Title
Tessellations

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TESSELLATIONS

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A *tessellation* may be defined as a division of a space into convex polygonal regions; divisions of the *plane* ($\mathbf{R}^2$) are most often discussed. While geometrists centuries ago reserved the term tessellation merely for divisions of the plane into *regular* polygons of equal size, most authors now define tessellations very broadly so as to include any arrangements of non-overlapping shapes covering R^d, as well as partitions of other metric spaces.

Tessellations arise quite naturally in numerous applications. In some situations, e.g. in geography, cellular biology or crystallography, one may wish to describe observed structures using models for tessellations. In many other applications, point process data are observed from which one may wish to characterize the mosaic of regional patterns. The Voronoi tessellation and its dual concept, the Delaunay tessellation, are commonly used in such circumstances. Chapter 1.2 of [14] provides a survey of the historical development of Voronoi diagrams and Delaunay tessellations, including early applications in diverse fields such as crystallography, ecology, meteorology, epidemiology, linguistics, economics, archaeology, and astronomy, dating back to Descartes in the early 17th century. References to numerous further examples are given in chapter 10.2 of [17], including examples in forestry, communication theory, geology, metallography, and zoology, and chapter 5.3 of [14] summarizes recent applications in a wide variety of disciplines.

VORONOI TESSELLATIONS

A classic example of a *randomly* generated tessellation is the Voronoi tessellation (also associated with the names Dirichlet and Thiessen), which separates a region into *cells* $\{D_i\}$ using a point process N.

(Figure 1 here)

The Voronoi tessellation is constructed as follows: for each point τ_i of the point process, let D_i be the area consisting of all locations in the space which are closer to τ_i than to any other point of N. This definition is applicable to Voronoi tessellations in any metric space. An example in $\mathbf{R}^2$ is depicted in Figure 1. Attention is usually restricted to non-degenerate cases to ensure that the Voronoi tessellation resulting from a point pattern is well defined. For example, one typically assumes that there are at least two points, that the points are all distinct, and that there are only finitely many in any bounded region.

Green and Sibson [3] provide the following delightfully intuitive description: "One might think of the points as being the locations of the lairs of competitive predators of equal strength; the region associated with each point is then the area available to the corresponding predator." Voronoi tessellations have thus been applied in models for populations of species such as plants and birds (see e.g. chapter 8.6 of [15]).

Certain geometric properties of the Voronoi tessellation in $\mathbf{R}^d$ are immediate. For instance, for any two adjacent cells D_i and D_j containing points τ_i and τ_j, the side common to both cells is a perpendicular bisector of the points τ_i and τ_j. Another example is that each vertex is the circumcenter of the points whose cells share the vertex. Further examples of general properties of Voronoi tessellations are discussed in chapter 2.3 of [14].

A special case is the *Poisson-Voronoi tessellation*, which is simply a Voronoi tessellation generated from a Poisson process [see point processes]. The Poisson process generating the tessellation is generally assumed to be stationary, with rate λ. Many properties of the (stationary) Poisson-Voronoi tessellation have been discovered: in the planar case, the expected number of vertices (or edges) in an arbitraily-chosen cell (e.g. the cell containing the origin) is 6, and its expected area is $1/\lambda$, for example. An excellent survey of these and similar results for Poisson-Voronoi tessellations in $\mathbf{R}^d$ is given in [10]; also see chapter 10.6 of [17] and chapters 5.2 and 5.5 of [14]. Poisson-Voronoi tessellations on the sphere are discussed in [1] and [9].

Voronoi tessellations originating from non-Poisson point processes have also been investigated. For instance the point process N may be modeled as compound Poisson, clustered, etc. A survey of results on planar and spatial tessellations generated by various special non-Poisson point processes is given in chapter 5.12 of [14].

Green and Sibson [3] provide an efficient computer algorithm for constructing a Voronoi tessellation from a point pattern. See chapter 4 of [14] and references therein for a description and comparison of the Green-Sibson and alternative construction methods. These algorithms, in conjunction with efficient routines for simulating point processes (e.g. [6,13]), make simulating a Voronoi tessellation a simple and speedy task.

DELAUNAY TESSELLATIONS

Given a point process with points $\{\tau_i\}$, an alternative type of tessellation can be formed by joining all neighboring points; by "neighboring" we mean pairs of points whose cells in the Voronoi tessellation share an edge. The tessellation resulting from this construction is the *Delaunay tessellation*. Under general conditions (see chapter 2.3 of [14]), the cells of a planar Delaunay tessellation are triangular. The Delaunay tessellation is also called the *dual* of the Voronoi tessellation. Note that like the Voronoi tessellation, the definition of the Delaunay tessellation applies not merely to R^d but to general metric spaces.

When the point process in this construction is a stationary Poisson process, the result is called a *Poisson-Delaunay tessellation*. Properties of Poisson-Delaunay tessellations are very well-known; for instance in the planar case the density function corresponding to the typical cell, in terms of its size and angles, has been derived [5,8]. Chapter 5.11 of [14] provides a nice summary of such properties, including results for Poisson-Delaunay tessellations on $\mathbf{R}^d$.

JOHNSON-MEHL TESSELLATIONS

Another important type of tessellation is the *Johnson-Mehl* tessellation, which is derived from a dynamic (e.g. spatial-temporal) point process N. Suppose that, after it is generated, each point of N is perceived to *grow* in every direction, with some constant speed. To a point in the point process, there corresponds a cell consisting of all spatial locations first hit by the growth of this point. The Johnson-Mehl tessellation is the collection of these cells.

In the special case where the points of N all occur at exactly the same time, the Johnson-Mehl tessellation reduces to a Voronoi tessellation. In general, however, Johnson-Mehl tessellations may be quite complex, containing cells that are not convex, for instance. Okabe et al. [14] note that the Johnson-Mehl tessellation is a special case of an additively weighted Voronoi tessellation, in which the weight associated with each point τ_i is simply the time t_i at which the point occurs.

Well-written treatments of properties of Johnson-Mehl tessellations are given in [11,12]. For further references on properties and applications in various fields including crystallography, metallurgy and biology, see page 314 of [17] or chapter 5.8 of [14].

HYPERPLANE TESSELLATIONS

An important type of tessellation in $\mathbf{R}^d$ is the hyperplane tessellation, i.e. the tessellation generated by dividing up $\mathbf{R}^d$ via $(d-1)$-dimensional hyperplanes. For instance, one may divide the plane up into cells using a random collection of lines.

An important model for a random collection of lines in the plane is the (undirected) Poisson line process in $\mathbf{R}^2$, which may be defined as a Poisson point process on the space $\mathbf{R} \times (0, \pi]$, with the convention that any point (t, θ) in $\mathbf{R} \times (0, \pi]$ corresponds to the line in $\mathbf{R}^2$ whose perpendicular distance from the origin is t and whose angle with the x-axis, measured counterclockwise, is θ. Note that the space $\mathbf{R} \times (0, \pi]$ represents the half-cylinder of unit radius in $\mathbf{R}^3$; see chapter 8.2 of [17] for elaboration.

The definition of the Poisson line process extends readily to the Poisson hyperplane processes, which generate the Poisson hyperplane tessellations. Properties of Poisson line tessellations and Poisson plane tessellations are summarized in chapter 10.5 of [17] and page 40 of [14]. For examples of other types of hyperplane processes, see pages 250-255 of [17].

FURTHER TESSELLATIONS

The definition of the Voronoi tessellation has been extended in various ways. An important example is the weighted Voronoi tessellation, which is the tessellation resulting from the construction identical to that of the ordinary Voronoi tessellation except that instead of Euclidean distance, a different metric (or non-metric function) is used. For example the distance function used may be a multiple of Euclidean distance, i.e. the distance between an arbitrary location x and a point τ_i of the generating point process N may be given by $w_i d(x, \tau_i)$, where d denotes Euclidean distance, and w_i is a scalar that depends on i. The resulting tessellation is called a multiplicatively weighted Voronoi tessellation. Alternatively, one may construct an additively weighted Voronoi tessellation, using a distance function of the form $w_i + d(x, \tau_i)$. For further examples of such tessellations and their applications, see chapter 3.1 of [14].

Another important generalization of the Voronoi tessellation is the order-k

Voronoi tessellation, in which a cell consists not of all locations closer to a *single* point τ_i of the point process N than to any other point τ_j, but of all locations closer to each of the k points $\{\tau_{i_1}, \ldots, \tau_{i_k}\}$ than to any other point of N. Alternatively one may modify the construction of the Voronoi tessellation by prescribing that cell i consist of locations for which point τ_i is kth (rather than first) in the list of distances from the points of N to that location, or in the case where N contains finitely many points, that cell i consist of all locations *furthest from* (rather than closest to) point τ_i. For properties of these and other extensions of the Voronoi tessellation, see chapter 3 of [14].

A construct closely related to the Johnson-Mehl tessellation is the dead leaves tessellation of Matheron [7], in which random shapes are placed in the plane (or higher-dimensional space) and centered at points $\{\tau_i\}$ of a stationary spatial-temporal Poisson process, with the convention that if two shapes overlap, the later shape supercedes the prior one. See [2] and pages 508-511 of [16] for various properties. The case where all the points τ_i fall simultaneously, i.e. where random shapes are placed with their centers at the points of a stationary spatial Poisson process, is called a Boolean model. Typically for the Boolean model, one allows all the shapes to overlap and investigates the properties of *clumps*, defined as connected clusters of overlapping shapes. See [4], chapter 13B of [16] or chapter 3 of [17] for properties and statistics for the Boolean model.

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Figure 1: Construction of Poisson-Voronoi tessellation