Containment control of multi-agent systems by exploiting the control inputs of neighbors

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SUMMARY

This paper studies the containment control problem for multi-agent systems consisting of multiple leaders and followers connected as a network. The objective is to design control protocols so that the leaders will converge to a certain desired formation while the followers converge to the convex hull of the leaders. A novel protocol is proposed by exploiting the control input information of neighbors. Both continuous-time and discrete-time systems are considered. For continuous-time systems, it is proved that the protocol is robust to any constant delays of the neighbors' control inputs. For discrete-time systems, a sufficient condition on the feedback gain for the containment control is given in terms of the time delay and graph information. Some numerical examples are given to demonstrate the results.

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1. INTRODUCTION

The consensus problem for multi-agent systems has gained more and more attentions in recent years [1–3]. The problem is widely encountered in the real world, for example, in distributed computation, flocking, formation flight, and traffic congestion control.

For the distributed consensus problem, we need to design a local control protocol for each agent such that the states of all the agents converge to a common value based on the information of the agent itself and that of its neighbors. The consensus problem can be classified into leader-following consensus and leaderless consensus. In the leader-following consensus, there is one single leader agent that does not have access to any information of other agents. All the other agents have to follow the leader agent and finally converge to the leader by locally exchanging information through the network (see [1, 4–8] for reference). In [1], the consensus on the heading angles based on the Vicsek model [9] is studied for both leaderless and leader-following cases. For second-order multi-agent systems, if the velocity of the leader is not available to one or more followers, an observer-based protocol is proposed in [4]. Peng and Yang [5] consider the leader-following problem for multi-agent systems with a dynamic leader and communication delays. For multi-agent systems with general linear dynamics, the leader-following problem is investigated in [6]. In [7], measurement noises are considered in communication channels. A consensus protocol that can maintain the communication connectivity is studied in [8].

The containment control problem where a collection of agents is to be driven to a certain compact set [10], has recently attracted much interest. The containment control may be considered as

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a leader-following problem with multiple leaders. The leaders may be connected to form a certain geometric formation. All the follower agents are to be controlled to converge to the convex hull of the leaders [11]. One motivation of this problem is to ensure that a collection of autonomous robots does not venture into hazardous areas. As such, some virtual/actual leaders are introduced to guide the robots to move around safe areas. Other motivations include distributed sensor localization [12]. In [13], it is shown that the followers asymptotically converge to the convex hull of the leaders if the graph is connected. When the followers cannot connect with the leaders all the time, the containment problem is considered in [14]. On the basis of the convex analysis, Shi and Hong in [15] study target aggregation of nonlinear multi-agent systems under switching topologies. They show that if the communication graph is jointly connected, then all the agents will converge to a convex set. A finite-time containment controller that involves a signum function is proposed in [16], which, however, may encounter the chattering problem in the implementation of the algorithm. When the communication links are randomly switching, a second-order multi-agent system guided by multiple leaders is studied in [17]. A distributed containment control protocol is proposed such that all the agents almost surely asymptotically converge to the static convex leader set. By introducing the notions of set input-to-state stability and set integral input-to-state stability, necessary and sufficient conditions on the connectivity are provided in [18] for containment tracking with moving leaders. For multi-agent systems with general linear dynamics, the containment problem has been investigated in [19].

In this paper, we consider the two-level containment control problem where the leaders will cooperate with each other to achieve certain formation while the followers are to be driven to the convex hull of the leaders. We note that in the single leader case, there are advantages in exploiting the neighbors' control input information in the protocol design [20]. For example, it has been shown in [20] that speed of convergence can be selected arbitrarily, independently of the graph topology. Moreover, the consensus performance with respect to a certain quadratic performance index is improved. In many distributed control systems, for example, Unmanned Aerial Vehicle (UAV) formation control, the relative state information can be obtained by on-board sensors such as laser scanner and camera, whereas the control input information of neighbors need to be communicated through wireless transmission channels that suffer from time delays. On the other hand, the calculation of the protocol with exploiting the neighbors' control inputs depends on each other, which makes the protocol not implementable. To address these problems, only delayed neighbors' control input information will be considered in the proposed protocol. In this paper, we shall extend the delayed neighbors' control input-based protocol into the containment control problem for multi-agent systems with both continuous-time and discrete-time dynamics. We show that in the continuous setting, all the agents will converge into the convex hull of the leaders asymptotically for any time delay in the neighbors' control inputs, whereas in the discrete-time setting, a sufficient condition on the feedback gain is derived to guarantee the containment of the agents by the multiple leaders.

Before closing this section, some notations should be introduced. $\mathbb{R}$ (respectively $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$) denotes the set of real numbers (respectively n-dimensional real vectors, $m \times n$ real matrices). $\mathbb{Z}^+$ means the set of positive integers and $\bar{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{0\}$. Given $A \in \mathbb{R}^{m\times n}$, $A^{i,j}$ means the element on the ith row and jth column of A. We use $\mathrm{col}\{x_1, x_2, \ldots, x_n\}$ and $\mathrm{diag}\{x_1, x_2, \ldots, x_n\}$ to denote a column vector and a diagonal matrix formed by $x_1, x_2, \ldots, x_n$, respectively. $\mathbf{1}_n$ denotes the n-dimensional column vector with all elements of 1. $R(\cdot)$ means the real component of a complex number.

Let $Z = \{z_1, \ldots, z_k\}$ be a finite set in $\mathbb{R}$. The convex hull of Z, denoted by $\mathrm{co}(Z)$, is defined as $\mathrm{co}(Z) = \{\sum_{i=1}^k \lambda_i z_i \mid \sum_{i=1}^k \lambda_i = 1,\ \lambda_i \geqslant 0\}$. $\mathrm{dist}(x, \mathcal{S})$ means the distance from $x \in \mathbb{R}^n$ to the set $\mathcal{S} \subseteq \mathbb{R}^n$ in the sense of Euclidean norm, that is,

$$\mathrm{dist}(x, \mathcal{S}) = \inf_{y\in\mathcal{S}} \|x - y\|_2.$$

2. PRELIMINARIES AND ASSUMPTIONS

In this section, some definitions and assumptions in graph theory are reviewed.

A directed graph, denoted by $\mathcal{G}$, with node set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V} \times \mathcal{V}$, is often used to model communications among agents. For convenience, we define $\mathcal{V} = \{1, 2, \ldots, N\}$, where N is the number of nodes. For any pair $(i, j) \in \mathcal{E}(\mathcal{G})$, i is called parent node whose information is transmitted to agent j. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \mid j \in \mathcal{V},\ (j, i) \in \mathcal{E}(\mathcal{G})\}$. We denote by $a_{i,j} \geqslant 0$ the weighting on the edge (j, i). $a_{i,j} > 0$ if and only if $j \in \mathcal{N}_i$. If $a_{i,j} = a_{j,i}$, the corresponding graph is called undirected graph. Usually, we use the triplet $\{\mathcal{V}, \mathcal{E}, A\}$ to describe a graph $\mathcal{G}$, where $A \in \mathbb{R}^{N \times N}$ is the adjacency matrix associated with $\mathcal{G}$ and $\forall i, j \in \mathcal{V}$, $A^{i,j} = a_{i,j}$. We call $d_i \triangleq \sum_{j \in \mathcal{V}} a_{i,j}$ the in-degree of node i. The Laplacian matrix L of the graph $\mathcal{G}$ is defined as follows:

$$L^{i,j} \triangleq \begin{cases} -a_{i,j}, & i \neq j, \\ d_i, & i = j. \end{cases}$$

In this paper, we assume that the agents can be classified into two groups, leader agents and follower agents. The leaders will not receive any information from the followers, and the followers will be controlled on the basis of local information to cooperate with the leaders. We consider N agents with n_l leaders and $n_\mathrm{f} = N - n_\mathrm{l}$ followers. We assume that n_l satisfies $1 < n_\mathrm{l} < N$, which implies that there are multiple leaders and at least one follower. Moreover, the followers connect with each other via an undirected graph. Without loss of generality, the followers are labeled as the first n_f agents, that is, $\mathcal{V}_\mathrm{f} = \{1, 2, \ldots, n_\mathrm{f}\}$ with $\mathcal{V}_\mathrm{f}$ as the node set of the followers. The node set of the leaders is denoted by $\mathcal{V}_\mathrm{l} = \{n_\mathrm{f} + 1, \ldots, N\}$. It is clear that $\mathcal{V}_\mathrm{f} \cup \mathcal{V}_\mathrm{l} = \mathcal{V}$ and $\mathcal{V}_\mathrm{f} \cap \mathcal{V}_\mathrm{l} = \emptyset$. Then, the adjacency matrix A and Laplacian matrix L can be partitioned in the following way

$$A = \begin{bmatrix} A_\mathrm{f} & A_\mathrm{l} \\ 0 & 0 \end{bmatrix},\ L = \begin{bmatrix} L_\mathrm{f} & -A_\mathrm{l} \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where

$$L_\mathrm{f} = D_\mathrm{f} - A_\mathrm{f},\ D_\mathrm{f} = \mathrm{diag}\{d_1, \ldots, d_{n_\mathrm{f}}\}. \tag{2}$$

$A_\mathrm{f} \in \mathbb{R}^{n_\mathrm{f} \times n_\mathrm{f}}$ reflects the communication topology of the followers, $A_\mathrm{l} \in \mathbb{R}^{n_\mathrm{f} \times n_\mathrm{l}}$ reflects how the followers connect to the leaders. Specifically, if $A_\mathrm{l}^{i,j} > 0$, agent i is connected to leader j.

We say that a graph has a united spanning tree if for any one of the followers, there exists at least one leader that has a directed path to that follower [21]. In this paper, we assume that the communication graph has a united spanning tree.

3. PROBLEM STATEMENT

We consider each agent with the following dynamics

$$\dot{x}_i(t) = u_i(t),\ i = 1, \ldots, N, \tag{3}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are, respectively, the state and the input of agent i. Two cases are studied. First is that all the leaders are stationary, that is, $u_i \equiv 0$, $i \in \mathcal{V}_\mathrm{l}$. Second is that the leaders are controlled to reach a desired formation. In this case, the topology of the leaders is assumed to contain a spanning tree. In both cases, the control inputs of the followers can be designed with access to only local information such as their own state and the states of the neighboring agents to make sure that the followers are asymptotically contained in the convex hull of the leaders, that is, $\forall i \in \mathcal{V}_\mathrm{f}$,

$$\lim_{t \to \infty} \mathrm{dist}(x_i(t), \mathrm{co}\{x_j(t), j \in \mathcal{V}_\mathrm{l}\}) = 0.$$

Two protocols of the followers are given as follows:

$$u_i(t) = \frac{1}{d_i}\left\{-\gamma \sum_{j \in \mathcal{V}} a_{i,j}[x_i(t) - x_j(t)] + \sum_{j \in \mathcal{V}_\mathrm{f}} a_{i,j} u_j(t)\right\},\ i \in \mathcal{V}_\mathrm{f}, \tag{4}$$

$$u_i(t) = -\frac{\gamma}{d_i} \sum_{j \in \mathcal{V}} a_{i,j}[x_i(t) - x_j(t)], \ i \in \mathcal{V}_{\rm f}, \tag{5}$$

where d_i is the in-degree of agent i, and $\gamma > 0$ is a constant. Protocol (5) can be found in [22–24] and others. Protocol (4) is an extension of the leader-following consensus protocol in [3, 20].

Under protocol (4), the combined system of the followers can be written into the following compact form

$$\dot{X}_{\rm f}(t) = -\gamma D_{\rm f}^{-1}[L_{\rm f}X_{\rm f}(t) - A_{\rm l}X_{\rm l}] + D_{\rm f}^{-1}A_{\rm f}\dot{X}_{\rm f}(t), \tag{6}$$

where

$$X_{\rm f}(t) = {\rm col}\{x_1(t), \ldots, x_{n_{\rm f}}(t)\}, \ X_{\rm l} = {\rm col}\{x_{n_{\rm f}+1}, \ldots, x_N\}. \tag{7}$$

After mathematical manipulation, system (6) can be simplified as

$$\dot{X}_{\rm f}(t) = -\gamma X_{\rm f}(t) + \gamma L_{\rm f}^{-1}A_{\rm l}X_{\rm l}. \tag{8}$$

In (8), we use the fact that $L_{\rm f}$ is positive definite if the communication graph has a united spanning tree. From (8), it is shown that if $X_{\rm f}(t)$ converges, $\lim_{t\to\infty} X_{\rm f}(t) = L_{\rm f}^{-1}A_{\rm l}X_{\rm l}$, where $L_{\rm f}^{-1}A_{\rm l}$ is a nonnegative matrix. According to the definition of $L_{\rm f}$, we have $A_{\rm l}\mathbf{1}_{n_{\rm l}} = L_{\rm f}\mathbf{1}_{n_{\rm f}}$. Note that

$$L_{\rm f}^{-1}A_{\rm l}\mathbf{1}_{n_{\rm l}} = L_{\rm f}^{-1}(A_{\rm l}\mathbf{1}_{n_{\rm l}}) = L_{\rm f}^{-1}(L_{\rm f}\mathbf{1}_{n_{\rm f}}) = \mathbf{1}_{n_{\rm f}},$$

which means that the row sums of nonnegative matrix $L_{\rm f}^{-1}A_{\rm l}$ are equal to 1. Hence, the steady states of the followers lie in the convex hull of the leaders. From (8), we can see that the convergence rate of the multi-agent system depends only on γ. Therefore, we can choose the convergence rate without the information of the network. By contrast, when protocol (5) is applied, the closed-loop system of the followers becomes

$$\dot{X}_{\rm f}(t) = -\gamma D_{\rm f}^{-1}L_{\rm f}X_{\rm f}(t) + \gamma D_{\rm f}^{-1}A_{\rm l}X_{\rm l},$$

and the convergence rate of the multi-agent system depends on not only γ but also the network connection. Furthermore, in [20], it has been proved that given a certain quadratic cost function, protocol (4) can lead to a better performance. This can be understood as that protocol (4) contains more information, that is, neighbors' control inputs.

It should be noted that protocol (4) is not implementable because the control inputs are coupled; that is, the control inputs depend on each other and none of them can be figured out. On the other hand, in many practical systems such as UAV formation control, the relative position can be directly obtained, but the control input information of neighbors needs to be communicated through wireless channels. In this case, communication delay is unavoidable during the transmission. In the rest of the paper, we shall solve this implementation problem and provide stability analysis for stationary leaders and dynamically moving leaders. Parallel results for discrete-time systems are also provided.

4. CONTAINMENT CONTROLLER DESIGN AND STABILITY ANALYSIS

4.1. Containment control for continuous-time systems

We study multi-agent systems with each agent being single integrator dynamics (3). We first consider that all the leaders are stationary. Later, we shall provide the result for two-level control, which means to control the leaders to form a formation meanwhile to control the followers to asymptotically converge into the convex hull formed by the leaders. As is analyzed in the last section, we prefer protocol (4) to (5). To make the protocol more practical, we consider a protocol based on (4) by replacing the second term with an outdated one, that is,

$$u_i(t) = \frac{1}{d_i}\left\{-\gamma \sum_{j \in \mathcal{V}} a_{i,j}[x_i(t) - x_j(t)] + \sum_{j \in \mathcal{V}_{\rm f}} a_{i,j}u_j(t-\tau)\right\}, \quad i \in \mathcal{V}_{\rm f}. \tag{9}$$

Protocol (9) is expected to approximate (4) when τ approaches 0. Then, we have the following result.

Theorem 4.1
For multi-agent systems (3), if the communication topology $\mathcal{G}$ contains a united spanning tree, then the followers with protocol (9) are asymptotically contained in $\mathrm{co}\{x_j(0), j \in \mathcal{V}_{\mathrm{l}}\}$ for any $\tau > 0$.

Before proving the result, the following lemma is needed.

Lemma 4.1
Given A_{f} and D_{f} defined in (1) and (2), we denote the eigenvalues of $D_{\mathrm{f}}^{-1}A_{\mathrm{f}}$ by μ_i, $i \in \mathcal{V}_{\mathrm{f}}$. If the communication graph $\mathcal{G}$ contains a united spanning tree, we have

$$-1 < \mu_i < 1, \ i \in \mathcal{V}_{\mathrm{f}}. \tag{10}$$

Proof
Define

$$P = \begin{bmatrix} I_{n_{\mathrm{f}}} & 0 \\ 0 & \mathbf{1}'_{n_{\mathrm{l}}} \end{bmatrix} \in \mathbb{R}^{(n_{\mathrm{f}}+1)\times N}, \ D = \begin{bmatrix} D_{\mathrm{f}} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n_{\mathrm{f}}+1)\times(n_{\mathrm{f}}+1)}.$$

It can be verified that

$$\tilde{L} \triangleq D^{-1}PLP' = \left[\begin{array}{c|c} I - D_{\mathrm{f}}^{-1}A_{\mathrm{f}} & -b \\ \hline 0 & 0 \end{array}\right]$$

is a valid Laplacian matrix, where L is the Laplacian matrix of graph $\mathcal{G}$, $b = D_{\mathrm{f}}^{-1}A_{\mathrm{l}}\mathbf{1}_{n_{\mathrm{l}}}$. Denote by $\tilde{\mathcal{G}}$ the graph corresponding to Laplacian matrix $\tilde{L}$. Note that graph $\mathcal{G}$ that contains a united spanning tree is equivalent to graph $\tilde{\mathcal{G}}$ that contains a spanning tree. Then, by recalling [25], we can see that

$$I - \tilde{L} = \left[\begin{array}{c|c} D_{\mathrm{f}}^{-1}A_{\mathrm{f}} & b \\ \hline 0 & 1 \end{array}\right]$$

is a stochastic indecomposable and aperiodic matrix. According to [1], it follows that all of the eigenvalues of $D_{\mathrm{f}}^{-1}A_{\mathrm{f}}$ lie strictly inside the unit disk. On the other hand, thanks to the undirected connection of the followers, we know that A_{f} and $D_{\mathrm{f}}^{-1/2}A_{\mathrm{f}}D_{\mathrm{f}}^{-1/2}$ are symmetric matrices. Because $D_{\mathrm{f}}^{-1}A_{\mathrm{f}}$ is similar to $D_{\mathrm{f}}^{-1/2}A_{\mathrm{f}}D_{\mathrm{f}}^{-1/2}$, all the eigenvalues of $D_{\mathrm{f}}^{-1}A_{\mathrm{f}}$ are real, which implies (10). □

Now, we are in the position to prove Theorem 4.1.

Proof
Define

$$\tilde{X}_{\mathrm{f}}(t) = X_{\mathrm{f}}(t) - L_{\mathrm{f}}^{-1}A_{\mathrm{l}}X_{\mathrm{l}}. \tag{11}$$

Then, we obtain

$$\dot{\tilde{X}}_{\mathrm{f}}(t) = D_{\mathrm{f}}^{-1}A_{\mathrm{f}}\dot{\tilde{X}}_{\mathrm{f}}(t-\tau) - \gamma D_{\mathrm{f}}^{-1}L_{\mathrm{f}}\tilde{X}_{\mathrm{f}}(t). \tag{12}$$

It is obvious that the containment control problem is solved if system (12) is asymptotically stable. Define $F_0 = -\gamma D_{\mathrm{f}}^{-1}L_{\mathrm{f}}$, $F_1 = D_{\mathrm{f}}^{-1}A_{\mathrm{f}}$. We can obtain the following conditions that guarantee the stability of system (12) by [26]:

(1) $\det\left[I - F_1e^{\iota\theta}\right] \neq 0, \ \forall\theta \in [0, 2\pi]$,
(2) $\mathrm{Re}\left(\lambda\left[(I - F_1)^{-1}F_0\right]\right) < 0$,
(3) $\det\left[\iota y(I - F_1e^{\iota\theta}) - F_0\right] \neq 0, \ \forall y \in \mathbb{R}/\{0\}, \ \forall\theta \in [0, 2\pi]$,

where $\iota = \sqrt{-1}$.

By Lemma 4.1, we know that condition 1 is satisfied.

$$(I - F_1)^{-1} F_0 = -\gamma(I - D_{\mathrm{f}}^{-1} A_{\mathrm{f}})^{-1} D_{\mathrm{f}}^{-1} L_{\mathrm{f}} = -\gamma(D_{\mathrm{f}} - A_{\mathrm{f}})^{-1} L_{\mathrm{f}} = -\gamma I,$$

which implies the satisfaction of condition 2. Now, we consider condition 3. We write

$$\begin{aligned}\iota y(I - F_1 e^{\iota\theta}) - F_0 &= \iota y[I - D_{\mathrm{f}}^{-1} A_{\mathrm{f}}(\cos\theta + \iota\sin\theta)] + \gamma D_{\mathrm{f}}^{-1} L_{\mathrm{f}} \\ &= \gamma I - \gamma D_{\mathrm{f}}^{-1} A_{\mathrm{f}} + y D_{\mathrm{f}}^{-1} A_{\mathrm{f}} \sin\theta + \iota y(I - D_{\mathrm{f}}^{-1} A_{\mathrm{f}} \cos\theta).\end{aligned}$$

Then, condition 3 is equivalent to that

$$\gamma - \gamma\mu_i + y\mu_i \sin\theta + \iota y(1 - \mu_i \cos\theta) \neq 0, \ i \in \mathcal{V}_{\mathrm{f}}. \tag{13}$$

From (10), we know that the imaginary part in (13) cannot go to 0, which verifies the third condition. Therefore, system (12) is stable, which implies $\lim_{t\to\infty} \tilde{X}_{\mathrm{f}}(t) = 0$. Denote by e_i a row vector with the ith entry being 1 and the rest of the entries being 0. Because $L_{\mathrm{f}}^{-1} A_{\mathrm{l}}$ is a stochastic matrix, one has $e_i X_{\mathrm{f}}(t) = x_i(t)$ and $e_i L_{\mathrm{f}}^{-1} A_{\mathrm{l}} X_{\mathrm{l}} \in \mathrm{co}\{x_j(t), j \in \mathcal{V}_{\mathrm{l}}\}$. Then, we have

$$\begin{aligned}\lim_{t\to\infty} \mathrm{dist}(x_i(t), \mathrm{co}\{x_j, j \in \mathcal{V}_{\mathrm{l}}\}) &\leqslant \lim_{t\to\infty} \mathrm{dist}(e_i X_{\mathrm{f}}(t), e_i L_{\mathrm{f}}^{-1} A_{\mathrm{l}} X_{\mathrm{l}}) \\ &\quad + \lim_{t\to\infty} \mathrm{dist}(e_i L_{\mathrm{f}}^{-1} A_{\mathrm{l}} X_{\mathrm{l}}, \mathrm{co}\{x_j, j \in \mathcal{V}_{\mathrm{l}}\}) \\ &= \lim_{t\to\infty} \left\| e_i \tilde{X}_{\mathrm{f}}(t) \right\|_2 \\ &= 0,\end{aligned}$$

which implies that all the followers are in the convex hull of the leaders asymptotically. □

Next, we shall consider dynamically moving leaders. In this case, all the leaders are connected as a network that is assumed to contain a spanning tree. They are controlled to form a given formation while moving along a direction with a same speed. To be specific, we assume that each leader is with the following dynamics

$$\dot{x}_i(t) = u_i(t) = \eta_c \sum_{j\in\mathcal{V}_{\mathrm{l}}} a_{i,j}[(x_j(t) - h_j) - (x_i(t) - h_i)] + v, \ i \in \mathcal{V}_{\mathrm{l}}, \tag{14}$$

where $a_{i,j} > 0$ when leader i can receive information from leader j and $a_{i,j} = 0$ otherwise; $\eta_c > 0$ is the control gain; h_i, $i \in \mathcal{V}_{\mathrm{l}}$ are the formation information of the leaders; v is a given common velocity of the leaders. Denote by L_{l} the Laplacian matrix of the network formed by all the leaders, which is defined in Section 2. Then, we have

$$\dot{X}_{\mathrm{l}}(t) = -\eta_c L_{\mathrm{l}}(X_{\mathrm{l}}(t) - H) + v\mathbf{1}_{n_{\mathrm{l}}}, \tag{15}$$

where X_{l} is defined in (7),

$$H = \mathrm{col}\{h_{n_{\mathrm{f}}+1}, \ldots, h_N\}. \tag{16}$$

Denote by ξ satisfying $\xi'\mathbf{1}_{n_{\mathrm{l}}} = 1$ the left eigenvector of L_{l} corresponding to eigenvalue 0 and define $\tilde{X}_{\mathrm{l}}(t) = X_{\mathrm{l}}(t) - H - vt\mathbf{1}_{n_{\mathrm{l}}}$. Then, according to (15), we have

$$\dot{\tilde{X}}_{\mathrm{l}}(t) = -\eta_c L_{\mathrm{l}} \tilde{X}_{\mathrm{l}}(t), \tag{17}$$

where we have applied the fact that $L_{\mathrm{l}}\mathbf{1}_{n_{\mathrm{l}}} = 0$. The solution of (17) can be written as

$$\tilde{X}_{\mathrm{l}}(t) = \xi' \tilde{X}_{\mathrm{l}}(0)\mathbf{1}_{n_{\mathrm{l}}} + O(e^{-\eta_c \lambda_2 t}), \tag{18}$$

where λ_2 is the eigenvalue of L_{l} with the second smallest real part. Because the graph of the leaders contains a spanning tree, we have $R(\lambda_2) > 0$. Then, (18) implies

$$\lim_{t\to\infty} [x_i(t) - h_i - vt - \xi' \tilde{X}_{\mathrm{l}}(0)] = 0,$$

which means that the leaders finally converge to the desired formation while moving with the same speed v. Then, we arrive at that

$$\dot{X}_{\mathrm{l}}(t) - \dot{X}_{\mathrm{l}}(t-\tau) = \dot{\tilde{X}}_{\mathrm{l}}(t) - \dot{\tilde{X}}_{\mathrm{l}}(t-\tau) = O(e^{-\eta_c \lambda_2 t}). \tag{19}$$

Now, we are in the position to propose the control protocol for dynamically moving leaders case.

$$u_i(t) = \frac{1}{d_i}\left\{-\gamma \sum_{j\in\mathcal{V}} a_{i,j}[x_i(t) - x_j(t)] + \sum_{j\in\mathcal{V}} a_{i,j} u_j(t-\tau)\right\}, \quad i \in \mathcal{V}_{\mathrm{f}}. \tag{20}$$

Different from (9), the leaders' control information is included in protocol (20). The result is given as follows.

Theorem 4.2
For multi-agent systems (3) with multiple moving leaders satisfying (14), if the communication topology contains a united spanning tree, then the followers with protocol (20) are asymptotically contained in $\mathrm{co}\{x_j(t), j \in \mathcal{V}_{\mathrm{l}}\}$ for any $\tau > 0$.

Proof
According to protocol (20), we have the following closed-loop system

$$\dot{X}_{\mathrm{f}}(t) = \gamma D_{\mathrm{f}}^{-1}[-L_{\mathrm{f}} X_{\mathrm{f}}(t) + A_{\mathrm{l}} X_{\mathrm{l}}(t)] + D_{\mathrm{f}}^{-1} A_{\mathrm{f}} \dot{X}_{\mathrm{f}}(t-\tau) + D_{\mathrm{f}}^{-1} A_{\mathrm{l}} \dot{X}_{\mathrm{l}}(t-\tau).$$

By introducing $\tilde{X}_{\mathrm{f}}(t)$ defined in (11), we have

$$\begin{aligned}\dot{\tilde{X}}_{\mathrm{f}}(t) &= -\gamma D_{\mathrm{f}}^{-1} L_{\mathrm{f}} \tilde{X}_{\mathrm{f}}(t) + D_{\mathrm{f}}^{-1} A_{\mathrm{f}} \dot{\tilde{X}}_{\mathrm{f}}(t-\tau) + D_{\mathrm{f}}^{-1} A_{\mathrm{f}} L_{\mathrm{f}}^{-1} A_{\mathrm{l}} \dot{X}_{\mathrm{l}}(t-\tau) + D_{\mathrm{f}}^{-1} A_{\mathrm{l}} \dot{X}_{\mathrm{l}}(t-\tau)\\ &\quad - L_{\mathrm{f}}^{-1} A_{\mathrm{l}} \dot{X}_{\mathrm{l}}(t)\\ &= -\gamma D_{\mathrm{f}}^{-1} L_{\mathrm{f}} \tilde{X}_{\mathrm{f}}(t) + D_{\mathrm{f}}^{-1} A_{\mathrm{f}} \dot{\tilde{X}}_{\mathrm{f}}(t-\tau) + e_c(t),\end{aligned} \tag{21}$$

where $e_c(t) = L_{\mathrm{f}}^{-1} A_{\mathrm{l}}[\dot{X}_{\mathrm{l}}(t-\tau) - \dot{X}_{\mathrm{l}}(t)]$ that is $O(e^{-\eta_c \lambda_2 t})$ according to (19). In (21), we use the fact $D_{\mathrm{f}}^{-1} A_{\mathrm{f}} L_{\mathrm{f}}^{-1} A_{\mathrm{l}} = (I - D_{\mathrm{f}}^{-1} L_{\mathrm{f}}) L_{\mathrm{f}}^{-1} A_{\mathrm{l}} = L_{\mathrm{f}}^{-1} A_{\mathrm{l}} - D_{\mathrm{f}}^{-1} A_{\mathrm{l}}$. Therefore, there exists a constant $C_1 > 0$ such that $\|e_c(t)\|_2 \leqslant C_1 e^{-\eta_c \lambda_2 t}$.

Define the Hilbert space $\mathcal{H} = \mathbb{R}^{n_{\mathrm{f}}} \times L_2\{-\tau, 0; \mathbb{R}^{n_{\mathrm{f}}}\}$, with the obvious inner product, where $L_2\{-\tau, 0; \mathbb{R}^{n_{\mathrm{f}}}\}$ denotes a square integrable vector function over $[-\tau, 0]$. A semigroup $\mathcal{S}(\cdot)$ over space $\mathcal{H}$ based on system (12) is introduced, which is defined as

$$\mathcal{X}(t) = \mathcal{S}(t)\mathcal{X}(0),$$

where $\mathcal{X}(t) \triangleq (\tilde{X}_{\mathrm{f}}(t), \tilde{X}_{\mathrm{f}}(t+\nu)) \in \mathcal{H}$, $\nu \in [-\tau, 0]$. It can be checked that $\mathcal{S}$ is differentiable, and the corresponding infinitesimal generator $\mathcal{A}$ is given as

$$\mathcal{A}(w, z) = (-\gamma D_{\mathrm{f}}^{-1} L_{\mathrm{f}} w + D_{\mathrm{f}}^{-1} A_{\mathrm{f}} z_\nu(-\tau), z_\nu),$$

where $z_\nu(-\tau) \triangleq \frac{dz}{d\nu}|_{\nu=-\tau}$. By recalling [27], the type of $\mathcal{S}$ is given as follows:

$$\omega(\mathcal{S}) = \inf_{t>0} \frac{\ln\|\mathcal{S}(t)\|}{t} = \lim_{t\to\infty} \frac{\ln\|\mathcal{S}(t)\|}{t}.$$

In the proof of Theorem 4.1, we know that system (12) is stable, which implies that $\omega < 0$. There exists a constant $C_2 > 0$ such that

$$\|\mathcal{S}(t)\| \leqslant C_2 e^{\omega t}.$$

Let $\phi(t) = (e_c(t), 0) \in \mathcal{H}$. Then, according to (21), we have

$$\begin{aligned}\|\mathcal{X}(t)\| &= \left\|\mathcal{S}(t)\mathcal{X}(0) + \int_0^t \mathcal{S}(t-s)\phi(s)\mathrm{d}s\right\| \\ &\leqslant \|\mathcal{S}(t)\|\,\|\mathcal{X}(0)\| + \int_0^t \|\mathcal{S}(t-s)\|\,\|\phi(s)\|\,\mathrm{d}s \\ &\leqslant C_2 e^{\omega t}\,\|\mathcal{X}(0)\| + \int_0^t C_2 e^{\omega(t-s)} C_1 e^{-\eta_c \lambda_2 s}\mathrm{d}s \\ &= \begin{cases} C_2 e^{\omega t}\,\|\mathcal{X}(0)\| + C_1 C_2 \frac{e^{\omega t} - e^{-\eta_c\lambda_2 t}}{\omega + \eta_c \lambda_2}, & \omega + \eta_c\lambda_2 \neq 0, \\ C_2 e^{\omega t}\,\|\mathcal{X}(0)\| + C_1 C_2 e^{\omega t} t, & \omega + \eta_c \lambda_2 = 0, \end{cases}\end{aligned}$$

which implies that $\lim_{t\to\infty} \tilde{X}_{\mathrm{f}}(t) = 0$. The rest of the proof follows from the same line of arguments as that of Theorem 4.1. □

Remark 4.1

Note that the calculation of the protocol with exploiting the neighbors' control inputs depends on each other, which makes protocol (4) not implementable. On the other hand, in many practical systems, the relative state information can be directly obtained, but the control input information of neighbors needs to be communicated through wireless channels with time delays. To overcome these problems, delayed neighbors' control input information instead of the instantaneous one is applied in the protocol. Theorems 4.1 and 4.2 show that the convergence can be guaranteed for any $\tau > 0$.

Remark 4.2

The dynamically moving leaders case has also been considered in [23] and [18]. In [18], the leaders are assumed to move with limited velocities. The containment tracking error is given in terms of the upper bound of the leaders' velocity and transmission disturbances. Different from [18], our focus is on developing a new protocol and proving the convergence and robustness to the time delay τ.

4.2. Containment control for discrete-time systems

In this section, we consider the agents with the following first-order discrete-time systems

$$x_i(k+1) = x_i(k) + u_i(k),\ i \in \mathcal{V}_{\mathrm{f}},\ k \in \bar{\mathbb{Z}}^+ \tag{22}$$

Similar to the continuous-time case, the control objective is

$$\lim_{k\to\infty} \mathrm{dist}(x_i(k), \mathrm{co}\{x_j(k), j \in \mathcal{V}_{\mathrm{l}}\}) = 0.$$

We first assume that the leaders are stationary, that is, $x_i(k) = x_i(0)$, $i \in \mathcal{V}_{\mathrm{l}}$. The following protocol is proposed

$$u_i(k) = \frac{1}{d_i}\left\{-\gamma \sum_{j\in\mathcal{V}} a_{i,j}[x_i(k) - x_j(k)] + \sum_{j\in\mathcal{V}_{\mathrm{f}}} a_{i,j} u_j(k-\tau)\right\},\ i \in \mathcal{V}_{\mathrm{f}}, \tag{23}$$

where $\gamma > 0$ is the feedback gain to be designed, and $\tau \in \mathbb{Z}^+$ is the time delay. Here, τ is strictly positive because the neighbors' control input information $u_j(k)$ is not available at time instant k. Next, given time delay τ, we need to find the condition of γ such that the follower is contained in the leaders' convex hull asymptotically. Denote

$$M = \frac{2\mu\left(1-(1-\theta)^\tau\right)}{\theta\left((1-\theta)^\tau - \mu\right)}, \tag{24}$$

where θ is any number satisfying $\theta \in (0, 1 - \sqrt[\tau]{\mu})$, and $\mu = \max\{|\mu_1|, \ldots, |\mu_{n_f}|\}$ and μ_i, $i \in \mathcal{V}_f$ are eigenvalues of $D_f^{-1} A_f$. From Lemma 4.1, we know that $\mu < 1$, which implies $M > 0$. Then, we have the following result.

Theorem 4.3
For multi-agent systems (22) with the communication topology containing a united spanning tree, the control protocol (23) with

$$\gamma < \min\left\{\frac{1}{M}, \frac{\theta}{2}, \frac{1}{\tau}\frac{1-\mu}{1+\mu}\right\} \tag{25}$$

guarantees that the followers converge to the convex hull of the leaders for any initial conditions.

Proof
By substituting (23) into (22), the closed-loop system can be written in the following compact form

$$X_f(k+1) = X_f(k) - \gamma D_f^{-1}(L_f X_f(k) - A_l X_l) + D_f^{-1} A_f X_f(k-\tau+1) - D_f^{-1} A_f X_f(k-\tau),$$

where X_f and X_l are defined in (7). By introducing

$$\tilde{X}_f(k) = X_f(k) - L_f^{-1} A_l X_l, \tag{26}$$

we have

$$\tilde{X}_f(k+1) = (I - \gamma D_f^{-1} L_f)\tilde{X}_f(k) + D_f^{-1} A_f \tilde{X}_f(k-\tau+1) - D_f^{-1} A_f \tilde{X}_f(k-\tau). \tag{27}$$

We can find an invertible matrix T such that $D_f^{-1} A_f = T^{-1}\Lambda T$, where $\Lambda = \mathrm{diag}\{\mu_1, \ldots, \mu_{n_f}\}$. Then, we introduce a linear transformation $T\tilde{X}_f = \mathrm{col}\{\tilde{x}_1, \ldots, \tilde{x}_{n_f}\}$. Realizing that $D_f^{-1} L_f = I - D_f^{-1} A_f$, system (27) can be decoupled in the following way

$$\tilde{x}_i(k+1) = (1 - \gamma(1-\mu_i))\tilde{x}_i(k) + \mu_i \tilde{x}_i(k-\tau+1) - \mu_i \tilde{x}_i(k-\tau), \; i \in \mathcal{V}_f. \tag{28}$$

What remains is to derive the condition on γ to guarantee the stability of system (28). However, (28) is a multiple-delay system, and the stability of which is difficult to be analyzed. Motivated by [28], we introduce the following auxiliary systems

$$\tilde{x}_i(k+1) = \alpha(k)\tilde{x}_i(k) + f_i(k), \tag{29}$$

$$f_i(k) = \sum_{j=1}^{\tau} \beta_j(k) f_i(k-j). \tag{30}$$

By comparing the coefficients of (28)–(30), one has the following relationships:

(1) When $\tau = 1$,

$$\alpha(k) + \beta_1(k) = 1 + \mu_i - \gamma(1-\mu_i), \tag{31}$$

$$\alpha(k-1)\beta_1(k) = \mu_i. \tag{32}$$

(2) When $\tau > 1$,

$$\alpha(k) + \beta_1(k) = 1 - \gamma(1-\mu_i), \tag{33}$$

$$\alpha(k-1)\beta_1(k) - \beta_2(k) = 0, \tag{34}$$

$$\vdots$$

$$\alpha(k-\tau+2)\beta_{\tau-2}(k) - \beta_{\tau-1}(k) = 0, \tag{35}$$

$$\alpha(k-\tau+1)\beta_{\tau-1}(k)-\beta_\tau(k)=-\mu_i, \tag{36}$$

$$\alpha(k-\tau)\beta_\tau(k)=\mu_i. \tag{37}$$

Equations (34) and (35) are ignored in case 2 when $\tau = 2$. There are infinite number of ways to decompose system (28) into (29) and(30). However, once the initial conditions of α are determined, the decomposition is fixed. Define $\Delta_\alpha(k) = \alpha(k) - (1-\gamma)$. We choose the initial conditions of α in the following way

$$\alpha(k) \in \mathcal{C}(M,\gamma) \triangleq \{\alpha(k) \in \mathbb{R} : |\Delta_\alpha(k)| \leqslant M\gamma^2\},\ k = -1,\ldots,-\tau, \tag{38}$$

where M is defined in (24). Now, we are going to prove $\alpha(k) \in \mathcal{C}(M,\gamma)$ for all $k \in \bar{\mathbb{Z}}^+$ by induction. On the basis of (31)–(37), we have

$$\alpha(k) = 1-\gamma(1-\mu_i)+\mu_i(\Pi^\alpha_{k-1,k-\tau})^{-1}(\alpha(k-\tau)-1),$$

where $\Pi^\alpha_{i,j} = \alpha(i)\alpha(i-1)\ldots\alpha(j),\ i \geqslant j$. Given $k \in \bar{\mathbb{Z}}^+$, we assume $\alpha(k-j) \in \mathcal{C}(M,\gamma)$, that is, $|\Delta_\alpha(k-j)| \leqslant M\gamma^2,\ j = 1,\ldots,\tau$ that holds when $k = 0$ in light of (38). Then, we can calculate the upper bound of $|\Delta_\alpha(k)|$.

$$\begin{aligned}
|\Delta_\alpha(k)| &= \mu_i\left[\gamma + (\Pi^\alpha_{k-1,k-\tau})^{-1}(-\gamma+\Delta_\alpha(k-\tau))\right] \\
&\leqslant \mu_i\gamma|(\Pi^\alpha_{k-1,k-\tau})^{-1}-1| + \mu_i(\Pi^\alpha_{k-1,k-\tau})^{-1}|\Delta_\alpha(k-\tau)| \\
&\leqslant \mu_i\gamma(\gamma+M\gamma^2)\frac{(1-\theta)^{-\tau}-1}{\theta} + \mu_i(1-\theta)^{-\tau}M\gamma^2 \\
&\leqslant 2\mu_i\gamma^2\frac{(1-\theta)^{-\tau}-1}{\theta} + \mu_i(1-\theta)^{-\tau}M\gamma^2 \\
&\leqslant M\gamma^2,
\end{aligned}$$

which leads to $\alpha(k) \in \mathcal{C}(M,\gamma)$. The second inequality is due to (25) that guarantees $\gamma + M\gamma^2 < 2\gamma < \theta$. The last inequality is due to the definition of M in (24). By induction, we can see that $\forall k \in \bar{\mathbb{Z}}^+$, $|\Delta_\alpha(k)| \leqslant M\gamma^2$, that is, $\alpha(k) \in \mathcal{C}(M,\gamma)$. Because $\gamma < 1/M$ according to (25), we have

$$\bar{\alpha} \triangleq \sup_{k\in\bar{\mathbb{Z}}^+} \alpha(k) < 1.$$

On the other hand, on the basis of condition (25) and the definition of θ, $\forall k \in \bar{\mathbb{Z}}^+$,

$$\alpha(k) \geqslant 1-\gamma-M\gamma^2 > 1-2\gamma > 1-\theta > \sqrt[\tau]{\mu} \geqslant 0.$$

We can see that system (29) is uniformly stable.

Now, we consider system (30). Because $\forall k \in \bar{\mathbb{Z}}^+$, $\alpha(k) > 0$, from (31)–(37), we find that $\beta_j(k) > 0$ for all $j = 1,\ldots,\tau$, $k \geqslant 0$. Moreover, when $\tau = 1$,

$$\beta_1(k) = \frac{\mu_i}{\alpha(k-1)} < \frac{\mu_i}{\mu} \leqslant 1,$$

which implies that $f_i(k)$ is contracting and $\lim_{k\to\infty} f_i(k) = 0$. When $\tau > 1$, we define

$$\bar{f_i}(k) = \max_{j=1,\ldots,\tau} |f_i(k-j)|.$$

On the basis of (33)–(37), we have the following inequalities:

$$\beta_1(k) = 1-\gamma+\gamma\mu-\alpha(k) = \gamma\mu - \Delta_\alpha(k) \leqslant \gamma(1+\mu), \tag{39}$$

$$\beta_2(k) = \alpha(k-1)\beta_1(k) \leqslant \beta_1(k) \leqslant \gamma(1+\mu), \tag{40}$$

$$\vdots$$

$$\beta_{\tau-1}(k) = \alpha(k-\tau+2)\beta_{\tau-2}(k) \leqslant \gamma(1+\mu), \tag{41}$$

$$\beta_\tau(k) = \frac{\mu_i}{\alpha(k-\tau)} < \frac{\mu}{1-2\gamma}. \tag{42}$$

Because $\gamma < \frac{1}{\tau}\frac{1-\mu}{1+\mu} \leqslant \frac{1}{2}\frac{1-\mu}{1+\mu}$, we have

$$\begin{aligned}\frac{\mu}{1-2\gamma} - \gamma(1+\mu) &= \mu + \gamma\left[\frac{2\mu}{1-2\gamma} - 1 - \mu\right] \\ &< \mu + \gamma\left[\frac{2\mu}{1-\frac{1-\mu}{1+\mu}} - 1 - \mu\right] \\ &= \mu.\end{aligned} \tag{43}$$

Then, we can reach the contraction of $\bar{f}_i$ based on the following inequality:

$$\begin{aligned}|f_i(k)| &\leqslant \sum_{j=1}^{\tau} \beta_j(k)|f_i(k-j)| \\ &\leqslant \bar{f}_i(k)\left[\gamma(\tau-1)(1+\mu) + \frac{\mu}{1-2\gamma}\right] \\ &< \bar{f}_i(k)\left[\gamma\tau(1+\mu)+\mu\right] \\ &< \bar{f}_i(k),\end{aligned}$$

where the second inequality comes from (39)–(42), the third inequality comes from (43), and the last inequality comes from (25). Similarly, we can arrive at that $f_i(k+j) < \bar{f}_i(k)$, $j > 0$, which implies the existence of $\lambda \in (\gamma\tau(1+\mu)+\mu, 1)$ such that

$$\bar{f}_i(k+\tau) < \lambda \bar{f}_i(k), \ \forall k \in \bar{\mathbb{Z}}^+.$$

In light of the definition of $\bar{f}_i$, we have $\lim_{k\to\infty} f_i(k) = 0$, which together with uniform stability of system (29) yields $\lim_{k\to\infty} \tilde{x}_i(k) = 0$. Then, we have

$$\lim_{k\to\infty} \tilde{X}_{\mathrm{f}}(k) = T^{-1} \lim_{k\to\infty} \mathrm{col}\{\tilde{x}_1(k), \ldots, \tilde{x}_{n_{\mathrm{f}}}(k)\} = 0,$$

which leads to

$$\lim_{k\to\infty} X_{\mathrm{f}}(k) = \lim_{k\to\infty} \tilde{X}_{\mathrm{f}}(k) + L_{\mathrm{f}}^{-1} A_{\mathrm{f}} X_{\mathrm{l}} = L_{\mathrm{f}}^{-1} A_{\mathrm{f}} X_{\mathrm{l}} \in \mathrm{co}\{x_j(0), j \in \mathcal{V}_{\mathrm{l}}\}.$$

□

Remark 4.3

When all the followers are directly connected to the leaders and no communication exists among the followers, it has $\mu = 0$, and consequently, $M = 0$. $1/M$ in (25) goes to infinity. In fact, the upper bound in (25) is not tight. In this particular case, we know that the second term in protocol (23) disappears, and the upper bound of γ is 1. Although the condition of γ given in (25) may be conservative in some particular scenarios, it provides a uniform condition for any time delays in neighbors' control input information.

Now, we consider dynamically moving leaders. Assume that the leaders are connected with a network and the corresponding graph contains a spanning tree. Each leader is with the following dynamics:

$$x_i(k+1) = x_i(k) + u_i(k), \ i \in \mathcal{V}_{\mathrm{l}}, \tag{44}$$

where

$$u_i(k) = \eta_d \sum a_{i,j}[(x_j(k) - h_j) - (x_i(k) - h_i)] + v;$$

$a_{i,j} > 0$ when leader i can receive information from leader j and $a_{i,j} = 0$ otherwise; $\eta_d \in (0, 1/d_{(\mathrm{l})})$ and $d_{(\mathrm{l})} = \max_i\{\sum_{j\in\mathcal{V}_\mathrm{l}, j\neq i} a_{i,j}\}$. Denote by L_l the Laplacian matrix of the graph corresponding to the leaders' network and by ξ satisfying $\xi'\mathbf{1}_{n_\mathrm{f}} = 1$ the left eigenvalue of L_l corresponding to the eigenvalue 0. Then, we have

$$X_\mathrm{l}(k+1) - H = (I - \eta_d L_\mathrm{l})(X_\mathrm{l}(k) - H) + v\mathbf{1}_{n_\mathrm{l}},$$

where H is defined in (16), and the solution to the aforementioned equation satisfies

$$X_\mathrm{l}(k) = H + \xi'(X_\mathrm{l}(0) - H)\mathbf{1}_{n_\mathrm{l}} + vk\mathbf{1}_{n_\mathrm{l}} + O((1-\eta_d\lambda_2)^k), \tag{45}$$

where λ_2 is the eigenvalue of L_l with the second smallest real part. Because the graph contains a spanning tree, we have $R(\lambda_2) > 0$. From (45), we can see that the leaders can asymptotically achieve a formation while moving along a direction at the same speed.

We propose the following controller

$$u_i(k) = \frac{1}{d_i}\left\{-\gamma\sum_{j\in\mathcal{V}} a_{i,j}[x_i(k) - x_j(k)] + \sum_{j\in\mathcal{V}} a_{i,j}u_j(k-\tau)\right\},\ i \in \mathcal{V}_\mathrm{f}, \tag{46}$$

which is protocol (23) by incorporating the delayed control input information of the leaders. Then, we have the following result.

Theorem 4.4

For multi-agent systems (22) with dynamically moving leaders satisfying (44), if the communication topology contains a united spanning tree, then the control protocol (46) with γ satisfying (25) leads the followers to the convex hull of the leaders for any initial conditions.

Proof

By substituting (46) into system (22), we have

$$\begin{aligned} X_\mathrm{f}(k+1) =& X_\mathrm{f}(k) - \gamma D_\mathrm{f}^{-1}(L_\mathrm{f}X_\mathrm{f}(k) - A_\mathrm{l}X_\mathrm{l}) + D_\mathrm{f}^{-1}A_\mathrm{f}X_\mathrm{f}(k-\tau+1) - D_\mathrm{f}^{-1}A_\mathrm{f}X_\mathrm{f}(k-\tau) \\ &+ D_\mathrm{f}^{-1}A_\mathrm{l}X_\mathrm{l}(k-\tau+1) - D_\mathrm{f}^{-1}A_\mathrm{l}X_\mathrm{l}(k-\tau). \end{aligned}$$

By introducing $\tilde{X}_\mathrm{f}(k)$ defined in (26), we have the closed-loop system of the error $\tilde{X}_\mathrm{f}(k)$ as follows:

$$\begin{aligned} \tilde{X}_\mathrm{f}(k+1) &= (I - \gamma D_\mathrm{f}^{-1}L_\mathrm{f})\tilde{X}_\mathrm{f}(k) + D_\mathrm{f}^{-1}A_\mathrm{f}[\tilde{X}_\mathrm{f}(k-\tau+1) - \tilde{X}_\mathrm{f}(k-\tau)] \\ &\quad - L_\mathrm{f}^{-1}A_\mathrm{l}[X_\mathrm{l}(k+1) - X_\mathrm{l}(k)] + D_\mathrm{f}^{-1}A_\mathrm{f}L_\mathrm{f}^{-1}A_\mathrm{l}[X_\mathrm{l}(k-\tau+1) - X_\mathrm{l}(k-\tau)] \\ &\quad + D_\mathrm{f}^{-1}A_\mathrm{l}[X_\mathrm{l}(k-\tau+1) - X_\mathrm{l}(k-\tau)] \\ &= (I - \gamma D_\mathrm{f}^{-1}L_\mathrm{f})\tilde{X}_\mathrm{f}(k) + D_\mathrm{f}^{-1}A_\mathrm{f}[\tilde{X}_\mathrm{f}(k-\tau+1) - \tilde{X}_\mathrm{f}(k-\tau)] + L_\mathrm{f}^{-1}A_\mathrm{l}e_d(k), \end{aligned} \tag{47}$$

where $e_d(k) = [X_\mathrm{l}(k-\tau+1) - X_\mathrm{l}(k-\tau)] - [X_\mathrm{l}(k+1) - X_\mathrm{l}(k)]$. Then, according to (45), we know that $\|e_d(k)\|_2 = O((1-\eta_d\lambda_2)^k)$. Denote $TL_\mathrm{f}^{-1}A_\mathrm{l}e_d(k) = \mathrm{col}\{e_1(k), \ldots, e_{n_\mathrm{l}}(k)\}$ and $T\tilde{X}(k) = \mathrm{col}\{\tilde{x}_1(k), \ldots, \tilde{x}_{n_\mathrm{f}}(k)\}$, where T is an invertible matrix such that $TD_\mathrm{f}^{-1}A_\mathrm{f}T^{-1}$ is diagonal. Then, (47) can be decomposed into the following systems:

$$\tilde{x}_i(k+1) = (1 - \gamma(1-\mu_i))\tilde{x}_i(k) + \mu_i\tilde{x}_i(k-\tau+1) - \mu_i\tilde{x}_i(k-\tau) + e_i(k),\ i = 1, \ldots, n_\mathrm{f}.$$

Similar to the proof of Theorem 4.3, we introduce the auxiliary systems

$$\tilde{x}_i(k+1) = \alpha(k)\tilde{x}_i(k) + f_i(k), \tag{48}$$

$$f_i(k) = \sum_{j=1}^{\tau}\beta_j(k)f_i(k-j) + e_i(k), \tag{49}$$

where α and β satisfy (31)–(37). It can be shown that if γ satisfies (25),

(1) $\alpha \geqslant 0$ and $\sup_{k\in\bar{\mathbb{Z}}^+} \alpha(k) < 1$,
(2) $\bar{f}_i(k+\tau) < \lambda \bar{f}_i(k) + \tau \max_{j=-\tau}^{\tau-1} |e_i(k+j)|, \forall k \in \bar{\mathbb{Z}}^+$, where $\bar{f}_i(k) = \max_{j=1,\dots,\tau} |f_i(k-j)|$, $\lambda \in (0,1)$.

According to the definition of e_i, one has $e_i(k) = O((1-\eta_d\lambda_2)^k)$. For any small number $\varepsilon > 0$, there must exist a number $N \in \mathbb{Z}^+$ such that

$$\forall k \geqslant N,\ \tau \max_{j=-\tau}^{\tau-1} |e_i(k+j)| < \varepsilon.$$

Then, according to 2, it follows that

$$\bar{f}_i(k+n\tau) < \lambda^n \bar{f}_i(k) + \sum_{m=0}^{n-1} \lambda^{n-m-1} \tau \max_{j=-\tau}^{\tau-1} |e_i(k+m\tau+j)|$$

and

$$\lim_{n\to\infty} \bar{f}_i(k+n\tau) < \lim_{n\to\infty} \left\{ \sum_{m=0}^{\bar{n}} \lambda^{n-m-1} \tau \max_{j=-\tau}^{\tau-1} |e_i(k+m\tau+j)| + \varepsilon \sum_{m=\bar{n}+1}^{n-1} \lambda^{n-m-1} \right\} = \frac{\varepsilon}{1-\lambda},$$

where $\bar{n} = \arg\min_j \{k + j\tau \geqslant N\}$. The arbitrariness of ε implies $\lim_{k\to\infty} \bar{f}_i(k) = 0$, which together with uniform stability of system (48) yields that $\lim_{k\to\infty} \tilde{X}_{\mathrm{f}}(k) = 0$. The rest of the proof follows from the same line of arguments as that of Theorem 4.3. □

Remark 4.4
The key problem is to select γ such that the time-delay system (28) is stable. The traditional approaches including state augmentation and z-transformation approaches cannot work. It is difficult to guarantee the stability of the system matrix of the augmentation system or the stability of the solution of character equation by selecting γ. Motivated by [28] and [29], we introduced the auxiliary systems (29) and (30). By carefully designing the initial condition of α, the stability problem can be solved.

5. NUMERICAL EXAMPLES

In this section, we shall provide some examples to demonstrate the results. We consider three leader agents and five followers in a two-dimensional space; that is, the state of each agent satisfies $x_i \in \mathbb{R}^2$. All the agents are connected as in Figure 1. We assume that $a_{i,j} = 1$ when $j \in \mathcal{N}_i$ and $a_{i,j} = 0$ otherwise. Then, we have $D_{\mathrm{f}} = \mathrm{diag}\{2,2,3,2,2\}$ and

$$A_{\mathrm{f}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ L_{\mathrm{f}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix},\ A_{\mathrm{l}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (50)$$

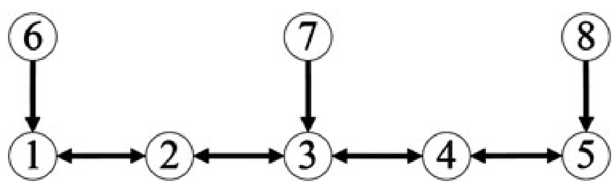

Figure 1. Communication graph $\mathcal{G}$.

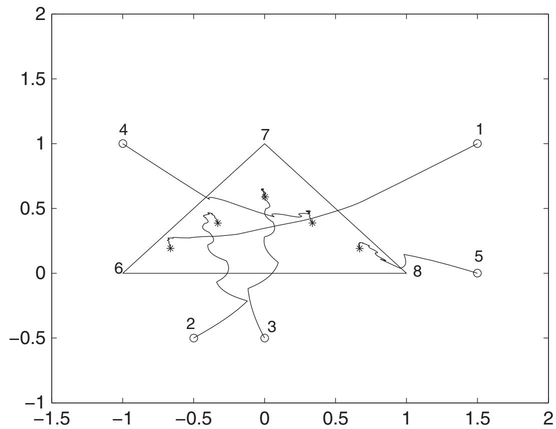

Figure 2. State trajectories of the followers.

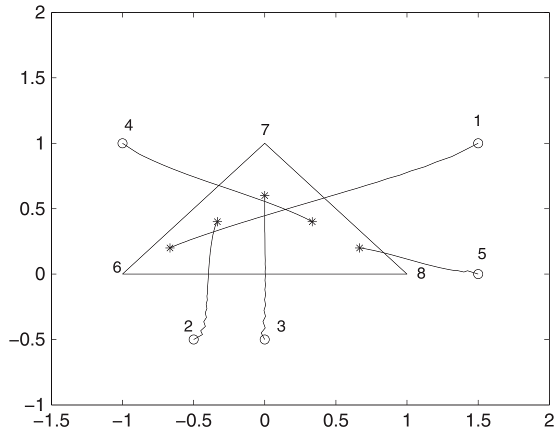

Figure 3. State trajectories of the followers.

The initial conditions are $x_1(0) = [1.5, 1]'$, $x_2(0) = [-0.5, -0.5]'$, $x_3(0) = [0, -0.5]'$, $x_4(0) = [-1, 1]'$, $x_5(0) = [1.5, 0]'$, $x_6(0) = [-1, 0]'$, $x_7(0) = [0, 1]'$, and $x_8(0) = [1, 0]'$.

First, we consider continuous-time system (3) with protocol (9). Let $\gamma = 1$ and $\tau = 0.5$. The trajectories of the followers are shown in Figure 2. The initial positions of the followers are marked with 'o', and the end positions are marked with '$*$'. It is clear that all the followers are finally enclosed in the triangular formed by the leaders.

Now, we consider the discrete-time system (22) with control protocol (23). According to (50), we have $\mu = 0.764$. Consider $\tau = 1$ and $\theta = 0.134$. In light of (25), γ can be chosen as 0.067. The state trajectories of the followers are illustrated in Figure 3.

6. CONCLUSION

We have studied the containment control problem by developing a delayed neighbors' control input information-based protocol. The stationary leaders case was first studied. It has been proved that all the followers asymptotically converge into a convex hull of the leaders. The moving leaders case has also been considered. In this case, all the leaders were controlled to asymptotically achieve a desired formation while all the followers can be controlled to achieve the containment tracking. The problems were studied for both continuous-time and discrete-time systems. The result for discrete-time systems is not a trivial extension from the result for continuous-time systems. A systematic method has been given to select the consensus gain γ based on the time delay τ and graph information μ. Numerical examples have been given to verify the algorithms.

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