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THE ROLE OF FLUCTUATIONS IN THE QUANTAL DESCRIPTION OF DAMPED MOTION*

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We review the construction of a frictional time-dependent Schrodinger equation for harmonic motion in a restricted framework, i.e., demanding the conservation of the gaussian shape of wave packets. We discuss the evolution of the quantal fluctuations in a time-independent model and show that such a situation does not correspond to damped harmonic oscillations. We discuss the role of fluctuations in providing dissipative behavior and analyze in detail the time evolution of arbitrary wave functions subject to damped motion as described by Schrödinger-Kostin equation.

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One of the most familiar problems in classical physics, the analysis of damped motion, gives rise to a significant amount of puzzling features when extended into the realm of quantum mechanics[1,2]. Recent developments in nuclear physics, especially in the field of heavy ion collisions and, to a lesser extent, nuclear fission, have stimulated both the necessity and the interest in the problem of quantal damped motion[3,4)]. The major approaches to the subject can be classified into three groups. In one of these we may consider treatments based on time-dependent perturbation theory[5,6,7)] that give rise to transport-like equations. This category includes the finite-temperature linear response theory[5,6)] that leads to a Fokker-Planck equation, the zero-temperature perturbation method[7)] that provides equations of motion for the fluctuations of the quantal trajectory, and the quantal master equation based on zero-temperature linear response theory[8)]. Another group could include the frictional Schrödinger equation that can be extracted from Liouville-Von Neumann equation[9)] with the help of perturbation methods. The third major category could contain the several attempts to construct a frictional Schrödinger equation with a non-linear Hamiltonian that provides a satisfactory description of the classical limit[1)].

This listing is far from exhaustive and the field is at the moment receiving contributions from various different standpoints (see [10)] for references regarding other methods of quantizing damped motion). Recently, the possibility of describing the relaxation of the charge asymmetry in a heavy ion collision via a damped, time-dependent, quantal harmonic oscillator[11] pointed out the convenience of reviewing these methods and selecting the most adequate for the problem[2]. In the case of interest, the nature of the system being studied casts some doubt on the legitimacy of perturbation-based treatments, due to the comparable time scales, or characteristic energies, of the damped collective mode and its thermal environment. Furthermore, the time-dependence of the inertial, potential and dissipation parameters of the oscillator prevents the use of formalisms that rely on frequency-dependent, rather than time-dependent, transport coefficients. Lastly, the appearance of overdamped oscillations and the need to achieve relaxation of the mode towards a steady state, as suggested by experiment[11], pose further restrictions on the picture to be selected. The non-linear Schrödinger equation derived by Kostin[12] and Kan and Griffin[13] proved to fulfill every requirement of the model for charge asymmetry equilibration[2,11].

The purpose of the present work is to reexamine the approaches towards the construction of a non-linear frictional Hamiltonian and discuss the significance of dissipation in the quantal limit. It will be shown that, although the correspondence principle indicates that the center of damped wave packets must move along the classical trajectories, the quantal picture does not correspond to a dissipative process unless the non-negligible fluctuations evolve towards asymptotic values. We will restrict ourselves to the study of the motion of a particle in presence of a conservative quadratic potential,

$$V(x) = \frac{1}{2}k(\hat{x} - x_0)^2 \qquad (1.1)$$

that includes both free and free fall motion, in addition to the harmonic one. The derivation, although based on a particular assumption - the conservation of the gaussian shape of wave packets undergoing damped motion - is illustrative enough for the purpose of understanding the role of quantal fluctuations in the nonlinear frictional Schrödinger equation. It should be borne in mind that all known frictional Hamiltonians conserve the gaussian shape[1)] and that this shape is suitable for a large number of applications regarding free, free fall and harmonic motion[1,2,4,11)].

In Section 2 we illustrate the construction of a frictional Hamiltonian that cause gaussian wave packets to move along the classical damped path. This derivation is valid for the full time-dependent problem in which the parameters k and x_0 in (1.1), as well as the inertia m and the friction coefficient γ, are given functions of time. In Section 3, we study in detail the consequences of assuming that the coefficients of the quadratic terms in the frictional Hamiltonian are time-independent. We show that a family of Hamiltonians can be derived and they do not correspond to harmonic motion. This destruction of the original system under analysis can be traced to the neglection of fluctuations in the frictional term. Section 4 summarizes the properties and consequences of Schrödinger-Kostin equation, which gives the proper dissipative behavior of the damped oscillator. We also discuss the evolution of time-dependent wave functions towards stationary oscillator states. The conclusions are presented in Section 5.

2. Derivation of the generalized frictional Hamiltonian

We will assume that the system in the quadratic potential well (1.1) is subject to dissipation via a linear frictional force, $F = -\gamma p$ and can be described, at all times, by a gaussian wave packet (GWP). Following Hasse[1,4] we write

$$\psi_0(\hat{X},t) = \frac{1}{(2\pi\chi)^{1/4}} \exp\left\{-\frac{\hat{X}^2}{2\alpha} + \frac{i}{\hbar}(p\hat{X} - \theta)\right\} \tag{2.1}$$

We utilize the following notation;

$$\hat{X} = \hat{x} - x \tag{2.2a}$$

$$\hat{P} = \hat{p} - p \tag{2.2b}$$

where

$$x = \langle \hat{x} \rangle \quad , \tag{2.3a}$$

$$p = \langle \hat{p} \rangle \tag{2.3b}$$

are the first moments of the wave function. The quantity θ is a real phase and α is a complex width whose relationship to the fluctuations or second moments,

$$\chi = \langle \hat{X}^2 \rangle \tag{2.4a}$$

$$\phi = \langle \hat{P}^2 \rangle \tag{2.4b}$$

$$\sigma = \frac{1}{2} \langle \{\hat{X}, \hat{P}\} \rangle \tag{2.4c}$$

can be summarized in the expressions,

$$\frac{1}{\alpha} = \frac{1}{2\chi} - \frac{i}{\hbar} \frac{\sigma}{\chi} = \frac{1}{\hbar} \sqrt{\frac{\phi}{\chi}} \exp\left(-i \operatorname{tg}^{-1} \frac{2\sigma}{\chi}\right) \tag{2.5a}$$

$$\alpha = \frac{\hbar^2}{2\phi} + i\hbar \frac{\sigma}{\phi} = \hbar \sqrt{\frac{\chi}{\phi}} \exp\left(i \operatorname{tg}^{-1} \frac{2\sigma}{\chi}\right) \tag{2.5b}$$

We assume that the characteristic GWP condition,

$$\chi\phi - \sigma^2 = \frac{\hbar^2}{4} \tag{2.6}$$

is preserved during the time evolution of ψ_0. Our aim is to find a frictional Hamiltonian

$$H = T + V + W \tag{2.7}$$

such that ψ_0 is a solution of the wave equation

$$i\hbar \dot{\psi}_0 = H\psi_0 \tag{2.8}$$

Substitution of eq. (2.1) into (2.8) yields the following constraint,

$$-i\hbar \frac{\dot{x}}{4x} + i\hbar \hat{X}^2 \frac{\dot{\alpha}}{2\alpha^2} - \dot{p}\,\hat{X} + \frac{p^2}{2m} + \dot{\theta} =$$

$$\frac{\hbar^2}{2m\alpha} - \frac{\hbar^2}{2m\alpha^2}\hat{X}^2 + V(\hat{x}) + \frac{W\psi_0}{\psi_0} \tag{2.9}$$

Eq. (2.9) shows that W has to be a second-degree polinomial in $\hat{X}$, $\hat{P}$. We choose to represent it as

$$W = \frac{\hat{P}^2}{2m^*} + \frac{\gamma^*}{2}\{\hat{X},\hat{P}\} + \frac{k^*}{2}\hat{X}^2 + \gamma p X - \frac{\phi}{2m^*} - \gamma^*\sigma - \frac{k^* X}{2} \tag{2.10}$$

where m^*, γ^* and k^* are unknown functions of γ. The coefficient of the linear term has been selected so that Ehrenfest's theorem,

$$\dot{p} = -\langle \frac{\partial}{\partial \hat{x}} (V + W)\rangle \tag{2.11}$$

becomes identical to the classical equation of motion,

$$\dot{p} = -k(x - x_0) - \gamma p \tag{2.12}$$

The independent term has been chosen to yield a vanishing average for W. This selection guarantees that the energy of the damped system (i.e., $E = \langle H\rangle$) coincides with the expectation value of the kinetic plus potential operators.

With the shorthand notation

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{m^*} \quad , \quad \kappa = k + k^* \quad , \quad W_0 = -\frac{\phi}{2m^*} - \gamma^* \sigma - \frac{k^* \chi}{2} \quad ,$$

we can extract the equations of motion for $\alpha(t)$ and $\theta(t)$ as follows:

$$i\hbar\dot{\alpha} = -\frac{\hbar^2}{\mu} + \kappa\alpha^2 + 2i\hbar\,\gamma^*\,\alpha \tag{2.13a}$$

$$\theta = \frac{\hbar^2}{2\mu\chi} - \frac{p^2}{2m} + \frac{k}{2}\,(x - x_0)^2 + W_0 \tag{2.13b}$$

Eq. (2.13a) is equivalent to the following system

$$\dot{\chi} = 2\gamma^*\chi + 2\frac{\sigma}{\mu} \tag{2.14a}$$

$$\dot{\phi} = -\,2\gamma^*\phi - 2\kappa\sigma \tag{2.14b}$$

$$\dot{\sigma} = -\,\kappa\chi + \frac{\phi}{\mu} \tag{2.14c}$$

In addition, the energy of the system varies in time according to the law, (disregarding the conservative variation associated with the time-dependence of κ and μ),

$$\dot{E} = \frac{d}{dt}\,\langle T + V\rangle \quad = -\,\gamma\,\frac{p^2}{m} + \frac{\dot{\phi}}{2m} + \frac{k\chi}{2} \tag{2.15}$$

In the correspondence limit, fluctuations vanish and we recover the classical law, $\dot{E} = -\gamma\frac{p^2}{m}$. In the quantal situation in which fluctuations are not negligible, according to eqs. (2.14) the energy rate takes the form

$$\dot{E} = -\gamma \frac{p^2}{m} - \gamma^* \left(\frac{\phi}{m} - k\chi \right) + \sigma \left(\frac{k}{\mu} - \frac{k}{m} \right) \tag{2.16}$$

which indicates that the time-evolution of the second moments of the wave packet is crucial to guarantee dissipative behavior, i.e., $\dot{E} \leq 0$ at all times. The importance of the role of fluctuations in dissipation has been stressed by Hasse[7,9]. One should keep in mind that for sufficiently long times, $\dot{E} > 0$ is to be expected. This observation corresponds to the fact that the system cannot decrease its energy beyond the ground-state.

3. The time-independent oscillator

Let us assume that the inertial, potential and frictional parameters of the oscillator under study are given constants. In this situation, the assumption that k^*, m^* and γ^* in W are also constants allows an analytical solution for the fluctuations as functions of time. The particular case $k^* = \frac{1}{m^*} = 0$ and γ^* proportional to γ has been considered by Hasse[1,4]. We readily find the general linear solution for the system (2.14),

$$\begin{pmatrix} \chi \\ \phi \\ \sigma \end{pmatrix} = \sum_{i=1}^{3} e^{\lambda_i t} \begin{pmatrix} \chi_i \\ \phi_i \\ \sigma_i \end{pmatrix} \tag{3.1}$$

with the eigenfrequencies $\lambda_i = o, \pm 2\omega$, where ω plays the role of the shifted frequency of a damped oscillator. In fact, if we introduce the "total" frequency Ω^*,

$$\Omega^{*} = \sqrt{\frac{\kappa}{\mu}} \tag{3.2}$$

we have

$$\omega = \sqrt{\Omega^{*2} - \gamma^{*2}} \tag{3.3}$$

This is identical to the shifted frequency of the classical damped oscillator when $k^{*} = \frac{1}{m^{*}} = 0$, $\gamma^{*} = \frac{\gamma}{2}$ (see [1,4]). We can see that the solutions (3.1) correspond to an undamped system, since $E(t\to\infty) \neq 0$ (cf. eq. (2.16)). This example indicates that damping in quantum mechanics demands damping of the wave function at least up to the second moments. In the situation we are considering, the wave function (GWP) is fully described by its first and second moments. However, the particular selection of the coefficients in the frictional term W gives rise to a nonlinear Hamiltonian that depends only on the first moments. We see from the above discussion that this partial dependence of the frictional Hamiltonian on the wave function is insufficient to yield the expected dissipative behavior. These considerations thus suggest that W should depend on on the second, as well as the first, moments. In such a case, one can expect the oscillating pattern depicted in eqs. (3.1) to change into an asymptotic evolution of the fluctuations towards steady states.

It is of interest to note that the system (2.14) admits the time-independent solution[4]

$$\chi = \frac{\hbar}{2\omega\mu} \tag{3.4a}$$

$$\phi = \frac{\hbar\kappa}{2\omega} \tag{3.4b}$$

$$\sigma = -\frac{\hbar\gamma^{*}}{2\omega} \tag{3.4c}$$

These values are to be compared with those of the ground-state of the undamped oscillator,

$$\chi = \frac{\hbar}{2m\Omega} \tag{3.5a}$$

$$\phi = \frac{\hbar k}{2\Omega} \tag{3.5b}$$

$$\sigma = 0 \tag{3.5c}$$

The quantities displayed in eqs. (3.5) should constitute the asymptotes of the damped fluctuations $\chi(t)$, $\phi(t)$ and $\sigma(t)$, in correspondence with the fact that damping cannot have any effect on the zero-point motion. In other words, one could say the process under study is characterized by a frictional force that vanishes asymptotically, as the particle approaches a halt. The frictional Hamiltonian thus becomes identical to the unperturbed one, except for an irrelevant constant, and the system must evolve towards the unperturbed ground-state.

Since the frictional term W with constant coefficients does not represent a quantal damped oscillator, it is of some interest, before giving up the subject, to understand which is the object described by this particular Hamiltonian. This is a simple project in Heisenberg representation: let $\hat{x}(t)$, $\hat{p}(t)$ be canonical conjugate Heisenberg operators. With the help of eqs. (2.2), (2.7) and (2.10) we can write the Hamiltonian as

$$H = \frac{1}{2} V^T(t) M V(t) + N(t) \cdot V(t) + C(t) \tag{3.6}$$

where

$$V(t) = \begin{pmatrix} \hat{p}(t) \\ \hat{x}(t) \end{pmatrix} \tag{3.7a}$$

$$M = \begin{pmatrix} \frac{1}{\mu} & \gamma^* \\ \gamma^* & \kappa \end{pmatrix} \tag{3.7b}$$

$$N = \begin{pmatrix} -\gamma^* x - \frac{p}{m^*} \\ kx_0 - k^* x + (\gamma - \gamma^*)\, p \end{pmatrix} \tag{3.7c}$$

$$C(t) = \frac{1}{2} k\, x_0^2 + \frac{1}{2} k^* x^2 + \frac{p^2}{2m^*} + (\gamma^* - \gamma) xp + W_0 \tag{3.7d}$$

The quadratic form (3.6) can be reduced to canonical form,

$$H = \frac{1}{2} V'^{T}(t) \; M V'(t) + C'(t) \tag{3.8}$$

with

$$V' = V - V_0 \tag{3.9a}$$

$$V_0(t) = \frac{1}{\omega^2} \begin{pmatrix} \gamma^* k(x - x_0) + [\frac{1}{m^*} + \gamma^*(\gamma - \gamma^*)]p \\ (\omega^2 - \frac{k}{\mu}) x + (\frac{\gamma^*}{m} - \frac{\gamma}{\mu})p + \frac{k x_0}{\mu} \end{pmatrix} \tag{3.9b}$$

$$C'(t) = \frac{1}{2} N(t) \cdot V_0(t) + C(t) \tag{3.9c}$$

and diagonalized afterwards. Let us choose the adimensional notation,

$$\hat{\xi}' = \sqrt{\frac{\mu\Omega^*}{\hbar}} \, \hat{x}' \tag{3.10a}$$

$$\hat{\pi}' = \frac{\hat{p}'}{\sqrt{\hbar \, \mu\Omega^*}} \tag{3.10b}$$

We easily obtain

$$H = \frac{\hbar}{2} \left[(\Omega^* - \gamma^*) \hat{\eta}^2 + (\Omega^* + \gamma^*) \hat{\zeta}^2 \right] \tag{3.11}$$

where the principal axis $\hat{\eta}$, $\hat{\zeta}$ are given by

$$\hat{\eta} = \frac{\hat{\xi}' - \pi'}{\sqrt{2}} \tag{3.12a}$$

$$\hat{\zeta} = \frac{\hat{\xi}^{\dagger} + \hat{\pi}^{\dagger}}{\sqrt{2}} \tag{3.12b}$$

A reduction to occupation number representation via the boson operators

$$\hat{a} = \sqrt{\frac{\Omega^{*} - \gamma^{*}}{2\omega}}\,\hat{\eta} + i\sqrt{\frac{\Omega^{*} + \gamma^{*}}{2\omega}}\,\hat{\zeta} \tag{3.13a}$$

$$\hat{a}^{+} = \sqrt{\frac{\Omega^{*} - \gamma^{*}}{2\omega}}\,\hat{\eta} - i\sqrt{\frac{\Omega^{*} + \gamma^{*}}{2\omega}}\,\hat{\zeta} \tag{3.13b}$$

allows us to write

$$H = \hbar\omega(\hat{a}^{+}\hat{a} + \tfrac{1}{2}) + C^{'}(t) \tag{3.14}$$

We realize that the system described by the Hamiltonian (3.6) is a harmonic oscillator with the shifted frequency ω, undamped, whose time-independent spectrum rests on the time-dependent reference level $C^{'}(t)$. For sufficiently long times, this reference level becomes identical to W_0 plus a constant (ef. eqs. (3.7) and (3.9)). These harmonic oscillations, however, take place along the "normal" coordinate $\hat{\eta}$, rather than along the physical coordinate $\hat{x}$. The explicit time-dependence for the Heisenberg operators $\hat{x}(t)$, $\hat{p}(t)$ can be recovered with the help of the equation of motion for $\hat{a}$,

$$i\hbar\dot{\hat{a}} = [\hat{a}, H] = \omega\hat{a} \tag{3.15}$$

and its hermitian conjugate. The solution reads, in terms of the displaced operators $\hat{x}^{'}(t)$, $\hat{p}^{'}(t)$,

$$\hat{x}'(t) = \left(\cos\omega t + \frac{\gamma^{\star}}{\omega}\sin\omega t\right)\hat{x}'(o) + \frac{\sin\omega t}{\omega\mu}\hat{p}'(o) \qquad (3.16a)$$

$$\hat{p}'(t) = -\frac{\mu\Omega^2}{\omega}\sin\omega t\,\hat{x}'(o) + \left(\cos\omega t - \frac{\gamma^{\star}}{\omega}\sin\omega t\right)\hat{p}'(o) \qquad (3.16b)$$

This result should be compared with the corresponding one for the undamped oscillator with frequence $\Omega = \sqrt{\frac{k}{m}}$,

$$\hat{x}(t) = \cos\Omega t\,\hat{x}(o) + \frac{\sin\Omega t}{\mu\Omega}\hat{p}(o) \qquad (3.17a)$$

$$\hat{p}(t) = -\mu\Omega\sin\Omega t\,\hat{x}(o) + \cos\Omega t\,\hat{p}(o) \qquad (3.17b)$$

It is interesting to quote here the solution obtained by Ford, Kac and Mazur[14] for the Heisenberg operators of an oscillator that undergoes Brownian motion. Disregarding the terms associated with the fluctuating part of Langevin force, these operators are, in the present notation,

$$\hat{x}(t) = \exp\left(-\frac{\gamma t}{2}\right)\left[\left(\cos\omega t + \frac{\gamma}{2\omega}\sin\omega t\right)\hat{x}(o) + \frac{\sin\omega t}{m\omega}\hat{p}(o)\right] \qquad (3.18a)$$

$$\hat{p}(t) = \exp\left(-\frac{\gamma t}{2}\right)\left[-m\omega\sin\omega t\,\hat{x}(o) + \left(\cos\omega t - \frac{\gamma}{2\omega}\sin\omega t\right)\hat{p}(o)\right] \qquad (3.18b)$$

The remarkable point here is that the Heisenberg operators themselves exhibit attenuation. This ought to be regarded as a characteristic feature of the quantal damped system and it could be considered as a "trial test" to decide whether a given Hamiltonian representation provides a sensible approach to the problem of dissipation in quantum mechanics.

4. Dissipative Hamiltonian and nonstationary wave functions

Kostin[12] put forward a nonlinear, frictional Schrödinger equation for a Brownian particle on the basis of the Langevin equation of motion for Heisenberg operators given by Ford et al.[14]. This approach was completed by Kan and Griffin[13] who obtained the same equation within a hydrodynamical picture and provided a clear physical interpretation of the significance of the frictional term in the Hamiltonian[15]. More recently, Skagerstam[16] derived an identical equation on the basis of stochastic mechanics. These independent derivations provide a solid foundation for this approach. Its usefulness has been recently tested in a variety of examples concerning variable mass systems[2] and in a full description of the time-dependent charge equilibration process in heavy ion collisions[11]. In this section we shall concentrate in the analysis of the asymptotic behavior of the wave functions that satisfy Schrödinger-Kostin (SK) equation.

This wave equation reads

$$i\hbar\dot{\psi} = \left[T+V - \frac{i\hbar\gamma}{2}\left(\ln\frac{\psi}{\psi^*} - \ln\langle\frac{\psi}{\psi^*}\rangle\right)\right]\psi \tag{4.1}$$

If the wave function ψ is expressed in complex notation as[13]

$$\psi = \phi\exp\left(-\frac{imS}{\hbar}\right) \tag{4.2}$$

the frictional contribution to the Hamiltonian takes the form

$$W = - m\gamma(S - \langle S \rangle) \tag{4.3}$$

In the case in which ψ is a GWP (eq. (2.1)) we obtain the explicit representation

$$W = \frac{\gamma\sigma\hat{X}^2}{2\chi} + \gamma p\hat{X} - \frac{\gamma\sigma}{2} \tag{4.4}$$

This frictional potential is of the general form (2.10) with

$$k^* = \frac{\gamma\sigma}{\chi} , \; \gamma^* = \frac{1}{m^*} = 0.$$

The equations of motion for the complex width, real phase and fluctuations can be easily extracted from eqs. (2.13) and (2.14). It is clear that the system (2.14) becomes non-linear and it has been shown[2] that these fluctuations evolve in time towards steady values, allowing the energy rate $\dot{E}$ to vanish asymptotically. When the variable parameters of the oscillator tend towards constant values, the final state of the GWP is the oscillator ground state associated with the asymptotic inertia and stiffness[2].

The GWP is a particular type of solution of the frictional Schrödinger equation[1]. Kan and Griffin[13] have shown that when the mass, stiffness and damping parameters are time-independent, the SK equation admits a complete set of stationary solutions, namely, those of the undamped harmonic oscillator. This puzzling feature becomes more understandable in view of the fact that harmonic oscillator wave functions whose centers move along the classical damped trajectory, are also solutions of SK equation[17]. These moving wave packets evolve in time towards the stationary, unperturbed oscillator eigenfunctions, that appear as temporal asymptotes for the damped motion. However, we still wonder about the type of wave functions that are to be expected for the oscillator with time-dependent parameters, including damping. It is a straightforward exercise to show that the functions

$$\psi_N(\hat{X},t) = \frac{1}{\sqrt{2^N N!\ \sqrt{2\pi\chi}}} \exp\left\{-\frac{\hat{X}^2}{2\alpha} + \frac{i}{\hbar}\left(p\hat{X} - \theta_N\right)\right\} H_N\left(\frac{\hat{X}}{\sqrt{2\chi}}\right) \quad , \qquad (4.5)$$

where $H_N(Z)$ is the N-th Hermite polynomial, are solutions of SK equation (4.1). Here α is the complex width that satisfies the equation (cf. eq. (2.13a)).

$$i\hbar\dot{\alpha} = -\frac{\hbar^2}{m(t)} + \left(k(t) + \frac{\gamma(t)\sigma}{\chi}\right)\alpha^2 \quad , \qquad (4.6)$$

χ is given by eq. (2.5a) and θ_N is related to θ in eq. (2.13b) by

$$\theta_N = \left(N + \frac{1}{2}\right)\frac{\hbar^2}{2m\chi} - \frac{p^2}{2m} + \frac{k}{2}(x-x_0)^2 - \frac{\gamma\sigma}{2} \qquad (4.7)$$

A complete and more sophisticated derivation of the solutions (4.5) can be traced to the theory of invariant operators[18,19] and is outlined in the Appendix. Since the system (2.14) that rules the evolution of the three fluctuations can be decoupled[4], we obtain a closed-form equation for χ^2),

$$x\ddot{x} - \frac{\dot{x}^2}{2} + x\dot{x}\left(\gamma(t) + \frac{\dot{m}(t)}{m(t)}\right) + 2\Omega^2(t)\,x = \frac{\hbar^2}{2m^2(t)} \tag{4.8}$$

For very long times, the solution of this equation tends to the value[2)]

$$x_\infty = \frac{\hbar}{2m_\infty \Omega_\infty} \tag{4.9}$$

Correspondingly, $\phi_\infty = \frac{\hbar k_\infty}{2\Omega_\infty}$ and $\sigma_\infty = 0$.

We see that the nonstationary wave functions (4.5), not only travel along the classical path, but the "oscillator length" associated with x also evolves towards a steady value. The function ψ_N possesses the following asymptote,

$$\phi_N = \frac{1}{\sqrt{2^N N!\;\sqrt{2\pi x}}}\exp\left\{-\frac{\hat{x}^2}{4x_\infty} - i(N+\tfrac{1}{2})\Omega_\infty t\right\} H_N\left(\frac{\hat{x}}{\sqrt{2x_\infty}}\right) \tag{4.10}$$

The noticeable aspect in these observations is the fact that in the quantal picture of damping by means of a frictional Hamiltonian, we can find a complete, time-dependent set of states that decay towards the unperturbed stationary states of the harmonic oscillator. This corresponds to the intuitive motion that damping cannot have any effect on any stationary state; it seems then necessary that such a set of sharp, well-defined states be eigenfunctions of the damped Hamiltonian for sufficiently long times.

The evolution of an arbitrary solution of the time-dependent SK equation can be analyzed with the help of the set $\{\psi_N\}$ given by eq. (4.5). The completeness of this set is obvious insofar as the time variable is regarded as a parameter. Accordingly, for each value of t we can write an expansion,

$$\psi(t) = \sum_{N=0}^{\infty} c_N \psi_N(t) \tag{4.11}$$

For this expansion to be valid at all times, we must allow a time-dependence in the set $\{c_N\}$. Now, the nonlinearity of SK equation prevents us from writing an explicit expression for $H(\psi)\psi_N$. However, we can represent this result, for every t, as

$$H\left(\psi(t)\right) \;\; \psi_N(t) = \sum_{M=0}^{\infty} d_M^N \left(\psi(t)\right) \;\; \psi_M(t) \tag{4.12}$$

with nonlinear coefficients $d_M^N(\psi)$. The wave equation $i\hbar\dot{\psi}(t) = H(\psi)\psi(t)$ gives rise to the following nonlinear, coupled equations for $c_N(t)$,

$$\begin{aligned} i\hbar\dot{c}_N(t) = & \left[\left(N + \frac{1}{2}\right)\left(k\chi + \frac{\phi}{m} + \gamma\upsilon\right) + \frac{p^2}{2m} + \frac{kx^2}{2}\right] c_N \\ & + \sqrt{\chi N}\left(\dot{p} - i\frac{\hbar}{\alpha}\dot{x}\right) c_{N-1} - \sqrt{\chi(N+1)}\left(\dot{P} + i\frac{\hbar}{\alpha^*}\dot{x}\right) c_{N+1} \\ & + i\hbar\frac{\chi}{2}\sqrt{N(N-1)}\;\frac{\partial}{\partial t}\left(\frac{1}{\alpha}\right) c_{N-2} \\ & - i\hbar\frac{\chi}{2}\sqrt{(N+1)(N+2)}\;\frac{\partial}{\partial t}\left(\frac{1}{\alpha^*}\right) c_{N+2} + \sum_{M=0}^{\infty} c_M\, d_N^M(c_M) \end{aligned} \tag{4.13}$$

A close look at eq. (4.13) leads to the conclusion that for a very long time the coefficients c_N become constants. Indeed, if we consider that in this limit the friction term $W(\psi)$ vanishes, $H(\psi)$ becomes linear and we can draw the following symbolic sequence,

$$H\psi_N \rightarrow i\hbar\psi_N \rightarrow \dot{\theta}_N\ \psi_N \rightarrow \hbar\Omega_\infty\ \phi_N \tag{4.14}$$

which upon substitution in (4.13) together with (4.7) and (4.9) brings cancellation of the r.h.s. This would indicate that in principle, the arbitrary wave packet would move towards a superposition of stationary states for the asymptotic Hamiltoniam. However, as pointed out by Kostin[20)] the condition $\dot{E} = 0$ that must occur for sufficiently long times is only consistent with a steady wave function. In the present stage, we cannot extract further information from eq. (4.13) and some more investigation is necessary to determine which is the final state of $\psi(t)$.

5. Conclusions

A quantal description of damping must exhibit dissipation features, i.e., $\dot{E} \leq o$, at any level, in the extreme quantal limit, as well as in the classical limit where the energy rate must coincide with the well-known expression $\dot{E} = -\gamma\frac{p^2}{m}$. When quantum numbers are small, i.e., near the ground-state or low excited states, the presence of non-zero quantal fluctuations may yield a different expression for the time-derivative of the expectation value of the energy operator.

In this case, the evolution of these fluctuations must take place in a way that guarantees $\dot{E} \leq 0$ at all times and $\dot{E} \to 0$ for very long times. A close look at eq. (2.15) indicates that the coordinate and momentum fluctuations must evolve towards constant values, i.e., the second moments must be damped. When the wave function under consideration is a gaussian wave packet, this "dissipation condition" is equivalent to the statement that the wave function itself must be damped and evolve towards the unperturbed ground-state. This consideration is further supported by the time-dependence of the Heisenberg operators for an oscillator exposed to a Langevin-type force[14)].

In agreement with these analysis, the Kostin[12)] and Kan and Griffin[13)] picture of quantal damped motion provides wave functions that move along the classical damped paths towards a stationary state of the asymptotically unperturbed Hamiltonian. This behavior is to be regarded as a peculiarity of quantal systems subject to loss mechanisms that provide, in the average, a linear frictional force.

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Appendix. The invariant operator.

One can easily verify that the following operator,

$$J = \frac{1}{2} \left\{ \chi \hat{P}^2 + \phi \hat{X}^2 - \sigma \{ \hat{X}, \hat{P} \} \right\} \qquad (A.1)$$

is an exact invariant of the motion for the frictional Kostin Hamiltonian $H = T + V + W(\psi_0)$, W given by eq. (4.4). It means, J satisfies the conservation equation[19]

$$i \frac{\partial J}{\partial t} + [J,H] = 0 \qquad (A.2)$$

Eq. (A.2) can be readily proven with the help of the set of equations (cf. eqs. (2.14) and 4.4))

$$\dot{\chi} = 2 \frac{\sigma}{m} \qquad (A.3a)$$

$$\dot{\phi} = -2k\sigma - 2\frac{\gamma\sigma^2}{\chi} \qquad (A.3b)$$

$$\dot{\sigma} = -k\chi + \frac{\phi}{m} - \gamma\sigma \qquad (A.3c)$$

in addition to the classical equation of motion $\dot{p} = -k(x-x_0) - \gamma p$, $\dot{x} = \frac{p}{m}$ and the commutators $[\hat{X},H]$, $[\hat{P},H]$,. It is interesting to recall that the operator J in (A.1) is an exact invariant as well for a time-dependent, undamped harmonic oscillator[21]. In fact, it can be shown that J is the invariant operator for any motion consistent with the gaussian shape[19], since the symmetry associated with this constant of the motion is, precisely, the minimality of the generalized uncertainty relationship $\chi\phi - \sigma^2$ (cf. eq. (2.6)).

With the help of the operator J it is a simple task to derive solutions of the time-dependent Schrödinger-Kostin equation. First, the eigenvalue equation

$$J\phi_N(\hat{X},t) = \lambda_N \phi_N(\hat{X},t) \tag{A.4}$$

gives us a complete set $\{\phi_N\}$ that contains the time as a parameter[18]. Secondly, it is possible to find real phases $\theta_N(t)$ such that the functions,

$$\psi_N(t) = \phi_N(t) \exp\left\{-\frac{i}{\hbar}\,\theta_N(t)\right\} \tag{A.5}$$

are solutions of the wave equation $i\hbar\dot{\psi}_N = H\psi_N$ (see [18]). These phases are given by

$$\dot{\theta}_N = \langle N \,|\, H - i\hbar\frac{\partial}{\partial t} \,|\, N\rangle \tag{A.6}$$

Eq. (A.4) can be solved according to the following steps. We diagonalize J in (A.1) by means of the boson operators,

$$D = \frac{\sqrt{\chi}}{\hbar}\,(\hat{P} - i\,\frac{\hbar}{\alpha}\,\hat{X}) \tag{A.7a}$$

$$D^{+} = \frac{\sqrt{\chi}}{\hbar}\left(\hat{P} + i\,\frac{\hbar}{\alpha^{*}}\,\hat{X}\right) \qquad (A.7b)$$

such that $[D,D^{+}] = 1$ (A.7c)

and obtain $J = \frac{\hbar^{2}}{2}\left(D^{+}D + \frac{1}{2}\right)$ (A.8)

We can now find the structure of the vacuum ϕ_0,

$$D\phi_0 = 0 \quad . \qquad (A.9)$$

With D given by (A.7a) and recalling the significance of the displaced operators $\hat{X}$ and $\hat{P}$ (ef. eqs. (2.2)) we get,

$$\phi_0 \propto \exp\left\{-\frac{\hat{X}^2}{2\alpha} + \frac{i}{k}\,P^{\hat{X}}\right\} \qquad (A.10)$$

The remaining eigenfunctions ϕ_N are

$$\phi_N = \langle\hat{X}|N\rangle \quad \text{with } |N\rangle = \frac{(D^{+})^{N}}{\sqrt{N!}}\,|0\rangle \qquad (A.11)$$

If we express

$$\langle X| = \sum_{N=0}^{\infty}\phi_N\,\langle N| \qquad (A.12)$$

and write the eigenvalue equation $\langle\hat{X}|\hat{X} = X\langle\hat{X}|$, with

$$\hat{X} = -\,i\sqrt{\chi}\,(D^{+} - D) \quad , \qquad (A.13)$$

we readily find the recurrence relationship,

$$\chi\phi_N - i\sqrt{\chi N}\ \phi_{N-1} + i\sqrt{\chi(N+1)}\ \phi_{N+1} = 0 \qquad (A.14)$$

In virtue of the recurrence properties of Hermite polynomials, this establishes that $_N$ must be of the form

$$\phi_N \propto \frac{i^N}{\sqrt{2^N N!}}\ H_N\left(\frac{\hat{X}}{\sqrt{2\chi}}\right) \qquad (A.15)$$

With the proper normalization, the eigenfunctions of J are found to be

$$\phi_N(\hat{X},t) = \frac{1}{\sqrt{2^N N!\ \sqrt{2\pi\chi}}}\exp\left\{-\frac{\chi^2}{2\alpha} + \frac{i}{\hbar}\,p\hat{X}\right\}\ H_N\left(\frac{\hat{X}}{\sqrt{2\chi}}\right) \qquad (A.16)$$

where the explicit time-dependence is displayed in χ, α, x and p. Expression (A.6) can also be evaluated using the properties of the ladder operators D, D^+, the inverse of transformation (A.7) and the representation

$$H = \frac{1}{2}\left(k + \frac{\gamma\sigma}{\chi}\right)\hat{X}^2 + \frac{\hat{P}^2}{2m} - \dot{p}X + \dot{x}\,\hat{P} + \frac{1}{2}k\ (x-x_0)^2 + \frac{p^2}{2m} - \frac{\gamma\sigma}{2} \qquad (A.17)$$

After some algebra, we find the diagonal matrix elements

$$\langle N|H|N\rangle = \left(N + \frac{1}{2}\right)\left(k\chi + \frac{\phi}{m} + \gamma\sigma\right) + \frac{1}{2}\,k(x-x_0)^2 + \frac{p^2}{2m} - \frac{\gamma\sigma}{2} \qquad (A.18)$$

and

$$\langle N|\frac{\partial}{\partial t}|N\rangle = -\frac{i}{\hbar} N \left(k\chi + \frac{\phi}{m} + \gamma\sigma - \frac{\hbar^2}{2m\chi}\right) + \langle 0|\frac{\partial}{\partial t}|0\rangle \qquad \text{(A.19)}$$

which leaves the phase of the vacuum undetermined. Since it is convenient to settle this phase according to the general equation (2.13b) (for this particular case, $\mu \doteq m$ and $W_0 = -\frac{\gamma\sigma}{2}$), combining this requirement with (A.18) and (A.19) we obtain the equation of motion,

$$\dot{\theta}_N = \left(N + \frac{1}{2}\right) \frac{\hbar^2}{2m\chi} - \frac{p^2}{2m} + \frac{k}{2}(x - x_0)^2 - \frac{\gamma\sigma}{2} \qquad \text{(A.20)}$$

which proves that the wave functions $\psi_N(t)$ in eq. (4.5) are the nonstationary solutions of SK wave equation.

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