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There is some interest at present in experiments involving the directions of the spins of particles moving with relativistic velocities. In these experiments there may be a sequence of processes each of which is conveniently described in its own center-of-mass frame. The actual measurements are performed in the laboratory frame. Although the relativistic transformation of tensor quantities between various frames is simple, the transformation of the spin direction, a three-vector quantity, has caused some uncertainty.

The purpose of this note is to try to clarify the situation by a short discussion on a practical level. The first part of the discussion is specifically related to spin-$\frac{1}{2}$ particles but the generalizations to higher spins are valid.

The spin of a Dirac particle is described by the expectation value of the antisymmetic tensor $\sigma_{\mu\nu}$. Considerations relating to conservation of angular momenta involve this tensor. However, in experiments concerning the asymmetries related to polarization effects, it is convenient to describe the state of polarization by a three-dimensional vector called the polarization vector. This vector is the expection value of the spin (in appropriate units) measured in the rest frame of the particle. In this frame the expectation value of $\sigma_{\mu\nu}$ reduces to a three-by-three antisymmetric tensor which may be expressed as an axial vector. The usefulness of the polarization vector lies partially in its simple relationship to the observed asymmetries. In the reaction-center-of-mass frame on the angular distribution of the decay products of a Dirac particle is of the form $(1 + \alpha \underset{\sim}{P} \cdot \underset{\sim}{V})$, where $\underset{\sim}{P}$ is the polarization vector of the decaying particle, $\underset{\sim}{V}$ is the velocity of the decay products, and $\alpha$ is a constant. A similar relationship obtains if the spin of the particle is detected by means of a scattering experiment (see below).

A second useful property of the polarization vector is that for particles emerging from some reaction--such as a production process or a scattering process, etc.--it is this vector that is expressed rather directly in terms of the basic parameters of the reaction process. For these two reasons it is convenient in the discussion of experiments of this nature to work directly with the polarization vectors, eliminating explicit reference to the $\sigma_{\mu\nu}$.

There is, however, a problem. The definition of the polarization vector given above is ambiguous; in order to define this vector (or more accurately the relationship between the vector and its components) a specific rest frame for the particle must be selected. That is, the orientation of the space axis must be determined. A natural choice for the rest frame is that frame which is obtained by a direct Lorentz transformation of the laboratory frame to the velocity of the particle. By a "direct" Lorentz transformation I mean that transformation corresponding to a certain change in velocity which leaves the space components perpendicular to this velocity unchanged. Let the rest frame defined in this manner be denoted by $\Sigma^{L}_{RF}$, where the superscript L indicates that it is the laboratory frame to which it is related by a direct Lorentz transformation. The complication arises because the simple formulae mentioned above that relate the polarization vector of a particle emerging from a reaction to the parameters of the reaction do not refer to the polarization vector as measured in $\Sigma_{RF}^{L}$, but rather to the polarization vector as measured in a different rest frame of the particle. This second rest frame, called $\Sigma_{RF}^{CM}$, is the one obtained by a direct Lorentz transformation to the velocity of the particle, not of the laboratory frame, but of the reaction center-of-mass frame. When the components of the polarization vector are measured in this frame, $\Sigma_{RF}^{CM}$, the relationships between the polarization vector and the parameters of the interaction are identical with the simple relationships obtained from a nonrelativistic treatment.

It is important to clearly distinguish between these two frames. For instance, in the decay of a polarized particle the experiments are usually analyzed by transforming the laboratory results to the rest frame of the decaying particle. More specifically one performs the direct Lorentz transformation of the laboratory measurements to the rest frame of the particle. This transforms the measurements to $\Sigma_{CM}^{L}$. On the other hand, the theoretical value of the polarization vector is given directly in the frame $\Sigma_{RF}^{CM}$. In order to transform this theoretical value into the frame $\Sigma_{RF}^{L}$ one may apply a sequence of transformations, the first taking $\Sigma_{RF}^{CM}$ to the center-of-mass frame $\Sigma_{CM}$, the next taking $\Sigma_{CM}$ to the laboratory frame $\Sigma_{L}$, and the last taking the laboratory frame to $\Sigma_{RF}^{L}$. In the intermediate stages the spin is described by the full antisymmetric tensor but at the end it again becomes expressible as a three-vector. In general this polarization vector, $\underset{\sim}{P}^{L}$ is different from the original polarization vector $\underset{\sim}{P}^{CM}$ (i.e., the components of the proper polarization in the two frames differ). However, the difference is only in direction, the product of the three Lorentz transformations being equivalent to a pure rotation. If the laboratory and center-of-mass frames are related by a direct Lorentz transformation then the product of the three direct Lorentz transformations ($\Sigma_{RF}^{CM} \to \Sigma_{CM} \to \Sigma_L \to \Sigma_{RF}^{L}$) is equivalent to a rotation whose magnitude $\Omega$ is given by

$$\sin|\Omega| = \frac{|\underset{\sim}{V}_1 \times \underset{\sim}{V}_2| (1 + \gamma_1 + \gamma_2 + \gamma_3)}{(1 + \gamma_1)(1 + \gamma_2)(1 + \gamma_3)} , \tag{1}$$

where 1, 2, and 3 refer to the three transformations (in order), the $\gamma_i$ are the relativistic contraction factors $(1 - \beta_i^{\,2})^{-1/2}$, and $\underset{\sim}{V}_i$ are the covariant velocities $d\underset{\sim}{x}/d\tau = \gamma\, d\underset{\sim}{x}/dt$. If it is the effective rotation of the polarization vector ($\underset{\sim}{P}^{CM} \to \underset{\sim}{P}^{L}$) that is considered (as opposed to a rotation of the coordinate system) then the sense of the rotation is the same as the rotation that takes the direction of motion of the particle from the center-of-mass frame to the laboratory frame. The use of this formula can often be avoided, however, as the examples to be given below show.

As a first practical example the asymmetry of the $\mu$ decay in the $\pi - \mu - e$ chain is considered for the case in which the $\pi$ is moving rapidly. In the decay of the $\pi$ the reaction center-of-mass frame, $\Sigma_{CM}$, is the $\pi$-rest frame. The frame $\Sigma_{RF}^{CM}$ is obtained by a Lorentz transformation of $\Sigma_{CM}$ to the velocity of the $\mu$ particle. In $\Sigma_{RF}^{CM}$ the polarization vector of the $\mu$ lies along the direction $(\Theta , \Phi)$ where these are the $\pi$-decay angles as measured in the $\pi$ center-of-mass frame $\Sigma_{CM}$. The experimental data on the $\mu$ decay could be analyzed by transforming the laboratory velocities of the $\mu$-decay products first to the $\pi$ rest frame and then to the frame $\Sigma_{RF}^{CM}$. After these two transformations the angular distribution of the $\mu$-decay products would show a peak in the direction $(\Theta, \Phi)$, the direction of motion of the $\mu$. Alternatively one may analyze the experiments by transforming the laboratory velocities <u>directly</u> to the $\mu$-rest frame (i.e., to $\Sigma_{RF}^{L}$).

The direction of the polarization vector in this latter frame may be obtained from the known direction in $\Sigma_{RF}^{CM}$ by means of Eq. (1). Alternatively the magnitude of the rotation $\Omega$ may be deduced by the following argument: In the laboratory frame, $\Sigma_L$, the direction of motion of the $\mu$ particle is given by $(\theta, \phi)$ with respect to a polar axis along the $\pi$ direction. Therefore seen from $\Sigma_{RF}^{L}$ the motion of the laboratory is along this same direction but with the opposite sense. This is an immediate consequence of the definition of $\Sigma_{RF}^{L}$. Likewise, seen from the frame $\Sigma_{RF}^{CM}$, the $\pi$-rest frame $\Sigma_{CM}$ moves along the direction $(\Theta, \Phi)$ in the negative sense. With respect to the $\pi$ rest frame the laboratory velocity is known and thus the angle measured in $\Sigma_{RF}^{CM}$ between the laboratory velocity and the velocity of the $\pi$ is easily computed by the addition-of-velocities formula. Let this angle be called $\tau$. Then as measured in $\Sigma_{RF}^{CM}$ the laboratory velocity is in the direction $(\Theta - \tau, \Phi)$, whereas in the frame $\Sigma_{RF}^{L}$ this direction is $(\theta, \phi)$, the sense being negative in both cases. Since $\phi$ and $\Phi$ are equal, the magnitude of the rotation that brings vectors in $\Sigma_{RF}^{CM}$ to their values in $\Sigma_{RF}^{L}$ must be $(\theta + \tau - \Theta)$. This is the magnitude of rotation $\Omega$. Since the direction of the spin in $\Sigma_{RF}^{CM}$ is $(\Theta, \Phi)$, the direction in $\Sigma_{RF}^{L}$ is $(\theta + \tau, \Phi)$. The difference in this frame between the direction of motion of the $\mu$ (relative to the laboratory) and the direction of the polarization vector is $\theta + \tau - \theta = \tau$.

As a second example the angular distribution from the decay of a spin-$\frac{3}{2}$ hyperon produced in a pion-nucleon collision is considered. For energies at which only final S waves contribute to the production process the hyperon has no vector polarization, but it does have a tensor polarization or alignment. This generalized type of polarization gives to the decay products an angular distribution of the form $(3 \cos^2 \Theta' + 1)$, where $\Theta'$ is is the angle between the decay products and the incident direction of the production process. This angle is given a meaning by the specification that the incident direction be measured in the center-of-mass frame of the production process and that the decay angle be measured in $\Sigma_{RF}^{CM}$, a rest frame of the hyperon. That is, the preferred direction of the decay process makes an angle of $\Theta$ with the velocity of the production center of mass as seen from the hyperon, where $\Theta$ is the production angle in the center-of-mass frame. Again the experiments may be analyzed by transforming the laboratory velocities of the decay products first to the center-of-mass frame and then to $\Sigma_{RF}^{CM}$. Alternatively the direct transformation to $\Sigma_{CM}^{L}$ may be used if the directions defined in $\Sigma_{RF}^{CM}$ are rotated by $\Omega = (\theta + \tau - \Theta)$ to bring them to $\Sigma_{CM}^{L}$. The preferred decay direction in $\Sigma_{RF}^{CM}$ is along the z axis if in $\Sigma_{CM}$ the incident direction of the production process is taken along the z axis. In the frame $\Sigma_{RF}^{L}$ this direction is at the polar angle $(\theta + \tau - \Theta)$. The angle between the laboratory direction of motion and the preferred direction of decay, both seen from $\Sigma_{RF}^{L}$, is then $(\tau - \Theta)$.

As a final example the detection of the direction of polarization by means of a scattering process is considered. If the target particle can be considered as infinitely heavy the situation is similar to the ones just considered. The differential cross section for a spin-$\frac{1}{2}$ particle scattered by an unpolarized target is of the form $(A(\theta) + B(\theta)\underset{\sim}{V}_{in} \times \underset{\sim}{V}_{out} \cdot \underset{\sim}{P})$, where $\underset{\sim}{P}$ is the polarization vector of the incident particle as measured in $\Sigma_{RF}^{CM'}$. The primed center-of-mass frame referred to by the superscript is the center-of-mass frame of the scattering process. If the target particle is at rest in the laboratory frame and can be considered as infinitely massive, then $\Sigma_{RF}^{CM'}$ is the same as $\Sigma_{RF}^{L}$ and the previous considerations are directly applicable. If the target is not sufficiently massive to justify this approximation the relationship between $\Sigma_{RF}^{L}$ and $\Sigma_{RF}^{CM'}$ can be established in the same way as the relationship between $\Sigma_{RF}^{L}$ and $\Sigma_{RF}^{CM}$ was established earlier.

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