathbb{Z}} z^{-\ell-1}\left[\sum_{k\in\mathbb{Z}} :\beta_k\gamma_{\ell-k}:\right]. \tag{5.11}$$

In addition to the $\mathbb{Z}_{\geq 0}$-grading given by L_0, the $\beta\gamma$ VOA is strongly $\mathbb{Z}$-graded (in the sense of [HLZ10-11, Definition 2.23]) with respect to J_0.

It will be handy for future computations to record some of the commutation relations and modes of $T(z)$ and $J(z)$

$$J_0 = \sum_{n\geq 0} \gamma_{-n}\beta_n + \sum_{n\geq 1} \beta_{-n}\gamma_n \tag{5.12}$$

$$L_0 = \sum_{k\geq 1} k\left[\beta_{-k}\gamma_k - \gamma_{-k}\beta_k\right] \tag{5.13}$$

$$L_{-1} = \sum_{k\geq 1} k\left[\beta_{-1-k}\gamma_k - \gamma_{-k}\beta_{k-1}\right] \tag{5.14}$$

$$\begin{aligned} [J_0, \beta_k] &= \beta_k \qquad [L_0, \beta_k] = -k\beta_k \\ [J_0, \gamma_k] &= -\gamma_k \qquad [L_0, \gamma_k] = -k\gamma_k \end{aligned} \tag{5.15}$$

The following $\beta\gamma$ automorphisms will frequently appear when discussing modules for $V_{\beta\gamma}$:

- Conjugation: $c(\beta_n) = \gamma_n \qquad c(\gamma_n) = -\beta_n$
- Spectral flow: $\sigma(\beta_n) = \beta_{n-1} \qquad \sigma(\gamma_n) = \gamma_{n+1}$

When combined with the $U(1)$ global symmetry described above, the existence of the conjugation automorphism tells us that $V_{\beta\gamma}$ actually has an $Sp(2)$ global symmetry. The spectral flow automorphism can be thought of as arising from a 1-form symmetry present in the 3d bulk theory (with a line operator) whose boundary algebra is $V_{\beta\gamma}$. We now present the boundary VOA one obtains by choosing opposite boundary conditions.

5.5 Example 4: $\widehat{\mathfrak{gl}(1|1)}$ VOA from twisted BCs

The affine Kac-Moody VOSA of $\mathfrak{gl}(1|1)$ was first studied by [RS92] in the context of applying supergroup Chern-Simons to reproduce certain topological polynomial invariants. To obtain the affine Kac-Moody VOSA of $\mathfrak{gl}(1|1)$ on ∂M, we take the B-twist of supersymmetric quantum electrodynamics (SQED) with a single hypermultiplet and impose 2d $\mathcal{N} = (0, 4)$

Dirichlet boundary conditions on the hypermultiplet and vector multiplet [CCG19, CR09].

The Lie superalgebra $\mathfrak{gl}(1|1)$ is defined as the endomorphism algebra of the superspace $\mathbb{C}^{1|1}$. This Lie algebra has basis

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5.16}$$

where N and E are even and $\psi^\pm$ are odd. The non-trivial commutation relations are

$$[N, \psi^\pm] = \pm\psi^\pm \qquad \{\psi^+, \psi^-\} = E. \tag{5.17}$$

There is a supersymmetric, even, non-degenerate, invariant bilinear form $\kappa(\cdot,\cdot)$ on $\mathfrak{gl}(1|1)$ whose non-zero values on basis elements are

$$\kappa(N, E) = \kappa(E, N) = 1 \qquad \kappa(\psi^+, \psi^-) = -\kappa(\psi^-, \psi^+) = 1. \tag{5.18}$$

Following the construction outlined in Section 4.3.4, we obtain a family of VOSAs $V_k(\widehat{\mathfrak{gl}(1|1)})$ parametrized by $k \in \mathbb{C}$. It turns out that they are all isomorphic for $k \neq 0$ [CMY20a], so without loss of generality, we take $k = 1$ and denote it by $V(\widehat{\mathfrak{gl}(1|1)})$.

We write out the conformal vector and associated L_0 mode for future convenience:

$$\omega = \frac{1}{2}(N_{-1}E_{-1} + E_{-1}N_{-1} - \psi^+_{-1}\psi^-_{-1} + \psi^-_{-1}\psi^+_{-1}) + \frac{1}{2}E^2_{-1} \tag{5.19}$$

$$L_0 = \sum_{r=1}^{\infty}\left(N_{-r}E_r + E_{-r}N_r - \psi^+_{-r}\psi^-_r + \psi^-_{-r}\psi^+_{-r} + E_{-r}E_r\right) \\ + (N_0 + E_0/2)E_0 - \frac{1}{2}(\psi^+_0\psi^-_0 - \psi^-_0\psi^+_0) \tag{5.20}$$

It also possesses spectral flow automorphisms defined by

$$\sigma^\ell(N_r) = N_r \qquad \sigma^\ell(E_r) = E_r - \ell\delta_{r,0} \qquad \sigma^\ell(\psi_r^\pm) = \psi_{r\mp\ell}^\pm \tag{5.21}$$

as well as a conjugation automorphism defined by

$$w(N_r) = -N_r \qquad w(E_r) = -E_r \qquad w(\psi_r^\pm) = \pm\psi_r^\mp. \tag{5.22}$$

5.6 Boundary conditions and mirror symmetry

As discussed in Section 3, mirror symmetry typically acts non-trivially on boundary conditions. In Section 2.3.2 we introduced the notion of a half-index, which is a quantity that essentially counts the boundary operators. These spaces of boundary operators must match under MS; in this section we perform checks on various statements of MS by computing half-indices for the mirror pairs considered above and demonstrating their equality. Before proceeding though, let us review the supersymmetric index in more detail.

Recall that the Witten index [Wit82] is the graded trace

$$\mathrm{tr}_{\mathcal{H}}\left((-1)^F\right) \tag{5.23}$$

over the Hilbert space $\mathcal{H}$, where F is the $\mathbb{Z}$-valued (or $\mathbb{Z}_2$-valued) fermion number operator. This index counts the difference between the bosonic and fermionic supersymmetric ground states in $\mathcal{H}$. When symmetries are present in a theory, such as supersymmetry/R-symmetry, flavor symmetry, or gauge symmetry, one can further refine the grading on $\mathcal{H}$ according to the charges of the states under the Cartan subalgebra of these symmetries. This equates to throwing additional fugacities and chemical potentials into the trace. The supersymmetric index, first introduced in [KMMR07] in the context of 4d theories, is a generalization of the Witten index in this exact manner. Subsequent generalizations of the supersymmetric index to various dimensions were studied in [Rom06, BBMR08], for example.

Many checks of MS have been performed for 3d theories without boundary by computing bulk indices; see [KWY10b, KWY20, IY11, Oka19b], for example. The simplest example of 3d mirror symmetry is that of SQED[1] and a free twisted 3d $\mathcal{N} = 4$ hypermultiplet, conjectured by [KS99, GW09]. Computation of the bulk indices of these theories was performed in [Oka19b]. Let us go through the process of checking that the half-indices match when we put appropriate pairs of BCs on these theories. We consider 2d $\mathcal{N} = (2, 2)$ boundary conditions, following the work of [Oka21]; see [Oka19a] for studies of MS on 2d $\mathcal{N} = (0, 4)$ boundary conditions.

The half-index is defined as

$$\mathrm{II}(t, x; q) := \mathrm{tr}\left[(-1)^F q^{J+\frac{H+C}{4}} t^{H-C} x^f\right] \tag{5.24}$$

where J measures the spin of the operator in the plane of the boundary, H and C measure the R-symmetry charges under $U(1)_H \subseteq SU(2)_H$ and $U(1)_C \subseteq SU(2)_C$ respectively, and f measures the charges under the Cartan subalgebra of the flavor symmetry algebra. While we view the parameters (t, x, q) as formal variables, the half-index originates from the supersymmetric localization of a partition function computation of the theory on the hemisphere times a circle $HS^2 \times S^1$ [DGP18], wherein these variables are actually numbers that depend on quantities such as the FI and mass parameters of the theory. The trace is taken over the set of boundary operators preserved by one of the supercharges of the 2d $\mathcal{N} = (2, 2)$ SUSY algebra, modulo the equivalence relation identifying operators which differ by a SUSY transformation.

Let us define the q-Pochhammer symbol and some associated notation used in the rest of this section:

$$\begin{aligned} (x;q)_n &:= \prod_{k=0}^{n-1}(1-xq^k) & (x;q)_\infty &:= \prod_{k=0}^{\infty}(1-xq^k) \\ \theta(x) &:= (x;q)_\infty(qx^{-1};q)_\infty & (x_1,\dots,x_m;q)_\infty &:= \prod_{k=1}^{m}(x_k;q)_\infty \end{aligned} \tag{5.25}$$

These combinatorial expressions encode the contribution of a field and its derivatives to the graded traces that form the index.

5.6.1 Half-indices of 3d $\mathcal{N}=4$ free twisted hypermultiplet with 2d $\mathcal{N}=(2,2)$ BCs

Before we can begin computing the half-index, we need to fix the R-charges of the fields in the multiplet. A twisted hypermultiplet contains the same content as a hypermultiplet but transforms under the R-symmetry in a slightly different way: we make use of an external automorphism to swap the $SU(2)_H$ and $SU(2)_C$ actions. We pick the following charge assignments for the hypermultiplet Φ

	$U(1)_H$	$U(1)_C$	$U(1)_{\text{flavor}}$
X	1	0	1
Y	1	0	-1
$\bar{\psi}_-^X$	0	1	1
$\bar{\psi}_-^Y$	0	1	-1

(5.26)

hence the representation table for the twisted hypermultiplet $\widetilde{\Phi}$ is

$$\begin{array}{c|ccc} & U(1)_H & U(1)_C & U(1)_{\text{flavor}} \\ \hline \widetilde{X} & 0 & 1 & 1 \\ \widetilde{Y} & 0 & 1 & -1 \\ \overline{\widetilde{\psi}}{}^X_- & 1 & 0 & 1 \\ \overline{\widetilde{\psi}}{}^Y_- & 1 & 0 & -1 \end{array} \tag{5.27}$$

The action for a free hypermultiplet enjoys a global $U(1)_{\text{flavor}}$ symmetry that rotates Φ with charge 1 and $\overline{\Phi}$ with charge -1. We work with the twisted hypermultiplet for the rest of this section.

There are 4 BCs we can put on a free hypermultiplet, some of which were discussed in Section 5.1:

1. $\mathcal{B}_+$: Dirichlet on $\widetilde{Y}$ and Neumann on $\widetilde{X}$. Preserving 2d $\mathcal{N} = (2,2)$ SUSY requires $\overline{\widetilde{\psi}}{}^Y_-\Big|_{\partial M} = \widetilde{\psi}^Y_+\Big|_{\partial M} = 0$. Therefore the only local operators present on the boundary are $\widetilde{X}, \overline{\widetilde{\psi}}{}^X_-$, and their holomorphic derivatives[5]:

$$\left[\bigotimes_{n=0}^{\infty}\bigoplus_{k=0}^{\infty}(\partial_z^n\widetilde{X})^k\right]\otimes\bigotimes_{n=0}^{\infty}\left(\mathbb{C}\oplus\mathbb{C}\partial_z^n\overline{\widetilde{\psi}}{}^X_-\right). \tag{5.28}$$

   Thus the half-index takes the form[6]

$$\mathrm{II}_{\mathcal{B}_+} = \left[\prod_{n\geq 0}\sum_{k\geq 0}(q^{n+\frac{1}{4}}t^{-1}x)^k\right]\left[\prod_{n\geq 0}\left(1+(-1)q^{\left(n+\frac{1}{2}\right)+\frac{1}{4}}tx\right)\right] \tag{5.29}$$

$$= \frac{1}{\prod_{n\geq 0}(1-q^{n+\frac{1}{4}}t^{-1}x)}(q^{\frac{3}{4}}tx^{-1};q)_\infty \tag{5.30}$$

$$\Longrightarrow \mathrm{II}_{\mathcal{B}_+} = \frac{(q^{\frac{3}{4}}tx;q)_\infty}{(q^{\frac{1}{4}}t^{-1}x;q)_\infty}. \tag{5.31}$$

[5] Anti-holomorphic derivatives do not contribute because they are Q-exact, and other fields do not contribute if they are not Q-closed.

[6] Recall the definition of the q-Pochhammer symbol from Section 2.3.2.

2. $\mathcal{B}_-$: Dirichlet on $\widetilde{X}$ and Neumann on $\widetilde{Y}$. This requires $\bar{\widetilde{\psi}}_-^X\Big|_{\partial M} = \widetilde{\psi}_+^X\Big|_{\partial M} = 0$. By a similar computation, we find

$$\mathrm{II}_{\mathcal{B}_-} = \frac{(q^{\frac{3}{4}}tx^{-1};q)_\infty}{(q^{\frac{1}{4}}t^{-1}x^{-1};q)_\infty}. \tag{5.32}$$

3. $\mathcal{B}_{+,c}$: Generic Dirichlet on $\widetilde{Y}$ (i.e. $\widetilde{Y}$ is set to a non-zero constant on ∂M) and Neumann on $\widetilde{X}$. Preserving 2d $\mathcal{N}=(2,2)$ SUSY requires $\bar{\widetilde{\psi}}_-^Y\Big|_{\partial M} = \widetilde{\psi}_+^Y\Big|_{\partial M} = 0$. The operator content is the same as in $\mathcal{B}_+$, but the non-zero value for $\widetilde{Y}$ breaks the symmetries of the theory to a subgroup. In terms of the $U(1)_H \times U(1)_C \times U(1)_{\text{flavor}}$ subgroup that matters for the index, we see that it is broken to the subgroup

$$\langle (e^{ia}, e^{ib}, e^{ic}) \mid b-(-c)=0 \rangle. \tag{5.33}$$

Identifying this subgroup with $U(1)_1 \times U(1)_2 = \langle (e^{ia}, e^{ib}, e^{-ib}) \rangle$, the effective charge of $\widetilde{Y}$ becomes $(0,0)$. This changes the result of the index computation:

$$\mathrm{II}_{\mathcal{B}_{+,c}} = \frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}. \tag{5.34}$$

A nifty trick to obtain $\mathrm{II}_{\mathcal{B}_{+,c}}$ from $\mathrm{II}_{\mathcal{B}_+}$ is to set the original fugacity of $\widetilde{Y}$, $q^{\frac{1}{4}}t^{-1}x^{-1}$, to 1, solve for the broken fugacity x, and substitute this into $\mathrm{II}_{\mathcal{B}_+}$. Setting the fugacity to 1 is the statement that $\widetilde{Y}$ must transform trivially under the entire symmetry group to preserve the generic Dirichlet BC.

4. $\mathcal{B}_{-,c}$: Generic Dirichlet on $\widetilde{X}$ and Neumann on $\widetilde{Y}$. This requires $\bar{\widetilde{\psi}}_-^X\Big|_{\partial M} = \widetilde{\psi}_+^X\Big|_{\partial M} = 0$. Deforming $\mathrm{II}_{\mathcal{B}_-}$ by $q^{\frac{1}{4}}t^{-1}x = 1 \longleftrightarrow x = q^{-\frac{1}{4}}t$, we obtain

$$\mathrm{II}_{\mathcal{B}_{-,c}} = \frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}. \tag{5.35}$$

5.6.2 Half-indices of SQED[1] with 2d $\mathcal{N} = (2,2)$ BCs

The charges of the fields from the vector multiplet that can contribute are

$$\begin{array}{c|ccc} & U(1)_H & U(1)_C & U(1)_{\text{flavor}} \\ \hline \sigma + iA_\perp & 0 & 0 & 0 \\ \varphi & 0 & 2 & 0 \\ \lambda_- & -1 & -1 & 1/2 \\ \bar{\eta}_- & 1 & -1 & 1/2 \end{array} \tag{5.36}$$

There are two 2d $\mathcal{N} = (2,2)$ BCs we can place on the vector multiplet:

1. N: the fields which survive on the boundary are φ, λ_-, and their holomorphic (gauge-covariant) derivatives.

2. D: the fields which survive on the boundary are $\sigma + iA_\perp, \bar{\eta}_-$, and their holomorphic (gauge-covariant) derivatives.

The charges of the hypermultiplet fields are the same as before, but now they additionally transform under $U(1)_{\text{gauge}}$ with charge $+1$ (i.e. X has charge 1 and Y has charge -1); we call the corresponding fugacity s. The BCs on the hypermultiplets are independent of the BCs placed on the vector multiplet, hence there are many half-indices to compute; we compute a select few for brevity.

1. N_+: the contribution from the hypermultiplet takes the form

$$\frac{(q^{\frac{3}{4}}t^{-1}sx;q)_\infty}{(q^{\frac{1}{4}}tsx;q)_\infty} \tag{5.37}$$

   We should only have gauge invariant operators on the boundary. Thus the formula for the half-index should involve a counter integral $\oint \frac{ds}{2\pi is}$ which picks out the gauge invariant combinations of the fields. By looking at the charges of the surviving matter, one can see that there no gauge invariant combinations can be formed, hence the result of the contour integral should be 1; indeed this is what one obtains from the computation. Thus the half-index in this case is

$$\mathrm{II}_{N_+} = \frac{(q)_\infty}{(q^{\frac{1}{2}}t^{-2};q)_\infty}. \tag{5.38}$$

2. D_+: Dirichlet boundary conditions on the vector multiplet support topologically nontrivial field configurations (monopoles) for the gauge field on the boundary [BDGH16]. Therefore the set of boundary operators is not simply the set of monomials created out of the fundamental fields, but rather splits up into sectors of monomials together with a monopole of some integer charge

$$\mathcal{O} = \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_m. \tag{5.39}$$

The presence of a monopole shifts the spins of the matter fields, which has the effect of shifting $s \mapsto q^m s$ in the index. The monopole itself also carries charge determined by the anomaly polynomial (see [DGP18, Section 3] for details). Lastly, the Dirichlet boundary condition breaks the gauge symmetry to a flavor symmetry on the boundary, so there is no contour integral needed to project to gauge-invariants. The result is

$$\mathrm{II}_{D_+} = \frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{(q^{m+\frac{3}{4}}t^{-1}s;q)_\infty}{(q^{m+\frac{1}{4}}ts;q)_\infty} \left(q^{\frac{1}{4}}t^{-1}x\right)^m \tag{5.40}$$

where x is the fugacity for the $U(1)$ topological Coulomb symmetry G_C [BDGH16, Section 1.2]. It turns out that this boundary condition breaks 2d $\mathcal{N}=(2,2)$ SUSY to 2d $\mathcal{N}=(0,2)$ because it only supports the topological A-twist; the B-twist is broken which can be seen by taking the limit $t \mapsto q^{-\frac{1}{4}}$. We will need the above expression for the half-index though for the next BC we consider.

3. $D_{c,+}$: Just as in the previous section, the generic Dirichlet boundary condition breaks

some of the boundary flavor symmetry, which has the effect of deforming a fugacity to some combination of the others. Since the gauge symmetry reduces to a flavor symmetry on the boundary, we deform $s \mapsto q^{\frac{1}{4}}t$. Thus we obtain

$$\mathrm{II}_{D_{c,+}} = \mathrm{II}_{D_+}\Big|_{s\mapsto q^{\frac{1}{4}}t} = \frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q)_\infty}\sum_{m\in\mathbb{Z}}\frac{(q^{m+1};q)_\infty}{(q^{m+\frac{1}{2}}t^2;q)_\infty}\left(q^{\frac{1}{4}}t^{-1}x\right)^m. \tag{5.41}$$

5.6.3 Matching

Recall that MS swaps $SU(2)_H \leftrightarrow SU(2)_C$. Therefore when matching indices, the exponent of q is preserved but $t \mapsto t^{-1}$. It was conjectured in [BDGH16] that MS swaps Neumann and generic Dirichlet BCs and maps Dirichlet BCs to themselves. This statement was slightly corrected in the recent paper [BZ21] by conjecturing (with some examples providing evidence) that Dirichlet BCs are actually exchanged with enriched Neumann BCs. These Neumann BCs are enriched in the sense that one couples the bulk theory to purely 2d boundary matter. To keep things simple, we only demonstrate that the Neumann and generic Dirichlet half-indices computed in the previous examples match. Indeed (5.34) is equal to (5.38). Additionally, after applying Ramanujan's summation formula

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \tag{5.42}$$

we see that (5.31) is equal to (5.41).

We now transition to computing half-indices for BCs preserving 2d $\mathcal{N}=(0,4)$ SUSY in the remaining two examples above, following [Oka19a].

5.6.4 Half-indices of 3d $\mathcal{N}=4$ free twisted hypermultiplet with 2d $\mathcal{N}=(0,4)$ BCs

The charges of the fundamental fields are the same as in (5.27) but the definitions of the 2d $\mathcal{N}=(0,4)$ BCs are different from the 2d $\mathcal{N}=(2,2)$ BCs discussed in that section.

We consider 2d $\mathcal{N}=(0,4)$ Neumann BCs in which both scalars X and Y are free to vary at the boundary and all of the fermions are killed (or are Q-exact). These are the only fundamental fields at one's disposal to construct local operators with, hence the half-index is easily computed to be

$$\mathrm{II}_{\mathcal{N}} = \frac{1}{(q^{\frac{1}{4}}t^{-1}x, q^{\frac{1}{4}}t^{-1}x^{-1}; q)_\infty}. \tag{5.43}$$

5.6.5 Half-indices of SQED[1] with 2d $\mathcal{N}=(0,4)$ BCs

We impose 2d $\mathcal{N}=(0,4)$ Dirichlet BCs on the boundary. From the matter fields, this preserves only the fermions on the boundary. From the vector multiplet, only the scalars σ and φ are free to vary and contribute to the half-index. Since the components of the gauge field A_μ along the boundary are constant, the gauge symmetry is broken to a flavor symmetry on the boundary. Just as in Section 5.6.2, this BC supports monopole operator configurations. The half-index is

$$\mathrm{II}_{\mathcal{D}} = \frac{1}{(q, q^{\frac{1}{2}}t^{-2}; q)_\infty} \sum_{m\in\mathbb{Z}} (q^{\frac{3}{4}+m}t^{-1}x, q^{\frac{3}{4}-m}t^{-1}x^{-1}; q)_\infty \, q^{\frac{m^2}{2}} (-1)^m x^m f^{-m}. \tag{5.44}$$

5.6.6 Matching

One may verify from a low order q-expansion that the half-indices (5.43) and (5.44) do not match. While these theories are indeed mirror in the bulk, the boundary conditions we imposed in the previous sections are not quite dual; one must enrich the Neumann BC on the free twisted hypermultiplet by a 2d $\mathcal{N}=(0,4)$ boundary Fermi multiplet [Oka19a]. In addition to the half-index considerations in this section, we will demonstrate the necessity of these extra fermions (corresponding to a bc ghost system) from the perspective of matching VOA module categories in Section 6.3.1.

The correct charge assignments for the Fermi multiplet Γ turn out to be

$$\begin{array}{c|cccc} & U(1)_H & U(1)_C & U(1)_{\text{flavor}} & U(1)_{\text{topological}} \\ \hline \Gamma & 0 & 0 & 1 & -1 \end{array} \tag{5.45}$$

contributing a factor of $\theta(q^{\frac{1}{2}}xf^{-1})$ to the half-index. Therefore the complete half-index for the free twisted hypermultiplet with enriched Neumann BCs is

$$\mathrm{II}_{\mathcal{N}_{\text{en}}} = \frac{\theta(q^{\frac{1}{2}}xf^{-1})}{(q^{\frac{1}{4}}t^{-1}x, q^{\frac{1}{4}}t^{-1}x^{-1}; q)_\infty}. \tag{5.46}$$

We observe that the half-indices (5.44) and (5.46) are equal after performing some algebraic manipulation and applying Ramanujan's summation formula (5.42).

6 Non-trivial MS check + consequences

In this section, we compute the category of line operators $\mathcal{L}_A$ in the A-twist of a free twisted hypermultiplet (recall the discussion in Section 3.4), demonstrate that it possesses the structure of a braided tensor category, and prove that this category is equivalent to the category of line operators $\mathcal{L}_B$ in the B-twist of SQED[1]. We study the former category by artificially cutting the theory and placing 2d $\mathcal{N} = (0, 4)$ Neumann BCs on the cut; analogously to the method reviewed in Sections 4.2 and 4.4, this establishes an equivalence between the category of line operators and the category of modules for the boundary VOA. The boundary VOA happens to be the $\beta\gamma$ VOA introduced in section 5.4. On the B-side, one obtains modules for the affine VOSA of $\widehat{\mathfrak{gl}(1|1)}$. The main result of this thesis is

Theorem 6.1. *The category $\mathcal{C}_{\beta\gamma}$ has the structure of a braided tensor category defined by $P(z)$-intertwining operators. Moreover, there is an equivalence between braided tensor categories:*

$$\mathcal{C}_{\beta\gamma} \cong KL^0/(\mathbb{Z} \times \mathbb{Z}_2). \tag{6.1}$$

where KL^0 is an appropriate category of $V(\widehat{\mathfrak{gl}(1|1)})$-modules. Upon taking the derived categories on both sides, we obtain $\mathcal{L}_A \cong \mathcal{L}_B$. Another major contribution of this work is the classification of this enlarged non-semisimple $\beta\gamma$ module category, as well as the computation the tensor structure in the category. Much of this section is adapted from [BN22].

6.1 The $\beta\gamma$ VOA and its representation category

In this section, we focus on the $\beta\gamma$ vertex operator algebra $V_{\beta\gamma}$. In Section 6.1.1, we recall basic examples of modules of $V_{\beta\gamma}$ following [AW20], and define the category of interest $\mathcal{C}_{\beta\gamma}$. In Section 6.1.2, we give examples of indecomposable modules in $\mathcal{C}_{\beta\gamma}$, and prove classification results in Theorem 6.2 and Theorem 6.3. In Section 6.1.3, we recall the notion of $P(z)$-intertwining operators and explain the difficulty in constructing monoidal structure for

$\mathcal{C}_{\beta\gamma}$.

6.1.1 Our large representation category

Various representation categories $V_{\beta\gamma}$ have been investigated by others (e.g. [RW15, AW20]). For reasons explained in Section 6.4.1, these categories are too small to correctly match the physics. Essentially, the self-extension group (i.e. the derived endomorphism algebra) of the vacuum module in these categories did not match with what one should obtain when computing the bulk local operators. Before we can define the category of physical interest in this thesis, we must first introduce some basic modules of the $\beta\gamma$ VOA.

Let $\beta\gamma_{\geq 0}$ be the unital subalgebra of $\beta\gamma$ generated by $\{\beta_n, \gamma_n, \mathbb{1}\}_{n\geq 0}$. The simplest module is the vacuum module

$$\mathcal{V} := \mathrm{Ind}_{\beta\gamma_{\geq 0}}^{\beta\gamma} \mathbb{C}[\gamma_0] \tag{6.2}$$

where β_0 acts as $-\frac{\partial}{\partial\gamma_0}$ on $\mathbb{C}[\gamma_0]$, and β_n and γ_n act as 0 for $n \geq 1$. As a vector space, the vacuum module $\mathcal{V}$ of $V_{\beta\gamma}$ coincides with $V_{\beta\gamma}$ as a module over itself.

Similarly, for $\mu \in \mathbb{C} \setminus \mathbb{Z}$, the so called *typical* modules are defined by

$$\mathcal{W}_\lambda := \mathrm{Ind}_{\beta\gamma_{\geq 0}}^{\beta\gamma} (\gamma_0)^\mu \mathbb{C}[\gamma_0, \gamma_0^{-1}] \tag{6.3}$$

where β_0 acts as $-\frac{\partial}{\partial\gamma_0}$, and β_n and γ_n act as 0 for $n \geq 1$. Here $\lambda = \mu + \mathbb{Z}$ and one can see that $\mathcal{W}_\lambda$ is independent (up to isomorphism) of the choice of $\mu \in \lambda$, hence these modules are parametrized by $(\mathbb{C} \setminus \mathbb{Z})/\mathbb{Z}$.

Both $\mathcal{V}$ and $\mathcal{W}_\lambda$ are simple objects. There are two distinct modules that are reducible but indecomposable, and they morally correspond to the different ways one can take the limit of $\mathcal{W}_\lambda$ as $\lambda \to \mathbb{Z}$. They are called *atypical* modules and are defined by

$$\mathcal{W}_0^+ := \mathrm{Ind}_{\beta\gamma_{\geq 0}}^{\beta\gamma} \mathbb{C}[\gamma_0, \gamma_0^{-1}] \qquad \mathcal{W}_0^- := \mathrm{Ind}_{\beta\gamma_{\geq 0}}^{\beta\gamma} \mathbb{C}[\beta_0, \beta_0^{-1}] \tag{6.4}$$

Given any $V_{\beta\gamma}$ module M, we can construct another module by twisting the action of the $\beta\gamma$ VOA with the spectral flow automorphism: for any $n \in \mathbb{Z}$, $\sigma^n M$ is the module that is equal to M as a set, but carries the action

$$\alpha \star v := \sigma^{-n}(\alpha) \cdot v \quad \text{for every } v \in \sigma^n M,\ \alpha \in \beta\gamma \tag{6.5}$$

where $\cdot$ is the action on M. For all of the modules above, $\sigma^n M \ncong \sigma^m M$ for $m \neq n$.

As a side note, the atypicals can equivalently be defined by the Loewy diagrams

$$\mathcal{W}_0^+ = \left(\mathcal{V} \longrightarrow \sigma^{-1}\mathcal{V}\right) \qquad \mathcal{W}_0^- = \left(\sigma^{-1}\mathcal{V} \longrightarrow \mathcal{V}\right) \tag{6.6}$$

In this thesis, a Loewy diagram $X \longrightarrow Y$ represents a module that is an extension of Y by X. One may have instead drawn such a module as $Y \longrightarrow X$, but we feel our convention better matches the visual appearance of the corresponding short exact sequence describing the extension.

With these examples in hand, we can now define the category that we study in our thesis, which we believe properly matches our physical systems of interest.

Definition 6.1. *Let $\mathcal{C}_{\beta\gamma}$ be the abelian subcategory of smooth, finite-length, $\beta\gamma$ VOA modules generated by $\mathcal{V}$, $\mathcal{W}_\lambda$, and their spectral flows, such that $\mathcal{C}_{\beta\gamma}$ is closed under taking extension.*

While the element L_0 (5.13) provides the $\mathbb{C}$-grading included in the definition of a VOA module, the element J_0 (5.12) provides an *additional* grading on the modules; see [HLZ10-11, Definition 2.25] for details. A main feature the reader should keep in mind from Definition 6.1 is that we do not exclude modules on which L_0 and J_0 act non-semisimply. Additionally, modules in $\mathcal{C}_{\beta\gamma}$ can be decomposed into a direct sum of finite-dimensional simultaneous generalized eigenspaces for L_0 and J_0. This should be contrasted with the representation category studied in, e.g., [AW20] wherein J_0 was required to act semisimply; the category that we study will consequentially be strictly larger. While $\mathcal{C}_{\beta\gamma}$ is strictly smaller than the category studied in [HR21], the authors do not provide a classification of their category, and more structure exists on $\mathcal{C}_{\beta\gamma}$ that we study that does not exist on their category. To the best of our knowledge, our present study of $\mathcal{C}_{\beta\gamma}$ is a new addition to the existing literature on $\beta\gamma$ representation categories.

6.1.2 Extension structure and classification results

To characterize the objects in $\mathcal{C}_{\beta\gamma}$, we must understand the new modules that are present when we demand closure under extension. For modules induced by polynomial representations of a 1D Weyl subalgebra of $\beta\gamma$, one can explicitly construct self-extensions by adjoining powers of a formal variable $\log \beta_0$ or $\log \gamma_0$ before inducing. For example, the first self-extension of $\mathcal{W}_0^-$, which we denote by $\mathcal{W}_0^{-,2}$, can be constructed by

$$\mathcal{W}_0^{-,2} := \mathrm{Ind}_{\beta\gamma_{\geq 0}}(\mathbb{C}[\beta_0, \beta_0^{-1}] \oplus \mathbb{C}[\beta_0, \beta_0^{-1}] \log \beta_0) \tag{6.7}$$

where $\gamma_0 \cdot \log \beta_0 = \beta_0^{-1}$. This module is also is described by the Loewy diagram

$$\sigma^{-1}\mathcal{V} \longrightarrow \mathcal{V} \longrightarrow \sigma^{-1}\mathcal{V} \longrightarrow \mathcal{V} \tag{6.8}$$

This is an object of $\mathcal{C}_{\beta\gamma}$ that does *not* carry a semisimple action of J_0, demonstrating that $\mathcal{C}_{\beta\gamma}$ is an enlargement of the representation category studied in [AW20]. We denote the $(n-1)^{\text{th}}$ iterated extension of $\mathcal{W}_0^-$ by itself as $\mathcal{W}_0^{-,n}$. A similar line of reasoning gives the definition of $\mathcal{W}_0^{+,n}$. For example, $\mathcal{W}_0^{+,2}$ looks like

$$\mathcal{V} \longrightarrow \sigma^{-1}\mathcal{V} \longrightarrow \mathcal{V} \longrightarrow \sigma^{-1}\mathcal{V} \tag{6.9}$$

We call these modules “chains”, and by taking submodule/quotient, we can form chains of odd length. We say a chain is *positive* (*negative*) if it is a quotient of some $\sigma^n\mathcal{W}_0^{+,k}$ ($\sigma^n\mathcal{W}_0^{-,k}$). To the best of our knowledge, these modules have not been studied in the literature.

Since every module can be expressed as a direct sum of indecomposable modules, it suffices to restrict our focus to nontrivial extensions. From the commutation relations in equation (5.15), we see that $\mathcal{C}_{\beta\gamma}$ admits a block decomposition

$$\mathcal{C}_{\beta\gamma} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{C}_{\beta\gamma,\lambda} \tag{6.10}$$

where $\mathcal{C}_{\beta\gamma,\lambda}$ is the full abelian subcategory of $\mathcal{C}_{\beta\gamma}$ that contains all $\beta\gamma$ modules such that the generalized eigenvalues of the representation of J_0 lie in λ. The morphisms and extensions between a module from $\mathcal{C}_{\beta\gamma,\lambda}$ and another from $\mathcal{C}_{\beta\gamma,\lambda'}$ are trivial for $\lambda \neq \lambda'$ because morphisms in $\mathcal{C}_{\beta\gamma}$ respect the generalized eigenvalues of L_0 and J_0, and equation (5.15) tells us that the $\beta\gamma$ modes can only shift them by an integer. Therefore we only need to study the extension structure *within* each $\mathcal{C}_{\beta\gamma,\lambda}$. Modules in $\mathcal{C}_{\beta\gamma,\lambda}$ for $[\lambda] \neq [0]$ are called typical modules while modules in $\mathcal{C}_{\beta\gamma,[0]}$ are called atypical modules.

This task is rather involved, essentially because there are not enough injectives and projectives in $\mathcal{C}_{\beta\gamma}$, so we outline our approach before getting into the technical details. It will follow from the definition of $\mathcal{C}_{\beta\gamma}$ and Lemma 6.1 that every module in $\mathcal{C}_{\beta\gamma,\lambda}$ can be constructed as an induced module of a representation for a finite-dimensional Weyl subalgebra of $\beta\gamma$. Consequentially, we will see that the extension structure in $\mathcal{C}_{\beta\gamma,\lambda}$ can be understood in terms of the extension structure between modules of finite-dimensional Weyl algebras. Therefore we begin by computing the latter, then we state a lemma that explains how to utilize finite-dimensional results to understand the extension structure in $\mathcal{C}_{\beta\gamma,\lambda}$, and finally we classify the indecomposable objects within each $\mathcal{C}_{\beta\gamma,\lambda}$. Along the way, we describe a useful way to visualize $\beta\gamma$ modules in terms their corresponding finite-dimensional Weyl algebra modules. Concrete examples will also be provided.

As a warm up, we start by computing extensions between some basic modules for the 1D Weyl algebra $H = \mathbb{C}[x, \partial]$ in the category of all H-modules. This category corresponds to a $\beta\gamma$ VOA module category strictly larger than $\mathcal{C}_{\beta\gamma}$, but the computations will nonetheless provide us with important information about $\mathcal{C}_{\beta\gamma}$. In particular, with a bit of extra work, we find that the computations of Ext^1 in this H-module category are the same as those in $\mathcal{C}_{\beta\gamma}$. Let us now begin.

The simple modules $\mathbb{C}[x], \mathbb{C}[\partial]$, and $x^\lambda \mathbb{C}[x, x^{-1}]$ $(\lambda \in \mathbb{C} \setminus \mathbb{Z})$ each have a 2-step free, hence projective, resolution by H. For example,

$$\cdots \longrightarrow 0 \longrightarrow H \xrightarrow{f_1} H \xrightarrow{f_0} \mathbb{C}[x] \longrightarrow 0 \tag{6.11}$$

is exact, where $f_1(1) = \partial$ and $f_0(1) = 1$. Using these resolutions to compute $\mathrm{Ext}^\bullet$, we arrive at the following results:

- $\mathrm{Ext}^k(\mathbb{C}[x], \mathbb{C}[x]) = \mathrm{Ext}^k(\mathbb{C}[\partial], \mathbb{C}[\partial]) = \mathbb{C}\,\delta_{k,0}$. In particular, neither $\mathbb{C}[x]$ nor $\mathbb{C}[\partial]$ have nontrivial self-extensions.
- $\mathrm{Ext}^k(\mathbb{C}[\partial], \mathbb{C}[x]) = \mathbb{C}\,\delta_{k,1}$, where the unique nontrivial extension is $\mathbb{C}[x, x^{-1}]$.
- $\mathrm{Ext}^k(\mathbb{C}[x], \mathbb{C}[\partial]) = \mathbb{C}\,\delta_{k,1}$, where the unique nontrivial extension is $\mathbb{C}[\partial, \partial^{-1}]$.
- $\mathrm{Ext}^k(x^\lambda \mathbb{C}[x, x^{-1}], x^\lambda \mathbb{C}[x, x^{-1}]) = \mathbb{C}\,\delta_{k,0} \oplus \mathbb{C}\,\delta_{k,1}$ where the unique self-extension is given by $\overline{\mathcal{W}}_\lambda^2 := x^\lambda \mathbb{C}[x, x^{-1}] \oplus x^\lambda \mathbb{C}[x, x^{-1}] \log x$. Furthermore the unique iterated self-extensions of $x^\lambda \mathbb{C}[x, x^{-1}]$ are

$$\overline{\mathcal{W}}_\lambda^k := x^\lambda \mathbb{C}[x, x^{-1}] \oplus \cdots \oplus x^\lambda \mathbb{C}[x, x^{-1}] \log^{k-1} x \tag{6.12}$$

- The extension algebra between $\overline{\mathcal{W}}_{\lambda_1}^{k_1}$ and $\overline{\mathcal{W}}_{\lambda_2}^{k_2}$ is zero for $\lambda_1 \neq \lambda_2$
- The extension algebra between $\overline{\mathcal{W}}_\lambda^k$ and $\mathbb{C}[x]$ or $\mathbb{C}[\partial]$ are both zero.

The following lemma explains why it suffices to compute the extensions between modules by restricting focus to their structure under a finite-dimensional Weyl subalgebra of $\beta\gamma$. For this, call

$$A_N = \mathbb{C}[\beta_k, \gamma_{-k}]_{-N \le k \le N}. \tag{6.13}$$

Lemma 6.1. *Let U, V, and W be in $\mathcal{C}_{\beta\gamma}$, and assume that they fit in the short exact sequence $0 \to U \to V \to W \to 0$. Suppose also that both U and W come from induction over A_N, namely*

$$U = \mathrm{Ind}^{\beta\gamma}_{A_N[\beta_j,\gamma_j]_{j>N}} U_N, \quad W = \mathrm{Ind}^{\beta\gamma}_{A_N[\beta_j,\gamma_j]_{j>N}} W_N. \tag{6.14}$$

Then V also comes from induction of an A_N-module V_N, and the above short exact sequence comes from the induction of the short exact sequence

$$0 \longrightarrow U_N \longrightarrow V_N \longrightarrow W_N \longrightarrow 0. \tag{6.15}$$

Proof. Given a module M of $\beta\gamma$, denote by $K_N(M)$ the kernel of all the β_k and γ_k for $k > N$. Such $K_N(M)$ is easily seen to be a module of A_N. When M comes from induction from some A_N-module M_N, we have $K_N(M) = M_N$. Applying this to the short exact sequence $0 \to U \to V \to W \to 0$, using the fact that taking kernel is left exact, we get

$$0 \longrightarrow U_N \longrightarrow K_N(V) \longrightarrow W_N. \tag{6.16}$$

We claim that $K_N(V) \to W_N$ is surjective. Given any $w \in W_N$, since $V \to W$ is onto, we may choose $v \in V$ such that it's image in W is w. Since V is a generalized VOA module, there exists K such that $\beta_k v = \gamma_k v = 0$ for $k > K$. We will adjust v in a way that it's image in W is still w, but it will be annihilated by β_j and γ_j for $N < j \le K$. For any such mode, say β_j, if $\beta_j v \neq 0$, then since it's image in W is zero, it must be in the kernel of $V \to W$, which is U. Thus $\beta_j v \in U$. By the fact that U comes from induction, there exists n such that $\beta_j^{n+1} v = 0$. Now using $[\beta_j^n, \gamma_{-j}] = -n\beta_j^{n-1}$, one gets

$$\beta_j^n \gamma_{-j} \beta_j v = -n\beta_j^n v. \tag{6.17}$$

In other words, $\beta_j^n(v+\frac{1}{n}\gamma_{-j}\beta_j v)=0$. The element $v+\frac{1}{n}\gamma_{-j}\beta_j v$ has the same image as v in W, since $\gamma_{-j}\beta_j v\in U$. Inductively, one can find $u\in U$ such that $v-u$ is in fact annihilated by β_j, and u is of the form $f(\beta_j,\gamma_{-j})\beta_j v$ for some polynomial f of two variables, where we always choose the normal ordering in the polynomial, namely γ_{-j} appears before β_j. Notice that if v is annihilated by β_k for $k\neq j$, then so is $v-f(\beta_j,\gamma_{-j})\beta_j v$. Applying this to all the β_j and γ_j for $N<j<K$, one gets v that is annihilated by all β_j and γ_j for $j>N$, namely $v\in K_N(V)$ whose image is w.

In conclusion, we have a short exact sequence

$$0 \longrightarrow U_N \longrightarrow K_N(V) \longrightarrow W_N \longrightarrow 0\ . \tag{6.18}$$

Induction gives us a short exact sequence that fits in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & \mathrm{Ind}^{\beta\gamma}_{A_N[\beta_j,\gamma_j]_{j>N}}(K_N(V)) & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \end{array} \tag{6.19}$$

The left and right down-arrows are isomorphisms, so by the snake lemma, the middle down-arrow is an isomorphism as well. This completes the proof. □

Let us demonstrate how one can combine the 1D results with Lemma 6.1 to compute extensions in $\mathcal{C}_{\beta\gamma}$. Suppose we have a nontrivial extension M of $\mathcal{V}$ by $\sigma^{-1}\mathcal{V}$. In the notation of the lemma,

$$\mathcal{V}=\mathrm{Ind}^{\beta\gamma}_{A_0[\beta_j,\gamma_j]_{j>0}}\mathbb{C}[\gamma_0] \tag{6.20}$$

Similarly,

$$\sigma^{-1}\mathcal{V}=\mathrm{Ind}^{\beta\gamma}_{A_0[\beta_j,\gamma_j]_{j>0}}\mathbb{C}[\beta_0] \tag{6.21}$$

By Lemma 6.1, M is the induction of some $A_0[\beta_j,\gamma_j]_{j>1}$-module M_0 that fits into the short exact sequence

$$0 \longrightarrow \mathbb{C}[\beta_0] \longrightarrow M_0 \longrightarrow \mathbb{C}[\gamma_0] \longrightarrow 0 \tag{6.22}$$

It must be that M_0 is a nontrivial extension, since otherwise

$$0 \longrightarrow \sigma^{-1}\mathcal{V} \longrightarrow M \longrightarrow \mathcal{V} \longrightarrow 0 \tag{6.23}$$

would split. From our computational results just below (6.11), it must be that $M_0 \cong \mathbb{C}[\beta_0, \beta_0^{-1}]$, hence

$$M \cong \mathrm{Ind}^{\beta\gamma}_{A_0[\beta_j,\gamma_j]_{j>0}} \mathbb{C}[\beta_0, \beta_0^{-1}] \tag{6.24}$$

This module indeed lies in $\mathcal{C}_{\beta\gamma}$ (i.e. a nontrivial extension of $\mathcal{V}$ by $\sigma^{-1}\mathcal{V}$ indeed exists in our category) and it is none other than $\mathcal{W}_0^-$ as we expected from equation (6.6)!

Before proceeding to characterize the objects in $\mathcal{C}_{\beta\gamma}$, we need to investigate how to compute extensions between induced objects a bit further. Suppose we have a module $M \in \mathcal{C}_{\beta\gamma}$ that comes from the induction of an A_N-module M'. Then since

$$\beta\gamma \simeq \bigotimes_{k\in\mathbb{Z}} \beta\gamma_k = A_N \otimes \left(\bigotimes_{|k|>N} \beta\gamma_k \right), \tag{6.25}$$

we have the decomposition

$$M \cong M' \otimes \left(\bigotimes_{|k|>N} M_k \right) \tag{6.26}$$

where each M_k is a $\beta\gamma_k$-module. In fact, each M_k is $\mathbb{C}[\partial_k]$ for $k > 0$ and $\mathbb{C}[x_{-k}]$ for $k < 0$. We visualize this data in terms of the following "column picture" for M:

$$\begin{array}{cccccccccccccc} \beta\gamma & \cdots & \otimes & \beta\gamma_{-N-2} & \otimes & \beta\gamma_{-N-1} & \otimes & A_N & \otimes & \beta\gamma_{N+1} & \otimes & \beta\gamma_{N+2} & \otimes & \cdots \\ M & \cdots & \otimes & M_{-N-2} & \otimes & M_{-N-1} & \otimes & M' & \otimes & M_{N+1} & \otimes & M_{N+2} & \otimes & \cdots \end{array} \tag{6.27}$$

Visualizing/decomposing induced modules in this way makes the proofs in this section easier to follow. Let us provide a concrete example of the column picture for $\mathcal{V}$. Recalling that $\mathcal{V} = \mathrm{Ind}^{\beta\gamma}_{\beta\gamma_{\geq 0}}\mathbb{C}[\gamma_0]$ (i.e. $\mathcal{V}$ comes from the induction of the A_0-module $\mathbb{C}[\gamma_0]$), its column picture is

$$\begin{array}{ccccccccccccc} \beta\gamma & \cdots & \otimes & \beta\gamma_{-2} & \otimes & \beta\gamma_{-1} & \otimes & \beta\gamma_0 & \otimes & \beta\gamma_1 & \otimes & \beta\gamma_2 & \otimes & \cdots \\ \mathcal{V} & \cdots & \otimes & \mathbb{C}[x_2] & \otimes & \mathbb{C}[x_1] & \otimes & \mathbb{C}[\partial_0] & \otimes & \mathbb{C}[\partial_1] & \otimes & \mathbb{C}[\partial_2] & \otimes & \cdots \end{array} \tag{6.28}$$

Note that spectral flow simply shifts the column picture of a module horizontally. For example, the column picture for $\sigma^{-1}\mathcal{V}$ can be obtained by shifting the column picture of $\mathcal{V}$ by one unit to the right. We typically relabel the indices on x_k and ∂_k after shifting to make it easier to remember which column they correspond to.

Let us demonstrate how the column picture can be used to compute the extensions between modules with an example. To compute the extensions of $\mathcal{V}$ by $\sigma\mathcal{V}$, we first stack the column pictures for $\mathcal{V}$ and $\sigma\mathcal{V}$:

$$\begin{array}{ccccccccccccc} \beta\gamma & \cdots & \otimes & \beta\gamma_{-2} & \otimes & \beta\gamma_{-1} & \otimes & \beta\gamma_0 & \otimes & \beta\gamma_1 & \otimes & \beta\gamma_2 & \otimes & \cdots \\ \mathcal{V} & \cdots & \otimes & \mathbb{C}[x_2] & \otimes & \mathbb{C}[x_1] & \otimes & \mathbb{C}[\partial_0] & \otimes & \mathbb{C}[\partial_1] & \otimes & \mathbb{C}[\partial_2] & \otimes & \cdots \\ \sigma\mathcal{V} & \cdots & \otimes & \mathbb{C}[x_2] & \otimes & \mathbb{C}[\partial_{-1}] & \otimes & \mathbb{C}[\partial_0] & \otimes & \mathbb{C}[\partial_1] & \otimes & \mathbb{C}[\partial_2] & \otimes & \cdots \end{array} \tag{6.29}$$

Notice that

$$\sigma\mathcal{V} = \mathrm{Ind}^{\beta\gamma}_{A_1[\beta_j,\gamma_j]_{j>1}}(\mathbb{C}[\partial_{-1}] \otimes \mathbb{C}[\partial_0] \otimes \mathbb{C}[\partial_1]) \tag{6.30}$$

where A_1 acts in the standard manner as a 3-dimensional Weyl algebra and β_j and γ_j act as 0 for $j > 1$. According to Lemma 6.1, it suffices to instead compute the extensions between the modules contained only in the middle 3 columns

$$\begin{array}{ccccc} \beta\gamma_{-1} & \otimes & \beta\gamma_0 & \otimes & \beta\gamma_1 \\ \mathbb{C}[x_1] & \otimes & \mathbb{C}[\partial_0] & \otimes & \mathbb{C}[\partial_1] \\ \mathbb{C}[\partial_{-1}] & \otimes & \mathbb{C}[\partial_0] & \otimes & \mathbb{C}[\partial_1] \end{array} \tag{6.31}$$

Applying a Künneth formula, the only non-zero contribution to $\mathrm{Ext}^k(\mathcal{V}, \sigma\mathcal{V})$ is in degree 1 and comes from the left column, which is $\mathbb{C}[\partial_{-1}, \partial_{-1}^{-1}]$. Thus $\mathrm{Ext}^k(\mathcal{V}, \sigma\mathcal{V}) = \mathbb{C}\,\delta_{k,1}$. The column picture representing the module corresponding to the degree-1 extension is

$$\begin{array}{ccccccccccccccc} \beta\gamma & \cdots & \otimes & \beta\gamma_{-2} & \otimes & \beta\gamma_{-1} & \otimes & \beta\gamma_0 & \otimes & \beta\gamma_1 & \otimes & \beta\gamma_2 & \otimes & \cdots \\ \big(\sigma\mathcal{V} \longrightarrow \mathcal{V}\big) & \cdots & \otimes & \mathbb{C}[x_2] & \otimes & \mathbb{C}[\partial_{-1}, \partial_{-1}^{-1}] & \otimes & \mathbb{C}[\partial_0] & \otimes & \mathbb{C}[\partial_1] & \otimes & \mathbb{C}[\partial_2] & \otimes & \cdots \end{array} \tag{6.32}$$

which is precisely $\sigma\mathcal{W}_0^+$!

Finally, we characterize the objects of $\mathcal{C}_{\beta\gamma}$:

Theorem 6.2. *Every indecomposable object in $\mathcal{C}_{\beta\gamma,\lambda}$ for $\lambda \neq \mathbb{Z}$ is isomorphic to $\sigma^n\mathcal{W}_\lambda^k$ for some $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.*

Proof. Let M be an indecomposable object in $\mathcal{C}_{\beta\gamma,\lambda}$ and pick any $\mu \in \lambda$. We induct on the length of M. If M is simple, it must be $\sigma^n\mathcal{W}_\lambda$ for some $n \in \mathbb{Z}$.

Now suppose M has length k and assume its length $k-1$ submodule (in any particular composition series) is isomorphic to $\sigma^n\mathcal{W}_\lambda^{k-1}$. Then $M/\sigma^n W_\lambda^{k-1}$ is a simple module in $\mathcal{C}_{\beta\gamma,\lambda}$, hence is isomorphic to $\sigma^m\mathcal{W}_\lambda$ for some $m \in \mathbb{Z}$. Thus M fits into the short exact sequence

$$0 \longrightarrow \sigma^n\mathcal{W}_\lambda^{k-1} \longrightarrow M \longrightarrow \sigma^m\mathcal{W}_\lambda \longrightarrow 0\ . \tag{6.33}$$

We now characterize all possible extensions of this type. In the column picture of $\sigma^n\mathcal{W}_\lambda^{k-1}$, there is a $(\partial_{-n})^\mu(\mathbb{C}[\partial_{-n}, \partial_{-n}^{-1}] \oplus \cdots \oplus \mathbb{C}[\partial_{-n}, \partial_{-n}^{-1}][\log^{k-2}\partial_{-n}])$ in the $\beta\gamma_{-n}$ column, and everything in the columns to its left and right are $\mathbb{C}[x]$'s and $\mathbb{C}[\partial]$'s, respectively. The column picture of $\sigma^m\mathcal{W}_\lambda$ has a $(\partial_{-m})^\mu\mathbb{C}[\partial_{-m}, \partial_{-m}^{-1}]$ in the $\beta\gamma_{-m}$ column, and the other columns are similarly $\mathbb{C}[x]$'s and $\mathbb{C}[\partial]$'s. Our results about the representation theory of H dictate that $m = n$ in order to have a nontrivial extension. Furthermore, when $m = n$, the same results tell us that the unique nontrivial extension is $(\partial_{-n})^\mu(\mathbb{C}[\partial_{-n}, \partial_{-n}^{-1}] \oplus \cdots \oplus \mathbb{C}[\partial_{-n}, \partial_{-n}^{-1}][\log^{k-1}\partial_{-n}])$. Thus $M \cong \sigma^n\mathcal{W}_\lambda^k$, finishing the induction. □

To characterize the indecomposables in $\mathcal{C}_{\beta\gamma,\mathbb{Z}}$, we introduce a new class of modules, called *roofs*, with the following property: each module in $\mathcal{C}_{\beta\gamma,\mathbb{Z}}$ can be covered by a finite direct sum of roofs. To construct a roof, one first takes the direct sum of a positive and a negative chain that have the same head, and then one takes the submodule generated by the diagonal of the head. For example, the heads of $\mathcal{W}_0^{-,n}$ and $\sigma\mathcal{W}_0^{+,m}$ are both $\mathcal{V}$, so the roof $\mathcal{R}_{2n,2m}$ is the unique submodule of $\mathcal{W}_0^{-,n} \oplus \sigma\mathcal{W}_0^{+,m}$ generated by the diagonal of the head $\mathcal{V}$. The Loewy diagram of $\mathcal{R}_{4,4}$, rotated 90° clockwise to fit better on the page, looks like

$$\begin{array}{ccccccc} \sigma^{-1}\mathcal{V} & \longrightarrow & \mathcal{V} & \longrightarrow & \sigma^{-1}\mathcal{V} & & \\ & & & & & \searrow & \\ & & & & & & \mathcal{V} \\ & & & & & \nearrow & \\ \sigma\mathcal{V} & \longrightarrow & \mathcal{V} & \longrightarrow & \sigma\mathcal{V} & & \end{array} \tag{6.34}$$

which is essentially diagrams (6.8) and a spectral flow of (6.9) pinched together at the head. This diagram looks like a tall roof, when unrotated, hence the name. The subscripts a and b on $\mathcal{R}_{a,b}$ represent the length of the left and right sides of the roof, respectively. When $a = b$, we drop the redundant subscript

$$\mathcal{R}_a := \mathcal{R}_{a,a} \tag{6.35}$$

Our proof of the characterization theorem uses results about the extensions of a chain by $\sigma^m\mathcal{V}$. We state the necessary results without proof, but one can easily compute these extension groups with inductive arguments and homological techniques similar to those used below.

$$\mathrm{Ext}^1(\sigma^n\mathcal{W}_0^{+,k}, \sigma^m\mathcal{V}) = \mathbb{C}\,\delta_{m,n-2} \oplus \mathbb{C}\,\delta_{m,n-1} \tag{6.36}$$

$$\mathrm{Ext}^1(\sigma^n\mathcal{W}_0^{-,k}, \sigma^m\mathcal{V}) = \mathbb{C}\,\delta_{m,n} \oplus \mathbb{C}\,\delta_{m,n+1} \tag{6.37}$$

Theorem 6.3. *Every indecomposable object in $\mathcal{C}_{\beta\gamma,\mathbb{Z}}$ is isomorphic to a quotient of a finite direct sum of $\sigma^n\mathcal{R}_k$ for various $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.*

Proof. Let M be a length ℓ indecomposable object in $\mathcal{C}_{\beta\gamma,\mathbb{Z}}$. We induct on ℓ. The statement holds for $\ell = 1$ since all simple modules in $\mathcal{C}_{\beta\gamma,\mathbb{Z}}$ are $\sigma^n\mathcal{V}$, which are equal to $\sigma^n\mathcal{R}_1$.

Take any composition series for M and suppose $\sigma^m\mathcal{V}$ is the first term of the series. By induction, $\widetilde{M} := M/\sigma^m\mathcal{V}$ is a quotient of $\bigoplus_i \sigma^{n_i}\mathcal{R}_{a_i,b_i}$. Since a roof can be covered by a longer roof, $\widetilde{M}$ is also a quotient of $\bigoplus_i \sigma^{n_i}\mathcal{R}_{k_i}$ for k_i sufficiently large. We will see that one can choose the k_i strategically to simplify the proof.

So far, we have the following exact diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & \sigma^m\mathcal{V} & \longrightarrow & M & \xrightarrow{\pi} & \widetilde{M} & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle \pi'} & & \\ & & & & & & \bigoplus_i \sigma^{n_i}\mathcal{R}_{k_i} & & \end{array} \tag{6.38}$$

Letting M' be the fiber product of π and π', the above exact diagram can be extended to

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \sigma^m\mathcal{V} & \longrightarrow & M & \xrightarrow{\pi} & \widetilde{M} & \longrightarrow & 0 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow{\scriptstyle \pi'} & & \\ 0 & \longrightarrow & \sigma^m\mathcal{V} & \longrightarrow & M' & \longrightarrow & \bigoplus_i \sigma^{n_i}\mathcal{R}_{k_i} & \longrightarrow & 0 \end{array} \tag{6.39}$$

The proof will be complete if we can show that M' is a quotient of roofs, so we must analyze the extensions of $\bigoplus_i \sigma^{n_i}\mathcal{R}_{k_i}$ by $\sigma^m\mathcal{V}$. Since

$$\mathrm{Ext}^1\left(\bigoplus_i \sigma^{n_i}\mathcal{R}_{k_i}, \sigma^m\mathcal{V}\right) \cong \bigoplus_i \mathrm{Ext}^1(\sigma^{n_i}\mathcal{R}_{k_i}, \sigma^m\mathcal{V}) \tag{6.40}$$

we will see that we can assume, without loss of generality, that $\widetilde{M}$ is covered by a single roof $\sigma^n\mathcal{R}_k$. We split the analysis into cases based on the value of m.

If $m = n$, then choose k to be odd. Let L be the length $k-1$ submodule of $\sigma^n \mathcal{R}_k$ constituting its "left half", i.e. L looks like

$$\sigma^n \mathcal{V} \longrightarrow \sigma^{n-1} \mathcal{V} \longrightarrow \sigma^n \mathcal{V} \longrightarrow \cdots \longrightarrow \sigma^{n-1} \mathcal{V} \tag{6.41}$$

Then $R := \sigma^n \mathcal{R}_k / L$ is the length k "right half" of the roof, which looks like

$$\sigma^n \mathcal{V} \longrightarrow \sigma^{n+1} \mathcal{V} \longrightarrow \sigma^n \mathcal{V} \longrightarrow \cdots \longrightarrow \sigma^n \mathcal{V} \tag{6.42}$$

Applying $\mathrm{Ext}(-, \sigma^m \mathcal{V})$ to

$$0 \longrightarrow L \longrightarrow \sigma^n \mathcal{R}_k \longrightarrow R \longrightarrow 0 \tag{6.43}$$

produces the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(R, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Hom}(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Hom}(L, \sigma^m \mathcal{V}) \\ &\longrightarrow \mathrm{Ext}^1(R, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Ext}^1(L, \sigma^m \mathcal{V}) \end{aligned} \tag{6.44}$$

which is

$$\begin{aligned} 0 &\longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0 \\ &\longrightarrow 0 \longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \longrightarrow 0 \end{aligned} \tag{6.45}$$

hence there are no nontrivial extensions. This means that $M' \cong \sigma^m \mathcal{V} \oplus \sigma^n \mathcal{R}_k$ in this case, which indeed is a (trivial) quotient of roofs.

If $m = n \pm 1$, then choose k to be even so that we have a short exact sequence

$$0 \longrightarrow \sigma^n \mathcal{R}_k \longrightarrow \sigma^n \mathcal{W}_0^{-,k/2} \oplus \sigma^{n+1} \mathcal{W}_0^{+,k/2} \longrightarrow \sigma^n \mathcal{V} \longrightarrow 0 \tag{6.46}$$

Applying $\mathrm{Ext}(-, \sigma^m \mathcal{V})$ to this yields

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}(\sigma^n \mathcal{V}, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Hom}(\sigma^n \mathcal{W}_0^{-,k/2} \oplus \sigma^{n+1} \mathcal{W}_0^{+,k/n}, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Hom}(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \\ &\longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{V}, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{W}_0^{-,k/2} \oplus \sigma^{n+1} \mathcal{W}_0^{+,k/n}, \sigma^m \mathcal{V}) \longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \\ &\longrightarrow \mathrm{Ext}^2(\sigma^n \mathcal{V}, \sigma^m \mathcal{V})\end{aligned} \tag{6.47}$$

which is

$$\begin{aligned}0 &\longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \\ &\longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \\ &\longrightarrow 0\end{aligned} \tag{6.48}$$

hence there are no nontrivial extensions.

If $m = n + 2$, then choose k to be odd and define L and R as in the $m = n$ case above. Starting from the same short exact sequence, the evaluation of the long exact sequence (6.44) for $m = n + 2$ gives

$$\begin{aligned}0 &\longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \\ &\longrightarrow 0 \longrightarrow \mathrm{Ext}^1(\sigma^n \mathcal{R}_k, \sigma^m \mathcal{V}) \longrightarrow 0\end{aligned} \tag{6.49}$$

hence there are no nontrivial extensions. The case $m = n - 2$ is similar but one must instead choose R to be the length $k - 1$ submodule involving $\sigma^{n+1}\mathcal{V}$ and L to be the corresponding length k quotient.

By a similar argument, there are no nontrivial extensions if $|m - n| > 2$.

Therefore, with a suitable choice for k, we have shown $\mathrm{Ext}^1(\sigma^{n_i} \mathcal{R}_k, \sigma^m \mathcal{V}) = 0$. Removing our assumption, the same technique can be used to show that equation (6.40) is zero (with suitably chosen k_i), hence M' is a direct sum of roofs, as desired. □

As a corollary of these two theorems, we now understand why $\mathcal{C}_{\beta\gamma}$ does not have enough projectives. Suppose we suspect an object P to be projective in $\mathcal{C}_{\beta\gamma,\lambda}$. By the previous two theorems, P is a quotient of a direct sum of chains C described by a surjective map

$f\colon C \to P$. Let C' be the direct sum of the same chains that appear in C but make them, say, 3 times as long. From our remarks earlier in the section, there exists another surjection $\pi\colon C' \to C$ that maps the top third of C' onto C and the bottom two-thirds of C' to 0. If P is projective, then there should exist some map g making the following commute

$$\begin{array}{ccccc} & & & & C' \\ & & & \overset{g}{\nearrow} & \Big\downarrow{\scriptstyle f\circ\pi} \\ C & \xrightarrow{f} & P & \xrightarrow{\mathbb{1}} & P \end{array} \tag{6.50}$$

Such a map does not exist because the image of $g \circ f$ is contained in the kernel of π: chains cannot map to composition factors in another chain that are "higher up the chain" (i.e. further from the bottom of the chain) than the length of the original chain, hence $g \circ f$ maps to the bottom two-thirds of C'. Thus P is not projective. The argument for objects in $\mathcal{C}_{\beta\gamma,\mathbb{Z}}$ is similar; simply replace each occurrence of "chain" with "roof". In conclusion, we have actually managed to show that $\mathcal{C}_{\beta\gamma}$ does not even contain a single projective object!

6.1.3 Tensor Structure

Our category $\mathcal{C}_{\beta\gamma}$ possesses the structure of a braided tensor category given by the $P(z)$-fusion product (this was defined in Section 4.3.5). It is not straightforward to prove that $\mathcal{C}_{\beta\gamma}$ satisfies the assumptions in the work of [HLZ10-11] since the modules fail to have bounded-from-below conformal weights, thus we cannot directly conclude that $P(z)$-fusion products actually define a tensor structure on $\mathcal{C}_{\beta\gamma}$. Moreover, performing computations in $\mathcal{C}_{\beta\gamma}$ with this universal definition is very difficult in practice. To circumvent this roadblock, we will use the idea of mirror symmetry to connect $\mathcal{C}_{\beta\gamma}$ to the category of modules for a simple current extension of the VOA associated to $\widehat{\mathfrak{gl}(1|1)}$. This approach was successfully executed in [AP19] to determine fusion rules in the subcategory of *weight* modules studied by [RW15]. The advantage of following this approach to study the larger category of modules of $\widehat{\mathfrak{gl}(1|1)}$ that we consider is that the grading restriction is automatically satisfied, hence the machinery of [HLZ10-11] can be applied. In Section 6.3, we demonstrate that $\mathcal{C}_{\beta\gamma}$ sits in a larger category $\mathrm{Rep}^0(\mathcal{V}_{ext})$ which is related to the category of modules of $\widehat{\mathfrak{gl}(1|1)}$. By the work of [CMY20b, CMY20a], we can show that $\mathrm{Rep}^0(\mathcal{V}_{ext})$ is a braided tensor category defined by $P(z)$-intertwining maps, which will lead to a braided tensor structure on $\mathcal{C}_{\beta\gamma}$. Moreover, using our classification results in Section 6.1.2, we will prove the 3d mirror symmetry statement, namely the second half of Theorem 6.1. Let us now turn to $\widehat{\mathfrak{gl}(1|1)}$, which is the next main ingredient of our story.

6.2 The affine Lie superalgebra $\widehat{\mathfrak{gl}(1|1)}$

In this section, we will study the representation theory of the affine Lie superalgebra $\widehat{\mathfrak{gl}(1|1)}$, the VOSA that naturally arose in Section 5.5. In Section 6.2.1, we describe a relevant category of finite-dimensional $\mathfrak{gl}(1|1)$-modules. In Section 6.2.4, we review the affine Lie superalgebra $\widehat{\mathfrak{gl}(1|1)}$, following the work of [CMY20a]. In Section 6.2.5, we describe the category KL, again following the work of [CMY20a]; we then proceed to prove in Proposition 6.1 that KL is closely related to the category of finite-dimensional modules of $\mathfrak{gl}(1|1)$. In Section 6.2.6, we use the result of [CMY20a] to compute the fusion product of indecomposable modules of $\widehat{\mathfrak{gl}(1|1)}$. We especially highlight how the structure of the representation categories of $\mathfrak{gl}(1|1)$ and $\widehat{\mathfrak{gl}(1|1)}$ are related to each other.

6.2.1 The relevant representation category of $\mathfrak{gl}(1|1)$

To make contact with $\mathcal{C}_{\beta\gamma}$, we must carefully choose the $\mathfrak{gl}(1|1)$ representation subcategory that we study. For now, let $\mathcal{C}$ be the supercategory of finite-dimensional modules for the Lie superalgebra $\mathfrak{gl}(1|1)$, enriched to contain morphisms of odd degree [Bru14]. We do not require N nor E to act semisimply on $\mathcal{C}$, unlike most of the literature on the representation theory of $\mathfrak{gl}(1|1)$. However, we will eventually restrict to a full subcategory wherein a specified linear combination of N and E *does* act semisimply. With this in mind, we organize modules into families parametrized by a $\mathbb{CP}^1$-valued parameter x, which indicates that $N - xE$ acts semisimply (the case $x = \infty$ is understood to mean that E acts semisimply). We drop the label when the family contains exactly one module.

In the following sections, one may notice that we have described module families and given proofs in separate cases based on the value of x. Let us briefly digress to explain why this was necessary. Our work will often utilize a certain linear combination of N and E that possesses a non-zero nilpotent part. If it were possible to provide the same module descriptions and proofs for every $x \in \mathbb{CP}^1$, we would need a continuous parameterization of the linear combinations $N - \alpha(x)E$, with $\alpha(x) \in \mathbb{CP}^1$, that acts non-semisimply on modules with label x. This is equivalent to finding a continuous map $\alpha\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\alpha(x) \neq x$ for all $x \in \mathbb{CP}^1$. The Brouwer fixed point theorem tells us this cannot be done.

6.2.2 Elementary $\mathfrak{gl}(1|1)$ modules

We introduce some basic objects in $\mathcal{C}$ that will be heavily used in this thesis.

1. **Singletons:** $A_n^k \quad (k \in \mathbb{Z},\, n \in \mathbb{C})$

   A_n^k is a k-dimensional module on which E and $\psi^{\pm}$ act as zero, and N has a rank k Jordan block with eigenvalue n.

2. **Typical chains:** $V_{n,e,x}^k \quad (k \in \mathbb{Z},\, n \in \mathbb{C},\, e \in \mathbb{C} \setminus \{0\})$

   The chain $V_{n,e,x}^k$ for $x \neq \infty$ is uniquely characterized (up to isomorphism) by the following property: there exists a vector v_1 such that

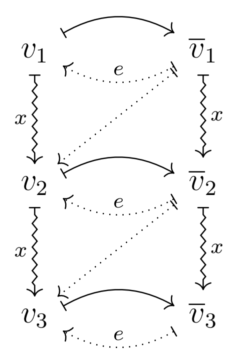
   - $v_1 \in \operatorname{geig}(N, n + \frac{1}{2})$
   - $\psi^+ v_1 = 0$
   - $(E-e)^k v_1 = 0$ but $(E-e)^{k-1} v_1 \neq 0$ and $(E-e)^{k-1}\psi^- v_1 \neq 0$
   - Defining $v_j = (E-e)^{j-1} v_1$ and $\overline{v}_j = \psi^- v_j$ for $1 \leq j \leq k$, $\{v_1, \overline{v}_1, \ldots, v_k, \overline{v}_k\}$ form a basis for $V_{n,e,x}^k$.
   - $N - xE$ acts semisimply

where $\mathrm{geig}(N, \lambda)$ denotes the generalized eigenspace of N corresponding to eigenvalue λ. One can use the Jordan-Chevalley decomposition to check that N is composed of a Jordan block of rank k corresponding to eigenvalue $n + \frac{1}{2}$ and a Jordan block of rank k corresponding to eigenvalue $n - \frac{1}{2}$.

The chain $V^k_{n,e,\infty}$ is uniquely characterized (up to isomorphism) by the following property: there exists a vector v_1 such that

- $\psi^+ v_1 = 0$
- $(N - (n + \frac{1}{2}))^k v_1 = 0$ but $(N - (n + \frac{1}{2}))^{k-1} v_1 \neq 0$ and $(N - (n - \frac{1}{2}))^{k-1} \psi^- v_1 \neq 0$
- Defining $v_j = (N - (n + \frac{1}{2}))^{j-1} v_1$ and $\overline{v}_j = \psi^- v_j$ for $1 \leq j \leq k$, $\{v_1, \overline{v}_1, \ldots, v_k, \overline{v}_k\}$ form a basis for $V^k_{n,e,\infty}$.
- E acts semisimply

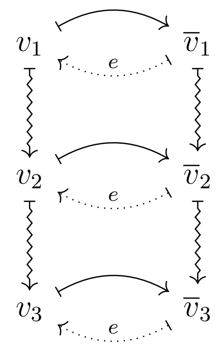
As a visual aid, we depict $V^3_{n,e,x}$ ($x \neq \infty$) here:



(6.51)

The squiggly lines represent the off-diagonal diagonal action of N; the semisimple part of N is $n + \frac{1}{2}$ on the left column, and $n - \frac{1}{2}$ on the right column. The dotted arrows represent the action of ψ^+ and the solid arrows represent the action of ψ^-.

For $x = \infty$, $V^3_{n,e,\infty}$ looks like



(6.52)

Note that $V^k_{n,e,x}$ is the unique $(k-1)^{\text{th}}$-iterated self extension of $V_{n,e,x}$.

3. **Atypical chains** $V^k_{n,0,\pm,x} \quad (k \in \mathbb{Z},\, n \in \mathbb{C})$

   There are two ways to take the heuristic limit $\lim_{e \to 0} V^k_{n,e,x}$, and each results in a distinct module. They look very similar to the typical chains, but we instead choose to describe them in terms of their Loewy diagram. For $x \neq \infty$, the *positive* chains look like

$$V^k_{n,0,+,x} := \;\; A^1_{n-\frac{1}{2}} \longrightarrow A^1_{n+\frac{1}{2}} \longrightarrow \cdots \longrightarrow A^1_{n-\frac{1}{2}} \longrightarrow A^1_{n+\frac{1}{2}} \tag{6.53}$$

   whereas the *negative* chains look like

$$V^k_{n,0,-,x} := \;\; A^1_{n+\frac{1}{2}} \longrightarrow A^1_{n-\frac{1}{2}} \longrightarrow \cdots \longrightarrow A^1_{n+\frac{1}{2}} \longrightarrow A^1_{n-\frac{1}{2}} \tag{6.54}$$

   Together with the semisimplicity condition on $N - xE$, these Loewy diagrams uniquely describe the atypical chains. The Loewy diagrams look slightly different when $x = \infty$, but this case will not be important for us.

4. **Diamonds** $P^k_{n,x} \quad (k \in \mathbb{Z}, n \in \mathbb{C})$

   The diamond $P^k_{n,x}$ for $x \neq \infty$ is uniquely characterized (up to isomorphism) by the following property: there exists a vector v_1 such that

   - $v_1 \in \mathrm{geig}(N, n)$

- $E^k v_1 = 0$
- defining

$$w_1 = \psi^+ v_1 \qquad v_j = \begin{cases} \psi^- v_{j-1} & j \text{ even} \\ \psi^+ v_{j-1} & j \text{ odd} \end{cases} \qquad w_j = \begin{cases} \psi^- w_{j-1} & j \text{ even} \\ \psi^+ w_{j-1} & j \text{ odd} \end{cases} \tag{6.55}$$

  for $2 \leq j \leq 2k$, we have that $\{v_1, w_1, \ldots, v_{2k}, w_{2k}\}$ forms a basis for $P^k_{n,x}$

- $N - xE$ acts semisimply

The diamond $P^k_{n,\infty}$ is uniquely characterized (up to isomorphism) by the following property: there exists a vector v_1 such that

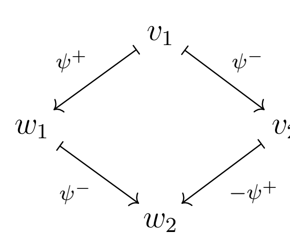
- $v_1 \in \mathrm{geig}(N, n)$
- defining

$$v_j = (N-n)^{j-1} v_1 \qquad x_j = \psi^+ v_j \qquad y_j = \psi^- v_j \qquad w_j = \psi^- \psi^+ v_j \tag{6.56}$$

  for $1 \leq j \leq k$, we have that $\{v_1, x_1, y_1, w_1, \ldots, v_k, x_k, y_k, w_k\}$ forms a basis for $P^k_{n,\infty}$

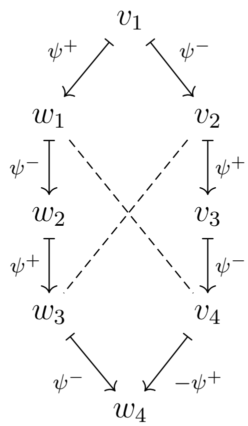
- $E = 0$ on the entire module

We first depict the diamonds when $x = 0$ to remove some clutter that might otherwise obfuscate their core structure. $P_{n,0}$ looks like



(6.57)

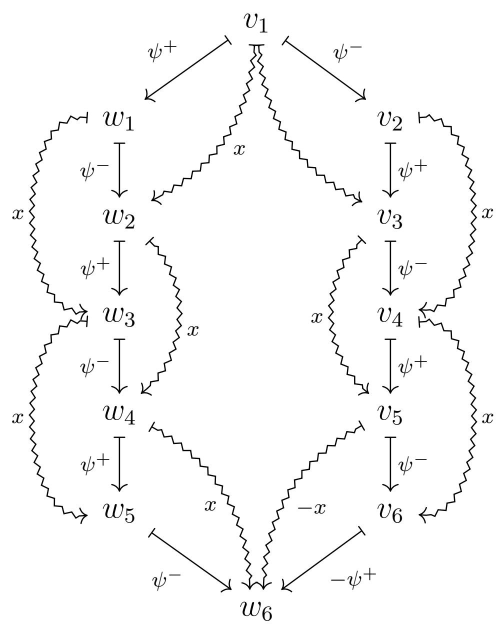
where $Nv_1 = nv_1$. There exist iterated self-extensions of $P_{n,0}$ called $P^k_{n,0}$. The positive integer k refers to the "number of $P_{n,0}$'s it contains". For example, $P^2_{n,0}$ is



(6.58)

The dashed lines here illustrate the $P_{n,0}$'s that $P^2_{n,0}$ contains as submodules and quotients. The submodule (bottom diamond) is generated by $w_2 + v_3$. When we quotient by this bottom diamond, the bottom vector of the quotient (top diamond) is given by the equivalence class of $w_2 - v_3$.

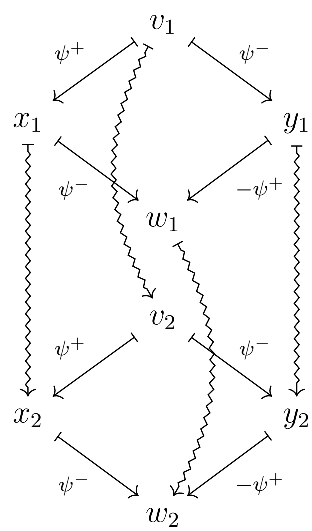
Restoring x, we may illustrate $P^3_{n,x}$ with $x \neq \infty$ as:



(6.59)

where the squiggly arrows represent the off-diagonal action of N.

The module $P^2_{n,\infty}$ is drawn as:



(6.60)

where the squiggly arrows represent the off-diagonal action of N.

At this point, the reader may wish to look back at the $\beta\gamma$ modules (their Loewy diagrams, in particular) introduced in Section 6.1.1 to get a sense of what the categorical equivalence will ultimately look like. A small detail that we have swept under the rug is the $\mathbb{Z}_2$ grading on the modules and morphisms in the supercategory $\mathcal{C}$. Given an object $X \in \mathrm{Ob}(\mathcal{C})$, its parity conjugate ΠX is also an object in $\mathcal{C}$. There is no corresponding notion of the parity-shifted version of a module in $\mathcal{C}_{\beta\gamma}$, therefore one might be concerned about the correctness of the equivalence. It turns out that X and ΠX are isomorphic in $\mathcal{C}$, albeit via an *odd* isomorphism, for which there is no analogue in $\mathcal{C}_{\beta\gamma}$. To foreshadow the resolution, we will instead find that our category matches $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$. This does not impede our ultimate goal because SVect has a nearly transparent effect on the tensor structure, so we can straightforwardly extract the fusion structure on $\mathcal{C}_{\beta\gamma}$.

6.2.3 Tensor structure on $\mathfrak{gl}(1|1)$

As we have mentioned before, we still need to pass through a few more categories and constructions before connecting with $\mathcal{C}_{\beta\gamma}$. However, much of the tensor structure on $\mathcal{C}_{\beta\gamma}$ can ultimately be obtained from the structure on $\mathcal{C}$, where computations are much easier. In this section, we give the tensor product decompositions that we computed that will be relevant for the remainder of this thesis.

The action of $\mathfrak{gl}(1|1)$ on a tensor product of super modules is

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + (-1)^{|x||v|} v \otimes (x \cdot w) \tag{6.61}$$

for homogeneous x and v. Also the map $\tau\colon V \otimes W \to W \otimes V$ given by $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$ is an isomorphism. Since we will eventually restrict to a subcategory labeled by a particular value of $x \in \mathbb{CP}^1$, we only compute tensor products between modules with the same x label and will ignore the $x = \infty$ case. By tedious but not conceptually challenging computations, one sees

$$V^s_{n,e,x} \otimes V^t_{m,f,x} \cong \begin{cases} \bigoplus_{k=0}^{\min(s,t)-1} \left(V^{s+t-1-2k}_{n+m+\frac{1}{2},e+f,x} \oplus V^{s+t-1-2k}_{n+m-\frac{1}{2},e+f,x} \right) & e+f \neq 0 \\ \bigoplus_{k=0}^{\min(s,t)-1} P^{s+t-1-2k}_{n+m,x} & e+f = 0 \end{cases} \tag{6.62}$$

$$V^s_{n,e,x} \otimes V^t_{m,0,\pm,x} \cong \bigoplus_{k=0}^{\min(s,t)-1} \left(V^{s+t-1-2k}_{n+m+\frac{1}{2},e,x} \oplus V^{s+t-1-2k}_{n+m-\frac{1}{2},e,x} \right) \tag{6.63}$$

$$V^s_{n,0,\epsilon_1,x} \otimes V^t_{m,0,\epsilon_2,x} \cong \begin{cases} V^t_{n+m+\frac{1}{2},0,\epsilon_1,x} \oplus V^t_{n+m-\frac{1}{2},0,\epsilon_2,x} & \epsilon_1 = \epsilon_2,\ s = 1 \\ \bigoplus_{k=0}^{\min(s,t)-1} P^{s+t-1-2k}_{n+m,x} & \epsilon_1 = -\epsilon_2 \end{cases} \tag{6.64}$$

The tensor products where $\epsilon_1 = \epsilon_2$ with $s > 1$ can generate new indecomposables that we will not write down here. Furthermore:

$$P^s_{n,x} \otimes P^t_{m,x} \cong \bigoplus_{j=0}^{\min(s,t)-1} \left(P^{s+t-1-2j}_{m+n+1,x} \oplus 2\, P^{s+t-1-2j}_{m+n,x} \oplus P^{s+t-1-2j}_{m+n-1,x} \right) \tag{6.65}$$

$$P^s_{n,x} \otimes V^t_{m,e,x} \cong \bigoplus_{j=0}^{\min(s,t)-1} \left(V^{s+t-1-2j}_{m+n+1,e,x} \oplus 2\, V^{s+t-1-2j}_{m+n,e,x} \oplus V^{s+t-1-2j}_{m+n-1,e,x} \right) \tag{6.66}$$

6.2.4 The vertex superalgebra $\widehat{\mathfrak{gl}(1|1)}$

Now we turn to the definition of the affine Lie superalgebra $\widehat{\mathfrak{gl}(1|1)}$ associated to the bilinear form κ. It is defined as the super vector space $\mathfrak{gl}(1|1) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ where $\mathbb{C}[t, t^{-1}]$ and $\mathbf{k}$ are even, together with the following nontrivial Lie brackets:

$$[N_r, E_s] = r\mathbf{k}\delta_{r+s,0}, \quad [N_r, \psi_s^{\pm}] = \pm\psi_{r+s}^{\pm}, \quad \{\psi_r^+, \psi_s^-\} = E_{r+s} + r\mathbf{k}\delta_{r+s,0} \tag{6.67}$$

where a_r denotes $a \otimes t^r$.

Given a module M of $\mathfrak{gl}(1|1)$, one may obtain a module of $\widehat{\mathfrak{gl}(1|1)}$ as follows: one first view M as a module of $\mathfrak{gl}(1|1) \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k}$ such that $\mathfrak{gl}(1|1) \otimes t\mathbb{C}[t]$ acts trivially and $\mathbf{k}$ acts as a number $k \in \mathbb{C}$; one may then define the induced module:

$$\widehat{M}^k = \mathcal{U}(\widehat{\mathfrak{gl}(1|1)}) \otimes_{\mathcal{U}(\mathfrak{gl}(1|1) \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k})} M \tag{6.68}$$

as a representation of $\widehat{\mathfrak{gl}(1|1)}$. When M is the trivial module, $\widehat{M}^k$ has the structure of a vertex operator superalgebra (VOSA), which we denote by $V_k(\widehat{\mathfrak{gl}(1|1)})$. For general M, $\widehat{M}^k$ becomes a module of $V_k(\widehat{\mathfrak{gl}(1|1)})$. The assignment $M \to \widehat{M}^k$ defines a functor $\mathcal{I}nd$, which we will call the induction functor.

Remark. It turns out, as explained in [CMY20a], that for different choice of $k \neq 0$, the vertex algebras $V_k(\widehat{\mathfrak{gl}(1|1)})$ are isomorphic to each other. Thus we will once and for all fix $k = 1$, and drop k from all notations.

The VOSA $V(\widehat{\mathfrak{gl}(1|1)})$ has the following conformal element:

$$\omega = \frac{1}{2}(N_{-1}E_{-1} + E_{-1}N_{-1} - \psi_{-1}^+\psi_{-1}^- + \psi_{-1}^-\psi_{-1}^+) + \frac{1}{2}E_{-1}^2 \tag{6.69}$$

with the associated Virasoro zero mode:

$$L_0 = \sum_{r>0} \left(N_{-r}E_r + E_{-r}N_r - \psi^+_{-r}\psi^-_r + \psi^-_{-r}\psi^+_{-r} \right) + \sum_{r>0} E_{-r}E_r \\ + (N_0 + E_0/2)E_0 - \frac{1}{2}(\psi^+_0\psi^-_0 - \psi^-_0\psi^+_0) \tag{6.70}$$

It also enjoys spectral flow symmetries σ^l:

$$\sigma^l(N_r) = N_r, \quad \sigma^l(E_r) = E_r - l\delta_{r,0}, \quad \sigma(\psi^\pm_r) = \psi^\pm_{r\mp l} \tag{6.71}$$

as well as the conjugation w:

$$w(N_r) = -N_r, \quad w(E_r) = -E_r, \quad w(\psi^+_r) = \psi^-_r, \quad w(\psi^-_r) = -\psi^+_r. \tag{6.72}$$

These can be used to twist representations to obtain new representations.

6.2.5 The Kazhdan-Lusztig category

In this section, we recall the Kazhdan-Lusztig category KL of representations of $V(\widehat{\mathfrak{gl}(1|1)})$ that we are interested in. This category is characterized by certain weight constraints. A generalized $V(\widehat{\mathfrak{gl}(1|1)})$ module W is said to be *finite-length* if it has a finite composition series of irreducible $V(\widehat{\mathfrak{gl}(1|1)})$ modules. W is called *grading restricted* if it is graded by generalized conformal weights (the generalized eigenvalues of L_0) and the generalized conformal weights are bounded from below. For more details, see [CKM17].

Definition 6.2. *The Kazhdan-Lusztig category KL is defined as the supercategory of finite-length grading-restricted generalized $V(\widehat{\mathfrak{gl}(1|1)})$ modules.*

Any simple module of this category is generated by its lowest conformal weight space, which is a finite-dimensional representation of $\mathfrak{gl}(1|1)$, and thus any simple module is a quotient of $\widehat{V}_{n,e}$ for $e \neq 0$ or $\widehat{A}_n$. In what follows, when we write $\widehat{M}$, we always mean the image of a module M of $\mathfrak{gl}(1|1)$ under the induction functor $\mathcal{I}nd$. The following is shown in [CR13a]:

- $\widehat{V}_{n,e}$ is irreducible iff $e \notin \mathbb{Z}$.

- When $e = 0$, $\widehat{A}_n$ is irreducible, and there are non-split exact sequences:

$$\begin{gathered} 0 \longrightarrow \widehat{A}_{n-\frac{1}{2}} \longrightarrow \widehat{V}_{n,0,+} \longrightarrow \widehat{A}_{n+\frac{1}{2}} \longrightarrow 0 \\ 0 \longrightarrow \widehat{A}_{n+\frac{1}{2}} \longrightarrow \widehat{V}_{n,0,-} \longrightarrow \widehat{A}_{n-\frac{1}{2}} \longrightarrow 0 \end{gathered} \tag{6.73}$$

- When $e \in \mathbb{Z} \setminus \{0\}$, there is additional simple modules $\widehat{A}_{n,e}$. They fit in the following short exact sequences:

$$\begin{gathered} 0 \longrightarrow \widehat{A}_{n+1,e} \longrightarrow \widehat{V}_{n,e} \longrightarrow \widehat{A}_{n,e} \longrightarrow 0 \qquad (e > 0) \\ 0 \longrightarrow \widehat{A}_{n-1,e} \longrightarrow \widehat{V}_{n,e} \longrightarrow \widehat{A}_{n,e} \longrightarrow 0 \qquad (e < 0) \end{gathered} \tag{6.74}$$

Remark. To unify the notation, we will write $\widehat{A}_{n,0}$ for $\widehat{A}_n$. The modules $\widehat{A}_{n,e}$ for $e \in \mathbb{Z}$ are called simple currents.

Since E_0 is a central element, any representation of $V(\widehat{\mathfrak{gl}(1|1)})$ can be decomposed into direct sums according to the generalized eigenvalues of E_0, which is possible by finite-length property. We may thus write

$$KL = \bigoplus_{e \in \mathbb{C}} KL_e \tag{6.75}$$

where KL_e is the subcategory such that the generalized eigenvalue of E_0 is e. From the description of the above simple modules, it is clear that KL_e for $e \notin \mathbb{Z}$ is generated by $\widehat{V}_{n,e}$ and for $m \in \mathbb{Z}$, KL_m is generated by $\widehat{A}_{n,m}$. There is a similar decomposition of $\mathcal{C}$, the category of finite-dimensional representations of $\mathfrak{gl}(1|1)$, so we write $\mathcal{C}_e$ to be the subcategory of $\mathcal{C}$ where the action of E has generalized weight e. Clearly, induction is a functor from $\mathcal{C}_e$ to KL_e. In fact, more is true about this induction:

Proposition 6.1. *When $e \notin \mathbb{Z}$ or $e = 0$, induction functor $\mathcal{I}nd$ gives an equivalence of abelian supercategories:*

$$\mathcal{C}_e \cong KL_e \tag{6.76}$$

Proof. We only consider the case $e \notin \mathbb{Z}$, since the proof for $e = 0$ is almost identical. By definition, any simple module in KL_e comes from induction. We now show that induction is essentially surjective. For this, choose $M \in KL_e$, we will use induction on the length of M. Choose a maximal sub-module $N \subseteq M$. Then by assumption N is induced from $\mathcal{C}_e$ and $M/N \cong \widehat{V}_{n,e}$ for some n. We claim that we may choose generators of M that are annihilated by all the positive modes of $\widehat{\mathfrak{gl}(1|1)}$. We may choose such generators for N and M/N, say $n_1, \dots, n_k$ and $\overline{m}$, that are generates the lowest-weight spaces for the respective module. Choosing a pre-image m of $\overline{m}$, we will adjust m step by step so that it is annihilated by all positive modes.

First, suppose there exists $t > 0$ such that $E_t m \neq 0$. By grading restriction, $E_t^l m = 0$ for some $l > 1$. Using $[E_t^l, N_{-t}] = (tl)E_t^{l-1}$, we see that $E_t^l N_{-t} m = (tl)E_t^{l-1} m$. Let $m' = m - \frac{1}{tl} E_t N_{-t} m$, then $E_t^{l-1} m' = 0$, and $\overline{m'} = \overline{m} - \frac{1}{l}\overline{m} \neq 0$ since $l > 1$. This is again a generator. Using this procedure, we may adjust m such that $E_t m = 0$ for all $t > 0$.

Suppose $\psi_t^+ m \neq 0$ for some $t > 0$, and assume $(\psi_t^+)^l m = 0$ for some $l > 1$. Since $\{(\psi_t^+)^l, \psi_{-t}^-\} = l(E_0 + t)(\psi_t^+)^{l-1}$, we have $(\psi_t^+)^l \psi_{-t}^- m = l(\psi_t^+)^{l-1}(E_0 + t)m$. Let $m' = l(E_0 + t)m - \psi_t^+ \psi_{-t}^- m$, then $(\psi_t^+)^{l-1} m' = 0$, and $\overline{m'} = (l-1)(e+t)\overline{m}$ which is still a generator since $e + t \neq 0$ and $l - 1 \neq 0$. If the starting m satisfies $\psi_j^\pm m = E_k m = 0$ for $j > t$ and $k > 0$, then after the adjustment, $\psi_j^\pm m' = E_k m' = 0$ for $j \geq t$ and $k > 0$. We may then adjust m downward from the largest t such that $\psi_t^\pm m \neq 0$, and obtain m that is annihilated by all $\psi_t^\pm, E_t$ for $t > 0$.

Finally we perform a similar procedure for N_t to obtain the final m. We may also assume that $N_0 \overline{m} = (n - 1/2)\overline{m}$ for some $n \in \mathbb{C}$, which means that $(N_0 - n + 1/2)m \in N$. This is an element annihilated by all positive modes of $\widehat{\mathfrak{gl}(1|1)}$, which then must be in the lowest conformal weight space of N.

Now consider the $\mathfrak{gl}(1|1)$ module V generated by m and the n_i. By the above consideration, this module is finite dimensional. The map $V \to M$ then induces a surjection $\widehat{V} \to M$ as $V(\widehat{\mathfrak{gl}(1|1)})$ modules. We claim that the kernel K must be of the form $\widehat{W}$ for some $W \subseteq V$. To prove the claim, we use an inductive argument on the number of composition factors of $\widehat{V}$. Choose a minimal sub-module $V_{n',e}$ of V and let U be the quotient $V/V_{n',e}$. We have a short exact sequence:

$$0 \longrightarrow \widehat{V}_{n',e} \longrightarrow \widehat{V} \longrightarrow \widehat{U} \longrightarrow 0 \tag{6.77}$$

If $K \cap \widehat{V}_{n',e} = 0$, then we are done by inductive hypothesis. Otherwise, K fits in the exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{V}_{n',e} & \longrightarrow & \widehat{V} & \longrightarrow & \widehat{U} & \longrightarrow & 0 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \widehat{V}_{n',e} & \longrightarrow & K & \longrightarrow & K/\widehat{V}_{n',e} & \longrightarrow & 0 \end{array} \tag{6.78}$$

By induction, $K/\widehat{V}_{n',e}$ is equal to $\widehat{L}$ for some $L \subseteq U$. Then by counting composition factors, it is clear that $K \cong \widehat{W}$ where W is the pre-image of L under the map $V \to U$. Thus M is of the form $\widehat{V/W}$ as desired. □

Because of this, the difficulty in the study of KL lies in understanding KL_n for $n \in \mathbb{Z} \setminus \{0\}$. We will see in the next section, that fusion product with simple currents gives a way to study them. The objects in KL_e for $e \in \mathbb{Z}$ are called atypical modules, while those in KL_e for $e \notin \mathbb{Z}$ are called typical modules(c.f. Section 6.1.2).

6.2.6 Fusion structure

By the work of [HLZ10-11], $P(z)$-intertwining operators define a monoidal structure on KL. In this section, we will compute the fusion products of modules by relating the fusion product to the tensor product of $\mathfrak{gl}(1|1)$ modules.

To start, we have the following statement, which is proved in [CMY20a]:

Theorem 6.4. *KL is a rigid braided tensor supercategory; moreover, it is a ribbon category with even natural twist $\theta = e^{2\pi i L_0}$.*

The fusion product (if it exists) of two generalized modules W_1 and W_2 of a VOSA V will be denoted by $W_1 \times_V W_2$, or simply $W_1 \times W_2$ if it is clear what the VOSA is. It turns out that one can understand the fusion structure of KL using the tensor structure of $\mathfrak{gl}(1|1)$ modules. Let M_1, M_2 and M_3 be finite-dimensional $\mathfrak{gl}(1|1)$ modules. Given an intertwiner operator $\mathcal{Y}$ of type $\binom{\widehat{M}_3}{\widehat{M}_1\,\widehat{M}_2}$, consider the assignment $\pi(\mathcal{Y})\colon M_1 \otimes M_2 \to M_3$ given by:

$$\pi(\mathcal{Y})(m_1 \otimes m_2) = \pi_0(\mathcal{Y}(m_1, 1)m_2) \tag{6.79}$$

where π_0 denotes projection onto the lowest conformal weight space. This $\pi(\mathcal{Y})$ is in fact a homomorphism of $\mathfrak{gl}(1|1)$ modules. The inverse of this is established in [CMY20a]. To state it, let's recall the definition of taking contragredient dual. For a full definition, see [CKM17]. Let M be a finite-dimensional module of $\mathfrak{gl}(1|1)$, then the linear dual $M^* = \mathrm{Hom}(M, \mathbb{C})$ has the structure of a $\mathfrak{gl}(1|1)$ module, induced by the action of $\mathfrak{gl}(1|1)$ on M. There is a similar operation on modules of $\widehat{\mathfrak{gl}(1|1)}$. Given a grading-restricted generalized $\widehat{\mathfrak{gl}(1|1)}$ module W. Write the decomposition of W into generalized conformal weight spaces as:

$$W = \bigoplus_{h \in \mathbb{C}} W_{[h]}. \tag{6.80}$$

Define W' to be the super vector space:

$$W' = \bigoplus_{h \in \mathbb{C}} W_{[h]}^*, \tag{6.81}$$

together with the action of $V(\widehat{\mathfrak{gl}(1|1)})$ by:

$$\langle Y_{W'}(v, z)w', w\rangle = \langle w', Y_W(e^{zL_1}(-z^2)^{L_0}v, z^{-1})w\rangle. \tag{6.82}$$

Here $\langle -,- \rangle$ is the natural pairing between W and W', and the above is well-defined by the grading restriction condition. The module W' is called the contragredient dual of W. The following is proved in [CMY20a]:

Proposition 6.2. *Let M_1, M_2 and M_3 be finite-dimensional $\mathfrak{gl}(1|1)$ modules, and let $f\colon M_1 \otimes M_2 \to M_3^*$ be a homomorphism of $\mathfrak{gl}(1|1)$ modules. Then there exists a unique intertwiner operator $\mathcal{Y}$ of type $\binom{\widehat{M_3'}}{\widehat{M_1}\,\widehat{M_2}}$ such that $\pi(\mathcal{Y}) = f$.*

This Proposition, together with rigidity and Proposition 6.1, can help us compute fusion product explicitly for any pair modules in KL. Let $S_b := \mathbb{Z} \setminus \{0\}$ and $S_g := \mathbb{C} \setminus S_b$.

Lemma 6.2. *$\widehat{M_e^*} \cong \widehat{M_e'}$ for any $e \in S_g$.*

Proof. By definition, the lowest conformal weight space of $\widehat{M_e'}$ is M_e^*, thus the identity map $M_e^* \to M_e^*$ induces a map of VOSA modules $\widehat{M_e^*} \to \widehat{M_e'}$. We claim that this map is injective. Suppose otherwise, then its kernel would contain a minimal module $\widehat{V}_{n-1/2,e}$ for some n in the case $e \neq 0$ or $\widehat{A}_{n,0}$ for the case $e = 0$. In any case we denote such minimal module by $\widehat{V}$. Since $\mathrm{Hom}(V, M_e^*) \cong \mathrm{Hom}(\widehat{V}, \widehat{M_e^*})$, the embedding $\widehat{V} \to \widehat{M_e^*}$ comes from the induces map of a injection $V \to M_e^*$, and the map $\widehat{V} \to \widehat{M_e'}$ then comes from the map $\widehat{V} \to \widehat{V^*}' \to \widehat{M_e'}$. The first map is an isomorphism by [CMY20a], and the second is nonzero and thus is an embedding, a contradiction. Since this map is injective, by counting the number of composition factors, this is also surjective, so it is an isomorphism. $\square$

As a corollary, we obtain the following:

Corollary 6.1. *In the case when $e, e' \in S_g$ and $e + e' \in S_g$, $\widehat{M_e \otimes M_{e'}} \cong \widehat{M_e} \times \widehat{M_{e'}}$.*

Proof. By Proposition 6.2 and Lemma 6.2, one has a map $\widehat{M_e} \times \widehat{M_{e'}} \to \widehat{M_e \otimes M_{e'}}$. This is surjective since $\widehat{M_e \otimes M_{e'}}$ is generated by $M_e \otimes M_{e'}$. Indeed, by Proposition 6.1, if the image of this intertwiner was contained in some sub-module, it would be a sub-module of the form $\widehat{W}$ for some $W \subseteq M_e \otimes M_{e'}$. This is contradicting the fact that $\pi(\mathcal{Y})$ is an isomorphism onto $M_e \otimes M_{e'}$. To conclude the proof, we only need to compute the number of composition factors on both sides. Rigidity of KL implies that fusion is exact, and so the number of composition series can be computed using fusion rules of irreducible modules as in [CMY20a, Theorem 3.2.4]. Comparing this with the composition series of $M_e \otimes M_{e'}$, we conclude that the map is an isomorphism. □

We can use Corollary 6.1 to also deal with fusion rules of M_e for $e \in S_b$. Indeed, since fusion product with simple currents preserves socle series [CR13b], the functor:

$$\widehat{A}_{n,l} \times -\colon KL_e \to KL_{e+l} \tag{6.83}$$

is an equivalence, with inverse given by $\widehat{A}_{-n,-l}$. In particular, for any $V \in KL_e$ where $e \in S_b$, we can "shift" V to be in KL_0 using simple currents

$$V = \widehat{A}_{n,e} \times (\widehat{A}_{-n,-e} \times V). \tag{6.84}$$

Suppose we want to compute $V \times W$ for $V \in KL_e$ and $W \in KL_f$. If either e, f or $e+f \in S_b$, then we can use simple currents to pull the computation into KL_0, so that we can apply Corollary 6.1. The question now becomes identifying $\widehat{A}_{n,e} \times V$ for various V and $e \in S_b$. It suffices to take V to be indecomposable. We introduce a collection of indecomposable modules that will be relevant for connecting to $\mathcal{C}_{\beta\gamma}$ (they either come from induction, or a spectral flow of a module from induction). Note that this is not a complete list of indecomposables in KL.

- $\widehat{A}_{n,l}$ for $l \in \mathbb{Z}$. When $l < 0$, $\widehat{A}_{n,l} \cong \sigma^l(\widehat{A}_{n+l+\frac{1}{2},0})$. When $l > 0$, $\widehat{A}_{n,l} \cong \sigma^l \circ w(\widehat{A}_{-n-l+\frac{1}{2},0}) \cong \sigma^l(\widehat{A}_{n+l-\frac{1}{2},0})$.
- $\widehat{V}^t_{n,0,\pm,x}$ for $x \in \mathbb{C}$ and $t \in \mathbb{N}$ as well as $\widehat{V}^t_{n,e,x}$ for $e \notin \mathbb{Z}$. These are the induced module from $V^t_{n,0,\pm,x}$ and $V^t_{n,e,x}$ respectively. For $l \in \mathbb{Z}$, define $\widehat{V}^t_{n,l,\pm,x}$ to be $\sigma^l(\widehat{V}^t_{n+l,0,\pm,x})$. One sees then that when $l < 0$, $\widehat{V}^t_{n,l,-,x} = \widehat{V}^t_{n,l,x}$, and when $l > 0$, $\widehat{V}^t_{n,l,+,x} = \widehat{V}^t_{n,l,x}$.

- $\widehat{P}^t_{n,x}$, which is induced from $P^t_{n,x}$. Define $\widehat{P}^t_{n-l-\frac{1}{2},l,x}$ for $l<0$ to be $\sigma^l(\widehat{P}^t_{n,x})$ and for $l>0$, define $\widehat{P}^t_{-n-l+\frac{1}{2},l,x}$ to be $\sigma^l \circ w(\widehat{P}^t_{n,x})$. Note that all these can be obtained by fusion product with simple currents.

We will also introduce a function $\epsilon(l)$ on $\mathbb{Z}$ given by:

$$\epsilon(l) = \begin{cases} -\frac{1}{2} & \text{if } l<0, \\ 0 & \text{if } l=0, \\ \frac{1}{2} & \text{if } l>0. \end{cases} \tag{6.85}$$

Define $\epsilon(l,l') = \epsilon(l) + \epsilon(l') - \epsilon(l+l')$.

Lemma 6.3. *Let $l' \in S_g$ and $l \in \mathbb{Z}$. One has the following fusion rule with simple currents:*

$$\widehat{A}_{n,l} \times \widehat{V}^t_{n',l',x} = \widehat{V}^t_{n'+n-\epsilon(l),l'+l,x} \tag{6.86}$$

Proof. One has a homomorphism of $V(\widehat{\mathfrak{gl}(1|1)})$ modules:

$$\mathcal{Y}\colon \widehat{V}_{n,l} \times \widehat{V}^t_{n',l',x} \to \widehat{V_{n,l} \otimes V^t_{n',l',x}}, \tag{6.87}$$

such that $\pi(\mathcal{Y})$ is an isomorphism. Since $l+l' \in S_g$, submodules of $\widehat{V_{n,l} \otimes V^t_{n',l',x}}$ are all induced from submodules of $V_{n,l} \otimes V^t_{n',l',x}$. By construction of $\mathcal{Y}$, if it were not surjective, then $\pi(\mathcal{Y})$ wouldn't be surjective. Thus the fusion rule is surjective. Now because $\widehat{V}_{n,l}$ is an extension of simple currents and fusion is right exact, the number of irreducible composition factors must not exceed two times the number of irreducible composition factors of $\widehat{V}^t_{n',l',x}$, and so by counting such factors one sees that this is an isomorphism.

Precomposing this with the embedding $\widehat{A}_{n+2\epsilon(l),l} \times \widehat{V}^t_{n',l',x} \to \widehat{V}_{n,l} \times \widehat{V}^t_{n',l',x}$ and using that fusion with simple currents $\widehat{A}_{n\pm 1,l}$ preserve socle series, one gets the desired image. □

Lemma 6.4. *Let $l \in \mathbb{Z}$. Then one has:*

$$\widehat{A}_{n,l} \times \widehat{V}^t_{n',-l,x} = \begin{cases} \widehat{V}^t_{n+n'-\frac{1}{2},0,-} & \text{if } l \geq 0, \\ \\ \widehat{V}^t_{n+n+\frac{1}{2},0,+} & \text{if } l < 0. \end{cases} \tag{6.88}$$

Proof. One has the following intertwiner from the isomorphism $V_{n,l} \otimes V^t_{n',-l,x} \to V_{n,l} \otimes V^t_{n',-l,x}$:

$$\mathcal{Y} \colon \widehat{V}_{n,l} \times \widehat{V}^t_{n',-l,x} \to V_{n,l}\widehat{\otimes V^t_{n',-l,x}}. \tag{6.89}$$

Since now $V_{n,l} \otimes V^t_{n',-l,x} \cong P^t_{n+n',0}$ belongs to S_g, one can again show that this $\mathcal{Y}$ must be an isomorphism.

When $l \geq 0$, the composition factors of $\widehat{A}_{n,l} \times \widehat{V}^t_{n',-l,x,-}$ are $\widehat{A}_{n+n'-1,0}$ and $\widehat{A}_{n+n',0}$ and has a unique minimal submodule $\widehat{A}_{n+n'-1,0}$. The only quotient of $\widehat{P}^t_{n+n',0}$ having this property and the right number of irreducible factors is $\widehat{V}^t_{n+n'-\frac{1}{2},0,-}$.

When $l < 0$ the composition factors of $\widehat{A}_{n,l} \times \widehat{V}^t_{n',-l,x,-}$ are $\widehat{A}_{n+n'+1,0}$ and $\widehat{A}_{n+n',0}$ and has a unique minimal submodule $\widehat{A}_{n+n'+1,0}$. The only quotient of $\widehat{P}^t_{n+n',0}$ having this property and the right number of irreducible factors is $\widehat{V}^t_{n+n'+\frac{1}{2},0,+}$. This completes the proof. ∎

Using this, we obtain the following fusion rules:

Corollary 6.2. *For $l, l' \in \mathbb{Z}$ and $m \geq m'$, one has the following fusion rule:*

$$\begin{aligned} \widehat{P}^m_{n,l,x} \times \widehat{P}^{m'}_{n',l',x} = & \bigoplus_{0 \leq t \leq m'-1} \widehat{P}^{m+m'-1-2t}_{n+n'+1-\epsilon(l,l'),l+l',x} \\ & \bigoplus_{0 \leq t \leq m'-1} 2\, \widehat{P}^{m+m'-1-2t}_{n+n'-\epsilon(l,l'),l+l',x} \\ & \bigoplus_{0 \leq t \leq m'-1} \widehat{P}^{m+m'-1-2t}_{n+n'-1-\epsilon(l,l'),l+l',x} \end{aligned} \tag{6.90}$$

For $l > 0$, one has:

$$\widehat{V}^{m}_{n,l,x} \times \widehat{V}^{m'}_{n',-l,x} = \bigoplus_{0\leq k\leq m'-1} \widehat{P}^{m+m'-1-2k}_{n+n',0,x} \tag{6.91}$$

Proof. When $l = 0$, these follow from the tensor product structure of $\mathfrak{gl}(1|1)$. For $l \neq 0$, we use the fact that $\widehat{P}^{m}_{n,l,x} = \widehat{A}_{n,l} \times \widehat{P}^{m}_{0,0,x}$. □

6.3 The mirror symmetry statement

In this section, we prove Theorem 6.1. In Section 6.3.1 we introduce the category KL^0, the de-equivariantization $KL^0/\mathbb{Z}$ as well as the lifting functor $\mathcal{F}$ from KL^0 to the category of modules of the VOSA $V_{\beta\gamma} \otimes V_{bc}$. The main ingredient here is a simple current extension described in [CR13b], as well as the machinery of [CKM17, CMY20b]. In Section 6.3.2, we recall free-field realizations of the VOSAs of our interest, following [CGN21, AW20], and show that they are compatible with the simple current extension; we use them to show that certain objects in KL do lie in KL^0, and so can be lifted by $\mathcal{F}$. As a consequence, we show the first part of Theorem 6.1, namely that $\mathcal{C}_{\beta\gamma}$ has the structure of a braided tensor category defined by $P(z)$-intertwiners. In Section 6.3.4, we use our knowledge of $\mathcal{C}_{\beta\gamma}$ and KL^0 established in Section 6.1 and Section 6.2 to complete the proof of Theorem 6.1. In Section 6.3.5, we compute fusion rules of indecomposable objects in $\mathcal{C}_{\beta\gamma}$ using the tensor equivalence $\mathcal{F}$.

6.3.1 $\beta\gamma$ as a simple current extension

As we have discussed earlier, due to the contribution of monopole operators, the physical boundary VOSA for T_B is a simple current extension of $V(\widehat{\mathfrak{gl}(1|1)})$. In [CR13b], the authors showed that the VOSA $V(\widehat{\mathfrak{gl}(1|1)})$ has many simple current extensions, which can often be identified as known VOSAs. A simple current extension is a direct sum of simple currents of $V(\widehat{\mathfrak{gl}(1|1)})$ such that the resulting module carries a VOSA structure that extends the VOSA structure of $V(\widehat{\mathfrak{gl}(1|1)})$. Among all the simple current extensions, the following is the one we consider:

$$\widehat{A}_{0,0} \oplus \left(\bigoplus_{m>0} \widehat{A}_{\frac{1-m}{2},m} \right) \oplus \widehat{A}_{\frac{m-1}{2},-m}. \tag{6.92}$$

The choice is made by comparing the indices. The index of this module is computed in [CR13b] to be

$$\sum_{m\in\mathbb{Z}} q^{m^2/2} s^{m/2} \prod_{i=1}^{\infty} \frac{(1+sq^{i+m})(1+s^{-1}q^{i-m})}{(1-q^i)^2}. \tag{6.93}$$

Here the power of s records the weights under the gauge group generator N_0 and the power of q records the conformal weights. This correctly reproduces the index computed in [DGP18, Equation 3.31].[7]

The module in equation (6.92) has the structure of a vertex operator superalgebra, and this VOSA is isomorphic to $V_{\beta\gamma} \otimes V_{bc}$, the tensor product of the $\beta\gamma$ VOA with a pair of free complex fermions. Let us denote by $\mathcal{V}_{ext}$ this extended VOSA.

The following is explained in [CKM17]: given a vertex operator superalgebra extension A in a VOA module category $\mathcal{C}$ where $P(z)$-intertwiners define a symmetric monoidal structure on $\mathcal{C}$, the category of local modules of A in $\mathcal{C}$ coincides with the category of generalized modules of the VOA A as braided tensor supercategories. However, in our situation (equation (6.92)) as well as in many other cases, the object A does not live in $\mathcal{C}$ but in a suitable completion of $\mathcal{C}$. Thus one needs to take a completion of $\mathcal{C}$ to allow infinite direct sums. This is explained in [CMY20b, Theorem 1.1]: under suitable circumstances, one can extend the symmetric monoidal structure from $\mathcal{C}$ to a completion called $\mathrm{Ind}(\mathcal{C})$, such that the object A is now contained in $\mathrm{Ind}(\mathcal{C})$. The authors then showed [CMY20b, Theorem 1.4] that the category of generalized A-modules in $\mathcal{C}$ also has a braided tensor supercategory structure defined via $P(z)$-intertwiners (see Section 6.1.3).

We apply this method to KL and $\mathcal{C}_{\beta\gamma}$. Denote now by $\mathrm{Ind}(KL)$ the completion of KL in the sense of [CMY20b], then $\mathcal{V}_{ext}$ is a VOSA object in $\mathrm{Ind}(KL)$. Denote by $\mathrm{Rep}^0(\mathcal{V}_{ext})$

[7]Note that the boundary condition has effective level $k_{\text{eff}} = 1$ since each hypermultiplet contributes to $\frac{1}{2}$ of the level.

the category of generalized modules of $\mathcal{V}_{ext}$ that lie in $\text{Ind}(KL)$. This is a braided tensor supercategory via the $P(z)$-intertwining maps, as was shown in [CMY20b, Theorem 1.4].

Given such an extension, for any object W inside $\text{Ind}(KL)$, the product $\mathcal{V}_{ext} \times W$ has the action by the mode algebra of the extended VOSA $\mathcal{V}_{ext}$. However, this action is not local in general. In [CKL20], the author explained that the monodromy determines whether the resulting action is local, thus becoming a generalized module of the VOSA $\mathcal{V}_{ext}$. More precisely, monodromy is defined by a composition of braiding isomorphisms:

$$M : \mathcal{V}_{ext} \times W \longrightarrow W \times \mathcal{V}_{ext} \longrightarrow \mathcal{V}_{ext} \times W. \tag{6.94}$$

The module $\mathcal{V}_{ext} \times W$ is local if and only if the map M is the identity morphism, or in other words, the monodromy acts trivially. Let us denote by KL^0 the full tensor subcategory of KL consisting of W such that monodromy on $\mathcal{V}_{ext} \times W$ is trivial. One has a functor:

$$\mathcal{F} \colon KL^0 \to \text{Rep}^0(\mathcal{V}_{ext}) \tag{6.95}$$

that maps an object W to $\mathcal{V}_{ext} \times W$ as an object in $\text{Ind}(KL)$ together with the natural structure of a $\mathcal{V}_{ext}$ module. The result of [CKL20] immediately implies:

Theorem 6.5. *The functor*

$$\mathcal{F} \colon KL^0 \to \text{Rep}^0(\mathcal{V}_{ext}) \tag{6.96}$$

is a tensor functor between braided tensor supercategories.

The simple currents $\widehat{A}_{0,\pm 1}$ generate a tensor subcategory of KL^0 isomorphic to $\text{Rep}(\mathbb{C}^*)$. This tensor subcategory lives in the center of KL^0. We denote by $KL^0/\mathbb{Z}$ the de-equivariantization of KL^0 by $\text{Rep}(\mathbb{C}^*)$ in the sense of [EGNO15, Theorem 8.23.3]. The functor $\mathcal{F}$ identifies the image of KL^0 with the de-equivariantization $KL^0/\mathbb{Z}$. The notation comes from the isomorphism $\text{Rep}(\mathbb{C}^*) \cong \text{Coh}(\mathbb{Z})$, and the action of $\text{Rep}(\mathbb{C}^*)$ on KL^0 can be understood as an action of $\mathbb{Z}$ on the objects of KL^0. The de-equivariantization can be understood as the quotient category, where objects that are related by $\mathbb{Z}$ are considered equivalent. There is an action of $\mathrm{Coh}(\mathbb{Z}_2)$ on $KL^0/\mathbb{Z}$ generated by parity shift, and the de-equivariantization $KL^0/(\mathbb{Z}\times\mathbb{Z}_2)$ is the category of line operators for T_B discussed in the introduction.[8]

The category $\mathrm{Rep}^0(\mathcal{V}_{ext})$ may seem abstract, or at least not immediately related to $\mathcal{C}_{\beta\gamma}$. However, since $\mathcal{V}_{ext}\cong V_{\beta\gamma}\otimes V_{bc}$, and the category of generalized modules for V_{bc} is equivalent to the category of super vector spaces, this category $\mathrm{Rep}^0(\mathcal{V}_{ext})$ is the Deligne product of a category of modules of the VOA $V_{\beta\gamma}$ with SVect, the category of super vector spaces. What we show in the following sections is that the image of $\mathcal{F}$ is identified with $\mathcal{C}_{\beta\gamma}\boxtimes\mathrm{SVect}$. The difficulty of analyzing this functor lies in the fact that computation of monodromy is complicated; it lacks concrete algebraic expressions. One way to go around this difficulty is through free-field realizations. This method is based on the observation that lifting modules from one free-field VOA to another is much easier to deal with. We now go on to introduce free-field realizations of $V(\widehat{\mathfrak{gl}(1|1)})$ as well as of $\mathcal{V}_{ext}$ that are compatible with the embedding $V(\widehat{\mathfrak{gl}(1|1)})\to\mathcal{V}_{ext}$. In what follows, when we compute fusion product of objects that are infinite direct sums, we always mean the fusion product in the sense of [CMY20b].

6.3.2 A free field realization of $V(\widehat{\mathfrak{gl}(1|1)})$

We start with describing the free-field realization of $V(\widehat{\mathfrak{gl}(1|1)})$ given in [CGN21]. Let X,Y,Z be a triple of free bosons with the non-degenerate pairing $(X,Y)=(Z,Z)=1$. The Heisenberg VOA generated by this triple is denote by $\mathbb{F}_0$. For any linear combination of X,Y,Z, say $\mu=aX+bY+cZ$, there is a simple module, denoted by $\mathbb{F}_\mu$, generated by the vacuum vector $|\mu\rangle$ whose weights under $\partial X,\partial Y$ and ∂Z are given by the non-degenerate pairing between μ and X,Y,Z. The module $\bigoplus_{n\in\mathbb{Z}}\mathbb{F}_{nZ}$ has the structure of a VOSA, generated

[8] We note that de-equivariantization by $\mathbb{Z}_2$ does not change much of the category, apart from identifying an object with its parity shift, and forgetting the parity of Hom spaces. In particular, no new homomorphism is introduced since the parity is generated by an internal symmetry of $V(\widehat{\mathfrak{gl}(1|1)})$.

by $\mathbb{F}_0$ together with the vertex operators $:e^{\pm Z(z)}:$, defined by:

$$:e^{\pm Z(z)}:= e^{\pm Z} z^{\pm Z_0} \prod_{m\geq 1} \exp\left(\pm \frac{Z_{-m}}{m} z^m\right) \exp\left(\pm \frac{-Z_m}{m} z^{-m}\right). \tag{6.97}$$

See the definition in [FB04] Section 5.2. We treat the modes of $:e^{\pm Z(z)}:$ as odd. This VOSA is denoted by V_Z. Through the Bose-Fermi correspondence (see [FB04], Section 5.3), the sub-algebra generated by $:e^{\pm Z(z)}:$ is isomorphic to the bc ghost VOSA V_{bc}, generated by two fermionic (odd degree) fields b, c with OPE:

$$b(z)c(z) \sim \frac{1}{z-w}. \tag{6.98}$$

Under this correspondence, $:e^{Z(z)}:\mapsto b(z)$ and $:e^{-Z(z)}:\mapsto c(z)$.

There is an embedding of VOSAs $V(\widehat{\mathfrak{gl}(1|1)}) \to V_Z$ given by

$$\begin{aligned} &E(z) \mapsto \partial Y(z), \quad N(z) \mapsto -:c(z)b(z): +\partial X(z) - \frac{\partial Y(z)}{2}, \\ &\psi^+(z) \mapsto b(z), \quad \psi^-(z) \mapsto c(z)\partial Y(z) + \partial c(z) \end{aligned} \tag{6.99}$$

For each linear combination of X and Y, say $\nu = aX + bY$, one obtain a simple module of V_Z whose underlying object is:

$$V_{\nu,Z} := V_Z \times_{\mathbb{F}_0} \mathbb{F}_\nu. \tag{6.100}$$

On the other hand, if ν involves cZ for $c \notin \mathbb{Z}$, the resulting module is not local with respect to $:e^Z:$. This is a special example of Theorem 6.5. We now introduce the screening operator, which helps us understand the embedding $V(\widehat{\mathfrak{gl}(1|1)}) \to V_Z$:

Definition 6.3. *Let us define an intertwiner $S(z)\colon V_{nY,Z} \to V_{(n-1)Y,Z}$ by*

$$S(z) =: e^{Z(z)-Y(z)}: \tag{6.101}$$

The screening operator is then defined as the residue:

$$S = \oint S(z)\mathrm{d}z. \tag{6.102}$$

It is shown in [CGN21] that $V(\widehat{\mathfrak{gl}(1|1)})$ is the kernel of $S\colon V_{0,Z} \to V_{-Y,Z}$. The following Proposition shows how we can identify modules of $V(\widehat{\mathfrak{gl}(1|1)})$ as restrictions of modules of V_Z:

Proposition 6.3. *Let $\mu = aX + bY$. When $a \in \mathbb{Z}$, $V_{\mu,Z} \cong \widehat{V}_{(2b-a-1)/2,a,-}$ as $V(\widehat{\mathfrak{gl}(1|1)})$ modules. When $a \notin \mathbb{Z}$, $V_{\mu,Z} \cong \widehat{V}_{(2b-a-1)/2,a}$ as $V(\widehat{\mathfrak{gl}(1|1)})$ modules.*

Proof. The case when $a \in S_g$ is proven in [CGN21]. For $a \in S_b$, we compute the conformal weight of the kernel of the screening operator using [CMY21]. To do so, we need to re-write the free-field realization using the following vectors: $\alpha = Y - Z, \beta = Z + X$ and $\gamma = Z + X - Y$. These are three orthogonal generators of the lattice. The screening operator corresponds to $\alpha_- = -\alpha$, and $\alpha_+ = 2\alpha$. In terms of these generators, the conformal weight vector is:

$$\frac{1}{2}(\alpha(-1)^2 + \beta(-1)^2 - \gamma(-1)^2) + \frac{\alpha(-2)}{2}, \tag{6.103}$$

and X, Y, Z can be written as:

$$X = \gamma + \alpha_+/2, \quad Y = \beta - \gamma, \quad Z = \beta - \gamma - \alpha_+/2. \tag{6.104}$$

So for a general $\mu = aX + bY$, the module $V_{\mu,Z}$ can be written in terms of the generators α, β and γ as:

$$\bigoplus_m \mathbb{F}_0 \left| \frac{a-m}{2}\alpha_+ + (b+m)\beta + (a-b-m)\gamma \right\rangle. \tag{6.105}$$

For each m, the lowest conformal weight of the kernel of the screening operator restricted to:

$$\mathbb{F}_0 \left| \frac{a-m}{2}\alpha_+ + (b+m)\beta + (a-b-m)\gamma \right\rangle \tag{6.106}$$

is given by $h = h_{\alpha_{1+m-a,1}} + h_{(b+m)\beta} - h_{(a-b-m)\gamma}$, where:

$$\begin{aligned} h_{(b+m)\beta} &= \frac{(b+m)^2}{2} \\ h_{(a-b-m)\gamma} &= \frac{(a-b-m)^2}{2} \end{aligned} \tag{6.107}$$

and as in [CMY21]:

$$h_{\alpha_{1+m-a,1}} = \begin{cases} \frac{(1+m-a)^2-(1+m-a)}{2} & \text{if } a \leq m, \\ \frac{(1+a-m)^2-(1+a-m)}{2} & \text{if } a > m. \end{cases} \tag{6.108}$$

Combining these we have an explicit formula for h:

$$h = \begin{cases} \frac{1}{2}(m^2 + m + 2ab - a) & \text{if } a \leq m, \\ \frac{1}{2}(m^2 - m + 2ab + a) & \text{if } a > m. \end{cases} \tag{6.109}$$

Now suppose $a < 0$, then if $m < a$, m cannot be $1/2$, so the minimum of h in this region is obtained at $m = a - 1$, giving $\frac{1}{2}(a^2 + 2ab - 2a + 2)$; when $m \geq a$, then the minimum can be taken at $m = 0$, which is $\frac{1}{2}(2ab - a)$. Since $a < 0$, $\frac{1}{2}(2ab - a)$ is smaller. This is exactly the minimum conformal weight of $\widehat{A}_{\frac{2b-a-1}{2},a}$. When $a > 0$, one can show that the minimum conformal weight is $\frac{1}{2}(2ab + a)$ which is the minimum conformal weight of $\widehat{A}_{\frac{2b-a+1}{2},a}$. We thus have embedding $\widehat{A}_{\frac{2b-a-1}{2},a} \subset V_{\mu,Z}$ when $a < 0$ and $\widehat{A}_{\frac{2b-a+1}{2},a} \subset V_{\mu,Z}$ when $a > 0$.

To finish the proof, we utilize the free-field fusion rule $\mathbb{F}_0 \times_{\mathbb{F}_0} \mathbb{F}_\mu \cong \mathbb{F}_\mu$, which after combining the embedding $\widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a} \to V_{\mu,Z}$ gives a $V(\widehat{\mathfrak{gl}(1|1)})$ intertwiner:

$$V_Z \times \widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a} \to V_{\mu,Z}. \tag{6.110}$$

Since V_Z is indecomposable as a $V(\widehat{\mathfrak{gl}(1|1)})$ module and $\widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a}$ is a simple current, by Proposition 2.5 of [CKLR19], the fusion $V_Z \times \widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a}$ is still indecomposable, and we only need to show that it maps isomorphically onto $V_{\mu,Z}$. If we restrict this intertwiner to the sub-module $\widehat{A}_{0,0} \subset V_Z$, this composition $\widehat{A}_{0,0} \times \widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a} \to V_{\mu,Z}$ is nothing but the action of the VOA $V(\widehat{\mathfrak{gl}(1|1)})$ on $\widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a}$, and so it is an isomorphism when restricted to this minimal-submodule. However, since $\widehat{A}_{0,0}$ is the unique irreducible sub-module of V_Z, the intertwiner in equation (6.110) must be injective. Now we see that it is also surjective since $V_{\mu,Z}$ only has two simple currents in its composition series. Hence $V_{\mu,Z}$ is in-decomposable and is isomorphic to $V_Z \times \widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a}$. Now using Lemma 6.4, we obtain the desired result. □

It is in fact possible to identify the chains $\widehat{V}^t_{n,e,x}$ for $e \notin \mathbb{Z}$ and $\widehat{V}^t_{n,e,-,x}$ for $e \in \mathbb{Z}$ as restrictions of modules of the VOSA V_Z. Let us consider the module of V_Z generated by the vacuum vector $(mX+nY)^{t-1}|\mu\rangle$ such that $n-m(x+\frac{1}{2})=0$, which we denote by $V^t_{\mu,x,Z}$. This module is an iterated self-extension of $V_{\mu,Z}$, and the action of ∂X and ∂Y have nontrivial Jordan blocks.

Proposition 6.4. *Let $\mu = aX + bY$. When $a \in \mathbb{Z}$, there is an isomorphism of $V(\widehat{\mathfrak{gl}(1|1)})$ modules: $\widehat{V}^t_{(2b-a-1)/2,a,x,-} \cong V^t_{\mu,x,Z}$. When $a \notin \mathbb{Z}$, there is an isomorphism of $V(\widehat{\mathfrak{gl}(1|1)})$ modules: $\widehat{V}^t_{(2b-a-1)/2,a,x} \cong V^t_{\mu,x,Z}$.*

Proof. For the case when $a \notin \mathbb{Z}$, we proceed in the same way as the proof in [CGN21]: note that there is a map of $\mathfrak{gl}(1|1)$ modules $V^t_{b-(a-1)/2,a,x} \to V^t_{\mu,x,Z}[0]$, where $V^t_{\mu,x,Z}[0]$ is the lowest conformal weight space. This morphism induces a morphism of $V(\widehat{\mathfrak{gl}(1|1)})$ modules: $\widehat{V}^t_{b-(a-1)/2,a,x,-} \to V^t_{\mu,x,Z}$. This is an isomorphism on the unique minimal submodule of $\widehat{V}^t_{b-(a-1)/2,a,x}$, thus is an embedding. It is thus an isomorphism by counting the number of irreducible factors. Similar situation holds for $a = 0$.

When $a \in S_b$, we have the free field logarithmic intertwiner $V_{\mu,Z} \times_{V_Z} V^t_{0,x,Z} \cong V^t_{\mu,x,Z}$. Restricting this to the submodule $\widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a} \subseteq V_{\mu,Z}$ one gets an intertwiner of $V(\widehat{\mathfrak{gl}(1|1)})$ modules:

$$\widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a} \times V^t_{0,x,Z} \to V^t_{\mu,x,Z}. \tag{6.111}$$

It is injective because it is non-zero when restricted to the unique irreducible sub-module

$\widehat{A}_{\frac{2b-a}{2}+\epsilon(a),a} \times \widehat{A}_{0,0}$. It is then an isomorphism by counting the number of composition factors. Thus comparing this with Lemma 6.4 we obtain the desired result. □

In conclusion, many modules of $V(\widehat{\mathfrak{gl}(1|1)})$ can be identified with restrictions of modules of V_Z. We want to remark here that although only $\widehat{V}^t_{n,e,-,x}$ show up in the above consideration, we can also find $\widehat{V}^t_{n,e,+,x}$ by pre-composing the embedding $V(\widehat{\mathfrak{gl}(1|1)}) \to V_Z$ with the automorphism w introduced in Section 6.2.4.

6.3.3 A free field realization of $\mathcal{V}_{ext}$

Let us return to the VOSA $\mathcal{V}_{ext}$. Our goal is to introduce a free-field realization of $\mathcal{V}_{ext}$ that is compatible with the free-field realization $V(\widehat{\mathfrak{gl}(1|1)}) \to V_Z$ described in the last section. Recall that $\mathcal{V}_{ext} \cong V_{\beta\gamma} \otimes V_{bc}$. It is well known (e.g. [AW20]) that $V_{\beta\gamma}$ has a free-field realization given by a lattice VOA generated by $\partial\psi, \partial\theta$ and $e^{\pm\psi+\theta}$, where $(\psi,\psi) = -(\theta,\theta) = 1$. There is a screening operator $S =: e^{\psi}:$ such that $V_{\beta\gamma}$ is the kernel of S. By Bose-Fermi correspondence, V_{bc} is isomorphic to a lattice VOSA of a single free boson. To conform with the free-field realization of $V(\widehat{\mathfrak{gl}(1|1)})$, we set $\psi = Z - Y$ and $\theta = Y - Z - X$, which means that $\psi + \theta = -X$. We again treat the modes of $:e^{\pm Z}:$ as odd. With this redefinition, $Z + X$ becomes an independent variable (it has zero pairing with ψ, θ), whose associated lattice VOSA is V_{bc}. Thus we can extend V_Z by the operator $:e^{\pm X}:$, or in other words, the V_Z module $\mathcal{V} = \bigoplus_{n\in\mathbb{Z}} V_{nX,Z}$ has the structure of a VOSA, and is the lattice VOSA associated to the lattice generated by X, Z. The embedding $\mathcal{V}_{ext} \to \mathcal{V}$ is given by

$$b \mapsto :e^{Z+X}: \qquad c \mapsto :e^{-Z-X}: \qquad \beta \mapsto :e^{-X}: \qquad \gamma \mapsto :(\partial Z - \partial Y)e^{X}: \tag{6.112}$$

and its image is the kernel of the screening operator $S = \oint_z :e^{\psi}: = \oint_z :e^{Z-Y}:$. We thus have:

Proposition 6.5. *There are free-field realizations compatible with the embedding* $V(\widehat{\mathfrak{gl}(1|1)}) \to$

$\mathcal{V}_{ext}$ *given by the following diagram*

$$\begin{array}{ccc} V(\widehat{\mathfrak{gl}(1|1)}) & \longrightarrow & V_Z \cong \bigoplus_n \mathbb{F}_{nZ} \\ \downarrow & & \downarrow \\ \mathcal{V}_{ext} & \longrightarrow & \mathcal{V} \cong \bigoplus_{m,n} \mathbb{F}_{mX+nZ} \end{array} \tag{6.113}$$

We now come back to the question of lifting modules via the functor $\mathcal{F}$. This can be done by lifting modules from V_Z to $\mathcal{V}$, and then restricting them to $\mathcal{V}_{ext}$ modules. The modules of V_Z are given by $V^t_{mX+nY,Z,x}$, and they are generated by the vacuum module $(mX+nY)^{t-1}|\mu\rangle$. However, not all of them can be lifted to $\mathcal{V}$. In order to lift to $\mathcal{V}$, $n = 0$ otherwise the action of $:e^X:$ is non-local. Since $n - m(x + \frac{1}{2}) = 0$, this requires that $x = -\frac{1}{2}$. Similarly, we cannot use any linear combination $\mu = aX + bY$, since otherwise the conformal weight of $:e^{-X}:$ is not an integer. To have integer conformal weight, we need that $b \in \mathbb{Z}$. Thus we require $b \in \mathbb{Z}$ and $x = -\frac{1}{2}$. We denote by $\mathcal{V}^t_\mu$ the resulting module of $\mathcal{V}$. On the other hand, given a module of $\mathcal{W}$ of $V_{\beta\gamma}$, we can view the Deligne product $\mathcal{W} \boxtimes V_{bc}$ as a module of $\mathcal{V}_{ext}$ using the isomorphism $\mathcal{V}_{ext} \cong V_{\beta\gamma} \otimes V_{bc}$. Recall the $V_{\beta\gamma}$ modules $\mathcal{W}^t_{[a]}$ and $(\mathcal{W}^\pm_0)^t$. When $t = 1$, the following is shown in [AW20, Proposition 2.12]:

Proposition 6.6. *Let $\mu = aX + bY$ with $b \in \mathbb{Z}$. When $a \notin \mathbb{Z}$, there are isomorphisms of $\mathcal{V}_{ext}$ modules:*

$$\mathcal{V}_\mu \cong \sigma^{-b+1}\mathcal{W}_{[-a]} \boxtimes V_{bc}. \tag{6.114}$$

When $a \in \mathbb{Z}$, there are isomorphism of $\mathcal{V}_{ext}$ modules:

$$\mathcal{V}_\mu \cong \sigma^{-b+1}(\mathcal{W}^-_{[0]}) \boxtimes V_{bc}. \tag{6.115}$$

We extend it to the following:

Proposition 6.7. *Let $\mu = aX + bY$ with $b \in \mathbb{Z}$. When $a \notin \mathbb{Z}$, there are isomorphisms of*

$\mathcal{V}_{ext}$ *modules*

$$\mathcal{V}_\mu^t \cong \sigma^{-b+1}\mathcal{W}_{[-a]}^t \boxtimes V_{bc}. \tag{6.116}$$

When $a \in \mathbb{Z}$, there are isomorphism of $\mathcal{V}_{ext}$ modules

$$\mathcal{V}_\mu^t \cong \sigma^{-b+1}(\mathcal{W}_{[0]}^-)^t \boxtimes V_{bc}. \tag{6.117}$$

Proof. When $t = 1$, this is simply [AW20, Proposition 2.12]. When $t > 1$, we only need to show that the module $\mathcal{V}_\mu^t$ is indecomposable, and the result will follow from Theorem 6.2 and Theorem 6.3: the only indecomposable modules having $\sigma^{-b+1}\mathcal{W}_{[-a]}$ or $\sigma^{-b+1}\mathcal{W}_{[0]}^-$ in its composition series are the long chains. To show that it is indecomposable, we compute the action of $:\gamma\beta:$ on the generator $X^{t-1}|\mu\rangle$. It is easy to see that there is a Jordan block of size t, and thus it is indecomposable. □

With this, we derive:

Proposition 6.8. *Let $b \in \mathbb{Z}$. If $a \notin \mathbb{Z}$, one has the following isomorphism of $\mathcal{V}_{ext}$ modules*

$$\mathcal{F}\left(\widehat{V}^t_{(2b-a-1)/2,a,x=-1/2}\right) \cong \sigma^{-b+1}\mathcal{W}_{[-a]}^t \boxtimes V_{bc}. \tag{6.118}$$

For $a \in \mathbb{Z}$, one has the following isomorphism of $\mathcal{V}_{ext}$ modules

$$\begin{aligned} \mathcal{F}\left(\widehat{V}^t_{(2b-a-1)/2,a,-,x=-1/2}\right) &\cong \sigma^{-b+1}(\mathcal{W}_0^-)^t \boxtimes V_{bc}, \\ \mathcal{F}\left(\widehat{V}^t_{(2b-a-1)/2,a,+,x=-1/2}\right) &\cong \sigma^{-b}(\mathcal{W}_0^+)^t \boxtimes V_{bc}. \end{aligned} \tag{6.119}$$

Proof. We have seen that $\mathcal{V} \times_{V_Z} V^t_{\mu,x=-1/2,Z} \cong \mathcal{V}^t_\mu$. The inclusion $\mathcal{V}_{ext} \to \mathcal{V}$ induces an isomorphism

$$\mathcal{F}\left(\mathcal{V}_\mu^t\right) \cong \mathcal{V} \times_{V_Z} V^t_{\mu,x=-1/2,Z} \tag{6.120}$$

as modules of the mode algebra of $\mathcal{V}_{ext}$. Comparing this with Proposition 6.4 and Proposition 6.7, we get the desired result for $\widehat{V}^t_{b-(a-1)/2,a,x=-1/2}$ when $a \notin \mathbb{Z}$ and $\widehat{V}^t_{b-\frac{a-1}{2},a,-,x=-1/2}$ when

$a \in \mathbb{Z}$. Twisting both free-field realizations of $V(\widehat{\mathfrak{gl}(1|1)})$ and $V_{\beta\gamma}$ by conjugation w gives the result for $\widehat{V}^t_{b-\frac{a-1}{2},a,+,x=-1/2}$. □

Remark. The requirement that $x = -1/2$ implies that $N_0 + E_0/2$ acts semisimply, and requiring that $b \in \mathbb{Z}$ implies that it has integer eigenvalues. The above Proposition implies that a sufficient condition for a $V(\widehat{\mathfrak{gl}(1|1)})$ module to be lifted to $\mathcal{V}_{ext}$ is that $N_0 + E_0/2$ acts semisimply with integer eigenvalues. We denote by $KL^{N+E/2}$ the subcategory of KL where $N_0 + E_0/2$ acts semisimply with integer eigenvalues.

As a corollary, we prove the first part of Theorem 6.1:

Corollary 6.3. *The category $\mathcal{C}_{\beta\gamma}$ has the structure of a braided tensor category defined by $P(z)$-intertwining operators.*

Proof. As discussed in Section 6.3.1, the supercategory $\mathrm{Rep}^0(\mathcal{V}_{ext})$ has the structure of a braided tensor supercategory defined by $P(z)$-intertwining operators. By Theorem 6.2, Theorem 6.3 as well as Proposition 6.8, we conclude that $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$ is a full subcategory of $\mathrm{Rep}^0(\mathcal{V}_{ext})$, so we only need to show that this subcategory is a closed under fusion product. By [CMY20a], we conclude that fusion product on KL, and consequently on $\mathrm{Rep}^0(\mathcal{V}_{ext})$, is exact. Thus by the definition of $\mathcal{C}_{\beta\gamma}$, we only need to show that given two modules in $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$, fusion product of their composition factors are still in $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$. This is computed in [AW20], and we conclude that $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$ is closed under fusion, and is thus a braided tensor subcategory. Thus $P(z)$-intertwining operators define a braided tensor category structure on $\mathcal{C}_{\beta\gamma}$. □

6.3.4 The equivalence

In this section, we complete the proof of Theorem 6.1. Since the work of [CKM17, CMY20b] already implies that the functor $\mathcal{F}$ is a braided tensor functor, we only need to show that the image of KL^0 under $\mathcal{F}$ coincides with $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$ as abelian supercategories. The procedure for the proof is the following:

1. Recall the category $KL^{N+E/2}$. We first conclude from Proposition 6.8 that $KL^{N+E/2}$ is a subcategory of KL^0.

2. We then use Theorem 6.2 and 6.3 to conclude that the lifting functor $\mathcal{F}$ restricted to $KL^{N+E/2}$ is essentially surjective onto $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$.

3. Finally, we conclude from the above two points that the functor $\mathcal{F}$ identifies $KL^0/\mathbb{Z}$ with $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$, and that KL^0 coincides with with $KL^{N+E/2}$.

Let us start with the following:

Lemma 6.5. *$KL^{N+E/2}$ is a subcategory of KL^0.*

Proof. We need to show that objects in $KL^{N+E/2}$ can be lifted to $\mathrm{Rep}^0(\mathcal{V}_{ext})$. Consider first $e \notin \mathbb{Z}$ with $\widehat{M} \in KL_e$ for some $\mathfrak{gl}(1|1)$ module M, and suppose $\widehat{M} \in KL^{N+E/2}$. To show that $\widehat{M}$ can be lifted to a local module of $\mathcal{V}_{ext}$, we only need to show that it is a quotient of a module that can be lifted, since if the monodromy is trivial on a module, it is trivial on all quotients. This is guaranteed if M is a quotient of a direct sum of $V^t_{n,e,x=-1/2}$ for various $n \in (1-e)/2 + \mathbb{Z}$ and t. Without loss of generality one may assume that M is generated by a single m such that $(E-e)^t m = 0$, and $(N + E/2)m = km$ for some $k \in \mathbb{Z}$. We may choose k to be the largest such value so that $\psi^+ m = 0$. It is clear then that the module M is spanned by the vectors coming from applying $\mathfrak{gl}(1|1)$ to m

$$m, \psi^- m, (E-e)m, (E-e)\psi^- m, \ldots, (E-e)^{t-1} m, (E-e)^{t-1}\psi^- m. \tag{6.121}$$

From the analysis of Section 6.2.1 there is a surjection:

$$V^t_{k-(e+1)/2,e,x=-1/2} \longrightarrow M \tag{6.122}$$

by mapping the highest weight vector of the generator to m. This is a well-defined map since the structure of $V^t_{k-(e+1)/2,e,x=-1/2}$ is determined by an element v with $(N+E/2)v = kv$, $\psi^+ v = 0$ as well as $(E-e)$ has order t. This shows that M is a quotient of $V^t_{k-(e+1)/2,e,x=-1/2}$, and so $\widehat{V}^t_{k-(e+1)/2,e,x=-1/2}$ maps onto $\widehat{M}$. Since $\widehat{V}^t_{k-(e+1)/2,e,x=-1/2}$ can be lifted to a local module, so can $\widehat{M}$.

A similar argument can be applied when $e = 0$ since any M such that N acts semisimply with integer eigenvalues and E acts nilpotently is a quotient of $P^t_{n,0,x=-1/2}$.

Finally, let us consider when $e \in S_b$. Let $W \in KL_e$, then $W = \widehat{A}_{-e/2+\epsilon(e),e} \times \widehat{M}$ for some M. This means that W can be extended if and only if $\widehat{M}$ can be. We must show that $N + E/2$ acts semisimply on M with integer eigenvalues. If this were not the case, two things can go wrong: first, $N + E/2$ does not have integer eigenvalues, or it does not act semisimply. Suppose it was the first case, which means that M has a sub-quotient $A_{n,0}$ such that $n \notin \mathbb{Z}$, but this means that W has a sub-quotient $\widehat{A}_{-e/2+\epsilon(e),e} \times \widehat{A}_{n,0} = \widehat{A}_{n-e/2+\epsilon(e),e}$, a sub-module of $\widehat{V}_{n-e/2-\epsilon(e),e}$ on which $N_0 + E_0/2$ acts semisimply with eigenvalues in $n+\mathbb{Z} \neq \mathbb{Z}$, a contradiction to our assumption on W. If $N + E/2$ does not act semisimply, then since $\widehat{A}_{-e/2+\epsilon(e),e} = \sigma^e(\widehat{A}_{e,0})$, one has

$$\widehat{A}_{-e/2+\epsilon(e),e} \times \widehat{M} \cong \sigma^e\left(\widehat{A_e \otimes M}\right). \tag{6.123}$$

By the definition of spectral flow, the action of $N_0 + E_0/2$ on $\sigma^e\left(\widehat{A_e \otimes M}\right)$ is the same as the action of $N_0 + E_0/2 + e_0/2$ on $\widehat{A_e \otimes M}$. If $N + E/2$ does not act semisimply on M, then $N_0 + E_0/2 + e_0/2$ can not act semisimply since they only differ by a scalar, giving a contradiction to our assumption on W.

This means that we may apply the previous argument to M, so that $\widehat{M}$ can be lifted, and thus so can W. This completes the proof. □

Next, we prove:

Proposition 6.9. *The functor $\mathcal{F}$ restricted to $KL^{N+E/2}$ is essentially surjective onto $\mathcal{C}_{\beta\gamma} \boxtimes$*

SVect.

Proof. To show that it is essentially surjective, by Theorem 6.2 and 6.3, we need only show that the image of $\mathcal{F}$ contains $\sigma^l \mathcal{W}^t_{[a]}$ for $a \notin \mathbb{Z}$ and $\sigma^l(\mathcal{W}^{\pm}_{[0]})^t$. These then follow from Proposition 6.8. □

Now we can finish the proof of Theorem 6.1:

Theorem 6.6. *The functor $\mathcal{F}$ provides an equivalence of braided tensor supercategories*

$$\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect} \cong KL^0/\mathbb{Z}. \tag{6.124}$$

Proof. Let us show that the image of KL^0 under $\mathcal{F}$ is $\mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect}$. For this, it is enough to show that the simple modules of KL^0 coincides with those of $KL^{N+E/2}$. First of all, consider $e \notin \mathbb{Z}$, in which case the simple modules are $\widehat{V}_{n,e}$. If monomdromy acts trivially on $\mathcal{V}_{ext} \times_{V(\widehat{\mathfrak{gl}(1|1)})} \widehat{V}_{n,e}$, then the monodromy of $\mathcal{V} \times_{V_Z} \widehat{V}_{n,e}$ would also be trivial. We have seen that this only happens if $n + e/2 \pm \frac{1}{2} \in \mathbb{Z}$, or in other words, when $\widehat{V}_{n,e} \in KL^{N+E/2}$. When $e \in \mathbb{Z}$, the simple modules are $\widehat{A}_{n,e}$, and we embed them into $\widehat{V}_{n,e,-}$. Since monodromy can be computed by using the twist element $e^{2\pi i L_0}$, and L_0 acts semisimply on $\widehat{V}_{n,e,-}, \mathcal{V}_{ext}$ as well as the fusion $\mathcal{V}_{ext} \times_{V(\widehat{\mathfrak{gl}(1|1)})} \widehat{V}_{n,e,-}$, we know that the monodromy map is semisimple. More-over, $\widehat{V}_{n,e,-}$ is indecomposable as a $V(\widehat{\mathfrak{gl}(1|1)})$ module, and so the monodromy must be scalar on each direct summand of $\mathcal{V}_{ext} \times_{V(\widehat{\mathfrak{gl}(1|1)})} \widehat{V}_{n,e,-}$. This means that if monodromy action on $\mathcal{V}_{ext} \times_{V(\widehat{\mathfrak{gl}(1|1)})} \widehat{V}_{n,e,-}$ is nontrivial, then it is nontrivial on $\mathcal{V}_{ext} \times_{V(\widehat{\mathfrak{gl}(1|1)})} \widehat{A}_{n,e}$ by naturality of commutativity constraints and exactness of fusion. Using similar argument as in the case of $e \notin \mathbb{Z}$, this means that $\widehat{V}_{n,e,-}$ is an object in $KL^{N+E/2}$, and so $\widehat{A}_{n,e}$ must also be in $KL^{N+E/2}$.

By the discussions of Section 6.3.1, the functor $\mathcal{F}$ identifies the image of KL^0 with the de-equivariantization. Thus we have an equivalence

$$KL^0/\mathbb{Z} \cong \mathcal{C}_{\beta\gamma} \boxtimes \mathrm{SVect} \tag{6.125}$$

as desired. ∎

Corollary 6.4. *$KL^{N+E/2}$ coincides with KL^0. Moreover, for each $e \in \mathbb{C}$, $\mathcal{F}$ gives an equivalence between abelian supercategories*

$$\mathcal{F} : KL_e^{N+E/2} = KL_e^0 \cong \mathcal{C}_{\beta\gamma,[e]} \boxtimes \mathrm{SVect} \tag{6.126}$$

Proof. We first show that $KL^{N+E/2}$ coincides with KL^0. Let $W \in KL_e^0$ for some e, then $\mathcal{F}(W)$ is an element in $\mathcal{C}_{\beta\gamma,[e]}$. By Corollary 6.9, there is an element $V \in KL_e^{N+E/2}$ such that

$$\mathcal{F}(V) \cong \mathcal{F}(W) \tag{6.127}$$

This is an equivalence as modules of $\mathcal{V}_{ext}$, so must be an equivalence as modules between $V(\widehat{\mathfrak{gl}(1|1)})$. Taking the generalized E_0 weight e part, we must have $W \cong V$. Thus $KL^{N+E/2}$ coincides with KL^0.

We now show that $\mathcal{F}$ restricts to an equivalence of abelian supercategories

$$\mathcal{F} : KL_e^{N+E/2} = KL_e^0 \cong \mathcal{C}_{\beta\gamma,[e]} \boxtimes \mathrm{SVect} \tag{6.128}$$

It is essentially surjective from Corollary 6.9, and we only need to show that it is fully-faithful. For any $W_1, W_2 \in KL_e^{N+E/2}$, if $f \colon W_1 \to W_2$ is nonzero, because $\mathcal{V}_{ext}$ is a direct sum of simple currents, the associated map

$$\mathcal{F}(W_1) \to \mathcal{F}(W_2) \tag{6.129}$$

is non-zero. Thus the functor $\mathcal{F}$ is faithful. To show that it is full, take any $W_1, W_2 \in KL_e^{N+E/2}$ and $f \in \mathrm{Hom}(\mathcal{F}(W_1), \mathcal{F}(W_2))$. It is a morphism as $\mathcal{V}_{ext}$ modules, so must be a morphism in $\mathrm{Ind}(KL)$. By restricting this morphism to the generalized E_0 weight e part, one obtains a morphism between W_1 and W_2. This provides natural bijections, and so the functor $\mathcal{F}$ is fully faithful when restricted to $KL_e^{N+E/2}$. ∎

6.3.5 The fusion structure of $\mathcal{C}_{\beta\gamma}$

Using the equivalence as well as the fusion structure of $V(\widehat{\mathfrak{gl}(1|1)})$ obtained in Section 6.2.6, we have the following fusion structure for $\beta\gamma$ VOA:

Corollary 6.5. *For any $\mathcal{M} \in \mathcal{C}_{\beta\gamma}$, denote by $\mathcal{M}_n$ the spectral flow $\sigma^n(\mathcal{M})$. We have the following fusion rules for $\beta\gamma$ modules*

$$\begin{aligned}
&\mathcal{P}_n^t \times \mathcal{P}_m^s \cong \bigoplus_{l=0}^{\min\{t,s\}-1} P_{n+m-1}^{s+t-1-2l} \oplus 2P_{n+m}^{s+t-1-2l} \oplus P_{n+m+1}^{s+t-1-2l}, \\
&\mathcal{W}_{[\lambda],n}^t \times \mathcal{W}_{[\mu],m}^s \cong \bigoplus_{l=0}^{\min\{t,s\}-1} \mathcal{W}_{[\lambda+\mu],n+m}^{t+s-1-2l} \oplus \mathcal{W}_{[\lambda+\mu],n+m-1}^{t+s-1-2l}, \text{ if } \lambda+\mu \notin \mathbb{Z}, \\
&\mathcal{W}_{[\lambda],n}^t \times \mathcal{W}_{[-\lambda],m}^s \cong \mathcal{W}_{+,n}^t \times \mathcal{W}_{-,m}^s \cong \bigoplus_{l=0}^{\min\{t,s\}-1} \mathcal{P}_{m+n-1}^{t+s-1-2l}
\end{aligned} \tag{6.130}$$

6.4 Quiver algebra and quantum group

In this section, we will focus on the subcategory of atypical modules $\mathcal{C}_{\beta\gamma,[0]}$. In Section 6.4.1, we will compute the endomorphism of the identity line operator, and compare the result with the computation in the smaller category studied in [AW20]. In Section 6.4.2, we show that $\mathcal{C}_{\beta\gamma,[0]}$ can be described as the category of modules of a quiver algebra, and show that this quiver algebra can be related to the quantum group $\overline{U}_q^H(\mathfrak{sl}(2))$.

6.4.1 The category of atypical modules

Recall the following decomposition in equation (6.10)

$$\mathcal{C}_{\beta\gamma} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{C}_{\beta\gamma,\lambda}. \tag{6.131}$$

When $\lambda = [0]$, the full subcategory $\mathcal{C}_{\beta\gamma,[0]}$ is called the category of atypical modules. This is a tensor subcategory of $\mathcal{C}_{\beta\gamma}$, and the tensor identity of $\mathcal{C}_{\beta\gamma}$, which is $\mathbb{1} = \mathcal{V} = V_{\beta\gamma}$, lies in $\mathcal{C}_{\beta\gamma,[0]}$.

On the other hand, by Theorem 6.1, there are equivalences of abelian categories

$$\mathcal{C}_{\beta\gamma,[0]} \cong KL_0^{N+E/2}/\mathbb{Z}_2 \cong \mathcal{C}_0^{N+E/2}/\mathbb{Z}_2, \tag{6.132}$$

where recall that $\mathcal{C}_0^{N+E/2}$ is the category of finite-dimensional modules of $\mathfrak{gl}(1|1)$ such that $N+E/2$ acts semisimply with integer eigenvalues and E acts nilpotently. Now consider the category $\mathcal{C}_0^{N+E/2}$. In the following, we will view the action of $N+E/2$ as a $\mathbb{C}^*$ grading on the objects of $\mathcal{C}_0^{N+E/2}$. Denote by A the algebra over $\mathbb{C}$ generated by $\psi^\pm$. The only relations the two generators satisfy are $(\psi^+)^2 = (\psi^-)^2 = 0$. The algebra A can be viewed as a super algebra if we view $\psi^\pm$ as odd elements. The adjoint action of $N+E/2$ gives a $\mathbb{C}^*$ grading under which $\psi^\pm$ have weight ± 1. Denote by $A-\mathrm{Mod}^{\mathbb{C}^*}$ the category of finite-dimensional $\mathbb{C}^*$-equivariant modules of the (non-super) algebra A, then it is clear by definition, that we have the following equivalence of abelian categories

$$\mathcal{C}^{N+E/2}/\mathbb{Z}_2 \cong A-\mathrm{Mod}^{\mathbb{C}^*}. \tag{6.133}$$

Indeed, the supercategory $\mathcal{C}^{N+E/2}$ is equivalent to the supercategory of $\mathbb{C}^*$-equivariant, finite-dimensional modules of the super algebra A. Since the superalgebra structure can be induced by the action of $\mathbb{C}^*$, the de-equivariantization simply forgets the superalgebra structure. In other words, a $\mathbb{C}^*$ equivariant module automatically has a compatible $\mathbb{Z}_2$ grading. From the above two equations we conclude

$$\mathcal{C}_{\beta\gamma,[0]} \cong A-\mathrm{Mod}_{nil}^{\mathbb{C}^*}. \tag{6.134}$$

Here $A-\mathrm{Mod}_{nil}^{\mathbb{C}^*}$ is the subcategory of $A-\mathrm{Mod}^{\mathbb{C}^*}$ where E acts nilpotently. Under this isomorphism, the identity object $\mathbb{1}$ corresponds to the trivial representation $\mathbb{C}$ of A, on which $\psi^\pm$ acts trivially. We are now ready to compute

Proposition 6.10. *The derived endomorphism algebra of the identity line in $\mathcal{C}_{\beta\gamma}$ is trivial*

$$\mathrm{End}_{D^b\mathcal{C}_{\beta\gamma}}(\mathbb{1}) \cong \mathbb{C}. \tag{6.135}$$

Here $D^b\mathcal{C}_{\beta\gamma}$ is the bounded derived category of the abelian category $\mathcal{C}_{\beta\gamma}$.

Proof. From the above, we only need to show that

$$\mathrm{End}_{D^bA-\mathrm{Mod}^{\mathbb{C}^*}}(\mathbb{C}) \cong \mathbb{C}. \tag{6.136}$$

The algebra A is well-known to be a Koszul algebra, and its Koszul dual is $A^! = \mathbb{C}[x,y]/(xy)$, a commutative algebra of two variables x, y satisfying $xy = 0$. Thus, the derived endomorphism of $\mathbb{C}$ as an ungraded A module is

$$\mathrm{End}_{D^bA-\mathrm{Mod}}(\mathbb{C}) \cong A^! = \mathbb{C}[x,y]/(xy). \tag{6.137}$$

Here we need to use a projective resolution of $\mathbb{C}$, which technically do not belong to $D^bA-\mathrm{Mod}$. This is not an issue for us since $D^bA - \mathrm{Mod}$ is a full subcategory of the derived category of finitely-generated A modules. The $\mathbb{C}^*$ action gives x and y weight 1 and -1 respectively. Taking $\mathbb{C}^*$-invariants, we find

$$\mathrm{End}_{D^bA-\mathrm{Mod}^{\mathbb{C}^*}}(\mathbb{C}) \cong (A^!)^{\mathbb{C}^*} = \mathbb{C}. \tag{6.138}$$

This completes the proof. □

We have finally proved the claim that $\mathcal{C}_{\beta\gamma}$ produces the correct Coulomb branch! In contrast, the category of modules considered in [AW20] will not produce the correct answer for our setting. Indeed, the category considered in [AW20] can be obtained from $\mathcal{C}_{\beta\gamma}$ by requiring J_0 to act semisimply. On the algebra A, this corresponds to a further restriction $E = 0$. We are thus restricted to the Grassmann algebra of two variables $B = \mathbb{C}[\epsilon_1, \epsilon_2]$. The Koszul dual is $B^! = \mathbb{C}[x, y]$, and we have

$$\mathrm{End}_{D^b B-\mathrm{Mod}^{\mathbb{C}^*}}(\mathbb{C}) \cong (B^!)^{\mathbb{C}^*} = \mathbb{C}[xy]. \tag{6.139}$$

The computation above produces a commutative algebra generated by a single variable xy, instead of the trivial algebra ($\mathbb{C}$) which we expect. The xy in equation (6.139) corresponds to a degree 2 extension of $V_{\beta\gamma}$ by itself, in the form of the following exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}_0^+ \longrightarrow \mathcal{W}_0^- \longrightarrow \mathcal{V} \longrightarrow 0. \tag{6.140}$$

This is trivialized in our category $\mathcal{C}_{\beta\gamma}$ due to the existence of the indecomposable module with the following Loewy diagram

$$\mathcal{V} \longrightarrow \sigma^{-1}\mathcal{V} \longrightarrow \mathcal{V}. \tag{6.141}$$

This is similar to the discussion in [CCG19], where the authors showed that the choice of the category changes the resulting endomorphism algebra.

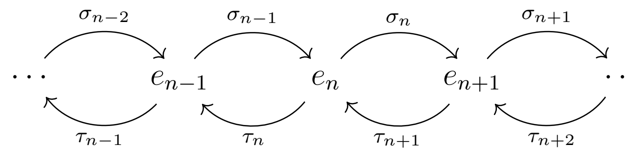
6.4.2 A quiver description

From the mirror symmetry statement, we have an equivalence

$$\mathcal{C}_{\beta\gamma,[0]} \boxtimes \mathrm{SVect} \cong KL_0^{N+E/2} \cong \mathcal{C}_0^{N+E/2}. \tag{6.142}$$

Let us analyze the category $\mathcal{C}_0^{N+E/2}$. Since the action of $N + E/2$ is semisimple, let e_n be the operator of projection onto the eigenspace with eigenvalue $n \in \mathbb{Z}$. Let $\sigma_n = \psi^+ \circ e_n$ and $\tau_n = \psi^- \circ e_n$, it is clear then $\psi^2 = 0$ now becomes $\sigma_n\sigma_{n-1} = \tau_n\tau_{n+1} = 0$. These e_n, σ_n and τ_n

is the path algebra of the following quiver



(6.143)

quotient by the quadratic relation $\sigma_n\sigma_{n-1} = \tau_n\tau_{n+1} = 0$. Let this algebra be denoted by Λ. It is graded once we give σ_n and τ_n degree 1. From the above description, the supercategory $\mathcal{C}_0^{N+E/2}$ is equivalent to the supercategory of graded finite-dimensional unitary modules of Λ. To get an ordinary category instead of a supercategory, we only need to choose a grading for each module. We say that a vector is even if it's in the image of e_n for n even. Once we do this, we find an equivalence between $\mathcal{C}_{\beta\gamma,[0]}$ with the category of finite-dimensional unitary modules of Λ (as an ungraded algebra) on which $\sigma_n\tau_{n+1}$ and $\tau_n\sigma_{n-1}$ act nilpotently for all n. The last condition comes from that E acts nilpotently for objects in $\mathcal{C}_0$. This category we denote by $\Lambda - \text{mod}_{nil} \cong \mathcal{C}_{\beta\gamma,[0]}$. The interesting thing about this category is that it has three in principle distinct braided tensor category structure. One of them comes from the identification with objects in $\mathcal{C}_0^{N+E/2}$, another from the identification with objects in $\mathcal{C}_{\beta\gamma,[0]}$. The third one is rather surprising, as it comes from a morphism from $\overline{U}_q^H(\mathfrak{sl}(2))$.

6.4.3 Relation to quantum group

Consider the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{sl}(2))$ at the fourth root of unity $q = i$. By this, we mean the algebra generated by $E, F, H, K^{\pm}$ with the relation

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \quad KE = -EK, \quad KF = -FK, \\ &[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = \frac{K - K^{-1}}{2i}, \quad E^2 = F^2 = 0. \end{aligned} \tag{6.144}$$

When considering modules, we also consider modules that satisfies relation $K = q^H = e^{\pi i H/2}$, although this is not a well-defined relation in the algebra itself.

Lemma 6.6. *Let $\sigma = \sum_n \sigma_n$, $\tau = \sum_n \tau_n$ and $L = \sigma\tau + \tau\sigma$. Let $f(x)$ be the Taylor series of* *the function $\frac{1-e^{-\pi i x}}{x}$. The assignment*

$$H \mapsto \left(\sum_n 2ne_n\right) + L, \quad K \mapsto e^{\frac{\pi i}{2}H}, \quad E \mapsto \sigma, \quad F \mapsto \frac{\tau}{2i} f(L)K \tag{6.145}$$

gives a well-defined action of $\overline{U}_q^H(\mathfrak{sl}(2))$ on finite-dimensional unitary modules of Λ, and satisfies $K = q^H$.

Proof. These give well-defined operators on any finite-dimensional unitary module of Λ, so we only need to show that this gives the right commutation relation, which is a simple algebraic check. We only show the relation $[E, F] = \frac{K-K^{-1}}{2i}$. We have that

$$[\sigma, \frac{\tau}{2i} f(L)K] = \frac{1}{2i}\{\sigma, \tau\} f(L)K = \frac{1}{2i} Lf(L)K. \tag{6.146}$$

By definition, $Lf(L) = 1 - e^{-\pi i L} = 1 - e^{-\pi i H} = 1 - K^{-2}$, and so $Lf(L)K = K - K^{-1}$, thus the relation. □

Remark. It can be shown that when restricted to the category of atypical modules of $\overline{U}_q^H(\mathfrak{sl}(2))$, this gives an equivalence to $\Lambda - \text{mod}_{nil}$, with a well-defined inverse assignment. This assignment should be, in some sense, similar to the induction functor.

It turns out, that one can transfer the structure of the braided tensor category from $\overline{U}_q^H(\mathfrak{sl}(2))$ to $\Lambda - \text{mod}$. For example, the coproduct Δ_t (t for twisted) is given by

$$\begin{aligned} &\Delta_t(e_n) = \sum_{p+q=n} e_p \otimes e_q, \quad \Delta_t(\sigma) = \sigma \otimes 1 + e^{\frac{\pi i}{2}H} \otimes \sigma, \quad \Delta_t(L) = L \otimes 1 + 1 \otimes L \\ &\Delta_t(\tau) = 2i(1 \otimes F + F \otimes e^{-\frac{\pi i}{2}H})(e^{-\frac{\pi i}{2}H} \otimes e^{-\frac{\pi i}{2}H}) \frac{1}{f(L)} (L \otimes 1 + 1 \otimes L) \end{aligned} \tag{6.147}$$

where $\frac{1}{f(x)}$ should be understood as the Taylor series of the quotient. This is very asymmetric compared to the coproduct Δ from $\mathfrak{gl}(1|1)$

$$\Delta(e_n) = \sum_{p+q=n} e_p \otimes e_q, \quad \Delta(\sigma) = \sigma \otimes 1 + e^{\pi i \sum_n n e_n} \otimes \sigma, \quad \Delta(\tau) = \tau \otimes 1 + e^{\pi i \sum_n n e_n} \otimes \tau. \tag{6.148}$$

Under the coproduct Δ, the braiding is trivial, but the braiding under Δ_t is nontrivial

$$R = e^{\pi i H \otimes H/4}(1 + 2iE \otimes F), \tag{6.149}$$

and so there is a nontrivial twist

$$\theta = K(e^{-\pi i H^2/4} - 2iKFe^{-\pi i H^2/4}E), \tag{6.150}$$

where H, E, F should be expressed using elements in Λ as in equation (6.145). These formulas can be found in [Oht02]. We expect that for any $M, N \in \Lambda-\text{mod}$, the two co-product $M \otimes N$ and $M \otimes_t N$ are isomorphic as Λ modules. However, we do not expect the two coproducts to give equivalent braided tensor structures. It remains a question whether the tensor structure Δ_t induced from $\overline{U}_q^H(\mathfrak{sl}(2))$ is equivalent to that coming from $\mathcal{C}_{\beta\gamma}$.

7 Generalization to arbitrary abelian theories

The entirety of the previous section was focused on the case of a single free twisted hypermultiplet and $U(1)$ gauge theory with a single hypermultiplet. We now summarize some aspects of our work in progress [BCDN22] that generalizes this equivalence to mirror pairs of theories with *arbitrary* abelian gauge groups.

Given a 3d $\mathcal{N} = 4$ abelian gauge theory with gauge group of rank r and n hypermultiplets, let q_{ai} denote the weight of the a^{th} hypermultiplet under the i^{th} factor of the gauge group. This defines a homomorphism $q\colon \mathbb{Z}^r \to \mathbb{Z}^n$. The charges of the matter in the mirror theory, which has gauge group of rank $n-r$ and n hypermultiplets, is determined (up to automorphism) by a map $p\colon \mathbb{Z}^n \to \mathbb{Z}^{n-r}$ such that the sequence

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\ q\ } \mathbb{Z}^n \xrightarrow{\ p\ } \mathbb{Z}^{n-r} \longrightarrow 0 \tag{7.1}$$

is exact [BHOOY97].

On the A side we consider $U(1)^r$ gauge theory with n free hypermultiplets transforming under the charge matrix q. We would like to impose Neumann BCs, which result in n copies of the $\beta\gamma$ VOA $V^n_{\beta\gamma}$ living on the boundary. However in order to cancel a gauge anomaly, we must enrich these BCs by adding 2d boundary matter just as we saw in Section 6.3.1. The correct choice will be to add n copies of the bc (i.e. free fermion) VOA V^n_{bc}. The Neumann BCs on the vector multiplets contribute another r copies of the bc system to the boundary, hence we find that the category of line operators in this theory is equivalent to an appropriate module category for $V^n_{\beta\gamma} \otimes V^{n+r}_{\text{bc}}$, in agreement with the prediction of [CG19].

The perturbative boundary VOA $V(\widehat{\mathfrak{g}_*(q)})$ that shows up on the B side after imposing Dirichlet BCs on the corresponding mirror theory is a particular subquotient of n copies of

$V(\widehat{\mathfrak{gl}(1|1)})$. It is defined by the following fields and OPEs

$$\begin{aligned} N^i(z)E^j(w) &\sim \frac{\delta_{ij}}{(z-w)^2} \qquad N^i(z)\psi^{a,\pm}(w) \sim \pm\frac{q_{ai}\psi^{a,\pm}(w)}{z-w} \\ \psi^{a,+}(z)\psi^{b,-}(w) &\sim \frac{\delta_{ab}}{(z-w)^2} + \frac{\delta_{ab}\sum_{j=1}^r q_{aj}E^j(w)}{z-w}. \end{aligned} \tag{7.2}$$

As in Section 5.6.2, these BCs support monopole operators which are not contained in $V(\widehat{\mathfrak{g}_*(q)})$, so we must find a suitable VOA extension that includes the monopole operators [CG19]. We achieve this by passing through a free field realization for $V(\widehat{\mathfrak{g}_*(q)})$ wherein finding extensions is much easier. To confirm our proposed extension is correct, we will compute its half-index and demonstrate that it matches the required form found via a physical argument by [DGP18].

7.1 Identifying the extension

We begin by describing the free field realization of $V(\widehat{\mathfrak{gl}(1|1)})$. Consider the $2r+n$ dimensional lattice L generated by X^i, Y^i, Z^a for $1 \le i \le r, 1 \le a \le n$ with symmetric bilinear form defined by

$$\langle X^i, Y^j\rangle = \delta_{ij} \qquad \langle Z^a, Z^b\rangle = \delta_{ab}. \tag{7.3}$$

Through the construction described in Section 4.3.3, this defines a lattice VOA H with fundamental fields satisfying the OPEs

$$\partial X^i(z)\partial Y^j(w) \sim \frac{\delta_{ij}}{(z-w)^2} \qquad \partial Z^a(z)\partial Z^b(w) \sim \frac{\delta_{ab}}{(z-w)^2}. \tag{7.4}$$

Let F be the lattice VOA extension of H along the sublattice $K_{\mathbb{Z}} := \langle Z^a\rangle_{\mathbb{Z}}$. In other words, we additionally have the vertex operators $Y(n|Z^a\rangle, z) =: e^{nZ^a(z)}:$ in the theory for $n \in \mathbb{Z}$.

The free field realization of $V(\widehat{\mathfrak{g}_*(q)})$ into F is as follows:

$$\begin{aligned}
N^i(z) &\mapsto \partial X^i(z) - \frac{1}{2}\sum_{a=1}^{n}\sum_{j=1}^{r} q_{aj}q_{ai}\partial Y^j(z) + \sum_{a=1}^{n} q_{ai}\partial Z^a(z) \\
E^i(z) &\mapsto \partial Y^i(z) \\
\psi^{a,+}(z) &\mapsto :e^{Z^a(z)}: \\
\psi^{a,-}(z) &\mapsto :\sum_{i=1}^{r} q_{ai}\partial Y^i(z) e^{-Z^a(z)}: + :\partial e^{-Z^a(z)}: .
\end{aligned} \tag{7.5}$$

This embedding is not surjective, so we must determine the image of the embedding to specify which operators in F contribute to the half-index. The result proven in [BCDN22] is that $V(\widehat{\mathfrak{g}_*(q)})$ is isomorphic to the intersection of the kernels of the screening operators

$$S^a := \oint dz \; :e^{Z^a(z) - \sum_{i=1}^{r} q_{ai} Y^i(z)}: \tag{7.6}$$

on F, i.e.

$$V(\widehat{\mathfrak{g}_*(q)}) := \bigcap_{a=1}^{n} \ker(S^a). \tag{7.7}$$

Now we can finally tackle the problem of adding in the monopole operators. Through anomaly matching considerations, one finds that one should extend F by the sublattice spanned by $X^i + \sum_{a=1}^{n} q_{ai} Z^a$; we call the resulting space F_{ext}. The action of the screening operators S^a on F naturally extends to an action on F_{ext}, therefore we claim that the correct extension of $V(\widehat{\mathfrak{g}_*(q)})$ by the monopole operators is given by the simultaneous kernel of S^a on F_{ext}:

$$V_{B,q} := \bigcap_{a=1}^{n} \ker\left(S^a\big|_{F_{\text{ext}}} \right). \tag{7.8}$$

We now compute the index of $V_{B,q}$, taking into account the fermion number, conformal weight, and global symmetry gradings/fugacities. This quantity is defined by the formula

$$\mathbb{I}_{V_{B,q}} := \mathrm{Tr}_{V_{B,q}} \left((-1)^F q^{L_0} s^{\tilde{N}_0} \right) \tag{7.9}$$

where $s^{\tilde{N}_0} := \prod_{i=1}^r s_i^{\tilde{N}_0^i}$. Below we jot down the mode expansions of these grading operators in the free field realization for later use:

$$\tilde{N}_0^i = x_0^i + \sum_{a=1}^n q_{ai} z_0^a \tag{7.10}$$

$$L_0 = \sum_{i=1}^r \left[\frac{1}{2}(x_0^i y_0^i + y_0^i x_0^i) + \sum_{m=1}^\infty (x_{-m}^i y_m^i + y_{-m}^i x_m^i) \right] + \sum_{a=1}^n \left[\frac{1}{2}(z_0^a)^2 + \sum_{m=1}^\infty z_{-m}^a z_m^a \right] + \frac{1}{2} \sum_{a=1}^n \left[z_0^a - \sum_{i=1}^r q_{ai} y_0^i \right]. \tag{7.11}$$

To compute the index, we compute the Verma module generated by each monopole (labeled by $\ell \in \mathbb{Z}^r$) acting on the vacuum

$$|\ell\rangle := \left| \sum_{i=1}^n \ell_i \left(X^i + \sum_{a=1}^n q_{ai} Z^a \right) \right\rangle = S_{\sum_{i=1}^n \ell_i \left(X^i + \sum_{a=1}^n q_{ai} Z^a \right)} |0\rangle. \tag{7.12}$$

The *shift operator* $S_{\sum \ell_i \cdots}$ on the RHS is a straightforward generalization of the operator $e^{m\sqrt{n}\tilde{x}}$ in equation (4.47); see [FB04, Equation 5.2.8].

In the rest of this section, whenever we write a mode belonging to the Lie algebra associated to $V(\widehat{\mathfrak{g}_*(q)})$, the image of this mode under the free field realization (7.5) should be implicitly understood. Thus we see that N_k^i and E_k^i act freely on $|\ell\rangle$ for $k < 0$ and act as (possibly vanishing) scalars for $k > 0$. This gives the same parabolic decomposition as the one obtained by analyzing the action on the vacuum $|0\rangle$; this pattern will not be quite true for $\psi^{a,\pm}(z)$, but the difference manifests itself as a sort of spectral flow depending on ℓ and the charge matrix q_{ai}. Note that

$$\psi^{a,+}(w)|\ell\rangle = S_{Z^a} w^{z_0^a} e^{\sum_{k<0} -\frac{1}{k} z_k^a w^{-k}} e^{\sum_{k>0} -\frac{1}{k} z_k^a w^{-k}} S_{\sum_{i=1}^r \ell_i (X^i + \sum_{b=1}^n q_{bi} Z^b)} |0\rangle$$
$$= S_{Z^a + \sum_{i=1}^r \ell_i (X^i + \sum_{b=1}^n q_{bi} Z^b)} w^{\sum_{i=1}^r \ell_i q_{ai}} e^{\sum_{k<0} -\frac{1}{k} z_k^a w^{-k}} |0\rangle. \tag{7.13}$$

When comparing the mode expansions of the LHS and RHS, the factor $w^{\sum_i \ell_i q_{ai}}$ effectively shifts which modes of $\psi^{a,+}(w)$ act freely vs. act as scalars, as compared to mode splitting when acting on $|0\rangle$. Defining $L_a := \sum_{i=1}^r \ell_i a_{ai}$, we find that $\psi_k^{a,+}$ acts freely for $k < -L_a$ and as a scalar for $k \geq -L_a$. A similar analysis reveals that $\psi_k^{a,-}$ acts freely for $k < L_a$ and otherwise acts as a scalar. Thus the Verma module built upon $|\ell\rangle$ has a PBW decomposition

$$\bigotimes_{a=1}^n \left[\left[\bigotimes_{k \leq -L_a - 1} (\mathbb{C} \oplus \mathbb{C}\psi_k^{a,+}) \right] \otimes \left[\bigotimes_{k \leq L_a - 1} (\mathbb{C} \oplus \mathbb{C}\psi_k^{a,-}) \right] \right]$$
$$\otimes \bigotimes_{i=1}^r \left[\bigotimes_{k \leq -1} \left(\bigoplus_{m \geq 0} \mathbb{C}(N_k^i)^m \otimes \bigoplus_{m \geq 0} \mathbb{C}(E_k^i)^m \right) \right] |\ell\rangle. \tag{7.14}$$

The contribution of this sector of $V_{B,q}$ to the index *solely from the mode algebra* is thus, after some straightforward algebraic manipulation,

$$\frac{1}{(q)_\infty^{2r}} \prod_{a=1}^n \left(q \prod_{i=1}^r (s_i q^{\ell_i})^{q_{ai}}, q \prod_{i=1}^r (s_i q^{\ell_i})^{-q_{ai}}; q \right)_\infty . \tag{7.15}$$

But we cannot forget that $|\ell\rangle$ itself has non-trivial grading under L_0 and $\tilde{N}_0$! Properly taking this into account when computing the index will yield an expression equal to equation (7.15) multiplied by an overall factor consisting of the fugacities of $|\ell\rangle$. Let us now calculate this factor:

$$
\begin{aligned}
L_0|\ell\rangle &= \frac{1}{2}\sum_{b=1}^{n}\left[(z_0^b)^2 + z_0^b - \sum_{j=1}^{r} q_{bj} y_0^j\right] S_{\sum_{i=1}^r \ell_i (X^i + \sum_{a=1}^n q_{ai} Z^a)}|0\rangle \\
&= \frac{1}{2}\sum_{i,j,a} \ell_i \ell_j q_{ai}^2 |\ell\rangle \\
&= \frac{1}{2}\ell^T q^T q \ell |\ell\rangle
\end{aligned}
$$

$$
\begin{aligned}
\tilde{N}_0^i|\ell\rangle &= \left[x_0^i + \sum_{a=1}^{n} q_{ai} z_0^a\right] S_{\sum_{j=1}^r \ell_j (X^j + \sum_{b=1}^n q_{bj} Z^b)}|0\rangle \\
&= \sum_{a,j} q_{ai} q_{aj} \ell_j |\ell\rangle \\
&= (q^T q \ell)_i |\ell\rangle
\end{aligned}
\tag{7.16}
$$

$$
\begin{aligned}
(-1)^F|\ell\rangle &= (-1)^{\langle \sum_{i=1}^r \ell_i (X^i + \sum_{a=1}^n q_{ai} Z^a), \sum_{j=1}^r \ell_j (X^j + \sum_{b=1}^n q_{bj} Z^b)\rangle}|\ell\rangle \\
&= (-1)^{\sum_{i,j,a} \ell_i q_{ai} q_{aj} \ell_j}|\ell\rangle \\
&= (-1)^{\ell^T q^T q \ell}|\ell\rangle.
\end{aligned}
$$

The missing factor is therefore

$$
(-1)^{\ell^T q^T q \ell} q^{\frac{1}{2}\ell^T q^T q \ell} \prod_{i=1}^{r} s_i^{(q^T q \ell)_i}. \tag{7.17}
$$

Summing over monopole sectors (i.e. Verma modules built upon each $|\ell\rangle$), we finally obtain the index

$$
\begin{aligned}
\mathbb{I}_{V_{B,q}} = \frac{1}{(q)_\infty^{2r}} \sum_{\ell \in \mathbb{Z}^r} &(-1)^{\ell^T q^T q \ell} q^{\frac{1}{2}\ell^T q^T q \ell} \left[\prod_{i=1}^{r} s_i^{(q^T q \ell)_i}\right] \\
&\times \prod_{a=1}^{n} \left(q \prod_{i=1}^{r} (s_i q^{\ell_i})^{q_{ai}}, q \prod_{i=1}^{r} (s_i q^{\ell_i})^{-q_{ai}}; q\right)_\infty.
\end{aligned}
\tag{7.18}
$$

This is precisely of the form derived in [CDG20, Equation 7.45] up to some minor fugacity redefinitions discussed in [DGP18, Footnote 12]! Therefore $V_{B,q}$ is precisely the extension of $V(\widehat{\mathfrak{g}_*(q)})$ we sought out.

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