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Weak Convergence and Rapidly Oscillating Pendula

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Abstract

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The ordinary differential equation for the motion of an inverted pendulum whose support is subjected to a vertical harmonic excitation with high frequency and small amplitude is studied. A mathematical insight is provided as asymptotic methods are employed to derive the limiting equation when the excitation frequency tends to infinity. These show that as the frequency becomes high and certain parametric conditions are satisfied, the inverted pendulum theoretically can be stabilized. A new and rigorous proof using weak convergence methods is given, to confirm the conclusions of the formal asymptotics. Then the analysis is generalized to a single inverted pendulum on an oscillatory base driven at other angles, a single inverted pendulum on a flywheel and a double inverted pendulum on a vertically oscillatory base and, in each case, if the pendulum or pendula can be stabilized, parametric conditions for stability are given.

Then we also use this method to shed light on the Paul Trap which is a method of levitating charged particles via an oscillating electric field. Last, PDE analogues are studied in the same manner.

Contents

CHAPTER 1

Introduction

1. Overview

1.1. Inverted pendula. A pendulum, if set upside down, cannot stand upright and will fall immediately. The inverted pendulum, which is a regular pendulum put upside down, is a widely used example in classical mechanics textbooks. Especially, an inverted pendulum whose pivot is subjected to small periodic (most often sinusoidal) oscillations of high frequency is commonly used an example of non-inertial reference frames and of the Lagrange equation in Lagrangian mechanics.

As is fairly well-known, an inverted pendulum can be stabilized if its pivot is simply oscillating periodically in certain patterns, for example, a fast enough vertical sinusoidal oscillation with a suitably large amplitude. There are numerous demonstrations in scientific literature. The most significant early contributions were made by Andrew Stephenson and Pyotr Kapitza.

Andrew Stephenson is believed to be the first to present this phenomenon in literature in his two papers [**Ste07**] and [**Ste08**] almost a century ago. He made theoretical predictions that an inverted pendulum or even “a number of freely jointed links” can be ”rendered stable by rapid vertical vibration of the point of support” when the frequency of vibration is higher than some threshold Stephenson’s work had been rather overlooked and no further explanation was given until 1951 when Russian Nobel laureate physicist Pyotr Kapitza physically explained the stabilization in terms of effective potentials in [**Kap65**] and [**Kap65**]. Kapitza’s work drew many researchers’ attention and consequently enormous research has been done on this intriguing phenomenon since then. Because of the significance of Stephenson and Kapitza’s work, the inverted pendulum is sometimes called Kapitza pendulum or Stephenson-Kapitza pendulum.

The oscillation of the pivot that has a stabilizing effect is, of course, not limited to a vertical sinusoidal type, as one would guess. For example, H. P. Kalmus in [**Kal70**] gave an illustration of the case when the suspension point is subjected to short impulses as a means to provide a more intuitive understanding of the phenomenon; A. B. Pippard showed in [**Pip87**] that stability conditions can also be obtained by replacing the sinusoidal motion of the support by a sawtooth or a parabolic approximation; A paper [**CSH**] by Y. Cheng, S. Shapero and R. Hayward also studied the stability of an inverted pendulum subjected to Jacobi elliptic forcing numerically and experimentally (with damping).

A much more mathematical treatment in [**Mit72**] by R. Mitchell in 1972 studied in detail more general base motions including almost periodic and stochastic ones and gave computer simulation for each case as well. While theoretical stability conditions were obtained for the case of sinusoidal and almost periodic base motions, similars conditions were predicted in computer simulation for stochastic base motion. The primary tool employed by the R. Mitchell in [**Mit72**] was the method of averaging seen in N. N. Bogoliubov, Y. A. Mitropolsky's [**BM61**], P. R. Sethna's [**Set67**], Y. A. Mitropolsky's [**Mit67**] and [**HS68**] by G. W. Hemp and P. R. Sethna.

Another approach is found in Mark Levi's 1988 paper [**Lev88**]. He showed the stability of the inverted pendulum on a vertically oscillatory base from a topological point of view and thus gave stability conditions qualitatively.

Blackburn, Smith and Gronbech Jensen, in 1992, summarized the two theoretical approaches to the problem in their paper [**BSGJ92**]: an approximation of the equation of motion by the Mathieu equation which was well-studied in literature, and a phenomenological model based on an effective potential roughly introduced by Landau and Lifshitz in [**LL76**], but both only on the fast time scale.

Acheson, in 1993, gave a general argument for the stability conditions for an inverted N-pendulum in [**Ach93**]. In his approach, he put the governing equations in terms of normal coordinates and thus reduced them also to Mathieu equations. Then the stability condition for the N-pendulum is equivalent to the parameters of each of the Mathieu equation lying in their stability region simultaneously.

In 2001, Eugene I. Butikov gave in [**But01**] a rather simple and effective explanation of the stability of the inverted pendulum on either a vertically oscillatory base or a horizontally oscillatory base by analyzing the torque of the force of inertia averaged over a period of oscillations which also yields the stability condition quantitatively.

Gustavo. J. Mata and E. Pestana in their 2004 paper [**MP04**] gave a rather neat alternative approach of obtaining the stability condition of the single inverted pendulum by averaging the fast time dependence of the Hamiltonian to obtain an effective Hamiltonian and thus finding the condition under which the effective Hamiltonian has a minimum when the inverted pendulum is at the upright position.

Although not a focus of our work, it is worth pointing out that, aside from stability, the dynamic behavior of the inverted pendulum is extremely complex. For some of the works on this, we refer the reader to [**But01**] and [**But05**] both by Eugene I. Butikov and [**YMA04**] by H. Yabuno, M. Miura and N. Aoshima.

We have seen so far that, among many effective approaches, the Mathieu equation is used as a primary tool in existing literature on the inverted pendulum. The canonical form for the Mathieu differential equation is

$$\frac{d^2y}{dx^2} + [a - 2q\cos(2x)]y = 0. \tag{1.1}$$

While analytical solutions of the Mathieu equation are given in power series and rather complicated to express, its stability region is well studied in literature.

Mathieu first solved this equation in 1868 using, among other methods, doubly periodic series, by transforming the governing equation for the inverted pendulum into the Mathieu equation, an analytical solution in terms of power series was given in Phelps and Hunter's 1965 paper [**PIHJ65**]. Besides the inverted pendulum on a vertically oscillatory base, the paper also included the cases when the base is subjected to a horizontal oscillation and attached to the rim of a flywheel which can then be easily obtained. Derivation of the governing equation in each case in elementary principles has also been given in [**Bli65**] by L. Blitzer.

We discussed the aforementioned cases in our study too as further applications of our method and obtained interesting stability results.

1.2. This thesis. The main interest here is to study the asymptotic behavior of the inverted pendulum as the frequency tends to infinity. The main result that the ordinary differential equation governing the motion of the inverted pendulum has a limiting equation only in the slow variable is not new in literature. For example, It has been introduced by Landau and Lifchitz in [**LL76**]. Eugene I. Butikov has obtained similar results by analyzing the averaged torque of the force of inertia over a period of oscillations in [**But01**]. However, no mathematical insight has been given in the framework of asymptotic analysis and weak convergence.

Our method here is essentially different from the major approach we have seen so far in literature which treats the governing equation as a Mathieu equation on the fast time scale and readily predicts stability conditions from stability diagram of the Mathieu equation. While this approach gives stability conditions with a high precision and is useful for choosing parameters practically in experiments. Our analysis studies the asymptotic behavior and predicts stability from the limiting equation on the slow time scale, which is a clearer and deeper mathematical explanation that fits in a much larger picture.

In the work presented here, the equation will be studied using asymptotic expansions with multiple scales (see A. Bensoussan, J.- L. Lions and G. Papanicolaoun's book [**BLP78**] for reference) and the convergence of the solution is proved rigorously. We point out that the application of asymptotic expansions with multiple scales to equations of inverted pendula has been used by I. M. Arkhipova, A. Luongo and A. P. Seyranian in [**ALS12**] in the study of stability of double inverted pendula although on different time scales than ours and without giving a proof for convergence results.

1.3. The Paul trap. The equation for the inverted pendulum can also be used to model what is called the 'Paul trap'.

An ion trap is a device that uses a combination of electric or magnetic fields to trap electrically charged particles. One the most common types of ion traps is the 'Paul trap' (3D quadrupole ion trap). The Paul trap was invented by Wolfgang Paul who shared the Nobel Prize in Physics in 1989 for this work. (See [**OPF58**] by O. Osberghaus, W. Paul, W and E. Fischer and [**Pau90**] by W. Paul for reference.)

In short, the Paul trap uses a combination of fast oscillating radio frequency and static electric fields. The electrostatic potential in the Paul trap is a harmonic function and has no minima. Thus no possible stable equilibria can be obtained in the electrostatic field. However, when a high frequency oscillation is applied, a minimum can be produced for the effective potential. (See [**Lev99**] and [**Lev05**] by M. Levi for reference.)

Motion of particles inside the Paul trap is described with Mathieu equation and thus is analogous to inverted pendula. In [**Lev99**] and [**Lev05**], Mark Levi used a method of averaging to study the mechanics of the Paul trap and the stability of inverted pendula. We will herein apply asymptotic expansions with multiple scales to study the convergence of the governing equations of the motion of particles in the Paul trap to give an alternative way of understanding this effect.

2. Outline

Chapter 2 is dedicated to single inverted pendula. Section 1.1 gives the complete derivation of the equation of motion of a single inverted pendulum on a vertically oscillatory base; section 1.2 uses asymptotic analysis with multiple scales to formally derive the limiting equation; section 1.3 then gives a rigorous proof for the convergence result and discusses the stability condition. This is the central method that is employed in this dissertation. Section 2-4 largely repeat the same procedure and approach for a single inverted pendulum on a horizontally oscillatory base, an oscillatory base driven at arbitrary angles and on a flywheel respectively. Section 5 studies the limiting equation and stability of double inverted pendula but only with a rigorous proof that is analogous to that of section 2-4. Section 6 follows the same scheme but is dedicated to a completely different case, the Paul trap in nuclear physics.

Chapter 3 then picks up the scheme of chapter 2. Section 1 and section 2 study a PDE and a system of PDE analogues, respectively.

CHAPTER 2

Inverted pendula

1. Single inverted pendulum on a vertically oscillatory base

1.1. Derivation of equation. We start with a regular ideal undamped pendulum and derive our main equation step by step. A mass m is attached to a weightless rigid rod of length l. Let θ be the angular coordinate of m measured counterclockwise from the downward position as shown in Figure 1.

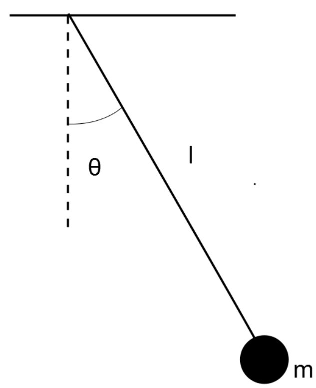

FIGURE 1. Pendulum

The kinetic energy T is thus given by

$$T = \frac{1}{2} m l^2 (\theta')^2,$$

where $'=d/dt$ denotes the time derivative. and potential energy V given by

$$V = -mgl\cos\theta.$$

The Lagrangian is $L = T - V$ and the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \frac{\partial L}{\partial \theta} = 0$$

reads

$$\theta'' + \frac{g}{l}\sin\theta = 0. \tag{1.1}$$

Equation (1.1) is of the form

$$\theta'' + \Phi'(\theta) = 0$$

for a potential function $\Phi(z) = -\cos z$. $\Phi''(0) > 0$ and thus the downward position, $\theta = 0$, is stable.

We next turn this pendulum into an invented one, that is, we put it upside down. The pendulum becomes unstable at the upward position. We let θ be the angular coordinate of m measured clockwise from the upward position as shown in Figure 2.

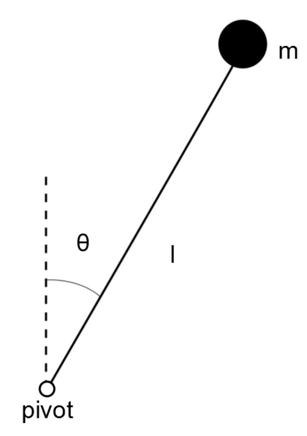

FIGURE 2. Inverted pendulum

Similarly to the previous case, the kinetic energy is given by $T = 1/2ml^2(\theta')^2$ and potential energy $V = mgl\cos\theta$. So the Lagrange equation writes

$$\theta'' - \frac{g}{l}\sin\theta = 0. \tag{1.2}$$

Similar analysis to the previous case shows that the upward position, also at $\theta = 0$, is unstable as the sign of the potential function is reversed. However, if put on an periodically oscillatory base, for appropriate frequencies and amplitudes, the inverted pendulum can be stabilized at the upward position as shown in enormous experiments in literature.

We now derive the governing equation for the motion of the inverted pendulum with a vertically oscillatory pivot. Set up the coordinate axes as shown in Figure 3. Let the pivot of the inverted pendulum be subjected to a vertical sinusoidal oscillation given by $y_\varepsilon = -\varepsilon bl\cos(t/\varepsilon)$ for some small positive parameter ε, so here εb is the dimensionless amplitude. Changing the sign of b has the same effect as shifting time by π and, as we will see later, does not change the limiting equation.

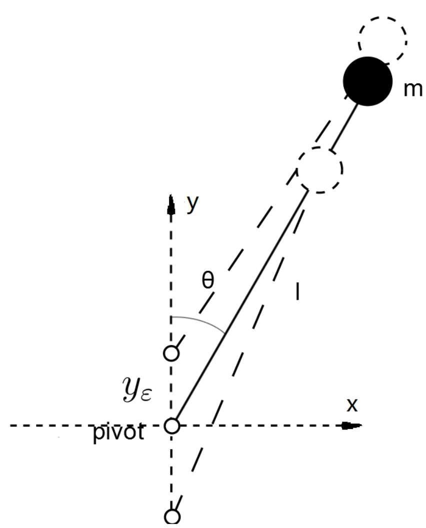

FIGURE 3. Inverted pendulum on a vertical sinusoidal oscillatory base

Different ways of derivations of equation for different types of inverted pendula can be seen in, for example, L. Blitzer's paper [**Bli65**], J. L.Trueba, J. P. Baltanas and M.A.F. Sanjuan's paper [**TBS03**] and G. J. VanDelan's paper [**Van04**]. We will use the method G. J. VanDelan used to derive the governing equation. General procedures for derivation of equations can be found in many classic mechanics textbooks, for example, [**Gre06**] by R. Douglas Gregory.

Let the position of the center of the mass m be (x, y). Then we have

$$x = l \sin\theta,$$
$$y = y_\varepsilon + l\cos\theta = -\varepsilon bl \cos\frac{t}{\varepsilon} + l\cos\theta,$$

and

$$x' = l\cos\theta\theta',$$
$$y' = bl\sin\frac{t}{\varepsilon} - l\sin\theta\theta'.$$

Then the kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}m(l\cos\theta\theta')^2 + \frac{1}{2}m\left(bl\sin\frac{t}{\varepsilon} - l\sin\theta\theta'\right)^2 \\ &= \frac{1}{2}ml^2(\theta')^2 + \frac{1}{2}mb^2l^2\sin^2\frac{t}{\varepsilon} - mbl^2\sin\theta\theta'\sin\frac{t}{\varepsilon}. \end{aligned} \tag{1.3}$$

The potential energy is given by

$$V = mg\left(l\cos\theta - \varepsilon bl\cos\frac{t}{\varepsilon}\right). \tag{1.4}$$

In this case, the Lagrange equation still holds with moving constraints. (See [**Gre06**] for reference.)

To write the Lagrange equation, we calculate $L = T - V$ and the following derivatives:

$$\frac{\partial L}{\partial\theta'} = ml^2\theta' - mbl^2\sin\theta\sin\frac{t}{\varepsilon}, \tag{1.5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\theta'}\right) = ml^2\theta'' - mbl^2\cos\theta\theta'\sin\frac{t}{\varepsilon} - m\frac{b}{\varepsilon}l^2\sin\theta\cos\frac{t}{\varepsilon}, \tag{1.6}$$

$$\frac{\partial L}{\partial\theta} = -mbl^2\cos\theta\theta'\sin\frac{t}{\varepsilon} + mlg\sin\theta. \tag{1.7}$$

Therefore the Lagrange equation writes

$$\theta'' - \left(\frac{g}{l} + \frac{b}{\varepsilon}\cos\frac{t}{\varepsilon}\right)\sin\theta = 0. \tag{1.8}$$

It is obviously unstable for $b = 0$ at the upward position $\theta = 0$. In the next two sections, we will show that it can be stabilized for large enough b, upon sending ε to 0.

1.2. Asymptotic expansions. For subsequent calculations in this section we will let $a = g/l > 0$ and and thus (1.8) becomes

$$\theta_\varepsilon'' - \left(a + \frac{b}{\varepsilon}\cos\frac{t}{\varepsilon}\right)\sin\theta_\varepsilon = 0. \tag{1.9}$$

We write θ as θ_ε to show explicitly its dependence on the parameter ε, which we will then send to 0.

We will derive the limiting equation for (1.9) and show stability. We will employ the method of asymptotic expansions using multiple scales (see A. Bensoussan, J.- L. Lions and G. Papanicolaoun's book [**BLP78**] for reference). We will introduce a slow and a fast variable. By doing this we can formally obtain the limiting equation. And the justification will be made rigorous in the next section. Both the formal asymptotics and the proof using weak convergence were suggested by Prof. John New who formerly worked at UC Berkeley who we owe many thanks to.

Formally, we let θ_ε be a function of the slow variable, the fast variable as well as the parameter ε, and thus

$$\theta_\varepsilon = \Theta(t, \frac{t}{\varepsilon}, \varepsilon),$$

where $\Theta = \Theta(t, s, \varepsilon)$ is 2π-periodic in s, Then its first and second derivatives will be as follows:

$$\theta_\varepsilon' = \Theta_t + \frac{1}{\varepsilon}\Theta_s$$

$$\theta_\varepsilon'' = \Theta_{tt} + \frac{2}{\varepsilon}\Theta_{ts} + \frac{1}{\varepsilon^2}\Theta_{ss}$$

We now plug the foregoing into (1.9), which in turn becomes

$$\Theta_{tt} + \frac{2}{\varepsilon}\Theta_{ts} + \frac{1}{\varepsilon^2}\Theta_{ss} - \left(a + \frac{b}{\varepsilon}\cos s\right)\sin\Theta = 0. \tag{1.10}$$

We look for the asymptotic expansion of θ_ε by letting

$$\Theta = \Theta^0 + \varepsilon\Theta^1 + \varepsilon^2\Theta^2 + \cdots , \tag{1.11}$$

where

$$\Theta^k = \Theta^k(t, s)$$

is 2π-periodic in s, $k = 0, 1, 2, \cdots$.

We will plug it in the equation and identify powers of ε. Then to derive the limiting equation, we will replace the fast variable t/ε by s and thereafter treat the slow variable t and the fast variable s as two independent ones. Plug (1.11) into (1.9) and find

$$\begin{aligned} \Theta^0_{tt} + \frac{2}{\varepsilon}\Theta^0_{ts} + \frac{1}{\varepsilon^2\Theta^0_{ss}} + \varepsilon\left(\Theta^1_{tt} + \frac{2}{\varepsilon}\Theta^1_{ts} + \frac{1}{\varepsilon^2}\Theta^1_{ss}\right) + \\ \varepsilon^2\left(\Theta^2_{tt} + \frac{2}{\varepsilon}\Theta^2_{ts} + \frac{1}{\varepsilon^2}\Theta^2_{ss}\right)\cdots \\ = \left(a + \frac{b}{\varepsilon}\cos s\right)\left(\sin\Theta^0 + \varepsilon\cos\Theta^0\Theta^1\right)\cdots . \end{aligned} \tag{1.12}$$

We identify powers of ε and get

(1.13)

$$\boldsymbol{O(\varepsilon^{-2})} : \qquad \Theta^0_{ss} = 0,$$

(1.14)

$$\boldsymbol{O(\varepsilon^{-1})} : \qquad 2\Theta^0_{ts} + \Theta^1_{ss} = b\cos s\sin\Theta^0,$$

(1.15)

$$\boldsymbol{O(1)} : \qquad \Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} = a\sin\Theta^0 + b\cos s\cos\Theta^0\Theta^1.$$

"Solving" the three equations above will yield the limiting equation of (1.9). $O(\varepsilon^{-2})$ terms: (1.13) and periodicity in s implies

$$\Theta^0 = \Theta^0(t); \tag{1.16}$$

that is, Θ^0 only depends on t.

$O(\varepsilon^{-1})$ terms: $\Theta^0_{ts} \equiv 0$ by (1.16) and so (1.14) implies

$$\Theta^1_{ss} = b\cos s\sin\Theta^0(t).$$

Thus

$$\Theta^1(t, s) = -b\cos s\sin\Theta^0(t). \tag{1.17}$$

$O(1)$ terms: using (1.17) in (1.15) we get

$$\Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} = a \sin \Theta^0 - b^2 \cos^2 s \sin \Theta^0 \cos \Theta^0.$$

Integrating with respect to s form 0 to 2π and recalling that Θ^1 is 2π-periodic in s, we get

$$2\pi \Theta^0_{tt} = 2\pi a \sin \Theta^0 - b^2 \sin \Theta^0 \cos \Theta^0 \int_0^{2\pi} \cos^2 s \, \mathrm{d}s. \tag{1.18}$$

Thus we get

$$\Theta^0_{tt} = a \sin \Theta^0 - \frac{b^2}{2} \sin \Theta^0 \cos \Theta^0, \tag{1.19}$$

and

$$\Theta^0_{tt} + \frac{b^2}{4} \sin 2\Theta^0 - a \sin \Theta^0 = 0. \tag{1.20}$$

This has the form

$$\Theta^0_{tt} + \Phi'(\Theta^0) = 0 \tag{1.21}$$

for

$$\Phi(z) := -\frac{b^2}{8} \cos 2z + a \cos z. \tag{1.22}$$

Now $\Phi''(0) > 0$ would be a sufficient condition for $\Theta^0 \equiv 0$ being stable for (1.20). Now

$$\Phi''(z) = \frac{b^2}{2} \cos 2z - a \cos z. \tag{1.23}$$

So indeed we have

$$\Phi''(0) = \frac{b^2}{2} - a > 0, \tag{1.24}$$

if and only if

$$|b| \geq \sqrt{2a}.$$

Thus we have formally obtained the limiting equation of (1.8) as well as the condition for stability of the solution $\theta \equiv 0$ when ε goes to 0. In the next section, we will give a rigorous proof for the convergence result.

1.3. Review of weak convergence. Throughout this section, we let E denote a real Banach space and O a bounded open subset of $\mathbb{R}^n$.

We first give the definition of weak convergence in a Banach space and a theorem about weak compactness and then restrict our attention to the special case of $L^2(O)$.

DEFINITION 1.1. A sequence $\{x_k\} \subset E$ is said to converge weakly to $x \in E$ if and only if

$$\langle x^*, x_k \rangle \to \langle x^*, x \rangle$$

for each bounded linear functional $x^* \in E^*$. We write

$$x_k \rightharpoonup x.$$

THEOREM 1.2. (Weak compactness) *Let E be a reflexive Banach space and let $\{x_k\}$ be a bounded sequence in E. Then there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and $x \in E$ such that*

$$x_{k_j} \rightharpoonup x \text{ in } E.$$

We will mainly deal with weak convergence in $L^2(O)$ which is a special case of the above definition of weak convergence. For $1 \leq p < \infty$, we identify the dual space of $L^p(O)$ as $L^q(O)$, where $1/p + 1/q = 1$ and $1 < q \leq \infty$.

Thus, a sequence $\{f_k\} \subset L^2(O)$ weakly converges to $f \in L^2(O)$, that is,

$$f_k \rightharpoonup f \text{ weakly in } L^2(O)$$

if and only if

$$\int_O g f_k \, \mathrm{d}x \to \int_O g f \, \mathrm{d}x$$

as $k \to \infty$, for all $g \in L^2(O)$.

For $1 < p < \infty$, since $L^p(O)$ is reflexive, Theorem 1.2 implies that we can extract a weakly convergent subsequence from any bounded sequence in $L^p(O)$. We will use this fact for the case of $L^2(O)$ many times in our discussion.

We will let $\mathbb{T}$ denote the interval $[0,1] \subset \mathbb{R}$ and thus $\mathbb{T}^n$ is the interval $[0,1] \times [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n$.

DEFINITION 1.3. Let f be a function defined a.e on $\mathbb{R}^n$. f is called $\mathbb{T}^n$-periodic if and only if

$$f(x + k\boldsymbol{e}_i) = f(x) \text{ a.e. on } \mathbb{R}^n,$$

for all $k \in \mathbb{Z}$ and each $i \in \{1, ..., n\}$, where $\{\boldsymbol{e}_1, ..., \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n$.

We will use the following examples of weak convergence of $\mathbb{T}^n$-periodic functions many times.

1. For any $\mathbb{T}^n$-periodic function $f \in C(\mathbb{R}^n)$, we let

$$\overline{f} := \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} f(y)\, \mathrm{d}y$$

denote the value of f averaged over one period. Then

$$f\left(\frac{x}{\varepsilon}\right) \rightharpoonup \overline{f} \quad \text{weakly in} \quad L^2(O) \quad \text{as } \varepsilon \to 0. \tag{1.25}$$

We give below some examples we will use in this case.

Example 1.4.

$$\sin\left(\frac{t}{\varepsilon}\right) \rightharpoonup 0 \quad \text{weakly in} \quad L^2(a,b) \quad \text{as } \varepsilon \to 0, \tag{1.26}$$

and

$$\cos\left(\frac{t}{\varepsilon}\right) \rightharpoonup 0 \quad \text{weakly in} \quad L^2(a,b) \quad \text{as } \varepsilon \to 0, \tag{1.27}$$

where $a,\, b \in \mathbb{R}$.

Example 1.5.

$$\sin^2\left(\frac{t}{\varepsilon}\right) \rightharpoonup \frac{1}{2} \quad \text{weakly in} \quad L^2(a,b) \quad \text{as } \varepsilon \to 0, \tag{1.28}$$

and

$$\cos^2\left(\frac{t}{\varepsilon}\right) \rightharpoonup \frac{1}{2} \quad \text{weakly in} \quad L^2(a,b) \quad \text{as } \varepsilon \to 0, \tag{1.29}$$

where $a,\, b \in \mathbb{R}$.

2. For any function $f(x,y) \in C(O \times \mathbb{R}^n)$, assume $f(x,y)$ is $\mathbb{T}^n$-periodic in y. Let $\overline{f}(x) := \dfrac{1}{|\mathbb{T}^n|} \displaystyle\int_{\mathbb{T}^n} f(x,y)\, \mathrm{d}y \in C(O)$ denote the value of $f(x,y)$ averaged over one period of y. Then

$$f\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup \overline{f}(x) \quad \text{weakly in} \quad L^2(O) \quad \text{as } \varepsilon \to 0. \tag{1.30}$$

Note that as Example 1.5 shows, in general, if $f_n \rightharpoonup f$ and $g_n \rightharpoonup g$ weakly in $L^2(O)$ as $n \to \infty$, it is not necessarily true that $f_n g_n \rightharpoonup fg$ weakly in $L^2(O)$ as $n \to \infty$.

1.4. Proof of convergence. This section gives a rigorous proof of the convergence result, that is, the solution θ_ε to (1.8) converges to the solution to the limiting equation (1.20) as ε goes to zero. We consider the following initial-value problem:

$$\begin{cases} \theta_\varepsilon'' = \left(a + \dfrac{b}{\varepsilon}\cos\dfrac{t}{\varepsilon}\right)\sin\theta_\varepsilon \\ \theta_\varepsilon(0) = \alpha \\ \theta_\varepsilon'(0) = \beta. \end{cases} \tag{1.31}$$

We shall show that θ_ε, as ε goes to 0, converges locally uniformly on any finite interval to the solution of the following initial-value problem:

$$\begin{cases} \theta'' + \dfrac{b^2}{4}\sin 2\theta - a\sin\theta = 0 \\ \theta(0) = \alpha \\ \theta'(0) = \beta. \end{cases} \tag{1.32}$$

We first need the following lemma for uniformly boundedness of θ_ε and θ_ε' to justify subsequent passage to limits through a subsequence.

Lemma 1.6. *For each $T > 0$,*

$$\max_{0\le t\le T} |\theta_\varepsilon|, |\theta_\varepsilon'| \le C_T \tag{1.33}$$

for some constant C_T that is independent of ε.

Proof. First rewrite (1.8) to get

$$\begin{aligned} \theta_\varepsilon'' &= a\sin\theta_\varepsilon + b\left(\sin\frac{t}{\varepsilon}\right)'\sin\theta_\varepsilon \\ &= a\sin\theta_\varepsilon + \left(b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon\right)' - b\sin\frac{t}{\varepsilon}\cos\theta_\varepsilon\theta_\varepsilon'. \end{aligned}$$

Thus

$$\left(\theta_\varepsilon' - b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon\right)' = a\sin\theta_\varepsilon - b\sin\frac{t}{\varepsilon}\cos\theta_\varepsilon\theta_\varepsilon'. \tag{1.34}$$

Integrate to find

$$|\theta_\varepsilon'(t)| \leq C_1 + C_2 \int_0^t |\theta_\varepsilon'| \, \mathrm{d}s,$$

for each $0 \leq t \leq T$ and constants C_1, $C_2 \leq 0$.

By Gronwall's inequality (integral form) (See [**Eva98**]),

$$|\theta_\varepsilon'(t)| \leq C_1 \left(1 + C_2 t e^{C_1 t}\right) \leq C_T, \tag{1.35}$$

for each $0 \leq t \leq T$ and a constant $C_T > 0$ that only depends on T. Thus $|\theta_\varepsilon'|$ and $|\theta_\varepsilon|$ are both bounded uniformly for $0 \leq t \leq T$ by the constant $C_T > 0$ which only depends on T. $\square$

Using (1.33) we have

$$\begin{cases} \theta_\varepsilon \to \theta \text{ uniformly on } [0,T] \\ \theta_\varepsilon' \rightharpoonup \theta' \text{ weakly in } L^2(0,T), \end{cases}$$

through a subsequence, as $\varepsilon \to 0$.

THEOREM 1.7. *θ solves* (1.32).

PROOF. Let $\phi = \phi(t)$ be smooth, vanishing near $t = 0$, $t = T$. Multiply (1.34) by ϕ and integrate from 0 to T, to get

$$\begin{aligned} -\int_0^T \left(\theta_\varepsilon' - b \sin \frac{t}{\varepsilon} \sin \theta_\varepsilon\right) \phi' \, \mathrm{d}t = \\ \int_0^T a \sin \theta_\varepsilon \phi \, \mathrm{d}t - \int_0^T b \sin \frac{t}{\varepsilon} \cos \theta_\varepsilon \theta_\varepsilon' \phi \, \mathrm{d}t. \end{aligned} \tag{1.36}$$

We will look at each integral in the above equation separately. Let

$$\begin{cases} A := -\displaystyle\int_0^T \left(\theta_\varepsilon' - b \sin \frac{t}{\varepsilon} \sin \theta_\varepsilon\right) \phi' \, \mathrm{d}t \\ B := \displaystyle\int_0^T a \sin \theta_\varepsilon \phi \, \mathrm{d}t \\ C := -\displaystyle\int_0^T b \sin \frac{t}{\varepsilon} \cos \theta_\varepsilon \theta_\varepsilon' \phi \, \mathrm{d}t. \end{cases} \tag{1.37}$$

Then we have

$$\lim_{\varepsilon \to 0} A = -\int_0^T \theta' \phi' \, \mathrm{d}t,$$

since $\sin \theta_\varepsilon \to \sin \theta$ and $\sin(t/\varepsilon) \rightharpoonup 0$ in $L^2(0,T)$ as $\varepsilon \to 0$.

It is easy to see that

$$\lim_{\varepsilon\to 0} B = \int_0^T a\sin\theta\phi\,\mathrm{d}t.$$

To compute $\lim\limits_{\varepsilon\to 0} C$, we first claim that

$$\sin\frac{t}{\varepsilon}\theta'_\varepsilon \rightharpoonup \frac{b}{2}\sin\theta, \text{ weakly in } L^2(0,T), \tag{1.38}$$

as $\varepsilon \to 0$.

To check this, we will show that, for all $\psi \in C_c^\infty[0,T]$,

$$\begin{aligned}
\int_0^T \psi\sin\frac{t}{\varepsilon}\theta'_\varepsilon\,\mathrm{d}t &= -\varepsilon\int_0^T \psi\left(\cos\frac{t}{\varepsilon}\right)'\theta'_\varepsilon\,\mathrm{d}t \\
&= \varepsilon\int_0^T \psi'\cos\frac{t}{\varepsilon}\theta'_\varepsilon\,\mathrm{d}t + \varepsilon\int_0^T \psi\cos\frac{t}{\varepsilon}\theta''_\varepsilon\,\mathrm{d}t \\
&= O(\varepsilon) + \int_0^T \psi\cos\frac{t}{\varepsilon}\left(\varepsilon a + b\cos\frac{t}{\varepsilon}\right)\sin\theta_\varepsilon\,\mathrm{d}t \\
&= O(\varepsilon) + \int_0^T b\psi\cos^2\frac{t}{\varepsilon}\sin\theta_\varepsilon\,\mathrm{d}t \\
&\to \int_0^t \frac{b}{2}\sin\theta\psi\,\mathrm{d}t
\end{aligned}$$

as $\varepsilon \to 0$. This proves (1.38).

Then we use (1.38) to deduce

$$\lim_{\varepsilon\to 0} C = -\int_0^t b\cos\theta\frac{b}{2}\sin\theta\phi\,\mathrm{d}t.$$

So, passing to limits in (1.36), we get

$$-\int_0^T \theta'\phi'\,\mathrm{d}t = \int_0^T\left(a\sin\theta - \frac{b^2}{2}\cos\theta\sin\theta\right)\phi\,\mathrm{d}t = \int_0^T\left(a\sin\theta - \frac{b^2}{4}\sin 2\theta\right)\phi\,\mathrm{d}t. \tag{1.39}$$

Thus we showed θ solves the ODE in (1.32), that is,

$$\theta'' = a\sin\theta - \frac{b^2}{4}\sin 2\theta, \tag{1.40}$$

in the weak sense.

In fact, if we integrate (1.34) from 0 to t for $0 < t < T$, we get

$$\theta_\varepsilon' - b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon(t) = \beta + \int_0^t a\sin\theta_\varepsilon - b\sin\frac{s}{\varepsilon}\cos\theta_\varepsilon\theta_\varepsilon'\,\mathrm{d}s. \tag{1.41}$$

Then we send ε to 0 and pass to weak limits, using (1.38) and uniform convergence of θ_ε, to get

(1.42)

$$\theta'(t) = \beta + \int_0^t a\sin\theta - \frac{b^2}{2}\cos\theta\sin\theta\,\mathrm{d}s = \beta + \int_0^t a\sin\theta - \frac{b^2}{4}\sin(2\theta)\,\mathrm{d}s$$

This not only shows that θ is in fact differentiable on $[0, T]$ and solves (1.40) but also yields the correct initial condition $\theta'(0) = \beta$. And since $\theta_\varepsilon \to \theta$ locally uniformly, we also have $\theta(0) = \alpha$.

Since the solution θ to the initial-value problem (1.32) is unique, we in fact further have that the entire sequence $\{\theta_\varepsilon\}$ converges to θ locally uniformly. $\square$

2. Single inverted pendulum on a horizontally oscillatory base

While a vertically oscillatory pivot can stabilize the inverted pendulum, it is intriguing to think about the effect of a horizontally oscillatory pivot. We will apply our previous analysis to this case and show that, as Eugene I. Butikov pointed out in [**But01**] , the oscillation has an aligning effect along its axis.

2.1. Derivation of equation. We first derive the governing equation for the motion of the inverted pendulum with a horizontally oscillatory pivot. Set up the coordinate axes as shown in Figure 4. The pivot of the inverted pendulum is subjected to a horizontal sinusoidal oscillation given by $x_\varepsilon = \varepsilon b l \cos(t/\varepsilon)$ for some small positive parameter ε, so here εb is the dimensionless amplitude.

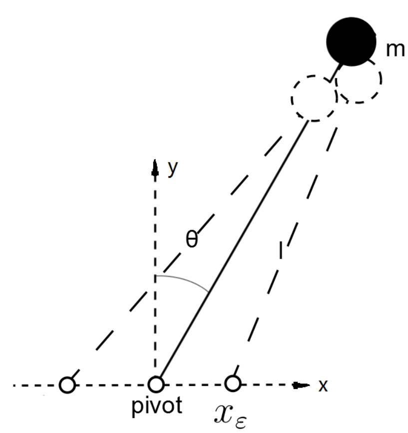

FIGURE 4. Single inverted pendulum on a horizontally oscillatory base

Let the position of the center of the mass m be (x, y), then we have

$$x = x_\varepsilon + l \sin\theta = \varepsilon b l \cos\frac{t}{\varepsilon} + l \sin\theta,$$

$$y = l\cos\theta,$$

and

$$x' = -bl \sin \frac{t}{\varepsilon} + l \cos \theta \theta',$$
$$y' = -l \sin \theta \theta'.$$

Then the kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2} m \left(bl \sin \frac{t}{\varepsilon} - l \cos \theta \theta' \right)^2 + \frac{1}{2} m (l \sin \theta \theta')^2 \\ &= \frac{1}{2} m l^2 (\theta')^2 + \frac{1}{2} m b^2 l^2 \sin^2 \frac{t}{\varepsilon} - m b l^2 \cos \theta \theta' \sin \frac{t}{\varepsilon}. \end{aligned} \tag{2.1}$$

The potential energy is given by

$$V = mgl \cos \theta. \tag{2.2}$$

To write the Lagrange equation, we calculate $L = T - V$ and the following derivatives:

$$\frac{\partial L}{\partial \theta'} = m l^2 \theta' - m b l^2 \cos \theta \sin \frac{t}{\varepsilon}, \tag{2.3}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \theta'} \right) = m l^2 \theta'' + m b l^2 \sin \theta \theta' \sin \frac{t}{\varepsilon} - m \frac{b}{\varepsilon} l^2 \cos \theta \cos \frac{t}{\varepsilon}, \tag{2.4}$$

$$\frac{\partial L}{\partial \theta} = m b l^2 \sin \theta \theta' \sin \frac{t}{\varepsilon} + m l g \sin \theta. \tag{2.5}$$

Therefore the Lagrange equation writes

$$\theta'' - \frac{g}{l} \sin \theta - \frac{b}{\varepsilon} \cos \frac{t}{\varepsilon} \cos \theta = 0. \tag{2.6}$$

In the next two subsections, we will discuss its stability conditions.

2.2. Asymptotic expansions. Similarly to the previous section, for subsequent calculations in this section we will also let $a = g/l > 0$ and thus (2.6) becomes

$$\theta'' - a \sin \theta - \frac{b}{\varepsilon} \cos \frac{t}{\varepsilon} \cos \theta = 0. \tag{2.7}$$

Perform asymptotic expansions, write θ as θ_ε to show explicitly its dependence on ε.

Formally, let θ_ε be a function of the slow variable t, the fast variable t/ε as well as the frequency ε,

$$\theta_\varepsilon = \Theta(t, \frac{t}{\varepsilon}, \varepsilon),$$

where $\Theta = \Theta(t, s, \varepsilon)$ is 2π-periodic in s,

Similarly to the previous section, calculate the first and second derivatives and plug in (2.7) which then becomes

$$\Theta_{tt} + \frac{2}{\varepsilon}\Theta_{ts} + \frac{1}{\varepsilon^2}\Theta_{ss} - a\sin\Theta - \frac{b}{\varepsilon}\cos s\cos\Theta = 0. \tag{2.8}$$

Write out the asymptotic expansion for Θ

$$\Theta = \Theta^0 + \varepsilon\Theta^1 + \varepsilon^2\Theta^2 + \cdots, \tag{2.9}$$

where $\Theta^k = \Theta^k(t, s)$ is 2π-periodic in s, $k = 0, 1, 2, \cdots$

We plug (2.9) into (2.7) and get,

$$\begin{aligned}\Theta^0_{tt} + \frac{2}{\varepsilon}\Theta^0_{ts} + \frac{1}{\varepsilon^2}\Theta^0_{ss} + \varepsilon\left(\Theta^1_{tt} + \frac{2}{\varepsilon}\Theta^1_{ts} + \frac{1}{\varepsilon^2}\Theta^1_{ss}\right) + \\ \varepsilon^2\left(\Theta^2_{tt} + \frac{2}{\varepsilon}\Theta^2_{ts} + \frac{1}{\varepsilon^2}\Theta^2_{ss}\right)\cdots \\ = a\left(\sin\Theta^0 + \varepsilon\cos\Theta^0\Theta^1\right) + \frac{b}{\varepsilon}\cos s\left(\cos\Theta^0 - \varepsilon\sin\Theta^0\Theta^1\right)\cdots.\end{aligned} \tag{2.10}$$

We replace t/ε with s and treat t and s as two independent variables. Identify powers of ε, to learn that,

$$\boldsymbol{O(\varepsilon^{-2})} \qquad \Theta^0_{ss} = 0 \tag{2.11}$$

$$\boldsymbol{O(\varepsilon^{-1})} \qquad 2\Theta^0_{ts} + \Theta^1_{ss} = b\cos s\cos\Theta^0 \tag{2.12}$$

$$\boldsymbol{O(1)} \qquad \Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} = a\sin\Theta^0 - b\cos s\sin\Theta^0\Theta^1. \tag{2.13}$$

First, we look at $O(\varepsilon^{-2})$ terms. (2.11) and periodicity in s implies

$$\Theta^0 = \Theta^0(t). \tag{2.14}$$

In $O(\varepsilon^{-1})$ terms, by (2.14), $\Theta^0_{ts} \equiv 0$. So (2.12) implies

$$\Theta^1_{ss} = b\cos s\cos\Theta^0(t).$$

Thus

$$\Theta^1(t, s) = -b\cos s\cos\Theta^0(t). \tag{2.15}$$

$O(1)$ terms, we plug (2.15) in (2.13) to get

$$\Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} = a \sin \Theta^0 + b^2 \cos^2 s \sin \Theta^0 \cos \Theta^0.$$

We integrate with respect to s form 0 to 2π. By the 2π-periodicity in s of Θ^1, we get

$$2\pi\Theta^0_{tt} = 2\pi a \sin \Theta^0 + b^2 \sin \Theta^0 \cos \Theta^0 \int_0^{2\pi} \cos^2 s \, \mathrm{d}s.$$

Thus

$$\Theta^0_{tt} = a \sin \Theta^0 + \frac{b^2}{4} \sin \Theta^0 \cos \Theta^0,$$

which we rewrite to read

$$\Theta^0_{tt} - \frac{b^2}{4} \sin 2\Theta^0 - a \sin \Theta^0 = 0. \tag{2.16}$$

This has the form

$$\Theta^0_{tt} + \Phi'(\Theta^0) = 0$$

for

$$\Phi(z) := \frac{b^2}{8} \cos 2z + a \cos z.$$

The potential Φ has two minima at $\pi \pm \cos^{-1}(2a/b^2)$, respectively. This means that the pendulum can be stabilized at these two symmetric positions, if and only if

$$b^2 > 2a.$$

A simple physical explanation was given in Eugene I. Butikov's paper [**But01**]: Once the motion is put in the noninertial reference frame associated with the pivot, "the torque of the force of inertia, averaged over a period of oscillations, tends to align the pendulum along the direction of constrained oscillations of the axis."

In the next subsection, we will give a rigorous proof for the convergence result.

2.3. Proof of convergence. Consider the following initial-value problem:

$$\begin{cases} \theta''_\varepsilon = a \sin \theta_\varepsilon + \dfrac{b}{\varepsilon} \cos \dfrac{t}{\varepsilon} \cos \theta_\varepsilon \\ \theta_\varepsilon(0) = \alpha \\ \theta'_\varepsilon(0) = \beta. \end{cases} \tag{2.17}$$

We will prove that the limiting equation, as ε goes to 0, is

$$\begin{cases} \theta'' - \dfrac{b^2}{4}\sin 2\theta - a\sin\theta = 0 \\ \theta(0) = \alpha \\ \theta'(0) = \beta. \end{cases} \tag{2.18}$$

LEMMA 2.1. *For each $T > 0$,*

$$\max_{0\le t\le T} |\theta_\varepsilon|, |\theta'_\varepsilon| \le C_T. \tag{2.19}$$

PROOF. We have

$$\begin{aligned} \theta''_\varepsilon &= a\sin\theta_\varepsilon + b\left(\sin\frac{t}{\varepsilon}\right)'\cos\theta_\varepsilon \\ &= a\sin\theta_\varepsilon + \left(b\sin\frac{t}{\varepsilon}\cos\theta_\varepsilon\right)' + b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon\theta'_\varepsilon. \end{aligned}$$

Rewrite the equation as

$$\left(\theta'_\varepsilon - b\sin\frac{t}{\varepsilon}\cos\theta_\varepsilon\right)' = a\sin\theta_\varepsilon - b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon\theta'_\varepsilon. \tag{2.20}$$

Integrate to find

$$|\theta'_\varepsilon(t)| \le C_1 + C_2\int_0^t |\theta'_\varepsilon|\,\mathrm{d}s,$$

for $0 \le t \le T$ and constants C_1, $C_2 \le 0$.

By Gronwall's inequality (integral form),

$$|\theta'_\varepsilon(t)| \le C_1\left(1 + C_2 t e^{C_1 t}\right) \le C_T,$$

for each $0 \le t \le T$ and a constant $C_T > 0$ that only depends on T.

Thus $|\theta'_\varepsilon|$ and $|\theta_\varepsilon|$ are both bounded uniformly for $0 \le t \le T$ by the appropriately chosen constant $C_T > 0$ which only depends on T. □

Thus by (2.19) we have

$$\begin{cases} \theta_\varepsilon \to \theta \text{ uniformly on } [0,T] \\ \theta'_\varepsilon \rightharpoonup \theta' \text{ weakly in } L^2(0,T), \end{cases}$$

through a subsequence, as $\varepsilon \to 0$.

THEOREM 2.2. *θ solves* (2.18).

PROOF. Let $\phi = \phi(t)$ be smooth, vanishing near $t = 0$ and $t = T$. We multiply (2.20) by ϕ and integrate to get

$$-\int_0^T \left(\theta_\varepsilon' - b\sin\frac{t}{\varepsilon}\cos\theta_\varepsilon\right)\phi'\,\mathrm{d}t = \int_0^T a\sin\theta_\varepsilon\phi\,\mathrm{d}t + \int_0^T b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon\theta_\varepsilon'\phi\,\mathrm{d}t. \tag{2.21}$$

Let

$$\begin{cases} A := -\displaystyle\int_0^T \left(\theta_\varepsilon' - b\sin\frac{t}{\varepsilon}\cos\theta_\varepsilon\right)\phi'\,\mathrm{d}t \\ B := \displaystyle\int_0^T a\sin\theta_\varepsilon\phi\,\mathrm{d}t \\ C := \displaystyle\int_0^T b\sin\frac{t}{\varepsilon}\sin\theta_\varepsilon\theta_\varepsilon'\phi\,\mathrm{d}t. \end{cases}$$

Then

$$\lim_{\varepsilon\to 0} A = -\int_0^T \theta'\phi'\,\mathrm{d}t,$$

since $\cos\theta_\varepsilon \to \cos\theta$ and $\sin(t/\varepsilon) \rightharpoonup 0$ in $L^2(0,T)$ as $\varepsilon \to 0$.

And

$$\lim_{\varepsilon\to 0} B = \int_0^T a\sin\theta\phi\,\mathrm{d}t.$$

We claim

$$\sin\frac{t}{\varepsilon}\theta_\varepsilon' \rightharpoonup \frac{b}{2}\cos\theta, \tag{2.22}$$

as $\varepsilon \to 0$.

To check this, for all $\psi \in C_c^\infty[0,T]$,

$$\begin{aligned}\int_0^T \psi\sin\frac{t}{\varepsilon}\theta_\varepsilon'\,\mathrm{d}t &= -\varepsilon\int_0^T \psi\left(\cos\frac{t}{\varepsilon}\right)'\theta_\varepsilon'\,\mathrm{d}t \\ &= \varepsilon\int_0^T \psi'\cos\frac{t}{\varepsilon}\theta_\varepsilon'\,\mathrm{d}t + \varepsilon\int_0^T \psi\cos\frac{t}{\varepsilon}\theta_\varepsilon''\,\mathrm{d}t \\ &= O(\varepsilon) + \int_0^T \psi\cos\frac{t}{\varepsilon}\left(\varepsilon a\sin\theta_\varepsilon + b\cos\frac{t}{\varepsilon}\cos\theta_\varepsilon\right)\,\mathrm{d}t \\ &= O(\varepsilon) + \int_0^T b\psi\cos^2\frac{t}{\varepsilon}\cos\theta_\varepsilon\,\mathrm{d}t\end{aligned}$$

$$\to \int_0^t \frac{b}{2}\cos\theta\psi\,\mathrm{d}t$$

as $\varepsilon \to 0$.

This proves (2.22), and therefore,

$$\lim_{\varepsilon\to 0} C = \int_0^t b\sin\theta\frac{b}{2}\cos\theta\phi\,\mathrm{d}t$$

So passing to the limit in (2.21) we get

$$(2.23)\quad -\int_0^T \theta'\phi'\,\mathrm{d}t = \int_0^T \left(a\sin\theta + \frac{b^2}{2}\cos\theta\sin\theta\right)\phi\,\mathrm{d}t = \int_0^T \left(a\sin\theta + \frac{b^2}{4}\sin 2\theta\right)\phi\,\mathrm{d}t.$$

Thus θ solves the ODE (2.16) in the weak sense. Following an argument similar to that from (1.41) to (1.42), we can also show that θ and solves the initial-value problem (2.18) and the entire sequence of $\{\theta_\varepsilon\}$ converges to θ locally uniformly. $\square$

3. Single inverted pendulum on an oscillatory base driven at arbitrary angles

The previous two sections are special cases of a single inverted pendulum driven at an arbitrary angle and and have closed form stability conditions. In general, we don't have a closed form of the stability condition although we know theoretically that as the frequency gets large enough the pendulum will oscillate about a stable angle other than the upright one because of the aligning effect of the oscillation, as pointed out in both Eugene I. Butikov's [**But01**] and G. J. VanDalen's [**Van04**].

3.1. Derivation of equation. We first derive the governing equation for the motion of an inverted pendulum driven an arbitrary angle. We set up the coordinate axes as shown in Figure 5. The pivot of the inverted pendulum is subjected to a sinusoidal oscillation driven at an angle of θ_d measured counterclockwise from the upward vertical with amplitude $x_\varepsilon = -\varepsilon bl\cos(t/\varepsilon)$ for some small positive parameter ε, so here εb is the dimensionless amplitude.

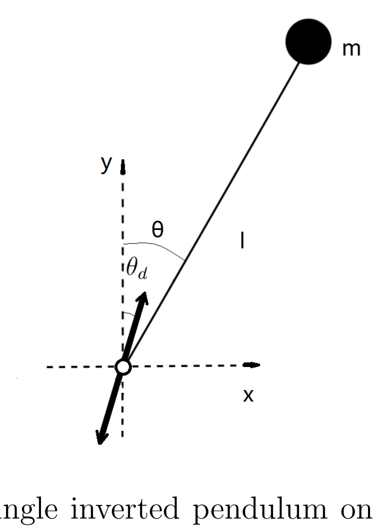

FIGURE 5. Single inverted pendulum on a horizontally oscillatory base

Let the position of the center of the mass m be (x, y), then we have

$$x = -\varepsilon bl \sin\theta_d \cos\frac{t}{\varepsilon} + l\sin\theta,$$

$$y = -\varepsilon bl \cos\theta_d \cos\frac{t}{\varepsilon} + l\cos\theta.$$

And

$$x' = bl\sin\theta_d \sin\frac{t}{\varepsilon} + l\cos\theta\theta',$$

$$y' = bl\cos\theta_d \sin\frac{t}{\varepsilon} - l\sin\theta\theta'.$$

Then the kinetic energy is given by

(3.1)

$$\begin{aligned}T &= \frac{1}{2}m\left(bl\sin\theta_d\sin\frac{t}{\varepsilon} + l\cos\theta\theta'\right)^2 + \frac{1}{2}m\left(bl\cos\theta_d\sin\frac{t}{\varepsilon} - l\sin\theta\theta'\right)^2\\ &= \frac{1}{2}ml^2(\theta')^2 + \frac{1}{2}mb^2l^2\sin^2\frac{t}{\varepsilon} + mbl^2\sin(\theta_d-\theta)\sin\frac{t}{\varepsilon}.\end{aligned}$$

The potential energy is given by

$$V = mg\left(-\varepsilon bl\cos\theta_d\cos\frac{t}{\varepsilon} + l\cos\theta\right). \tag{3.2}$$

To write the Lagrange equation, we calculate $L = T - V$ and the following derivatives:

$$\frac{\partial L}{\partial\theta'} = ml^2\theta' + mbl^2\sin(\theta_d-\theta)\sin\frac{t}{\varepsilon}, \tag{3.3}$$

(3.4)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\theta'}\right) = ml^2\theta'' - mbl^2\cos(\theta_d-\theta)\theta'\sin\frac{t}{\varepsilon} + m\frac{b}{\varepsilon}l^2\sin(\theta_d-\theta)\cos\frac{t}{\varepsilon},$$

$$\frac{\partial L}{\partial\theta} = -mbl^2\cos(\theta_d-\theta)\theta'\sin\frac{t}{\varepsilon} + mlg\sin\theta. \tag{3.5}$$

Therefore the Lagrange equation is

$$\theta'' - \frac{g}{l}\sin\theta + \frac{b}{\varepsilon}\sin(\theta_d-\theta)\cos\frac{t}{\varepsilon} = 0. \tag{3.6}$$

In the next two subsections, we will discuss its stability conditions.

3.2. Asymptotic expansions. Similarly to the previous section, for subsequent calculations in this section we will also let $a = g/l > 0$ and thus (3.6) becomes

$$\theta'' - a\sin\theta + \frac{b}{\varepsilon}\sin(\theta_d - \theta)\cos\frac{t}{\varepsilon} = 0. \tag{3.7}$$

Perform asymptotic expansions, write θ as θ_ε to show explicitly its dependence on ε.

Formally, let θ_ε be a function of the slow variable t, the fast variable t/ε as well as the frequency ε,

$$\theta_\varepsilon = \Theta(t, \frac{t}{\varepsilon}, \varepsilon),$$

where $\Theta = \Theta(t, s, \varepsilon)$ is 2π-periodic in s,

Similarly to the previous section, calculate the first and second derivatives and plug in (3.7) which then becomes

$$\Theta_{tt} + \frac{2}{\varepsilon}\Theta_{ts} + \frac{1}{\varepsilon^2}\Theta_{ss} - a\sin\Theta + \frac{b}{\varepsilon}\sin(\theta_d - \Theta)\cos s = 0. \tag{3.8}$$

Write out the asymptotic expansion for Θ

$$\Theta = \Theta^0 + \varepsilon\Theta^1 + \varepsilon^2\Theta^2 + \cdots, \tag{3.9}$$

where $\Theta^k = \Theta^k(t, s)$ is 2π-periodic in s, $k = 0, 1, 2, \cdots$

Plug (3.9) into (3.7) to get,

$$\begin{aligned} &\Theta^0_{tt} + \frac{2}{\varepsilon}\Theta^0_{ts} + \frac{1}{\varepsilon^2}\Theta^0_{ss} + \varepsilon\left(\Theta^1_{tt} + \frac{2}{\varepsilon}\Theta^1_{ts} + \frac{1}{\varepsilon^2}\Theta^1_{ss}\right) + \\ &\qquad \varepsilon^2\left(\Theta^2_{tt} + \frac{2}{\varepsilon}\Theta^2_{ts} + \frac{1}{\varepsilon^2}\Theta^2_{ss}\right) + \cdots \\ &= a\left(\sin\Theta^0 + \varepsilon\cos\Theta^0\Theta^1\right) - \frac{b}{\varepsilon}\cos s\left[\sin(\theta_d - \Theta^0) - \varepsilon\cos(\theta_d - \Theta^0)\Theta^1\right] + \cdots \end{aligned} \tag{3.10}$$

Replace t/ε with s and treat t and s as two independent variables. Identify powers of ε and get,

(3.11)

$$\boldsymbol{O(\varepsilon^{-2})}: \qquad \Theta^0_{ss} = 0$$

(3.12)

$$\boldsymbol{O(\varepsilon^{-1})}: \qquad 2\Theta^0_{ts} + \Theta^1_{ss} = -b\cos s \sin(\theta_d - \Theta^0)$$

(3.13)

$$\boldsymbol{O(1)}: \qquad \Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} = a\sin\Theta^0 + b\cos s\cos(\theta_d - \Theta^0)\Theta^1.$$

Look at $O(\varepsilon^{-2})$ terms. (3.11) and periodicity in s implies

(3.14)
$$\Theta^0 = \Theta^0(t).$$

In $O(\varepsilon^{-1})$ terms, by (3.14), $\Theta^0_{ts} \equiv 0$. So (3.12) implies

$$\Theta^1_{ss} = -b\cos s\sin(\theta_d - \Theta^0).$$

Thus

(3.15)
$$\Theta^1(t,s) = b\cos s\sin(\theta_d - \Theta^0).$$

$O(1)$ terms, plug (3.15) in (3.13) to get

$$\Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} = a\sin\Theta^0 + b^2\cos^2 s\sin(\theta_d - \Theta^0)\cos(\theta_d - \Theta^0).$$

Integrate with respect to s form 0 to 2π and by the 2π-periodicity in s of Θ^1.

$$2\pi\Theta^0_{tt} = 2\pi a\sin\Theta^0 + b^2\sin(\theta_d - \Theta^0)\cos(\theta_d - \Theta^0)\int_0^{2\pi}\cos^2 s\,\mathrm{d}s.$$

Thus

$$\Theta^0_{tt} = a\sin\Theta^0 + \frac{b^2}{4}\sin(\theta_d - \Theta^0).$$

Rewrite as

(3.16)
$$\Theta^0_{tt} - a\sin\Theta^0 - \frac{b^2}{4}\sin(\theta_d - \Theta^0) = 0.$$

This has the form

$$\Theta^0_{tt} + \Phi'(\Theta^0) = 0$$

for

$$\Phi(z) := a\cos z - \frac{b^2}{8}\cos[2(\theta_d - z)].$$

The stable values of z can be solved numerically as a function of θ_d (See [**Van04**]). And we can see that section 2 and section 3 are two special cases of this general scenario for $\theta_d = 0$ and $\theta_d = -\pi/2$ respectively.

In the next subsection, we will give a rigorous proof for the convergence result.

3.3. Proof of convergence. Consider the following initial-value problem:

$$\begin{cases} \theta_\varepsilon'' = a\sin\theta_\varepsilon - \dfrac{b}{\varepsilon}\cos\dfrac{t}{\varepsilon}\sin(\theta_d - \theta_\varepsilon) \\ \theta_\varepsilon(0) = \alpha \\ \theta_\varepsilon'(0) = \beta. \end{cases} \tag{3.17}$$

We will prove that the limiting equation, as ε goes to 0, is

$$\begin{cases} \theta'' - a\sin\theta - \dfrac{b^2}{4}\sin[2(\theta_d - \theta)] = 0 \\ \theta(0) = \alpha \\ \theta'(0) = \beta. \end{cases} \tag{3.18}$$

Lemma 3.1. *For each $T > 0$,*

$$\max_{0\le t\le T} |\theta_\varepsilon|, |\theta_\varepsilon'| \le C_T. \tag{3.19}$$

Proof. We have

$$\begin{aligned} \theta_\varepsilon'' &= a\sin\theta_\varepsilon - b\left(\sin\frac{t}{\varepsilon}\right)'\sin(\theta_d - \theta_\varepsilon) \\ &= a\sin\theta_\varepsilon - \left[b\sin\frac{t}{\varepsilon}\sin(\theta_d - \theta_\varepsilon)\right]' - b\sin\frac{t}{\varepsilon}\cos(\theta_d - \theta_\varepsilon)\theta_\varepsilon'. \end{aligned}$$

Rewrite the equation as

$$\left[\theta_\varepsilon' + b\sin\frac{t}{\varepsilon}\sin(\theta_d - \theta_\varepsilon)\right]' = a\sin\theta_\varepsilon - b\sin\frac{t}{\varepsilon}\cos(\theta_d - \theta_\varepsilon)\theta_\varepsilon'. \tag{3.20}$$

Integrate to find

$$|\theta_\varepsilon'(t)| \le C_1 + C_2\int_0^t |\theta_\varepsilon'|\,\mathrm{d}s,$$

for $0 \le t \le T$ and constants C_1, $C_2 \le 0$.

By Gronwall's inequality (integral form),

$$|\theta_\varepsilon'(t)| \le C_1 \left(1 + C_2 t e^{C_1 t}\right) \le C_T,$$

for each $0 \le t \le T$ and a constant $C_T > 0$ that only depends on T.

Thus $|\theta_\varepsilon'|$ and $|\theta_\varepsilon|$ are both bounded uniformly for $0 \le t \le T$ by the appropriately chosen constant $C_T > 0$ which only depends on T. $\square$

Thus by (3.19) we have

$$\begin{cases} \theta_\varepsilon \to \theta \text{ uniformly on } [0,T] \\ \theta_\varepsilon' \rightharpoonup \theta' \text{ weakly in } L^2(0,T), \end{cases}$$

through a subsequence, as $\varepsilon \to 0$.

THEOREM 3.2. *θ solves* (3.18).

PROOF. Let $\phi = \phi(t)$ be smooth, vanishing near $t = 0$ and $t = T$. Multiply (3.20) by ϕ and integrate to get

$$(3.21) \quad -\int_0^T \left[\theta_\varepsilon' + b \sin\frac{t}{\varepsilon} \sin(\theta_d - \theta_\varepsilon)\right] \phi' \,\mathrm{d}t = \int_0^T a \sin\theta_\varepsilon \phi \,\mathrm{d}t - \int_0^T b \sin\frac{t}{\varepsilon} \cos(\theta_d - \theta_\varepsilon)\theta_\varepsilon' \phi \,\mathrm{d}t.$$

Let

$$\begin{cases} A := -\int_0^T \left[\theta_\varepsilon' + b \sin\frac{t}{\varepsilon} \sin(\theta_d - \theta_\varepsilon)\right] \phi' \,\mathrm{d}t \\ B := \int_0^T a \sin\theta_\varepsilon \phi \,\mathrm{d}t \\ C := -\int_0^T b \sin\frac{t}{\varepsilon} \cos(\theta_d - \theta_\varepsilon)\theta_\varepsilon' \phi \,\mathrm{d}t. \end{cases}$$

Then

$$\lim_{\varepsilon \to 0} A = -\int_0^T \theta' \phi' \,\mathrm{d}t,$$

since $\theta_\varepsilon \to \theta$ and $\sin(t/\varepsilon) \rightharpoonup 0$ in $L^2(0,T)$ as $\varepsilon \to 0$.

And

$$\lim_{\varepsilon \to 0} B = \int_0^T a \sin\theta \phi \,\mathrm{d}t.$$

We claim

$$\sin\frac{t}{\varepsilon}\theta'_\varepsilon \rightharpoonup -\frac{b}{2}\sin(\theta_d-\theta), \tag{3.22}$$

as $\varepsilon \to 0$.

To check this, for all $\psi \in C_c^\infty[0,T]$,

$$\begin{aligned}
\int_0^T \psi \sin\frac{t}{\varepsilon}\theta'_\varepsilon \,\mathrm{d}t &= -\varepsilon\int_0^T \psi\left(\cos\frac{t}{\varepsilon}\right)'\theta'_\varepsilon\,\mathrm{d}t\\
&= \varepsilon\int_0^T \psi'\cos\frac{t}{\varepsilon}\theta'_\varepsilon\,\mathrm{d}t + \varepsilon\int_0^T \psi\cos\frac{t}{\varepsilon}\theta''_\varepsilon\,\mathrm{d}t\\
&= O(\varepsilon) + \int_0^T \psi\cos\frac{t}{\varepsilon}\left[\varepsilon a\sin\theta_\varepsilon - b\cos\frac{t}{\varepsilon}\sin(\theta_d-\theta_\varepsilon)\right]\,\mathrm{d}t\\
&= O(\varepsilon) - \int_0^T b\psi\cos^2\frac{t}{\varepsilon}\sin(\theta_d-\theta_\varepsilon)\,\mathrm{d}t\\
&\to -\int_0^t \frac{b}{2}\sin(\theta_d-\theta)\psi\,\mathrm{d}t
\end{aligned}$$

as $\varepsilon \to 0$.

This proves (3.22).

By (3.22),

$$\lim_{\varepsilon\to 0} C = \int_0^t b\cos(\theta_d-\theta)\frac{b}{2}\sin(\theta_d-\theta)\phi\,\mathrm{d}t$$

So passing to the limit we get

$$\begin{aligned}
-\int_0^T \theta'\phi'\,\mathrm{d}t = \int_0^T &\left[a\sin\theta + \frac{b^2}{2}\cos(\theta_d-\theta)\sin(\theta_d-\theta)\right]\phi\,\mathrm{d}t =\\
&\int_0^T\left\{a\sin\theta + \frac{b^2}{4}\sin[2(\theta_d-\theta)]\right\}\phi\,\mathrm{d}t.
\end{aligned} \tag{3.23}$$

Thus θ solves the ODE (3.16) in the weak sense. Following an argument similar to that from (1.41) to (1.42), we can also show that θ and solves the initial-value problem (3.18) and the entire sequence of $\{\theta_\varepsilon\}$ converges to θ locally uniformly. $\square$

4. Single inverted pendulum on a flywheel

The single inverted pendulum whose support is attached to the rim of a flywheel is given as a special case of the pendulum driven in two dimensions in L. Blitzer's [**Bli65**]. While many types of fast oscillations of the pivot have a stabilizing or aligning effect on the inverted pendulum, We shall see that, as it turns out, a fast spinning flywheel does not stabilize the inverted pendulum.

4.1. Derivation of equation. Now we derive the governing equation for the motion of the inverted pendulum with its pivot attached to the rim of a flywheel. The pivot of the inverted pendulum is placed εbl units away from the center of the flywheel for some small positive parameter ε. Set up the coordinate axes as shown in Figure 6.

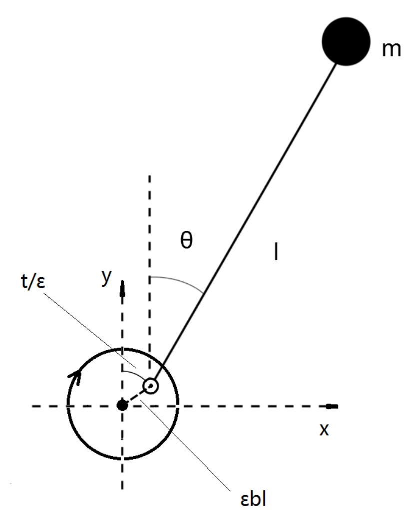

FIGURE 6. Single inverted pendulum on a flywheel

Let the position of the center of the mass m be (x, y). Then we have

$$x = \varepsilon bl \sin \frac{t}{\varepsilon} + l \sin \theta,$$

$$y = \varepsilon bl \cos \frac{t}{\varepsilon} + l \cos \theta.$$

And

$$x' = bl\cos\frac{t}{\varepsilon} + l\cos\theta\theta',$$
$$y' = -bl\sin\frac{t}{\varepsilon} - l\sin\theta\theta'.$$

Then the kinetic energy is given by

$$T = \frac{1}{2}m\left(bl\sin\frac{t}{\varepsilon} + l\sin\theta\theta'\right)^2 + \frac{1}{2}m(bl\cos\frac{t}{\varepsilon} + l\cos\theta\theta')^2 \tag{4.1}$$
$$= \frac{1}{2}ml^2(\theta')^2 + \frac{1}{2}mb^2l^2 + mbl^2\cos\left(\theta - \frac{t}{\varepsilon}\right)\theta'.$$

The potential energy is given by

$$V = mg\varepsilon bl\cos\frac{t}{\varepsilon} + mgl\cos\theta. \tag{4.2}$$

To write the Lagrange equation, we calculate $L = T - V$ and the following derivatives:

$$\frac{\partial L}{\partial\theta'} = ml^2\theta' + mbl^2\cos\left(\theta - \frac{t}{\varepsilon}\right), \tag{4.3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\theta'}\right) = ml^2\theta'' - mbl^2\sin\left(\theta - \frac{t}{\varepsilon}\right)\left(\theta' - \frac{1}{\varepsilon}\right) \tag{4.4}$$

$$\frac{\partial L}{\partial\theta} = -mbl^2\sin\left(\theta - \frac{t}{\varepsilon}\right)\theta' + mlg\sin\theta. \tag{4.5}$$

Therefore the Lagrange equation writes

$$\theta'' + \frac{b}{\varepsilon}\sin\left(\theta - \frac{t}{\varepsilon}\right) - \frac{g}{l}\sin\theta = 0. \tag{4.6}$$

In the next two subsections, we will discuss its stability conditions.

4.2. Asymptotic expansions. Adopting our notations from previous sections, the equation of motion for a single pendulum on a base attached to a flywheel is

$$\theta'' + \frac{b}{\varepsilon}\sin\left(\theta - \frac{t}{\varepsilon}\right) - a\sin\theta = 0. \tag{4.7}$$

Perform asymptotic expansions, write θ as

$$\theta_\varepsilon = \Theta(t, \frac{t}{\varepsilon}, \varepsilon),$$

where $\Theta = \Theta(t, s, \varepsilon)$ is 2π-periodic in s. Plug in (4.7) and get

$$\Theta_{tt} + \frac{2}{\varepsilon}\Theta_{ts} + \frac{1}{\varepsilon^2}\Theta_{ss} - a\sin\Theta - \frac{b}{\varepsilon}\cos s\cos\Theta = 0. \tag{4.8}$$

Write out the asymptotic expansion for Θ

$$\Theta = \Theta^0 + \varepsilon\Theta^1 + \varepsilon^2\Theta^2 + \cdots, \tag{4.9}$$

where $\Theta^k = \Theta^k(t, s)$ is 2π-periodic in s, $k = 0, 1, 2, \cdots$.

Plug (4.9) into (4.8) to find,

$$\begin{aligned}\Theta^0_{tt} + \frac{2}{\varepsilon}\Theta^0_{ts} + \frac{1}{\varepsilon^2\Theta^0_{ss}} + \varepsilon\left(\Theta^1_{tt} + \frac{2}{\varepsilon}\Theta^1_{ts} + \frac{1}{\varepsilon^2}\Theta^1_{ss}\right) +& \\ \varepsilon^2\left(\Theta^2_{tt} + \frac{2}{\varepsilon}\Theta^2_{ts} + \frac{1}{\varepsilon^2}\Theta^2_{ss}\right)\cdots& \\ = a\left(\sin\Theta^0 + \varepsilon\cos\Theta^0\Theta^1\right) - \frac{b}{\varepsilon}\cos s\left(\sin\Theta^0 - \varepsilon\cos\Theta^0\Theta^1\right)& \\ + \frac{b}{\varepsilon}\sin s\left(\cos\Theta^0 - \varepsilon\sin\Theta^0\Theta^1\right)\cdots.&\end{aligned} \tag{4.10}$$

Replace t/ε with s. Treat t and s as two independent variables. Identify powers of ε,

(4.11)

$$\boldsymbol{O(\varepsilon^{-2})}: \qquad \Theta^0_{ss} = 0$$

(4.12)

$$\boldsymbol{O(\varepsilon^{-1})}: \qquad 2\Theta^0_{ts} + \Theta^1_{ss} = b\sin s\cos\Theta^0 - b\cos s\sin\Theta^0 = b\sin(s - \Theta^0)$$

$$\begin{aligned}\boldsymbol{O(1)}: \qquad \Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} + b\cos s\cos\Theta^0\Theta^1& \\ + b\sin s\sin\Theta^0\Theta^1 - a\sin\Theta^0 = 0.&\end{aligned} \tag{4.13}$$

Look at $O(\varepsilon^{-2})$ terms. (4.11) and periodicity in s implies

$$\Theta^0 = \Theta^0(t). \tag{4.14}$$

In $O(\varepsilon^{-1})$ terms, by (2.14), $\Theta^0_{ts} \equiv 0$. So (4.12) implies

$$\Theta^1_{ss} = b\sin(s - \Theta^0).$$

Thus

$$\Theta^1(t,s) = b\sin(\Theta^0 - s). \tag{4.15}$$

$O(1)$ terms, plug (4.15) in (4.13) to get

$$\Theta^0_{tt} + 2\Theta^1_{ts} + \Theta^2_{ss} + b^2\sin(\Theta^0 - s)\cos(\Theta^0 - s) - a\sin\Theta^0 = 0.$$

Integrate with respect to s form 0 to 2π and divide through by 2π, we get

$$\Theta^0_{tt} - a\sin\Theta^0 = 0. \tag{4.16}$$

This is just a regular single pendulum. Intuitively, this is because a flywheel has no preference in any direction. A rigorous proof for the convergence of the equation is provided in the next subsection.

4.3. Proof of convergence. Consider the following initial-value problem:

$$\begin{cases} \theta_\varepsilon'' + \dfrac{b}{\varepsilon}\sin\left(\theta_\varepsilon - \dfrac{t}{\varepsilon}\right) - a\sin\theta_\varepsilon = 0 \\ \theta_\varepsilon(0) = \alpha \\ \theta_\varepsilon'(0) = \beta, \end{cases} \tag{4.17}$$

where we have written θ as θ_ε to show the dependence of the solution on ε. We will prove that the limiting equation, as ε goes to 0, is

$$\begin{cases} \theta'' - a\sin\theta = 0 \\ \theta(0) = \alpha \\ \theta'(0) = \beta. \end{cases} \tag{4.18}$$

We begin with an analogue of Lemma 2.19 and get the following lemma.

Lemma 4.1. For each $T > 0$,

$$\max_{0\le t\le T} |\theta_\varepsilon|, |\theta_\varepsilon'| \le C_T \tag{4.19}$$

Thus we also have

$$\begin{cases} \theta_\varepsilon \to \theta \text{ uniformly on } [0,T] \\ \theta_\varepsilon' \rightharpoonup \theta' \text{ weakly in } L^2(0,T) \end{cases}$$

through a subsequence, as $\varepsilon \to 0$.

Theorem 4.2. *θ solves* (4.18).

PROOF. We rewrite (4.17) as

$$\theta_\varepsilon'' - b\left[\cos\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\right]' + b\sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\theta_\varepsilon' - a\sin\theta_\varepsilon = 0. \tag{4.20}$$

Therefore

$$\left[\theta_\varepsilon' - b\cos\left((\theta_\varepsilon - \frac{t}{\varepsilon}\right)\right]' + b\sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\theta_\varepsilon' - a\sin\theta_\varepsilon = 0. \tag{4.21}$$

Let $\phi = \phi(t)$ be smooth, vanishing near $t = 0$ and $t = T$. Multiply (4.21) by ϕ and integrate to get

$$\begin{aligned} -\int_0^T \left[\theta_\varepsilon' - b\cos\left((\theta_\varepsilon - \frac{t}{\varepsilon}\right)\right]\phi'\,\mathrm{d}t = \\ -\int_0^T b\sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\theta_\varepsilon'\phi\,\mathrm{d}t + \int_0^T a\sin\theta_\varepsilon\phi\,\mathrm{d}t. \end{aligned} \tag{4.22}$$

Let

$$\begin{cases} A := -\displaystyle\int_0^T \left[\theta_\varepsilon' - b\cos\left(\frac{t}{\varepsilon} - \theta_\varepsilon\right)\right]\phi'\,\mathrm{d}t \\ B := -\displaystyle\int_0^T b\sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\theta_\varepsilon'\phi\,\mathrm{d}t \\ C := \displaystyle\int_0^T a\sin\theta_\varepsilon\phi\,\mathrm{d}t. \end{cases}$$

Then

$$\lim_{\varepsilon\to 0} A = -\int_0^T \theta'\phi'\,\mathrm{d}t,$$

since $\cos\theta_\varepsilon \to \cos\theta$ and $\sin(t/\varepsilon)$, $\cos(t/\varepsilon) \rightharpoonup 0$ in $L^2(0,T)$ as $\varepsilon \to 0$. Similarly

$$\lim_{\varepsilon\to 0} C = \int_0^T a\sin\theta\phi\,\mathrm{d}t.$$

$$\begin{aligned} B &= -\int_0^T b\sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\theta_\varepsilon'\phi\,\mathrm{d}t \\ &= -b\int_0^T \varepsilon\sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)(\theta_\varepsilon')^2\phi - \varepsilon\left[\cos\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\right]'\theta_\varepsilon'\phi\,\mathrm{d}t \\ &= O(\varepsilon) - b\varepsilon\int_0^T \left[\cos\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\right]'\theta_\varepsilon'\phi\,\mathrm{d}t \\ &= O(\varepsilon) + b\varepsilon\int_0^T \cos\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)(\theta_\varepsilon''\phi + \theta_\varepsilon'\phi')\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= O(\varepsilon) + b\varepsilon \int_0^T \cos(\theta_\varepsilon - \frac{t}{\varepsilon}) \left[a \sin \theta_\varepsilon - \frac{b}{\varepsilon} \sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right)\right] \phi \,\mathrm{d}t \\
&= O(\varepsilon) - b \int_0^T \cos\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right) \sin\left(\theta_\varepsilon - \frac{t}{\varepsilon}\right) \phi \,\mathrm{d}t \\
&= O(\varepsilon) - \frac{b}{2} \int_0^T \sin\left(2\theta_\varepsilon - 2\frac{t}{\varepsilon}\right) \phi \,\mathrm{d}t \\
&= O(\varepsilon) - \frac{b}{2} \int_0^T \left[\sin(2\theta_\varepsilon) \cos\left(\frac{2t}{\varepsilon}\right) - \cos(2\theta_\varepsilon) \sin\left(\frac{2t}{\varepsilon}\right)\right] \phi \,\mathrm{d}t \\
&\to 0,
\end{aligned}
$$

as $\varepsilon \to 0$. Thus passing to the limit we get

$$
-\int_0^T \theta' \phi' \,\mathrm{d}t = \int_0^T a \sin \theta \phi \,\mathrm{d}t.
$$

Thus θ solves the ODE (4.16) in the weak sense. Following an argument similar to that from (1.41) to (1.42), we can also show that θ and solves the initial-value problem (4.18) and the entire sequence of $\{\theta_\varepsilon\}$ converges to θ locally uniformly. $\square$

5. Double inverted pendula

Compared to single inverted pendula, stability of an inverted N-pendulum is relatively not very widely-studied. It can be easily understood, as the literature on single inverted pendula has already provided the foundation needed. For example, as we mentioned earlier, Acheson gave a general result for stability condition of an inverted N-pendulum in [**Ach93**]. As a generalization of the single inverted pendulum, we will herein look at the case of double inverted pendulum on a vertically oscillatory base, and shall show that it can be stabilized in a similar fashion. Derivation of equations and numerical analysis has been carried out in, for example, S. Nikolov's paper [**Nik12**] and I. M. Arkhipova and A. Luongo and A. P. Seyranian's paper [**ALS12**]. Especially, in [**ALS12**] I. M. Arkhipova, A. Luongo and A. P. Seyranian have used multiple scale methods to obtain the generating equations of the perturbation process and from there given parametric stability conditions.

Our work here will give an argument for finding the limiting equations and giving a proof at the same time.

5.1. Derivation of equation. A regular double inverted pendulum is set up as shown in Figure 7.

Put it on a vertically oscillatory base given by

$$y_\varepsilon = -\varepsilon b \cos\frac{t}{\varepsilon},$$

as shown in Figure 8.

Let the center of mass of m_1 and m_2 be (x_1, y_1) and (x_2, y_2) respectively. Then we have

$$x_1 = l_1 \sin\theta_1^\varepsilon,$$
$$y_1 = l_1 \cos\theta_1^\varepsilon - \varepsilon b \cos\frac{t}{\varepsilon},$$
$$x_2 = l_1 \sin\theta_1^\varepsilon + l_2 \sin\theta_2^\varepsilon,$$
$$y_2 = l_1 \cos\theta_1^\varepsilon + l_2 \cos\theta_2^\varepsilon - \varepsilon b \cos\frac{t}{\varepsilon}.$$

Then

$$x_1' = l_1 \cos\theta_1^\varepsilon (\theta_1^\varepsilon)',$$
$$y_1' = -l_1 \sin\theta_1^\varepsilon (\theta_1^\varepsilon)' + b \sin\frac{t}{\varepsilon},$$

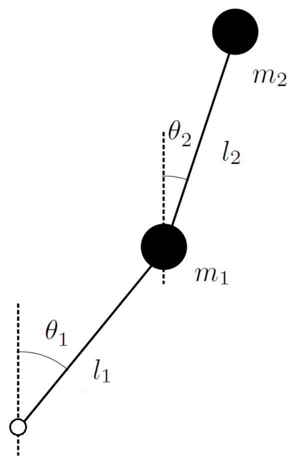

FIGURE 7. Double inverted pendulum

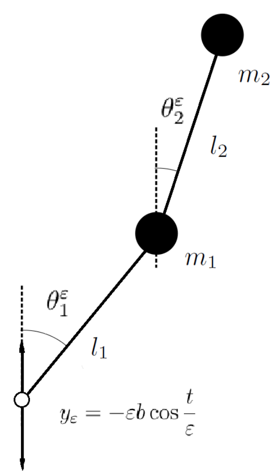

FIGURE 8. Double inverted pendulum on a vertically oscillatory base

$$x_2' = l_1 \cos \theta_1^\varepsilon (\theta_1^\varepsilon)' + l_2 \cos \theta_2^\varepsilon (\theta_2^\varepsilon)',$$

$$y_2' = -l_1 \sin\theta_1^\varepsilon(\theta_1^\varepsilon)' - l_2 \sin\theta_2^\varepsilon(\theta_2^\varepsilon)' + b\sin\frac{t}{\varepsilon}.$$

The potential energy of the system is given by

(5.1)

$$V = m_1 g\left(l_1\cos\theta_1^\varepsilon - \varepsilon b\cos\frac{t}{\varepsilon}\right) + m_2 g\left(l_1\cos\theta_1^\varepsilon + l_2\cos\theta_2^\varepsilon - \varepsilon b\cos\frac{t}{\varepsilon}\right).$$

And the kinetic energy is given by

$$\begin{aligned}T = {}& \frac{1}{2}m_1\left(l_1^2((\theta_1^\varepsilon)')^2 + b^2\sin^2\frac{t}{\varepsilon} - 2bl_1(\theta_1^\varepsilon)'\sin\theta_1^\varepsilon\sin\frac{t}{\varepsilon}\right)\\ &+ \frac{1}{2}m_2\Big(l_1^2((\theta_1^\varepsilon)')^2 + l_2^2((\theta_2^\varepsilon)')^2 + 2l_1l_2\cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)'(\theta_2^\varepsilon)'\\ &\qquad -2bl_1(\theta_1^\varepsilon)'\sin\theta_1^\varepsilon\sin\frac{t}{\varepsilon} - 2bl_2(\theta_2^\varepsilon)'\sin\theta_2^\varepsilon\sin\frac{t}{\varepsilon}\Big).\end{aligned}$$

Thus the Lagrangian is

$$\begin{aligned}L = {}& T - V\\ = {}& \frac{1}{2}m_1\left(l_1^2((\theta_1^\varepsilon)')^2 + b^2\sin^2\frac{t}{\varepsilon} - 2bl_1(\theta_1^\varepsilon)'\sin\theta_1^\varepsilon\sin\frac{t}{\varepsilon}\right)\\ &+ \frac{1}{2}m_2\Big(l_1^2((\theta_1^\varepsilon)')^2 + l_2^2((\theta_2^\varepsilon)')^2 + 2l_1l_2\cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)'(\theta_2^\varepsilon)'\\ &\qquad -2bl_1(\theta_1^\varepsilon)'\sin\theta_1^\varepsilon\sin\frac{t}{\varepsilon} - 2bl_2(\theta_2^\varepsilon)'\sin\theta_2^\varepsilon\sin\frac{t}{\varepsilon}\Big)\\ &- m_1 g\left(l_1\cos\theta_1^\varepsilon - \varepsilon b\cos\frac{t}{\varepsilon}\right) - m_2 g\left(l_1\cos\theta_1^\varepsilon + l_2\cos\theta_2^\varepsilon - \varepsilon b\cos\frac{t}{\varepsilon}\right).\end{aligned}$$

We calculate $\dfrac{\partial L}{\partial(\theta_1^\varepsilon)'}, \dfrac{\partial L}{\partial\theta_1^\varepsilon}, \dfrac{\partial L}{\partial(\theta_1^\varepsilon)'}, \dfrac{\partial L}{\partial\theta_2^\varepsilon}, \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial(\theta_1^\varepsilon)'}\right)$ and $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial(\theta_1^\varepsilon)'}\right)$ and get

$$\begin{aligned}\text{(5.2)}\quad \frac{\partial L}{\partial(\theta_1^\varepsilon)'} = {}& (m_1+m_2)l_1^2(\theta_1^\varepsilon)'\\ &- (m_1+m_2)bl_1\sin\theta_1^\varepsilon\sin\frac{t}{\varepsilon} + m_2l_1l_2\cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_2^\varepsilon)',\end{aligned}$$

(5.3)

$$\begin{aligned}\frac{\partial L}{\partial\theta_1^\varepsilon} = {}& -(m_1+m_2)bl_1\cos\theta_1^\varepsilon\sin\frac{t}{\varepsilon}(\theta_1^\varepsilon)' - m_2l_1l_2\sin(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)'(\theta_2^\varepsilon)'\\ &+ (m_1+m_2)l_1 g\sin\theta_1^\varepsilon,\end{aligned}$$

$$\frac{\partial L}{\partial(\theta_2^\varepsilon)'} = m_2 l_2^2 \theta_2^\varepsilon + m_2 l_1 l_2 \cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)' - m_2 b l_2 \sin\theta_2^\varepsilon \sin\frac{t}{\varepsilon}, \tag{5.4}$$

$$\frac{\partial L}{\partial \theta_2^\varepsilon} = m_2 l_1 l_2 \sin(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)'(\theta_2^\varepsilon)' - m_2 b l_2 \cos\theta_2^\varepsilon \sin\frac{t}{\varepsilon}(\theta_2^\varepsilon)' + m_2 l_2 g \sin\theta_2^\varepsilon, \tag{5.5}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial(\theta_1^\varepsilon)'}\right) = (m_1+m_2) l_1^2 (\theta_1^\varepsilon)'' &- m_1 b l_1 \cos\theta_1^\varepsilon \sin\frac{t}{\varepsilon}(\theta_1^\varepsilon)' \\ &- m_1\frac{b}{\varepsilon} l_1 \sin\theta_1^\varepsilon \cos\frac{t}{\varepsilon} - m_2 l_1 l_2 \sin(\theta_1^\varepsilon \\ - \theta_2^\varepsilon)((\theta_1^\varepsilon)' - (\theta_2^\varepsilon)')(\theta_2^\varepsilon)' &+ m_2 l_1 l_2 \cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_2^\varepsilon)'' \\ &- m_2\frac{b}{\varepsilon} l_1 \sin\theta_1^\varepsilon \cos\frac{t}{\varepsilon} - m_2 b l_1 \cos\theta_1^\varepsilon \sin\frac{t}{\varepsilon}(\theta_1^\varepsilon)',\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial(\theta_2^\varepsilon)'}\right) &= m_2 l_2^2 (\theta_2^\varepsilon)'' + m_2 l_1 l_2 \cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)'' \\ &\quad - m_2 l_1 l_2 \sin(\theta_1^\varepsilon - \theta_2^\varepsilon)((\theta_1^\varepsilon)' - (\theta_2^\varepsilon)')(\theta_1^\varepsilon)' \\ &\quad - m_2\frac{b}{\varepsilon} l_2 \sin\theta_2^\varepsilon \cos\frac{t}{\varepsilon} - m_2 b l_2 \cos\theta_2^\varepsilon \sin\frac{t}{\varepsilon}(\theta_2^\varepsilon)'.\end{aligned}$$

So the Lagrange equations are

$$\begin{cases} (m_1+m_2) l_1 (\theta_1^\varepsilon)'' + m_2 l_2 \cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_2^\varepsilon)'' & (A^\varepsilon) \\ \qquad + m_2 l_2 \sin(\theta_1^\varepsilon - \theta_2^\varepsilon)((\theta_2^\varepsilon)')^2 & \\ \qquad = \left[(m_1+m_2) g + \frac{b}{\varepsilon}(m_1+m_2)\cos\frac{t}{\varepsilon}\right]\sin\theta_1^\varepsilon & \\ l_2(\theta_2^\varepsilon)'' + l_1\cos(\theta_1^\varepsilon - \theta_2^\varepsilon)(\theta_1^\varepsilon)'' - l_1 \sin(\theta_1^\varepsilon - \theta_2^\varepsilon)((\theta_1^\varepsilon)')^2 & (B^\varepsilon) \\ \qquad = \left(g + \frac{b}{\varepsilon}\cos\frac{t}{\varepsilon}\right)\sin\theta_2^\varepsilon. & \end{cases}$$

5.2. Proof of convergence. For the purpose of this expository work, we suppose that the magnitude and velocity of the motion remain bounded, that is, we make the convenient assumption that for $T > 0$,

$$\max_{0\le t\le T} |(\theta_1^\varepsilon)'|, |(\theta_2^\varepsilon)'| \le C_T. \tag{5.6}$$

Passing through a subsequence, we have, as $\varepsilon \to 0$,

$$\text{(5.7)} \qquad \begin{cases} \theta_1^\varepsilon \to \theta_1, \theta_2^\varepsilon \to \theta_2 \text{ uniformly on } [0,T] \\ (\theta_1^\varepsilon)' \rightharpoonup {\theta_1}', (\theta_1^\varepsilon)' \rightharpoonup {\theta_1}' \text{ weakly in } L^2(0,T). \end{cases}$$

To find the limiting equations of A^ε and B^ε, we will need to compute the weak limit of $(\theta_1^\varepsilon)' \sin(t/\varepsilon)$, $(\theta_2^\varepsilon)' \sin(t/\varepsilon)$ and $(\theta_1^\varepsilon)'(\theta_2^\varepsilon)'$. And the results are given by Lemma 5.1, Lemma 5.2 and Lemma 5.4 below.

Lemma 5.1.

$$(\theta_1^\varepsilon)' \sin \frac{t}{\varepsilon} \rightharpoonup \frac{b}{2l_1} \eta_1(\theta_1, \theta_2) \textit{ weakly in } L^2(0,T),$$

as $\varepsilon \to 0$, *where*

$$\text{(5.8)} \quad \eta_1(x,y) = \frac{(m_1+m_2)\sin x - m_2 \sin y \cos(x-y)}{m_1+m_2-m_2\cos^2(x-y)} \quad \textit{or } \frac{m_1 \sin x + m_2 \cos y \sin(x-y)}{m_1 + m_2 \sin^2(x-y)}.$$

Proof. For all $\psi(t) \in C_c^\infty[0,T]$, we have

(5.9)

$$\int_0^T (\theta_1^\varepsilon)' \sin \frac{t}{\varepsilon} \psi \, dt = -\int_0^T (\theta_1^\varepsilon)' \left(\varepsilon \cos \frac{t}{\varepsilon} \right)' \psi \, dt = -\int_0^T \varepsilon (\theta_1^\varepsilon)'' \cos \frac{t}{\varepsilon} \psi \, dt + O(\varepsilon).$$

To find $(\theta_1^\varepsilon)''$ using A^ε and B^ε, we calculate the following:

$$\frac{1}{(m_1+m_2)l_1 - m_1 l_1 \cos^2(\theta_1^\varepsilon - \theta_2^\varepsilon)} \cdot \left[A^\varepsilon - m_2 \cos(\theta_1^\varepsilon - \theta_2^\varepsilon) B^\varepsilon \right],$$

thus

$$\text{(5.10)} \qquad (\theta_1^\varepsilon)'' = O(1) + \frac{b}{l_1 \varepsilon} \cos \frac{t}{\varepsilon} \eta_1(\theta_1^\varepsilon, \theta_2^\varepsilon).$$

Therefore

$$\text{(5.11)} \qquad \varepsilon(\theta_1^\varepsilon)'' = O(\varepsilon) + \frac{b}{l_1} \cos \frac{t}{\varepsilon} \eta_1(\theta_1^\varepsilon, \theta_2^\varepsilon).$$

Plug (5.11) in (5.9). Since $\cos^2(t/\varepsilon) \rightharpoonup 1/2$ weakly in $L^2(0,T)$ as $\varepsilon \to 0$ and since (5.7) holds, we get

$$\text{(5.12)} \qquad \int_0^T (\theta_1^\varepsilon)' \sin \frac{t}{\varepsilon} \psi \, dt \to \int_0^T \frac{b}{2l_1} \eta_1(\theta_1, \theta_2) \psi \, dt.$$

$\square$

LEMMA 5.2.

$$(\theta_2^\varepsilon)' \sin\frac{t}{\varepsilon} \rightharpoonup \frac{b}{2l_2}\eta_2(\theta_1,\theta_2) \text{ weakly in } L^2(0,T),$$

as $\varepsilon \to 0$, *where*

$$\eta_2(x,y) = \frac{(m_1+m_2)\sin y - (m_1+m_2)\cos(y-x)\sin x}{m_1+m_2-m_2\cos^2(y-x)} \quad \textit{or} \quad \frac{(m_1+m_2)\sin(y-x)\cos x}{m_1+m_2\sin^2(x-y)}. \tag{5.13}$$

PROOF. For all $\psi(t) \in C_c^\infty[0,T]$, we have

$$\int_0^T (\theta_2^\varepsilon)' \sin\frac{t}{\varepsilon}\psi\, dt = -\int_0^T (\theta_2^\varepsilon)'\left(\varepsilon\cos\frac{t}{\varepsilon}\right)'\psi\, dt = -\int_0^T \varepsilon(\theta_2^\varepsilon)''\cos\frac{t}{\varepsilon}\psi\, dt + O(\varepsilon). \tag{5.14}$$

To find $(\theta_2^\varepsilon)''$ using A^ε and B^ε, calculate the following,

$$\frac{1}{m_2 l_2\cos^2(\theta_1^\varepsilon-\theta_2^\varepsilon)-(m_1+m_2)l_2}\cdot\left[\cos(\theta_1^\varepsilon-\theta_2^\varepsilon)A^\varepsilon-(m_1+m_2)B^\varepsilon\right],$$

which yields

$$(\theta_2^\varepsilon)'' = O(1) + \frac{b}{l_2\varepsilon}\cos\frac{t}{\varepsilon}\eta_2(\theta_2^\varepsilon,\theta_2^\varepsilon). \tag{5.15}$$

Thus

$$\varepsilon(\theta_2^\varepsilon)'' = O(\varepsilon) + \frac{b}{l_2}\cos\frac{t}{\varepsilon}\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon). \tag{5.16}$$

Plug (5.16) in (5.14). Since $\cos^2(t/\varepsilon) \rightharpoonup 1/2$ weakly in $L^2(0,T)$ as $\varepsilon \to 0$ and by (5.7), we get

$$\int_0^T (\theta_2^\varepsilon)'\sin\frac{t}{\varepsilon}\psi\, dt \to \int_0^T \frac{b}{2l_2}\eta_2(\theta_1,\theta_2)\psi\, dt. \tag{5.17}$$

$\square$

To prove the weak convergence of $(\theta_1^\varepsilon)'(\theta_2^\varepsilon)'$, we need an application of the Div-Curl Lemma (see [**Eva90**]) to the simple case of $L^2(0,T)$.

We firstly briefly introduce convergence in the sense of distribution and refer the reader to D. Cioranescu and P. Donato's book [**CD99**] where the following definitions are quoted from.

Let O be an open set, we denote by $D(O)$ the set of infinitely differentiable functions whose support is a compact subset of $\mathbb{R}^N$ contained in O. For a sequence $\{\psi_n\}$ in $D(O)$, we say that ψ_n converges to an element $\psi \in D(O)$, if and only if: (i) there exists a compact set $K \subset O$ such that, for any $n \in \mathbb{N}$, supp $\psi_n \subset K$; (ii) for any $\alpha \in \mathbb{N}^N$, $\partial^\alpha \psi_n$ converges uniformly to $\partial^\alpha \psi$ on K.

We say a map $T : D(O) \to \mathbb{R}$ is a distribution on O, if and only if:
(i) T is linear, that is, for all

$$\lambda_1, \lambda_2 \in R,\ \psi_1, \psi_2 \in D(O),\ T(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1 T(\psi_1) + \lambda_2 T(\psi_2),$$

(ii) T is continuous on sequences, that is,

$$(\psi_n \to \psi \text{ in } D(O)) \Rightarrow (T(\psi_n) \to T(\psi)).$$

We denote by $D'(O)$ the set of distributions on O.

A sequence T_n in $D'(O)$ is said to converge (in the sense of distributions) to an element $T \in D'(O)$ if and only if

$$\langle T_n, \psi\rangle_{D'(O),D(O)} \to \langle T, \psi\rangle_{D'(O),D(O)}, \text{ for all } \psi \in D(O).$$

We denote this convergence by

$$T_n \to T \text{ in } D'(O).$$

Lemma 5.3. (Div-Curl Lemma [**Eva90**]) *Assume that* $\{v_k\}_{k=1}^\infty$, $\{w_k\}_{k=1}^\infty$ *are two bounded sequences in* $L^2(U;\mathbb{R}^N)$ *such that*

(i) $\{\operatorname{div} v_k\}_{k=1}^\infty$ *lies in a compact subset of* $W^{-1,2}(U)$,

(ii) $\{\operatorname{curl} w_k\}_{k=1}^\infty$ *lies in a compact subset of* $W^{-1,2}(U; M^{N\times N})$. *Suppose further* $v_k \rightharpoonup v$, $w_k \rightharpoonup w$ *in* $L^2(U;\mathbb{R}^N)$. *Then*

$$v_k \cdot w_k \to v \cdot w$$

in the sense of distributions.

With the aid of the Div-Curl Lemma we will be able to prove the next lemma.

Lemma 5.4.

$$(\theta_1^\varepsilon)'(\theta_2^\varepsilon)' \rightharpoonup {\theta_1}'{\theta_2}' + \frac{b^2}{2l_1l_2}\eta_1(\theta_1,\theta_2)\eta_2(\theta_1,\theta_2) \text{ weakly in } L^2(0,T),$$

as $\varepsilon \to 0$.

Proof. It is easy to see from (5.10) that,

$$\left[(\theta_1^\varepsilon)' - \frac{b}{l_1}\sin\frac{t}{\varepsilon}\eta_1(\theta_1^\varepsilon,\theta_2^\varepsilon)\right]' = O(1) \tag{5.18}$$

and

$$\left[(\theta_2^\varepsilon)' - \frac{b}{l_2}\sin\frac{t}{\varepsilon}\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon)\right]' = O(1). \tag{5.19}$$

Since $\sin(t/\varepsilon) \rightharpoonup (1/2)$ weakly in $L^2(0,T)$ as $\varepsilon \to 0$, by (5.7) we also have

$$(\theta_1^\varepsilon)' - \frac{b}{l_1}\sin\frac{t}{\varepsilon}\eta_1(\theta_1^\varepsilon,\theta_2^\varepsilon) \rightharpoonup {\theta_1}' \tag{5.20}$$

and

$$(\theta_2^\varepsilon)' - \frac{b}{l_2}\sin\frac{t}{\varepsilon}\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon) \rightharpoonup {\theta_2}', \tag{5.21}$$

weakly in $L^2(0,T)$ as $\varepsilon \to 0$.

Combining (5.18), (5.19), (5.20) and (5.21) together, and using the Div-Curl Lemma, we see that

$$\left[(\theta_1^\varepsilon)' - \frac{b}{l_1}\sin\frac{t}{\varepsilon}\eta_1(\theta_1^\varepsilon,\theta_2^\varepsilon)\right]\cdot\left[(\theta_2^\varepsilon)' - \frac{b}{l_2}\sin\frac{t}{\varepsilon}\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon)\right] \rightharpoonup {\theta_1}'{\theta_2}', \tag{5.22}$$

weakly in $L^2(0,T)$ as $\varepsilon \to 0$.

But

(5.23)

$$\begin{aligned}(\theta_1^\varepsilon)'(\theta_2^\varepsilon)' = &\left[(\theta_1^\varepsilon)' - \frac{b}{l_1}\sin\frac{t}{\varepsilon}\eta_1(\theta_1^\varepsilon,\theta_2^\varepsilon)\right]\left[(\theta_2^\varepsilon)' - \frac{b}{l_2}\sin\frac{t}{\varepsilon}\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon)\right] + \\ &\frac{b}{l_1}\left((\theta_2^\varepsilon)'\sin\frac{t}{\varepsilon}\right)\eta_1(\theta_1^\varepsilon,\theta_2^\varepsilon) + \frac{b}{l_2}\left((\theta_1^\varepsilon)'\sin\frac{t}{\varepsilon}\right)\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon) - \\ &\frac{b^2}{l_1 l_2}\sin^2\frac{t}{\varepsilon}\eta_1(\theta_1^\varepsilon,\theta_2^\varepsilon)\eta_2(\theta_1^\varepsilon,\theta_2^\varepsilon).\end{aligned}$$

Thus by (5.24), Lemma 5.1, Lemma 5.2 and the fact that

$$\sin^2\left(\frac{t}{\varepsilon}\right) \rightharpoonup \frac{1}{2}, \tag{5.24}$$

weakly in $L^2(0,T)$ as $\varepsilon \to 0$, we conclude that

(5.25)
$$
\begin{aligned}
(\theta_1^\varepsilon)'(\theta_2^\varepsilon)' \rightharpoonup & {\theta_1}'{\theta_2}' + \frac{b}{l_1}\left[\frac{b}{2l_2}\eta_2(\theta_1,\theta_2)\right]\eta_1(\theta_1,\theta_2) \\
& + \frac{b}{l_2}\left[\frac{b}{2l_1}\eta_1(\theta_1,\theta_2)\right]\eta_w(\theta_1,\theta_2) - \frac{b^2}{2l_1l_2}\eta_1(\theta_1,\theta_2)\eta_2(\theta_1,\theta_2) \\
& = {\theta_1}'{\theta_2}' + \frac{b^2}{2l_1l_2}\eta_1(\theta_1,\theta_2)\eta_2(\theta_1,\theta_2),
\end{aligned}
$$

weakly in $L^2(0,T)$ as $\varepsilon \to 0$. $\square$

Now we proceed to finding the limiting equations of A^ε and B^ε. Multiply A^ε by a test function $\phi(t) \in C_c^\infty[0,T]$ and integrate from 0 to T. Upon integration by parts, we get

(5.26)
$$
\begin{aligned}
& -\underbrace{\int_0^T (m_1+m_2)l_1(\theta_1^\varepsilon)'\phi'\,dt}_{\mathrm{C}} - \underbrace{\int_0^T m_2l_2\cos(\theta_1^\varepsilon-\theta_2^\varepsilon)(\theta_2^\varepsilon)'\phi'\,dt}_{\mathrm{D}} \\
& + \underbrace{\int_0^T m_2l_2\sin(\theta_1^\varepsilon-\theta_2^\varepsilon)(\theta_1^\varepsilon)'(\theta_2^\varepsilon)'\phi\,dt}_{\mathrm{E}} = \underbrace{\int_0^T (m_1+m_2)g\sin\theta_1^\varepsilon\phi\,dt}_{\mathrm{F}} \\
& - \underbrace{\int_0^T b(m_1+m_2)\sin\frac{t}{\varepsilon}\cos\theta_1^\varepsilon(\theta_1^\varepsilon)'\phi\,dt}_{\mathrm{G}} - \underbrace{\int_0^T b(m_1+m_2)\sin\frac{t}{\varepsilon}\sin\theta_1^\varepsilon\phi'\,dt}_{\mathrm{H}}.
\end{aligned}
$$

By (5.7), we can easily get the following convergence results.

$$
(5.27) \qquad \begin{cases}
\mathrm{C} \to \int_0^T (m_1+m_2)l_1{\theta_1}'\phi'\,dt \\
\mathrm{D} \to \int_0^T m_2l_2\cos(\theta_1-\theta_2)\theta_2'\;\phi'\,dt \\
\mathrm{F} \to \int_0^T (m_1+m_2)g\sin\theta_1\phi\,dt \\
\mathrm{H} \to 0,
\end{cases}
$$

as $\varepsilon \to 0$.

By (5.7) and Lemma 5.4, we get

(5.28)
$$\mathrm{E} \to \int_0^T m_2 l_2 \sin(\theta_1 - \theta_2) \left[{\theta_1}' {\theta_2}' + \frac{b^2}{2 l_1 l_2} \eta_1(\theta_1, \theta_2) \eta_2(\theta_1, \theta_2) \right] \phi \, dt,$$

as $\varepsilon \to 0$.

Using (5.7) and Lemma 5.1, we obtain

$$\mathrm{G} \to \int_0^T b(m_1 + m_2) \cos\theta_1 \cdot \frac{b}{2 l_1} \eta_1(\theta_1, \theta_2) \cdot \phi \, dt, \tag{5.29}$$

as $\varepsilon \to 0$. Thus as $\varepsilon \to 0$, (5.26) converges to

$$\begin{aligned}
\int_0^T & -(m_1 + m_2) l_1^2 {\theta_1}' \phi' - m_2 l_1 l_2 \cos(\theta_1 - \theta_2) {\theta_2}' \phi' \\
& + m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \left[{\theta_1}' {\theta_2}' + \frac{b^2}{2 l_1 l_2} \eta_1(\theta_1, \theta_2) \eta_2(\theta_1, \theta_2) \right] \phi \, dt \\
= & \int_0^T (m_1 l_1 + m_2 l_1) g \sin\theta_1 \phi - \frac{b^2}{2} (m_1 + m_2) \cos\theta_1 \eta_1(\theta_1, \theta_2) \phi \, dt.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
(A) \quad & (m_1 + m_2) l_1^2 {\theta_1}'' + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) {\theta_2}'' \\
& + m_2 l_1 l_2 \sin(\theta_1 - \theta_2) ({\theta_2}')^2 + \frac{b^2}{2} m_2 \sin(\theta_1 - \theta_2) \eta_1(\theta_1, \theta_2) \eta_2(\theta_1, \theta_2) \\
& = (m_1 + m_2) l_1 g \sin\theta_1 - \frac{b^2}{2} (m_1 + m_2) \cos\theta_1 \eta_1(\theta_1, \theta_2).
\end{aligned}$$

Similarly, using (5.7), Lemma 5.2 and Lemma 5.4, we can find the limiting equation of B^ε, which is as following,

$$\begin{aligned}
(B) \quad & l_2^2 {\theta_2}'' + l_1 l_2 \cos(\theta_1 - \theta_2) {\theta_1}'' + l_1 l_2 \sin(\theta_1 - \theta_2) ({\theta_1}')^2 + \\
& \frac{b^2}{2} \sin(\theta_1 - \theta_2) \eta_1(\theta_1, \theta_2) \eta_2(\theta_1, \theta_2) \\
& = l_2 g \sin\theta_2 - \frac{b^2}{2} \cos\theta_2 \eta_2(\theta_1, \theta_2).
\end{aligned}$$

Then we will examine the stability of the solutions.

Assuming $|\theta_1|, |\theta_2|, |\theta_1{}'|, |\theta_2{}'|$ are $O(\varepsilon)$, write $c := b^2/2$ and the linearization of A and B together is

(5.30)

$$\begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ l_1 l_2 & l_2^2 \end{pmatrix} \begin{pmatrix} \theta_1{}'' \\ \theta_2{}'' \end{pmatrix} =$$

$$\begin{pmatrix} (m_1 l_1 + m_2 l_1)g - c(m_1+m_2)^2/m_1 & \dfrac{c(m_1+m_2)m_2}{m_1} \\ \dfrac{c(m_1+m_2)}{m_1} & l_2 g - \dfrac{c(m_1+m_2)}{m_1} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \\ \theta_2 \end{pmatrix}.$$

Solve for $\theta_1{}'', \theta_2{}''$ and (5.30) becomes

$$\begin{pmatrix} \theta_1{}'' \\ \theta_2{}'' \end{pmatrix} = M \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{5.31}$$

where

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{5.32}$$

for

$$M_{11} = \frac{m_1+m_2}{m_1 l_1} g - \frac{(m_1+m_2)^2}{m_1^2 l_1^2} c - \frac{(m_1+m_2)m_2}{m_1^2 l_1 l_2} c, \tag{5.33}$$

$$M_{12} = \frac{(m_1+m_2)m_2}{m_1^2 l_1^2} c + \frac{(m_1+m_2)m_2}{m_1^2 l_1 l_2} c - \frac{m_2}{m_1 l_2} g, \tag{5.34}$$

$$M_{21=} \frac{(m_1+m_2)^2}{m_1^2 l_1 l_2} c + \frac{(m_1+m_2)^2}{m_1^2 l_2^2} c - \frac{m_1+m_2}{m_1 l_2} g, \tag{5.35}$$

$$M_{22} = \frac{m_1+m_2}{m_1 l_2} g - \frac{(m_1+m_2)m_2}{m_1^2 l_1 l_2} c - \frac{(m_1+m_2)^2}{m_2^2 l_2^2} c. \tag{5.36}$$

Now (5.31) has stable solutions for small initial conditions if and only if M has two distinct negative eigenvalues which is equivalent to the condition

(5.37)

$$c > \frac{g(m_1+m_2)(l_1+l_2) + g\sqrt{m_1+m_2}\sqrt{m_1(l_1-l_2)^2 + m_2(l_1+l_2)^2}}{2(m_1+m_2)},$$

that is,

(5.38)

$$b^2 > \frac{g(m_1+m_2)(l_1+l_2) + g\sqrt{m_1+m_2}\sqrt{m_1(l_1-l_2)^2 + m_2(l_1+l_2)^2}}{m_1+m_2}.$$

6. The Paul trap

6.1. Formulation and asymptotic expansions. Consider the motion of a particle $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ subject to a time-dependent force field of the following special form:

$$\boldsymbol{x}'' = a(t, \varepsilon)\boldsymbol{f}(\boldsymbol{x}), \tag{6.1}$$

where the 'acceleration' a is given by

$$a(t, \varepsilon) = \frac{1}{\varepsilon} A\left(\frac{t}{\varepsilon}\right) \tag{6.2}$$

where $A(s) : \mathbb{R} \mapsto \mathbb{R}$ is 1-periodic. We assume $\boldsymbol{f}$ is C^2 and write the gradient of $\boldsymbol{f} = (f^1, \dots, f^n) \in \mathbb{R}^n$ as

$$D\boldsymbol{f} = \begin{pmatrix} f^1_{x_1} & \cdots & f^1_{x_n} \\ \vdots & \ddots & \vdots \\ f^n_{x_1} & \cdots & f^n_{x_n} \end{pmatrix}. \tag{6.3}$$

Heuristically, let

$$\boldsymbol{x} = \boldsymbol{X}^0\left(t, \frac{t}{\varepsilon}\right) + \varepsilon \boldsymbol{X}^1\left(t, \frac{t}{\varepsilon}\right) + \varepsilon^2 \boldsymbol{X}^2\left(t, \frac{t}{\varepsilon}\right) + \cdots,$$

where

$$\boldsymbol{X}^k = \boldsymbol{X}^k(t, s)$$

is 1-periodic in s, $k = 0, 1, 2, \cdots$ and plug into (6.1).

We get

$$\begin{aligned} \boldsymbol{X}^0_{tt} + 2\varepsilon^{-1}\boldsymbol{X}^0_{ts} + \varepsilon^{-2}\boldsymbol{X}^0_{ss} + 2\boldsymbol{X}^1_{ts} + \varepsilon^{-1}\boldsymbol{X}^1_{ss} + \boldsymbol{X}^2_{ss} + o(1) \\ = \varepsilon^{-1} A\boldsymbol{f}(\boldsymbol{X}^0) + A\, D\boldsymbol{f}(\boldsymbol{X}^0)\boldsymbol{X}^1 + o(1). \end{aligned} \tag{6.4}$$

Then we identify powers of ε.

For $O(\varepsilon^{-2})$ terms, we get $\boldsymbol{X}^0_{ss} = 0$ which implies $\boldsymbol{X}^0$ does not depend on s.

For $O(\varepsilon^{-1})$ terms, we get $\boldsymbol{X}^1_{ss} = A\boldsymbol{f}(\boldsymbol{X}^0)$.

To solve for $\boldsymbol{X}^1$ in terms of $\boldsymbol{X}^0$, the solvability condition is there exists a 1-periodic function $C(s)$ s.t. $A(s) = C''(s)$ and thus $\overline{A} = 0$.

We solve for $\boldsymbol{X}^1$, and get

$$\boldsymbol{X}^1(t,s) = C(s)\boldsymbol{f}(\boldsymbol{X}^0(t)), \tag{6.5}$$

where

$$C(s) = \int_{s_0}^{s} B(\nu)\mathrm{d}\nu, \tag{6.6}$$

where

$$B(\nu) = \int_{\nu_0}^{\nu} A(\mu)\,\mathrm{d}\mu \tag{6.7}$$

for some appropriately chosen s_0 and ν_0.

For $O(1)$ terms, we get

$$\boldsymbol{X}^0_{tt} + 2\boldsymbol{X}^1_{ts} + \boldsymbol{X}^2_{ss} = A\,D\boldsymbol{f}(\boldsymbol{X}^0)\boldsymbol{X}^1. \tag{6.8}$$

We now plug (6.5) in (6.8), take the average with respect to s over one period and get

$$\boldsymbol{X}^0_{tt} = \overline{AC}\,D\boldsymbol{f}(\boldsymbol{X}^0)\boldsymbol{f}(\boldsymbol{X}^0) = -\overline{B^2}\,D\boldsymbol{f}(\boldsymbol{X}^0)\boldsymbol{f}(\boldsymbol{X}^0). \tag{6.9}$$

6.2. Proof of convergence. As in previous chapters, we want to prove the convergence result we get from asymptotic expansion, namely show that the solution to (6.1) converges to the solution to (6.9) as ε goes to zero. We look at the following initial-value problem:

$$\begin{cases} \boldsymbol{x}''_\varepsilon = \varepsilon^{-1} A\left(\dfrac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon) \\ \boldsymbol{x}_\varepsilon(0) = \boldsymbol{\gamma} \\ \boldsymbol{x}'_\varepsilon(0) = \boldsymbol{\delta}. \end{cases} \tag{6.10}$$

We will show that, $\boldsymbol{x}_\varepsilon$, the solution to (6.10), as ε goes to 0, converges locally uniformly on any finite interval to the solution to the following initial-value problem:

$$\begin{cases} \boldsymbol{x} = \overline{AC}\,D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) = -\overline{B^2}\,D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{x}_\varepsilon(0) = \boldsymbol{\gamma} \\ \boldsymbol{x}'_\varepsilon(0) = \boldsymbol{\delta}. \end{cases} \tag{6.11}$$

Lemma 6.1. *For each $T > 0$,*

$$\max_{0\le t\le T} |\boldsymbol{x}_\varepsilon|, |\boldsymbol{x}'_\varepsilon| \le C_T \tag{6.12}$$

PROOF.

$$\begin{aligned}\boldsymbol{x}_\varepsilon'' &= \varepsilon^{-1} A\left(\frac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\\ &= \left[B\left(\frac{t}{\varepsilon}\right)\right]'\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\\ &= \left[B\left(\frac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\right]' - B\left(\frac{t}{\varepsilon}\right)D\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\boldsymbol{x}_\varepsilon'.\end{aligned}$$

We rewrite the equation in (6.10) as

$$\left[\boldsymbol{x}' - B\left(\frac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\right]' = -B\left(\frac{t}{\varepsilon}\right)D\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\boldsymbol{x}_\varepsilon'. \tag{6.13}$$

Integrate from 0 to t to find

$$|\boldsymbol{x}_\varepsilon'(t)| \leq C_1 + C_2\int_0^t |\boldsymbol{x}_\varepsilon'|\,\mathrm{d}s,$$

for $0 \leq t \leq T$ and appropriately chosen constants C_1, $C_2 \leq 0$.

By Gronwall's inequality (integral form),

$$|\boldsymbol{x}_\varepsilon'(t)| \leq C_1\left(1 + C_2 t e^{C_1 t}\right) \leq C_T,$$

for each $0 \leq t \leq T$ and a constant $C_T > 0$ that only depends on T.

Thus $|\boldsymbol{x}_\varepsilon'|$ and $|\boldsymbol{x}_\varepsilon|$ are both bounded uniformly for $0 \leq t \leq T$ by some appropriately chosen constant $C_T > 0$ which only depends on T. □

Thus using (6.12) we have

$$\begin{cases}\boldsymbol{x}_\varepsilon \to \boldsymbol{x} \text{ uniformly on } [0,T]\\ \boldsymbol{x}_\varepsilon' \rightharpoonup \boldsymbol{x}' \text{ weakly in } L^2(0,T),\end{cases}$$

through a subsequence, as $\varepsilon \to 0$.

THEOREM 6.2. *$\boldsymbol{x}$ solves* (6.11).

PROOF. 1. Let $\boldsymbol{\phi} = \boldsymbol{\phi}(t) \in C_c^\infty([0,T];\mathbb{R}^n)$ be a test function, that is, smooth and vanishing near $t = 0$, $t = T$. Multiply (6.13) by $\boldsymbol{\phi}$ and integrate to get

(6.14)
$$\int_0^T -\boldsymbol{x}_\varepsilon' \cdot \boldsymbol{\phi}' + B\left(\frac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\phi}'\,\mathrm{d}t = -\int_0^T B\left(\frac{t}{\varepsilon}\right) D\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\boldsymbol{x}_\varepsilon'\cdot\boldsymbol{\phi}\,\mathrm{d}t.$$

The left hand side is

$$\int_0^T -\boldsymbol{x}_\varepsilon' \cdot \boldsymbol{\phi}' + B\left(\frac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\phi}'\,\mathrm{d}t \to \int_0^T -\boldsymbol{x}'\cdot\boldsymbol{\phi}'\,\mathrm{d}t, \tag{6.15}$$

since $\boldsymbol{x}_\varepsilon' \rightharpoonup \boldsymbol{x}'$, $\boldsymbol{f}(\boldsymbol{x}_\varepsilon) \to \boldsymbol{f}(\boldsymbol{x})$ and $B(t/\varepsilon) \rightharpoonup 0$ weakly in $L^2(0,T)$ as $\varepsilon \to 0$.

2. We claim

$$-B\left(\frac{t}{\varepsilon}\right)\boldsymbol{x}_\varepsilon' \rightharpoonup \overline{AC}\boldsymbol{f}(\boldsymbol{x}) \tag{6.16}$$

as $\varepsilon \to 0$.

To check this, for all $\boldsymbol{\psi} \in C_c^\infty([0,T];\mathbb{R}^n)$,

$$\begin{aligned}
\int_0^T -B\left(\frac{t}{\varepsilon}\right)\boldsymbol{x}_\varepsilon'\cdot\boldsymbol{\psi}\,\mathrm{d}t &= \int_0^T -\varepsilon\left[C\left(\frac{t}{\varepsilon}\right)\right]'\boldsymbol{x}_\varepsilon'\cdot\boldsymbol{\psi}\,\mathrm{d}t\\
&= \int_0^T \varepsilon C\left(\frac{t}{\varepsilon}\right)\boldsymbol{x}_\varepsilon''\cdot\boldsymbol{\psi} + \varepsilon C\left(\frac{t}{\varepsilon}\right)\boldsymbol{x}_\varepsilon'\cdot\boldsymbol{\psi}'\,\mathrm{d}t\\
&= \int_0^T A\left(\frac{t}{\varepsilon}\right)C\left(\frac{t}{\varepsilon}\right)\boldsymbol{f}(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\psi} + O(\varepsilon)\\
&\to \int_0^t \overline{AC}\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{\psi}\,\mathrm{d}t,
\end{aligned}$$

as $\varepsilon \to 0$.

This proves (6.16).

3. In summary, pass to the limit and we get

$$\int_0^T -\boldsymbol{x}'\cdot\boldsymbol{\phi}'\,\mathrm{d}t = \int_0^T \overline{AC}\,D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}),$$

which is the integral form of (6.9).

Thus $\boldsymbol{x}$ solves the ODE (6.9) in the weak sense. Following an argument similar to that from (1.41) to (1.42), we can then show that $\boldsymbol{x}$ and solves the initial-value problem (6.11) and the entire sequence of $\{\boldsymbol{x}_\varepsilon\}$ converges to θ locally uniformly. This completes the proof of the theorem. $\square$

CHAPTER 3

PDE analogues

1. PDE analogues

1.1. Asymptotic expansions. In this chapter, we will look at a PDE analogue of the ODE we analyzed. Suppose U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. We will look at a second-order elliptic equation of the following form:

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{1}{\varepsilon} c\left(\dfrac{x}{\varepsilon}, u^\varepsilon\right) = f(x) & \text{in } U \\ u^\varepsilon = 0 & \text{on } \partial U, \end{cases} \tag{1.1}$$

where $y \mapsto c(y,z)$ is $\mathbb{T}^n$-periodic and

$$\int_{\mathbb{T}^n} c(y,z)\,\mathrm{d}y = 0. \tag{1.2}$$

We have seen that ODE in this form have uniformly bounded solutions despite the seemingly dangerous ε^{-1} term. And as ε goes to 0, although the ε^{-1} term blows up, the solution actually converges to a "homogenized" equation. And there is an "effective potential" in the limiting equation for the ε^{-1} term. We hope to obtain a similar result for (1.1) and look for an "effective potential" term for $1/\varepsilon c\,(x/\varepsilon, u^\varepsilon)$.

Let $u^\varepsilon = U\,(x, x/\varepsilon, \varepsilon)$, where $U = U(x,y,\varepsilon)$ is $\mathbb{T}^n$-periodic in y, then

$$\Delta u^\varepsilon = \Delta_{xx}U + \frac{2}{\varepsilon}\Delta_{xy}U + \frac{1}{\varepsilon^2}\Delta_{yy}U,$$

where

$$\Delta_{xx}U = \sum_{i=1}^n \frac{\partial^2 U}{\partial {x_i}^2},$$

$$\Delta_{xy}U = \sum_{i=1}^n \frac{\partial^2 U}{\partial x_i \partial y_i},$$

and

$$\Delta_{yy} U = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial {y_i}^2}.$$

Then (1.1) becomes

$$\Delta_{xx} U + \frac{2}{\varepsilon} \Delta_{xy} U + \frac{1}{\varepsilon^2} \Delta_{yy} U = \frac{1}{\varepsilon} c\left(\frac{x}{\varepsilon}, U\right) - f. \tag{1.3}$$

Let $U = U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + \cdots$, where $U^k = U^k(x, y)$ is $\mathbb{T}^n$-periodic in y, $k = 0, 1, 2, \cdots$

Plug into (1.3),

(1.4)

$$\Delta_{xx} U^0 + \frac{2}{\varepsilon} \Delta_{xy} U^0 + \frac{1}{\varepsilon^2} \Delta_{yy} U^0 + \varepsilon \left(\Delta_{xx} U^1 + \frac{2}{\varepsilon} \Delta_{xy} U^1 + \frac{1}{\varepsilon^2} \Delta_{yy} U^1\right) +$$
$$\varepsilon^2 \left(\Delta_{xx} U^2 + \frac{2}{\varepsilon} \Delta_{xy} U^2 + \frac{1}{\varepsilon^2} \Delta_{yy} U^2\right) + \cdots =$$
$$- f + \frac{1}{\varepsilon} c(y, U^0) + \frac{1}{\varepsilon} c_z(y, U^0) \varepsilon U^1 + \cdots,$$

where

$$c_z = \frac{\partial}{\partial z} c(y, z).$$

Identify powers of ε,

(1.5)

$\boldsymbol{O(\varepsilon^{-2})}$: $$\Delta_{yy} U^0 = 0$$

(1.6)

$\boldsymbol{O(\varepsilon^{-1})}$: $$2\Delta_{xy} U^0 + \Delta_{yy} U^1 = c(y, U^0)$$

(1.7)

$\boldsymbol{O(1)}$: $$\Delta_{xx} U^0 + 2\Delta_{xy} U^1 + \Delta_{yy} U^2 = -f + c_z(y, U^0) U^1.$$

$O(\varepsilon^{-2})$ terms, (1.5) and periodicity in y imply

$$U^0 = U^0(x). \tag{1.8}$$

$O(\varepsilon^{-1})$ terms, by (1.8) $\Delta_{xy} U^0 \equiv 0$ and so (1.6) implies

$$\Delta_{yy} U^1 = c\left(y, U^0\right). \tag{1.9}$$

In (1.8), the derivatives are taken with respect to y only and x is a parameter. And since U_1 is $\mathbb{T}^n$-periodic in y, U_1 solves

$$\begin{cases} \Delta_{yy} U^1(y,x) = c\left(y, U^0(x)\right) \\ U^1(y,x) \text{ is } \mathbb{T}^n\text{-periodic in } y, \end{cases} \tag{1.10}$$

where x is a parameter. (1.10) admits a unique solution (up to an additive constant) since we assumed (1.2) (see [**BLP78**] for reference). (1.10) is the cell problem for (1.1) and the solution to the limiting equation of (1.1) would involve the solution to (1.10).

Let $v = v(y,z)$ solve

$$\begin{cases} \Delta_{yy} v(y,z) = c(y,z) \\ v \text{ is } \mathbb{T}^n\text{-periodic}, \end{cases} \tag{1.11}$$

where z is a parameter. And let

$$\boldsymbol{w}(y,z) = D_y v(y,z). \tag{1.12}$$

Then we have

$$c(y,z) = \Delta_y v(y,z) = \operatorname{div}_y \boldsymbol{w}(y,z) \tag{1.13}$$

and

$$U_1(y,x) = v(y, U^0(x)). \tag{1.14}$$

$O(1)$ terms, using (1.8), (1.9) and periodicity, integrate (1.7) with respect to y over $\mathbb{T}^n$, we get

$$-\Delta U^0 + \int_{\mathbb{T}^n} c_z(y, U^0) v(y, U^0)\, \mathrm{d}y = f. \tag{1.15}$$

Then we will prove this is the actual limiting equation.

1.2. Proof of convergence. Consider the boundary-value problem

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{1}{\varepsilon} c\left(\dfrac{x}{\varepsilon}, u^\varepsilon\right) = f(x) & \text{in } U \\ \qquad\qquad\qquad\quad u^\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{1.16}$$

with the assumption that f is continuous and that condition (1.11) and (1.12) hold.

For this expository work, we add the assumption about uniform L^2-boundedness of $\{u^\varepsilon\}$, that is

$$\|u^\varepsilon\|_{L^2} \leq C, \tag{1.17}$$

where C is independent of ε. Then we can get that the sequence $\{u^\varepsilon\}$ is bounded uniformly in $H_0^1(U)$ from the following the lemma.

Lemma 1.1. *We have*

$$\|u^\varepsilon\|_{H_0^1(U)} \leq C,$$

C is independent of ε.

Proof. Using (1.13), we have

$$\frac{1}{\varepsilon}c\left(\frac{x}{\varepsilon}, u^\varepsilon\right) = \frac{1}{\varepsilon}\mathrm{div}_y\, \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right) = \mathrm{div}\, \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - \boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \cdot Du^\varepsilon. \tag{1.18}$$

Plug into (1.16) to find

$$-\Delta u^\varepsilon + \mathrm{div}\, \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - \boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \cdot Du^\varepsilon = f. \tag{1.19}$$

Next, multiply by u^ε and integrate:

$$\int_U |Du^\varepsilon|^2 \,\mathrm{d}x = \int_U \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \cdot Du^\varepsilon + \boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \cdot Du^\varepsilon u^\varepsilon + f u^\varepsilon \,\mathrm{d}x. \tag{1.20}$$

Thus

$$\int_U |Du^\varepsilon|^2 \,\mathrm{d}x \leq \frac{1}{2}\int_U |Du^\varepsilon|^2 \,\mathrm{d}x + C_1 \int_U |u^\varepsilon|^2 \,\mathrm{d}x + C_2. \tag{1.21}$$

Therefore

$$\int_U |Du^\varepsilon|^2 \,\mathrm{d}x \leq C_3 \int_U |u^\varepsilon|^2 \,\mathrm{d}x + C_4.$$

And the conclusion follows from (1.17). □

We can then extract a weakly convergent subsequence of $\{u^\varepsilon\}$ in $H_0^1(U)$. We can in fact extract a further subsequence that converges strongly in $L^2(U)$. To do this, we introduce the concept of compact embedding and the Rellich-Kondrachov Compactness Theorem both of which we quote from [**Eva98**].

In case some readers are not familiar, we first give the definition of precompactness.

Definition 1.2. Let X, Y be Banach spaces, $X \subset Y$. (i) A bounded sequence x in X is called precompact (or pre-compact) in Y if it has a convergent subsequence in Y. (ii) We say that X is compactly embedded in Y, written

$$X \subset\subset Y,$$

provided

(a) $\|x\|_Y \leq c\|x\|_X$ $(x \in X)$ for some constant C, and

(b) each bounded sequence in X is precompact in Y.

Theorem 1.3. (Rellich-Kondrachov Compactness Theorem) *Assume U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1 \leq p < n$. Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leq q < p^*$.

Remark 1.4. (See [**CD99**].) It can be proved that Theorem (1.3) holds for $W_0^{1,p}(U)$ as well without any regularity assumption on ∂U.

Using Lemma 1.1, Theorem 1.3 and Remark 1.4, we have

$$u^\varepsilon \to u \text{ in } L^2(U) \tag{1.22}$$

and

$$Du^\varepsilon \rightharpoonup Du \text{ weakly in } L^2(U), \tag{1.23}$$

through a subsequence, as $\varepsilon \to 0$, for $u \in H_0^1(U)$.

Next we prove our main convergence result. Before we begin, we introduce the following lemma.

Lemma 1.5. *Let $\psi(x,y)$ be bounded and continuous for $x \in U$ and $\mathbb{T}^n$-periodic in y, then*

$$\psi(x, \frac{x}{\varepsilon}) \rightharpoonup \int_{\mathbb{T}^n} \psi(x,y)\,\mathrm{d}y$$

weakly in $L^2(U)$, as $\varepsilon \to 0$.

For proof of Lemma 1.5, see G. Allaire's [**All93**].

THEOREM 1.6. *u solves*

$$
\begin{cases} -\Delta u + \overline{vc_z}(u) = f & \text{in } U \\ \qquad\qquad u = 0 & \text{on } \partial U \end{cases}, \tag{1.24}
$$

where

$$
\overline{vc_z}(z) = \int_{\mathbb{T}^n} v(y,z)c_z(y,z)\,\mathrm{d}y.
$$

Note that (1.24) is exactly (1.15).

PROOF. Let $\phi \in C_c^\infty(U)$ be a test function, multiply (1.19) through and integrate,

$$
\int_U Du^\varepsilon \cdot D\phi\,\mathrm{d}x = \int_U \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\cdot D\phi + \boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\cdot Du^\varepsilon\phi + f\phi\,\mathrm{d}x. \tag{1.25}
$$

Let

$$
\begin{cases} A := \displaystyle\int_U \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\cdot D\phi\,\mathrm{d}x \\ B := \displaystyle\int_U \boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\cdot Du^\varepsilon\phi\,\mathrm{d}x \end{cases}
$$

Claim: $\lim\limits_{\varepsilon\to 0} A = 0$.

In fact,

$$
A = \int_U \left[\boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - \boldsymbol{w}\left(\frac{x}{\varepsilon}, u\right)\right]\cdot D\phi + \boldsymbol{w}\left(\frac{x}{\varepsilon}, u\right)\cdot D\phi\,\mathrm{d}x.
$$

The first term on the right hand side goes to zero because of (1.22) and our assumptions on $\boldsymbol{w}$. The second term converges to zero by Lemma 1.5. This concludes the proof for our claim that $\lim\limits_{\varepsilon\to 0} A = 0$.

An alternative proof is also possible given our definition of $\boldsymbol{w}$ and is given below. In fact,

$$
\boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right) = D_y v\left(\frac{x}{\varepsilon}, u^\varepsilon\right) = \varepsilon\left[Dv\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - v_z Du^\varepsilon\right]. \tag{1.26}
$$

Thus

$$
\begin{aligned}
A &= \int_U \boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\cdot D\phi\,\mathrm{d}x \\
&= \varepsilon\int_U Dv\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\cdot D\phi - v_z Du^\varepsilon\cdot D\phi\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon \int_U -v\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\Delta\phi - v_z Du^\varepsilon \cdot D\phi \, \mathrm{d}x \\
&\to 0,
\end{aligned}
$$

as $\varepsilon \to 0$, by (1.11) and Lemma 1.1.

Claim: $\lim_{\varepsilon\to 0} B = -\int_U \overline{v_z C}(u)\phi \, \mathrm{d}x$. In fact,

$$
\boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) = D_y v_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) = \varepsilon\left[Dv_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - v_{zz}Du^\varepsilon\right]. \tag{1.27}
$$

Thus

$$
\begin{aligned}
B &= \int_U \boldsymbol{w}_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \cdot Du^\varepsilon \phi \, \mathrm{d}x \\
&= \int_U \varepsilon\left[Dv_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - v_{zz}Du^\varepsilon\right] \cdot Du^\varepsilon \phi \, \mathrm{d}x \\
&= \int_U -\varepsilon v_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \operatorname{div}\left(Du^\varepsilon\phi\right) \mathrm{d}x + O(\varepsilon) \\
&= \int_U -\varepsilon v_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \Delta u^\varepsilon \phi \, \mathrm{d}x + O(\varepsilon) \\
&= \int_U -v_z\left(\frac{x}{\varepsilon}, u^\varepsilon\right) c\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \phi \, \mathrm{d}x + O(\varepsilon) \\
&= \int_U -v_z\left(\frac{x}{\varepsilon}, u\right) c\left(\frac{x}{\varepsilon}, u\right) \phi \, \mathrm{d}x + o(1) \\
&\to \int_U -\overline{v_z C}(u)\phi \, \mathrm{d}x
\end{aligned}
$$

as $\varepsilon \to 0$, since

$$
v_z\left(\frac{x}{\varepsilon}, z\right) c\left(\frac{x}{\varepsilon}, z\right) \rightharpoonup \int_{\mathbb{T}^n} v_z(y,z)c(y,z)\, \mathrm{d}y \text{ weakly in } L^2(U).
$$

And we have used Lemma 1.1. Thus $-\Delta u + \overline{v_z c}(u) = f$.

But also,

$$
\begin{aligned}
\overline{v_z C}(z) &= \int_{\mathbb{T}^n} v_z(y,z)c(y,z)\, \mathrm{d}y \\
&= \int_{\mathbb{T}^n} v_z(y,z)\Delta_y v(y,z)\, \mathrm{d}y \\
&= \int_{\mathbb{T}^n} \Delta_y v_z(y,z) v(y,z)\, \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^n} c_z(y,z)v(y,z)\,\mathrm{d}y \\
&= \overline{vc_z}(z)
\end{aligned}
$$

Thus u solves $-\Delta u + \overline{vc_z}(u) = f$. Since $u \in H_0^1(U)$, it also satisfies the boundary condition in (1.24). Thus u solves the boundary-value problem (1.24). $\square$

2. Systems

2.1. Asymptotic expansions. Suppose U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Consider a completely similar boundary-value problem to (1.1) for systems:

$$\begin{cases} -\Delta u_l^\varepsilon + \dfrac{1}{\varepsilon} c_l\left(\dfrac{x}{\varepsilon}, u^\varepsilon\right) = f_l(x) & \text{in } U \qquad l = 1, \dots, m \\ \qquad\qquad u_l^\varepsilon = 0 & \text{on } \partial U, \end{cases} \tag{2.1}$$

where the unknown is $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon) : U \to \mathbb{R}^m$. The functions $c(y,z) = (c_1, \dots, c_m) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^m$ with $y \mapsto c(y,z)$ being $\mathbb{T}^n$-periodic and $f(x) = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ are given. A necessary condition for solvability of the cell problem is

$$\int_{\mathbb{T}^n} c_l(y,z)\,\mathrm{d}y = 0 \qquad (l = 1, \dots, m). \tag{2.2}$$

Let $u_l^\varepsilon = U_l\left(x, x/\varepsilon, \varepsilon\right)$ where $U_l(x,y,\varepsilon)$ is $\mathbb{T}^n$-periodic in y for $l = 1, \dots, m$. Then

$$\Delta u_l^\varepsilon = \Delta_{xx} U_l + \frac{2}{\varepsilon}\Delta_{xy} U_l + \frac{1}{\varepsilon^2}\Delta_{yy} U_l \qquad (l = 1, \dots, m).$$

So (2.1) becomes

$$\Delta_{xx} U_l + \frac{2}{\varepsilon}\Delta_{xy} U_l + \frac{1}{\varepsilon^2}\Delta_{yy} U_l = \frac{1}{\varepsilon} c_l\left(\frac{x}{\varepsilon}, U\right) - f_l \qquad (l = 1, \dots, m). \tag{2.3}$$

Let

$$U_l = U_l^0 + \varepsilon U_l^1 + \varepsilon^2 U_l^2 + \cdots \qquad (l = 1, ..., m),$$

where $U_l^k = U_l^k(x,y)$ is $\mathbb{T}^n$-periodic in y for $l = 1, \dots, m$ and $k = 0, 1, 2, \dots$

Plug into (2.3),

(2.4)

$$\begin{aligned} \Delta_{xx} U_l^0 + \frac{2}{\varepsilon}\Delta_{xy} U_l^0 + \frac{1}{\varepsilon^2}\Delta_{yy} U_l^0 + \varepsilon\left(\Delta_{xx} U_l^1 + \frac{2}{\varepsilon}\Delta_{xy} U_l^1 + \frac{1}{\varepsilon^2}\Delta_{yy} U_l^1\right) + \\ \varepsilon^2\left(\Delta_{xx} U_l^2 + \frac{2}{\varepsilon}\Delta_{xy} U_l^2 + \frac{1}{\varepsilon^2}\Delta_{yy} U_l^2\right) = \\ - f_l + \frac{1}{\varepsilon} c_l(y, U^0) + \frac{1}{\varepsilon} D_z c_l(y, U^0) \cdot \varepsilon U^1 + O(\varepsilon) \\ (l = 1, \dots, m). \end{aligned}$$

(2.5)
$$\boldsymbol{O(\varepsilon^{-2})}: \qquad \Delta_{yy}U_l^0 = 0 \qquad (l = 1, \dots, m).$$
(2.6)
$$\boldsymbol{O(\varepsilon^{-1})}: \qquad 2\Delta_{xy}U_l^0 + \Delta_{yy}U_l^1 = c_l(y, U^0) \qquad (l = 1, \dots, m).$$

(2.7)
$$\boldsymbol{O(1)}: \qquad \begin{aligned} &\Delta_{xx}U_l^0 + 2\Delta_{xy}U_l^1 + \Delta_{yy}U_l^2 \\ &\quad = -f_l + D_z c(y, U^0) \cdot U^1 \qquad (l = 1, \dots, m). \end{aligned}$$

$O(\varepsilon^{-2})$ term, (2.5) and periodicity in y imply

$$U_l^0 = U_l^0(x) \qquad (l = 1, \dots, m). \tag{2.8}$$

$O(\varepsilon^{-1})$ term, by (2.8) $\Delta_{xy}U_l^0 \equiv 0$ for $l = 1, \dots, m$ and so (2.6) implies

$$\Delta_{yy}U_l^1 = c_l\left(y, U^0\right) \qquad (l = 1, \dots, m). \tag{2.9}$$

(2.2) ensures that we can solve for U_l^1 as a function of y with x being parameters for $l = 1, \dots, m$ (up to an additive constant). (See [**BLP78**] for reference.)

Let $v(y, z) = (v_1, \dots, v_m) : \mathbb{T}^n \times \mathbb{R}^m \to \mathbb{R}^m$ solve

$$\begin{cases} \Delta_{yy}v_l(y, z) = c_l(y, z) \\ \\ v_l \ \mathbb{T}^n - \text{periodic} \end{cases} \qquad l = 1, \dots, m. \tag{2.10}$$

for $z \in \mathbb{R}^m$. And let $\boldsymbol{w}_l(y, z) = D_y v_l(y, z)$ for $l = 1, \dots, m$. Then

$$c_l(y, z) = \Delta_y v_l(y, z) = \operatorname{div}_y \boldsymbol{w}_l(y, z) \qquad (l = 1, \dots, m). \tag{2.11}$$

and $U^1(y, x) = v(y, U^0(x))$.

$O(1)$ term, using (2.8) and periodicity, integrate (2.7) with respect to y over $\mathbb{T}^n$ to get,

$$-\Delta U_l^0 + \int_{\mathbb{T}^n} D_z c_l(y, U^0) \cdot U^1 \, \mathrm{d}y = f_l \qquad (l = 1, \dots, m). \tag{2.12}$$

2.2. Proof of convergence. Consider the initial-value problem (2.1). We assume as before that f is continuous and (2.10) and (2.11) hold. Moreover, we make the further assumption that

$$\|u^\varepsilon\|_{L^2(U;\mathbb{R}^m)} \le C, \tag{2.13}$$

C is independent of ε.

Then we can get the following estimate for $\{u_\varepsilon\}$.

LEMMA 2.1. $\|u^\varepsilon\|_{H_0^1(U;\mathbb{R}^m)} \le C$, *$C$ is independent of* ε.

PROOF. Using (2.11), we have

$$\begin{aligned} \frac{1}{\varepsilon} c_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) &= \frac{1}{\varepsilon}\mathrm{div}_y\, \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \\ &= \mathrm{div}\, \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - D_z\boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) : Du^\varepsilon \qquad (l = 1, \ldots, m). \end{aligned} \tag{2.14}$$

where $A : B = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}$ for $A = ((a_{ij})), B = ((b_{ij})) \in \mathbb{M}^{m\times n}$.

Plug into (2.1) and get

$$-\Delta u_l^\varepsilon + \mathrm{div}\, \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - D_z\boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) : Du^\varepsilon = f_l \qquad (l = 1, \ldots, m). \tag{2.15}$$

Now multiply by u^ε, integrate and sum over l:

$$\begin{aligned} (Du^\varepsilon, Du^\varepsilon)_{L^2(U;\mathbb{M}^{m\times n})} &= \left(\boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right), Du^\varepsilon\right)_{L^2(U;\mathbb{M}^{m\times n})} \\ &+ \sum_{l=1}^{m}\left(D_z\boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right), Du^\varepsilon u_l^\varepsilon\right)_{L^2(U;\mathbb{M}^{m\times n})} + (f, u^\varepsilon)_{L^2(U;\mathbb{R}^m)}. \end{aligned} \tag{2.16}$$

Thus

$$\|Du^\varepsilon\|^2_{L^2(U;\mathbb{M}^{m\times n})} \le \frac{1}{2}\|Du^\varepsilon\|^2_{L^2(U;\mathbb{M}^{m\times n})} + C_1\|u^\varepsilon\|^2_{L^2(U;\mathbb{R}^m)} + C_2. \tag{2.17}$$

Therefore

$$\|Du^\varepsilon\|^2_{L^2(U;\mathbb{M}^{m\times n})} \le C_3\|u^\varepsilon\|^2_{L^2(U;\mathbb{R}^m)} + C_4.$$

And (2.13) concludes our proof. $\square$

Using Lemma 2.1, the Rellich-Kondrachov Compactness Theorem and Remark 1.4, we have

$$u^\varepsilon \to u \text{ in } L^2(U;\mathbb{R}^m), \tag{2.18}$$

and

$$Du^\varepsilon \rightharpoonup Du \text{ weakly in } L^2(U;\mathbb{M}^{m\times n}), \tag{2.19}$$

through a subsequence, as $\varepsilon \to 0$, for $u \in H_0^1(U;\mathbb{R}^m)$.

THEOREM 2.2. *u solves*

$$\begin{cases} -\Delta u_l + \overline{D_z c_l \cdot v}(u) = f_l & \text{in } U \qquad\qquad l = 1,\dots,m. \\ u_l^\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{2.20}$$

Here (2.20) is exactly (2.12).

PROOF. Let $\phi \in C_c^\infty(U;\mathbb{R}^m)$ be a test function. Multiply (2.15) through, integrate and sum over l,

$$\begin{aligned} (Du^\varepsilon, D\phi)_{L^2(U;\mathbb{M}^{m\times n})} &= \left(\boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right), D\phi\right)_{L^2(U;\mathbb{M}^{m\times n})} \\ &+ \sum_{l=1}^m \left(D_z \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right), Du^\varepsilon \phi_l\right)_{L^2(U;\mathbb{M}^{m\times n})} + (f,\phi)_{\boldsymbol{L}^2(U;\mathbb{R}^m)}. \end{aligned} \tag{2.21}$$

Let

$$\begin{cases} A = \left(\boldsymbol{w}\left(\frac{x}{\varepsilon}, u^\varepsilon\right), D\phi\right)_{L^2(U;\mathbb{M}^{m\times n})} \\ B := \sum_{l=1}^m \left(D_z \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right), Du^\varepsilon \phi_l\right)_{L^2(U;\mathbb{R}^m)}. \end{cases}$$

Claim: $\lim_{\varepsilon\to 0} A = 0$. In fact,

$$\begin{aligned} \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) &= D_y v_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \\ &= \varepsilon\left[Dv_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - (D_z v_l)^T Du^\varepsilon\right] \qquad (l = 1,\dots,m). \end{aligned} \tag{2.22}$$

Thus

$$A = \sum_{l=1}^m \int_U \boldsymbol{w}_l\left(\frac{x}{\varepsilon}, u^\varepsilon\right) \cdot D\phi_l \,\mathrm{d}x$$

$$= \varepsilon \sum_{l=1}^{m} \left[\int_U Dv_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) \cdot D\phi_l - (D_z v_l)^T Du^\varepsilon (D\phi_l)^T \,\mathrm{d}x \right]$$
$$= \varepsilon \sum_{l=1}^{m} \left[\int_U -v_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) \Delta\phi_l - (D_z v_l)^T Du^\varepsilon (D\phi_l)^T \,\mathrm{d}x \right]$$
$$\to 0,$$

as $\varepsilon \to 0$, by (2.10) and Lemma 2.1.

Claim: $\lim_{\varepsilon \to 0} B = -\sum_{l=1}^{m} \int_U \overline{D_z v_l \cdot C}(u) \phi_l \,\mathrm{d}x$. In fact,

$$D_z \boldsymbol{w}_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) = D_{zy} v_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) = \varepsilon \left[D_{zx} v_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) - D_{zz} v_l Du^\varepsilon \right] \qquad (l = 1, \dots, m). \tag{2.23}$$

Thus

$$\begin{aligned} B &= \sum_{l=1}^{m} \int_U D_z \boldsymbol{w}_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) : Du^\varepsilon \phi_l \,\mathrm{d}x \\ &= \sum_{l=1}^{m} \int_U \varepsilon \left[D_{zx} v_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) - D_{zz} v_l Du^\varepsilon \right] : Du^\varepsilon \phi_l \,\mathrm{d}x \\ &= \sum_{l=1}^{m} \int_U \varepsilon D_{zx} v_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) : Du^\varepsilon \phi_l \,\mathrm{d}x + O(\varepsilon) \\ &= \sum_{l=1}^{m} \int_U -\varepsilon D_z v_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) \cdot \Delta u^\varepsilon \phi_l \,\mathrm{d}x + O(\varepsilon) \\ &= \sum_{l=1}^{m} \int_U -Dv_l \left(\frac{x}{\varepsilon}, u^\varepsilon \right) \cdot c \left(\frac{x}{\varepsilon}, u^\varepsilon \right) \phi_l \,\mathrm{d}x + O(\varepsilon) \\ &= \sum_{l=1}^{m} \int_U -zv_l \left(\frac{x}{\varepsilon}, u \right) \cdot c \left(\frac{x}{\varepsilon}, u \right) \phi_l \,\mathrm{d}x + o(1) \\ &\to \sum_{l=1}^{m} \int_U -\overline{D_z v_l \cdot C}(u) \phi_l \,\mathrm{d}x, \end{aligned}$$

as $\varepsilon \to 0$, since

$$D_z v_l \left(\frac{x}{\varepsilon}, z \right) \cdot c \left(\frac{x}{\varepsilon}, z \right) \rightharpoonup \int_{\mathbb{T}^n} D_z v_l(y, z) \cdot c(y, z) \,\mathrm{d}y$$

weakly in $L^2(U)$ for $l = 1, \dots, m$ and $z \in \mathbb{R}^m$, as $\varepsilon \to 0$, and using Lemma 2.1.

Thus $-\Delta u_l + \overline{D_z v_l \cdot C}(u) = f_l$, for $l = 1, \dots, m$.

But

$$
\begin{aligned}
\overline{D_z v_l \cdot C}(z) &= \int_{\mathbb{T}^n} D_z v_l(y,z) \cdot c(y,z)\,\mathrm{d}y \\
&= \int_{\mathbb{T}^n} D_z v_l(y,z) \cdot \Delta_y v(y,z)\,\mathrm{d}y \\
&= \int_{\mathbb{T}^n} D_z \Delta_y v_l(y,z) \cdot v(y,z)\,\mathrm{d}y \\
&= \int_{\mathbb{T}^n} D_z c_l(y,z) \cdot v(y,z)\,\mathrm{d}y \\
&= \overline{D_z C_l \cdot v}(z) \qquad (l = 1, \dots, m).
\end{aligned}
$$

Thus u solves (2.20). $\square$

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