Title

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Publication Date

1981-12-01

REPORT NO.
UCB/SESM-81/08

STRUCTURAL ENGINEERING AND
STRUCTURAL MECHANICS

GLOBAL ANALYSIS METHODS FOR THE SOLUTION OF THE ELASTOPLASTIC AND VISCOPLASTIC DYNAMIC PROBLEMS

by

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DECEMBER 1981

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BERKELEY, CALIFORNIA

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ABSTRACT

The elastoplastic dynamic problem is first formulated in a functional form that facilitates the application of the techniques of nonlinear semigroup theory. Existence and uniqueness of the solution is then proved by showing that the equations of motion define a contraction semigroup in an adequate Hilbert space. Product formulas are discussed that are seen to result in computationally efficient algorithms. Finally, the approximation of solutions by means of linear and nonlinear viscoplastic models is studied.

July 8, 1982

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1. Introduction

A number of numerical techniques have been proposed for the approximation of the solution of the infinitesimal elasto-perfectly plastic dynamic problem. For instance, a method that has been widely used in the past [16] consists of solving a viscoplastic problem having the same elastic domain and a small viscosity. The viscoplastic solution is then observed to approach the elastoplastic one as the viscosity is made to tend to zero. More recently, a number of step-by-step algorithms have been proposed [6,14] that consist of solving, for every time step, an incremental elastic problem and subsequently projecting the solution so obtained onto the elastic domain by means of a suitable "return mapping". These algorithms are computationally very efficient as they bypass the need to repeatedly form and factorize the tangent stiffness matrix. These methods have been successfully applied in a number of applications, but a formal mathematical treatment as yet is unavailable in the literature. This paper attempts to provide some advance in this direction.

The nonlinear theory of semigroups, on the other hand, has experienced a great deal of progress in recent years, and has been successfully applied to a variety of problems in mathematical physics. It is the purpose of this paper to apply these techniques, is conjunction with convex analysis, to the study of the elastoplastic dynamic problem. This task is greatly facilitated by a functional formulation of the elastoplastic equations of motion. A preeminent role is played in this respect by the concept of subdifferential, introduced by Moreau in his pioneering work [8], and subsequently applied to the study of plastic and viscoplastic materials [9,10,11,12]. In Moreau's formalism, the plastic constitutive mapping is defined to be the subdifferential of the indicator function of a functional elastic domain, thus rendering applicable all the available results on semigroups generated by subdifferential operators [2]. The equations of motion resulting from this approach are set-valued, due to the set-valuedness of the plastic constitutive mapping. It is shown in Section 2, however, that the classical consistency condition of plasticity has the effect of restricting these equations of motion to their canonical restriction, thus rendering them single-valued. Existence and uniqueness of the solution are then proved by showing that the elastoplastic equations of motion define a contraction semigroup in a suitable Hilbert space.

In subsequent Sections, a number of approximation techniques are discussed within the framework of nonlinear semigroup theory, that provide a formal background for the aforementioned numerical methods. In Section 4, for instance, some results of convex analysis in Hilbert space are used to show that the return mapping algorithms can be identified with the classical product formulas of semigroup theory. In Section 5, the approximation of the elastoplastic solution by means of viscoplastic models is discussed. A suitable relaxation property for nonlinear viscoplasticity is formulated that insures convergence to the elastoplastic solution in the limit of small viscosities.

2. The Elastoplastic Equations of Motion

In this Section we formulate the elastoplastic dynamic problem in a functional form which is suitable for the application of the techniques of the theory of nonlinear semigroups. For instance, the distribution of plastic strain rates over the body is expressed as the subdifferential of the indicator function of a functional elastic domain. This leads to equations of motion that involve a set-valued operator, **A**. The consistency condition is then shown to restrict this operator to its canonical restriction $\mathbf{A}^o$, rendering the equations of motion single-valued.

Consider an elastoplastic body occupying a bounded open region Ω in R^N. Denote by $\mathbf{v}$ and σ the velocity and stress field over Ω, respectively. Then, the unforced equations of motion can be expressed as:

$$\frac{d\mathbf{v}}{dt} = \frac{1}{\rho}\nabla\cdot\sigma$$
$$\frac{d\sigma}{dt} = D\cdot(\nabla\mathbf{v} - \dot{\epsilon}^p) \qquad (1)$$

For simplicity, the boundary conditions will be taken to be homogeneous and of the Dirichlet type. Here, $\dot{\epsilon}^p$ denotes the plastic deformation rate, ρ and D the mass density and elastic compliance tensor of the material, respectively. The evolution of the system is assumed to take place in a Hilbert space $H = H_u \times H_\tau$, where H_u denotes some closed linear subspace of $L^2(\Omega)$ consisting of those velocity fields that satisfy the displacement boundary conditions, with the inner product

$$<\mathbf{v}_1,\mathbf{v}_2>_{H_u} = <\rho\mathbf{v}_1,\mathbf{v}_2>_{L^2} \qquad (2)$$

and H_τ, signifies the set of stress fields in $L^2(\Omega)$ with the inner product

$$<\sigma_1,\sigma_2>_{H_\tau} = <D^{-1}\cdot\sigma_1,\sigma_2>_{L^2} \qquad (3)$$

This endows H with the following inner product

$$<\begin{Bmatrix}\mathbf{v}_1\\ \sigma_1\end{Bmatrix},\begin{Bmatrix}\mathbf{v}_2\\ \sigma_2\end{Bmatrix}> = <\mathbf{v}_1,\mathbf{v}_2>_{H_u} + <\sigma_1,\sigma_2>_{H_\tau} = <\rho\mathbf{v}_1,\mathbf{v}_2>_{L^2} + <D^{-1}\cdot\sigma_1,\sigma_2>_{L^2} \qquad (4)$$

or energy inner product, which proves convenient for the application at hand. Since $\rho>0$ and D^{-1} is symmetric and positive definite, this inner product is equivalent to the usual L^2 product. The symbol $\frac{d}{dt}$ denotes the Frechet derivative of a function of time. In other words, we view eqs. (1) as ordinary differential eqs. in the (infinite-dimensional) vector space H.

In order to have a complete set of equations, one has to supplement eqs. (1) with some constitutive relations for $\dot{\epsilon}^p$. For an elastic-perfectly plastic material, the existence of a closed convex set C in stress space S, or elastic region, is usually assumed such that, at every point $\omega \in \Omega$

$$\dot{\epsilon}^{p}(\omega) = \lambda n \quad \text{if } \sigma(\omega) \in \partial C$$
$$\dot{\epsilon}^{p}(\omega) = 0 \quad \text{if } \sigma(\omega) \in Int(C) \tag{5}$$

where n is the normal to ∂C at $\sigma(\omega)$, $\lambda \geqslant 0$ but otherwise indeterminate, and $Int(C) \equiv C - \partial C$ denotes the interior of C. Let us formally express this relation as

$$D \cdot \dot{\epsilon}^{p}(\omega) \in T\sigma(\omega) \tag{6}$$

Thus, T: $S \rightarrow S$ is a nonlinear, set-valued mapping in stress space.

Note that the definition (5) of the plastic constitutive relations presupposes the existence of the normal $\mathbf{n}$ to ∂C, i.e., applies only to the case in which the elastic region has a smooth boundary. A more general definition of T that applies equally well to any closed convex elastic region can be obtained by introducing the concept of subdifferential.* To this end, let $< , >_S$ denote the following inner product in stress space

$$<\alpha,\beta>_S = \alpha \cdot D^{-1} \cdot \beta \qquad \alpha,\beta \in S \tag{7}$$

Let $\sigma \in C$ and $\beta \in S$. Then, we shall say that $\beta \in T\sigma$ if and only if $<\beta,\sigma - \gamma>_S \geqslant 0$, for all γ in the elastic domain C. Note that if $\sigma \in Int(C)$ then $T\sigma = \{0\}$, and that if the boundary of the elastic region happens to be smooth and $\sigma \in \partial C$ then the set $T\sigma$ consists of elements that are normal to ∂C and that point outside the elastic region. Also note that T is not defined for values of σ outside C. For a general closed convex elastic region, the set $T\sigma$, $\sigma \in \partial C$, defined above is a closed convex cone which is commonly termed the vertex of C at σ [15]. Any element of $T\sigma$ is then said to be a normal to C at σ. Furthermore, it can be shown [1,2] that T is the subdifferential of the indicator function I_C of C:

$$I_C(\sigma) = 0 \quad \textit{if } \sigma \textit{ belongs to } C$$
$$I_C(\sigma) = \infty \quad \textit{if } \sigma \textit{ does not belong to } C \tag{8}$$

By comparison with the simple result that the gradient of a smooth function is normal to its level contours, the fact that $T = \partial I_C$ can be again interpreted as a generalized statement of the

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* The subdifferential ∂f of a convex, lower semicontinuous function f:$H \rightarrow R$ is the set [8]

$$\partial f = \{(x,y) \in H \times H \text{ s.t. } f(u) \geqslant f(x) + <y, u-x>, \textit{ for all } u \in H\}$$

If f is Frechet-differentiable it turns out that $\partial f = Df \equiv$ Frechet-derivative of f.

normality rule (5).

The mapping T, when applied pointwise to a stress field $\sigma \in H_\tau$ (modulo sets of measure zero), defines a mapping from H_τ into itself that we shall denote $\mathbf{T}_\tau$. Let us also define the set $\mathbf{C}_\tau = \{\sigma \in H_\tau \ s.t. \ \sigma(\omega) \in C \ a.e. \ in \ \Omega\}$. This is clearly a closed convex subset of H_τ Then, it is clear that for stress fields such that $\sigma \in Int(\mathbf{C})$ the body behaves elastically as a whole. On the other hand, $\mathbf{T}_\tau\sigma$ gives the distribution of plastic strain rates over the body. It therefore makes sense to interpret $\mathbf{C}_\tau$ and $\mathbf{T}_\tau$ as some "functional" elastic domain and platic constitutive mapping, respectively. The following proposition shows that the normality rule carries over to the functional level.

Proposition For every $\sigma \in \mathbf{C}_\tau$, $\beta \in \mathbf{T}_\tau\sigma$, iff $<\beta,\sigma - \gamma>_{H_\tau} \geqslant 0$ for all $\gamma \in \mathbf{C}_\tau$, i.e., $\mathbf{T}_\tau = \partial I_{\mathbf{C}_\tau}$.

Proof: Let $\sigma \in \mathbf{C}_\tau$ and assume $\beta \in \mathbf{T}_\tau\sigma$. Then, by the definition of $\mathbf{T}_\tau$, $\beta(\omega) \in T\sigma(\omega)$ a.e. in Ω, i.e., $<\beta(\omega),\sigma(\omega) - \gamma(\omega)>_S \geqslant 0$, a.e. in Ω and for all $\gamma \in \mathbf{C}_\tau$. Hence, by the properties of the Lebesgue integral, $<\beta,\alpha - \gamma>_{H_\tau} \geqslant 0$.

Assume $\beta \in H_\tau$ is such that $<\beta,\sigma - \gamma>_{H_\tau} \geqslant 0$, for all $\gamma \in \mathbf{C}_\tau$ and that there exists a subset E of Ω of non-zero measure such that $<\beta(\omega),\sigma(\omega) - \gamma(\omega)>_S < 0$ for all $\omega \in E$. Denote by γ_E and σ_{CE} the restrictions of γ and σ to E and $\Omega - E$, respectively. Then, by the definition of $\mathbf{C}_\tau$, it is clear that $\bar{\gamma} = \gamma_E + \sigma_{CE} \in \mathbf{C}_\tau$. But now $<\beta,\sigma - \bar{\gamma}>_{H_\tau} < 0$, which contradicts the assumption. Hence, $<\beta(\omega),\sigma(\omega) - \gamma(\omega)>_S \geqslant 0$ a.e. and for all $\gamma \in \mathbf{C}_\tau$, and $\beta \in \mathbf{T}_\tau\sigma$.////

Let us introduce the notation $\mathbf{x} \equiv \begin{Bmatrix} \mathbf{v} \\ \sigma \end{Bmatrix} \in H$. The mapping $\mathbf{T}_\tau$ can be trivially extended to a mapping $\mathbf{T}$ from H into H by setting

$$\mathbf{T}\mathbf{x} = \begin{Bmatrix} 0 \\ \mathbf{T}_\tau\sigma \end{Bmatrix}, \ for \ all \ \mathbf{x} = \begin{Bmatrix} \mathbf{v} \\ \sigma \end{Bmatrix} \in H \tag{9}$$

If we now define $\mathbf{C} = H_u(\Omega) \times \mathbf{C}_\tau$, it is then apparent that $\mathbf{C}$ is a closed convex subset of H and $\mathbf{T} = \partial I_{\mathbf{C}}$. With this notation, the equations of motion (1) can be rephrased in the more compact fashion

$$\frac{d}{dt}\mathbf{x}(t) \in \mathbf{A}\mathbf{x}(t) \quad in\ H \tag{10}$$

where $\mathbf{A}$ is a nonlinear, set-valued operator on H defined as $\mathbf{A} = \mathbf{W} - \mathbf{T}$, with

$$\mathbf{W} \equiv \begin{bmatrix} 0 & \frac{1}{\rho}\nabla\cdot \\ D\cdot\nabla & 0 \end{bmatrix} \tag{11}$$

being the linear elasticity wave operator.

It is easily checked that, with the choice (4) of inner product for H, $\mathbf{W}$ is a skewadjoint operator, i.e., $\mathbf{W}^* = -\mathbf{W}$, and therefore, by Stone's theorem [13], it generates a unitary group in H which we shall denote by $S_{\mathbf{W}}(t)$.

It is also a well-known fact [13], that the domain of $\mathbf{W}$, say $\mathbf{D}(\mathbf{W})$, is $H_u^1(\Omega)\times H_\tau^1(\Omega)$, where $H_u^1(\Omega)$ $(H_\tau^1(\Omega))= \{\mathbf{f}\in H_u\ (H_\tau)\ s.t.\ \nabla\mathbf{f}\in H_u\ (H_\tau)\}$. Thus, $\mathbf{W}$ is densely defined in H, i.e., $\overline{\mathbf{D}(\mathbf{W})} = H$. On the other hand, the domain of $\mathbf{T}$, say $\mathbf{D}(\mathbf{T})$, is $\mathbf{C}$. Therefore it follows that the domain of $\mathbf{A}$ is $\mathbf{D}(\mathbf{A})= \mathbf{D}(\mathbf{W}) \cap \mathbf{D}(\mathbf{T})= \mathbf{D}(\mathbf{W}) \cap \mathbf{C}$, which is a convex subset of H whose closure $\overline{\mathbf{D}(\mathbf{A})} = \mathbf{C}$.

Set-valued operators, as in (10), are common place in the theory of nonlinear semigroups. Here, the set-valuedness of $\mathbf{A}$ arises from the indeterminacy of the plastic strain rates. This indeterminacy is resolved, in practice, by imposing the so called "consistency condition" on the stress rates. In stress space, this can be stated as

$$\begin{aligned} &\dot{\sigma}(\omega)\cdot n = 0 && \text{if } \sigma(\omega)\in\partial C \\ &\dot{\sigma}(\omega) \ \ unconstrained && \text{if } \sigma(\omega)\in Int(C) \end{aligned} \tag{12}$$

for every $\omega\in\Omega$, where n denotes the normal to ∂C at $\sigma(\omega)$. This condition insures that the stress path does not wander out of the elastic region. Substituting (1b) into (12) and making use of (5) the following value of λ is obtained

$$\lambda = \frac{n\cdot D\cdot\dot{\epsilon}(\omega)}{n\cdot D\cdot n} \tag{13}$$

which substituted into (1b) yields

$$\dot{\sigma}(\omega) = \left[D - \frac{(n\cdot D)(n\cdot D)}{n\cdot D\cdot n}\right]\cdot\dot{\epsilon}(\omega) \tag{14}$$

This formulation of the consistency condition is often too restrictive since it assumes smoothness of the elastic region. For the present discussion, the following alternative approach proves more convenient. It is easily checked that the plastic strain rate resulting from imposing the consistency condition is the element of $T\sigma(\omega)$ that minimizes the norm of $\dot{\sigma}(\omega)$

$$||\dot{\sigma}(\omega)||_S^2 = <\dot{\sigma}(\omega),\dot{\sigma}(\omega)>_S = (\dot{\epsilon}(\omega) - \lambda n)\cdot D\cdot(\dot{\epsilon}(\omega) - \lambda n) \equiv f(\lambda) \tag{15}$$

In fact,

$$f'(\lambda) = 0 \quad \textit{iff} \; 0 = -2n\cdot D\cdot(\dot{\epsilon}(\omega) - \lambda \mathbf{n}) = -2n\cdot\dot{\sigma}(\omega) \tag{16}$$

which is the consistency condition. Thus, the consistency condition can be alternatively formulated as follows: of all the elements of $T\sigma(\omega)$ in stress space pick the one that renders

$$<D\cdot\nabla v(\omega) - T\sigma(\omega), D\cdot\nabla v(\omega) - T\sigma(\omega)>_S \quad \textit{minimum} \tag{17}$$

This minimal element always exists and is unique due to the fact that $T\sigma(\omega)$ is a closed convex cone. Note also the validity of this definition regardless of the smoothness of ∂C. In the context of H_τ, the consistency condition (17) can be generalized to

$$||\dot{\boldsymbol{\sigma}}||_{H_\tau}^2 = <D\cdot(\nabla \mathbf{v} - \mathbf{T}_\tau\boldsymbol{\sigma}), D\cdot(\nabla\mathbf{v} - \mathbf{T}_\tau\boldsymbol{\sigma})>_{H_\tau} \quad \textit{minimum} \tag{18}$$

This uniquely determines an element of the closed convex cone $\mathbf{T}_\tau\boldsymbol{\sigma}$, which we denote by $\mathbf{T}_\tau^o\boldsymbol{\sigma}$. Clearly, $\mathbf{T}_\tau^o\boldsymbol{\sigma}$ is the projection of $D\cdot\nabla\mathbf{v}$ onto the vertex $\partial I_{\mathbf{C}_\tau}(\boldsymbol{\sigma}) \equiv \mathbf{T}_\tau\boldsymbol{\sigma}$ [15] and $\dot{\boldsymbol{\sigma}} = D\cdot\nabla\mathbf{v} - \mathbf{T}_\tau^o\boldsymbol{\sigma}$ is its orthogonal complement and it therefore belongs to the dual cone of $\partial I_{\mathbf{C}_\tau}(\boldsymbol{\sigma})$, i.e., to the support cone of $\mathbf{C}_\tau$ at $\boldsymbol{\sigma}$. Thus, (18) is in effect an abstract generalization of the consistency condition that constrains the stress rate field over the body $\dot{\boldsymbol{\sigma}}$ to have zero component onto the "normal" $\partial I_{\mathbf{C}_\tau}$ to the functional elastic region. On the other hand, recalling the result [15] that $\boldsymbol{\beta} \in H_\tau$ is an element of the support cone of $\mathbf{C}_\tau$ at $\boldsymbol{\sigma}$ if and only if there is a curve $\boldsymbol{\sigma}(t)$ contained in $\mathbf{C}_\tau$ such that

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma} + t\boldsymbol{\beta} + o(t) \quad \textit{as } t \to 0^+ \tag{19}$$

it becomes apparent that the abstract consistency condition also serves the purpose of insuring that the trajectory of the stresses do not come out of the functional elastic region.

The mapping $\mathbf{T}_\tau^o$ can be extended to a map $\mathbf{T}^o$ on H as in (9). Then, it is readily checked that the element $\mathbf{A}^o\mathbf{x} \equiv (\mathbf{W} - \mathbf{T}^o)\mathbf{x}$ is the element of minimum norm of $\mathbf{A}\mathbf{x}$. The (single-valued) mapping A^o that assigns to $x \in D(A) = D(A^o)$ the element of least norm in Ax is often referred to as the minimal section or the canonical restriction of A (see [3] for further properties). From our previous discussion it also follows that the operator A^o can be expressed as

$$A^o x = \Pi_C(x) W x \quad \text{for } all \ x \in D(A) \tag{20}$$

where $\Pi_C(x)$ denotes the projection onto the support cone of C at $x \in C$.

We see, therefore, that the consistency condition has the effect of restricting the (set-valued) operator A in (10) to its canonical restriction A^o. The equations of motion now read

$$\frac{d}{dt} x(t) = A^o x(t) \quad in\ H \tag{21}$$

Note that the inclusion sign in (10) can now be replaced by the equality sign, due to the single-valuedness of A^o.

Equation (21) may be regarded as an abstract expression of the equations of motion of an elastoplastic body. The rest of the paper is devoted to studying the properties of (21) within the framework or the nonlinear theory of semigroups.

3. Existence and Uniqueness of the Solution of the Elastoplastic Equations of Motion.

The question we next ask ourselves is whether the equations of motion (10) define a contraction semigroup $S(t)$, i.e., if there exists a one parametric family $S(t)$ of (nonlinear) mappings from some subset E of H into itself, with t taking values in R^+, such that

$$\begin{aligned} &a)\ S(t+s) = S(t)S(s) \\ &b)\ S(0) = I \equiv \textit{identity mapping} \\ &c)\ S(t)x \ \textit{continuous in } t \\ &d)\ \|S(t)x - S(t)y\| \leqslant \|x - y\| \\ &e)\ \lim_{t \to 0+} \frac{S(t)x - x}{t} \in Ax \end{aligned} \tag{22}$$

for all $t, s \in R^+$ and $x, y \in E$. A trajectory $x(t) = S(t)x$ in H is then termed a (strong) solution of (10) (in the sense of (22)) with initial value x.

Let us first recall some concepts from the theory of semigroups in Hilbert space. A set A in $H \times H$ is said to be monotone if $\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0$ for every (x_1, y_1), (x_2, y_2) in A. A monotone set which is not properly contained in any other monotone set is called maximal monotone. A monotone set A is maximal monotone if and only if the range condition $\mathbf{R}(\mathbf{I} + \lambda \mathbf{A}) = H$ is satisfied for every $\lambda > 0$, [3]. Let A be a maximal monotone set. A subset A′ of A is a principal section of of A if A′ is single-valued, $\mathbf{D}(\mathbf{A}') = \mathbf{D}(\mathbf{A})$ and A has the following property: If $\mathbf{x} \in \overline{\mathbf{D}(\mathbf{A})}$ and $< \mathbf{y} - \mathbf{A}'\mathbf{z}, \mathbf{x} - \mathbf{z} > \geqslant 0$ for every $\mathbf{z} \in \mathbf{D}(\mathbf{A})$, then $(\mathbf{x},\mathbf{y}) \in \mathbf{A}$. For instance, the canonical restriction $\mathbf{A}^o$ of a maximal monotone set A is a principal section of A, [3].

Monotone sets play an important role in the theory of nonlinear semigroups in Hilbert space. For instance, a classical result by Kōmura, [5], states that if $-\mathbf{A}$ is maximal monotone then A generates a semigroup of contractions $\mathbf{S}(t)$ on $\overline{\mathbf{D}(\mathbf{A})}$. In fact, this semigroup satisfies the stronger equation in which the operator A is replaced by its canonical restriction $\mathbf{A}^o$, [3,4]. In terms of the problem at hand, this implies that the solutions of the multi-valued elastoplastic equations of motion (10) of necessity satisfy the consistency condition and are, therefore, solutions of the restricted equations of motion (21).

We next show that the negative of the elastoplastic operator A in (10) is in fact maximal monotone. For this, we first note that I_C is a proper, convex lower semicontinuous function on H. Therefore, its subdifferential T is maximal monotone [7]. The fact that $-\mathbf{A} = \mathbf{T} - \mathbf{W}$ is maximal monotone then follows immediately from a result by Crandall and Pazy [4] that states that given two maximal monotone operators A and B such that the intersection $(Int\, \mathbf{D}(\mathbf{A})) \cap \mathbf{D}(\mathbf{B})$ is not empty, the operator $\mathbf{A} + \mathbf{B}$ is maximal monotone. Thus, $-\mathbf{A}$ is the unique maximal extension of the restriction of the negative linear elastic wave operator $-\mathbf{W}$ to C [1]. This illustrates the fact that perfect plasticity is the only way of restricting the linear elasticity wave operator to a closed convex set of admissible stresses retaining the maximal monotonicity property. Finally, the existence and uniqueness of the solutions of (10) and (21) (in the semigroup sense), for any initial conditions in C, follows from the aforementioned result by Kōmura.

4. The Approximation of the Solutions of the Elastoplastic Equations of Motion by Means of Step-by-Step Algorithms

An important aspect of the theory of semigroups is that of the approximation of the solutions of equations of evolution such as (10). This problem can be approached in a variety of different ways. For instance, one can attempt to replace the operator A under consideration by a one parametric family of operators A_λ, with $\lambda>0$, say, that lead to equations that are more easily solvable, and whose solutions $x_\lambda(t)$ tend to the solution $x(t)$ corresponding to the operator A as $\lambda \to 0$. We shall return to this question in the next Section. Here, a different approach will be pursued. The idea is to approximate the solution by means of a step-by-step integration scheme, in a manner which is reminiscent of the numerical techniques which are commonly used to solve ordinary differential equations.

An important result in this direction that illustrates also the nature of this approach is one due to Brézis and Pazy, [1], that states that given a maximal monotone set A in $H \times H$ generating a semigroup $S(t)$, and a family of contractions $F(t)$ in $\overline{D(A)}$, for every $t>0$, if

$$\lim_{t\to 0} \frac{F(t)x - x}{t} = A'x \quad \text{for } \textit{every } x \in D(A) \tag{25}$$

where A' is a principal section of A, then

$$\lim_{n\to\infty} \left[F(t/n)\right]^n x = S(t)x \quad \text{for } \textit{every } x \in \overline{D(A)} \text{ and } t \geqslant 0 \tag{26}$$

where the limit is uniform in t on every bounded interval. Note that (25) implies that $F(0)$ be equal to I. One such family of contractions $F(t)$ is commonly referred to as an "algorithm" (for A). The exponential formula in (26), can be viewed as an abstract representation of the step-by-step integration procedure, in which the algorithm $F(t)$ is repeatedly applied n times to the initial value x, with a time step $h = \frac{t}{n}$, in order to approximate the value of the solution at time t. With this terminology, the theorem states that the approximate solution tends to the exact one uniformly as the time step in made to tend to 0.

We next turn our attention to the elastoplastic equations of motion and discuss some useful algorithms that can be used to approximate the solution. Let us denote by $P_C{:}H \to C$ the "closest point" mapping onto $\mathbf{C}$. The existence and uniqueness of $P_C x$ for all $x \in H$ is a well-known fact in the geometry of Hilbert space. Moreover, $\mathbf{C}$ is the set of fixed points of P_C, and so $\mathbf{C}$ and P_C determine each other. Thus, $P_C^2 = P_C$, i.e., P_C is a nonlinear projection.

A deeper fact about projections onto closed convex sets in Hilbert space is due to Zarantonello [15] and states that for every $x \in \mathbf{C}$ the directional derivative of $P_C(x)$ along $y \in H$ is given by

$$DP_C(x;y) \equiv \frac{d}{d\epsilon}\Big(P_C(x + \epsilon\, y)\Big)_{\epsilon=0} = \Pi_C(x)y \quad \text{for } every\ y \in H \tag{27}$$

In other words, given a differentiable curve $x(t)$ in H, with $x(0) = x \in \mathbf{C}$ and $\dot{x}(0) = \dot{x}$, it follows that

$$\left[\frac{d}{dt} P_C x(t)\right]_{t=0} = \Pi_C(x)\dot{x} \tag{28}$$

Consider now the algorithm

$$F(t) = P_C S_W(t) \tag{29}$$

Clearly, $F(t)$ maps $\mathbf{C}$ into itself. Furthermore, it is easy to check that this algorithm is a contraction. For this, we first note [15] that P_C is a non-expansive retraction, i.e., $||P_C x - P_C y|| \leqslant ||x - y||$, for every $x, y \in H$. Therefore, recalling that the group $S_W(t)$ is unitary, we obtain

$$\begin{aligned} &||F(t)x - F(t)y|| = ||P_C S_W(t)x - P_C S_W(t)y|| \leqslant \\ &||S_W(t)x - S_W(t)y|| = ||S_W(t)(x - y)|| = ||x - y|| \end{aligned} \tag{30}$$

for all $x, y \in H$, i.e., $F(t)$ defines a contraction from $\mathbf{C}$ into itself. On the other hand, recalling [13] that the curve $S_W(t)x$ is differentiable for every $x \in D(W)$, and making use of (20) and (28), it follows that

$$\lim_{t \to 0+} \frac{F(t)x - x}{t} = \lim_{t \to 0+} \frac{P_C S_W(t)x - P_C S_W(0)x}{t} = \Pi_C(x)Wx = A^o x \tag{31}$$

for every $x \in \mathbf{C} \bigcap D(W) = D(A)$. Therefore, by the aforementioned result by Br*'ezis and Pazy [1] it follows that

$$\lim_{n \to \infty}\Big[P_C S_W(t/n)\Big]^n x = S(t)x \quad \text{for } every\ x \in \overline{D(A)} = \mathbf{C} \tag{32}$$

where the convergence is uniform in t on bounded intervals.

This result, spelled out in more detail, asserts that the solution of the elastoplastic equations of motion can be uniformly approximated by means of a step-by-step integration scheme, in which at every time step one solves an incremental linear elastic problem the result of which is then projected onto the closest point in the elastic domain. Naturally, if the result of the linear elastic problem is within the elastic domain, the projection does not alter it.

The result expressed in (32) can be useful in cases in which a close form linear elastic solution is available. However, in most applications this is not the case, and the linear elastic problem also requires a numerical treatment. Thus, it would be of practical interest to know whether the same result holds true upon replacing $\mathbf{S}_W(t)$ in (32) by an algorithm $\mathbf{F}_W(t)$ for the linear elastic wave operator. To this effect, let us define now the algorithm $\mathbf{F}(t)$ as

$$\mathbf{F}(t) = \mathbf{P}_C\mathbf{F}_W(t) \tag{33}$$

where $\mathbf{F}_W(t)$ is a contraction such that the curve $\mathbf{x}(t) = \mathbf{F}_W(t)\mathbf{x}$ is differentiable, for every $\mathbf{x} \in \mathbf{D}(\mathbf{W})$, and such that consistency with $\mathbf{W}$ is satisfied in the sense expressed in (25). Then, the fact that $\mathbf{F}(t)$ is a contraction follows as in (30). Moreover, by (20) and (28)

$$\lim_{t \to 0} \frac{\mathbf{F}(t)\mathbf{x} - \mathbf{x}}{t} = \lim_{t \to 0} \frac{\mathbf{P}_C\mathbf{F}_W(t)\mathbf{x} - \mathbf{P}_C\mathbf{F}_W(0)\mathbf{x}}{t} = \Pi_C(\mathbf{x})\mathbf{W}\mathbf{x} = \mathbf{A}^o\mathbf{x} \tag{34}$$

for every $\mathbf{x} \in \mathbf{C} \cap \mathbf{D}(\mathbf{W})$. Hence, the following exponential formula

$$\lim_{n \to \infty} \left[\mathbf{P}_C\mathbf{F}_W(t/n)\right]^n \mathbf{x} = \mathbf{S}(t)\mathbf{x} \quad \textit{for every } \mathbf{x} \in \mathbf{C} \tag{35}$$

also holds in this case.

It is interesting to note that, from a numerical point of view, the practicality or this method lies in the fact that it bypasses completely the need to compute the tangent stiffness at every time step. In fact, only the linear elastic stiffness matrix needs to be formed, and all the necessary matrix manipulations can be carried out, once and for all, at the beginning of the integration process.

As a specific example, let us consider the case where $\mathbf{F}_W(t)$ is chosen to be the resolvent of $\mathbf{W}$, i.e., the mapping $\mathbf{J}_t^{\mathbf{W}} \equiv (\mathbf{I} + t\mathbf{W})^{-1}$. In finite dimensions, this would correspond to the use of a fully implicit algorithm for $\mathbf{W}$. It is a well-known fact [3] that the resolvent of a monotone operator A is a contraction with domain $\mathbf{D}(J_t^A) = \mathbf{R}(I + tA)$. Therefore, if A is maximal monotone it follows that $\mathbf{D}(J_t^A) = H$ i.e., the resolvent is everywhere defined. The divided difference operator

$$A_t = \frac{I - J_t^A}{t} \qquad t>0 \tag{36}$$

is, in this case, the so called "Yoshida approximation" to the operator A. It is also a well-known fact [3] that the Yoshida approximation satisfies $\lim_{t\to 0^+} A_t x = A^o x$, and, therefore, the resolvent mapping J_t^A is consistent with A in the sense (25). Thus, the product formula (35) does hold in this case. In fact, this particular case can be proven directly by using a result by Brézis and Pazy [1] stating that given two maximal monotone sets, A and B, such that $\overline{A + B}$ is also maximal monotone, and if $S_{\overline{A+B}}(t)$ is the semigroup generated by $\overline{A + B}$, then

$$S_{\overline{A+B}}x = \lim_{n\to\infty}\left[\left(I + \frac{t}{n}A\right)^{-1}\left(I + \frac{t}{n}B\right)^{-1}\right]^n x = \lim_{n\to\infty} J_{t/n}^A J_{t/n}^B x \tag{37}$$

for every $x \in \overline{D(A) \cap D(B)}$, and the limit is uniform in t on every bounded interval. Now, eq. (35) follows directly from (36) by substituting W for B, T for A and noting [2] that the resolvent of $T = \partial I_C$ is precisely P_C.

The closest point projection P_C arises naturally as a return mapping in a product formula like (35) as a result of the fact that it is the resolvent of the plastic constitutive mapping T. Nevertheless, this is by no means the only possible choice of a return mapping. Consider a contraction $\tilde{P}_C$ with a closed convex domain $D(\tilde{P}_C)$ properly containing the elastic region C, such that its range and set of fixed points is C, and satisfying the directional derivative condition (28). In this setting, this condition can be interpreted as the requirement that $\tilde{P}_C$ behave asymptotically as the closest point projection onto C, as one approaches the elastic domain. By exactly the same arguments leading to product formula (35), it can be now readily checked that

$$\lim_{n\to\infty}\left[\tilde{P}_C S_W(t/n)\right]^n x = S(t)x \quad \text{for } \textit{every } x \in C \tag{38}$$

where the convergence is uniform in t on bounded intervals. Eq. (38) shows that a broad class of return mappings $\tilde{P}_C$ can in fact be used to approximate the elastoplastic solution, as has been suggested in numerical applications [6,14].

5. The Viscoplastic Equations of Motion and the Viscoplastic Approximation to the Elastoplastic Equations of Motion

In this Section, we turn our attention to an alternative method of approximation. The idea is to replace the elastoplastic operator A by a family of related and more tractable operators, say A_λ that yield solutions that approximate the elastoplastic one as λ is made to tend to 0.

A commonly used approximation that has been thoroughly studied in the past, (for a collection of basic results, see [3]) is the Yoshida approximation, defined in (36). In the case of the plastic constitutive mapping it follows that J_λ^T, P_C= and the Yoshida approximation reads

$$T_\lambda = \frac{I - J_\lambda^T}{\lambda} = \frac{I - P_C}{\lambda} \quad \lambda > 0 \tag{39}$$

Thus, in this case, the Yoshida approximation T_λ coincides with the linear viscoplastic constitutive mapping, with elastic region C and viscosity λ.

The equations of motion of a linear viscoplastic material can therefore be expressed as

$$\frac{d}{dt}x(t) = A_\lambda x(t) \tag{40}$$

where $A_\lambda = W - T_\lambda$. This equation, apart from presenting some interest of its own, has been frequently used in practice to approximate the solutions of the corresponding elastoplastic problem, due to its computational advantage that it only requires the elastic stiffness matrix and bypasses the need of computing the tangent stiffness matrix [16]. In this context, eq. (40) is sometimes referred to as the linear viscoplastic approximation to (10), and, the method of solution, the penalty method.

From the properties of the Yoshida approximation [3], it follows that T_λ is a Lipschitz mapping with constant λ^{-1}, it is monotone and $D(A_\lambda)= D(J_\lambda^T)= D(P_C) = H$. Recalling [2] that the mapping $I - P_C$ satisfies

$$(I - P_C)x = \partial\left[\frac{1}{2}||(I - P_C)x||^2\right] \quad \textit{for every } x \in H \tag{41}$$

it then follows from (39) that

$$\mathbf{T}_\lambda = \partial\left[\frac{1}{2\lambda}||(\mathbf{I} - \mathbf{P}_C)\mathbf{x}||^2\right] \equiv \partial\psi_\lambda(\mathbf{x}) \quad \textit{for every } \mathbf{x} \in H \tag{42}$$

i.e., the linear viscoplastic constitutive mapping $\mathbf{T}_\lambda$ is the subdifferential of the function ψ_λ. By exactly the same arguments stated for the alasto-perfectly plastic case, Section 3, it follows here that the operator $-\mathbf{A}_\lambda = \mathbf{T}_\lambda - \mathbf{W}$ is maximal monotone. Therefore, it generates a semigroup of contractions in H, by Kōmura's theorem, that we shall denote $\mathbf{S}_\lambda(t)$. This proves the existence and uniqueness of the solution of the linear viscoplastic equations of motion. The same result follows directly from a theorem by Crandall and Pazy [3] that states that if $\mathbf{A}$ is maximal monotone and $\mathbf{B}$ is monotone and Lipschitz continuous, then $\mathbf{A} + \mathbf{B}$ is maximal monotone.

We next turn our attention to the problem of whether, in fact, the solution of the linear viscoplastic problem approximates the solution of the elastoplastic one, as the viscosity tends to zero. From the fact that the operator $\mathbf{T}_\lambda$ is the subdifferential of the proper convex, lower semicontinuous function ψ_λ defined in (42), we conclude that it is maximal monotone. Therefore, $-\mathbf{T}_\lambda$ generates a contraction semigroup in H, say $\mathbf{S}_{\mathbf{T}_\lambda}(t)$, by Kōmura's theorem. In fact, it is a simple matter to write down explicitly an expression for $\mathbf{S}_{\mathbf{T}_\lambda}(t)$. To this effect, let us recall the fact that projections onto closed convex sets in Hilbert space are "sunny projections", i.e., if $\mathbf{P}_C\mathbf{x} = \mathbf{z}$ and $\mathbf{x},\mathbf{y}$ and $\mathbf{z}$ are collinear, then $\mathbf{P}_C\mathbf{y} = \mathbf{z}$. Using this fact, it can be readily checked that

$$\mathbf{S}_{\mathbf{T}_\lambda}(t)\mathbf{x} = \exp\left(-\frac{t}{\lambda}\right)\mathbf{x} + \left[1 - \exp\left(-\frac{t}{\lambda}\right)\right]\mathbf{P}_C\mathbf{x} \quad \textit{for every } \mathbf{x} \in H \tag{44}$$

Note that

$$\mathbf{S}_{\mathbf{T}_\lambda}(t)\mathbf{x} \rightarrow \mathbf{P}_C\mathbf{x} \quad \textit{as}\ \lambda \rightarrow 0, \qquad t>0 \tag{45}$$

where the convergence is actually uniform in t on every bounded interval.

On the other hand, by a nonlinear version of the Trotter-Neveu-Kato theorem proved by Brézis and Pazy [1], the following exponential formula holds

$$\mathbf{S}_\lambda(t)\mathbf{x} = \lim_{n\rightarrow\infty}\left[\mathbf{S}_{T_\lambda}(t/n)\mathbf{S}_\mathbf{W}(t/n)\right]^n\mathbf{x} \quad \textit{for every } \mathbf{x} \in H \tag{46}$$

where the limit is uniform in t on every bounded interval. Taking the limit as $\lambda \rightarrow 0$, we obtain

$$\lim_{\lambda \to 0} S_\lambda(t)\mathbf{x} = \lim_{\lambda \to 0}\lim_{n \to \infty}\left[S_{T_\lambda}(t/n)S_W(t/n)\right]^n \mathbf{x} \tag{47}$$

But since the limits are uniform, their order can be interchanged, which leads to

$$\begin{aligned} \lim_{\lambda \to 0} S_\lambda(t)\mathbf{x} &= \lim_{n \to \infty}\lim_{\lambda \to 0}\left[S_{T_\lambda}(t/n)S_W(t/n)\right]^n \mathbf{x} = \\ \lim_{n \to \infty}\left[\mathbf{P}_C S_W(t/n)\right]^n \mathbf{x} &= S(t)\mathbf{x} \quad \textit{for every } \mathbf{x} \in \mathbf{C} \end{aligned} \tag{48}$$

by (32) and (45), where the convergence is uniform in t on every bounded interval.

The exponential formula (46) indicates that the viscoplastic solution can be approximated by means of a step-by-step integration in which one solves an incremental linear elastic problem over some time step, the solution so obtained being then allowed to relax according to the viscoplastic constitutive law during the same amount of time. Physically, eq. (48) then shows that the convergence of the linear viscoplastic solution to the elastoplastic one is due to the fact that the effect of the viscoplastic relaxation on the incremental elastic solutions becomes closer and closer to a projection onto the elastic domain, as in (32), as the viscosity tends to zero (relaxation property).

Although so far the results in this Section have been obtained for the linear viscoplastic approximation, it is a straightforward matter to state conditions under which the same results carry over to a general nonlinear viscoplastic approximation. Consider a return mapping $\tilde{\mathbf{P}}_C$ as defined in Section 4. We then define a viscoplastic approximation to $\mathbf{T}$ as a one parameter family of single-valued, monotone, Lipschitz continuous mappings $\tilde{\mathbf{T}}_\lambda$, $\lambda > 0$, with domains $\mathbf{D}(\tilde{\mathbf{T}}_\lambda)$ containing $\mathbf{D}(\tilde{\mathbf{P}}_C)$, and generating a family of contraction semigroups, $S_{\tilde{T}_\lambda}(t)$, and such that

$$\lim_{\lambda \to 0} S_{\tilde{T}_\lambda}(t)\mathbf{x} \to \tilde{\mathbf{P}}_C\mathbf{x}, \quad t>0 \quad \textit{for every } \mathbf{x} \in \mathbf{D}(\tilde{\mathbf{P}}_C) \tag{49}$$

uniformly in t for every bounded interval (relaxation property). By repeating the same arguments as in the linear viscoplastic approximation case, it is readily shown that the operator $\tilde{\mathbf{A}}_\lambda = \mathbf{W} - \tilde{\mathbf{T}}_\lambda$ generates a contraction semigroup that uniformly approximates the solutions of the elastoplastic problem, as in (48).

Acknowledgements

We are indebted to Prof. J. Lubliner for helpful discussions. The results reported here were prepared with financial assistance from the National Science Foundation, Grant PFR-7908984 and by grants from the Lawrence Livermore National Laboratory and General Motors Research Laboratories.

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