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The Design and Study of a Learning Environment to Support Growth and  
Change in Students' Knowledge of Fraction Multiplication

by

Rozy Brar

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Science and Mathematics Education

in the

GRADUATE DIVISION

of the

University of California, Berkeley

Committee in charge:

Professor Alan H. Schoenfeld, chair

Professor Andrea A. diSessa

Professor Geoffrey B. Saxe

Fall 2010

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## Abstract

## The Design and Study of a Learning Environment to Support Growth and Change in Students' Knowledge of Fraction Multiplication

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Doctor of Philosophy in Science and Mathematics Education

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Professor Alan H. Schoenfeld, Chair

There is a strong push from within mathematics education reform to incorporate representations in math classrooms (Behr, Harel, Post, & Lesh, 1993; Kieren, 1993; NCTM, 2000). However, questions regarding what representations should be used (for a given topic) and how representations should be used (such that students gain a deep understanding of that topic and a deep understanding of the representations) remain largely unanswered. Hence, we need a well-specified and general theoretical treatment of how students co-develop domain and representational competence.

In this dissertation study, I use design-based research (DBR) to investigate and support growth and change in students' knowledge of rational number operations. "Among all the topics in K-12 curriculum, rational numbers arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, and the most essential to success in higher mathematics and science" (Lamon, 2007). In order to shed some light on the domain of rational number operations, I designed a learning environment centered on the Area Model for Fraction Multiplication (AM-FM) representation, a computer-based tool intended to help students develop a deep understanding of fraction multiplication.

Data for the dissertation were collected from an urban school with a racially and socio-economically diverse student population. I met with ten students once a week for four weeks. During the first and last session students were asked to think-aloud through a pretest and posttest. The second and third sessions consisted of semi-structured clinical interviews during which students were asked to solve fraction multiplication problems using the AM-FM representation. All sessions were videotaped and transcribed. Two students were chosen to serve as cases of knowledge growth and change.

Findings indicate that both students followed a particular learning trajectory for making sense of fraction multiplication when using the AM-FM representation and their emergent knowledge was context sensitive. Furthermore, DBR is predicated on (a) design refinement and (b) local theory development (diSessa & Cobb, 2004; Schoenfeld, 2006). With respect to design, the AM-FM representation and the clinical interview protocol was refined based on analysis of the data. With respect to local theory, I offered a decomposition of

competence with fraction multiplication (i.e., domain competence) and the AM-FM representation (i.e., representational competence). Local theory was also refined based on an analysis of the data.

To my soon to be first-born, Rishan Kumar Vig.

## Table of Contents

<b>LIST OF TABLES AND FIGURES.....</b>	<b>V</b>
<b>CHAPTER 1: INTRODUCTION.....</b>	<b>1</b>
<b>CHAPTER 2: LITERATURE REVIEW .....</b>	<b>5</b>
2.1 CHAPTER OVERVIEW .....	5
2.2 KEY CONCEPTS .....	5
2.3 STUDENT DIFFICULTIES.....	6
2.3.1 <i>Multiple subconstructs</i> .....	6
2.3.2 <i>Whole number bias</i> .....	7
2.3.3 <i>Unitizing</i> .....	7
2.3.4 <i>Multiple representations</i> .....	8
2.4 DESIRED LEARNING GOALS .....	8
2.5 THREE PERSPECTIVES ON REPRESENTATIONS.....	9
<b>CHAPTER 3: DESIGN .....</b>	<b>11</b>
3.1 CHAPTER OVERVIEW .....	11
3.2 THE PROBLEM CONTEXT .....	11
3.3 PHYSICAL MEDIA: THE PAPER CUTOUTS AND THE NUMBER CHART .....	11
3.4 COMPUTER-BASED MEDIUM: THE AM-FM REPRESENTATION.....	14
3.4.1 <i>How it works</i> .....	14
3.4.2 <i>The medium</i> .....	20
3.4.3 <i>Range and domain</i> .....	20
3.4.4 <i>Scaling</i> .....	22
3.4.5 <i>Tile size</i> .....	22
3.4.6 <i>Sub-grid view</i> .....	22
3.4.7 <i>Student difficulties</i> .....	23
<b>CHAPTER 4: THEORY .....</b>	<b>25</b>
4.1 CHAPTER OVERVIEW .....	25
4.2 TRACY NARRATIVE: BEFORE EXPOSURE TO THE DESIGNED LEARNING ENVIRONMENT.....	26
4.3 IDEALIZED HYPOTHETICAL INITIAL STATE OF STUDENT UNDERSTANDING (S(I)).....	27
4.3.1 <i>Area model understanding (i1)</i> .....	28
4.3.2 <i>Number line understanding (i2)</i> .....	29
4.3.3 <i>Fraction notation understanding (i3)</i> .....	30
4.3.4 <i>Conceptual understanding (i4)</i> .....	31
4.4 TRACY NARRATIVE: AFTER EXPOSURE TO THE DESIGNED LEARNING ENVIRONMENT.....	33
4.5 IDEALIZED HYPOTHETICAL EXIT STATE OF STUDENT UNDERSTANDING (S(E)).....	35
4.5.1 <i>Fraction multiplication as stretching/shrinking (e1)</i> .....	35
4.5.2 <i>Number sense with fraction multiplication (e2)</i> .....	36
4.5.3 <i>Representational fluency for fraction multiplication (e3)</i> .....	36
4.5.4 <i>Representational fluency for fraction equivalence and fraction order, (e4 &amp; e5)</i> .....	37
4.6 IDEALIZED HYPOTHETICAL LEARNING TRAJECTORY FROM S(I) TO S(E).....	38
4.6.1 <i>Multiplication as stretching/shrinking (e1)</i> .....	39
4.6.2 <i>Number sense with fraction multiplication (e2)</i> .....	40
4.6.3 <i>Representational fluency for fraction multiplication (e3)</i> .....	40
<b>CHAPTER 5: METHODS .....</b>	<b>42</b>
5.1 CHAPTER OVERVIEW .....	42
5.2 CONTEXT .....	42
5.3 PARTICIPANTS.....	42
5.4 DATA COLLECTION.....	42

5.5 INTERVIEW PROTOCOL .....	43
5.6 DATA ANALYSIS .....	44
<b>CHAPTER 6: THE CASE OF NEATO.....</b>	<b>47</b>
6.1 CHAPTER OVERVIEW .....	47
6.2 FRACTION MULTIPLICATION AS STRETCHING/SHRINKING (E1): NEATO'S AM-FM CONSTRUCTION .....	47
6.3 NUMBER SENSE WITH FRACTION MULTIPLICATION (E2): NEATO'S PREDICTIONS AND JUSTIFICATIONS .....	49
6.4 REPRESENTATIONAL FLUENCY FOR FRACTION MULTIPLICATION (E3): NEATO'S AM-FM AND NUMBER CHART INTERPRETATION .....	51
6.4.1 <i>Neato's learning trajectory for interpreting final area output: Knowledge coordination across representations.</i> .....	52
6.4.2 <i>Area model and number line: Context sensitivity of knowledge coordination.</i> .....	60
6.5 REPRESENTATIONAL FLUENCY FOR FRACTION EQUIVALENCE (E4) .....	70
6.6 REPRESENTATIONAL FLUENCY FOR FRACTION ORDER (E5).....	77
6.7 CASE STUDY DISCUSSION .....	80
6.7.1 <i>Fraction multiplication as stretching/shrinking.</i> .....	80
6.7.2 <i>Number sense with fraction multiplication.</i> .....	81
6.7.3 <i>Representational fluency for fraction multiplication.</i> .....	82
6.7.4 <i>Representational fluency for fraction equivalence and fraction order.</i> .....	84
<b>CHAPTER 7: THE CASE OF OSCAR.....</b>	<b>85</b>
7.1 CHAPTER OVERVIEW .....	85
7.2 FRACTION MULTIPLICATION AS STRETCHING/SHRINKING (E1): OSCAR'S AM-FM CONSTRUCTION .....	85
7.3 NUMBER SENSE WITH FRACTION MULTIPLICATION (E2): OSCAR'S PREDICTIONS AND JUSTIFICATIONS .....	89
7.4 REPRESENTATIONAL FLUENCY FOR FRACTION MULTIPLICATION (E3): OSCAR'S AM-FM INTERPRETATION .....	91
7.4.1 <i>Oscar's learning trajectory for interpreting final area output: Knowledge coordination within a representation.</i> .....	91
7.4.2 <i>Subgrid view and the unit whole: Context sensitivity of knowledge coordination.</i> .....	96
7.5 REPRESENTATIONAL FLUENCY FOR FRACTION EQUIVALENCE (E4) .....	101
7.6 REPRESENTATIONAL FLUENCY FOR FRACTION ORDER (E5).....	107
7.7 CASE STUDY DISCUSSION .....	110
7.7.1 <i>Fraction multiplication as stretching/shrinking.</i> .....	111
7.7.2 <i>Number sense with fraction multiplication.</i> .....	111
7.7.3 <i>Representational fluency for fraction multiplication.</i> .....	112
7.7.4 <i>Representational fluency for fraction equivalence and fraction order.</i> .....	114
<b>CHAPTER 8: CONCLUSION.....</b>	<b>115</b>
8.1 CHAPTER OVERVIEW .....	115
8.2 THEORY IMPLICATION 1A .....	115
8.3 DESIGN IMPLICATION 1B.....	115
8.4 THEORY IMPLICATION 2A .....	116
8.5 DESIGN IMPLICATION 2B.....	117
8.6 THEORY IMPLICATION 3A .....	117
8.7 DESIGN IMPLICATION 3B.....	118
8.8 THEORY IMPLICATION 4A .....	119
8.9 DESIGN IMPLICATION 4B.....	119
8.10 CAVEATS .....	121
8.11 FUTURE WORK.....	121
<b>REFERENCES.....</b>	<b>123</b>
<b>APPENDIX A: TEST ITEMS.....</b>	<b>127</b>
<b>APPENDIX B: SUMMARY OF PRETEST TO POSTTEST GAINS.....</b>	<b>133</b>



<b>APPENDIX C: NEATO'S CONTENT LOG.....</b>	<b>135</b>
<b>APPENDIX D: NEATO'S CONTENT LOG SUMMARY TABLE .....</b>	<b>145</b>
<b>APPENDIX E: NEATO'S PRELIMINARY ANALYSIS TABLE .....</b>	<b>152</b>
<b>APPENDIX F: NEATO'S NARRATIVE SUMMARY .....</b>	<b>156</b>
<b>APPENDIX G: TRANSCRIPT OF NEATO'S CLINICAL INTERVIEW.....</b>	<b>160</b>
<b>APPENDIX H: TRANSCRIPT OF OSCAR'S CLINICAL INTERVIEW .....</b>	<b>229</b>

## List of Tables and Figures

TABLE 1. THE NUMBER CHART.....	13
TABLE 2. S(I) FOR AREA MODEL .....	29
TABLE 3. S(I) FOR NUMBER LINE.....	30
TABLE 4. S(I) FOR FRACTION NOTATION.....	31
TABLE 5. S(I) FOR CONCEPTUAL UNDERSTANDING .....	33
TABLE 6. S(E) FOR FRACTION MULTIPLICATION.....	37
TABLE 7. S(E) FOR FRACTION EQUIVALENCE AND FRACTION ORDER.....	38
TABLE 8. SUMMARY OF NEATO'S PREDICTIONS .....	50
TABLE 9. SUMMARY OF NEATO'S INTERPRETATION OF A(F) .....	52
TABLE 10. SUMMARY OF NEATO'S INTERPRETATION OF A(I).....	62
TABLE 11. SUMMARY OF NEATO'S INTERPRETATION OF FRACTION EQUIVALENCE.....	71
TABLE 12. SUMMARY OF NEATO'S INTERPRETATION OF FRACTION ORDER .....	78
TABLE 13. SUMMARY OF OSCAR'S PREDICTIONS.....	90
TABLE 14. SUMMARY OF OSCAR'S INTERPRETATION OF A(I) AND A(F).....	93
TABLE 15. SUMMARY OF OSCAR'S INTERPRETATION OF FRACTION EQUIVALENCE .....	102
TABLE 16. SUMMARY OF OSCAR'S INTERPRETATION OF FRACTION ORDER.....	107
FIGURE 1. USING THE AM-FM REPRESENTATION IN THE CASE OF $2/3 \times 1/2$ .....	2
FIGURE 2. THREE PAPER CUTOUTS. ....	12
FIGURE 3. A PAPER CUTOUT ARRANGEMENT IN THE CASE OF $3 \times 3/2$ . ....	12
FIGURE 4. THE AM-FM REPRESENTATION.....	14
FIGURE 5. HOW TO USE THE AM-FM REPRESENTATION IN THE CASE OF $2/3 \times 1/2$ . ....	15
FIGURE 6. ILLUSTRATION OF THE AM-FM MERGE/SPLIT FUNCTION WITH $3/3 \times 2/4$ AREA. ....	17
FIGURE 7. ILLUSTRATION OF THE AM-FM TILE MOVE FUNCTION WITH $4/3 \times 2/5$ AREA. ....	19
FIGURE 8. ILLUSTRATION OF THE AM-FM ZOOM FUNCTION WITH $3/8 \times 7/8$ AREA. ....	21
FIGURE 9. ILLUSTRATION OF THE AM-FM SUBGRID VIEW WITH $4/3 \times 2/5$ AREA. ....	23
FIGURE 10. TRANSFORMATION FOR S(I) TO S(E) FOR FRACTION MULTIPLICATION.....	39
FIGURE 11. TRANSFORMATION FROM S(I) TO FRACTION MULTIPLICATION AS STRETCHING/SHRINKING (E1). ....	40
FIGURE 12. TRANSFORMATION FROM S(I) TO NUMBER SENSE WITH FRACTION MULTIPLICATION (E2).....	40
FIGURE 13. TRANSFORMATION FROM S(I) TO REPRESENTATIONAL FLUENCY FOR FRACTION MULTIPLICATION (E3). .....	41
FIGURE 14. THE PREFERRED CONSTRUCTION PROCESS PRESENTED FOR CASE 8 ( $2/3$ OF $3/4$ ). ....	48
FIGURE 15. NEATO'S CONSTRUCTION PROCESS FOR CASE 9 ( $1/2$ OF $1/2$ ).....	48
FIGURE 16. NEATO'S CONSTRUCTION PROCESS FOR CASE 13 ( $3/5$ OF $3/4$ ).....	49
FIGURE 17. ILLUSTRATION OF DIALOGUE AND GESTURES WHEN ASKING FOR PREDICTIONS. ....	50
FIGURE 18. EXAMPLE OF $2/3$ OF $3/7$ WHERE A(I)= $3/7$ (AREA/NUMBER LINE CORRESPONDENCE) AND A(F)= $6/21$ (NO CORRESPONDENCE). ....	60
FIGURE 19. OSCAR'S INITIAL CONSTRUCTION FOR CASE 9 ( $1/2$ OF $1/2$ ). ....	86
FIGURE 20. RB'S CONSTRUCTION OF A $1/2$ OF $1/2$ AREA MODEL AND A $1/3$ AREA MODEL. ....	87
FIGURE 21. OSCAR'S FINAL CONSTRUCTION FOR CASE 9 ( $1/2$ OF $1/2$ ). ....	87
FIGURE 22. OSCAR'S INITIAL CONSTRUCTION FOR CASE 12 ( $2/3$ OF $1/3$ ). ....	88
FIGURE 23. RB'S AREA MODEL CONSTRUCTION FOR CASE 12 ( $2/3$ OF $1/3$ ). ....	88
FIGURE 24. OSCAR'S FINAL CONSTRUCTION FOR CASE 12 ( $2/3$ OF $1/3$ ). ....	88
FIGURE 25. ILLUSTRATION OF MATHEMATICS NOTATION CAPTURED IN THE X-DIVISIONS BOX AND Y-DIVISIONS BOX. ....	94
FIGURE 26. RB'S CONSTRUCTION OF A NUMBER LINE DURING INSTANCE Q1. ....	104
FIGURE 27. ILLUSTRATION OF FRACTION NOTATION INSCRIBED ALONG THE AXES OF THE AM-FM REPRESENTATION. ....	116
FIGURE 28. ILLUSTRATION OF THE $1 \times 1$ UNIT WHOLE WITH BACKGROUND SHADING. ....	120

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## Chapter 1: Introduction

There is a strong push from within mathematics education reform to incorporate representations in math classrooms (Behr, Harel, Post, & Lesh, 1993; Kieren, 1993; NCTM, 2000). One argument in support of this push is having students learn with understanding by providing them with opportunities to participate in practices that include the use of representations as tools *of* and tools *for* thought (Ball, 1993; Cobb, 2002; Lampert, 2001; Stevens & Hall, 1998). While a number of reform oriented documents (e.g., the 2000 NCTM *Principles and Standards*) describe the need for and utility value associated with using representations in the classroom, these documents provide little in the way of prescription. Questions regarding what representations should be used (for a given topic) and how representations should be used (such that students gain a deep understanding of that topic and a deep understanding of the representations) remain largely unanswered. In order to move beyond broad-brush statements about the need for and utility value of incorporating representations in the mathematics classroom, researchers and educators need to work together to address the practical as well as the theoretical issues associated with such practices. One route to seeing reform efforts come to fruition is via the use of design-based research (DBR) that attempts to bridge the gap between theory and practice (Abrahamson, 2009; Brown, 1992; Collins, 1992; Confrey, 2006; Schoenfeld, 2007).

In this dissertation study, I use DBR to investigate and support growth and change in students' knowledge of rational number operations. "Among all the topics in K-12 curriculum, rational numbers arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, and the most essential to success in higher mathematics and science" (Lamon, 2007). Operating with rational numbers means making sense of a wide range of concepts (e.g., equivalence, order, and unit) and doing so requires the meaningful use of a number of different representations (e.g., discrete sets, area models, and number lines). The difficulty lies in that the affordances of different representations differ and often these affordances are not adequately explicated. As a result there is little to no consensus regarding which representations should be used in the mathematics classroom much less how those representations should be used to adequately build on and extend students' understanding. In order to shed some light on the domain of rational number operations, I designed a learning environment centered on the use of the Area Model for Fraction Multiplication (AM-FM) representation, a computer-based tool intended to help students develop a deep understanding of fraction multiplication.

The AM-FM representation combines two canonical representations for fractions: area model and number line. The area model is depicted in the form of a coordinate grid and the number line is depicted in the form of coordinate axes. See Figure 1 for an illustration of how the AM-FM representation works in the case of  $\frac{2}{3} \times \frac{1}{2}$ .

Figure 1. Using the AM-FM representation in the case of  $2/3 \times 1/2$ .

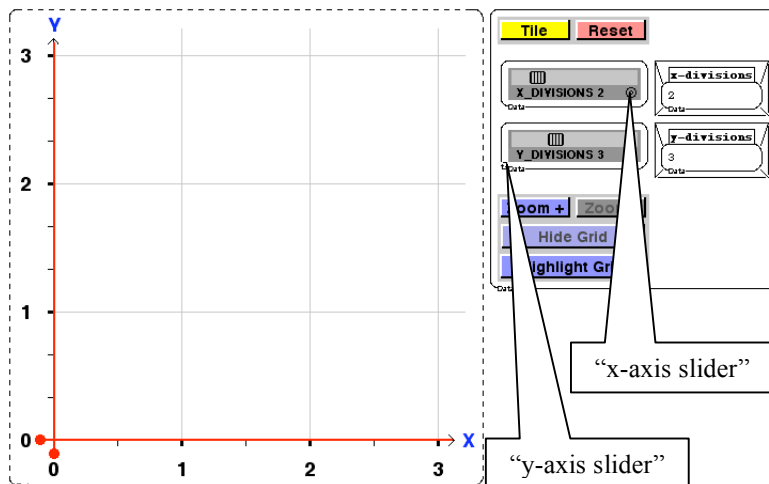


Figure 1a. The x-axis slider is moved to 2 resulting in x-axis sub-divisions of halves. The y-axis slider is moved to 3 resulting in y-axis sub-divisions of thirds.

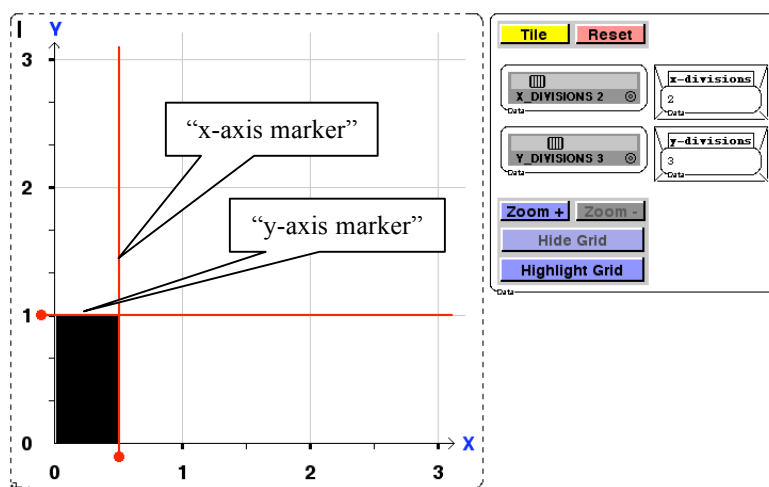


Figure 1b. The x-axis marker is moved to  $x=1/2$  (note: y-axis marker automatically jumps to  $y=1$ ) resulting in an area model representation of  $1/2$ .

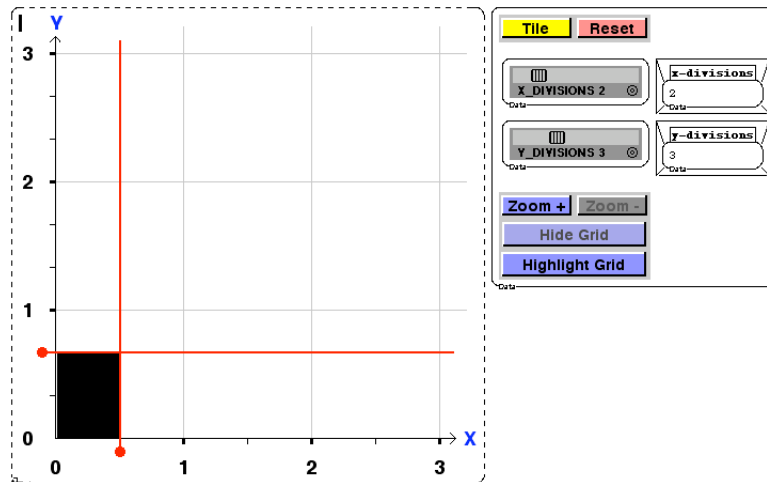


Figure 1c. The y-axis marker is moved to  $y=2/3$  resulting in an area model representation of  $2/3 \times 1/2$ .

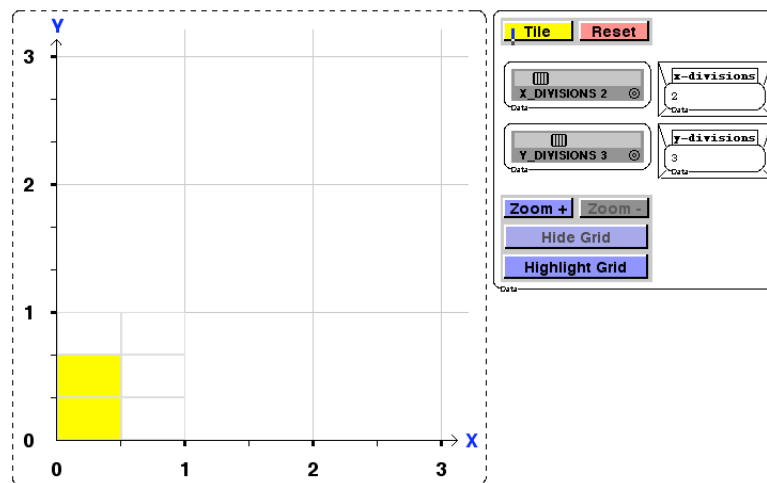


Figure 1d. The final area is tiled resulting in an area model representation of  $2/6$  in the sub-grid view. Note, the 2 vertically stacked tiles can be horizontally aligned and the x-axis slider moved from 2 to 1 resulting in an area model representation of  $1/3$  (equivalent to  $2/6$ ).

DBR is predicated on (a) design refinement and (b) local theory development (diSessa & Cobb, 2004; Schoenfeld, 2006). With respect to design, the AM-FM representation and the clinical interview protocol will be refined based on analysis of the data. With respect to theory, there are two theories of interest. The first is a global theory that determines my orientation toward learning, teaching, and research. The second is local theory that explicates (a) what student understandings I expect to see and (b) how I expect those understandings to change as the student makes sense of fraction multiplication while exposed to the designed learning environment. At the level of global theory, I draw on diSessa's knowledge-in-pieces perspective (1988). At the level of local theory, I offer a decomposition of competence with fraction multiplication (i.e., domain competence) and the AM-FM representation (i.e., representational competence). I presented concept maps of what I consider to be an ideal

initial state of student understanding (before exposure to the designed learning environment), an ideal exit state of student understanding (after exposure to the designed learning environment), and a hypothetical learning trajectory from ideal initial state to ideal exit state. As is the case with design, local theory will be refined based on an analysis of the data.

Data for the dissertation were collected from an urban school with a racially and socio-economically diverse student population. I met with ten students once a week for four weeks. During the first and last sessions students were asked to think-aloud through a pretest and posttest. The second and third sessions consisted of semi-structured clinical interviews during which students were asked to solve fraction multiplication problems using the AM-FM representation. All sessions were videotaped and transcribed. Two students were chosen to serve as case studies of knowledge growth and change. At a top level, the following research questions frame this dissertation study.

- What sense do the students make of rational numbers and fraction multiplication as they work within the designed learning environment?
- How does the sense-making<sup>1</sup> process emerge as the students work within the designed learning environment to (a) understand rational numbers and fraction multiplication (i.e. develop domain competence) and (b) understand the affordances and constraints of the AM-FM representation (i.e. develop representational competence)?
- What are the implications (theoretical and practical) for future design study iterations? More specifically, how will (local) theory and the design (tools and clinical interview protocol) used in this dissertation study be refined to better inform our understanding of growth and change in students' knowledge of fraction multiplication.

---

<sup>1</sup> I use the term “sense-making” to refer to a process by which students construct knowledge and come to understand mathematics. This construction is an individual and collective process located in students’ purposeful and socially, culturally, and historically situated mathematical activities.

## Chapter 2: Literature Review

### 2.1 Chapter Overview

Design-based research is dependent on local theory development and design refinement. In this chapter I present a literature view that supports both. I will discuss (a) the concepts that are central to rational numbers and fraction multiplication, (b) typical difficulties students experience in attempting to make sense of rational numbers and fraction multiplication, and (c) the desired learning goals for student understanding of rational numbers and fraction multiplication. This is followed by a brief literature review of three perspectives on representations.

It should be noted that the concepts, difficulties, and desired learning goals discussed here are not meant to be an exhaustive list nor are they necessarily mutually exclusive. They were chosen because they are particularly relevant in situating the design rationale to be discussed in Chapter 3. With respect to the brief literature review of representations the purpose is not to present three incommensurate perspectives, but rather, to highlight three differing views regarding the nature and function of representations. I will show how the three perspectives guide my own thinking as a designer, teacher, and researcher.

### 2.2 Key Concepts

There are a number of ways to parse key element (see for example Smith, 1999). I will present what I consider to be three central concepts in supporting students understanding of rational numbers and fraction multiplication: equivalence, order, and unit.

With respect to the concept of equivalence, if  $p$  and  $q$  are whole numbers and  $r$  is a whole number with  $r \neq 0$  then  $pr/qr = p/q$  and the two fractions  $pr/qr$  and  $p/q$  are said to be equivalent fractions. Given that  $r$  can vary, there are an infinite number of equivalent fractions for  $p/q$ . For example, fractions equivalent to  $4/12$  include  $8/24$  ( $r = 2$ ),  $1/3$  ( $r = 1/4$ ), and  $6/18$  ( $r = 1/2$ ) to name but a few. In the context of fraction multiplication, students often arrive at a final product that can be reduced to an equivalent fraction in which the numerator and denominator have no common factors.

The second concept is that of order, which involves comparing two fractions to see if one is greater than or equal to the other. There are many ways by which fractions can be ordered or compared. I will discuss two of them here (the first of which is conceptually opaque while the second is somewhat conceptually grounded). In the first method, given two fractions  $a/b$  and  $c/d$ , one can check the cross-products  $ad$  and  $bc$  in order to compare  $a/b$  to  $c/d$  (assuming  $c$  and  $d$  are positive, order of inequality is preserved when multiplying by  $cd$ ). For example, the cross-products of  $4/5$  and  $5/7$  are 28 and 25. Therefore  $4/5$  is greater than  $5/7$ . A second method by which to compare two fractions is to use a common reference point such as  $1/2$  or 1. For example,  $1/2$  can be used as a reference point to compare  $4/9$  to  $6/11$ . Since  $4/9$  is less than  $1/2$  (because half of 9 is 4.5 and 4 is less than 4.5) and  $6/11$  is greater than  $1/2$  (because half of 11 is 5.5 and 6 is greater than 5.5) one can conclude that  $4/9$  is less than  $6/11$ . In the context of fraction multiplication, the operation of multiplication often results in a product less than the two given fractions (assuming you started with two fractions



less than one). In order to see this, the student must have some means by which to order the product and the two given fractions.

The third and final concept to be discussed is that of unit. The unit refers to the point of reference for a fraction. Generally the unit is assumed to be “one”. Given a fraction,  $a/b$ , it is considered  $a/b$  of one whole unit. When multiplying two fractions the unit undergoes change. Given,  $a/b \times c/d$ , one can interpret the operation of multiplication as taking  $c/d$  of a unit of one whole, followed by taking  $a/b$  of a unit of  $c/d$ . For example, one interpretation of  $2/3 \times 1/2$  is “ $2/3$  of  $1/2$  of 1.” This hidden unit is sometimes referred to as the “ghost one” highlighting what is generally implicit in fraction notation.

While designing the learning environment, I attempted to make the three key concepts of equivalence, order, and unit explicit for discussion and exploration.

### 2.3 Student Difficulties

It is well documented that rational numbers are difficult for students to understand (Behr et al., 1993; Izsak, 2005; Kieren, 1993; Lehrer, 2003; Moss & Cass, 1999; Saxe, Taylor, McIntosh, & Gearhart, 2005). I present four commonly identified difficulties that students are said to experience in making sense of rational numbers and some of the implicit and/or explicit instructional implications associated with each. These four difficulties ground some of the design rationale for the development of the AM-FM representation to be discussed in Chapter 3. Similarly, the instructional implications associated with these difficulties ground much of the design rationale for the clinical interview protocol that accompanied the use of the AM-FM representation also to be discussed in Chapter 3.

#### 2.3.1 Multiple subconstructs.

One well-established reason for students’ difficulty with rational numbers is that there exist multiple subconstructs (i.e., interpretations) associated with rational numbers. These include: part-whole, operator, measure, rate, and ratio (Behr et al., 1993; Kieren, 1993). Based on the form and function of the representations used in the learning environment, the three subconstructs most pertinent to this dissertation study include fractions as part-whole comparisons, fractions as operators, and fractions as measures. Which subconstruct(s) is most salient to students will depend on what features of the learning environment they are attending to, how they are attending to those features, and the context under which they are attending to those features.

To illustrate the meaning of each of these subconstructs, consider the fraction  $3/5$ . According to the interpretation of fractions as part-whole comparisons,  $3/5$  means 3 parts out of a unit made up of 5 equal parts. Part-whole comparisons often surface in the context of representing fractions using area models. According to the interpretation of fractions as operators,  $3/5$  gives a rule that tells how to operate on a unit, that is, you multiply by 3 (apply a stretching metaphor) and divide your result by 5 (apply a shrinking metaphor), or you divide by 5 and multiply your result by 3. The operator subconstruct often surfaces in the context of fraction multiplication. Finally, according to the interpretation of fractions as measures,  $3/5$  means to iterate a  $1/5$  piece three times. The measure subconstruct often surfaces in the context of representing fraction using number lines.

In addition to the relevance of the representational context there is also the context of application. For example, in the context of a partitioning activity that involves taking  $3/5$  of

10 cookies, one can think of partitioning the cookies into 5 sets of 2 cookies and taking 3 of those sets, which results in 6 cookies. Alternatively one could triple the set of 10 cookies to arrive at 30 cookies and partitioning that set into 5 sets consisting of 6 cookies each. Within this context of application it is the operator subconstruct that appears to be most prevalent.

In terms of teaching and learning, not only is there disagreement about whether it is necessary for students to understand all subconstructs, there is also uncertainty regarding the sequence in which various subconstructs should be introduced. The instructional implications associated with this difficulty include (a) the recognition of multiple subconstructs elicited in curriculum, instruction, and assessment and (b) the examination of short-term and long-term tradeoffs associated with choosing to emphasize one subconstruct over another (Schoenfeld, 1986). For example, focusing exclusively on part-whole comparisons might make sense in the context of working with equivalent fractions but may not be appropriate when considering fraction multiplication in which the “whole” (or unit) varies.

### 2.3.2 Whole number bias.

A second reason students experience difficulty with rational numbers is referred to as the whole number bias, that is, the “tendency in students to use single-unit counting schemes associated with whole numbers to interpret fractions” (Ni & Zhou, 2005, p. 28). In other words, students’ prior understanding of whole numbers as discrete quantities interfere with their ability to make sense of rational numbers as continuous quantities. One example of the whole number bias is when students argue that  $\frac{1}{5}$  is greater than  $\frac{1}{3}$  because 5 is greater than 3. In this instance, students are attending to the individual whole numbers that make up each fraction without considering the multiplicative relationship between the numerator and denominator for each fraction.

Instructional implications associated with the whole number bias entail replacing a part-whole approach to the topic of rational numbers with a fair-share or measurement approach and/or introducing rational numbers in the early grades to support the co-development of whole number and rational number understanding (Ni et al., 2005). Research also shows that student understanding of fractions starts with  $\frac{1}{2}$  followed by other unit fractions ( $\frac{1}{3}$ ,  $\frac{1}{4}$ , etc.), non-unit fractions ( $\frac{2}{3}$ ,  $\frac{6}{12}$ ,  $\frac{5}{5}$ , etc.), improper fractions ( $\frac{3}{2}$ ,  $\frac{7}{5}$ , etc.), and mixed numbers ( $1\frac{1}{2}$ ,  $2\frac{2}{5}$ , etc.) and therefore instruction should proceed in this sequence (Behr et al, 1993; Piaget, Inhelder, & Szeminska, 1960).

### 2.3.3 Unitizing.

A third difficulty students experience in trying to make sense of rational numbers is a result of the complex processes associated with unitizing, that is, identifying the unit of a given fraction (Lamon, 1996; Piaget et al., 1960; Steffe, 2003). As an illustration consider the following question: which is greater,  $\frac{3}{5}$  or  $\frac{3}{4}$ . The answer depends on the unit. Generally the unit for each fraction is assumed to be one in which case the comparison can be made by finding the cross products, using a reference point like  $\frac{1}{2}$ , dividing the numerators by the denominators and comparing the decimal answers, or using number sense (e.g., the numerator is the same and 5 is great than 4 so  $\frac{1}{5} < \frac{1}{4}$  and  $\frac{3}{5} < \frac{3}{4}$ ). In the context of fraction multiplication, what students take to be the unit can change as they proceed through the problem (see the example provided in Section 2.2 on the unit concept). Because students have

had little to no experience with different size units for fractions, they are ill prepared to handle a shifting unit for fraction multiplication.

One instructional implication for supporting students' understanding of unitizing is to provide them with partitioning activities in which the unit is made to vary thereby problematizing the very concept of unit (Lamon, 1996; Piaget et al., 1960; Streefland, 1993). For example, you can have students partition one cookie among two people (unit = 1 cookie) and then have them partition two cookies among two people (unit = 2 cookies). In each case you have asked the students to distribute one-half of the total amount but because the unit differs (one cookie versus two cookies) the amount distributed will also differ (half of 1 cookie versus 2 halves of 2 cookies). It is argued that when students are engaged in partitioning activities, fraction multiplication problems should be sequenced as follows: (a) fractions in which the numerator of one is identical to the denominator of the other, (b) fractions in which the numerator of one is identical to the denominator of the other (e.g.,  $1/2$  and  $2/3$ ), (c) fractions in which the numerator of one and the denominator of the other are composites of each other (e.g.,  $3/4$  and  $8/10$ ), and (d) fractions in which each pair of numerator and denominator are prime (e.g.,  $2/5$  and  $3/7$ ) (Behr et al, 1993; Mack, 2001). For example, in the first case,  $1/2 \times 2/3$ , visualize an area model representation. If you start by taking  $2/3$  of 1 whole, then when you take  $1/2$  of  $2/3$ , your  $2/3$  unit has already been partitioned into halves and there is no need for a second partitioning. In the second case,  $3/4 \times 8/10$ , again visualize an area model representation. If you start by taking  $8/10$  of 1 whole, then you can pair up the shaded parts such that you have  $4/4$  (of  $8/10$ ) shaded and you can easily take  $3/4$  of  $8/10$ . A second partitioning is not required. You can simply rearrange the shaded region to see  $3/4$ . Finally, in the third case,  $2/5 \times 3/7$ , again visualize an area model representation. If you start by taking  $3/7$  of 1 whole, then when you take  $2/5$  of  $3/7$ , you must further partition the  $3/7$  shaded region into 15 parts (other numbers work but 15 is the smallest number that works) such that you have  $15/15$  (of  $3/7$ ) shaded. Then you can take  $2/5$  of  $3/7$  by taking  $6/15$  (equivalent to  $2/5$ ) of the shaded part.

#### 2.3.4 Multiple representations.

A fourth reason that students may experience difficulty with rational numbers is the existence of multiple representations for any given interpretation (Behr et al., 1993; Kieren, 1993; Taber, 2001). Questions about which representations should be used and how representations should be used (so that students gain a deep understanding of rational numbers as the representations themselves) remain largely unanswered (diSessa, 2004; Levin & Brar, 2010; Saxe et al., 2010).

One instructional argument that attempts to address this difficulty supports practices that provide students with an opportunity to translate within and among different representations to develop deep understanding of ideas associated with rational numbers. Lesh, Post, and Behr (1987), for example, refer to the translation and transformation of ideas with respect to five representational systems: experience based scripts, manipulative models, pictures or diagrams, spoken language, and written symbols.

## **2.4 Desired Learning Goals**

The difficulty associated with learning rational numbers is exacerbated by the introduction of operations on fractions (Lamon, 1996; Steffe, 2003). Research on operations

with fractions tends to disproportionately focus on addition/subtraction and division. Part of the reasoning for this focus is based on the argument that the algorithms associated with these operations are counter-intuitive for students when considering their prior experience with whole numbers (i.e., why do we not add denominators and why do we flip and multiply). The issue at hand is not exclusively about supporting students understanding of counter-intuitive algorithms (e.g., syntactic understanding) but it also entails helping students understanding the conditions under which a particular algorithm should be applied (e.g., semantic understanding) (Hiebert, 1994; Kaput, 1987; Schoenfeld, 1991).

Ultimately we want students to “recognize nuances in meaning; to associate each meaning with appropriate situations and operations; and to develop insight, comfort, and flexibility in dealing with rational numbers” (Lamon, 2007). In general they should be comfortable in reasoning, computing, and problem solving in the domain of rational numbers. Students who have developed rational number sense have “an intuitive feel for the relative sizes of rational numbers and the ability to estimate; to think both additively and multiplicatively; to recognize and understand shifts in units; to move flexibly between interpretations, representations, and different contexts; and to make sound decisions and reasonable judgments during the process of problem solving” (Lamon, 2007).

With respect to fraction multiplication, at a macro level, it is my goal that students comes to develop deep conceptual and procedural understanding of fraction multiplication, fluency within and across different representations in making sense of fraction multiplication, and the ability to reflect on and explain their thinking of fraction multiplication.

It was in keeping the key concepts, student difficulties, and desired learning goals in mind that I designed the learning environment around the use of the AM-FM representation.

## 2.5 Three Perspectives on Representations

I think about representations<sup>2</sup> as tools of and tools for thought. How students use (construct and interpret) representations provides insight into what students know while potentially pushing their understandings further. In addition, particular representations offer affordances for certain aspects of cognition while constraining other aspects of cognition. Deep understanding entails moving flexibly between representations in order to make optimal use of the affordances of each. Therefore a key design criterion of mine was to build representations that provided a conceptual underpinning that I thought and the literature said is central to understanding fraction multiplication.

Depending on one’s theoretical orientation the function of a representation varies. I offer three broad perspectives on representations and situate the proposed dissertation study in the intersection of all three. I have chosen to present the view of three researchers in order to depict three general theoretical orientations (cognitive, constructive, and situated) toward representation use.

The function of the representation from a cognitive perspective is to accurately represent information from the world (Palmer, 1978). The information, once contained in the representation, is there for the learner to process. The function of the representation from a constructive perspective is to elicit interpretation (Von Glassersfeld, 1987). In other words, a

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<sup>2</sup> The term representation will be used to refer to only external representations and not mental or internal representations unless otherwise noted.

representation is only a representation once it has been interpreted and interpretation depends on the learner's experience. Finally, the function of the representation from a situative perspective is to mediate activity among a community of individuals over time (Roth & McGinn, 1998). The dissertation study is situated in the intersection of these three perspectives. As a designer, I attempt to create a representing world for rational numbers and fraction multiplication. As a teacher, I attempt to understand where a student is in terms of his/her understandings of rational number and fraction multiplication and then use this knowledge to push that student's understanding forward. As a researcher, I attempt to focus my analysis on the interaction between the student, the learning environment, and myself over a brief period of time.

## Chapter 3: Design

### 3.1 Chapter Overview

Design-based research (DBR) bridges the gap between theory and practice by “engineering particular forms of learning and developing local theories while systematically studying those forms of learning and the means of supporting them” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). In other words, DBR involves (a) the construction of a learning environment and (b) the development of theory. A detailed discussion of theory will be presented in Chapter 4. In this chapter I discuss the design rationale that guided the construction of the learning environment: the problem context (i.e., fair share), the physical media (i.e., paper cutouts and number chart), and the computer-based medium (i.e., the AM-FM representation). The discussion will be heavily grounded in the literature review presented in Chapter 2.

### 3.2 The Problem Context

The following problem context was verbally presented to each student:

You just got a summer job as a lab assistant. Your main task is to feed cheese to the lab rats. On your first day at the job the head scientist shows you around the lab and gives you instructions about the amount of cheese you have to feed the rats. Because different rats are used for different kinds of experiments, some get more or less cheese than others. You will need to give each rat the exact amount of cheese specified by the scientist. The scientist has also asked that you keep a record of how much cheese you use. You must turn this record in at the end of each day. Do you understand your job?

This problem context allows students to engage in a fair-share activity in that they must distribute equal amounts of cheese to a specified number of rats. For example, a student might be asked to distribute one fourth of a slice of cheese among five rats. The problem context was intended to leverage students’ fair-share practices in order to ground the mathematics activity in students lived experience thereby helping to support the co-development of syntactic and semantic understanding.

### 3.3 Physical Media: The Paper Cutouts and the Number Chart

The cheese introduced in the problem context was physically embodied in the form of paper cutouts. The cutouts were approximately four inches by four inches in size. See Figure 2 for an illustration of three paper cutouts.

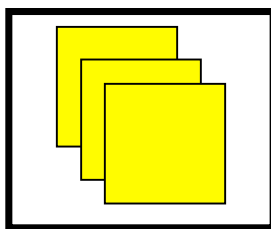


Figure 2. Three paper cutouts.

Each problem was introduced as a case. For example, in case 6 there were 3 rats and the students were asked to distribute  $\frac{3}{2}$  slices of cheese per rat (i.e.,  $3 \times \frac{3}{2}$ ). They were given paper cutouts and asked to produce an arrangement for each case to show how much total cheese was distributed per case. See Figure 3 for an illustration of a paper cutout arrangement in the case of  $3 \times \frac{3}{2}$ . A student might, for example, choose to fold the paper cutouts in half and then distribute 3 folded paper cutouts to each of the rats to produce the arrangement shown in Figure 3a. To determine how much total cheese was distributed, the student might rearrange the paper cutouts as shown in Figure 3b (putting together 2 of the 3 one-half slices for each rat to arrive at a distribution of  $1\frac{1}{2}$  slices for each rat) and then rearrange the paper cutouts again as shown in Figure 3c (putting together 2 of the 3 left over one-half slices to arrive at a total distribution of  $4\frac{1}{2}$  slices).

Figure 3. A paper cutout arrangement in the case of  $3 \times \frac{3}{2}$ .

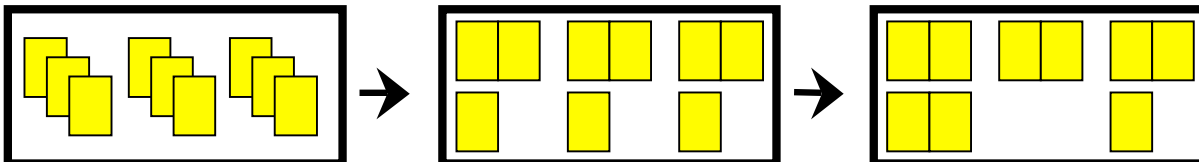


Figure 3a. Three one-half slices of cheese for three rats.

Figure 3b. One and one-half slices of cheese for three rats.

Figure 3c. Four and one-half slices of cheese for three rats.

The rationale behind the use of the paper cutouts is grounded in the literature on rational numbers. By providing students with the opportunity to engage in a partitioning activity, the paper cutouts allow for the exploration of equivalence and unit. In case 6 ( $3 \times \frac{3}{2}$ ), for example, students grapple with the equivalence of 3 one-half slices and  $1\frac{1}{2}$  slices as they move from the arrangement in Figure 3a to Figure 3b. The concept of unit is explored when students shift the unit from one-half to one whole as they move from the arrangement in Figure 3b to Figure 3c. Furthermore, the activity with the paper cutouts serves as an intermediate transition from the problem context to the computer-based medium.

In addition to the paper cutouts, students were also presented with a four-column table, which I refer to as the number chart. They used (constructed and interpreted) this chart while working with both the paper cutouts and the AM-FM representation. After creating an arrangement using the paper cutouts and/or the AM-FM representation, students were asked to record the total distribution of cheese used per case into the output column of the number chart. See Table 1 for an illustration of the number chart as presented to the students.

Table 1. The Number Chart

Case #	INPUT (units)	INPUT (units)	OUTPUT (units)
0	4 rats	3 slices/rat	
1	3 rats	4 slices/rat	
2	3 rats	2 slices/rat	
3	3 rats	1 slice/rat	
4	3 rats	1/2 slice/rat	
5	3 rats	4/5 slices/rat	
6	3 rats	3/2 slices/rat	
7	3 rats	6/5 slices/rat	
8	1 rat	2/3 of $\frac{3}{4}$ slice/rat	
9	1 rat	1/2 of $\frac{1}{2}$ slice/rat	
10	1 rat	1/2 of $\frac{1}{3}$ slice/rat	
11	1 rat	1/3 of $\frac{1}{2}$ slice/rat	
12	1 rat	2/3 of $\frac{1}{3}$ slice/rat	
13	1 rat	3/5 of $\frac{3}{4}$ slice/rat	
14	1 rat	5/6 of $\frac{2}{5}$ slice/rat	
15	1 rat	4/3 of $\frac{2}{5}$ slice/rat	
16	1 rat	2 $\frac{3}{5}$ of $\frac{2}{5}$ slice/rat	
17	1 rat	1 $\frac{2}{5}$ of $\frac{6}{4}$ slice/rat	
Etc.			

The case sequence proceeds from whole numbers, to  $\frac{1}{2}$ , to unit fractions, to non-unit fractions, to improper fractions, and finally to mixed numbers as suggested by the literature (i.e., Behr et al, 1993; Piaget et al, 1960). In the initial cases students were presented with two whole numbers, one representing number of rats and the other representing the amount of cheese slices to be distributed per rat. This was followed by cases with one whole number (representing number of rats) and one fraction (representing the amount of cheese to be distributed per rat). Next were cases in which the number of rats was fixed at one and the amount of cheese per rat was represented using two fractions. For example, in case 8 the student was presented with 1 rat and asked to distribute  $\frac{2}{3}$  of  $\frac{3}{4}$  of a slice per rat. The relationship between the denominator of one fraction and the numerator of another was also taken into consideration when choosing a pair of fraction for fraction multiplication. The case sequence begins with fractions in which the numerator of one was identical to the denominator of the other (i.e.,  $\frac{1}{2}$  and  $\frac{2}{3}$ ), followed by fractions in which the numerator of one and the denominator of the other were composites of each other (i.e.,  $\frac{3}{4}$  and  $\frac{8}{10}$ ), and



fractions in which each pair of numerator and denominator was prime (i.e.,  $2/5$  and  $3/7$ ) as suggested by the literature (i.e., Behr et al, 1993; Mack, 2001). Improper fractions and mixed numbers were introduced once it was determined that students had a grasp of multiplication with proper fractions.

The process of constructing and interpreting the number chart is intended to provide students with a context for pattern recognition, symbolic manipulation, and exploration of number and operation sense. The number chart allows for the exploration of equivalence, order, and unit depending on how students choose to attend to it. Furthermore, the number chart is intended to provide a link between the problem context, the physical medium (i.e., the paper cutouts), and the computer-based medium (i.e., the AM-FM representation). I want students to operate with conventional mathematical symbols such that the symbolic manipulation carried out is conceptually grounded. I believe that students will develop procedural understanding that is conceptually grounded as they use the number chart while they move first between the problem context and paper cutouts and later between the problem context and the AM-FM representation.

### 3.4 Computer-based Medium: The AM-FM Representation

I have discussed the design rationale for two components of the learning environment, the problem context and the physical media (i.e., paper cutouts and number chart). Next, I present a similar discussion regarding the AM-FM representation. I begin with a general overview of the AM-FM representation and the ways in which it can be used. Then I address the specific design rationale that led to its development. This discussion will also be grounded in the literature review presented in Chapter 2.

#### 3.4.1 How it works.

The AM-FM representation is a computer-based tool, which combines two canonical representations for fractions: area model and number line. See Figure 4 for an illustration of the AM-FM representation.

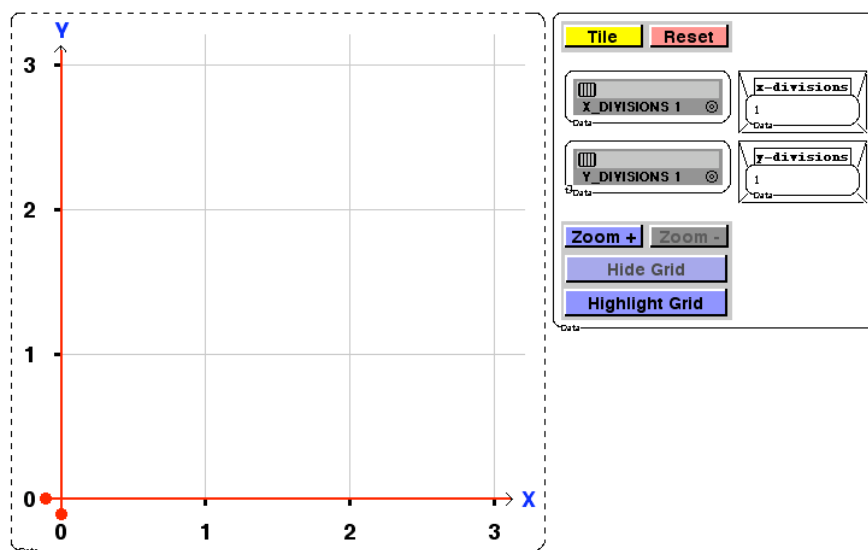


Figure 4. The AM-FM representation.

The area model is depicted in the form of the coordinate grid and the number line is depicted in the form of the coordinate axes. The red vertical line positioned at  $x=0$  and the red horizontal line positioned at  $y=0$  are referred to as the x-axis and y-axis marker lines, respectively. The two gray buttons on the right labeled “X\_DIVISIONS” and “Y\_DIVISIONS” are referred to as the x-axis and y-axis sliders, respectively. The remaining buttons are referred to by the button name. An illustration of how the AM-FM representation can be used in the case of  $2/3 \times 1/2$  is provided in Figure 5.

Figure 5. How to use the AM-FM representation in the case of  $2/3 \times 1/2$ .

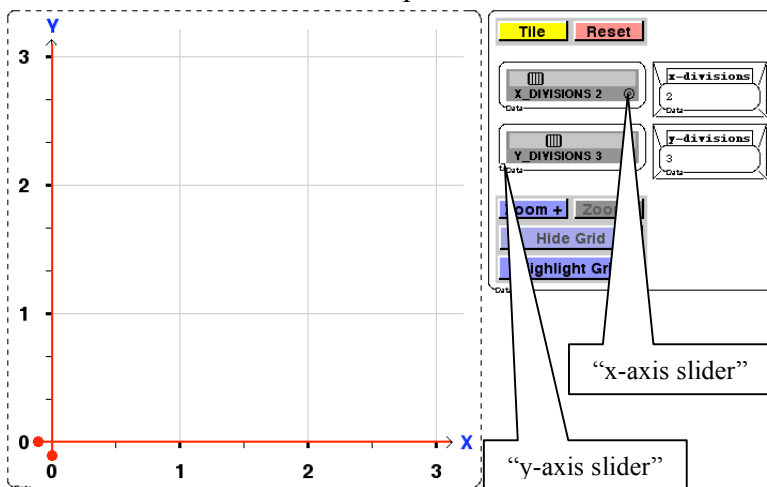


Figure 5a. The x-axis slider is moved to 2 resulting in x-axis sub-divisions of halves. The y-axis slider is moved to 3 resulting in y-axis sub-divisions of thirds.

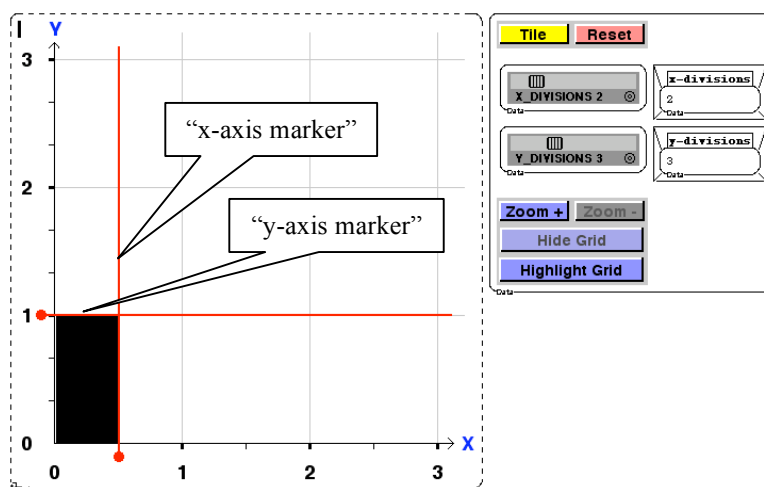


Figure 5b. The x-axis marker is moved to  $x=1/2$  (note: y-axis marker automatically jumps to  $y=1$ ) resulting in an area model representation of  $1/2$ .

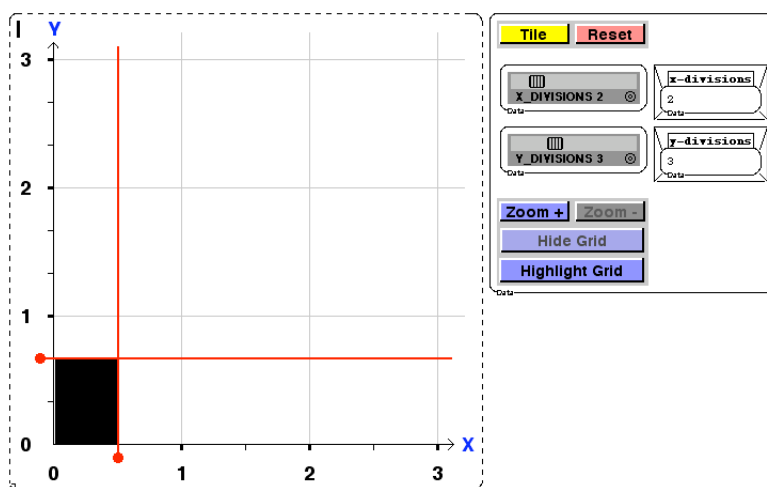


Figure 5c. The y-axis marker is moved to  $y=2/3$  resulting in an area model representation of  $2/3 \times 1/2$ .

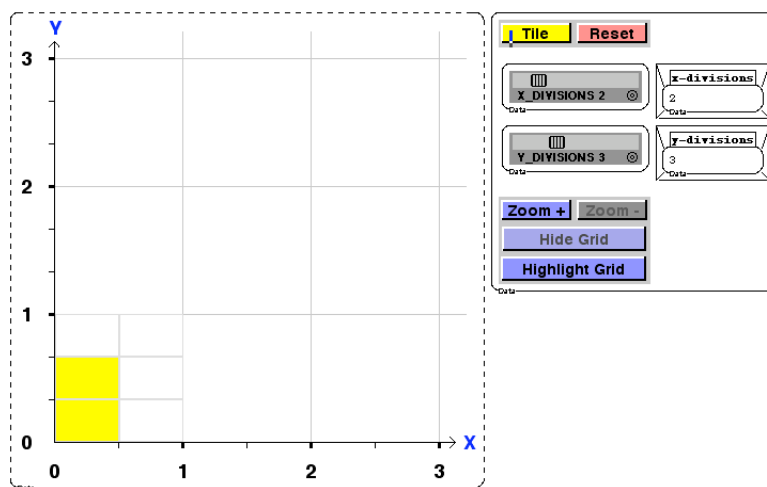


Figure 5d. The final area is tiled resulting in an area model representation of  $2/6$  in the sub-grid view. Note, the 2 vertically stacked tiles can be horizontally aligned and the x-axis slider moved from 2 to 1 resulting in an area model representation of  $1/3$  (equivalent to  $2/6$ ).

Traditionally either the number line *or* the area model is used to teach fractions in school so why use both in the AM-FM representation. Number lines are commonly associated with the measure subconstruct of fractions (Davydov & Tsvetkovich, 1991; Kieren, 1980). They have particular affordances in terms of teaching students about the relationship between whole numbers and fractions (e.g., improper fractions fall to the left of one), order (e.g.,  $1/3$  is more than  $1/4$  because  $1/3$  lies to the right of  $1/4$  on the number line) and equivalence (e.g.,  $3/6 = 1/2$  because they refer to the same number line marker (assuming all markers are equally spaced)). In contrast, area models are most commonly associated with the part-whole subconstruct of fractions (Kieren, 1980; Lamon, 1996; Piaget et al, 1960). They have particular affordances in terms of teaching students about the concept of unit and can support

students as they transition from representing fractions to representing operations with fractions. To compare two fractions using area models one generally presupposes the same size unit for each. For example,  $2/3$  is more than  $1/2$  because  $2/3$  of *one* is more than  $1/2$  of *one*. However,  $2/3$  is less than  $1/2$  if the unit is three and six, respectively, that is,  $2/3$  of *three* (which equals 2) is less than  $1/2$  of *six* (which equals 3). Shifting the unit provides students with an opportunity to grapple with the ghost one (the assumption that the unit is one) and leads naturally to a discussion of fraction operations (what does it mean to take  $2/3$  of some unit). Together the number line and area model provide a rich context for the exploration of fraction multiplication. The number line in the AM-FM representation is intended to serve as an index for fraction and whole number relations and foregrounds fraction order and fraction equivalence while the area model is intended to foreground the shifting unit as students multiply two fractions. The AM-FM representation (when used as illustrated in Figure 5) is most closely associated with the operator subconstruct of fractions (Behr et al, 1993; Kieren, 1980).

Other important functions of the AM-FM representation include merging tiles, splitting tiles, and moving tiles. See Figure 6 for an illustration of the merge and split function with  $3/3 \times 2/4$  area. The initial output is  $6/12$ , which can be reduced to  $3/6$ ,  $2/4$ , and  $1/2$ . The sliders can be used to adjust the divisions for each axis such that tiles can be merged and/or subdivided to see equivalent fractions.

Figure 6. Illustration of the AM-FM merge/split function with  $3/3 \times 2/4$  area.

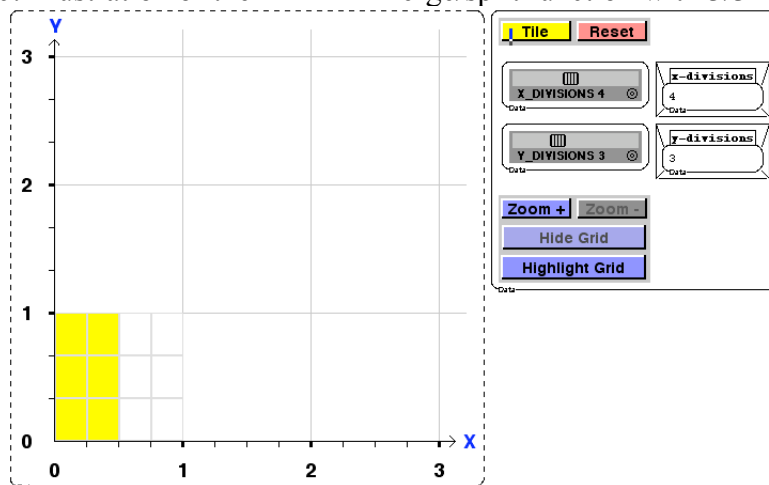


Figure 6a. Illustration of  $3/3 \times 2/4$  depicting tiled area of  $6/12$ .

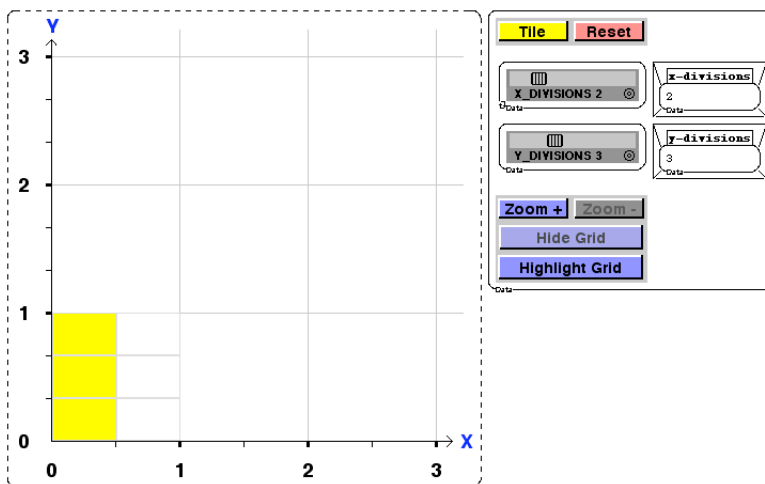


Figure 6b. The x-axis slider is moved from 4 to 2, the y-axis slider remains at 3, depicting a tiled area of  $3/6$ .

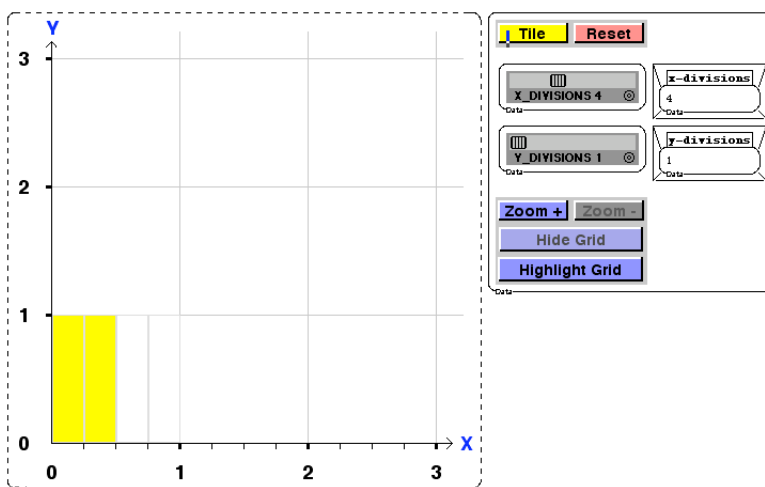


Figure 6c. The y-axis slider is moved from 3 to 0, the x-axis slider is moved from 2 back to 4, depicting a tiled area of  $2/4$ .

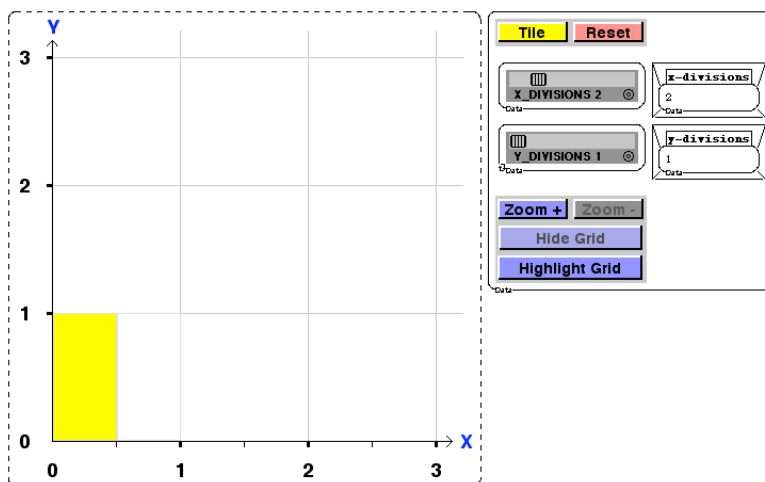


Figure 6d. The x-axis slider is moved from 4 to 2, the y-axis slider remains at 0, depicting a tiled area of  $1/2$ .

When operating with improper fractions and/or mixed numbers, the tiles will fall into two or more  $1 \times 1$  unit whole such that the wholes are not completely tiled. In order to correctly interpret the enclosed area using the AM-FM representation, it helps to move the tiles into a single  $1 \times 1$  unit whole. An illustration of how tiles can be moved is provided in the Figure 7 with  $4/3 \times 2/5$  area. The conceptual understanding associated with the procedure for moving tiles is related to the notion of unit. The name of a tile depends on what is treated as the unit. In the case of  $4/3 \times 2/5$ , if the student treats the  $1 \times 1$  square as the unit, each tile represents a *fifteenth* of that  $1 \times 1$  unit. The procedure for moving tiles is grounded in the need to conceptually support this interpretation of unit. If the student treats the  $2 \times 1$  rectangle as the unit, each tile represents a *thirtieth* of that  $2 \times 1$  unit and there is no need to move tiles.

Figure 7. Illustration of the AM-FM tile move function with  $4/3 \times 2/5$  area.

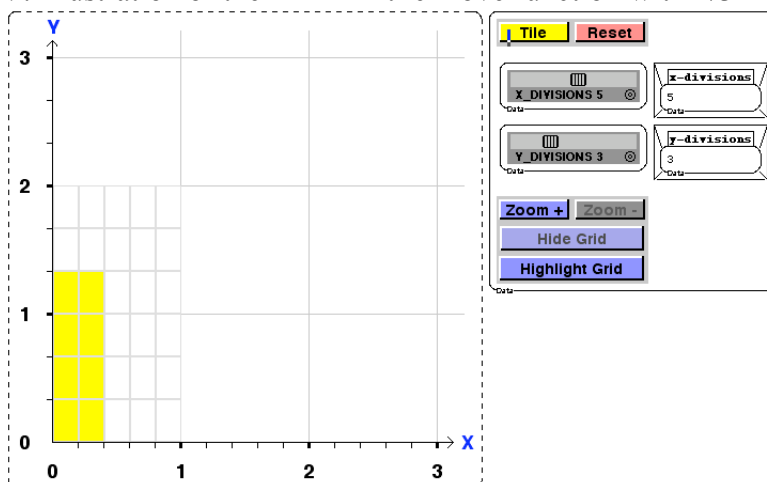


Figure 7a. Illustration of  $4/3 \times 2/5$ .

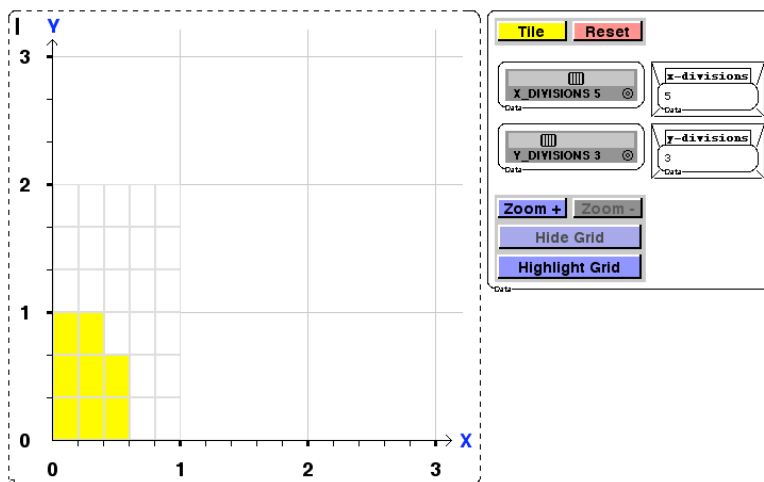


Figure 7b. Two  $\frac{1}{15}$  tile pieces have been moved into the corner  $1 \times 1$  unit whole depicting tiled area of  $\frac{8}{15}$ .

Having discussed the various functions of the AM-FM representation, I turn now to the design rationale as it pertains to (a) the use of a computer-based medium, (b) the choice of range and domain for the  $x$ - $y$  coordinate axes, (c) the choice of scaling for the  $x$ - $y$  coordinate axes, (d) the use of equal-sized tiles, and (e) the sub-grid view. This will be followed by a discussion of how the literature on student difficulties was taken into account when designing this learning environment and the AM-FM representation in particular. Of course these design choices are not exhaustive, in that, there were many others that lead to the development of the AM-FM representation. Nor are these design choices mutually exclusive, in that the choices made regarding one aspect of the design may impact or be impacted by choices regarding others. These design choices are being discussed because they have particular relevance to the ways in which students made sense of the AM-FM representation, rational numbers, and fraction multiplication.

### 3.4.2 The medium.

In developing the AM-FM representation, a computer-based interface was used to (1) introduce what for many students is a topic from third grade in a novel context so as to support student engagement with the content, (2) reduce the start-up costs associated with having to construct the coordinate grid of the AM-FM representation using other media (e.g, paper and pencil), and (3) reduce the start-up costs associated with students' need to revise their constructions.

### 3.4.3 Range and domain.

The range and domain of the  $x$ - $y$  coordinate axes are each set at zero to three because students were mostly presented with problems that involve proper fractions, improper fractions less than three, and mixed numbers less than three. Having a range and domain that extends beyond three would decrease the length of the line segment from one whole number to the next thereby limiting the number of partitions visible on the display. Even with the zero to three setting, representing fractions with large denominators is problematic. As a solution to this problem, the  $x$ - and  $y$ -divisions maximum was set at eighths and a zoom function was

incorporated into the representation. For an illustration of the zoom function with  $3/8 \times 7/8$  area, see Figure 8. If there was a need to use numbers larger other media (paper and pencil) could be employed.

Figure 8. Illustration of the AM-FM zoom function with  $3/8 \times 7/8$  area.

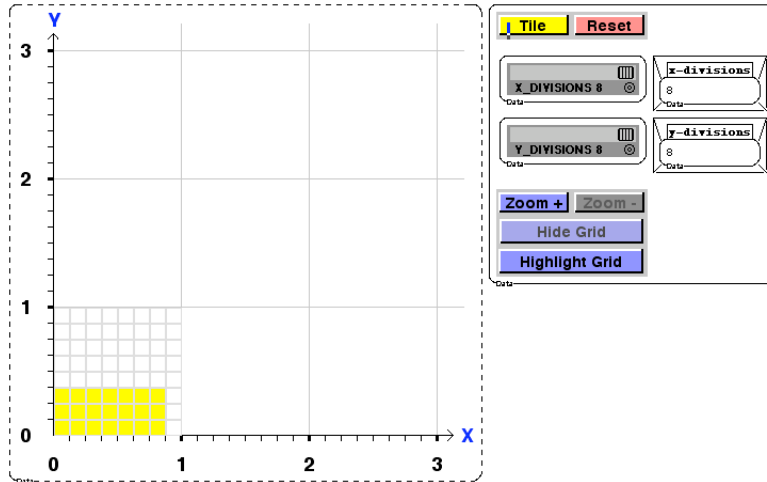


Figure 8a. Illustration of  $3/8 \times 7/8$  zoomed out to the maximum to see the 3 by 3 grid.

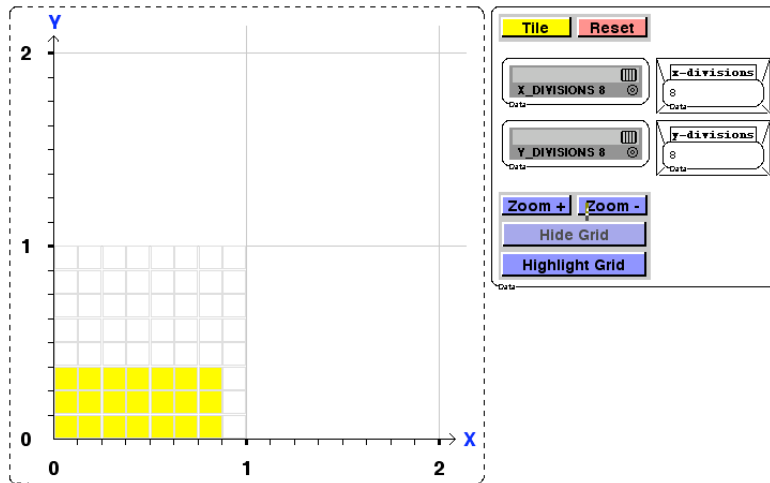


Figure 8b. Illustration of  $3/8 \times 7/8$  zoomed in to see the 2 by 2 grid.



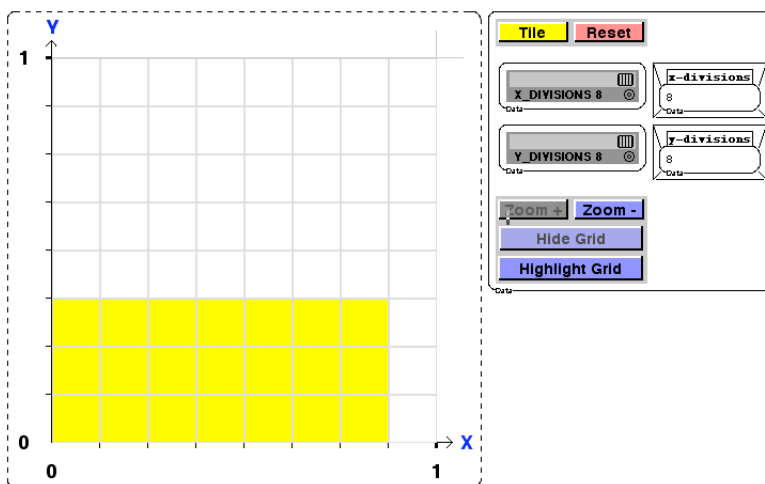


Figure 8c. Illustration of  $\frac{3}{8} \times \frac{7}{8}$  zoomed in to the maximum to see the 1 by 1 grid.

#### 3.4.4 Scaling.

There are two design aspects associated with the issue of scaling. First, the distance from 0 to 1, 1 to 2, and 2 to 3 for both the  $x$ -axis and the  $y$ -axis is preset. The rationale behind this design aspect is based on the assumption that by the seventh-grade students can correctly place whole numbers on a number line. Second, the scale for the  $x$ -axis is preset to be identical to the scale for the  $y$ -axis. This was done to (a) avoid the distraction that may be caused by different scales, and (b) to link the AM-FM representation to the problem context and the physical medium of the paper cutouts. The tradeoff of the preset scales is that it does not provide an opportunity for students to grapple with notions of scaling. This is a nuance that could be dealt with using an alternative problem context and media after students have a somewhat robust understanding of fraction multiplication.

#### 3.4.5 Tile size.

Closely related to scaling is the issue of equal-sized tiles. The fact that subdividing the axes results in partitions that produce area parts of equal size is somewhat problematic. The literature on students' partitioning activity points to the difficulties students experience in interpreting and constructing area models with and without equal area parts (Lamon, 1996; Piaget et al, 1960; Pothier & Sawada, 1983; Saxe et al., 2005). Most of this literature is based on empirical work with early elementary students and not seventh-grade students. While the AM-FM representation does not allow students to make mistakes related to constructing equal area parts, the physical medium of the paper cutouts does. The context of the paper cutouts allows me to test the assumption that seventh-grade students can partition area into equal parts. It further allows me to determine how strongly students adhere to ideal constructions and under what conditions (if any) they regress to less than ideal constructions.

#### 3.4.6 Sub-grid view.

When multiplying two proper fractions the sub-grid foregrounds the  $1 \times 1$  unit whole and students experience little difficulty with interpreting the enclosed area. When multiplying with improper or mixed numbers the sub-grid extends backgrounds the  $1 \times 1$  unit whole by

extending to the next whole number. This can cause students considerable difficulty in interpreting the enclosed area. They may shift what they initially considered to be the unit (the  $1 \times 1$  unit whole) to a larger unit that corresponds to the sub-grid view. For example, when multiplying  $\frac{4}{3}$  and  $\frac{2}{5}$ , the student may interpret the final output to be  $\frac{8}{30}$  as opposed to  $\frac{8}{15}$ . See Figure 9 for an illustration. This challenge associated with the sub-grid view provides students the opportunity to grapple with the concept of unit and allows me to test their understanding.

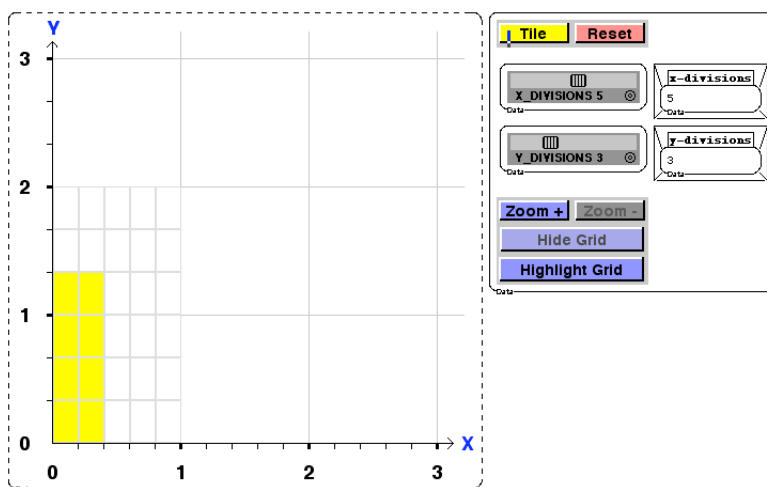


Figure 9. Illustration of the AM-FM subgrid view with  $\frac{4}{3} \times \frac{2}{5}$  area.

#### 3.4.7 Student difficulties.

Consider the literature on student difficulty mentioned earlier (i.e., multiple subconstructs, whole number bias, unitizing, and multiple representations). The AM-FM representation was designed to support the use of multiple subconstructs. Number lines are commonly associated with the measure subconstruct of fractions while area models are generally associated with the part-whole subconstruct. Furthermore, the operation of multiplication embodied in the use of the AM-FM representation is most closely associated with the operator subconstruct of fractions. Depending on what and how students attend to the various features of the representation, they can grapple with any one of the three subconstructs. The use of both whole numbers and fractions (depicted along the coordinate axes and also used in the early cases of the number chart) allows students the opportunity to grapple with the whole number bias. Unitizing is addressed via the problem context of a fair-share activity and is problematized via the use of the AM-FM representation when students are presented with cases involving improper fractions or mixed numbers. Finally, in working with the AM-FM representation, students have the opportunity to explore two canonical representations for fractions (i.e., area model and number line). Furthermore, the designed learning environment itself is centered on the use of multiple representations (i.e., problem context, paper cutouts, number chart, and AM-FM representation) and therefore provides students with the opportunity to translate and transform ideas across those representations.

In summary, different features of the learning environment offer potential affordances for certain understandings. The design of the representations and the interview protocol is

intended to help make sure those affordances become salient to students and are leveraged by students in the process of making sense of rational numbers and fraction multiplication.

## Chapter 4: Theory

### 4.1 Chapter Overview

At a macro level, it is my goal that students come to develop deep conceptual and procedural understanding of fraction multiplication, fluency within and across different representations in making sense of fraction multiplication, and the ability to reflect on and explain their thinking of fraction multiplication. In this chapter, I present the narrative of Tracy at a micro level. I will convey my conjectures about what Tracy can and cannot do with respect to fraction multiplication before, after, and during exposure to the designed learning environment. I will abstract from the narrative to discuss: (a) the idealized hypothetical initial state of student understanding before exposure to the designed learning environment, (b) the idealized hypothetical exit state of student understanding after exposure to the designed learning environment, and (c) the idealized hypothetical learning trajectory from the initial state of student understanding to the exit state of student understanding.<sup>3</sup> In the discussion of (a) and (b), I include what I consider to be relevant student knowledge within the initial and exit states (i.e., knowing) and the ways in which students might demonstrate that knowledge within the initial and exit states (i.e., doing). The idealized hypothetical learning trajectory will include a discussion of (a) and (b) as well as a discussion of the knowledge construction and reorganization I believe the designed learning environment is intended to support. The hypothetical initial and exit states are idealized in the sense that I expect a typical seventh-grade student, such as Tracy, to possess a subset of the knowing and doing described here. Similarly, the hypothetical learning trajectory is idealized in the sense that I expect a typical seventh-grade student, such as Tracy, to undergo some but not necessarily all of the knowledge construction and/or reorganization described here. My conjectures about the initial state of student understanding and what Tracy can do before exposure to the learning environment are based on my personal experience working with seventh-grade students.<sup>4</sup> My conjectures about the exit state of student understanding and what Tracy can do after exposure to the learning environment are based on my learning goals for students. And my conjectures about the learning trajectory and what Tracy can do during exposure to the learning environment are based on my design rationale, which is grounded in the literature presented in Chapter 2. Furthermore, I assume knowing and doing is contextual. What a student knows and does in one context may not transfer to another context. Similarly, what a student knows and does in one context may not be stable within that context. Finally, my conjectures will be tested against the data presented in this dissertation (to the extent the data allows) and revised in an attempt to build a robust local theory of growth and change in students' knowledge of fraction multiplication.

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<sup>3</sup> The designed learning environment consists primarily of the AM-FM representation together with the number chart.

<sup>4</sup> I served as a participant observer in a seventh-grade classroom for three years. During this time I administered test items related to fraction multiplication and conducted clinical interviews on students' understanding of fraction multiplication.

#### 4.2 Tracy Narrative: Before Exposure to the Designed Learning Environment

Consider the problem:  $1/2 \times 3/4$  (the product of two proper fractions). I hypothesize that prior to being exposed to the designed learning environment Tracy can do all of the following:

1. Apply the algorithm for fraction multiplication to arrive at the product  $3/8$  but cannot predict or justify whether the product would be less than, equal to, or greater than either  $1/2$  or  $3/4$ ;
2. Construct and interpret area models for  $1/2$  and for  $3/4$  but cannot use the area models to arrive at or justify a product of  $3/8$ ;
3. Construct and interpret  $1/2$  and  $3/4$  on a number line but cannot use the number line to arrive at or justify a product of  $3/8$ ;
4. Apply the algorithm for producing equivalent fractions to find a fraction equivalent to  $3/8$  and use the inverse operation of division to determine that  $3/8$  cannot be reduced but cannot use area models or number lines to arrive at or justify fraction equivalence or lack of fraction equivalence; and
5. Apply the algorithm for producing equivalent fractions with like denominators to arrive at the fraction order  $3/8$ ,  $1/2$ , and  $3/4$  but cannot use area models or number lines to arrive at or justify fraction order.

Now consider the problem:  $3/4 \times 3/2$  (the product of a proper fraction and improper fraction). I hypothesize that prior to being exposed to the designed learning environment Tracy can do all of the following:

1. Apply the algorithm for fraction multiplication to arrive at the product  $9/8$  but cannot predict or justify whether the product would be less than, equal to, or greater than either  $3/4$  or  $3/2$ ;
2. Construct and interpret area models for  $3/4$  but cannot construct or interpret area models for  $3/2$  to arrive at or justify a product of  $9/8$ ;
3. Construct and interpret  $3/4$  on a number line but cannot construct or interpret  $3/2$  on a number line to arrive at or justify a product of  $9/8$ ;
4. Apply the algorithm for converting improper fractions to mixed numbers to find a fraction equivalent to  $3/2$  and equivalent to the product  $9/8$  but cannot use area models or number lines to arrive at or justify fraction equivalence; and
5. Apply the algorithm for producing equivalent fractions with like denominators to arrive at the fraction order  $3/4$ ,  $9/8$ , and  $3/2$  but cannot use area models or number lines to arrive at or justify fraction order.

Finally, consider the problem:  $3/4 \times 1\frac{1}{2}$  (the product of a proper fraction and mixed number).<sup>5</sup> I hypothesize that prior to being exposed to the designed learning environment Tracy can do all of the following:

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<sup>5</sup> I did not offer the following problems as examples due to redundancy: the product of two improper fraction, the product of two mixed numbers, and the product of a mixed number and improper fraction.

1. Apply the algorithms for converting mixed number to improper fraction to find a fraction equivalent to  $1\frac{1}{2}$  but cannot use area models or number lines to arrive at or justify fraction equivalence;
2. Apply the algorithm for fraction multiplication after having converted  $1\frac{1}{2}$  to  $\frac{3}{2}$  to arrive at the product  $\frac{9}{8}$  but cannot arrive at a product without having converted  $1\frac{1}{2}$  to  $\frac{3}{2}$  and cannot predict or justify whether the product would be less than, equal to, or greater than either  $\frac{3}{4}$  or  $1\frac{1}{2}$ ;
3. Construct and interpret area models for  $\frac{3}{4}$  and  $1\frac{1}{2}$  but cannot use the area models to arrive at or justify a product of  $\frac{9}{8}$ ;
4. Construct and interpret  $\frac{3}{4}$  and  $1\frac{1}{2}$  on a number line but cannot use the number line to arrive at or justify a product of  $\frac{9}{8}$ ;
5. Apply the algorithm for converting improper fractions to mixed numbers to find a fraction equivalent to the product  $\frac{9}{8}$  but cannot use area models or number lines to arrive at or justify fraction equivalence; and
6. Apply the algorithm for producing equivalent fractions with like denominators after having converted  $1\frac{1}{2}$  to  $\frac{3}{2}$  to arrive at the fraction order  $\frac{3}{4}$ ,  $\frac{9}{8}$ , and  $1\frac{1}{2}$  but cannot use area models or number lines to arrive at or justify fraction order.

In summary, before exposure to the designed learning environment Tracy will know and be able to apply various procedures across different representational contexts to make “some” sense of fractions, fraction equivalence, fraction order, and fraction multiplication. However, Tracy will lack conceptual understanding to be able to (a) predict and justify whether any product will be less than, equal to, or greater than the two given fractions, and (b) move flexibly across different representational contexts to make sense of fraction equivalence, fraction order, and fraction multiplication. Fraction equivalence, fraction order, and fraction multiplication will make sense to Tracy when working within the representational context of fraction notation. While Tracy will be able to use area models and number lines to construct and interpret fractions, she will not be able to leverage these two representational contexts in the service of making sense of fraction equivalence, fraction order, or fraction multiplication. This is not to say that Tracy lacks conceptual understanding. Rather, the conceptual understanding Tracy has at her disposal is grounded in her prior understanding of whole numbers (see Chapter 2). In the case of fraction multiplication, I hypothesize that Tracy views the operation of multiplication as repeated addition and sees multiplication as making bigger. This prior understanding will fail Tracy in the domain of fraction multiplication. With respect to the macro learning goals established in Chapter 2, Tracy has yet to develop deep conceptual and procedural understanding of fraction multiplication, her fluency within and across different representations in making sense of fraction multiplication is limited, and her ability to reflect on and explain her thinking of fraction multiplication is emergent.

### **4.3 Idealized Hypothetical Initial State of Student Understanding (S(i))**

I abstract from the narrative of Tracy and present the idealized hypothetical initial state of student understanding (S(i)); understanding before the student is exposed to the designed learning environment. Knowing and doing in S(i) is discussed along four

dimensions of understanding: (i1) area model, (i2) number line, (i3) fraction notation, and (i4) conceptual.<sup>6</sup> I use the term “understanding” loosely in that what is presented here is not meant to be exhaustive. The first three dimensions highlight what I consider to be the three primary representations with which a typical seventh-grade student has had experience. The nature of knowing and doing within the first three dimensions is procedural. The fourth dimension of conceptual understanding spans all three representational contexts. I find it important to separate the fourth dimension from the first three dimensions because I believe the distinction between procedural and conceptual understanding is an important one (although I recognize and appreciate that the distinction is often blurry).

#### 4.3.1 Area model understanding (i1).

With respect to the first dimension of S(i), area model understanding, I hypothesize that a seventh-grade student knows that: (i1Ka) all area model wholes are of equal size, (i1Kb) the number of shaded area model wholes correspond to the whole number, (i1Kc) the parts of an area model are of equal size, (i1Kd) the parts of an area model exhaust the whole, (i1Ke) the number of shaded parts in an area model correspond to the fraction numerator, and (i1Kf) the total number of parts in an area model correspond to the fraction denominator. If a student knows all of the above about area models, I hypothesize the student can do the following: (i1Dx) given fractions  $a/b$  and  $M a/b$  (where  $a < b$ ,  $b \neq 0$ , and  $M a/b$  is some mixed number with whole number  $M$ ), the student can correctly construct an area model representation of  $a/b$  and  $M a/b$ , and (i1Dy) given area model representations of  $a/b$  and  $M a/b$  (conventional or unconventional), the student can correctly interpret the representations as the fractions  $a/b$  and  $M a/b$  (by adding and/or deleting partition lines to produce equal size parts, if necessary). See the summary of S(i) for area model in Table 2.

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<sup>6</sup> Each dimension in the initial state is indexed with a lowercase i. What a student knows with respect to a given dimension is indexed with a capital K. What a student can do with respect to a given dimension is indexed with a capital D.

Table 2. S(i) for Area Model

	<i>What student knows:</i>	<i>What student can do:</i>
<i>i1. Area Model Understanding</i>	(i1Ka) all area model wholes are of equal size (i1Kb) the number of shaded area models wholes correspond to the whole number (i1Kc) the parts of the whole are of equal size (i1Kd) the parts exhaust the whole (i1Ke) the number of shaded parts correspond to the fraction numerator (i1Kf) the number of total parts correspond to the fraction denominator	<i>Given (i1Ka), (i1Kb), (i1Kc), (i1Kd), (i1Ke), and (i1Kf)...</i> (i1Dx) construct an area model of $a/b$ and $M a/b$ (i1Dy) interpret an area model as $a/b$ and $M a/b$

#### 4.3.2 Number line understanding (i2).

With respect to the second dimension of S(i), number line understanding, I hypothesize that a seventh-grade student knows that: (i2Ka) all units of a number line are of equal size, (i2Kb) the number of units from zero to some whole number correspond to that whole number, (i2Kc) the subunits within a given number line unit are of equal size, (i2Kd) the subunits within a given number line unit exhaust that unit, (i2Ke) the number of subunits to some fraction within a given number line unit correspond to the fraction numerator, and (i2Kf) the number of subunits within a given number line unit correspond to the fraction denominator. If a student knows all of the above about number lines, I hypothesize the student can do the following: (i2Dx) given fractions  $a/b$  and  $M a/b$  (where  $a < b$ ,  $b \neq 0$ , and  $M a/b$  is some mixed number with whole number  $M$ ), the student can correctly construct number line representations of  $a/b$  and  $M a/b$ , and (i2Dy) given number line representations of  $a/b$  and  $M a/b$  (conventional or unconventional), the student can correctly interpret the representations as the fractions  $a/b$  and  $M a/b$  (by adding and/or deleting partition marks to produce equal size subunits, if necessary). See the summary of S(i) for number line in Table 2.



Table 3. S(i) for Number Line

	<i>What student knows:</i>	<i>What student can do:</i>
<i>i2. Number Line Understanding</i>	(i2Ka) all units of a number line are of equal size (i2Kb) the number of units from zero to some whole number correspond to that whole number (i2Kc) the subunits in the unit are of equal size (i2Kd) the subunits in the unit exhaust the unit (i2Ke) the number of subunits to some fraction correspond to the fraction numerator (i2Kf) the number of total subunits in the unit correspond to the fraction denominator	<i>Given (i2Ka), (i2Kb), (i2Kc), (i2Kd), (i2Ke), and (i2Kf)...</i> (i2Dx) construct a number line of $a/b$ and $M a/b$ (i2Dy) interpret a number line as $a/b$ and $M a/b$

#### 4.3.3 Fraction notation understanding (i3).

With respect to the third dimension of S(i), fraction notation understanding, I hypothesize that a seventh-grade student knows: (i3Ka) the algorithm for fraction multiplication and the commutative property, (i3Kb) the algorithm for producing equivalent proper fractions, (i3Kc) the algorithm for converting an improper fraction to a mixed numbers, (i3Kd) the algorithm for converting a mixed number to an improper fraction, (i3Ke) the algorithm for ordering fraction, and (i3Kf) how to use a reference point for ordering fractions.

If the student knows (i3Ka), I hypothesize the student can do the following: (i3Da) given  $a/b \times c/d$  (where  $b \neq 0$  and  $d \neq 0$ ), the student can correctly arrive at the product  $a/b \times c/d = ac/bd = c/d \times a/b$ . If the student knows (i3Kb), I hypothesize the student can do the following: (i3Db) given some fraction  $a/b$  (where  $b \neq 0$ ), the student can correctly generate an equivalent fraction  $an/bn$  (where  $n \neq 0$ ) and a reduced equivalent fraction  $(a/m)/(b/m)$  if such an  $m$  exists. If the student knows (i3Kc), I hypothesize the student can do the following: (i3Dc) given some improper fraction  $b/a$  (where  $a < b$  and  $a \neq 0$ ), the student can correctly divide the numerator by the denominator to find the quotient  $M$  and remainder  $r$  to arrive at an equivalent mixed number  $M r/a$ . If the student knows (i3Kd), I hypothesize the student can do the following: (i3Dd) given some mixed number  $M a/b$ , the student can correctly multiply the denominator by the whole number and add the numerator to arrive at an equivalent improper fraction  $(Mb+a)/b$ . If the student knows (i3Ke), I hypothesize that student can do the following: (i3De) given two fractions  $a/b$  and  $c/d$  (where  $b \neq 0$  and  $d \neq 0$ ), the student can generate equivalent fraction with like denominators,  $ad/bd$  and  $cb/bd$ , and correctly conclude  $a/b < c/d$  if  $ad < cb$  and  $a/b > c/d$  if  $ad > cb$ . Finally, if the student knows (i3Kf), I hypothesize that the student can do the following: given  $a/b$  and  $c/d$ , if the student can find a

fraction  $r/s$  such that  $a/b < r/s < c/d$  then the student can correctly conclude  $a/b < c/d$  and alternatively if the student can find a fraction  $r/s$  such that  $a/b > r/s > c/d$  then the student can correctly conclude  $a/b > c/d$  (this also holds in the case where  $r/s$  is a whole number). See the summary of S(i) for fraction notation in Table 4.

Table 4. S(i) for Fraction Notation

	<i>What student knows:</i>	<i>What student can do:</i>
<i>i3. Fraction Notation Understanding</i>	(i3Ka) the algorithm for fraction multiplication and the commutative property (i3Kb) the algorithm for producing equivalent proper fractions (i3Kc) the algorithm for converting an improper fraction to a mixed numbers (i3Kd) the algorithm for converting a mixed number to an improper fraction (i3Ke) the algorithm for ordering fractions (i3Kf) using a reference point for ordering fractions	<i>Given (i3Ka)...</i> (i3Da) $a/b \times c/d = ab/cd = c/d \times a/b$ <i>Given (i3Kb)...</i> (i3Db) $a/b \times n/n = an/bn$ , $(a/m)/(b/m) = c/d$ <i>Given (i3Kc)...</i> (i3Dc) $b/a = M r/a$ , where M is the quotient of b and a and r is the remainder <i>Given (i3Kd)...</i> (i3Dd) $M a/b = (Mb+a)/b$ <i>Given (i3Ke)...</i> (i3De) $a/b < c/d$ if $ad < cb$ and $a/b > c/d$ if $ad > cb$ <i>Given (i3Kf)...</i> (i3Df) if there exists a fraction $r/s$ such that $a/b < r/s < c/d$ then $a/b < c/d$ , alternatively if $a/b > r/s > c/d$ then $a/b > c/d$ (also holds in the case where $r/s$ is a whole number)

#### 4.3.4 Conceptual understanding (i4).

The fourth and final dimension of S(i) is conceptual understanding. Conceptual understanding will be discussed with respect to fractions and then with respect to the operation of multiplication. Furthermore, I include both the productive knowledge students might bring to the designed learning environment as well as non-productive knowledge. Because the designed learning environment is intended to leverage students' prior understandings in order to support the development of deep conceptual and procedural understanding of fraction multiplication, it is important to consider both productive and non-productive conceptual understandings they brings to bear in working within that environment.

With respect to conceptual understanding of fractions, I hypothesize that a seventh-grade student knows the following three subconstructs: (i4Ka) fraction as a part-whole

relation, (i4Kb) fraction as measure, and (i4Kc) fractions as quotient.<sup>7</sup> The representational context will in part determine which of the rational number subconstructs the student will draw upon to make sense of fractions. In the case of area model construction and interpretation, the part-whole subconstruct may be most salient. In the case of number line construction and interpretation, the measure subconstruct may be most salient. Finally, in the case of fraction notation and in particular the application of the algorithm for converting between improper fraction and mixed number, the quotient subconstruct may be most salient.

With respect to conceptual understanding of fraction multiplication, I hypothesize that a seventh-grade student knows: (i4Kd) multiplication as repeated addition, and (i4Ke) multiplication makes bigger. Note that both (i4Kd) and (i4Ke) are non-productive knowledge in the context of understanding fractions and fraction multiplication. I hypothesize that this non-productive knowledge will be present as the student engages within the designed learning environment and will serve as a sight for knowledge construction and knowledge reorganization.

Next, I discuss what a student can do given what he knows within the dimension of conceptual understanding. If the student knows (i4Ka), I hypothesize the student can do the following: (i4Da) see the fraction  $a/b$  as  $a$  parts out of a unit made up of  $b$  equal size parts.<sup>8</sup> If the student knows (i4Kb), I hypothesize the student can do the following: (i4Db) see the fraction  $a/b$  as a  $1/b$  piece of the unit iterated  $a$  times.<sup>9</sup> If the student knows (i4Kc), I hypothesize the student can do the following: (i4Dc) see the fraction  $a/b$  as  $a$  divided by  $b$ .

If the student knows (i4Kd), I hypothesize the student can do the following: (i4Dd) given  $p \times q$ , add  $q$  groups of  $p$  (or  $p$  groups of  $q$ ) to arrive at the product  $pq$  but given  $a/b \times c/d$ , the student cannot add  $c/d$  groups of  $a/b$  (or  $a/b$  groups of  $c/d$ ) to arrive at the product  $ab/cd$ . If the student knows (i4Ke), I hypothesize the student can do the following: (i4De) correctly predict and justify that the product of two whole numbers is greater than the two given whole numbers (assuming neither of the whole numbers is one or zero) but the student cannot correctly predict and justify whether the product of two rational numbers is less than, equal to, or greater than the two given rational numbers. See the summary of S(i) for conceptual understanding in Table 5.

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<sup>7</sup> The two remaining subconstructs, fraction as operator and fraction as ratio, are not part of the hypothetical initial state of student understanding but fraction as operator will be part of the hypothetical exit state of student understanding.

<sup>8</sup> The term unit here refers to the area model representation of unit whole.

<sup>9</sup> The term unit here refers to the number line representation of unit length.

Table 5. S(i) for Conceptual Understanding

	<i>What student knows:</i>	<i>What student can do:</i>
<i>i4. Conceptual Understanding</i>	(i4Ka) fraction as part-whole (i4Kb) fraction as measure (i4Kc) fraction as quotient (i4Kd) multiplication as repeated addition (i4Ke) multiplication makes bigger	<i>Given (i4Ka)...</i> (i4Da) see $a/b$ as a parts out of unit of $b$ equal parts <i>Given (i4Kb)...</i> (i4Db) see $a/b$ as a $1/b$ piece of the unit iterated a times <i>Given (i4Ka)...</i> (i4Db) see $a/b$ as a divided by $b$ <i>Given (i4Kd)...</i> (i4Dd) given $p \times q$ , add $q$ groups of $p$ to arrive at $pq$ but given $a/b \times c/d$ , cannot add $c/d$ groups of $a/b$ to arrive at $ab/cd$ <i>Given (i4Ke)...</i> (i4De) predict and justify the product of two whole numbers is greater than the two whole numbers but cannot predict and justify whether the product of two rational numbers is less than, equal to, or greater than the two rational numbers

#### 4.4 Tracy Narrative: After Exposure to the Designed Learning Environment

Having discussed the hypothetical initial state of student understanding, S(i), before exposure to the designed learning environment, I return now to the narrative of Tracy and the problems discussed at the start of this chapter: (a)  $1/2 \times 3/4$ , (b)  $3/4 \times 3/2$ , and (c)  $3/4 \times 1\frac{1}{2}$ . I will present what I believe Tracy can do with these problems after exposure to the designed learning environment.

First reconsider the problem:  $1/2 \times 3/4$ . I hypothesize that after being exposed to the designed learning environment Tracy can do all of the following:

1. Apply the algorithm for fraction multiplication to arrive at the product  $3/8$ ;
2. Predict and justify that the product is less than both  $1/2$  and  $3/4$  (because the product can be interpreted as  $1/2$  of  $3/4$  or  $3/4$  of  $1/2$  and both  $1/2$  and  $3/4$  are less than 1);
3. Construct and interpret area models for  $1/2$ ,  $3/4$ , and  $1/2 \times 3/4$ ;
4. Construct and interpret  $1/2$  on one number line,  $3/4$  on another number line, and  $1/2 \times 3/4$  on the coordination of the two number lines;

5. Apply the algorithm for producing equivalent fractions to find a fraction equivalent to  $\frac{3}{8}$  and use the inverse operation of division to determine that  $\frac{3}{8}$  cannot be reduced;
6. Use area models and number lines to arrive at and justify fraction equivalence or lack of fraction equivalence;
7. Apply the algorithm for producing equivalent fractions with like denominators to arrive at the fraction order  $\frac{3}{8}$ ,  $\frac{1}{2}$ , and  $\frac{3}{4}$ ; and
8. Use area models and number lines to arrive at and justify fraction order.

Next, reconsider the problem:  $\frac{3}{4} \times \frac{3}{2}$ . I hypothesize that after being exposed to the designed learning environment Tracy can do all of the following:

1. Apply the algorithm for fraction multiplication to arrive at the product  $\frac{9}{8}$ ;
2. Predict and justify that the product is greater than  $\frac{3}{4}$  (because the product can be interpreted as  $\frac{3}{2}$  of  $\frac{3}{4}$  and  $\frac{3}{2}$  is greater than 1) but less than  $\frac{3}{2}$  (because the product can be interpreted as  $\frac{3}{4}$  of  $\frac{3}{2}$  and  $\frac{3}{4}$  is less than 1);
3. Construct and interpret area models for  $\frac{3}{4}$ ,  $\frac{3}{2}$ , and  $\frac{3}{4} \times \frac{3}{2}$ ;
4. Construct and interpret  $\frac{3}{4}$  on one number line,  $\frac{3}{2}$  on another number line, and  $\frac{3}{4} \times \frac{3}{2}$  on the coordination of the two number lines;
5. Apply the algorithm for converting improper fractions to mixed numbers to find a fraction equivalent to  $\frac{3}{2}$  and equivalent to the product  $\frac{9}{8}$ ;
6. Use area models and number lines to arrive at and justify fraction equivalence;
7. Apply the algorithm for producing equivalent fractions with like denominators to arrive at the fraction order  $\frac{3}{4}$ ,  $\frac{9}{8}$ , and  $\frac{3}{2}$ ; and
8. Use area models and number lines to arrive at and justify fraction order.

Finally, reconsider the problem:  $\frac{3}{4} \times 1\frac{1}{2}$ . I hypothesize that after being exposed to the designed learning environment Tracy can do all of the following:

1. Apply the algorithms for converting mixed numbers to improper fractions to find a fraction equivalent to  $1\frac{1}{2}$ ;
2. Apply the algorithm for fraction multiplication after having converting  $1\frac{1}{2}$  to  $\frac{3}{2}$  to arrive at the product  $\frac{9}{8}$ ;
3. Predict and justify that the product is greater than  $\frac{3}{4}$  (because the product can be interpreted as  $1\frac{1}{2}$  of  $\frac{3}{4}$  and  $1\frac{1}{2}$  is greater than 1) but less than  $1\frac{1}{2}$  (because the product can be interpreted at  $\frac{3}{4}$  of  $1\frac{1}{2}$  and  $\frac{3}{4}$  is less than 1);
4. Construct and interpret area models for  $\frac{3}{4}$ ,  $1\frac{1}{2}$ , and  $\frac{3}{4} \times 1\frac{1}{2}$ ;
5. Construct and interpret  $\frac{3}{4}$  on one number line,  $1\frac{1}{2}$  on another number line, and  $\frac{3}{4} \times 1\frac{1}{2}$  on the coordination of the two number lines;
6. Apply the algorithm for converting improper fractions to mixed numbers to find a fraction equivalent to the product  $\frac{9}{8}$ ;
7. Use area models and number lines to arrive at and justify fraction equivalence;
8. Apply the algorithm for producing equivalent fractions with like denominators after having converted  $1\frac{1}{2}$  to  $\frac{3}{2}$  to arrive at the fraction order  $\frac{3}{4}$ ,  $\frac{9}{8}$ , and  $1\frac{1}{2}$ ; and

9. Use area models and number lines to arrive at and justify fraction order.

In summary, after exposure to the designed learning environment Tracy will be able to apply various procedures within and across different representational contexts, the procedures will be grounded in deep conceptual understanding, and Tracy will demonstrate these understandings through her ability to reflect on and explain her thinking. Tracy will have at her disposal the necessary conceptual understanding to be able to (a) predict and justify whether the product of two fractions is less than, equal to, or greater than the two given fractions and (b) move flexibly within and across a number of different representational contexts to make sense of fraction equivalence, fraction order, and fraction multiplication.

#### 4.5 Idealized Hypothetical Exit State of Student Understanding (S(e))

I abstract from the narrative of Tracy to present the idealized hypothetical exit state of student understanding (S(e)); understanding after the student is exposed to the designed learning environment. As was the case with S(i), I will discuss both knowing and doing in S(e). Knowing and doing in S(e) is presented along five dimensions: (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, (e3) representational fluency for fraction multiplication, (e4) representational fluency for fraction equivalence, and (e5) representational fluency for fraction order.<sup>10</sup> The first three dimensions are the primary concepts the learning environment is intended to support and are specific to fraction multiplication. These three dimensions map onto the macro learning goals identified in Chapter 2 and at the beginning of this chapter. More specifically, (e1) and (e2) address the development of conceptual and procedural understanding of fraction multiplication while (e3) addresses fluency within and across multiple representations in making sense of fraction multiplication. The ability to reflect on and explain one's thinking of fraction multiplication is implicit in the act of doing within the designed learning environment. The two remaining dimensions highlight the secondary concepts the learning environment is intended to support and are specific to fraction equivalence and fraction order, respectively.

##### 4.5.1 Fraction multiplication as stretching/shrinking (e1).

With respect to the first dimension of S(e), fraction multiplication as stretching/shrinking, I hypothesize that a seventh-grade student knows that: (e1Ka) multiplication as stretching/shrinking makes more sense in the case of fraction multiplication than does multiplication as repeated addition. If the student knows (e1Ka), I hypothesize the student can do the following: (e1Da) given  $a/b \times c/d$  (where  $a/b$ ,  $c/d$  are any combination of proper fraction, improper fraction, and/or mixed number), the student will start with some unit U, shrink (or stretch) the unit U by  $c/d$ , shrink (or stretch) the unit  $c/d$  (of U) by  $a/b$ , and interpret the product of  $a/b \times c/d$  relative to the original unit U (alternatively the student can operate on U with  $a/b$  followed by  $c/d$ ).

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<sup>10</sup> Each dimension in the exit state is indexed with a lowercase e. What a student knows with respect to a given dimension is indexed with a capital K. What a student can do with respect to a given dimension is indexed with a capital D.

#### 4.5.2 Number sense with fraction multiplication (e2).

With respect to the second dimension of S(e), number sense with fraction multiplication, I hypothesize that a seventh-grade student knows that: (e2Ka) fraction multiplication does *not always* make bigger. If the student knows (e2Ka) together with (e1Ka), I hypothesize the student can do the following: (e2Dx) given  $a/b \times c/d$  (where  $a/b$ ,  $c/d$  are any combination of proper fraction, improper fraction, and/or mixed number) the student can correctly predict and justify whether the product is (a) less than, equal to, or greater than  $a/b$  and (b) less than, equal to, or greater than  $c/d$ , and (e2Dx) given  $a/b \times c/d$  (where  $a/b$ ,  $c/d$  are any combination of proper fraction, improper fraction, and/or mixed number) the student can correctly predict and justify whether the product is less than, equal to, or greater than  $a/b \times r/s$  where  $r/s$  is any given fraction (alternatively for  $r/s \times a/b$ ) and (b) less than, equal to, or greater than  $r/s \times c/d$  where  $r/s$  is any given fraction (alternatively for  $c/d \times r/s$ ).

#### 4.5.3 Representational fluency for fraction multiplication (e3).

With respect to the third dimension of S(e), representational fluency for fraction multiplication, I hypothesize that a seventh-grade student knows that: (e3Ka) multiple representations can be used (constructed and interpreted) in order to make sense of fraction multiplication. If the student knows (e3Ka), I hypothesize the student can do the following: (e3Da) see the operation of fraction multiplication embodied in his/her use of the AM-FM representation and the number chart.<sup>11</sup> See the summary of S(e) for fraction multiplication in Table 6.

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<sup>11</sup> The AM-FM representation and the number chart constitute the two primary representational contexts that makeup the designed learning environment.

Table 6. S(e) for Fraction Multiplication

	<i>What student knows:</i>	<i>What student can do:</i>
<i>e1. Fraction Multiplication as Stretching/Shrinking</i>	(e1Ka) multiplication as stretching/shrinking makes more sense in the case of fraction multiplication than multiplication as repeat addition	<i>Given (e1Ka) and <math>a/b \times c/d</math> (where <math>a/b</math>, <math>c/d</math> are proper fractions, improper fractions, or mixed numbers)...</i> (e1Da) starts with some unit U, shrink (or stretch) the unit U by $c/d$ , shrink (or stretch) the unit $c/d$ (of U) by $a/b$ , and interpret the product of $a/b \times c/d$ relative to the original unit U (alternatively can operate on U with $a/b$ followed by $c/d$ )
<i>e2. Number Sense with Fraction Multiplication</i>	(e2Ka) fraction multiplication does <i>not always</i> make bigger.	<i>Given (e2Ka), (e1Ka) and <math>a/b \times c/d</math> (where <math>a/b</math>, <math>c/d</math> are proper fractions, improper fractions, or mixed numbers)...</i> (e2Dx) correctly predict and justify whether the product is (a) less than, equal to, or greater than $a/b$ and (b) less than, equal to, or greater than $c/d$ (e2Dy) correctly predict and justify whether the product is less than, equal to, or greater than $a/b \times r/s$ and (b) less than, equal to, or greater than $r/s \times c/d$
<i>e3. Representational Fluency for Fraction Multiplication</i>	(e3Ka) multiple representations can be used (constructed and interpreted) in order to make sense of fraction multiplication	<i>Given (e3Ka)...</i> (e3Da) see the operation of fraction multiplication embodied in use of the AM-FM representation and the number chart

#### 4.5.4 Representational fluency for fraction equivalence and fraction order, (e4 & e5).

Next, consider the two secondary concepts the learning environment is intended to support. With respect to the fourth dimension of S(e), representational fluency for fraction equivalence, I hypothesize that a seventh-grade student knows that: (e4Ka) multiple representations can be used (constructed and interpreted) in order to make sense of fraction equivalence. If the student knows (e4Ka), I hypothesize the student can do the following:



(e4Dx) correctly name equivalence fractions across his/her use of the AM-FM representation and the number chart and (e4Dy) correctly name equivalent fractions across his/her use of the area model and number line features of the AM-FM representation.

With respect to the fifth and final dimension of S(e), representational fluency for fraction order, I hypothesize a seventh-grade student knows that: (e5Ka) multiple representations can be used (constructed and interpreted) in order to make sense of fraction order. If the student knows (e5Ka), I hypothesize the student can do the following: (e5Dx) correctly order fractions across his/her use of the AM-FM representation and the number chart and (e5Dy) correctly order fraction across his/her use of the area model and number line features of the AM-FM representation. See Table 7 for a summary of S(e) for the secondary concepts the learning environment is intended to support.

Table 7. S(e) for Fraction Equivalence and Fraction Order

	<i>What student knows:</i>	<i>What student can do:</i>
<i>e4. Representational Fluency for Fraction Equivalence</i>	(e4Ka) multiple representations can be used (constructed and interpreted) in order to make sense of fraction equivalence	<i>Given (e4Ka)...</i> (e4Dx) correctly name equivalence fractions across use of the AM-FM representation and the number chart (e4Dy) correctly name equivalent fractions across use of the area model and number line features of the AM-FM representation
<i>e5. Representational Fluency for Fraction Order</i>	(e5Ka) multiple representations can be used (constructed and interpreted) in order to make sense of fraction order	<i>Given (e5Ka)...</i> (e5Dx) correctly order fractions across use of the AM-FM representation and the number chart (e5Dy) correctly order fractions across use of the area model and number line features of the AM-FM representation

#### 4.6 Idealized Hypothetical Learning Trajectory from S(i) to S(e)

I began with a discussion of the idealized hypothetical initial state of student understanding, S(i), before the student is exposed to the learning environment. Knowing and doing in S(i) was presented along four dimensions: (i1) area model understanding, (i2) number line understanding, (i3) fraction notation understanding, and (i4) conceptual understanding. This was followed by a discussion of the idealized hypothetical exit state of student understanding, S(e), after the student is exposed to the learning environment. Knowing and doing in S(e) was presented along five dimensions: (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, (e3) representational

fluency for fraction multiplication, (e4) representational fluency for fraction equivalence, and (e5) representational fluency for fraction order. Next, I return to the narrative of Tracy to exemplify an idealized hypothetical learning trajectory from the initial state of student understanding to the final state of student understanding.<sup>12</sup> I will draw on aspect of the learning environment that I believe will activate Tracy's prior knowledge as discussed in S(i) and result in the knowledge discussed in S(e). The trajectory is idealized in the sense that I expect a typical seventh-grade student to undergo some but not necessarily all of knowledge construction and/or reorganization described here.

I will present the idealized hypothetical trajectory along the three primary concepts the learning environment is intended to support: (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, and (e3) representational fluency for fraction multiplication.<sup>13</sup> Figure 10 illustrates the transformation from hypothetical initial state of student understanding to hypothetical final state of student understanding for fraction multiplication. Note, i1, i2, i3, and i4 in the initial state consist of a subset of knowledge resources that are not depicted in Figure 10. Furthermore, the connections between e1, e2, and e3 in the final state result from overlap among these knowledge resources. For example, e1 is connected to e2 because both draw on i1, i2, and some subset of knowledge resource from i4.

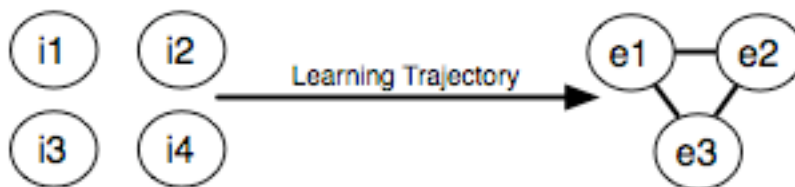


Figure 10. Transformation for S(i) to S(e) for fraction multiplication.

#### 4.6.1 Multiplication as stretching/shrinking (e1).

I begin first with the idealized hypothetical learning trajectory with respect to multiplication as stretching/shrinking, (e1). As Tracy works with the AM-FM representation and proceeds through a particular sequence of problems (from proper fraction  $\times$  proper fraction, to proper fraction  $\times$  improper fraction, to proper fraction  $\times$  mixed number, to improper  $\times$  mixed number), I hypothesize she will construct and coordinate her knowledge of the area model and number line features of the AM-FM representations and in doing come to understand multiplication as stretching/shrinking. Tracy will see area shrink as she starts the construction process with the  $1 \times 1$  unit whole, takes  $a/b$  amount of the unit whole (where  $a/b$  is some proper fraction), and then takes  $c/d$  amount of  $a/b$  (where  $c/d$  is some proper fraction). Similarly, Tracy will see area stretch when working with improper fraction and mixed numbers. Tracy will draw on her prior understandings of area model, number line, fraction as part-whole, fraction as measure, and multiplication as repeat addition in order to see

<sup>12</sup> Learning is defined roughly as any change to the individual's knowledge structure and capacity to do mathematics.

<sup>13</sup> The two secondary concepts the learning environment is intended to support will not be discussed in the narrative about the idealized hypothetical learning trajectory, however, they will be discussed in the two case study chapters.

multiplication as stretching/shrinking (see Figure 11). Tracy's non-productive knowledge of multiplication as repeat addition will be re-organized such that it is specific to whole number multiplication and not fraction multiplication.

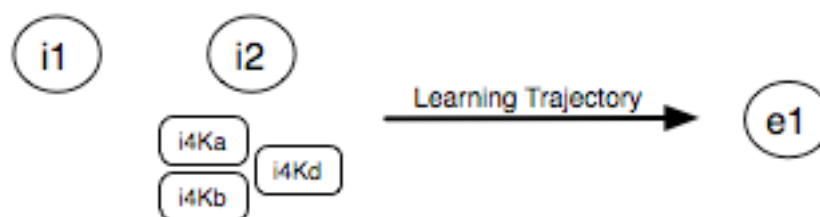


Figure 11. Transformation from S(i) to fraction multiplication as stretching/shrinking (e1).

#### 4.6.2 Number sense with fraction multiplication (e2).

Next, consider the idealized hypothetical learning trajectory with respect to number sense with fraction multiplication, (e2). As Tracy uses the AM-FM representation to proceed through a particular sequence of problems (from proper fraction  $\times$  proper fraction, to proper fraction  $\times$  improper fraction, to proper fraction  $\times$  mixed number, to improper  $\times$  mixed number), completes the number chart, and makes predictions and justifications about the output (as to whether an output will be more or less than the output in the previous case), I hypothesize Tracy will come to see that multiplication does not always make bigger. Tracy's predictions and justifications will proceed from incorrect to correct as she validates the result using the AM-FM representation. Ultimately, Tracy's predictions and justifications will be grounded in her visualization of the problems and an ability to justify her claims about fraction multiplication in AM-FM terms. Further evidence will be gathered as Tracy is asked to detect patterns across the output column in the number chart. Tracy will draw on her understandings of area model, number line, algorithm for fraction multiplication, algorithm for fraction equivalence, algorithm for fraction order, fraction as part-whole, fraction as measure, fraction as quotient, and multiplication makes bigger in order to see that multiplication does not always make bigger (see Figure 12). Tracy's non-productive knowledge that multiplication makes bigger will be re-organized such that it is specific to whole number multiplication and not fraction multiplication.



Figure 12. Transformation from S(i) to number sense with fraction multiplication (e2).

#### 4.6.3 Representational fluency for fraction multiplication (e3).

Finally, consider the idealized hypothetical learning trajectory with respect to representational fluency for fraction multiplication, (e3). As Tracy proceeds through the sequence of problems using the AM-FM representation and records her results in the number

chart, I hypothesize that she will come to see the operation of multiplication embodied in her use of not only the number chart but also the AM-FM representation. As Tracy proceeds through the sequence of problems, she will be asked to detect patterns in the number chart. Given Tracy's prior understandings of fraction notation and the algorithm for fraction multiplication, she will see multiplication as the operation embodied in her use of the number chart. This new knowledge of the number chart will lead Tracy to see the operation of multiplication embodied in her use of the AM-FM representation. Without the number chart, Tracy may view her use of the AM-FM representation in terms of division (as opposed to multiplication) given the heavy focus on partitioning. But after detecting the algorithm for fraction multiplication embedded as a pattern in the number chart, Tracy will come to see her use of the AM-FM representation in terms of multiplication. Furthermore, with respect to the AM-FM representation itself, Tracy will see multiplication embodied in her interpretation of final area. Tracy will multiply the number of vertical partitions by the number of horizontal partitions within the 1x1 unit whole to arrive at a denominator output and multiply the number of vertical partitions by the number of horizontal partitions within the shaded area to arrive at a numerator output. In arriving at this new knowledge, Tracy will be drawing on her understandings of area model, number line, algorithm for fraction multiplication, fraction as part-whole, fraction as measure, multiplication as repeated addition, and multiplication makes bigger (see Figure 13).



Figure 13. Transformation from S(i) to representational fluency for fraction multiplication (e3).

Having discussed (a) the idealized hypothetical initial state of student understanding before exposure to the designed learning environment, (b) the idealized hypothetical exit state of student understanding after exposure to the designed learning environment, and (c) the idealized hypothetical learning trajectory from the initial state of student understanding to the exit state of student understanding I turn now to methods.

## Chapter 5: Methods

### 5.1 Chapter Overview

The purpose of this section is to discuss: (a) the broader context of the design study, (b) the participants and the means by which they were selected, (c) the data collection procedures and the resulting data sources, and (d) the process by which those data sources were analyzed.

### 5.2 Context

The research for this dissertation study occurred during the 2005-2006 academic school year. It took place in an urban middle school with a racially and socio-economically diverse student population. Two seventh-grade teachers agreed to have me serve as a participant-observer in one of their class periods. Both teachers were members of a district-wide professional development collaboration with university researchers and demonstrated a strong commitment to supporting mathematics education research. Both had been teaching for approximately 4 years. They had each taught seventh-grade pre-algebra previously using College Preparatory Mathematics (CPM) curriculum, a reform-oriented curriculum that contained a unit on probability and fraction multiplication.

### 5.3 Participants

The participants in the design experiment were ten seventh-grade students. Selection was limited to students who had signed and returned both student and parent permission slips for participation in the study and who had demonstrated good attendance. In an attempt to increase the likelihood of data triangulation, the majority of the 10 students were selected from the two class periods in which I served as a participant-observer.

### 5.4 Data Collection

I engineered a learning environment centered on the use of the AM-FM representation to investigate growth and change in students' knowledge of fraction multiplication. The 10 students selected to participate in the study met with me once a week for four weeks. The four sessions each lasted approximately 90-minutes. During the first and last session each student took the pretest and posttest in a think-aloud format. The pretest and posttest consisted of 30 identical test items. The items were adapted from the literature on rational number and fraction multiplication. See Appendix A for complete list of test items. The second and third sessions consisted of semi-structured clinical interviews (Ginsburg, 1997). Students were presented with various problems related to rational number operations. They were asked to construct and interpret representations of fraction multiplication problems using physical and computer-based media. One camera was used during the pretest and posttest. The camera was zoomed in to capture all student work. Two cameras were used during the clinical interviews. One camera captured the student and myself seated at the desk and the other was zoomed in to capture the physical and computer-based media being used by the student and myself. All student work generated during the four sessions was collected.

Of the 10 students who participated in the study two were selected for the focus of detailed case studies. Both students were given pseudonyms in accord with human subjects protocol. The first case study will focus on Neato (NP) who demonstrated the greatest gains from pretest to posttest. See Appendix B for a summary of pretest to posttest gains for all 10 students.<sup>14</sup> Neato was also the only student to use paper and pencil to reconstruct the AM-FM representation during the posttest to answer questions that he could not answer on the pretest. As such, he is the best example of a student who profited from the environment; the case analysis will explore how and why he did. The second case study will focus on Oscar (OA). Oscar also demonstrated solid gains from pretest to posttest but the areas in which he made gains differed from Neato (see Appendix B). Together the two cases will provide a rich context for exploring the differential impact of the learning environment and will offer evidence to support the development of local theory and the refinement of design.

Video data of all four sessions were transcribed for both case study students. The transcription included video screen shots in order to capture what students were attending to at various points in time. For example, any time a student was asked to interpret area using the AM-FM representation, a screen shot of the student's AM-FM construction was included. Similarly, if a student looked to fractions inscribed in the number chart to justify his/her interpretation of area, a zoomed in screen shot of the number chart was included along with a note such as, "[Student looks to the number chart]."

### 5.5 Interview Protocol

My primary role during the think-aloud pretest and posttest sessions was to ask the student to talk aloud as they proceeded through each problem and provide clarification of problems if necessary. If there was a long period of silence the student was asked, "what are you thinking?" If the student could not answer a problem and wanted to skip to the next problem he/she was asked, "what do you find difficult about this problem?" The student's primary role during the think-aloud pretest and posttest sessions was to talk aloud as he/she worked through each problem, to ask for clarification of problems if necessary, and to provide an explanation of problem difficulty if he/she chose to skip a problem.

My primary role during the clinical interview sessions was to present the student with appropriate problems based on the thinking he/she had demonstrated, encourage the student to use the physical and computer-based media in order to solve problems, encourage the student to return to his/her prior understandings when presented with novel and challenging problems with which he/she experienced difficulty, and provide instructional assistance when requested or when I felt it was appropriate. Of course there were a number of instances when I made mistakes (i.e., asked leading questions, asked the wrong question, or forget to ask a question all together). The student's primary role during the clinical interview sessions was to attempt to solve problems using the physical and computer-based media, communicate his/her thinking as he/she proceeded through each problem, and ask questions regarding problems and request assistance as needed.

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<sup>14</sup> A number of students demonstrated little to no gains from pretest to posttest due to ceiling effects and were therefore not considered for case selection. The ceiling effects were not surprising given the students' 7<sup>th</sup>-grade level and the nature of the mathematical content covered on the test.

The clinical interview began with the use of the paper cutouts. After introducing the problem context to each student, I went through one example using the cutouts. The students were then asked to proceed through the remaining cases (i.e. problems). I gave the students input values verbally, they recorded the inputs on the number chart, made the appropriate arrangement using the paper cutouts, explained their arrangement, found the output (total amount of cheese used per case), and recorded the output on the number chart. Prior to making an arrangement with the cutouts for any case, students were asked to make predictions about whether the output would be more or less than the output of the previous case. The predictions were then tested and discussed. At the end of the session students were asked to look over the number chart and discuss any patterns they noticed (if time permitted). When working with the paper cutouts became tedious and/or cumbersome I transitioned the students to the AM-FM representation.

The majority of the clinical interview centered on the use of the AM-FM representation and proceeded in a similar manner. I reintroduced the problem context and went through one example using the representation. The students then continued with the remaining cases. The students were given input values in verbal and written form by me, they made the appropriate arrangement using the AM-FM representation, explained their arrangement, found the output values (total amount of cheese used per case), and I recorded the output on the number chart. All recording of input and output values was done by me in order to prevent having to move the camera focus away from the laptop screen. Prior to making an arrangement with the AM-FM representation for any case, students were asked to make predictions about whether the output would be more or less than the output of the previous case. These predictions were tested and discussed. When explaining their final arrangement for a case, students were often asked to name the value for  $x$  and  $y$  at various points along the marked number lines, to name each tile, to find an output value, and to explain their output value relative to the predictions made at the start of the case. At the end of the session students were asked to look over the completed number chart and discuss any patterns they noticed (if time permitted). The cases were sequenced using the research literature on rational numbers in order to highlight particular patterns.

## 5.6 Data Analysis

Data analysis addressed the research questions identified at the beginning of this proposal:

- What sense do the students make of rational numbers and fraction multiplication as they work within the designed learning environment?
- How does the sense-making process emerge as the students work within the designed learning environment to (a) understand rational numbers and fraction multiplication (i.e. develop domain competence) and (b) understand the affordances and constraints of the AM-FM representation (i.e. develop representational competence)?
- What are the implications (theoretical and practical) for future design study iterations? More specifically, how will (local) theory and the design (tools and clinical interview protocol) used in this dissertation study be refined to better inform our understanding of growth and change in students' knowledge of fraction multiplication.

In order to make claims regarding knowledge growth and change, I drew primarily on transcript analysis of the 90-minute clinical interview sessions with each student. I focused on language, gesture, and gaze as each student worked with different representations within the designed learning environment. To exemplify what I did, I will discuss in detail the analytical methods employed in the case of Neato.

Once Neato was chosen to serve as a focal case for analysis, I did a thorough read of his two clinical interview transcripts. In the first transcript, Neato worked exclusively with the paper cutouts. In the second transcript, he worked exclusively with the AM-FM representation. My primary interest was students' sense-making as it pertained to their use of the AM-FM representation. The paper cutouts were intended only as a transitional representation between the problem context (distributing cheese to lab rats) and the primary representation of interest (the AM-FM representation) and were therefore not expected to be either a focus of significant learning or a focus of analysis. But before I could devote my analytic attention exclusively to the second transcript, I had to be certain that Neato did not show significant knowledge growth and change as a function of having worked with the paper cutouts. I needed to know if there were things Neato did with the cutouts (and alternatively if there were things he did not do with the cutouts) that then had implications for his emergent understandings while working with the AM-FM representation? An analytic pass through the two transcripts indicated that Neato understood the problem context across his use of both representations and made no explicit or implicit reference to the paper cutouts when working with the AM-FM representation.

Having determined the content of the first transcript to be less relevant to addressing the research questions of interest, I focused my analytic attention on the second transcript. I read through the second transcript a few more times and created a content log (see Appendix C). The content log was condensed into a table that summarized what happened as Neato worked through each case (see Appendix D). In analyzing the cells of the table (which were fairly dense), I produced a second table in which the following six column headings emerged: predictions, prediction justifications, construction, interpretation, order and equivalence, unit and operation. The rows headings corresponded to the case numbers (see Appendix E). These headings intuitively made sense given the kind of questions I asked Neato across the different cases and given the kind of activity Neato engaged in while attempting to answer those questions. It was also not surprising to see equivalence, order, unit, and operation in the column headings, as these were the key concepts identified in Chapter 2.

Having arrived at a somewhat digestible table, I attempted to produce a 1-page summary of Neato's knowledge growth and change. The attempt resulted instead in a 3-page summary (see Appendix F). While writing the summary, I repeatedly asked myself what was changing. More specifically: What was Neato attending to and how was this changing as he worked within the designed learning environment? I relied on Neato's activity within the learning environment, as well as his language, gestures, and gaze to make claims about what he was attending to at any given moment. Once I had summarized what I considered to be change in Neato's knowledge, I asked what this change might say about (a) his emergent understanding of rational numbers and fraction multiplication and (b) his emergent understanding of the affordances and constraints of the AM-FM representation.

A similar 3-page summary was developed for Oscar. A number of different story lines emerged within and across the two case studies. I chose to present those story lines for which



I had the richest evidentiary warrants and those which spoke to the conjectures presented in Chapter 4 regarding growth and change in students' knowledge of fraction multiplication. Once the story lines were determined, I work my way back from the analysis table to the transcript in order to produce the content of the two analytic chapters, which I will present next.

## Chapter 6: The Case of Neato

### 6.1 Chapter Overview

Neato was chosen to serve as a case of knowledge growth and change primarily because he showed the greatest gains from pretest to posttest (see Appendix B). The following analysis considers how Neato's knowledge gets coordinated while using (constructing and interpreting) the AM-FM representation and the number chart. To support my claims regarding knowledge growth and change, I will draw on transcript of Neato's clinical interview session during which he worked with the AM-FM representation and the number chart.<sup>15</sup> The analysis will be presented along the five dimensions of the idealized hypothetical exit state of student understanding, (S(e)): (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, (e3) representational fluency for fraction multiplication, (e4) representational fluency for fraction equivalence, and (e5) representational fluency for fraction order. The first part focuses on Neato's construction of fraction multiplication as stretching/shrinking. The second part focuses on Neato's predictions and justifications for final area output that reveal his emergent number sense with fraction multiplication. The third part constitutes the bulk of the analysis and focuses on Neato's representational fluency for fraction multiplication. I present analysis of Neato's learning trajectory for naming final area output and demonstrate Neato's knowledge coordination of fraction multiplication across his use of the AM-FM representation and the number chart. This will be followed by analysis of the context sensitivity of Neato's knowledge coordination as it pertains to his representational fluency for fraction multiplication. Finally, in parts four and five I present the two secondary concepts the learning environment is intended to support: fraction equivalence and fraction order (respectively).

### 6.2 Fraction Multiplication as Stretching/Shrinking (e1): Neato's AM-FM Construction

Each problem is presented to Neato as a case. At the start of the clinical interview, I introduce Neato to a particular AM-FM construction process using case 8 ( $\frac{2}{3}$  of  $\frac{3}{4}$ ) as an example. See Figure 14 for screenshots of the preferred construction process presented to Neato for case 8.<sup>16</sup> The construction process involves: (a) representing the unit whole by moving the x-axis marker line from  $x=0$  to  $x=1$  (which automatically moves the y-axis marker line from  $y=0$  to  $y=1$  thereby resulting in an area model of a  $1 \times 1$  unit whole), (b) representing the second fraction by setting the x-axis divisions at 4 and moving the x-axis marker line from  $x=\frac{4}{4}$  to  $x=\frac{3}{4}$  (thereby resulting in an area model representation of  $\frac{3}{4}$  of 1), and (c) representing the first fraction by setting the y-axis divisions at 3 and moving the y-axis marker line from  $y=\frac{3}{3}$  to  $y=\frac{2}{3}$  (thereby resulting in an area model representation of  $\frac{2}{3}$  of  $\frac{3}{4}$  of 1). In terms of fraction multiplication as stretching/shrinking, you start with the  $1 \times 1$  unit whole area, shrink that unit whole area by  $\frac{3}{4}$ , and then shrink that  $\frac{3}{4}$  area by  $\frac{2}{3}$ .

<sup>15</sup> I will draw on the language, gesture, and gaze captured in the transcript to make claims regarding growth and change in Neato's knowledge.

<sup>16</sup> Recall from Chapter 3 that given  $a/b$  of  $c/d$ , the preferred AM-FM construction process entails taking  $c/d$  of the  $1 \times 1$  unit whole and then taking  $a/b$  of  $c/d$ .

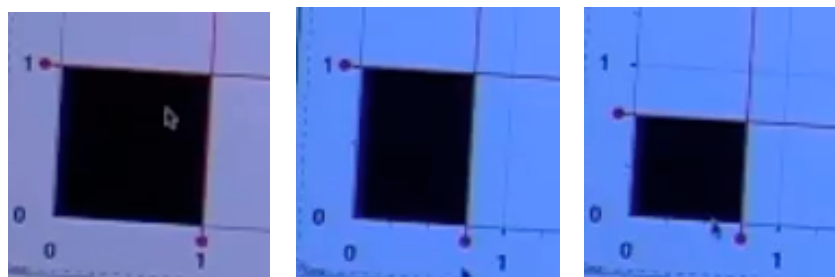


Figure 14. The preferred construction process presented for case 8 ( $2/3$  of  $3/4$ ).

The majority of students (8 of 10) followed this process during their constructions. Neato and Oscar, however, demonstrated a different construction process (Oscar will be discussed in Chapter 7). Neato's process of construction is presented below.

The first case following the example case 8 ( $2/3$  of  $3/4$ ) is case 9 ( $1/2$  of  $1/2$ ). Neato's task in case 9 is to use the AM-FM representation to construct  $1/2$  of  $1/2$  slices of cheese and interpret the final area to arrive at the total amount of cheese distributed in case 9 (i.e.,  $1/4$  slice of cheese). See Figure 15 for screenshots of Neato's construction process for case 9. Neato's construction process proceeds as follows: he represents the unit whole, sets both the x-axis divisions and the y-axis divisions at 4, and moves the x-axis marker line from  $4/4$  to  $1/2$  to  $1/4$ .

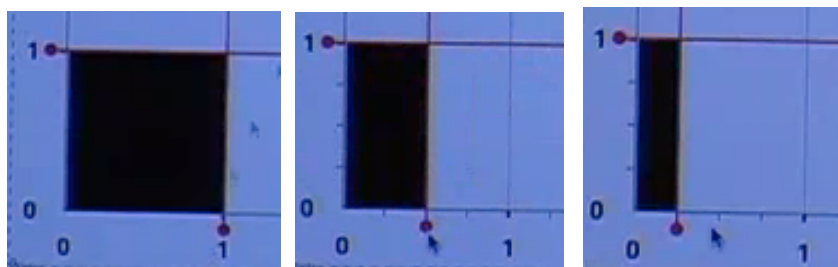


Figure 15. Neato's construction process for case 9 ( $1/2$  of  $1/2$ ).

This is a different construction process than the preferred construction process presented to Neato in case 8. This difference demonstrates Neato's independent thought process. Furthermore, the act of setting the divisions at 4 at the start of the process embodies the operation of denominator multiplication. After Neato completes his AM-FM construction for case 9, I intervene and reintroduce the preferred construction process for case 9.

In case 10 and case 12, Neato represents the second fraction on the x-axis and then the first fraction on the y-axis (similar to the preferred construction process). However, in case 11 ( $1/3$  of  $1/2$ ), Neato represents the first fraction on the y-axis and then the second fraction on the x-axis, taking  $1/2$  of  $1/3$  instead of  $1/3$  of  $1/2$ . Case 13 is interesting in that it reveals the context sensitivity of Neato's nascent understanding of the AM-FM construction process and the commutative property. In case 13 ( $3/5$  of  $3/4$ ), Neato represents the second fraction ( $3/4$ ) on the y-axis and concludes he "did it backwards" incorrectly thinking he made a mistake in his construction. Neato goes on to represent the first fraction ( $3/5$ ) on the y-axis and the second fraction ( $3/4$ ) on the x-axis. See Figure 16 for screenshots of Neato's construction process for case 13. While Neato notices that the final area output of  $a/b$  of  $c/d$  is the same as

the final area output of  $c/d$  of  $a/b$  in the context of using the number chart, the extent to which the commutative property is salient to Neato in the AM-FM representational context is unclear.

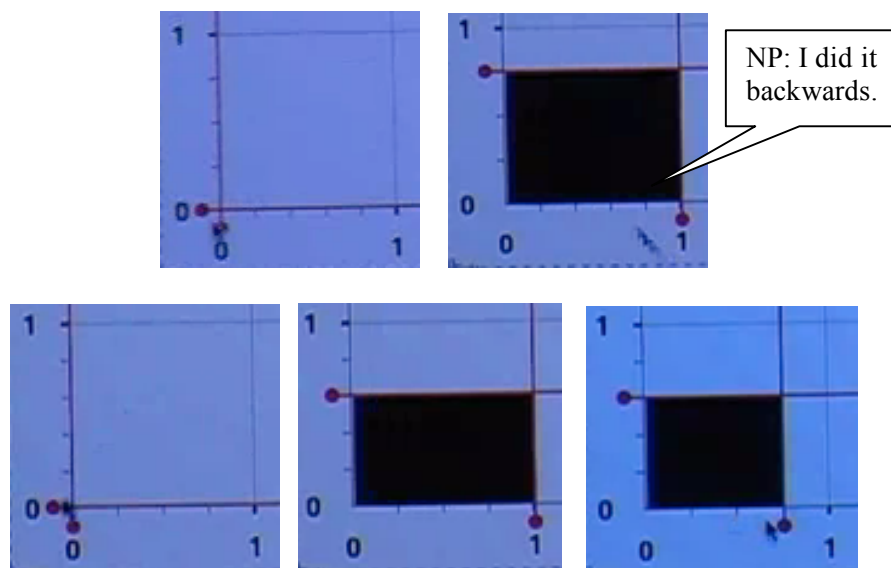


Figure 16. Neato's construction process for case 13 ( $3/5$  of  $3/4$ ).

Following case 13, Neato always represents the first fraction on the y-axis followed by the second fraction on the x-axis just as he did in case 11. Given  $a/b$  of  $c/d$ , Neato chooses to construct  $c/d$  of  $a/b$  by first representing  $a/b$  on the y-axis (stretching or shrinking the unit whole to  $a/b$ ) and then taking  $c/d$  of  $a/b$  by representing  $c/d$  on the x-axis (stretching or shrinking  $a/b$  by  $c/d$ ).

### 6.3 Number Sense with Fraction Multiplication (e2): Neato's Predictions and Justifications

As Neato proceeds through each case he is asked to make predictions. The predictions are always made in comparison to the previous case. For example, I would say to Neato, "We just finished case 9. Now before you use the AM-FM representation to work through case 10, do you think you'll use more or less cheese in case 10 [I point to the number chart input column for case 10] than you did in case 9 [I point to the output value recorded in the number chart output column for case 9]?" See Figure 17 for an illustration of the dialogue and gestures.

Case #	INPUT (units)	INPUT (units)	OUTPUT (units)
8	1 rat	$\frac{2}{3}$ of $\frac{3}{4}$ slice/rat	$\frac{1}{2}$ slice
9	1 rat	$\frac{1}{2}$ of $\frac{1}{2}$ slice/rat	$\frac{1}{4}$ slice
10	1 rat	$\frac{1}{2}$ of $\frac{1}{3}$ slice/rat	
11	1 rat	$\frac{1}{3}$ of $\frac{1}{2}$ slice/rat	
12	1 rat	$\frac{2}{3}$ of $\frac{1}{3}$ slice/rat	
13	1 rat	$\frac{3}{5}$ of $\frac{3}{4}$ slice/rat	
14	1 rat	$\frac{5}{6}$ of $\frac{2}{5}$ slice/rat	
15	1 rat	$\frac{4}{3}$ of $\frac{2}{5}$ slice/rat	
16	1 rat	$2\frac{3}{5}$ of $\frac{2}{5}$ slice/rat	
17	1 rat	$1\frac{2}{5}$ of $\frac{6}{4}$ slice/rat	

“Do you think you’ll use more or less cheese in case 10 than...”

“...you did in case 9?”

Figure 17. Illustration of dialogue and gestures when asking for predictions.

In addition to providing predictions, Neato is also required to justify his predictions. The results of this analysis are presented in Table 8.

Table 8. Summary of Neato’s Predictions

Case	Input	Predictions	Prediction Justifications
8	$\frac{2}{3}$ of $\frac{3}{4}$ slice/rat	N/A	N/A
9	$\frac{1}{2}$ of $\frac{1}{2}$ slice/rat	N/A	N/A
10	$\frac{1}{2}$ of $\frac{1}{3}$ slice/rat	C: < Case 9	C: $\frac{1}{3} < \frac{1}{2}$ using AMFM
11	$\frac{1}{3}$ of $\frac{1}{2}$ slice/rat	C: = Case 10	C: ODM when multiplying
12	$\frac{2}{3}$ of $\frac{1}{3}$ slice/rat	C: > Case 11	C: $\frac{2}{3} > \frac{1}{2}$ using ODM & AMFM
13	$\frac{3}{5}$ of $\frac{3}{4}$ slice/rat	C: > Case 12	I: there are more slices in 5ths (than in 3rds)
14	$\frac{5}{6}$ of $\frac{2}{5}$ slice/rat	I: > Case 13	I: $\frac{5}{6} > \frac{3}{5}$ because only 1 piece from whole (& $\frac{3}{4} > \frac{2}{5}$ because 5ths are smaller than 4ths)
15	$\frac{4}{3}$ of $\frac{2}{5}$ slice/rat	C: > Case 14	C: $\frac{4}{3} = 1\frac{1}{3}$ & $1\frac{1}{3} > \frac{5}{6}$
16	$2\frac{3}{5}$ of $\frac{2}{5}$ slice/rat	C: > Case 15	C: $\frac{4}{3} = 1\frac{1}{3}$ & $2\frac{3}{5} > 1\frac{1}{3}$
17	$1\frac{2}{5}$ of $\frac{6}{4}$ slice/rat	N/A	N/A

KEY: C=Correct; I=Incorrect; N/A=Not Addressed; ODM=Order Doesn’t Matter

Neato’s predictions are correct in 6 out of 7 cases. Neato’s prediction justifications are correct in 5 of the 7 cases. In case 13, Neato makes a correct prediction but gives an incorrect justification for his prediction.

Neato makes correct predictions when only one of the fraction inputs changes from the previous case (cases 10, 12, 15, and 16). In case 10 and case 12, the justifications are correct and grounded in Neato’s use of the AM-FM representation to show area model

representations for the two differing fractions. For example, in comparing case 10 ( $1/2$  of  $1/3$ ) to case 9 ( $1/2$  of  $1/2$ ), Neato uses the AM-FM representation to show that an area model of  $1/3$  is less than an area model of  $1/2$  and therefore case 10 ( $1/2$  of  $1/3$ ) is less than case 9 ( $1/2$  of  $1/2$ ). In case 15 and case 16, the justifications are correct and grounded in Neato's use of a reference point to compare fractions. For example, in comparing case 15 ( $4/3$  of  $2/5$ ) to case 14 ( $5/6$  of  $2/5$ ), Neato converts  $4/3$  to  $1\frac{1}{3}$  and concludes that since  $5/6$  is less than one,  $4/3$  must be greater than  $5/6$  and therefore case 15 ( $4/3$  of  $2/5$ ) is more than case 14 ( $5/6$  of  $2/5$ ). For case 11, the two fraction inputs remain the same and Neato references the commutative property to correctly justify his prediction.

Neato struggles to make correct predictions and justification when both fraction inputs change from the previous case (cases 13 and 14). In case 13, Neato correctly predicts that case 13 ( $3/5$  of  $3/4$ ) would be more than case 12 ( $2/3$  of  $1/3$ ). The justification for his correct prediction is that there are more slices in fifths than in thirds. Neato's justification is based on a comparison of fraction denominators for the first set of fraction inputs (i.e., 5 from  $3/5$  in case 13 compared to 3 from  $2/3$  in case 12) without considering the role of numerators, the second fraction inputs, or the operation of multiplication. In case 14, Neato incorrectly predicts that case 14 ( $5/6$  of  $2/5$ ) would be more than case 13 ( $3/5$  of  $3/4$ ). The justification for his incorrect prediction is that  $5/6$  is greater than  $3/5$  because in  $5/6$  you are only one piece away from one whole (whereas in  $3/5$  you are two pieces away from one whole). Again, Neato only considers the first set of fraction inputs in making the comparison (i.e.,  $5/6$  from case 14 compared to  $3/5$  from case 13) without considering the role of the second fraction inputs or the operation of multiplication. When prompted to consider the second set of fraction inputs (i.e.,  $2/5$  from case 14 and  $3/4$  from case 13) Neato concludes that  $3/4$  is greater than  $2/5$  because fifths are smaller than fourths. While Neato's statements regarding fraction order are correct the justifications for why  $5/6 > 3/5$  and why  $3/4 > 2/5$  are incorrect. The justification for the first statement ( $5/6 > 3/5$ ) is based on an additive relationship between the numerator and denominator. The justification for the second statement ( $3/4 > 2/5$ ) is based on a comparison of fraction denominators.

#### **6.4 Representational Fluency for Fraction Multiplication (e3): Neato's AM-FM and Number Chart Interpretation**

I have discussed Neato's AM-FM construction process, which embodies fraction multiplication as stretching/shrinking and Neato's predictions and justifications for final area output, which reveal his number sense with fraction multiplication. I move now to a discussion of Neato's representational fluency for fraction multiplication. In working with the AM-FM representation, Neato reveals a particular learning trajectory for naming final area outputs. I will present this trajectory to highlight Neato's knowledge coordination across two representations: the AM-FM representation and the number chart. This will be followed by a second analysis in which I discuss the context sensitivity of Neato's emergent knowledge coordination as it pertains to the area model and number line features of the AM-FM representation.

#### 6.4.1 Neato's learning trajectory for interpreting final area output: Knowledge coordination across representations.

The final area output,  $A(f)$ , refers to the area produced after both of the given fraction inputs have been represented on the axes of the AM-FM representation but before the shaded area is tiled. Neato interprets  $A(f)$  for cases 9 through 17. See Table 9 for a summary of Neato's interpretation of  $A(f)$ . In all but three of the cases (i.e., cases 12, 13, and 17) Neato is able to correctly name  $A(f)$ . In case 12 ( $2/3$  of  $1/3$ ), Neato self-corrected his interpretation of  $A(f)$  from  $2/6$  to  $2/9$ . Similarly, in case 13 ( $3/5$  of  $3/4$ ), Neato interprets  $A(f)$  as being  $6/20$  and self-corrected to  $9/20$  while providing a justification for his initial interpretation of  $6/20$ . Finally, in case 17 ( $1\frac{2}{5}$  of  $6/4$ ), Neato struggles to name the correct  $A(f)$ .

Table 9. Summary of Neato's Interpretation of  $A(f)$

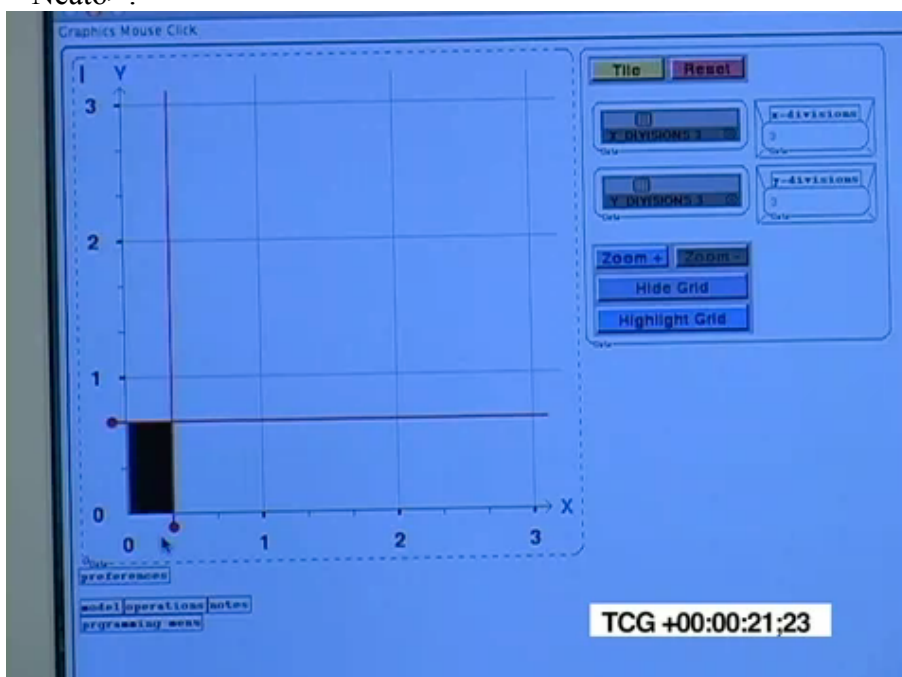
Case	Input	Name and Justification for $A(f)$	Representation Used
8	$2/3$ of $3/4$ slice/rat	N/A	N/A
9	$1/2$ of $1/2$ slice/rat	<u>C</u> : $1/4$ , because you cut whole into 4 and you can turn 1 slice into 4 [pieces]	AM-FM
10	$1/2$ of $1/3$ slice/rat	<u>C</u> : $1/6$ , because 6 of them [black tiles] will make 1 [whole]	AM-FM
11	$1/3$ of $1/2$ slice/rat	<u>C</u> : $1/6$ , because it's equal to case 10	Commutative Property
12	$2/3$ of $1/3$ slice/rat	<u>I</u> then <u>C</u> : $2/6$ to $2/9$ [self-corrects immediately], because there are 2 black tiles and 9 total tiles	AM-FM
13	$3/5$ of $3/4$ slice/rat	<u>I</u> then <u>C</u> : $6/20$ to $9/20$ , [self-correct during justification] because 5 times 4 is 20, oh, no it's $9/20$ [not $6/20$ ]	Number Chart
14	$5/6$ of $2/5$ slice/rat	<u>C</u> : $10/30$ , because 5 times 2 is 10 (also because you could imagine a vertical line at $x = 1/5$ which would cut the black shading in half giving you 10 black tiles) and 6 times 5 is 30	Number Chart & AM-FM
15	$4/3$ of $2/5$ slice/rat	<u>C</u> : $8/15$ , because 4 vertical parts and 2 horizontal parts [makes 8] and 5 times 3 because 5 horizontal parts and 3 vertical parts [makes 15]	AM-FM
16	$2\frac{3}{5}$ of $2/5$ slice/rat	<u>C</u> : $26/25$ , because 13 pieces going up and 2 going across so $13 + 13 = 26$ and 5 times 5 [makes vertical and horizontal motion with arm]	AM-FM
17	$1\frac{2}{5}$ of $6/4$ slice/rat	<u>I</u> then <u>I</u> then <u>C</u> : $2\frac{3}{4}$ to $2\frac{3}{4}$ to $2\frac{3}{20}$	AM-FM

KEY: C=Correct; I=Incorrect; N/A=Not Addressed

Neato's learning trajectory for interpreting  $A(f)$  will demonstrate Neato's knowledge coordination across the AM-FM representation and the number chart. Neato's initial approach entails attending exclusively to features of the AM-FM representation to name and justify  $A(f)$  (see cases 9, 10, and 12)<sup>17</sup>. Neato's second approach entails attending exclusively to features of the number chart to name and justify  $A(f)$  (see case 13). Case 14 marks an interesting shift in Neato's approach in that he begins to attend to features of both the AM-FM representation and the number chart to name and justify  $A(f)$ . This highlights the emergence of Neato's knowledge coordination across the two representations. Finally, in case 15 and case 16, Neato's approach once again entails attending exclusively to features of the AM-FM representation to name and justify  $A(f)$ . However, the way in which Neato attends to those features reveals the completion of Neato's knowledge coordination across the two representations (the AM-FM representation and the number chart). In what follows, I reveal Neato's learning trajectory for interpreting  $A(f)$ . I present an analysis of case 12 to highlight his first approach, case 13 to highlight his second approach, case 14 to highlight a transition in approach, and case 15 to highlight the new approach. The transcript of Neato's justification for  $A(f)$  in case 12 (2/3 of 1/3) is provided below. See Appendix G for the complete transcript of Neato's clinical interview session. This segment of transcript corresponds to transcript lines 1001 through 1023 in Appendix G.

*Neato's Clinical Interview Transcript: Case 12 (2/3 of 1/3)*

RB: Okay. So now um, how much, what is, how much is that? How much cheese did you end up using <references final area of  $2/9$  produced by Neato>?



NP: Um, two sixths. Wait. No. One ninth.

<sup>17</sup> In case 11, Neato appeals to the commutative property to justify his interpretation of  $A(f)$ .



RB: One ninth. How are you getting one ninth?

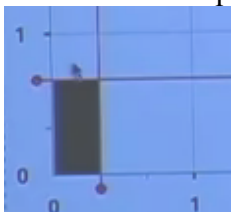
NP: I mean two ninths.

RB: Two ninths. How are you getting two ninths?

NP: Because that's one square <moves y-axis marker line down from  $y=2/3$  to  $y=1/3$ >



and that's two squares <moves y-axis marker line from  $y=1/3$  to  $y=2/3$ >.



RB: Um Hmm.

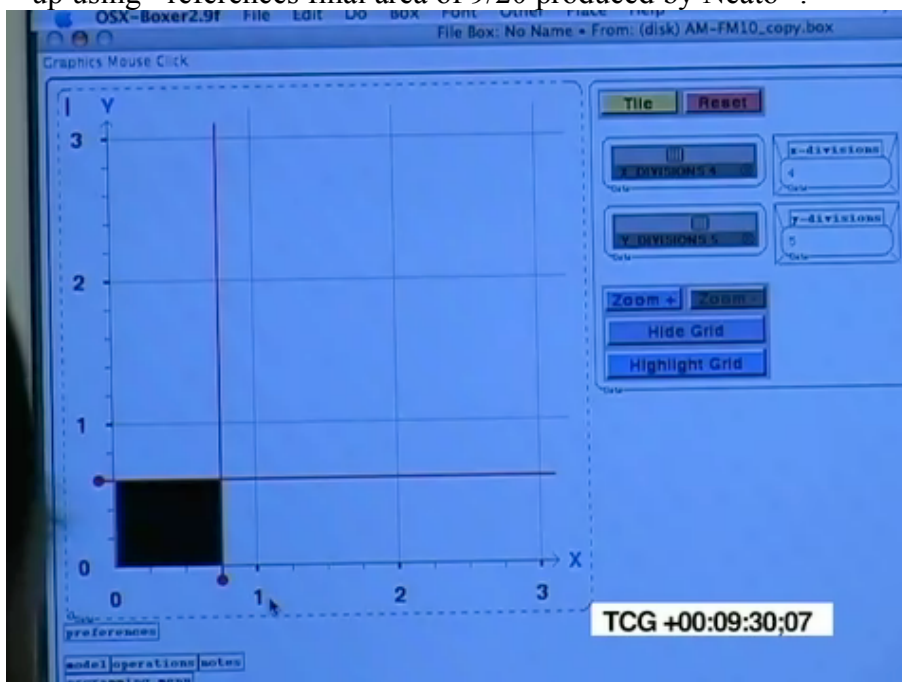
NP: Three square, four squares, five squares <counts on from the 2 tile pieces that make up the shaded region to the remaining tile pieces that make up the 1x1 unit whole>, six squares, seven squares, eight squares....

RB: Nine squares.

In Case 12, Neato names and justifies  $A(f)$  by attending to the “tile” feature of the AM-FM representation. He visualizes and counts shaded tile pieces to arrive at a numerator output of 2 and visualizes and counts total number of tiles pieces that constitute the 1x1 unit whole to arrive at a denominator output of 9. He engages in a similar practice for case 9 and case 10. In case 13, in Neato attends exclusively to features of the number chart to interpret  $A(f)$ . The transcript of Neato’s justification for  $A(f)$  in case 13 ( $3/5$  of  $3/4$ ) is provided below. This segment of transcript corresponds to transcript lines 1277 through 1296 in Appendix G.

*Neato's Clinical Interview Transcript of Case 13 (3/5 of 3/4)*

RB: Okay, okay. Um, so what's our final output? How much cheese do we end up using <references final area of  $9/20$  produced by Neato>?



NP: <looks up into space> Six. Twentieths?

RB: Six twentieths. How did you get that?

NP: That's just a guess.

RB: Six twentieth, how did you guess that? That's an interesting number to just randomly guess.

NP: Well because <looks at number chart> five times four is twenty, Oh no, it's nine twentieths.

RB: Nine twentieths.

NP: It should be nine twentieths.

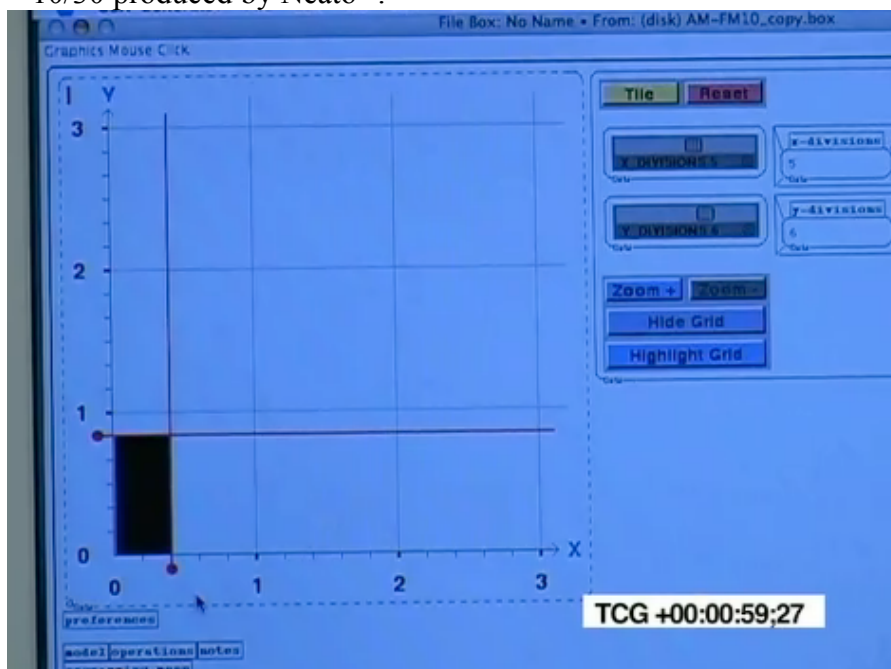
RB: Okay.

In case 12, Neato attends exclusively to the AM-FM representation to name and justify  $A(f)$ . In case 13, Neato attends to the two fraction inputs presented on the number chart. He multiplies the numerators of the two fraction inputs (5 times 4) and the denominators of the two fraction inputs (3 times 3) to name and justify  $A(f)$ . After tiling in case 13, Neato confirms his denominator output by multiplying the number of vertical and horizontal tiles that constitute the  $1 \times 1$  unit whole (see transcript lines 1290 through 1291 in

Appendix G). Case 14, marks a shift in Neato's approach for interpreting  $A(f)$  and highlights the emergence of Neato's knowledge coordination of multiplication across his use of the AM-FM representation and the number chart. The transcript of Neato's justification for  $A(f)$  in case 14 ( $5/6$  of  $2/5$ ) is provided below. This segment of transcript corresponds to transcript lines 1581 through 1617 in Appendix G.

*Neato's Clinical Interview Transcript: Case 14 (5/6 of 2/5)*

RB: Okay, excellent. Um, so now, how much is that <references final area of  $10/30$  produced by Neato>?



NP: That's um, <17 second pause> five sixths of two fifths.

RB: Um hmm. So how much cheese did we use? What's our output?

NP: Um, <10 second pause, looks to number chart> ten, it wouldn't be ten because <looks to AM-FM representation> yeah, no, yeah, yeah, ten thirtieths.

	INPUT (units)	OUTPUT (units)
rat	$\frac{2}{3}$ of $\frac{3}{4}$ slice/rat	$\frac{1}{2}$ slice
rat	$\frac{1}{2}$ of $\frac{1}{2}$ slice/rat	$\frac{1}{4}$ slice
1 rat	$\frac{1}{2}$ of $\frac{1}{3}$ slice/rat	$\frac{1}{6}$ slice
1 rat	$\frac{1}{3}$ of $\frac{1}{2}$ slice/rat	$\frac{1}{6}$ slice
1 rat	$\frac{2}{3}$ of $\frac{1}{3}$ slice/rat	$\frac{2}{9}$ slice
1 rat	$\frac{3}{5}$ of $\frac{3}{4}$ slice/rat	$\frac{9}{20}$ slice
1 rat	$\frac{5}{6}$ of $\frac{2}{5}$ slice/rat	$\frac{10}{30} = \frac{1}{3}$

RB: So I noticed that you looked over here first <points to the number chart> and you were looking at these numbers and you said ten. So were you multiplying across?

NP: Yeah, multiplying across.

RB: And so then you went back <points at shaded region> and said, it can't be ten, but then...

NP: But then I looked at this line <uses cursor to point to  $X=1/5$  and the imaginary vertical line that would result from that point> because I forgot that line was there and I was like yeah, it's going to be ten. Because you have five going down, cut it in half and so you have ten.

RB: So ten. Out of how many?

NP: <looks to number chart> Thirty.

RB: Thirty. And now again you looked at these numbers <points to number chart> when you said thirty.

NP: Yeah.

RB: So how did you know that? How did you get thirty?

NP: Because <looks up into space> six times five is thirty.

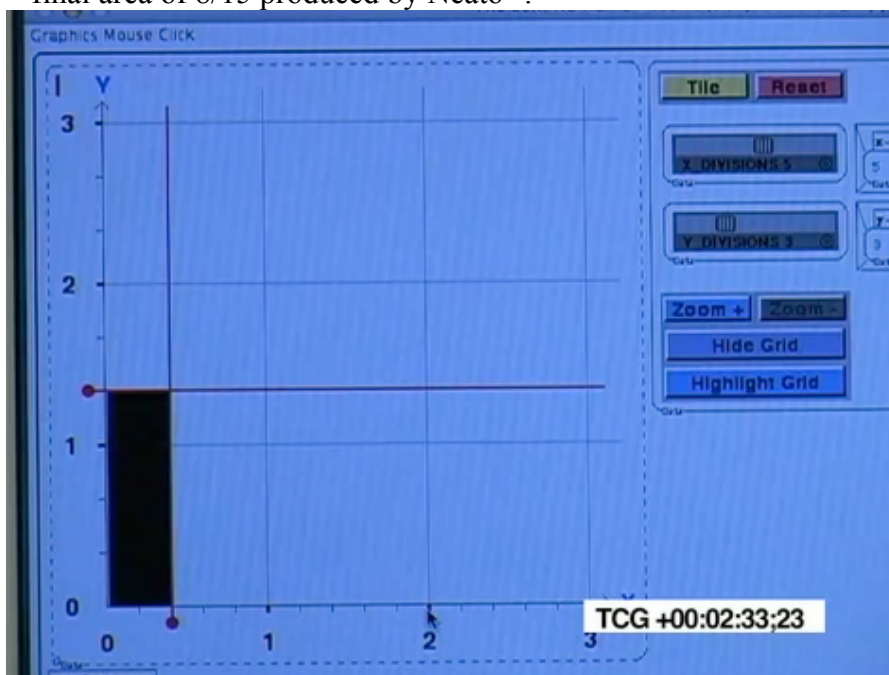
RB: Six times five is thirty, okay.

Case 14 is interesting in that Neato continues to look to the number chart and multiply the denominators of the two fraction inputs (6 times 5) to justify the denominator output of 30

(similar to case 13) but he provides two justifications for the numerator output of 10. He looks first to the number chart and multiplies the numerators of the two fraction inputs (2 times 5) to arrive at 10. He then looks to the AM-FM representation to confirm this result. At first he doesn't notice the vertical partition at  $x=1/5$  and visualizes and counts the shaded area as 5 tile pieces that move up the y-axis. In the process of correcting this error Neato arrives at a new way of attending to the features of the AM-FM representation. Neato counts the 5 vertical tile pieces and the 2 horizontal tile pieces that make up the shaded area and multiplies the counts to arrive at a numerator output of 10 that is consistent with the result obtained from multiplying the numerators of the two fraction inputs on the number chart. Neato appears to be on the verge of coordinating his knowledge of multiplication across both representations. In case 15, Neato demonstrates a new approach for naming and justifying  $A(f)$  by again attending exclusively to features of the AM-FM representation. This new approach highlights the completion of Neato's knowledge coordination of multiplication across his use of the AM-FM representation and the number chart. The transcript of Neato's justification for  $A(f)$  in case 15 ( $4/3$  of  $2/5$ ) is provided below. This segment of transcript corresponds to transcript lines 2160 through 2180 in Appendix G.

*Neato's Clinical Interview Transcript: Case 15 ( $4/3$  of  $2/5$ )*

RB: Right? Um, okay, so how much cheese did we end up using <references final area of  $8/15$  produced by Neato>?



NP: <10 second pause> Let's see. So... One two three four <uses cursor to point out where the yellow shaded tiles would be moving up the Y-axis> and this is cut in half <referencing the  $1/5$  mark that would split the tiles in half>, It would equal eight. Eight, eight something.

RB: Of what?

NP: This is, um, eight fifteenths. Yeah.

RB: Eight fifteenths. How did you get fifteen?

NP: Because five <moves cursor across x-axis which is partitioned into fifths> times, well <moves cursor up and down y-axis which is partitioned into thirds from  $y=0$  to  $y=1$ >...

RB: Five times three?

NP: Three is fifteen.

In case 15, Neato visualizes, counts, and multiplies the 2 horizontal tile pieces and the 4 vertical tile pieces that constitute the shaded region in order to arrive at a numerator output of 8. Similarly, Neato visualizes, counts, and multiplies the 5 horizontal tile pieces and the 3 vertical tile pieces that constitute the  $1 \times 1$  unit whole in order to arrive at a denominator output of 15. Neato tackles case 16 ( $2\frac{3}{5}$  of  $2/5$ ) in a similar manner. While Neato returns to attending exclusively to features of the AM-FM representation to name and justify  $A(f)$ , the way in which he attends to those features reveals coordination of Neato's knowledge of the operation of multiplication as embodied in his use of both the AM-FM representation and the number chart.

Before I move to a discussion of the context sensitivity of Neato's knowledge coordination, I provide a summary of the analysis presented above. Cases 12, 13, 14 and 15 demonstrate Neato's trajectory for interpreting  $A(f)$ . In phase one (case 12), Neato attends exclusively to the AM-FM representation. He visualizes and counts shaded tiles to total tiles that constitute the  $1 \times 1$  unit whole to name and justify  $A(f)$ . In phase two (case 13), Neato attends exclusively to the number chart. He multiplies the two fraction inputs to name and justify  $A(f)$ . In phase three (case 14), Neato attends to features of both the AM-FM representation and the number chart. He looks to the number chart to name and justify his denominator output by multiplying the denominators of the two fraction inputs. In order to name and justify his numerator output, he looks first to the number chart and multiplies the numerators of the two fraction inputs. Then he looks to the AM-FM representation to confirm the result. Similar to the approach he used in case 12, Neato attempts to visualize and count shaded tiles to arrive at a numerator output. However, the count produces a result that contradicts the numerator output Neato arrived at while attending to the number chart. In order to make sense of these different results, Neato develops a new way of attending to features of the AM-FM representation. Neato visualizes, counts, and multiplies the number of horizontal and vertical tile pieces that constitute the shaded region to arrive at a numerator output. He arrives at a result that confirms the results achieved by multiplying the numerators of the two fraction inputs on the number chart. Finally, in phase four (case 15), Neato visualizes, counts, and multiplies the number of horizontal and vertical tile pieces that constitute the shaded region to arrive at a numerator output and he visualizes, counts, and multiplies the total number of horizontal and vertical tile pieces that constitute the  $1 \times 1$  unit whole to arrive at a denominator output. Neato appears to recognize and apply the operation

of multiplication in his use of both the number chart and the AM-FM representation. However, Neato's emergent knowledge for interpreting  $A(f)$  is context sensitive.

#### 6.4.2 Area model and number line: Context sensitivity of knowledge coordination

Next, I present analysis that demonstrates the context sensitivity of Neato's knowledge coordination of fraction multiplication across the AM-FM representation and the number chart. I will show how the area model and number line features of the AM-FM representation constrain Neato's ability to correctly coordinate his knowledge of fraction multiplication across the two given representations.

When Neato works with the AM-FM representation, there are two instances during which he is asked to interpret area. The first is the initial area output,  $A(i)$ , which corresponds to an area model representation of one of the given fraction inputs. The second is the final area output,  $A(f)$ , which corresponds to an area model representation of the product of the two given fraction inputs. In the first instance, the shaded area corresponds to the location of the marker line. For example, if you move the y-axis marker line from the default position  $y=0$  to  $y=3/7$ , the total shaded area will be  $3/7$  (the x-axis marker line jumps from the default position  $x=0$  to  $x=1$ , similarly for the y-axis marker line if the x-axis marker line is moved from  $x=0$  to  $x=3/7$ ). However, in the second instance, the shaded area does NOT correspond to the location of a marker line (unless one or both marker lines are positioned at 1). For example, if after moving the y-axis marker line to  $y=3/7$  you move the x-axis marker line from  $x=1$  to  $x=2/3$  (taking  $2/3$  of  $3/7$  of 1) the total area is neither  $3/7$  nor  $2/3$  but  $6/21$  ( $2/3$  of  $3/7$ ). See Figure 18 for an illustration of the correspondence between the area model and number line features of the AM-FM representation with  $2/3$  of  $3/7$ . In the case of  $A(i)=3/7$  there exists a direct correspondence between area and the location of the y-axis marker line ( $y=3/7$ ). In the case of  $A(f)=6/21$  there exists no correspondence between area and the location of either marker lines ( $y=3/7$  and  $x=2/3$ ).

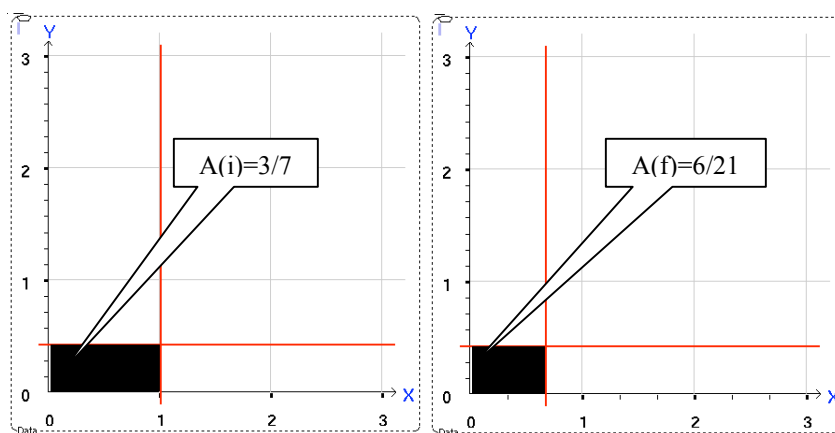


Figure 18. Example of  $2/3$  of  $3/7$  where  $A(i)=3/7$  (area/number line correspondence) and  $A(f)=6/21$  (no correspondence).

Understanding the relationship between the area model (as represented in the form of the coordinate grid) and the number line (as represented in the form of the x-axis and y-axis) is pivotal in order to coordinate knowledge of fraction multiplication across the AM-FM representation and the number chart. The analysis below will be presented in two parts. The

first part will focus on case 13 and how Neato comes to coordinate his knowledge of the area model and number line features of the AM-FM representation when naming A(i). The second part will focus on how Neato's emergent knowledge coordination of the area model and number line features of the AM-FM representation in the context of naming A(i) constrains his ability to correctly name A(f) in case 17 and coordinate his knowledge of fraction multiplication across the AM-FM representation and the number chart.

There are 3 instances in which Neato is asked to interpret A(i) (cases 13, 14, and 15). See Table 10 for a summary of Neato's interpretation of A(i). This is an extension of Table 9 presented previously.<sup>18</sup> The column of interest is highlighted.

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<sup>18</sup> Neato was not asked to justify his interpretation of A(i) as he was for A(f), thereby limiting the data available and constraining a more complete analysis.



Table 10. Summary of Neato's Interpretation of A(i)

Case	Input	Name for A(i)	Name and Justification for A(f)	Representation Used
8	2/3 of 3/4 slice/rat	N/A	N/A	N/A
9	1/2 of 1/2 slice/rat	N/A	<u>C</u> : 1/4, because you cut whole into 4 and you can turn 1 slice into 4 [pieces]	AM-FM
10	1/2 of 1/3 slice/rat	N/A	<u>C</u> : 1/6, because 6 of them [black tiles] will make 1 [whole]	AM-FM
11	1/3 of 1/2 slice/rat	N/A	<u>C</u> : 1/6, because it's equal to case 10	Commutative Property
12	2/3 of 1/3 slice/rat	N/A	<u>I then C</u> : 2/6 to 2/9 [self-corrects immediately], because there are 2 black tiles and 9 total tiles	AM-FM
13	3/5 of 3/4 slice/rat	<u>I</u> : 2/3 (not 3/5)	<u>I then C</u> : 6/20 to 9/20, [self-correct during justification] because 5 times 4 is 20, oh, no it's 9/20 [not 6/20]	Number Chart
14	5/6 of 2/5 slice/rat	<u>C</u> : 5/6	<u>C</u> : 10/30, because 5 times 2 is 10 (also because you could imagine a vertical line at $x=1/5$ which would cut the black shading in half giving you 10 black tiles) and 6 time 5 is 30	Number Chart & AM-FM
15	4/3 of 2/5 slice/rat	<u>C</u> : $1 \frac{1}{3}$	<u>C</u> : 8/15, because 4 vertical parts and 2 horizontal parts [makes 8] and 5 times 3 because 5 horizontal parts and 3 vertical parts [makes 15]	AM-FM
16	$2 \frac{3}{5}$ of 2/5 slice/rat	N/A	<u>C</u> : 26/25, because 13 pieces going up and 2 going across so $13 + 13 = 26$ and 5 times 5 [makes vertical and horizontal motion with arm]	AM-FM
17	$1 \frac{2}{5}$ of 6/4 slice/rat	N/A	<u>I then I then C</u> : $2 \frac{6}{4}$ to $2 \frac{3}{4}$ to $2 \frac{2}{20}$	AM-FM

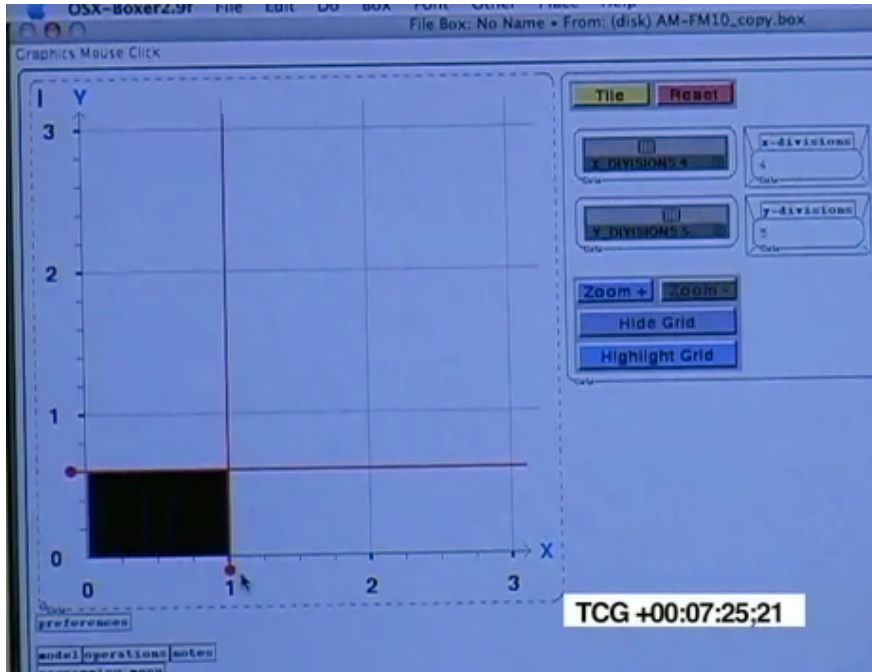
KEY: C=Correct; I=Incorrect; N/A=Not Addressed

In case 13, Neato positions the y-axis marker line at  $y=3/5$  and incorrectly interprets A(i) as  $2/3$  instead of  $3/5$ . Neato eventually arrives at a correct interpretation of  $3/5$ . I refer to this as the first “oh” moment.<sup>19</sup> In the next two cases, case 14 and case 15, Neato is again asked to interpret A(i) and shows no difficulty in understanding the one-to-one correspondence between the location of the marker line and A(i). Transcript of Neato's interpretation of A(i) in case 13 ( $3/5$  of  $3/4$ ) and the first “oh” moment is provided below. This segment of transcript corresponds to transcript lines 1196 through 1223 in Appendix G.

<sup>19</sup> The second “oh” moment will be discussed in section 6.5.

*Neato's Clinical Interview Transcript: Case 13 (3/5 of 3/4) and the First "Oh" Moment*

RB: What is that piece called <references initial area of 3/5 produced by Neato>?



NP: It would be, you have.

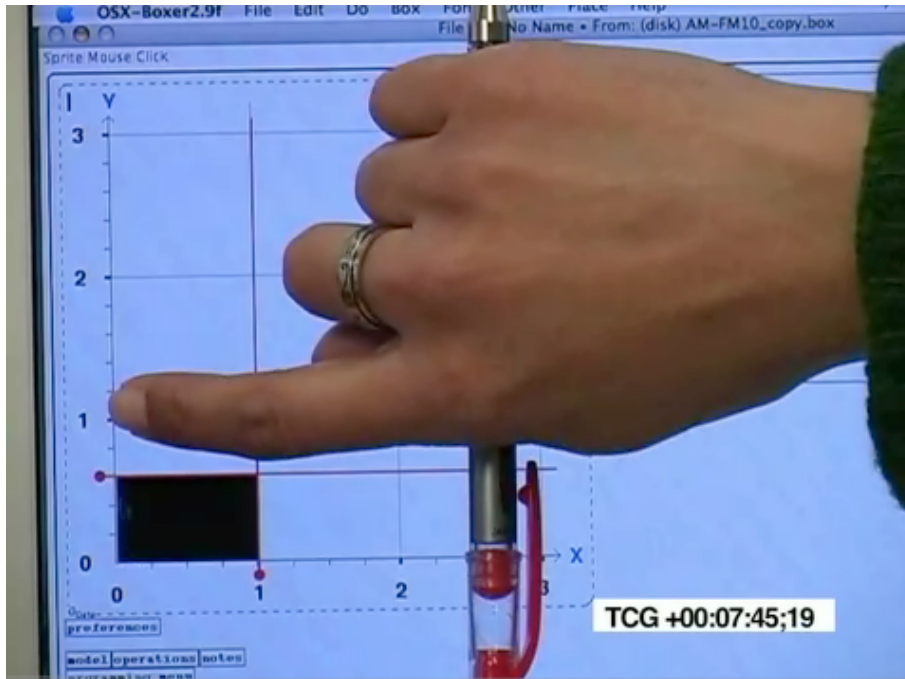
RB: How much of a slice?

NP: Two thirds, I think.

RB: Two thirds? Why is it called two thirds? So this axis <points to Y-axis> represents slices right?

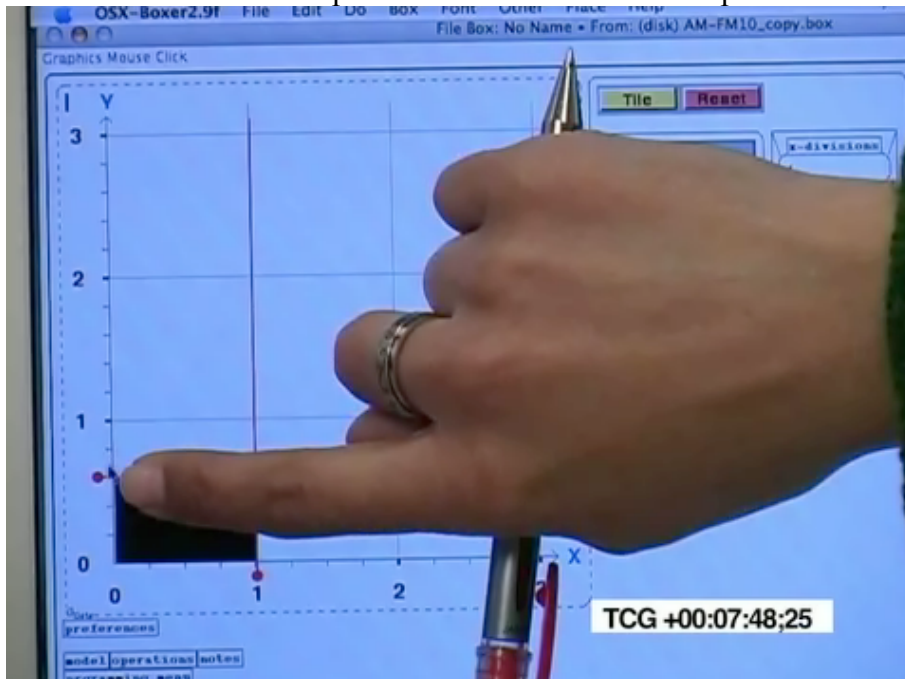
NP: Right.

RB: So if it went all the way up here <points to Y=1> it would be one.



NP: One.

RB: How much is this? <points to  $Y=3/5$ > What is this point called?



NP: The point is called three fifths right now.

RB: Three fifths? So how many slices did you take?

NP: Three fifths?

RB: Three fifths.

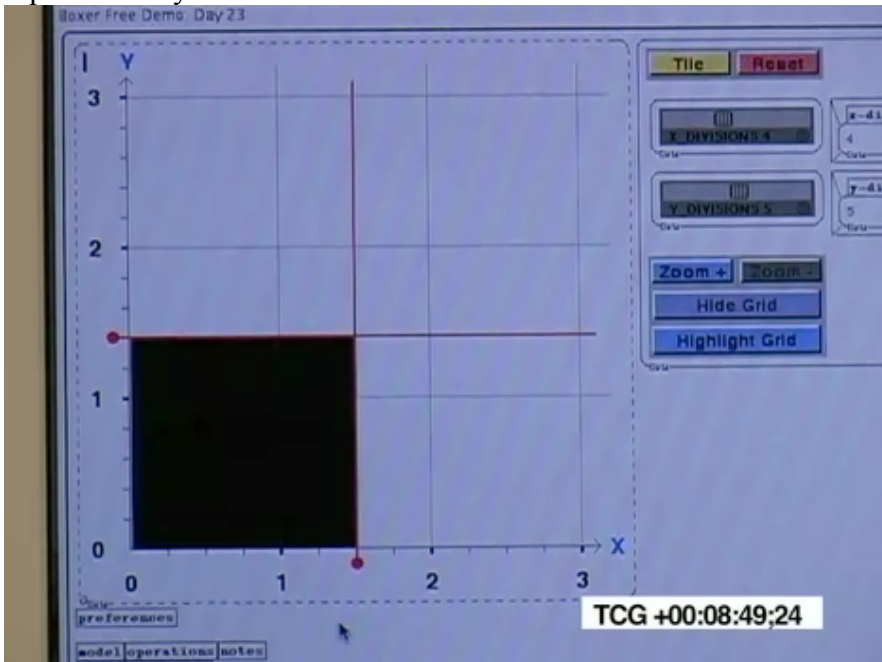
NP: Oh <chuckles>.

In case 13, Neato incorrectly names  $A(i)$  as  $2/3$  after which point I intervene. First, I reference the one-to-one correspondence between the location of the marker line and  $A(i)$  when  $y=1$ . Then, I change the initial question of “what is that piece called” to “what is this point called.” Following Neato’s response of “the point is called  $3/5$  right now” I return to the question of “how many slices did you take.” Neato arrives at an  $A(i)$  interpretation of  $3/5$  but remains uncertain of the correctness of his response (as is evident by his questioning tone). Once I confirm his response, Neato responds with “oh” and a slight chuckle. Following case 13, Neato is able to correctly name  $A(i)$  each time he is asked to do so. See case 14 and case 15 in Table 10. In case 14, Neato is asked to represent  $5/6$  of  $2/5$ . He follows his general construction process of representing the first fraction ( $5/6$ ) on the  $y$ -axis and correctly interprets  $A(i)$  as  $5/6$ . Similarly, in case 15, Neato is asked to represent  $4/3$  of  $3/5$  and correctly interprets  $A(i)$  as  $1\frac{1}{3}$ . In the context of naming  $A(i)$ , Neato’s knowledge of the relationship between the area model and number line features of the AM-FM representation appear to be coordinated. Neato appears to demonstrate emergent understanding of the one-to-one correspondence between the location of the marker line and  $A(i)$ .

Before I continue with the analysis of Neato’s AM-FM interpretation, I would like to take stock. I began this analysis with Neato’s learning trajectory for naming  $A(f)$  and highlighted Neato’s knowledge coordination of the operation of multiplication across his use of the AM-FM representation and the number chart. This was followed by an analysis of case 13, in which I demonstrated Neato’s knowledge coordination of the area model and number line features of the AM-FM representation when naming  $A(i)$ . Next, I will discuss the context sensitivity of Neato’s emergent knowledge coordination of fraction multiplication across the AM-FM representation and the number chart. I will highlight how Neato’s knowledge coordination of the area model and number line features of the AM-FM representation when naming  $A(i)$  constrains his ability to correctly name  $A(f)$  in case 17. The transcript of Neato’s justification for  $A(f)$  in case 17 ( $1\frac{2}{5}$  of  $6/4$ ) is provided below. This segment of transcript corresponds to transcript lines 2616 through 2672 in Appendix G.

*Neato's Clinical Interview Transcript: Case 17 (1 2/5 of 6/4)*

RB: Okay, so how much cheese do we give out <references final area of 42/20 produced by Neato>?



NP: <chuckles> Um, that's a lot <12 second pause>.

RB: <RB makes side comment to camera person>.

NP: <chuckles> I don't know, unless I count all the boxes.

RB: Unless you count all the boxes?

NP: Yeah.

RB: Is it more than one?

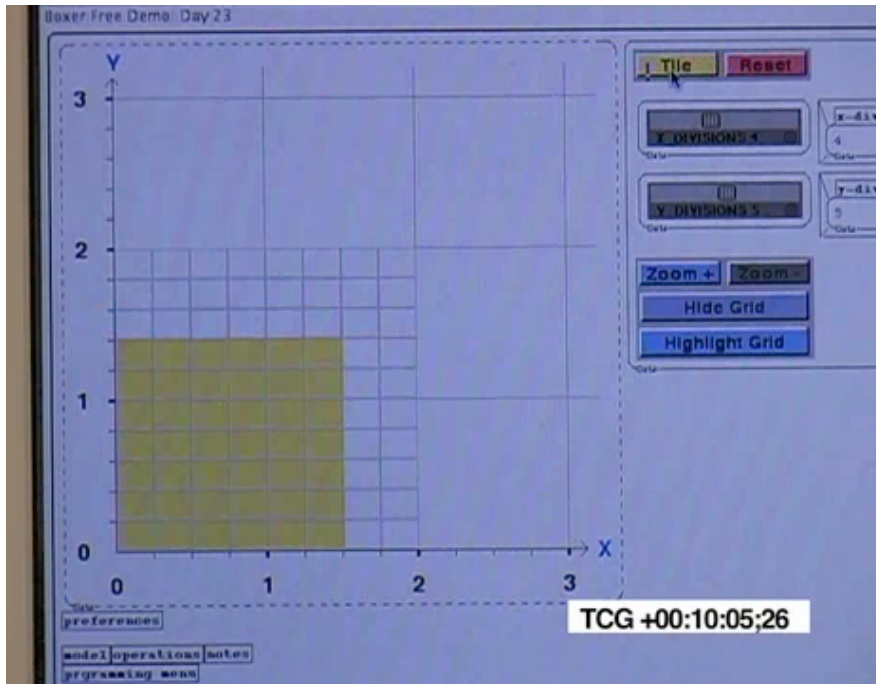
NP: Yeah, it's more than one.

RB: Do you think it's going to be more than two?

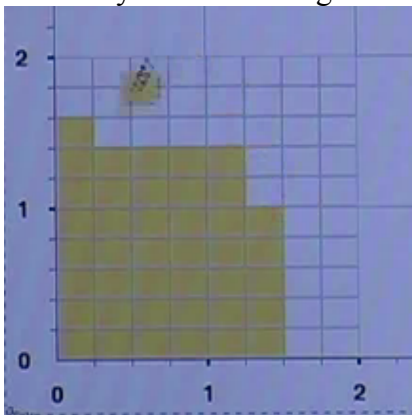
NP: No.

RB: No? No. Okay, why don't you tile?

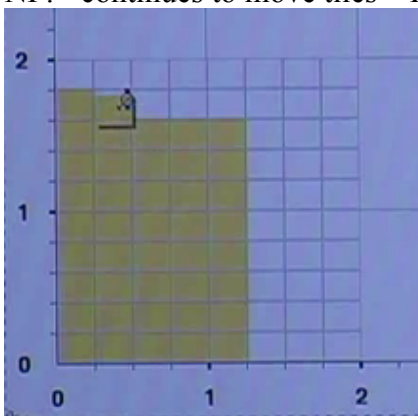
NP: <hits tile button>



NP: Okay <starts moving a few tiles>. Maybe it is more than two.



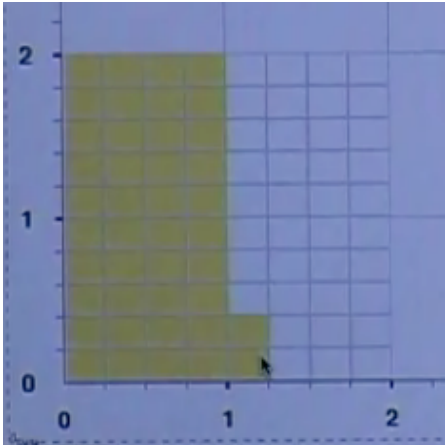
NP: <continues to move tiles> Yeah, it's more than two.



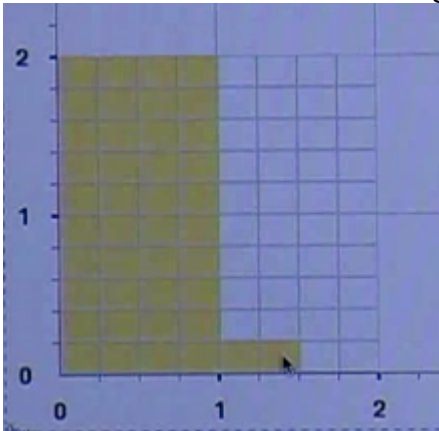
RB: It is?

NP: <finishes moving tiles> Yeah.

RB: So how much is that?



NP: So it's two and <moves a single tile>.



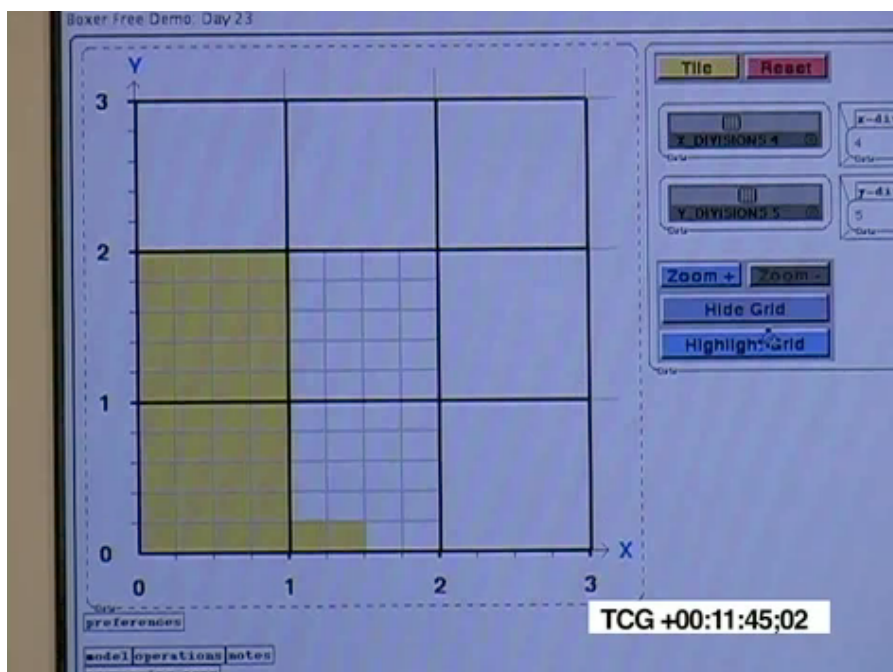
N: And <8 second pause> six fourths. Two and six fourths.

RB: Two and six fourths. Okay, how are you getting the two and six fourths?

NP: Because you have two <uses cursor to point out the two tiled wholes>, right and then one, two, three, four, five, six <counts the six  $\frac{1}{4}$  line segments across the x-axis from first position of first tile to position of last tile>.

RB: So that's how much cheese you gave out?

NP: Two and six fourths <hits highlight grid button>.



RB: How much cheese did you give out?

NP: Oh, two and two fourths.

RB: Two fourths.

NP: Yeah.

In case 17 ( $1\frac{2}{5}$  of  $6/4$ ), Neato is unable to name  $A(f)$  before tiling. After tiling he believes  $A(f)$  is greater than one but less than two. While in the process of moving tile pieces he self-corrects and states that  $A(f)$  will be greater than two. Prior to case 17, Neato demonstrates no difficulty in naming  $A(f)$  before tiling. In case 16 ( $2\frac{3}{5}$  of  $2/5$ ), Neato arrives at a correct interpretation of  $26/25$ . He visualizes, counts, and multiplies the number of horizontal and vertical tile pieces that constitute the shaded region to arrive at a numerator output of 26 and he visualizes, counts, and multiplies the total number of horizontal and vertical tile pieces that constitute the  $1 \times 1$  unit whole to arrive at a denominator output of 25. In case 17, Neato does not apply the same knowledge. Instead he wants to count all the tile pieces so he tiles the area, moves the tiles to fill two wholes, and places the two remaining tiles into a third whole. In the process of naming  $A(f)$ , Neato un-stacks the two vertically stacked tiles in the third whole so they lay flat across the x-axis. Neato then goes on to name  $A(f)$  as  $2\frac{2}{4}$ . He arrives at 2 from the 2 tiled wholes. He arrives at  $6/4$  by counting the six line segments (fourths) from where the first tile starts along the x-axis ( $x=0$ ) to where the last tile ends along the x-axis ( $x=6/4$ ). When I repeat the same question, Neato spontaneously uses the highlight grid function and changes his answer to  $2\frac{2}{4}$ . Again he arrives at 2 from the 2 tiled wholes. He arrives at  $2/4$  by counting the two line segments (fourths) starting from  $x=4/4$  and ending at  $x=6/4$ . In the first interpretation,  $A(f)=2\frac{2}{4}$ , Neato appears to apply his emergent knowledge of the one-to-one correspondence between location of the marker line and  $A(i)$  to



the incorrect context of interpreting  $A(f)$ . After Neato moves the tile pieces, the last tile across the x-axis ends at  $x=6/4$ . In the second interpretation,  $A(f)=2\frac{2}{4}$ , Neato uses the highlight grid function of the AM-FM representation. He shifts the zero location of the x-axis by one unit and again applies his knowledge of the one-to-one correspondence between the location of the marker line and  $A(i)$  to the incorrect context of interpreting  $A(f)$ . While this was appropriate knowledge to invoke in the context of interpreting  $A(i)$ , the one-to-one correspondence does not hold in the context of interpreting  $A(f)$  (unless one or both marker lines are positioned at 1) and it does not hold once  $A(f)$  is tiled and tiled pieces are moved.

### **6.5 Representational Fluency for Fraction Equivalence (e4)**

As Neato proceeds through each case he uses the AM-FM representation to name equivalent fractions. A summary of Neato's interpretation of fraction equivalence is presented in Table 11. Because fraction equivalence is often explored in the context of locating and naming fractions using the AM-FM representation, a summary of fractions names generated by Neato is also included in Table 11. To make sense of fraction equivalence, Neato attends either to the number line feature of the AM-FM representation (marked "NL" in Table 11) or the area model feature of the AM-FM representation (marked "AM" in Table 11).

Table 11. Summary of Neato's Interpretation of Fraction Equivalence

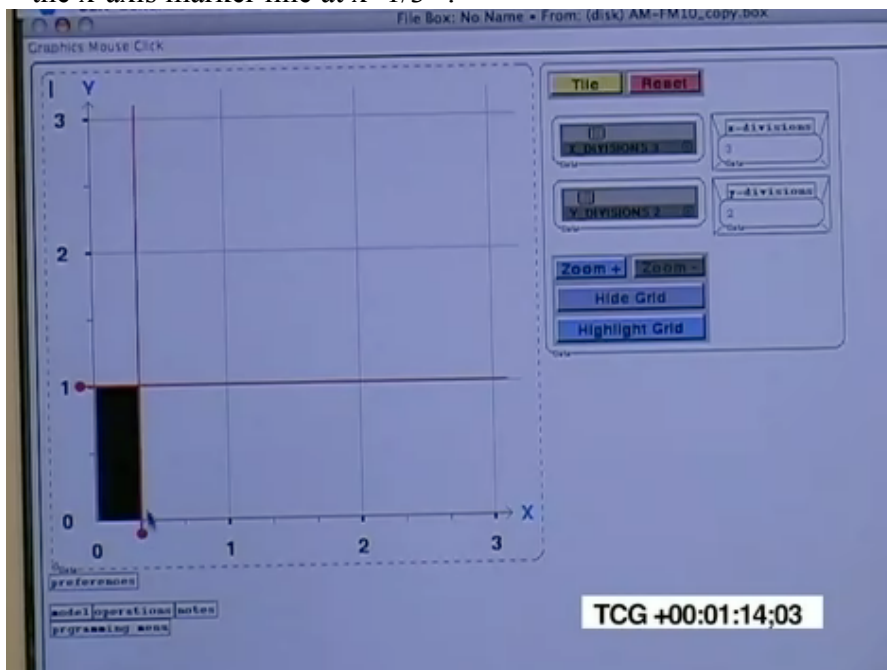
Case	Input	C	I	Representation Used
9	1/2 of 1/2 slice/rat			N/A
10	1/2 of 1/3 slice/rat	X X X X X X X	X	NL: 1/3 NL: 2/3 NL: 3/3=1 NL: 1 1/3 NL 1 1/3=? => Intervention NL: 4/3=1 1/3 NL: 5/3 NL: 6/3=2
11	1/3 of 1/2 slice/rat			N/A
12	2/3 of 1/3 slice/rat			N/A
13	3/5 of 3/4 slice/rat			N/A
14	5/6 of 2/5 slice/rat	X X X	X	NL: 6/6 NL: 8/6=1 1/3 AM: 10/30 ≠ 1/3 => Intervention AM: 10/30=1/3
15	4/3 of 2/5 slice/rat	X X X X X X X		NL: 1/5 NL: 0/5 NL: 5/8 NL: 6/5 NL: 2=10/5 NL: 15/5 NL: 4/3=1 1/3
16	2 3/5 of 2/5 slice/rat	X		AM: 26/25=1 1/25
17	1 3/5 of 6/4 slice/rat			N/A

KEY: C=Correct; I=Incorrect; N/A=Not Addressed; AM=Area Model; NL=Number Line

There are three instances during which Neato is asked to name points on the number line (cases 10, 14, and 15). In case 10 (1/2 of 1/3), Neato correctly names the points 1/3, 2/3, 3/3, and 1 1/3. He also recognizes that 3/3 is equivalent to 1 whole. But he struggles to give a fraction equivalent to 1 1/3 (i.e., 4/3). This results in an intervention designed to support Neato's understanding of number lines. During this intervention Neato has a second "oh" moment. In case 14 and case 15, Neato is again asked to name points on a number line and give equivalent fractions for certain points. In both cases Neato shows no difficulty in naming equivalent fractions. I will present transcript of case 10 (1/2 of 1/3) as an example of how Neato comes to use the number line feature of the AM-FM representation to make sense of fraction equivalence. This segment of transcript corresponds to transcript lines 361 through 448 in Appendix G.

*Neato's Clinical Interview Transcript: The Second "Oh" Moment ( $1\frac{1}{3} = 4/3$ )*

RB: Okay, and where is that place you positioned it <references the position of the x-axis marker line at  $x=1/3$ >?



NP: In...it's one third.

RB: It's one third, and what is this point called <points to  $2/3$  on X-axis> on the number line?

NP: Two thirds.

RB: Two thirds, and what would this point be called if we were using thirds <point to one on the X-axis>?

NP: One whole but or three thirds.

RB: Three thirds. And this point would be called <points to one and one third>.

NP: One and one third.

RB: And if I wanted it as an improper fraction? If I wanted it...so this is <points to one third> one third, <points to two thirds> two thirds, <points to one> three thirds, <points to one and one fourth>...

NP: One and one third.

RB: Which could be called, what's another name for one and one third?

NP: Um.

RB: Is there another name for that fraction?

NP: Um, not that I know of.

[Intervention]

RB: So one and one third <takes a piece of paper and writes “ $1\frac{1}{4}$ ” instead of  $1\frac{1}{3}$ >, right? So we have a number line, <draws a number line on the paper> we have zero, we have one, two, three <locates “0”, “1”, “2”, and “3” on the number line>.

NP: You put four instead of three, you put one and one fourth <points to a “ $1\frac{1}{4}$ ” that was written on the paper>?

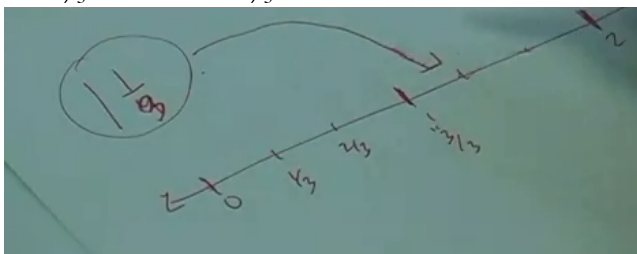
RB: What do you mean I put... <circles the “ $1\frac{1}{4}$ ” that was written on the paper> oh no, this is just a fraction. So we have, so you have it split, this into thirds, right <references the partitioning on the x-axis of the AM-FM representation>?

NP: Right.

RB: This is one third, this is two thirds, this you said is equal to three thirds <partitions the line segment between zero and one into thirds and marks them as “ $1/3$ ”, “ $2/3$ ”, and  $1 = 3/3$ >. Right?

NP: Right.

RB: So, this <points to “ $1\frac{1}{4}$ ”> oh I see what you are saying. Thank you. One and one thirds, right <changes “ $1\frac{1}{4}$ ” to “ $1\frac{1}{3}$ ” and draws an arrow from “ $1\frac{1}{3}$ ” to where  $1\frac{1}{3}$  would be located on the number line>?



NP: Right.

RB: Okay, but if I wanted it like in this form <points to “ $3/3$ ”> instead of having a mixed...this is called a mixed number, right <points to “ $1\frac{1}{3}$ ”>?

NP: Right.

RB: Because you have a whole number <points to the “1” in “ $1\frac{1}{3}$ ”> and you have a fraction, a proper fraction <points to “ $\frac{1}{3}$ ” in “ $1\frac{1}{3}$ ”>. And these are just <circles “ $\frac{1}{3}$ ”, “ $\frac{2}{3}$ ”, “ $\frac{3}{3}$ ” on the number line> called proper fractions, right?

NP: Right.

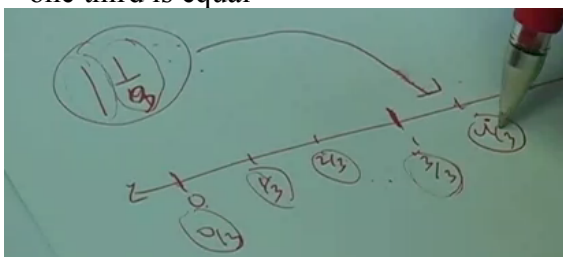
RB: But it’s just two numbers, one over the other. If I wanted that kind of number here <points to “ $1\frac{1}{3}$ ” on number line> what would it be? So look at the pattern. And this would be...how many thirds would this be <points to zero>?

NP: It would be zero.

RB: Zero thirds <writes “ $\frac{0}{3}$ ” under “0”>

NP: Oh, it would be, um, four thirds.

RB: It would be four thirds <writes “ $\frac{4}{3}$ ” on the number line> See that one and one third is equal-



NP: To four-

RB: To four thirds.

Prior to the intervention, I verbalize a pattern for naming  $\frac{1}{3}$ ,  $\frac{2}{3}$ ,  $\frac{3}{3}$  in an attempt to get Neato to see  $1\frac{1}{3}$  as  $\frac{4}{3}$ . But Neato fails to name a fraction equivalent to  $1\frac{1}{3}$ . While working with a drawn number line representation I again verbalize a pattern for naming  $\frac{1}{3}$ ,  $\frac{2}{3}$ ,  $\frac{3}{3}$  and explicitly ask Neato to attend to the pattern. When I name and labels  $\frac{0}{3}$ , Neato responds with “Oh, it would be, um,  $\frac{4}{3}$ .” Neato goes on to name  $\frac{5}{3}$  as well as  $\frac{6}{3}=2$ . Following the “oh” moment in case 10, Neato demonstrates no difficulty in naming fractions on the number line or generating equivalent fractions for various points on the number line (see cases 14 and 15).

Neato not only uses the number line feature of the AM-FM representation to name equivalent fractions, he also uses the area model feature. See cases 14, 15, and 16 in Table 11 regenerated below.

Table 11. Summary of Neato's Interpretation of Fraction Equivalence

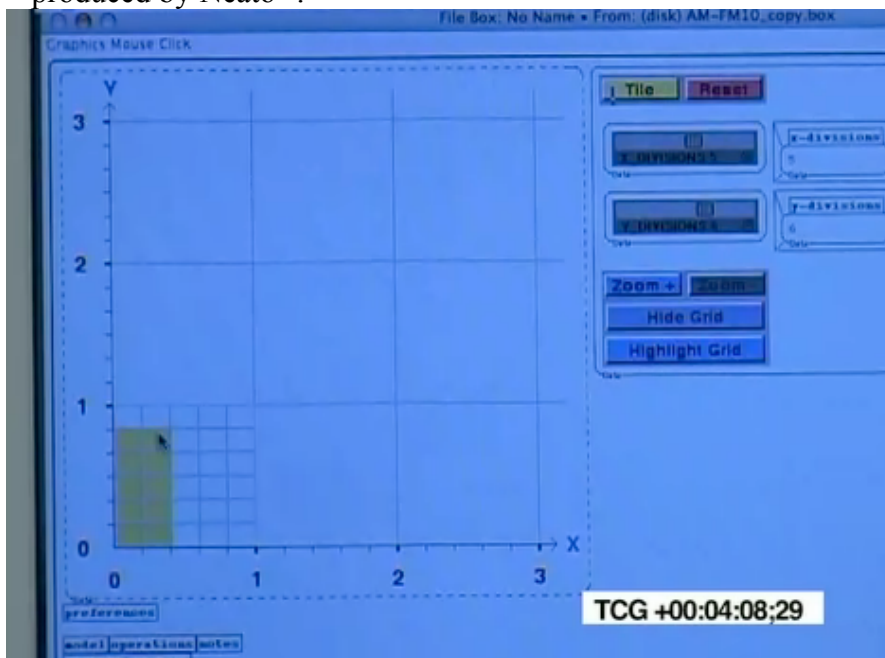
Case	Input	C	I	Representation Used
9	1/2 of 1/2 slice/rat			N/A
10	1/2 of 1/3 slice/rat	X X X X X X X	X	NL: 1/3 NL: 2/3 NL: 3/3=1 NL: 1 1/3 NL 1 1/3=? => Intervention NL: 4/3=1 1/3 NL: 5/3 NL: 6/3=2
11	1/3 of 1/2 slice/rat			N/A
12	2/3 of 1/3 slice/rat			N/A
13	3/5 of 3/4 slice/rat			N/A
14	5/6 of 2/5 slice/rat	X X X	X	NL: 6/6 NL: 8/6=1 2/6 AM: 10/30 ≠ 1/3 => Intervention AM: 10/30=1/3
15	4/3 of 2/5 slice/rat	X X X X X X X		NL: 1/5 NL: 0/5 NL: 5/8 NL: 6/5 NL: 2=10/5 NL: 15/5 NL: 4/3=1 1/3
16	2 3/5 of 2/5 slice/rat	X		AM: 26/25=1 1/25
17	1 3/5 of 6/4 slice/rat			N/A

KEY: C=Correct; I=Incorrect; N/A=Not Addressed; AM=Area Model; NL=Number Line

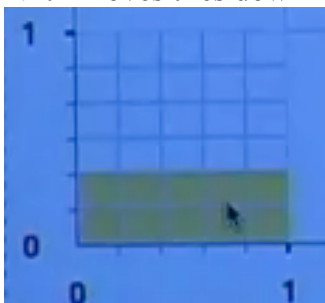
It is often the case that  $A(f)$  can be reduced. To arrive at an equivalent fraction for  $A(f)$ , Neato moves tiles either within the  $1 \times 1$  unit whole or across two  $1 \times 1$  unit wholes. In case 14 ( $5/6$  of  $2/5$ ), Neato uses the move function of the AM-FM representation to reduce  $10/30$  to  $1/3$  by counting tiles in groups of 10. In case 16 ( $2 \frac{3}{5}$  of  $2/5$ ), Neato uses the move function to reduce  $26/25$  to  $1 \frac{1}{25}$  by tiling one  $1 \times 1$  unit whole with one tile left over in another  $1 \times 1$  unit whole. In case 15 ( $4/3$  of  $2/3$ ), Neato correctly states that  $A(f)=8/15$  cannot be reduced and uses the move feature of the AM-FM representation to show that the tiles cannot be chunked into groups such that the size of the group covers (exactly) both the 8 tiles and the 15 total spaces that constitute the  $1 \times 1$  unit whole. I present transcript of case 14 as an example of how Neato uses the area model feature of the AM-FM representation to make sense of fraction equivalence. This segment of transcript corresponds to transcript lines 1657 through 1682 in Appendix G.

Neato's Clinical Interview Transcript: Case 14 (5/6 of 2/5 = 10/30 = 1/3)

RB: I want to know if you can call that area something else, other than ten thirtieths. Can you call it something else <references tiled area of 10/30 produced by Neato>?



NP: <moves tiles down>



NP: Two thirds

RB: Two thirds.

NP: Yeah.

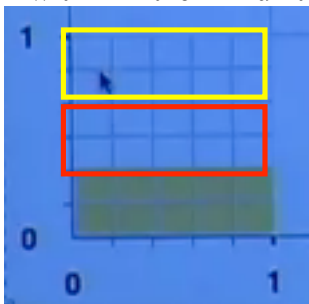
RB: How did you get that?

NP: Because you have, this is one <uses cursor to point to the two tiled rows that run from X=0 to X=1>.



RB: Um hmm.

NP: And then this is two, like these two rows would be like another one <uses cursor to point the two rows above the two tiled rows within the 1x1 unit>, these two count as another one <uses cursor to point to the top two rows within in the 1x1 unit>, so it would be, one, one, one third. Yeah, one third.



RB: One third.

NP: Yeah, one third.

In case 14, Neato initially interprets  $A(f)=10/30$  as being equivalent to  $2/3$  but in providing his justification for this statement he concludes the  $10/30$  is equivalent to  $1/3$  and not  $2/3$ . Neato moves tiles within the  $1 \times 1$  unit whole such that he can count tiles in groups of 10. He reasons that since there are 3 groups of 10 in 30 and only 1 of those groups is tiled,  $10/30$  is equivalent to  $1/3$ . Neato demonstrates no difficulty in generating equivalent fractions using the area model feature of the AM-FM representation (see also cases 15 and 16).

### 6.6 Representational Fluency for Fraction Order (e5)

Having discussed equivalence across Neato's use of both the number line and area model features of the AM-FM representation, I turn now to the concept of fraction order. See Table 12.



Table 12. Summary of Neato's Interpretation of Fraction Order

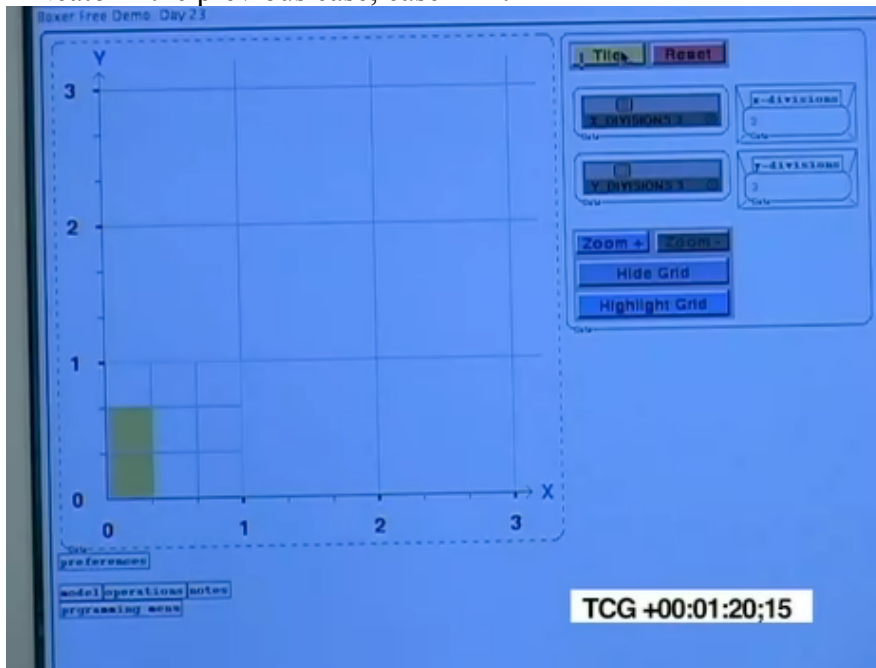
Case	Input	C	I	Representation Used
9	1/2 of 1/2 slice/rat			N/A
10	1/2 of 1/3 slice/rat			N/A
11	1/3 of 1/2 slice/rat			N/A
12	2/3 of 1/3 slice/rat	X X X X X		U: $2/3 > 1/3$ U: $1/2 > 1/3$ U: $2/3 = \text{half of } 3/3 \Rightarrow \text{Self-Corrects}$ NL&AM: $2/3 > 1/2$ NL&AM: $2/9 > 1/6$
13	3/5 of 3/4 slice/rat	X X X	X	U: $3/4 > 1/3$ U: $3/4 = 1/3$ NL&AM: $3/4 > 1/3$ NL&AM: $3/4 > 2/3$
14	5/6 of 2/5 slice/rat	X		U: $5/6 > 2/5$
15	4/3 of 2/5 slice/rat			N/A
16	2 3/5 of 2/5 slice/rat			N/A
17	1 3/5 of 6/4 slice/rat			N/A

KEY: C=Correct; I=Incorrect; N/A=Not Addressed; AM=Area Model; NL=Number Line; U=Representational Context is Unclear

When comparing two fractions Neato uses both the number line and area model features of the AM-FM representation (see case 12 and case 13). For example, after correctly constructing and interpreting A(f) for case 12 ( $2/3$  of  $1/3$ ) Neato is asked if his prediction was correct. In other words, is A(f) in case 12 ( $2/3$  of  $1/3$ ) greater than A(f) in case 11 ( $1/3$  of  $1/2$ ). Neato coordinates his use of the number line and area model features of the AM-FM representation show that A(f)= $2/9$  is indeed greater than A(f)= $1/6$ . Transcript of Neato's prediction verification for case 12 ( $2/3$  of  $1/3$ ) is provided below. This segment of transcript corresponds to transcript lines 1017 through 1040 in Appendix G.

Neato's Clinical Interview Transcript: Case 12 Prediction Verification ( $2/3$  of  $1/3 = 2/9 > 1/6 = 1/3$  of  $1/2$ )

RB: Pretty nice, excellent. Okay, I'll write that  $2/9$  down? So was that more or less cheese than one sixth  $\langle$ RB references final area of  $1/6$  produced by Neato in the previous case, case 11 $\rangle$ ?



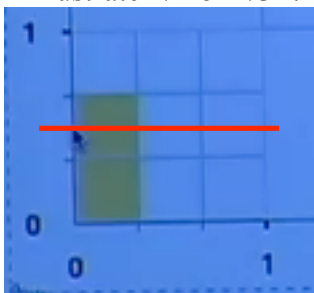
NP: One...wait. More.

RB: It was more?

NP: Yeah.

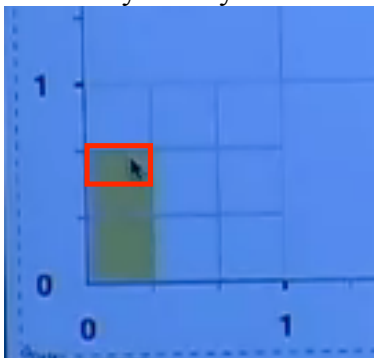
RB: How do you know  $\langle$ that  $2/9$  is greater than  $1/6$  $\rangle$ ?

NP: Because it's...because half would be like there  $\langle$ using the cursor to draw an imaginary horizontal line at  $Y=1/2$  through the  $1 \times 1$  unit whole in order to illustrate  $1/2$  of  $1/3$  $\rangle$ .



RB: Um hmm.

NP: And you have this little... <uses cursor to point out the tile half that is left over if you only took 1/2 of 1/3 instead of 2/3 of 1/3>.



RB: Little strip left over.

NP: Little strip left over.

In case 12 (2/3 of 1/3), Neato works with the AM-FM representation to generate  $A(f)=2/9$  by setting the y-axis marker line at 2/3 and the x-axis marker line at 1/3. When asked if his output was indeed greater than  $A(f)=1/6$  from case 11 (1/3 of 1/2), he indexes an imaginary y-axis marker line at  $y=1/2$  to show that 1/6 would result in less shaded area than 2/9 because there would be a “little strip left over.”

Of course growth and change in knowledge is context sensitive; what one “knows” in one context is not necessarily used in another. Neato demonstrates this phenomenon when asked to compare case 12 (2/3 of 1/3) to case 11 (1/3 of 1/2) and then again when asked to compare case 13 (3/5 of 3/4) to case 12 (2/3 of 1/3). In the first instance, Neato incorrectly states that 2/3 (the first fraction input from case 12) is half of 3/3 but quickly self-corrects. In the second instance, Neato correctly states that  $3/4 > 1/3$  and then incorrectly states that  $3/4 = 1/3$ . When making these statements it is unclear what Neato is attending to.

## 6.7 Case Study Discussion

In this discussion, I consider how the analysis above addresses the larger research questions of interest. In other words, what does the analysis say about (a) Neato’s understanding of rational numbers and fraction multiplication (i.e. the development of domain competence) and (b) Neato’s understanding of the affordances and constraints of the AM-FM representation (i.e. the development of representational competence). The discussion will be presented in four parts: (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, (e3) representational fluency for fraction multiplication, and (e4/e5) representational fluency for fraction equivalence and fraction order. The discussion will be revisited in Chapter 8, where I consider the broader implications of design-based research: (a) local theory development and (b) refinement of the designed learning environment.

### 6.7.1 Fraction multiplication as stretching/shrinking.

Neato developed his own construction process for arriving at  $A(f)$  that differed from the preferred construction process presented to him. He initially used only the x-axis to

represent both fraction inputs and in doing so embodied the operation of denominator multiplication. This revealed a certain level of independent thought process in Neato's use of the AM-FM representation. However, I intervened in Neato's construction process by re-introducing the preferred construction process. The intervention resulted in Neato's use of the y-axis to represent the first fraction input and the x-axis to represent the second fraction input. In other words, given  $a/b$  of  $c/d$ , Neato constructed  $c/d$  of  $a/b$  by first representing  $a/b$  on the y-axis (stretching or shrinking the unit whole to  $a/b$ ) and then taking  $c/d$  of  $a/b$  by representing  $c/d$  on the x-axis (stretching or shrinking  $a/b$  by  $c/d$ ). When asked about this switch, Neato was able to correctly articulate that it did not matter which fraction you represented first as both construction processes would result in the same  $A(f)$ . This recognition indexed Neato's emergent understanding of the commutative property, but this understanding was context sensitive. In one case, when given  $a/b$  of  $c/d$ , Neato started by representing  $c/d$  along the x-axis and claimed, "I did it backwards."

Recall Figure 11 presented in Chapter 4. In making sense of fraction multiplication as stretching/shrinking, Neato appeared to draw on his emergent knowledge of area models (i1), number lines (i2), fraction as part-whole (i4Ka), and fraction as measure (i4Kb). Data did not reveal knowledge of multiplication as repeat addition (i4Kd).<sup>20</sup>

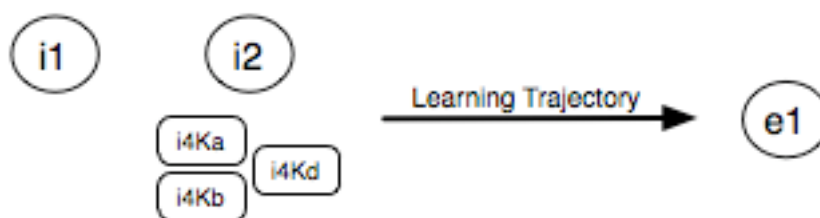


Figure 11. Transformation from S(i) to fraction multiplication as stretching/shrinking (e1).

### 6.7.2 Number sense with fraction multiplication.

When only one fraction changed from one case to the next, Neato provided correct predictions and justifications by adequately using the AM-FM representation or by applying his knowledge of a reference point such as one. Neato struggled to make accurate justifications when both fractions changed from one case to the next. In such instances, Neato's justifications were based on a comparison of the first fraction inputs from each case (without considering the role of the second set of fraction inputs or the operation of multiplication). Furthermore, while Neato was correct in his comparison of two fraction inputs, his justifications for arriving at a particular comparison were incorrect. His reasoning was based on either an additive understanding of the relationship between numerator and denominator or a comparison of the two denominators (without considering the role of the numerators).

Recall Figure 12 from Chapter 4. Neato's number sense with fraction multiplication appeared to draw on his emergent knowledge of area models (i1), number lines (i2), algorithm for fraction multiplication (i3Ka), algorithm for fraction equivalence (i3Kb), algorithm for

<sup>20</sup> This was due (in part) to the fact that I did not provide students with an opportunity to explicitly express their knowledge of fraction as repeat addition within the designed learning environment.

fraction order (i3Kc), fraction as part-whole (i4Ka), and fraction as measure. Data did not reveal knowledge of fraction as quotient (i4Kc) and multiplication makes bigger (i4Ke).<sup>21</sup>



Figure 12. Transformation from S(i) to number sense with fraction multiplication (e2).

### 6.7.3 Representational fluency for fraction multiplication.

Neato demonstrated a particular trajectory for naming A(f). In phase one, Neato attended to the AM-FM representation. He visualized and counted shaded tiles to total tiles that constituted the 1x1 unit whole. In phase two, Neato attended to the number chart. He multiplied the two fraction inputs. In phase three, Neato attended to both the AM-FM representation and number chart. First, he looked to the number chart and multiplied the denominators of the two fraction inputs to arrive at a denominator output. Next, he looked to the AM-FM representation to confirm the result. When Neato attempted to visualize and count shaded tiles to arrive at a numerator output the count produced a result that contradicted Neato's previous result. In order to make sense of the discrepancy, Neato developed a new way of attending to features of the AM-FM representation. Neato visualized, counted, and multiplied the number of horizontal and vertical tile pieces that constituted the shaded region to arrive at a numerator output. In doing so, he confirmed the results he obtained from multiplying the numerators of the two fraction inputs while attending to the number chart. Finally, in phase four, Neato again attended to the AM-FM representation. He visualized, counted, and multiplied the number of horizontal and vertical tile pieces that constituted the shaded region to arrive at a numerator output and he visualized, counted, and multiplied the total number of horizontal and vertical tile pieces that constituted the 1x1 unit whole to arrive at a denominator output.

This trajectory highlighted Neato's knowledge coordination of fraction multiplication as embodied in his use of two representations, the AM-FM representation and the number chart. The AM-FM representation afforded Neato the opportunity to (a) apply his knowledge of the part-whole subconstruct in the context of counting shaded tiles to total tiles (that make up the 1x1 unit) to arrive at A(f), and (b) recognize and apply the operation of multiplication in the context of multiplying vertical and horizontal shaded tiles to vertical and horizontal total tiles (within the 1x1 unit whole) to arrive at A(f). The number chart afforded Neato the opportunity to apply his prior knowledge of an algorithm for fraction multiplication to the given fraction inputs in order to arrive at A(f). In the process of using the AM-FM

<sup>21</sup> This was due (in part) to the fact that I did not provide students with many opportunities to explicitly express their knowledge of fraction as quotient and fraction makes bigger within the designed learning environment.

representation and the number chart to name and justify  $A(f)$ , Neato came to coordinate his knowledge of fraction multiplication.

Neato also came to coordinate his knowledge of the area model and number line features of the AM-FM representation when naming  $A(i)$ , which later constrained his ability to correctly name and justify  $A(f)$ . Recall that when naming  $A(i)$ , there exists a one-to-one correspondence between shaded area and the location of the marker line. However, this correspondence does not hold when naming  $A(f)$ , unless one or both the marker lines are positioned at one and the tiles remain unmoved.

When Neato was first asked to name  $A(i)=3/5$  he struggled to recognize the one-to-one correspondence between the shaded area,  $A(i)=3/5$ , and the location of the y-axis marker line at  $y=3/5$  (the x-axis marker line was at  $x=1$ ). I intervened with a series of leading questions, an example in the case of  $y=1$ , and the use of phrases like “how much [area] is that” followed immediately by “what is that point called.” The intervention resulted in an “oh” moment after which point Neato demonstrated no difficulty in naming  $A(i)$ . Later, in case 17 ( $1\frac{2}{5}$  of  $6/4$ ), when Neato was asked to interpret  $A(f)=2\frac{2}{20}$ , he tiled  $A(f)$ , moved tiles to fill two unit wholes with two tiles left over in the third unit whole, and incorrectly interpreted  $A(f)=2\frac{2}{4}$ . Neato arrived at  $A(f)=2\frac{2}{4}$  by attending to the two tiled wholes (i.e., attending to the area model feature of the AM-FM representation) and by attending to the location along the x-axis where the tiles ended,  $x=6/4$  (i.e., attending to the number line feature of the AM-FM representation).

Neato’s knowledge coordination of the area model and number line features of the AM-FM representation was emergent and context sensitive. The context sensitive of his knowledge coordination implied that Neato did not have a firm grasp on how to adequately use both the area model and number line features of the AM-FM representation to arrive at  $A(f)$ . Neato was more effective at naming  $A(f)$  when he attended only to the area model features by (a) counting shaded tile to total number of tiles that constitute the  $1\times 1$  unit whole or (b) visualizing, counting, and multiplying the vertical and horizontal shaded tiles to the vertical and horizontal total tiles that constitute the  $1\times 1$  unit whole. It appeared that the area model feature afforded Neato a better opportunity to attend to the  $1\times 1$  unit whole which led Neato to arrive at correct interpretations of  $A(f)$ .

Recall Figure 13 from Chapter 4. Neato’s representational fluency with fraction multiplication appeared to draw on his emergent knowledge of area models (i1), number lines (i2), algorithm for fraction multiplication (i3Ka), fraction as part-whole (i4Ka), and fraction as measure (i4Kb). Data did not reveal knowledge of multiplication as repeated addition (i4Kd) and multiplication makes bigger (i4Ke).<sup>22</sup>

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<sup>22</sup> This was due (in part) to the fact that I did not provide students with many opportunities to explicitly express their knowledge of fraction as repeated addition and fraction makes bigger within the designed learning environment.



Figure 13. Transformation from S(e) to representational fluency for fraction multiplication (e3).

#### 6.7.4 Representational fluency for fraction equivalence and fraction order.

When first asked to name fractions on the number line Neato could not give a fraction equivalent name for  $1\frac{1}{3}$  (i.e.,  $\frac{4}{3}$ ). During an intervention, Neato had a second “oh” moment after which point he demonstrated no difficulty in naming equivalent fractions using the number line feature of the AM-FM representation. Neato appeared to demonstrate knowledge coordination of improper fractions, mixed numbers, and equivalence when using the number line feature of the AM-FM representation. Equivalence was also explored using the area model feature of the AM-FM representation. When A(f) can be reduced Neato was able to rearrange tile pieces in such a ways as to afford seeing equivalence. Neato appeared to demonstrate knowledge coordination of the part-whole subconstruct and equivalence when using the area model feature of the AM-FM representation. Finally, Neato used both the number line and area model features of the AM-FM representation to order fractions. In case 12 ( $\frac{2}{3}$  of  $\frac{1}{3}$ ), Neato worked with the AM-FM representation to generate  $A(f)=\frac{2}{9}$  by setting the y-axis marker line at  $\frac{2}{3}$  and the x-axis marker line at  $\frac{1}{3}$ . In comparing  $\frac{2}{9}$  to  $A(f)=\frac{1}{6}$  from case 11 ( $\frac{1}{3}$  of  $\frac{1}{2}$ ), Neato indexed an imaginary y-axis marker line at  $y=\frac{1}{2}$  to show that  $\frac{1}{6}$  is less than  $\frac{2}{9}$  because there would be a “little strip left over.” Neato appeared to be coordinating his knowledge of fraction order, use of the number line feature of the AM-FM representation, and use of the area model feature of the AM-FM representation.

## Chapter 7: The Case of Oscar

### 7.1 Chapter Overview

After Neato was selected as the subject of the first case study, Oscar was chosen to serve as a contrasting case of knowledge growth and change. Like Neato, Oscar demonstrated gains from pretest to posttest (see Appendix B). However, the content areas in which Oscar demonstrated gains differed from Neato. Furthermore, both Neato and Oscar engaged in construction processes with the AM-FM representation that differed from the preferred construction process presented to them by me. In the case of Neato, I chose to intervene by re-introducing the preferred construction, which had particular implications for Neato's learning trajectory. In the case of Oscar, I chose not to intervene, which had different implications for Oscar's learning trajectory. The following analysis considers how Oscar's knowledge gets coordinated while using (constructing and interpreting) the AM-FM representation. To support my claims regarding knowledge growth and change, I will draw on transcript of Oscar's clinical interview session during which he worked with the AM-FM representation.<sup>23</sup> The analysis will be presented along the five dimensions of the idealized hypothetical exit state of student understanding, (S(e)): (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, (e3) representational fluency for fraction multiplication, (e4) representational fluency for fraction equivalence, and (e5) representational fluency for fraction order. The first part focuses on Oscar's construction of fraction multiplication as stretching/shrinking. The second part focuses on Oscar's predictions and justifications for final area output that reveal his emergent number sense with fraction multiplication. The third part constitutes the bulk of the analysis and focuses on Oscar's representational fluency for fraction multiplication. I present analysis of Oscar's learning trajectory for naming final area output and demonstrate Oscar's knowledge coordination of fraction multiplication across his use of different features of the AM-FM representation. This will be followed by analysis of the context sensitivity of Oscar's knowledge coordination as it pertains to his representational fluency for fraction multiplication. Finally, in parts four and five (respectively) I present the two secondary concepts the learning environment is intended to support: fraction equivalence and fraction order.

### 7.2 Fraction Multiplication as Stretching/Shrinking (e1): Oscar's AM-FM Construction

Oscar's initial construction process entails using only one axis of the AM-FM representation to construct final area output,  $A(f)$ <sup>24</sup>. When confronted with a case in which the product of the denominators is greater than 8, Oscar is forced to use both axes of the AM-FM representation to construct  $A(f)$  (recall from Chapter 3 that the maximum partitions allowed

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<sup>23</sup> I will draw on the language, gesture, and gaze captured in the transcript to make claims regarding growth and change in Oscar's knowledge.

<sup>24</sup> By final area output, I am referring to the area produced after both of the given fraction inputs have been represented on the axes of the AM-FM representation but before the shaded area is tiled.



on an axis of the AM-FM representation is 8). Next, I present an analysis of how Oscar makes the transition from using a single axis to construct  $A(f)$  to using both axes to construct  $A(f)$ .

Each problem is presented to Oscar as a case. At the start of the clinical interview, I introduce Oscar to a particular AM-FM construction process using case 8 ( $2/3$  of  $3/4$ ) as an example. The process involves: (a) representing the unit whole by moving the x-axis marker line from  $x=0$  to  $x=1$  (which automatically moves the y-axis marker line from  $y=0$  to  $y=1$  thereby resulting in an area model of a  $1 \times 1$  unit whole), (b) representing the second fraction by setting the x-axis divisions at 3 and moving the x-axis marker line from  $x=3/3$  to  $x=2/3$  (thereby resulting in an area model representation of  $2/3$  of 1), and (c) representing the first fraction by setting the y-axis divisions at 4 and moving the y-axis marker line from  $y=4/4$  to  $y=3/4$  (thereby resulting in an area model representation of  $3/4$  of  $2/3$  of 1). In terms of fraction multiplication as stretching/shrinking, you start with the  $1 \times 1$  unit whole area, shrink that unit whole area by  $2/3$ , and then shrink that  $2/3$  area by  $3/4$ . The majority of students (8 out of 10) followed this process during their constructions. Oscar and Neato were the two exceptions. Oscar's construction process is presented below.

The first case following the example case 8 ( $2/3$  of  $3/4$ ) is case 9 ( $1/2$  of  $1/2$ ). Oscar's task in case 9 is to use the AM-FM representation to construct  $1/2$  of  $1/2$  slices of cheese and interpret the final area to arrive at the total amount of cheese distributed in case 9 (i.e.,  $1/4$  slice of cheese). Oscar's initial construction process proceeds as follows: he represents the unit whole, sets the y-axis divisions at 3, and moves the y-axis marker line from  $3/3$  to  $1/3$ . See Figure 19 for screenshots of Oscar's initial construction for case 9.

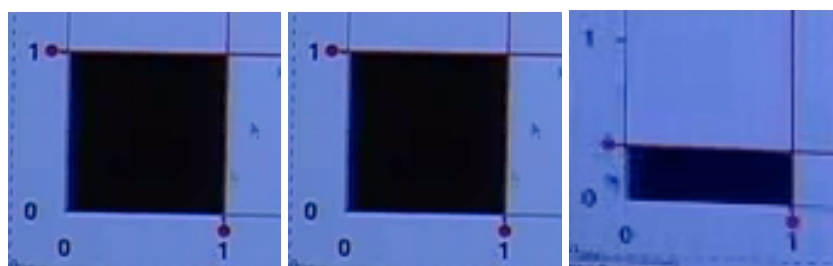


Figure 19. Oscar's initial construction for case 9 ( $1/2$  of  $1/2$ ).

After Oscar sets the y-axis division at 3, I ask Oscar to explain this choice. Oscar states, "Because half has to be around there somewhere <points to  $y=1/2$ > and half of that would be like right there somewhere <points to  $y=1/4$ >." I ask Oscar to repeat himself. Oscar follows with, "Like half of it, if I made it two <OA moves the y-axis division to 2>, it would be right there <points to  $y=1/2$ >. If I made it three <OA moves the y-axis division to 3>, it would be half of that, half of half." Oscar appears to be engaged in a form of estimation when partitioning the single axis of the AM-FM representation.

Following Oscar's initial construction for case 9, I intervene by drawing two area model representations, one depicting  $1/2$  of  $1/2$  and the other depicting  $1/3$ . See Figure 20 for a screenshot of the two area models.

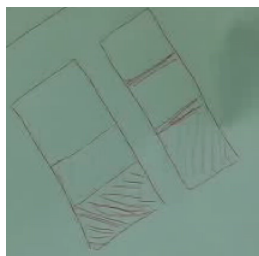


Figure 20. RB's construction of a  $\frac{1}{2}$  of  $\frac{1}{2}$  area model and a  $\frac{1}{3}$  area model.

Oscar concludes that the  $\frac{1}{2}$  of  $\frac{1}{2}$  area is less than the  $\frac{1}{3}$  area and goes on to correctly reconstruct  $\frac{1}{2}$  of  $\frac{1}{2}$  using the AM-FM representation. Oscar's final construction process proceeds as follows: he represents the unit whole, sets the y-axis divisions at 4, and moves the y-axis marker line from  $\frac{4}{4}$  to  $\frac{2}{4}$  to  $\frac{1}{4}$ . See Figure 21 for screenshots of Oscar's final construction for case 9.



Figure 21. Oscar's final construction for case 9 ( $\frac{1}{2}$  of  $\frac{1}{2}$ ).

Both Oscar's initial and final construction process in case 9 are different from the preferred construction process demonstrated in case 8. Rather than intervene by reintroducing the preferred construction process for case 9 (as was done with Neato), I choose to allow Oscar to pursue his construction process. In case 10 ( $\frac{1}{3}$  of  $\frac{1}{2}$ ), Oscar continues to use the y-axis to represent both fraction inputs. He represents the unit whole, sets the y-axis divisions at 6, and moves the y-axis marker line from  $\frac{6}{6}$  to  $\frac{3}{6}$  to  $\frac{1}{6}$ . Given  $a/b$  of  $c/d$ , Oscar represents the unit whole, sets the y-axis divisions at  $bd$ , and moves the y-axis marker from  $\frac{bd}{bd}$  (1) to  $\frac{bc}{bd}$  ( $c/d$ ) to  $\frac{ac}{bd}$  ( $a/b$  of  $c/d$ ). In other words, Oscar represents the unit whole, uses the y-axis to represent  $c/d$  (stretching or shrinking the unit whole to  $c/d$ ) followed by  $a/b$  (stretching or shrinking  $c/d$  by  $a/b$ ).

Oscar construction process of using the y-axis to represent both fraction inputs becomes problematic in case 12 ( $\frac{2}{3}$  of  $\frac{1}{3}$ ) where the product of the denominators exceeds 8. Recall from Chapter 3 that the maximum divisions allowed along the axes of the AM-FM representation are 8. In case 12, Oscar cannot set the y-axis divisions at 9 and struggles to arrive at a correct  $A(f)$ . Foreseeing this difficulty, I had intervened in Neato's construction process, but I choose not to intervene in Oscar construction process. Below, I present an analysis that demonstrates Oscar's struggle to arrive at a correct  $A(f)$  in case 12. This struggle has particular implications for Oscar's representational fluency.

In case 12 ( $\frac{2}{3}$  of  $\frac{1}{3}$ ), Oscar struggles to construct  $A(f)$ . In his first attempt, Oscar tries to set the y-divisions at 9 and realizes it does not go beyond 8. He sets the y-axis divisions at 3, moves the y-axis marker from  $\frac{3}{3}$  to  $\frac{1}{3}$ , changes the y-axis division from 3 to 6 resulting in an area shading of  $\frac{2}{6}$ , changes the y-axis division from 6 to 8 resulting in an

area shading of  $3/8$ , and moves the y-axis marker from  $3/8$  to  $2/8$ .<sup>25</sup> See Figure 22 for screenshots of Oscar's initial construction process for case 12.



Figure 22. Oscar's initial construction for case 12 (2/3 of 1/3).

Following Oscar's initial construction, I draw an area model representation of  $2/3$  of  $1/3$ . See Figure 23 for screenshots of the area model construction for case 12.

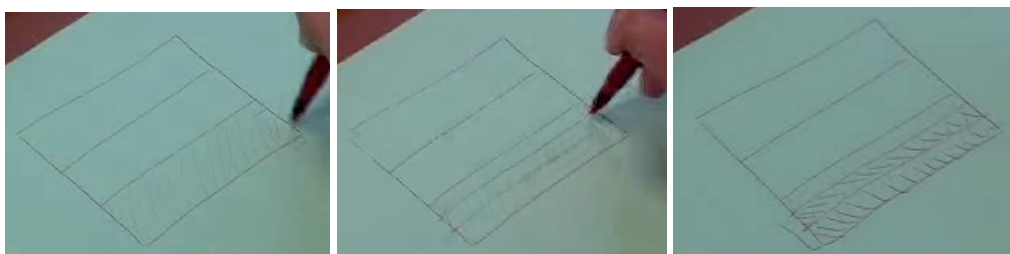


Figure 23. RB's area model construction for case 12 (2/3 of 1/3).

Oscar adds partitions lines to the final area model construction in Figure 23 to create an area model with 9 equal parts and correctly interprets the shaded area as  $2/9$ . When Oscar returns to the AM-FM representation, he continues to struggle to arrive at a correct  $A(f)$  for case 12. In his next attempt, Oscar repeats his previous construction process, but this time he uses the x-axis instead of the y-axis. At this point, I suggest to Oscar that he use both axes of the AM-FM representation. Following the suggestion, Oscar sets the x-division at 3, sets the y-division at 3, moves the y-axis marker line from  $3/3$  to  $1/3$ , moves the x-axis marker line from  $3/3$  to  $2/3$ , and concludes that the final area output is  $2/9$ . See Figure 24 for screenshots of Oscar's final construction process for case 12.

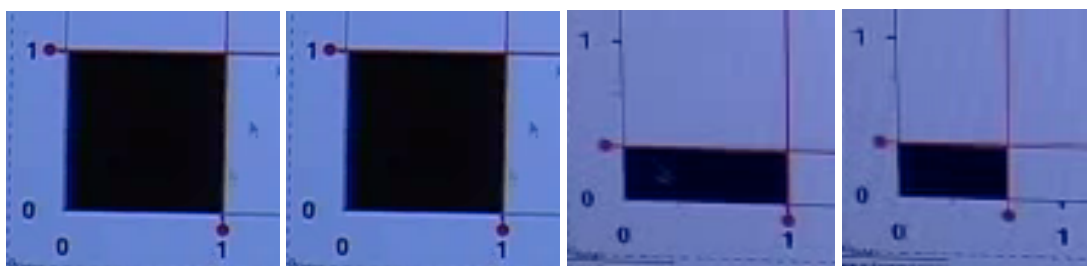


Figure 24. Oscar's final construction for case 12 (2/3 of 1/3).

<sup>25</sup> Given an AM-FM representation of  $a/b$ , if you change the partitions from  $b$  to some number  $x$ , that is not a multiple of  $b$  then the marker line will automatically move to one of the following positions:  $a/x$ ,  $(a+1)/x$ , or  $(a-1)/x$  (depending on which of the three positions is closest to  $a/b$ ).

In the cases following case 12, Oscar uses both axes of the AM-FM representation to construction A(f) and does so without difficulty. Given  $a/b$  of  $c/d$ , Oscar represents the unit whole, then  $c/d$  (stretching or shrinking the unit whole to  $c/d$ ) and then  $a/b$  (stretching or shrinking  $c/d$  by  $a/b$ ). He generally represents  $c/d$  on the y-axis and  $a/b$  on the x-axis. The exceptions are case 13 and case 14 in which Oscar represents  $c/d$  on the x-axis and  $a/b$  on the y-axis. Recall that Neato struggled to move flexibly between using the two axes of the AM-FM representation, and at one point mistakenly thought he had started a construction ‘backwards.’

### 7.3 Number Sense with Fraction Multiplication (e2): Oscar’s Predictions and Justifications

As Oscar proceeds through each case he is asked to make predictions. The predictions are always made in comparison to the previous case. For example, I would say to Oscar, “We just finished case 9. Now before you use the AM-FM representation to work through case 10, do you think you’ll use more or less cheese in case 10 [I point to the number chart input column for case 10] than you did in case 9 [I point to the output value recorded in the number chart output column for case 9]?” See Figure 17 regenerated below.

Case #	INPUT (units)	INPUT (units)	OUTPUT (units)
8	1 rat	$2/3$ of $3/4$ slice/rat	$1/2$ slice
9	1 rat	$1/2$ of $1/2$ slice/rat	$1/4$ slice
10	1 rat	$1/2$ of $1/3$ slice/rat	
11	1 rat	$1/3$ of $1/2$ slice/rat	
12	1 rat	$2/3$ of $1/3$ slice/rat	
13	1 rat	$3/5$ of $3/4$ slice/rat	
14	1 rat	$5/6$ of $2/5$ slice/rat	
15	1 rat	$4/3$ of $2/5$ slice/rat	
16	1 rat	$2\frac{3}{5}$ of $2/5$ slice/rat	
17	1 rat	$1\frac{3}{5}$ of $6/4$ slice/rat	

“Do you think you’ll use more or less cheese in case 10 than...”

“...you did in case 9?”

Figure 17. Illustration of dialogue and gestures when asking for predictions.

In addition to providing predictions, Oscar is also required to justify his predictions. The results of this analysis are presented in Table 13.

Table 13. Summary of Oscar's Predictions

Case	Input	Predictions	Prediction Justifications
8	$2/3$ of $3/4$ slice/rat	N/A	N/A
9	$1/2$ of $1/2$ slice/rat	N/A	N/A
10	$1/2$ of $1/3$ slice/rat	C: < Case 9	C: pictures splitting whole in half and taking half of that versus splitting whole in 3 and taking half of that
11	$1/3$ of $1/2$ slice/rat	C: = Case 10	C: the order is just switched
12	$2/3$ of $1/3$ slice/rat	C: > Case 11 (self-corrects)	C: $2/3 > 1/2$ using AMFM
13	$3/5$ of $3/4$ slice/rat	I: < Case 12	I: it's split into 5 and 4 makes more boxes so it's gonna be less
14	$5/6$ of $2/5$ slice/rat	I: > Case 13	I: $5/6 > 3/5$
15	$4/3$ of $2/5$ slice/rat	C: > Case 14 (self-corrects)	C: $4/3 > 1$ & $5/6 < 1$
16	$2\frac{3}{5}$ of $2/5$ slice/rat	C: > Case 15	C: $2\frac{3}{5} > 2$ and $1 < 4/3 < 2$
17	$1\frac{3}{5}$ of $6/4$ slice/rat	C: > Case 16	I: $1\frac{3}{5}$ and $6/4$ are both mixed numbers

KEY: C=Correct; I=Incorrect; N/A=Not Addressed

Oscar's predictions are correct in 6 out of 8 cases. Oscar's prediction justifications are correct in 5 of the 8 cases. In case 17, Oscar makes a correct prediction but gives an incorrect justification for his prediction.

Like Neato, Oscar makes correct predictions when only one of the fraction inputs changes from the previous case (cases 10, 12, 15, and 16). In case 10 and case 12, the justifications are correct and grounded in the use of the AM-FM representation. For example, in comparing case 12 ( $2/3$  of  $1/3$ ) to case 11 ( $1/3$  of  $1/2$ ), Oscar uses the AM-FM representation to show that an area model of  $2/3$  is more than an area model of  $1/2$  and therefore case 12 ( $2/3$  of  $1/3$ ) is more than case 11 ( $1/3$  of  $1/2$ ). In case 15 and case 16, the justifications are correct and grounded in the use of a reference point to compare two fractions. For example, in comparing case 15 ( $4/3$  of  $2/5$ ) to case 14 ( $5/6$  of  $2/5$ ), Oscar concludes that  $4/3$  is an improper fraction and therefore greater than one while  $5/6$  is less than one, and therefore case 15 ( $4/3$  of  $2/5$ ) is more than case 14 ( $5/6$  of  $2/5$ ). For case 11, the two fraction inputs remain the same and Oscar references the commutative property to correctly justify his prediction.

Like Neato, Oscar struggles to make correct predictions and justification when both fraction inputs change from the previous case (cases 13, 14, and 17). In case 13, Oscar incorrectly predicts that case 13 ( $3/5$  of  $3/4$ ) would be less than case 12 ( $2/3$  of  $1/3$ ). The justification for his incorrect prediction is that case 13 will have more total boxes than case 12 and therefore case 13 will have less area than case 12. Oscar's justification is based on an AM-FM visualization in which he compares the total number of parts for each case (i.e.,  $5 \times 4 = 20$  in case 13 compared to  $3 \times 3 = 9$  in case 12) without considering the role of the numerators for each case, the second set of fraction inputs for each case, or the operation of multiplication. In case 14, Oscar incorrectly predicts that case 14 ( $5/6$  of  $2/5$ ) would be more than case 13 ( $3/5$  of  $3/4$ ). The justification for his incorrect prediction is that  $5/6$  is great than

$3/5$  and therefore case 14 will have more area than case 13. Oscar's justification is based on his AM-FM visualization of the two areas. While Oscar's statement of and justification for fraction order (i.e.,  $5/6 > 3/5$ ) is correct, Oscar only considers the first set of fraction inputs in making the comparison between the two cases (i.e.,  $5/6$  from case 14 compared to  $3/5$  from case 13) without considering the second set of fraction inputs or the operation of multiplication. In case 17, Oscar correctly predicts that case 17 ( $1\frac{2}{5}$  of  $6/4$ ) would be more than case 16 ( $2\frac{3}{5}$  of  $2/5$ ). The justification for his correct prediction is that  $1\frac{2}{5}$  and  $6/4$  are both mixed numbers and therefore case 17 will have more area than case 16. Oscar's justification is based on an incorrect application of his knowledge of using a reference point to compare fractions without considering the operation of multiplication.

#### **7.4 Representational Fluency for Fraction Multiplication (e3): Oscar's AM-FM Interpretation**

I have discussed Oscar's AM-FM construction process, which embodies fraction multiplication as stretching/shrinking, and Oscar's predictions and justifications for final area output, which reveal his number sense with respect to fraction multiplication. I move not to a discussion of Oscar's representational fluency for fraction multiplication. In working with the AM-FM representation, Oscar reveals a particular learning trajectory for naming  $A(f)$ . I will present this trajectory to highlight Oscar's knowledge coordination of fraction multiplication as he attends to different features of the AM-FM representation. This will be followed by a second analysis in which I discuss the context sensitivity of Oscar's knowledge coordination as it pertains to the subgrid view of the AM-FM representation and the unit whole.

##### 7.4.1 Oscar's learning trajectory for interpreting final area output: Knowledge coordination within a representation.

Neato's learning trajectory for naming  $A(f)$  highlighted Neato's knowledge coordination of fraction multiplication across his use of two representations, the AM-FM representation and the number chart. Unlike Neato, Oscar works exclusively with the AM-FM representation to name  $A(f)$ . Oscar's learning trajectory for naming  $A(f)$  highlights Oscar's knowledge coordination of fraction multiplication across his use of different features of the AM-FM representation. Moreover, the operation of multiplication (with respect to fraction denominators) is embodied in Oscar's initial construction process of using a single axis of the AM-FM representation to construction  $A(f)$  and later in the context of using the x-divisions box and y-divisions box of the AM-FM representation to name  $A(f)$ ; a feature that I will argue serves a function similar to the function served by the number chart in Neato's case.

Oscar interprets  $A(f)$  in eight cases, cases 9 through 17 (excluding case 11). See Table 14 for a summary of Oscar's interpretation of  $A(i)$  and  $A(f)$ .<sup>26</sup> In a number of these cases Oscar's initial interpretation of  $A(f)$  is either partially correct or incorrect (see cases 9, 12, 15, and 17). However, Oscar does eventually arrive at a correct interpretation of  $A(f)$  in all eight cases. Case 9 ( $1/2$  of  $1/2$ ) and case 12 ( $2/3$  of  $1/3$ ) are marked as partially correct because Oscar provides a correct interpretation of  $A(f)$  that is incorrectly constructed. In both instances, I intervene with drawn area model representations and Oscar is able to correctly

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<sup>26</sup> By initial area output, I am referring to the area produced after only one of the given fraction inputs has been represented on the axis of the AM-FM representation.

construct and correctly interpret  $A(f)$ . In case 15 and case 17, Oscar constructs the correct  $A(f)$  but initially provides an incorrect interpretation of  $A(f)$ . In Case 15 ( $4/3$  of  $2/5$ ), Oscar interprets  $A(f)$  as  $4/15$  then  $8/15$  then  $8/30$  and then back to  $8/15$ . In case 17 ( $1\frac{2}{5}$  of  $6/4$ ), Oscar interprets  $A(f)$  as 2 and then as  $2\frac{2}{20}$ . In contrast to Neato, Oscar correctly interprets initial area output,  $A(i)$ , in all instances in which he was asked to do so and in some instances he did so spontaneously.

Table 14. Summary of Oscar's Interpretation of A(i) and A(f)

Case	Input	Name for A(i)	Name and Justification for A(f)	Representation Used
8	$2/3$ of $3/4$ slice/rat	N/A	N/A	N/A
9	$1/2$ of $1/2$ slice/rat	N/A	<u>PC to C</u> : $1/3$ [justification not asked, RB draws area model to show $1/3$ is too much area]; $1/4$ , because you split them into 4 and give them 1 of those	AM-FM
10	$1/3$ of $1/2$ slice/rat	<u>C</u> : $1/2$	<u>C</u> : $1/6$ , because it's 6 boxes and there is 1 shaded	AM-FM
11	$1/2$ of $1/3$ slice/rat	<u>C</u> : $1/3$	N/A	N/A
12	$2/3$ of $1/3$ slice/rat	<u>C</u> : $1/3$	<u>PC to C</u> : $2/8$ [justification not asked, RB suggest using both AM-FM axes to construct area]; $2/9$ , because I split it into 9 total and 2 of them are shaded	AM-FM
13	$3/5$ of $3/4$ slice/rat	<u>C</u> : $3/4$	<u>C</u> : $9/20$ , because 4 boxes times 5 boxes is 20 and 3 spaces times 3 spaces is 9	AM-FM
14	$5/6$ of $2/5$ slice/rat	<u>C</u> : $2/5$	<u>C</u> : $10/30$ , because 10 shaded boxes and 6 times 5 <points to sliders> is 30, which is the same as multiplying spaces [that make up the unit whole]	AM-FM
15	$4/3$ of $2/5$ slice/rat	<u>C</u> : $6/15$	<u>I then C then I then C</u> : $4/15$ , because 4 spaces [RB asks if Oscar is counting spaces or all boxes]; $8/15$ [justification not asked but RB asks OA to tile]; $8/30$ [RB asks for name of tile piece relative to a unit of 1 slice and OA moves tiles into a single unit whole]; $8/15$ [justification not asked]	AM-FM
16	$2 \frac{3}{5}$ of $2/5$ slice/rat	N/A	<u>C</u> : $1 \frac{1}{25}$ , because if I move the black shading here [into a single unit whole] I have 1 left over so it's... [RB asks name of tile piece] $1 \frac{1}{25}$	AM-FM
17	$1 \frac{2}{5}$ of $6/4$ slice/rat	N/A	<u>I then C</u> : 2 because the black shading [that falls outside the primary unit whole] can fit in the top [unit whole], [RB has OA tile and OA moves tiles]; $2 \frac{2}{20}$ , because there's 2 shaded wholes and 2 tiles left out of 20ths out of the whole	AM-FM

KEY: C=Correct; I=Incorrect; PC=Partially Correct; N/A=Not Addressed



Oscar's learning trajectory for interpreting  $A(f)$  will demonstrate Oscar's knowledge coordination of fraction multiplication across his use of different features of the AM-FM representation. Like Neato's, Oscar's first approach for naming and justifying  $A(f)$  entails visualizing and counting shaded tile pieces to arrive at a numerator output and visualizing and counting total number of tiles pieces that constitute the  $1 \times 1$  unit whole to arrive at a denominator output (see cases 9, 10, and 12). Oscar's second approach for naming and justifying  $A(f)$  entails visualizing, counting, and multiplying the horizontal tile pieces and the vertical tile pieces that constitute the shaded region in order to arrive at a numerator output and visualizing, counting, and multiplying the horizontal tile pieces and the vertical tile pieces that constitute the  $1 \times 1$  unit whole in order to arrive at a denominator output (see case 13). This approach is identical to the final approach used by Neato. Oscar's transition from his first approach to his second approach demonstrates knowledge coordination of fraction multiplication as embodied in his use of the tile pieces. Oscar does not appear to attend to the number chart as he transitions from his first approach to his second approach. Oscar does attend to the mathematical notation in the x-divisions box and the y-divisions box of the AM-FM presentation (See Figure 25). More specifically, in Oscar's third approach for naming and justifying  $A(f)$ , he returns to visualizing and counting shaded tile pieces to arrive at a numerator output but looks to and multiplies the number in the x-axis divisions box by the number in the y-axis divisions box to arrive at a denominator output (see case 14).

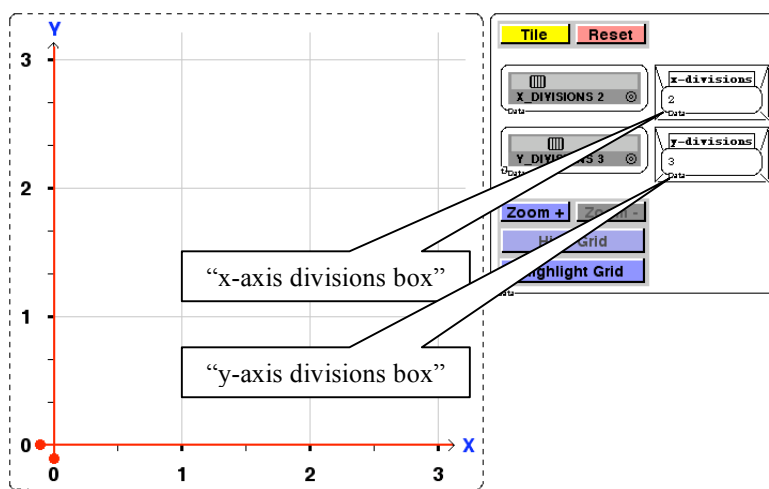


Figure 25. Illustration of mathematics notation captured in the x-divisions box and y-divisions box.

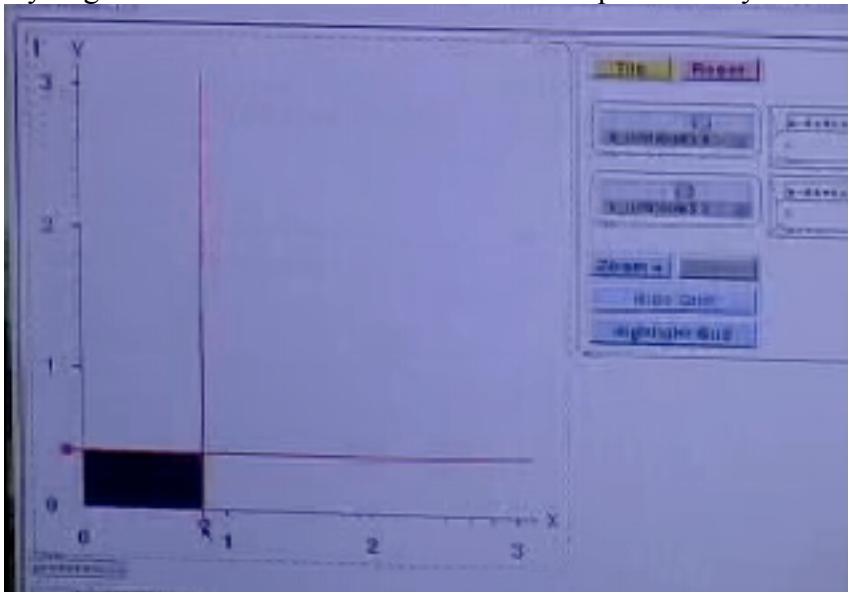
Oscar's final approach for naming and justifying  $A(f)$  is identical to his initial approach. It entails visualizing and counting shaded tile pieces to arrive at a numerator output and visualizing and counting total number of tiles pieces that constitute the  $1 \times 1$  unit whole to arrive at a denominator output (see cases 16 and 17). This return to the initial approach makes sense given the difficulty Oscar experiences in case 15.

I provide transcript analysis of Oscar's third approach for interpreting  $A(f)$ . The other approaches were discussed in Chapter 6 and will not be discussed here. In presenting analysis of Oscar's third approach, I hope to highlight a feature of the AM-FM representation that becomes salient for Oscar and come to serve a function similar to that of the number chart for

Neato. The transcript of Oscar's justification for  $A(f)$  in case 14 (5/6 of 2/5) is provided below. See Appendix H for the complete transcript of Oscar's clinical interview session. This segment of transcript corresponds to transcript lines 1741 through 1768 in Appendix H.

*Oscar's Clinical Interview Transcript: Case 14 (5/6 of 2/5)*

RB: Slices, slices, okay, good. So, how much is that? How much cheese did you give out <references final area of  $10/30$  produced by Oscar>?



OA: Ten...ten thirtieths.

RB: How did you get that so quick?

OA: I did, one, two, three, four, five, six, seven, eight, nine, ten <OA points out and counts aloud the total number of shaded boxes>.

RB: Uhum.

OA: And then six times five <points to the sliders> is thirtieths-thirty.

RB: So six times five you pointed to the sliders when you did that?

OA: Yeah. It's the same thing as right here <points to the x-axis and y-axis>.

RB: Same thing is right where?

OA: Right with these numbers right here <OA points to the y-axis from 1 to zero and points to the x-axis from zero to 1>.

RB: So the number of little-

OA: -Cubes.

RB: Cubes. I see.

In Case 14, Oscar names and justifies  $A(f)$  by attending to the numbers in each divisions box of the AM-FM representation. He multiplies the number “6” that appears in the x-axis divisions box by the number “5” that appears in y-axis divisions box to arrive at a denominator output of 30. Oscar states that multiplying the numbers in the x-axis divisions box and the y-axis divisions box is the same thing as multiplying the tile pieces across the x-axis and y-axis within a 1x1 unit whole. He visualizes and counts shaded tile pieces to arrive at a numerator output of 10. Oscar’s third approach demonstrates coordinate of his knowledge of multiplication across his use of two different features of the AM-FM representation, the horizontal tiles and vertical tiles across the 1x1 unit axes, and the x-axis divisions box and y-axis divisions box. This is similar to Neato’s knowledge coordination of multiplication across his use of the number chart and the AM-FM representation. In the case of Oscar, the coordination is incomplete as it occurs only in the context of the fraction denominator multiplication and not fraction numerator multiplication.

#### 7.4.2 Subgrid view and the unit whole: Context sensitivity of knowledge coordination.

Next, I demonstrate the context sensitivity of Oscar’s knowledge coordination of fraction multiplication across his use of the different features of the AM-FM representation. I present transcript analysis of case 15 ( $4/3$  of  $2/5$ ) to demonstrate how the subgrid view constrains Oscar’s ability to see the unit whole and correctly name  $A(f)$ . Prior to case 15, Oscar demonstrates no difficulty in identifying the unit whole and correctly interpreting  $A(f)$  (See Table 14 reproduced below).

Table 14. Summary of Oscar's Interpretation of A(i) and A(f)

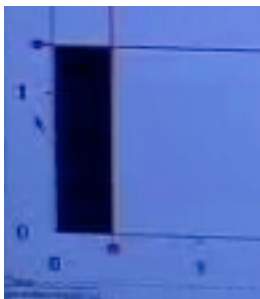
Case	Input	Name for A(i)	Name and Justification for A(f)	Representation Used
8	$2/3$ of $3/4$ slice/rat	N/A	N/A	N/A
9	$1/2$ of $1/2$ slice/rat	N/A	<u>PC to C</u> : $1/3$ [justification not asked, RB draws area model to show $1/3$ is too much area]; $1/4$ , because you split them into 4 and give them 1 of those	AM-FM
10	$1/3$ of $1/2$ slice/rat	<u>C</u> : $1/2$	<u>C</u> : $1/6$ , because it's 6 boxes and there is 1 shaded	AM-FM
11	$1/2$ of $1/3$ slice/rat	<u>C</u> : $1/3$	N/A	N/A
12	$2/3$ of $1/3$ slice/rat	<u>C</u> : $1/3$	<u>PC to C</u> : $2/8$ [justification not asked, RB suggest using both AM-FM axes to construct area]; $2/9$ , because I split it into 9 total and 2 of them are shaded	AM-FM
13	$3/5$ of $3/4$ slice/rat	<u>C</u> : $3/4$	<u>C</u> : $9/20$ , because 4 boxes times 5 boxes is 20 and 3 spaces times 3 spaces is 9	AM-FM
14	$5/6$ of $2/5$ slice/rat	<u>C</u> : $2/5$	<u>C</u> : $10/30$ , because 10 shaded boxes and 6 times 5 <points to sliders> is 30, which is the same as multiplying spaces [that make up the unit whole]	AM-FM
15	$4/3$ of $2/5$ slice/rat	<u>C</u> : $6/15$	<u>I then C then I then C</u> : $4/15$ , because 4 spaces [RB asks if Oscar is counting spaces or all boxes]; $8/15$ [justification not asked but RB asks OA to tile]; $8/30$ [RB asks for name of tile piece relative to a unit of 1 slice and OA moves tiles into a single unit whole]; $8/15$ [justification not asked]	AM-FM
16	$2 \frac{3}{5}$ of $2/5$ slice/rat	N/A	<u>C</u> : $1 \frac{1}{25}$ , because if I move the black shading here [into a single unit whole] I have 1 left over so it's... [RB asks name of tile piece] $1 \frac{1}{25}$	AM-FM
17	$1 \frac{2}{5}$ of $6/4$ slice/rat	N/A	<u>I then C</u> : 2 because the black shading [that falls outside the primary unit whole] can fit in the top [unit whole], [RB has OA tile and OA moves tiles]; $2 \frac{2}{20}$ , because there's 2 shaded wholes and 2 tiles left out of 20ths out of the whole	AM-FM

KEY: C=Correct; I=Incorrect; PC=Partially Correct; N/A=Not Addressed

During case 15 (4/3 of 2/5), Oscar interprets  $A(f)$  as being  $4/15$  after which I draw Oscar's attention to the shaded area. Oscar visualizes and counts the tiles that would make up the shaded area and changes his interpretation from  $4/15$  to  $8/15$ . Following the tiling process, Oscar changes his interpretation again from  $8/15$  to  $8/30$ . This change reveals the context sensitivity of Oscar's emergent knowledge of unit when  $A(f)$  is greater than one. I intervene and Oscar goes on to correctly identify the unit to once again arrive at  $A(f)=8/15$ . The transcript of Oscar's interpretation for  $A(f)$  in case 15 is provided below. This segment of transcript corresponds to transcript lines 2098 through 2176 in Appendix H.

*Oscar's Clinical Interview Transcript: Case 15 (4/3 of 2/5)*

RB: So how much cheese did you just use? How much cheese did you give out  
<references final area of  $8/15$  produced by Oscar>?



OA: Four, four fifteenths.

RB: Four fifteenths?

OA: Yeah.

RB: How are you getting four fifteenths?

OA: No wait...oh yeah four fift-no...four fifteenths yeah.

RB: And where is the four fifteenths coming from? Where is the number four coming from?

OA: Well, one, two, thee, four <points to four  $1/3$  line segments from  $y=0$  to  $y=4/3$ >.

RB: -Uhum-

OA: -And then out of fifteen <points to  $1 \times 1$  unit whole>.

RB: So what are counting, just this little, these little segments <RB points to segments on  $y$ -axis> or are you counting boxes?

OA: The...boxes.

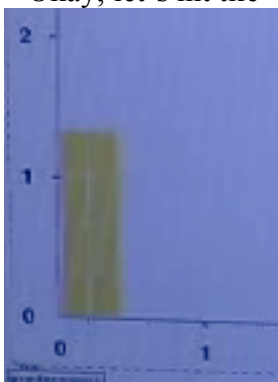
RB: And there is-

OA: -Four of them.

RB: So there is just this box here <points to the black shading from  $x=0$  to  $x=2/5$  and  $y=0$  to  $y=1/3$ >-

OA: -No, wait, wait, wait <appears to be counting to himself> eight-eight fifteenths.

RB: Eight fifteenths. So there is eight boxes and each box is called a fifteenth? Okay, let's hit tile <OA hits "Tile">. Is that eight?



OA: Uhum.

RB: So it's eight fifteenths, do you wanna write that down?

OA: -No wait... eight thirtieths.

RB: Okay, so first you said eight fifteenths and now you are saying eight thirtieths. Why eight thirtieths?

OA: Because, that's fifteen <OA points to first  $1 \times 1$  unit whole from  $y=0$  to  $y=3/3$ > and that's fifteen <OA points to second  $1 \times 1$  unit whole from  $y=3/3$  to  $y=6/3$ >, so that's thirty.

RB: So is this piece called a thirtieth <lifts up a single  $1/15$  tile piece>?

OA: Yeah.

RB: It's not a fifteenth.

OA: No.

RB: Okay. It's a thirtieth of how many slices?

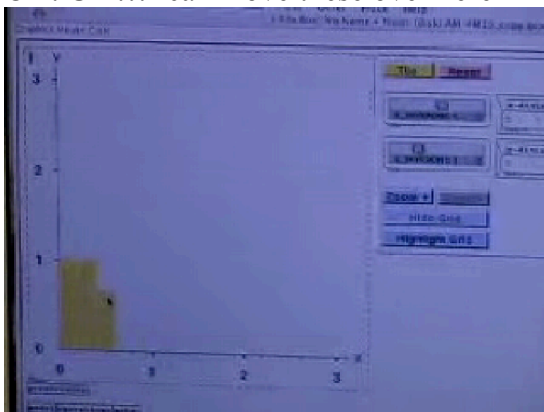
OA: Um, two.

RB: Okay, I want the unit to be one.

OA: No-one.

RB: It's a thirtieth of one slice? Remember, I always want the unit to be one.

OA: Um...I can move these over here <moves tiles into a single 1x1 unit>.



RB: So how much cheese did you give out?

OA: Eight fifteenths.

RB: Eight fifteenths. So back to eight fifteenths, are you happy with that?

OA: Yeah.

Case 15 ( $4/3$  of  $2/5$ ) is the first instance in which Oscar is confronted with an improper fraction. Oscar's first interpret of  $A(f)$  is  $4/15$ . In the previous case (case 14:  $5/6$  of  $2/5$ ) Oscar's approach for interpreting  $A(f)$  entailed visualizing, counting, and multiplying the horizontal tile pieces and the vertical tile pieces that constitute the shaded region in order to arrive at a numerator output and visualizing, counting, and multiplying the horizontal tile pieces and the vertical tile pieces that constitute the 1x1 unit whole in order to arrive at a denominator output. In case 15, Oscar attends only to the vertical tile pieces that constitute the shaded region in order to arrive at a numerator output. He fails to see the two  $1/5$  horizontal tile pieces from  $x=0$  to  $x=2/5$ . I intervene by asking Oscar whether he's attending to the tiles that would eventually constitute the shaded region or the line segments along the axes. Oscar responds with tiles at which point I draw Oscar's attention to what would constitute a single tile (from  $x=0$  to  $x=2/5$  and  $y=0$  to  $y=1/3$ ) according to his interpretation of  $A(f)$  as  $4/15$ . Oscar engages in the act of visualizing and counting all the shaded tile pieces rather than just the vertical tile pieces from  $y=0$  to  $y=4/3$  and changes his interpretation of  $A(f)$  from  $4/15$  to  $8/15$ . Recall that this was part of Oscar's first approach in naming and justifying  $A(f)$ .

Furthermore, Oscar does not appear to be applying the operation of multiplication in the context of arriving at a correct numerator output despite having done so previously.

Once Oscar arrives at the correct interpretation for A(f) in case 15, I ask Oscar to tile after which point he changes his interpretation from  $8/15$  to  $8/30$ . This time Oscar attends to the “two” unit wholes that have each been subdivided into 15 tile pieces to arrive at a denominator output of 30. I intervene by asking for the name of a single tile piece. Oscar responds with thirtieths of two slices. I tell Oscar I want the unit to be one slice. He spontaneously moves the tiles pieces into the primary  $1 \times 1$  unit whole and changes his interpretation of A(f) from  $8/30$  back to  $8/15$ . Case 16 ( $2\frac{3}{5}$  of  $2/5$ ) and case 17 ( $1\frac{2}{5}$  of  $6/4$ ) each involve improper fractions and in both cases Oscar is able to arrive at a correct denominator outputs by attending to a single  $1 \times 1$  unit whole.

### **7.5 Representational Fluency for Fraction Equivalence (e4)**

As Oscar proceeds through each case he uses the AM-FM representation to name equivalent fractions. A summary of Oscar’s interpretation of fraction equivalence is presented in Table 15. Because fraction equivalence is often explored in the context of locating and naming fractions across different representations, a summary of the fractions names generated by Oscar and the representational context used by Oscar is also included in Table 15. To make sense of fraction equivalence, Oscar generally attends to one of the following: the number line feature of the AM-FM representation (marked “NL” in Table 15), the area model feature of the AM-FM representation (marked “AM” in Table 15), or the number line representation drawn using paper and pencil (marked “Drawn NL” in Table 15).



Table 15. Summary of Oscar's Interpretation of Fraction Equivalence

Case	Input	C	I	Representation Used
9	1/2 of 1/2 slice/rat			N/A
10	1/3 of 1/2 slice/rat	X X X X X X  X X X X X X X X	X	[NOTE: Y-axis divisions set at 6] NL: 1/2 NL: 4/6 NL: 2/6 NL: 3/6 NL: 0 NL: 6/6 NL: $1\frac{2}{6}=8/12 \Rightarrow$ Int [NOTE: Drawn NL partitions into 6ths] Drawn NL: 4/6 Drawn NL: $\frac{1}{2}=3/6$ Drawn NL: 6/6 Drawn NL: $8/6=1\frac{2}{6}$ Drawn NL: $10/6=1\frac{4}{6}$ Drawn NL: $1\frac{5}{6}=11/6$ Drawn NL: $2=1\frac{6}{6} \Rightarrow$ Int Drawn NL: $2=12/6$
11	1/2 of 1/3 slice/rat			N/A
12	2/3 of 1/3 slice/rat			N/A
13	3/5 of 3/4 slice/rat	X X		[NOTE: X-axis divisions set at 5 & Y-axis divisions set at 4] NL: $6/5=1\frac{1}{5}$ NL: $1\frac{3}{4}=7/4$
14	5/6 of 2/5 slice/rat	X	X	AM: $10/30 > 1/3 \Rightarrow$ Int AM: $10/30=1/3$
15	4/3 of 2/5 slice/rat			N/A
16	$2\frac{3}{5}$ of 2/5 slice/rat	X		[NOTE: Y-axis divisions set at 5] NL: $2\frac{3}{5}=13/5$
17	$1\frac{2}{5}$ of 6/4 slice/rat			N/A

KEY: C=Correct; I=Incorrect; N/A=Not Addressed; AM=Area Model; NL=Number Line; Int=Intervention

Oscar is able to correctly locate and name fractions and equivalent fractions in all but two instances. One of those instances occurs during case 10 and one occurs during case 14. I will discuss the instance in case 10 (which I will call Instance Q1) and the instance in case 14

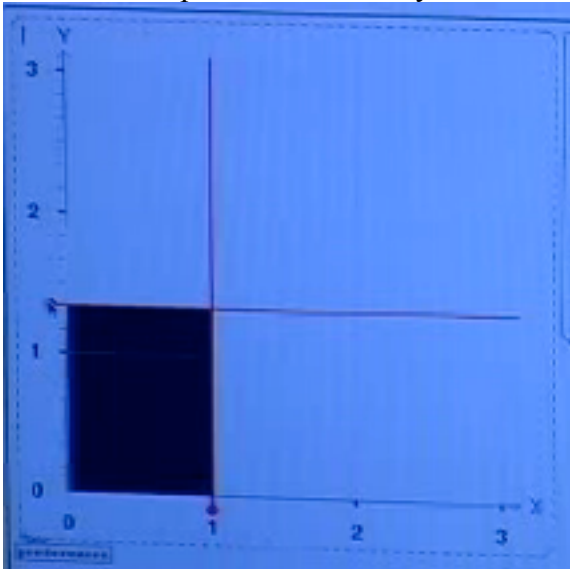
(which I will call Instance Q2) to highlight Oscar's knowledge coordination of fraction equivalence across different features of the AM-FM representation.

Prior to Instance Q1, Oscar demonstrates no difficulty in naming fractions on the y-axis of the AM-FM representation. The y-axis divisions slider of the AM-FM representation is set at 6. I ask Oscar, "What is the fraction name for this point?" as I move the y-axis marker line to various locations up and down the y-axis. When I position the y-axis marker line at a particular location,  $a/b$ , the result is an area enclosed of  $a/b$ . Both the number line and area model features of the AM-FM representation are highlighted as Oscar is asked to name fractions on the y-axis of the AM-FM representation. When asking Neato to name fractions on the axes of the AM-FM representation I did not move the y-axis or x-axis marker lines in order to enclose area but simply pointed to various locations on the marker lines. And unlike Oscar, Neato demonstrated some difficulty in correcting naming  $A(i)$  when first asked to do so. This will be considered further in the discussion section of this chapter.

Returning to Instance Q1, Oscar correctly names the point  $1\frac{2}{6}$  on the y-axis of the AM-FM representation but when asked to give another fraction name for  $1\frac{2}{6}$  Oscar gives  $\frac{8}{12}$ . Transcript of Instance Q1 is provided below. This segment of transcript corresponds to transcript lines 543 through 559 in Appendix H.

*Oscar's Clinical Interview Transcript: Fraction Equivalence Instance Q1 ( $1\frac{2}{6} = 12/6$ )*

RB: And this point <RB moves y-axis marker from  $6/6$  to  $8/6$ >?



OA: Um, one and two sixth.

RB: One and two sixth. Now, that's called a mixed number, right? One and two sixths, cause you have the one which is a whole number and you have a fraction with it. If I wanted this just as a regular fraction, no mixed number, what would I call this?

OA: Um...eight twelfths.

RB: Eight twelfths? Why eight twelfths?

OA: Cause I counted this six <points to the six  $\frac{1}{6}$  pieces that make up the shaded  $1 \times 1$  unit whole> and then six <points to the six  $\frac{1}{6}$  pieces that make the second  $1 \times 1$  unit whole which has  $\frac{2}{6}$  shaded> so that's twelve and then six <the shaded  $\frac{1}{6}$  pieces from the first unit whole> plus two <the shaded  $\frac{1}{6}$  pieces from the second unit whole> that's eight. So that's eight twelfths.

When interpreting  $1\frac{2}{6}$  as  $\frac{8}{12}$ , Oscar is visualizing the tiled area and seeing the two unit wholes as each being partitioned into twelfths with eight tiled pieces. It appears that the subgrid view is constraining Oscar's ability to correctly interpret fraction equivalence. I intervene by drawing in number line from zero to three with each unit partitioned into sixths. See Figure 26 for a screenshot of the drawn number line produced during Instance Q1.



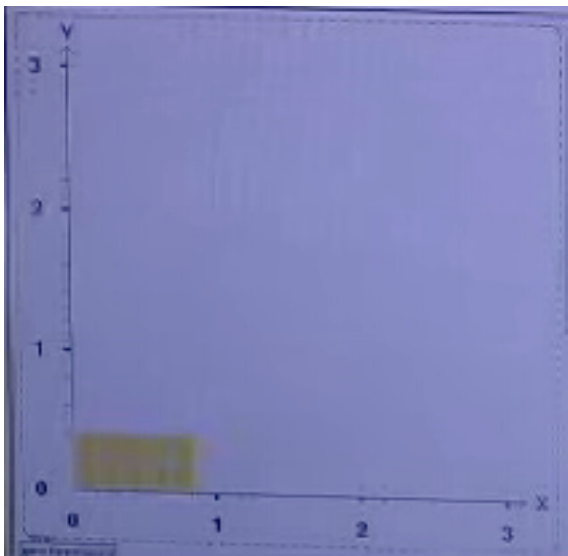
Figure 26. RB's construction of a number line during Instance Q1.

When working with the drawn number line, Oscar demonstrates no difficulty in naming equivalent fractions (see Table 15). Following the intervention with the drawn number line, Oscar demonstrates little difficulty in naming equivalent fractions using the AM-FM representation. The one exception is Instance Q2, which I will discuss next.

In Instance Q2, Oscar is asked to name equivalent fractions for an area output rather than for a point along an axis of the AM-FM representation or on a drawn number line. Instance Q2 takes place during case 14 ( $\frac{5}{6}$  of  $\frac{2}{5}$ ) after Oscar correctly interprets A(f) as being  $\frac{10}{30}$  and concludes that  $\frac{10}{30}$  can be reduced to  $\frac{1}{3}$ . I ask Oscar to hit the tile button and show why  $\frac{10}{30}$  is equivalent to  $\frac{1}{3}$ . As Oscar struggles to proceed, I suggest he consider moving the tile pieces. Oscar moves a single tile piece and concludes that the tiled area is actually greater than  $\frac{1}{3}$ . I intervene after which point Oscar moves the tile pieces to show that  $\frac{10}{30}$  is equivalent to  $\frac{1}{3}$ . Transcript of Instance Q2 is provided below. This segment of transcript corresponds to transcript lines 1831 through 1877 in Appendix H.

*Oscar's Clinical Interview Transcript: Fraction Equivalence Instance Q2 (10/30 > 1/3)*

OA: To make one third...

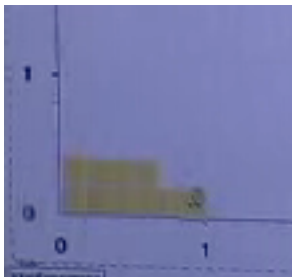


RB: Can we move the tiles? Somehow?

OA: Um...So...to make one third?

RB: Uhum.

OA: To make one third <moves a single tile>. It's already one third.



RB: What do you mean it's already one third?

OA: Well it's more than one third.

RB: What's more than one third?

OA: This-like-...wait, wait, wait, wait, wait. Well because when I did it I got rid of these things <OA points to 1/5 markers from y=0 to y=5/5>.

RB: Uhum.

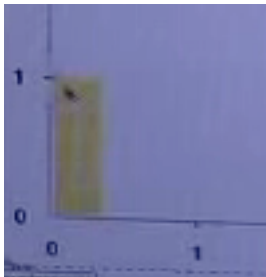
OA: And got rid of this one, this one and this one <OA points the x=1/6, x=3/6, x=5/6 markers>. So that's thirds.

RB: Uhum.

OA: And these two and this one that's one <point to two tile pieces laying across x-axis from  $x=0$  to  $x=2/6=1/3$ > that's one third.

RB: But would it be the whole column, or would it be just the-

OA: -No it would be the whole column. So <OA moves tile pieces up to fill first  $1/3$  column from  $x=0$  to  $x=2/6=1/3$ >.



RB: I see what you did. Okay, and why is that one third?

OA: Um, because that's one, that's two and that's three <OA points to the  $1/6$  line segments from  $x=0$  to  $x=6/6$  in pairs of two>.

RB: Uhum.

OA: So that's thirds, and these aren't here <points to the five  $1/5$  line segments from  $y=0$  to  $y=5/5$ >.

RB: Uhum-

OA: -So this would be like-this is one third.

Given a tiled area of  $10/30$ , Oscar moves a single tile to complete a tiled row from  $x=0$  to  $x=1$  and concludes that the tiled area is more than  $1/3$ . In this instance, Oscar incorrectly attends to the number line and area model features of the AM-FM representation and applies his knowledge of the one-to-one correspondence between the location of the marker line and area to an incorrect context. Oscar sees that there are tiles that fall to the right of  $x=2/6=1/3$  and concludes that the area is therefore more than  $1/3$ . He fails to recognize that in order for there to be a one-to-one correspondence between the location  $2/6=1/3$  on the x-axis and shaded area the tiles need to fill the two  $1/6$  columns from  $y=0$  to  $y=1$ . I intervene with, "But would it be the whole column or..." at which point Oscar self-corrects his initial interpretation of the tiled area being greater than  $1/3$ . Oscar moves tiles into the first two  $1/6$  columns of the  $1 \times 1$  unit whole and concludes the area is indeed equivalent to  $1/3$ . Neato demonstrated a similar context sensitivity when interpreting  $A(f)$ .

## 7.6 Representational Fluency for Fraction Order (e5)

Having discussed equivalence across Oscar's use of both the number line and area model features of the AM-FM representation, I turn now to the concept of order. See Table 16. Note the addition of fraction notation (marked "FN" in Table 16) to the representational context.

Table 16. Summary of Oscar's Interpretation of Fraction Order

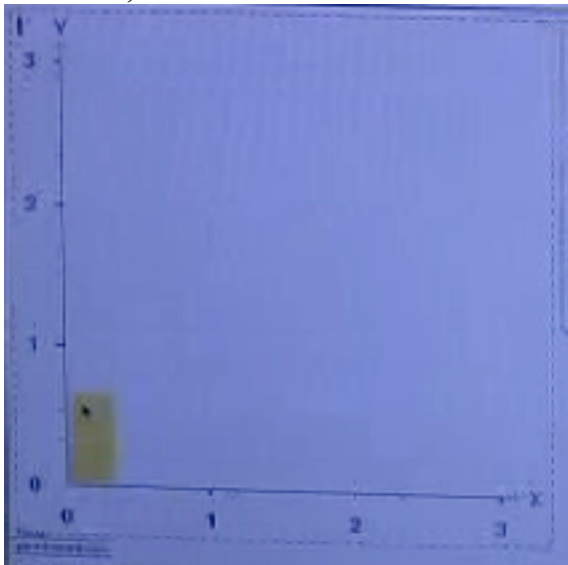
Case	Input	C	I	Representation Used
9	1/2 of 1/2 slice/rat			N/A
10	1/3 of 1/2 slice/rat	X		NL/AM: $1/6 < 1/4$
11	1/2 of 1/3 slice/rat			N/A
12	2/3 of 1/3 slice/rat	X		NL&AM: $2/3 < 1/2$ (self-corrects) $2/3 > 1/2$
			X	NL&AM: $2/9 < 1/6 \Rightarrow$ Int
			X	NL&AM: $2/9 > 1/6$
13	3/5 of 3/4 slice/rat	X		NL&AM: $9/20 < 1/2$
		X		NL&AM: $9/20 > 2/9$
14	5/6 of 2/5 slice/rat	X		U: $5/6 > 3/5$
15	4/3 of 2/5 slice/rat	X		FN: $4/3 < 5/6$ (self-corrects) $4/3 > 5/6$
16	$2\frac{3}{5}$ of 2/5 slice/rat	X		FN: $2\frac{3}{5} > 4/3$
17	$1\frac{2}{5}$ of 6/4 slice/rat			N/A

KEY: C=Correct; I=Incorrect; N/A=Not Addressed; AM=Area Model; NL=Number Line; FN=Fraction Notation; U=Representational Context is Unclear; Int=Intervention

Oscar is able to correctly order fractions in all but one instance (which I will call Instance R1). When working with  $A(f)$  less than one, Oscar attends to the number line and area model features of the AM-FM representation. When working with  $A(f)$  greater than one, Oscar applies his knowledge of fraction notation. Instance RI occurs during case 12 ( $2/3$  of  $1/3$ ) when Oscar incorrectly concludes that  $2/9$  area is less than  $1/6$  area. I intervene at which point Oscar correctly concludes that  $2/9$  area is more than  $1/6$  area and goes on to show that it is exactly half of a  $1/9$  tile more area. Transcript of Instance R1 is provided below. This segment of transcript corresponds to transcript lines 1831 through 1877 in Appendix H.

*Oscar's Clinical Interview Transcript: Fraction Order Instance R1 ( $2/9 < 1/6$ )*

RB: Now, was that more or less than one sixth?



OA: Umm...less.

RB: Two ninths is less than one sixth? Why?

OA: Because, one sixth was like right here <OA gestures to a horizontal area across the x-axis from zero to 1> where this line is at <looks to be pointing at  $y=1/3$ >.

RB: One sixth was right, where?

OA: Like right here <OA points to what looks to be  $y=1/3$ >.

RB: That's a third, right? This line is one third <RB points to  $y=1/3$ >. How did we get one sixth? What were the two fractions we were working with to get the one sixth? What were the two fractions we were working?

OA: One half of one third.

RB: Uhum. So.

OA: We got more.

RB: So we got more where? With two ninths or-

OA: -Right here.

RB: We have more with two ninths.

OA: Uhum.

RB: Okay. Where would the line be for one sixth.

OA: Right there-half of it <OA points to the y-axis between zero and  $1/3$ >.

RB: Right here <points to  $y=1/6$ >.

OA: Half of one third. It's like right here <points to  $y=1/6$ >.

RB: It would be right there <points to  $y=1/6$ >.

OA: Yeah.

RB: And your shading would go how far?

OA: Here. <OA points to  $x=1/3$ >.

RB: It would.

OA: No.... It would go all of this <OA points from  $x=0$  to  $x=1$ >.

RB: It would go all of that, right? Cause you didn't split this part <the x-axis> up. So it would go all of that.

OA: Yeah.

RB: So, then how many things would it fill up? How many boxes would it fill up? If there was a line right here <RB references an imaginary horizontal line at  $y=1/6$ > The bottom three <tiles> would be shaded <half way>, right?

OA: There is, this one has more.

RB: This one has more. How much more of a box?

OA: One sixth more.

RB: So does it have a complete extra box shaded?

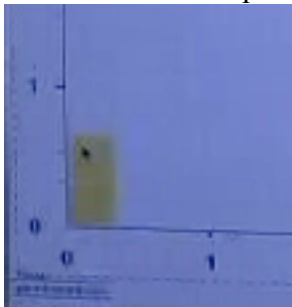
OA: No, wait. Well, what I did was, I split this in half <points to bottom  $1/9$  tile>, and then I put, I split this box in half <points to bottom  $1/9$  tile again> so I left one half there <points to tile area from  $x=0$  to  $x=1/3$ > and I put the other half there <points to tile area from  $x=1/3$  to  $x=2/3$ > and then I split this half <points to the top  $1/9$  tile>, put one half there <points to tile area from



$x=2/3$  to  $x=3/3$ >, put the other one up here <points to tile area from  $x=1/3$  to  $x=2/3$  and  $y=1/3$  to  $y=2/3$ >. So it would be, it would be...um...

RB: So how much yellow do we have here, in two ninths?

OA: Half of that <points to top  $1/9$  tile>.



RB: Half a box?

OA: Yeah.

Oscar's justification for why  $2/9$  area is less than  $1/6$  is based on an incorrect visualization of  $1/6$  area. This is due in large part to the shift in Oscar's construction process. Prior to case 12, Oscar had only used the y-axis to construct  $A(f)$ . Case 12 is the first instance in which he uses both axes of the AM-FM representation to construct  $A(f)$ . During Instance R1, Oscar visualizes a  $1/6$  area as being a  $1/3$  area enclosed by  $x=0$  to  $x=1$  and  $y=0$  to  $y=1/3$ . When I reference Oscar's previous construction process, Oscar concludes that  $2/9$  area is greater than  $1/6$  area. This time he visualizes a  $1/6$  area as being  $1/18$  area enclosed by  $x=0$  to  $x=1/3$  and  $y=0$  to  $y=1/6$ . When I ask if the shading would stop at  $x=1/3$ , Oscar correctly concludes that it would go from  $x=0$  to  $x=1$ . Next, I ask how much more  $2/9$  is than  $1/6$ . Oscar explains in detail that half of tile is left over when you place the  $2/9$  area into the  $1/6$  area he is visualizing. Following Instance R1, Oscar demonstrates no difficulty in correctly order fractions using the number line and area model feature of the AM-FM representation.

## 7.7 Case Study Discussion

In this discussion, I consider how the analysis above addresses the larger research questions of interest. In other words, what does the analysis say about (a) Oscar's understanding of rational numbers and fraction multiplication (i.e. the development of domain competence) and (b) Oscar's understanding of the affordances and constraints of the AM-FM representation (i.e. the development of representational competence)? The discussion will be presented in four parts: (e1) fraction multiplication as stretching/shrinking, (e2) number sense with fraction multiplication, (e3) representational fluency for fraction multiplication, and (e4/e5) representational fluency for fraction equivalence and fraction order. The discussion will be revisited in Chapter 8, where I consider the broader implications of design-based research: (a) local theory development and (b) refinement of the designed learning environment.

### 7.7.1 Fraction multiplication as stretching/shrinking.

Oscar developed his own construction process for arriving at A(f), a process that differed from the preferred construction process presented to him. Oscar's first approach was to estimate. In the case of  $\frac{1}{2}$  of  $\frac{1}{2}$ , Oscar set the y-axis marker line at  $\frac{1}{2}$  and proceeded to partition the y-axis into thirds in order to arrive at  $\frac{1}{4}$ . He had yet to discover the one-to-one correspondence between the product of the fraction denominators and the total number of parts in the unit whole. Following an intervention this error in estimate was corrected. Oscar's construction process of using a single axis to represent both fraction inputs embodied the operation of denominator multiplication. As in Neato's case, this construction process revealed a certain level of independent thought process in Oscar's use of the AM-FM representation. I did not intervene in Oscar's construction process by re-introducing the preferred construction process as I did with Neato. When confronted by a problem that required more partitions than the AM-FM representation affords on any one axis, I worked with Oscar to re-discover using both axes to represent the two fraction inputs. In such instances, given  $a/b$  of  $c/d$ , Oscar represented the unit whole, then  $c/d$  (stretching or shrinking the unit whole to  $c/d$ ) and then  $a/b$  (stretching or shrinking  $c/d$  by  $a/b$ ). Given  $a/b$  of  $c/d$ , Oscar's construction process corresponded to taking  $a/b$  of  $c/d$ . Recall that Neato construction process corresponded to taking  $c/d$  of  $a/b$ . Finally, Oscar used the axes of the AM-FM representation flexibly in that at time he represented  $a/b$  on the x-axis and other times he represented  $a/b$  on the y-axis. This is in contrast to Neato who struggled with the construction process when he switched from using the y-axis to represent  $a/b$  to using the x-axis to represent  $a/b$ .

Recall Figure 11 presented in Chapter 4. In making sense of fraction multiplication as stretching/shrinking, Oscar appeared to draw on his emergent knowledge of area models (i1), number lines (i2), fraction as part-whole (i4Ka), and fraction as measure (i4Kb). Data did not reveal knowledge of multiplication as repeat addition (i4Kd).<sup>27</sup>

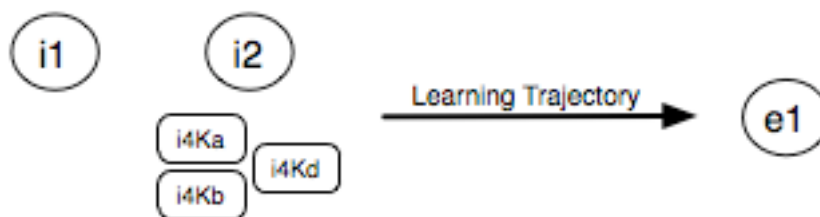


Figure 11. Transformation from S(i) to fraction multiplication as stretching/shrinking (e1).

### 7.7.2 Number sense with fraction multiplication.

When only one fraction changed from one case to the next, Oscar provided correct predictions and justifications by adequately using the AM-FM representation or by applied his knowledge of a reference point such as one. Oscar struggled to make accurate justifications when both fractions changed from one case to the next. In such instances, Oscar's justifications were based on a comparison of total number of parts for each case (without

<sup>27</sup> This was due (in part) to the fact that I did not provide students with an opportunity to explicitly express their knowledge of fraction as repeat addition within the designed learning environment.

considering the role of the numerators for each case, the second set of fraction inputs for each case, or the operation of multiplication) or a comparison of the first fraction input from each case (without considering the second set of fraction inputs for each case or the operation of multiplication). Finally, while Oscar correctly applied his knowledge of a reference point to compare cases when only one fraction was changing from one case to the next, his knowledge was applied incorrectly to the context in which both fractions changed from one case to the next.

Recall Figure 12 from Chapter 4. Oscar's number sense with fraction multiplication appeared to draw on his emergent knowledge of area models (i1), number lines (i2), algorithm for fraction multiplication (i3Ka), algorithm for fraction equivalence (i3Kb), algorithm for fraction order (i3Kc), fraction as part-whole (i4Ka), and fraction as measure. Data did not reveal knowledge of fraction as quotient (i4Kc) and multiplication makes bigger (i4Ke).<sup>28</sup>

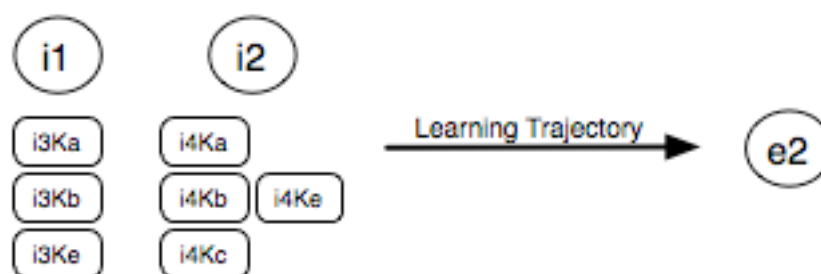


Figure 12. Transformation from S(e) to number sense with fraction multiplication (e2).

### 7.7.3 Representational fluency for fraction multiplication.

Oscar demonstrated a particular trajectory for naming A(f). In phase one, Oscar attended to the AM-FM representation. He visualized and counted shaded tiles to total tiles that constituted the 1x1 unit whole. This was identical to Neato first phase. In phase two, Oscar again attended to the AM-FM representation. He visualized, counted, and multiplied the number of horizontal and vertical tile pieces that constituted the shaded region to arrive at a numerator output and he visualized, counted, and multiplied the total number of horizontal and vertical tile pieces that constituted the 1x1 unit whole to arrive at a denominator output. This was identical to the final phase in Neato's learning trajectory. In phase three, Oscar continued to attend to the AM-FM representation. He returned to visualizing and counting shaded tile pieces to arrive at a numerator output but looks to and multiplies the number in the x-axis divisions box by the number in the y-axis divisions box to arrive at a denominator output. This was similar to phase three in Neato's learning trajectory when he attended to both the AM-FM representation and the number chart. The numbers in each divisions box served the same function for Oscar as the denominators in number chart served for Neato. The operation of fraction multiplication (denominator only) was embodied in Oscar's use of the x-axis divisions box and y-axis divisions box. The final phase in Oscar's learning trajectory was identical to Oscar's first phase during which he attended to the AM-FM representation. Oscar visualized and counted shaded tiles to total tiles that constituted the 1x1 unit whole. The

<sup>28</sup> This was due (in part) to the fact that I did not provide students with many opportunities to explicitly express their knowledge of fraction as quotient and fraction makes bigger within the designed learning environment.

return to phase one made sense given the difficulty Oscar experienced in naming A(f) between phase three and phase four (see case 15).

This trajectory highlighted Oscar's knowledge coordination of fraction multiplication as embodied in his use of different features of the AM-FM representation. The AM-FM representation afforded Oscar the opportunity to (a) apply his knowledge of the part-whole subconstruct in the context of counting shaded tiles to total tiles (that make up the 1x1 unit) to arrive at A(f), and (b) recognize and apply the algorithm for fraction multiplication in the context of multiplying vertical and horizontal shaded tiles to vertical and horizontal total tiles (within the 1x1 unit whole) to arrive at A(f). Furthermore, the x-axis divisions box and y-axis divisions box afforded Oscar the opportunity to apply his prior knowledge of the algorithm for fraction multiplication to the given fraction denominator inputs in order to arrive at the correct denominator output for A(f). When using the numbers in each divisions box, the coordination of Oscar's knowledge of fraction multiplication was incomplete in that the algorithm for fraction multiplication was not salient to Oscar when using the AM-FM representation to work with the fraction numerator inputs.

Oscar also came to coordinate his knowledge of the area model and number line features of the AM-FM representation when naming A(i). Recall that when naming A(i), there exists a one-to-one correspondence between shaded area and the location of the marker line. However, this correspondence did not hold when naming A(f), unless one or both the marker lines are positioned at one and tiles remain unmoved. Unlike Neato, Oscar was able to correctly name A(i) in all cases in which he was asked to do so. This is due to part to (a) Oscar's initial construction process of using a single axis of the AM-FM representation for both fraction inputs, and (b) when assessing Oscar's understanding of fraction equivalence, I moved the marker line to enclose the corresponding area making both the area model and number line features of the AM-FM representation salient to Oscar.

Like Neato, Oscar's knowledge was emergent and context sensitive. When A(f) is less than one, Oscar was able to use the subgrid view of the AM-FM representation to correctly interpret the unit whole and name A(f). When A(f) is greater than one, the subgrid view constrained Oscar's ability to correctly interpret the unit whole and name A(f).

Recall Figure 13 from Chapter 4. Oscar's representational fluency with fraction multiplication appeared to draw on his emergent knowledge of area models (i1), number lines (i2), algorithm for fraction multiplication (i3Ka), fraction as part-whole (i4Ka), and fraction as measure (i4Kb). Data did not reveal knowledge of multiplication as repeated addition (i4Kd) and multiplication makes bigger (i4Ke).<sup>29</sup>



Figure 13. Transformation from S(i) to representational fluency for fraction multiplication (e3).

<sup>29</sup> This was due (in part) to the fact that I did not provide students with many opportunities to explicitly express their knowledge of fraction as repeated addition and fraction makes bigger within the designed learning environment.

#### 7.7.4 Representational fluency for fraction equivalence and fraction order.

Oscar was asked to name fractions and fraction equivalence on the axes of the AM-FM representation and a drawn number line. When using the AM-FM representation, I positioned the marker line at a particular location,  $a/b$ , which resulted in an area enclosure of  $a/b$ . I did not move the marker line and enclose corresponding area when working with Neato. Subsequently, Oscar showed little difficulty in locating and naming fractions and equivalent fractions across the number line feature of the AM-FM representation. Oscar appeared to demonstrate knowledge coordination of improper fractions, mixed numbers, and equivalence when using the number line feature of the AM-FM representation.

However, Oscar's knowledge of fraction equivalence was context sensitive. When working with the number line feature of the AM-FM representation, Oscar stated that  $1\frac{2}{6}$  was equivalent to  $8/12$ . Because I had enclosed an area of  $1\frac{2}{6}$ , it appears the subgrid view once again constrained Oscar's ability to correctly interpret the unit whole and in this case name an equivalent fraction. After I intervened with a drawn number line, Oscar showed no further difficulty in naming equivalent fractions across his use of the number line feature of the AM-FM representation. The intervention with the drawn number line appeared to help Oscar correctly attend to the number line features of the AM-FM representation.

Equivalence was also explored using the area model feature of the AM-FM representation. When  $A(f)$  can be reduced Oscar was asked to name an equivalent fraction for the enclosed area which entailed rearrange tile pieces in such a ways as to afford seeing fraction equivalence. The first time Oscar was asked to use the area model feature to produce an equivalent fraction, he incorrectly applied his knowledge of the one-to-one correspondence between area and the location of the marker line to the context of naming  $A(f)$ . Neato demonstrated similar context sensitivity when asked to name  $A(f)$ . Following an intervention, Oscar demonstrated no further difficulty in naming equivalent fractions across his use of the area model feature of the AM-FM representation. Oscar appeared to demonstrate knowledge coordination of the part-whole subconstruct and equivalence when using the area model feature of the AM-FM representation.

Finally, Oscar used both the number line and area model features of the AM-FM representation to order fractions. Oscar was able to correctly order fraction in all but one case. The difficulty occurred due to a shift in Oscar's construction process from using a single axis to using both axes of the AM-FM representation. Following a quick reminder of the shift in the construction process, Oscar was able to correctly order the two fractions as well as provide a correct justification for the ordering. Oscar appeared to be coordinating his knowledge of fraction order and use of the number line and area model features of the AM-FM representation.

## Chapter 8: Conclusion

### 8.1 Chapter Overview

Design-based research is predicated on local theory development and design refinement. In this chapter I discuss both. Specifically, I will present four findings associated with growth and change in students' knowledge of fraction multiplication while exposed to the designed learning environment (i.e., local theory development) and propose five related changes to the AM-FM representation and/or the clinical interview protocol (i.e. design refinement). I end this chapter with some caveats and concluding remarks regarding directions for future work.

### 8.2 Theory Implication 1A

At the start of their AM-FM construction process, both of the case study students tried to use a single axis of the AM-FM representation to construct fraction multiplication. This was despite the fact that I had presented both students with an example problem in which I explicitly highlighted the use of both axes. From a local theory perspective, I had assumed students would come to discover multiplication in the process of use both axes of AM-FM representation to construct fraction multiplication. I had assumed student would progress through a particular learning trajectory for fraction multiplication that first entailed counting shaded tiles to counting total tiles in a unit whole and later arrive at the process of multiplying shaded horizontal tile by shaded vertical shaded tiles to total horizontal tiles by total vertical tiles in a unit whole. In the act of exclusively using a single axis of the AM-FM representation, a student can come to discover multiplication much more quickly than I had anticipated. For example, to construct  $1/2$  of  $1/3$ , the student begins by constructing an area model representation of  $1/3$  using vertical partitions.<sup>30</sup> If the student continues to use vertical partitions he/she must determine the number of partitions necessary to take  $1/2$  of  $1/3$ . The student must partition each  $1/3$  area into  $1/2$  thereby resulting in a unit whole partitioned vertically into six equal parts, which is three times two. The operation of multiplication (i.e., fraction denominator multiplication) is embodied in the act of partitioning when using a single axis of the AM-FM representation.

### 8.3 Design Implication 1B

Oscar, who used a single axis of the AM-FM to construct fraction multiplication, initially struggled to arrive at a final partition of four when given the problem  $1/2$  of  $1/2$ . He constructed an area model of  $1/2$  and changed the partitions from two to three in an attempt to construct  $1/2$  of  $1/2$ . While Oscar was able to correctly point out the approximate location of  $1/2$  of  $1/2$  along the y-axis (i.e.,  $1/4$ ), he incorrectly set the partitions at three (rather than four) believing that this would result in the correct final area. In order to support students in the process of discovering fraction denominator multiplication when using a single axis of the

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<sup>30</sup> Alternatively the student can begin the construction process with  $1/2$  and the student can use either vertical or horizontal partitions to arrive either construction.

AM-FM representation, I propose they make a final partition estimation and provide a justification for that estimation before constructing the final area model representation. Furthermore, if a student struggles to justify an estimation, I propose he/she be given a drawn area model representation to use in the process of their justification. This proved successful in the case of Oscar. I used a drawn area model representation of  $\frac{1}{3}$  contrasted with an area model representation of  $\frac{1}{2}$  of  $\frac{1}{2}$  to draw Oscar's attention to  $\frac{1}{4}$ . Finally, while fraction denominator multiplication is embodied in the student's process of partitioning when using a single axis of the AM-FM representation, fraction numerator multiplication is less salient during this process. To make fraction multiplication more salient to students when using the AM-FM representation, I propose inscribing both numerators and denominators into the AM-FM representation. As area is shaded along the x-axis, area will be inscribed in the form of fraction notation along that axis. See Figure 27. An additional result of this change to design may be that students come to see fraction multiplication (numerator as well as denominator) within features of the AM-FM representation and may not need to use the number chart as a bridge between fraction notation and the AM-FM representation (as was the case with Neato).

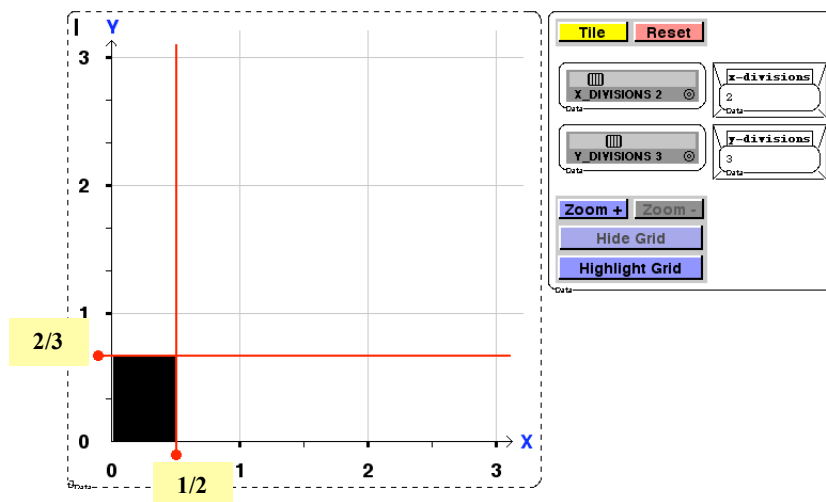


Figure 27. Illustration of fraction notation inscribed along the axes of the AM-FM representation.

#### 8.4 Theory Implication 2A

As part of the clinical interview protocol students were asked to make predictions about whether the final area output in one case would be more or less than the final area output in the previous case. Neato and Oscar struggled to make accurate predictions when both fractions changed from one case to the next. When asked to compare  $a/b$  of  $c/d$  with  $p/q$  of  $r/s$ , the students often resorted to comparing  $a/b$  to  $p/q$  or  $c/d$  to  $r/s$  without considering the role of the multiplication operation. Furthermore, when making such comparisons the students would often compare numerators or denominators and not the relationship between the two. From the perspective of local theory development, I assumed fraction as quantity as prerequisite knowledge but did not explicitly account for it in my theory. To compare  $a/b$  of  $c/d$  to  $p/q$  of  $r/s$ , students must first see each fraction as representing an amount (in this case an amount of area). They need to attend to the operation associated with “of” and come to see fraction multiplication as stretching/shrinking. While students could see fractions as quantity

and fraction multiplication as stretching/shrinking when attending to the AM-FM representation, when both fractions changed from one case to the next students instead attended to fraction notation to make their comparisons at lost their hold on fraction as quantity. Their emergent knowledge is contextual (Levin & Brar, 2010). Note, while there are alternative ways by which students can compare  $a/b$  of  $c/d$  to  $p/q$  of  $r/s$  (e.g., comparing  $ac/bd$  to  $pr/qs$  by generating equivalent fractions with a common denominator) because my objective in asking student to make such comparisons was to foster understanding of fraction multiplication as stretching/shrinking, I focus instead on what I consider to be the stepping stones to that understanding (i.e., fraction as quantity), which in this case I failed to count for in local theory.

### 8.5 Design Implication 2B

To support student understanding of fraction as quantity across the AM-FM representation and fraction notation I propose a brief tutorial session at the start of the clinical interview protocol. The tutorial would begin with the question, “What picture comes to mind when you hear  $2/5$ ?” I would like students to say something along the lines of, “A candy bar split into five (equal) pieces with only two pieces left.” Next, I would ask for a drawing of their example, talk about how fractions can be used to represent amounts of something, and provide additional examples. If the student struggles to answer the initial question, I would change the fraction from  $2/5$  to  $1/2$ . If the student still struggles, I would ask him/her to show me  $1/2$  using pizza as an example, talk about how fractions can be used to represent amounts of something, and provide additional examples. Once I felt the student had a grasp of fraction as quantity, I would ask the student to compare two fractions keeping one of their self-generated examples in mind. I would end with a brief summary of the student’s activity during the tutorial and move on to working with the AM-FM representation. Later, if the student struggles in making fraction multiplication comparisons using the AM-FM representation, I would refer back to the tutorial in order to get the student to once again see fraction as quantities while he/she is in the process of developing an understanding of fraction multiplication as stretching/shrinking.

### 8.6 Theory Implication 3A

To accurately interpret fraction multiplication as stretching/shrinking using the AM-FM representation students must see area as being stretched or shrunk during the construction process. Before student can attend to changes in area, they must first learn to correctly interpret area using the AM-FM representation. In other words, student must first coordinate their understanding of area models (as depicted in the coordinate grid of the AM-FM representation) and their understanding of number lines (as depicted along the x- and y-axes of the AM-FM representation). For example, given  $2/5$  of  $1/2$  (based on my theoretical assumptions), the student starts with an area model representation of  $1/2$  by partitioning the x-axis into halves and moving the x-axis marker line from zero to  $1/2$ . The location of the x-axis marker line ( $x=1/2$ ) now corresponds to the initial area output,  $A(i)=1/2$ . Next, the student represents  $2/5$  of  $1/2$  by partitions the y-axis into fifths and moving the y-axis marker line from one down to  $2/5$ . In the process of taking  $2/5$  of  $1/2$ , the area is shrunk from  $1/2$  of the unit whole to  $2/5$  of  $1/2$  of the unit whole. Furthermore, neither the location of the x-axis marker line ( $x=1/2$ ) nor the location of the y-axis marker line ( $y=2/5$ ) correspond to the final



area output,  $A(f)=2/10$ . Rather, to interpret  $A(f)=2/10$  the student must recognize that the final area output is the product of  $2/5$  and  $1/2$  and is less than both  $2/5$  and  $1/2$ . From the perspective of local theory development, I assumed my students would attend to changes in area as they were asked to construct fraction multiplication using the AM-FM representation. What I failed to account for was the role the number line feature of the AM-FM representation would play in how students came to interpret area.

### 8.7 Design Implication 3B

Neato struggled to correctly coordinate his emergent understanding of the area model and number line features the AM-FM represented. At one point he incorrectly attended to the x-axis of the AM-FM representation to interpret  $A(f)$ . Oscar, on the other hand, showed no such difficulties. One key difference in the clinical interview protocol for Neato and Oscar was that Oscar was explicitly asked to identify fractions, fraction equivalence, and  $A(i)$  using both features of the AM-FM representation. As such, Oscar came to correctly interpret area and see the one-to-one correspondence between  $A(i)$  and the location of a marker number line and the lack of such a correspondence in the case of  $A(f)$ . To support similar understanding among students I propose that both the area model and number line features of the AM-FM representation be made salient early in the clinical interview protocol. Students would be asked to represent fraction, show fraction equivalence, and identify  $A(i)$  using both the x- and y-axes of the AM-FM representation and the coordinate grid of the AM-FM representation. I also propose the use of direct counter-suggestions to test the strength of students' emergent understanding. For example, given an AM-FM representation of  $2/5$  of  $1/2$ , I would ask, "Another student said the final area output is  $2/5$ . Do you agree or disagree? Why or why not?"

Finally, while I had assumed student would attend to changes in area as they were asked to construct fraction multiplication using the AM-FM representation, the clinical interview protocol did more to support students in attending to area than to changes in area. For example, at the end of each case students were asked to interpret final area. Occasionally students were asked to compare the final area to the previous case and validate their prediction. I found this did little to support student understanding of fraction multiplication as stretching/shrinking because so much time had passed between when the initial prediction was made and justified to when the final area output was constructed, interpreted, and validated. To support students in seeing fraction multiplication as stretching/shrinking, I propose that changes in area be made salient not just across cases but also within a case. For example, given an AM-FM representation of  $2/5$  of  $1/2$ , I would present students with  $1/5$  of  $1/2$  and ask for a prediction of which would result in more area, a justification for the prediction, and a construction of  $1/5$  of  $1/2$  if necessary (i.e., if the prediction or justification is incorrect). Next, I would present students with  $4/5$  of  $1/2$  and proceed through the same protocol. If the student provides correct predictions and justifications for both  $1/5$  of  $1/2$  and  $4/5$  of  $1/2$ , I would move to the next case. A correct prediction and justification would be something along the lines of, "I think  $1/5$  of  $1/2$  would be less area because  $1/5$  is less than  $2/5$  so you're taking less of the same amount." If the student struggles, I would have the student construct  $1/5$  of  $1/2$  and  $4/5$  of  $1/2$  in order to validate his/her incorrect predictions or incorrect justifications.

## 8.8 Theory Implication 4A

The concept of unit is key to student understanding of fraction multiplication. In using the AM-FM representation to see fraction multiplication as stretching/shrinking students must grapple with a shifting unit. For example, given  $2/5$  of  $1/2$ , first the student operates on the  $1 \times 1$  unit whole by shading  $1/2$  of the unit whole, next the student operates on  $1/2$  by shading  $2/5$  of  $1/2$ , and lastly to interpret the final area output,  $A(f)$ , the student once again attends to the  $1 \times 1$  unit whole by comparing the final shaded area relative to the total area that make up the unit whole. To help students attend to the shifting unit when constructing fraction multiplication using the AM-FM representation I conducted a brief tutorial on unit at the start of the clinical interview protocol. The tutorial involved drawn area model representations of different size and shape and asking students to interpret and compare area across different units. Unfortunately, interpreting area using drawn area models did not transfer to the context of interpreting area using the AM-FM representation. Despite the use of the tutorial on unit, both students struggled when asked to interpret areas that involved improper fractions and/or mixed numbers. Recall from Chapter 3 that when given an improper fraction or mixed number the subgrid view extends to two or more unit whole. See Figure 9 regenerated below. Rather than compare the shaded area to the  $1 \times 1$  unit whole to arrive at  $A(f)=8/15$ , Oscar compared the shaded area to the  $2 \times 1$  unit whole and arrived at  $A(f)=8/30$ . While the extended subgrid view did provide me the opportunity to assess the strength of students' emergent knowledge of fraction multiplication and the concept of unit, it also made the act of interpreting area problematic for students. The unit was not as salient to students as I would have expected.

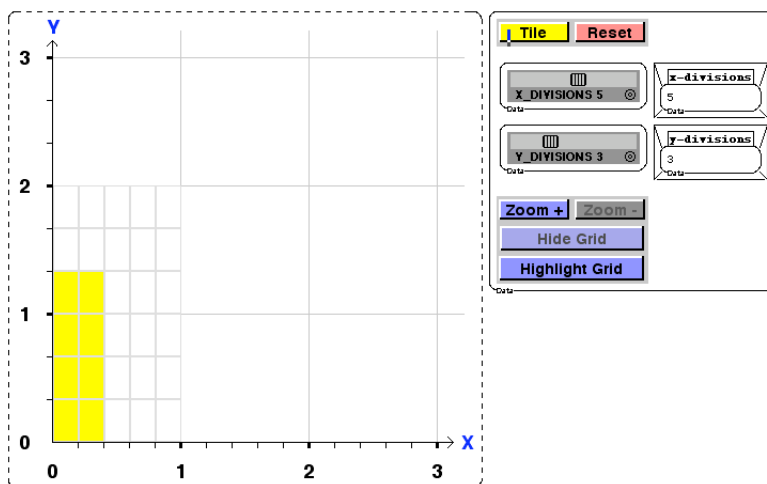


Figure 9. Illustration of the AM-FM subgrid view with  $4/3 \times 2/5$  area.

## 8.9 Design Implication 4B

To better support student understanding of unit I propose removal of the tutorial on unit, two changes to the clinical interview protocol, and a change to the AM-FM representation. Rather than use drawn area model representations to explore unit at the start of the clinical interview, I propose the use of counter-suggestions as students are presented the first case involving an improper fraction. For example, given an AM-FM representation of  $4/3$  of  $2/5$ , if the student interprets  $A(f)$  as being  $8/15$  and gives an accurate justification for the

interpretation, I would ask, “Another student said the final area is  $\frac{8}{30}$  because there are 8 shaded pieces and 30 pieces in total. What do you think of that reasoning?” If the student provides an accurate explanation for why  $\frac{8}{30}$  is an incorrect interpretation, I would move on to the next case. If the student initially interprets  $A(f)$  as being  $\frac{8}{30}$  and gives an appropriate justification for the interpretation, I would ask, “Another student said the final area is  $\frac{8}{15}$  because there are 8 shaded pieces and 15 total pieces in the  $1 \times 1$  unit whole. What do you think of that reasoning?” If the student changes his/her interpretation from  $\frac{8}{30}$  to  $\frac{8}{15}$  based on an accurate explanation for why  $\frac{8}{30}$  is an incorrect interpretation, I would move on to the next case. If the student fails to see a conflict between the two interpretation or continues to view  $A(f)$  as being  $\frac{8}{30}$ , I would lead the student through a discussion of the different implications of attending to a  $1 \times 1$  unit whole versus attending to a  $2 \times 1$  unit whole.

The second change I propose to the clinical interview protocol is the addition of the question, “What do you call this single tile piece?” following every final area output interpretation and justification. I did this when students struggled with interpreting area and I found that it helped students attend back to the  $1 \times 1$  unit. I would lift a single tiled piece of area and ask for a name for that tile. If the student failed to answer I provided example names, “Would you call it a fifth? An eighth? An eleventh?” Such examples were not only sufficient in getting students to answer correctly but it also lead students to provide correct justification for their interpretation. Once a student gave a name for the tile piece, say  $\frac{3}{5}$ , I would follow with, “ $\frac{3}{5}$  of what?”

The final change I would make to the AM-FM representation is to highlight the primary  $1 \times 1$  unit whole in the bottom right corner by shading the background light grey. This will allow for the  $1 \times 1$  unit whole to be salient throughout the construction process (from initial shading to final shading to tiling) and may help students to attend to the  $1 \times 1$  unit whole in cases where one or more of the fraction inputs is greater than one. See Figure 28 for a screenshot of the proposed change.

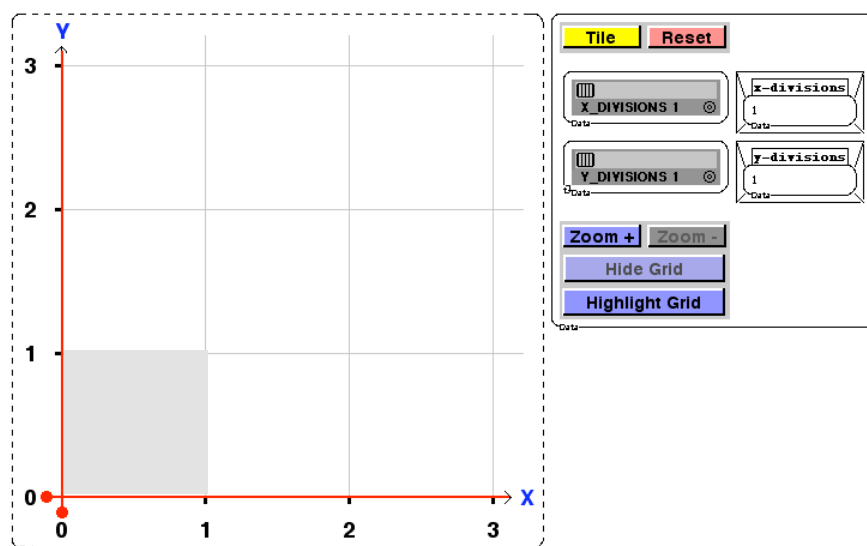


Figure 28. Illustration of the  $1 \times 1$  unit whole with background shading.

## 8.10 Caveats

The implications for theory are specific to the AM-FM representation and the designed learning environment. However, it would be incorrect to assume that any one representation alone is sufficient in supporting student competence with fraction multiplication. A student could, for example, come to see fraction multiplication as stretching/shrinking without the use of the AM-FM representation. In the context of my learning environment, one could argue that area may (from the perspective of students) emerge somewhat magically as the intersection between two number lines, and given the tendency of both my two case study students to operate with a single axis of the AM-FM representation in their initial constructions, it might be more appropriate to use the number line as a representation to help students see fraction multiplication as stretching/shrinking. As I stated in Chapter 1, there exist multiple representations for supporting student understanding a given topic and the affordances and constraints of different representations differ. In this dissertation, I attempted to explicate some of these for the AM-FM representation. I did so, in order to understand how best to use the representation to support particular aspects of fraction understanding. But the question of how that knowledge will "fit" with students' general understandings of fractions remains to be worked out. The point was not to position the AM-FM representation against other representations.

The implications for design may appear to focus on supporting student performance. However, performance is not the goal. The goal is to support growth in change in students' knowledge of fraction multiplication; knowledge this is conceptually as well as procedurally rich. For example, in the case of supporting student understanding of unit, the objective is not to have the student simply arrive at the correct interpretation of  $A(f)$ . The objective is to have the student develop sensitivity for unit selection and recognize the implications of attending to one unit over another. As such, the proposed changes to design do not attempt to eliminate the extended subgrid view of the AM-FM representation because it shifts students' attention away from the  $1 \times 1$  unit whole. Rather, I propose (a) the use of counter-suggestions with all students (including those who attend to the  $1 \times 1$  unit whole), (b) the use of additional questions to better understand how students are attending to particular features of the representation related to the concept of unit (e.g., "What do you call this single tile pieces?"), and (c) changes to the background shade of the  $1 \times 1$  unit whole so that both the extended subgrid view and the  $1 \times 1$  unit whole is visible when working with fractions larger than one. In summary, I do not propose to make changes to the designed learning environment to eliminate aspects of the design that offer students the opportunity to grapple with conceptual hurdles. Such opportunities provide a means by which to support growth and change in student knowledge and are therefore necessary to gain a more complete and accurate snapshot of student competence.

## 8.11 Future Work

The most immediate future work entails (a) the development of a more robust learning theory and (b) a better conceptualization of the design space and how to move more optimally within it.

In Chapter 4, I introduced my conjectures in the form of (a) the idealized hypothetical initial state of student understanding, (b) the idealized hypothetical exit state of student understanding, and (c) the idealized hypothetical learning trajectory from the initial state of

student understanding to the exit state of student understanding. One next step would be to more systematically map an experimental student into the idealized hypothetical exist state, subtract the difference in competence between the ideal case and the experimental case, and use what is left as potential error to guide the refinement of both theory and design.

In the caveat section, I discussed the value of conceptual hurdles elicited by the inclusion of more complex contexts within a designed learning environment. One next step would be to develop principles for designing in such circumstances. When and how do we give students the opportunity to grapple with conceptual hurdles by including slightly more complex contexts?

In later work, I plan to scale up from tutorial settings to small group settings to whole classroom settings. The scale up process will be gradual because each new context introduces new variables. The introduction of these new variables necessitates the use of additional theoretical perspectives in order to gain a more complete understanding of the complex relationship between the design of innovative learning environments and mathematics learning and teaching. In addition to scaling up, I also plan to shift content focus from topics in early elementary mathematics to topics in geometry and statistics. Designing learning environments for new content will allow me to test my theoretical and empirical assumptions and gain a deeper understanding of DBR as a methodology. Finally, I plan to use what I learn through DBR to investigate and support teacher learning. I will use the tools and local theories of student learning developed and refined through DBR as a means for teachers to gain insight into the practice of using representations to elucidate and guide student thinking.

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### Appendix A: Test Items

- 1) Is  $\frac{2}{3} \times \frac{5}{7}$  more or less than  $\frac{5}{7}$ ? Explain why.
- 2) What is  $\frac{1}{2} \times \frac{2}{3}$ ? Show all work.
- 3) What is  $2\frac{1}{2} \times \frac{2}{3}$ ? Show all work.
- 4) Convert the improper fraction  $\frac{9}{5}$  into a mixed number. Show all work.

[Item 5: Adapted from Moss & Case, 1999]

- 5) Which fraction is greater,  $\frac{5}{8}$  or  $\frac{5}{4}$ ? Explain why.

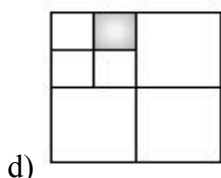
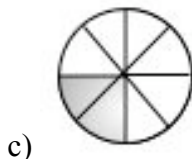
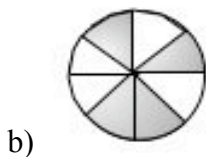
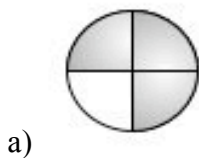
[Item 6: Adapted from Behr, Lesh, Post, & Silver, 1983]

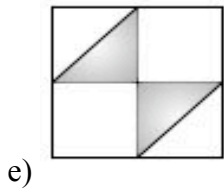
- 6) Shade  $\frac{3}{4}$  of the rectangle below.



[Item 7: Adapted from Saxe, Taylor, McIntosh, & Gearhart, 2005]

- 7) For each problem below, write a fraction to show which part is gray:





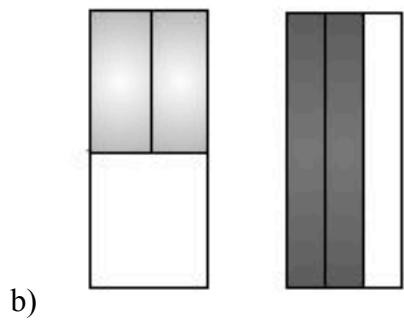
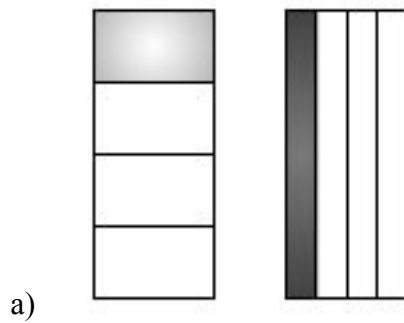
[Item 8: Adapted from Behr, Lesh, Post, & Silver, 1983]

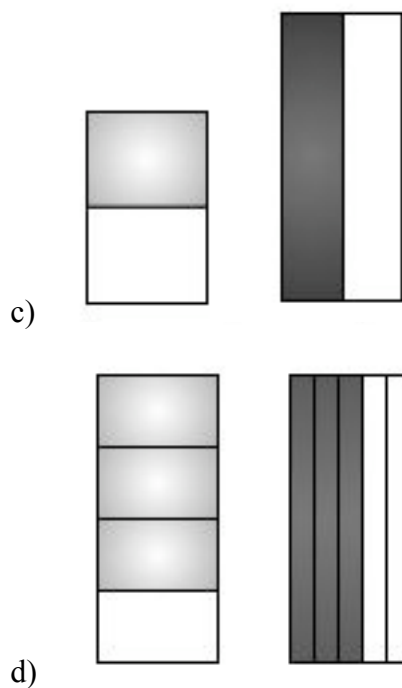
8) Shade  $\frac{3}{4}$  of the rectangle below.



[Item 9: Adapted from Armstrong, Larson, 1995]

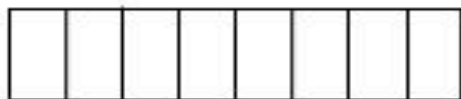
9) The rectangles represent two cakes. The shaded part is what is left of each cake. Which rectangle represents more cake. If they are the same amount write same next to the two cakes.



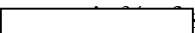


[Item 10: Adapted from Behr, Lesh, Post, & Silver, 1983]

10) Shade  $\frac{3}{4}$  of the rectangle below.

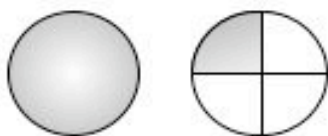


[Item 11: Adapted from Cramer, Post, & deMas, 2002]

11)  some area. Draw the whole area below.

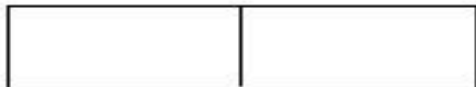
[Item 12: Adapted from Thompson & Saldanha, 2003; Mack, 1993]

12) See figure below. According to Mo it represents the fraction  $1\frac{1}{4}$ . But according to Sam it represents the fraction  $\frac{5}{4}$ . Mark says they are both right. Do you agree with the Mark? Why or why not?



[Item 13: Adapted from Behr, Lesh, Post, & Silver, 1983]

- 13) Shade  $\frac{3}{4}$  of the rectangle below.



[Item 14: Adapted from Streefland, 1997]

- 14) (a) If you have to divide 3 chocolate bars among 4 children equally, would each person get more or less than half a chocolate bar? Explain why.  
 (b) Draw a picture of the situation described above to explain why a person would get more or less than half of a chocolate bar.  
 (c) What fraction of a **whole chocolate bar** did each person get?

[Item 15: Adapted from Mack, 1995]

- 15) (a) Your friend gives you one-half of a candy bar. You decide to eat two-thirds of it as an after dinner snack. What fraction of the **whole candy bar** did you eat? Explain why. Draw a picture of the situation if that helps.  
 (b) Write a math problem (number sentence) of the situation described in part (a) by using fractions and an operation? Examples of operations include: addition, subtraction, multiplication, and division.  
 (c) Now solve the problem you just wrote in part (b).

[Item 16: Adapted from Lamon, 1999]

- 16) (a) If 4 people share **two – 6** packs of cola equally, would each person get more or less than 4 colas each? Explain why.  
 (b) Draw a picture of the situation described above to show how many colas each person gets.  
 (c) What fraction of **all** the colas (the two – 6 packs) did each person get?

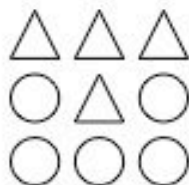
[Item 17: Adapted from Behr, Lesh, Post, & Silver, 1983]

- 17) Shade  $\frac{3}{4}$  of the rectangle below.



[Item 18: Adapted from Lamon, 1999]

- 18)



In the figure above the area of the triangles is equal to the area of the circles. Use the figure to answer the following questions.

- (a) The triangles (all of them) are what fraction of the whole picture?
- (b) Two triangles (  $\triangle \triangle$  ) are what fraction of the whole picture?
- (c) Two triangles (  $\triangle \triangle$  ) are what fraction of the set of triangles?

[Item 19: Adapted from Lamon, 1999]

19. Locate  $\frac{1}{4}$  on the number line given below.



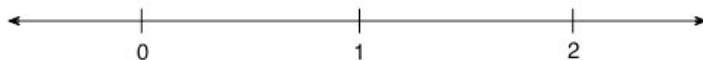
[Item 20: Adapted from Lamon, 1999]

20. Locate  $\frac{3}{9}$  on the number line given below.



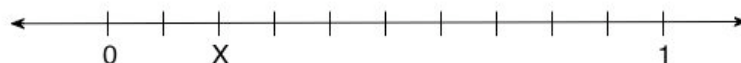
[Item 21: Adapted from Lamon, 1999]

21. Locate  $\frac{7}{5}$  on the number line given below.



[Item 22: Adapted from Lamon, 1999]

22. Find the fraction represented by "X".



- 23) Is  $\frac{3}{2} \times \frac{5}{7}$  more or less than  $\frac{5}{7}$ ? Explain why.
- 24) Give two fractions between  $\frac{1}{3}$  and  $\frac{2}{3}$ . Show all work.
- 25) What is  $\frac{3}{5} \times \frac{1}{3}$ ? Show all work.
- 26) What is another fraction that is equivalent (equal) to  $\frac{3}{4}$ ? Show all work.
- 27) Convert the mixed number  $3\frac{3}{4}$  into an improper fraction. Show all work.

[Item 28: Adapted from Moss & Case, 1999]

28) Which fraction is greater  $\frac{3}{7}$  or  $\frac{4}{7}$ ? Explain why.

[Item 29: Adapted form Moss & Case, 1999]

29) Which is more  $\frac{1}{3}$  of 15 lollypops or  $\frac{3}{2}$  of 6 lollypops? Explain why.

[Item 30: Adapted form Moss & Case, 1999]

30) Which is more  $\frac{5}{8}$  of 1 cup of sugar or  $\frac{5}{4}$  of 1 cup of sugar? Explain why.

## Appendix B: Summary of Pretest to Posttest Gains

Areas:	Test Items:	NP	S1	S2	OA	S3	S4	S5	S6	S7	S8
construction: am	6: $3/4$ w/ whole	C-C	C-C	I-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	8: $3/4$ w/ $3/3$	C-C	C-I	I-I	C-C	I-I	C-C	C-C	C-C	C-C	C-C
	10: $3/4$ w/ $8/8$	C-C	I-I	C-C	C-C	I-C	C-C	C-C	C-C	C-C	C-C
	11: unit	I-C	I-I	C-C	C-C	I-I	I-I	I-I	C-C	C-C	C-C
	13: $3/4$ w/ $1/2$	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	17: $3/4$ w/ $4/4$	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
construction: nl	19: $1/4$ w/ no ref	I-C	I-I	I-C	C-C	I-I	C-C	C-C	C-C	C-C	C-C
	20: $3/9$ w/ 0,1	I-C	I-I	I-C	C-C	I-I	I-C	C-C	C-C	C-C	C-C
	21: $7/5$ w/ 0,1,2	I-C	I-C	I-C	C-C	I-I	C-C	C-C	C-C	C-C	C-C
interpretation: am	7a: $3/4$	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	7b: $4/8$	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	7c: $2/8$	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	7d: $1/16$	I-I	I-I	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	7e: $1/4$	C-C	I-I	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	9a: $1/4=1/4$	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	9b: $2/4 < 2/3$	C-I	C-C	I-C	I-C	C-C	I-C	C-C	C-C	C-C	C-C
	9c: $1/2 < 1/2$	C-I	C-I	C-C	I-I	C-C	I-I	I-I	C-C	C-C	C-C
	9d: $3/4 > 3/5$	I-I	C-C	C-I	I-I	I-C	C-C	C-C	C-C	C-C	C-C
	12: $1 \frac{1}{4} = 5/4$	C-C	I-I	I-I	C-C	I-I	C-C	C-C	C-C	C-C	C-C
	18: discrete set	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
interpretation: nl	22: $2/10$	I-C	I-C	I-C	C-C	I-I	I-I	C-C	C-C	C-C	C-C
equivalence	4: $9/5 = 1 \frac{4}{5}$	I-I	I-C	C-I	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	26: $3/4 = ?$	C-C	C-C	I-C	C-C	I-I	C-C	C-C	C-C	C-C	C-C
	27: $3 \frac{3}{4} = 15/4$	I-C	C-C	I-I	I-C	C-C	C-C	C-C	C-C	C-C	C-C
order	1: $2/3 \times 5/7, 5/7$	I-C	I-C	C-I	I-I	I-C	I-I	I-C	C-C	C-C	C-C
	5: $5/8, 5/4$	I-C	C-C	I-I	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	23: $3/2 \times 5/7, 5/7$	I-C	I-C	C-C	C-I	I-C	C-C	I-C	I-I	C-C	C-C
	24: between $1/3, 2/3$	I-I	I-I	I-I	C-C	I-I	I-I	C-C	I-C	C-C	I-C
	28: $3/7, 4/7$	C-C	I-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
multiplication	2: $1/2 \times 2/3$	C-C	I-C	C-C	I-C	C-C	C-C	C-C	C-C	C-C	C-C
	3: $2 \frac{1}{2} \times 2/3$	I-I	C-C	I-I	I-C	I-I	I-C	C-C	I-C	I-C	C-C
	25: $3/5 \times 1/3$	C-C	C-C	C-C	I-C	C-C	C-C	C-C	C-C	C-C	C-C
unit	14: 3 bars, 4 kids	I-C	C-C	I-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	15: $2/3 \times 1/2$	C-C	I-C	C-C	C-C	I-I	C-C	C-C	C-C	C-C	C-C
	16: 2-6packs, 4 kids	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C	C-C
	29: $1/3 \times 15, 3/2 \times 6$	I-C	I-I	I-I	C-I	I-I	I-I	I-C	I-I	C-C	C-C
	30: $5/8 \times 1, 5/4 \times 1$	C-C	C-C	C-C	C-C	I-I	C-C	C-C	C-C	C-C	C-C

**KEY:**

am = area model, nl = number line



GREEN = I-C = incorrect on pretest and correct on posttest  
RED = C-I = correct on pretest and incorrect on posttest  
YELLOW = I-I = incorrect on both pretest and posttest  
BLUE = C-C = correct on both pretest and posttest  
Note: omissions were considered incorrect

### Appendix C: Neato's Content Log

[day 2 of clinical interview]  
[student = n; interview = rb]

2:00

rb: do you remember how we did  $2/3$  of  $3/4$ ?

n: no

2:30

case 8a:  $1/2$  of  $1/2$

n: it will be  $1/4$

rb asks n to use am-fm

rb asks n to start off with whole

n partitions x axis into 4ths and then y axis into 4ths

n says x axis represents slices

n moves x axis marker to half and then to  $1/4$

n: you give them half of  $1/2$  and that's  $1/4$

rb: how do you know that's  $1/4$ ?

n: I guess I just know. like cause you cut it up into four. because you can turn one slice into four pieces?

rb: because 4 pieces fill the whole?

n: 4 pieces would fill the whole?

rb: why did you split x axis into 4s?

n: how did I know or why did I do that?

rb: why?

n: will you had to start out with whole

reset the problem

rb asks to see one

rb: it says to half of it and then half of that so why did you split it into 4s? why didn't you split it into halves?

n: actually I don't know but 4s would make it easier cause it would be already cut up.

rb shows method using both markers and partitioning both axes into halves

rb: so what did you think of that method?

n: it's a good method.

rb: so you kind of knew that answer was going to be  $1/4$  so it that why you split it into 4s>?

n: yeah.

tiles

rb: if I move this piece here it's still called?

n: a fourth

reset

8:45

8b =  $\frac{1}{2}$  of  $\frac{1}{3}$  slices/rat

prediction: less than 8a; cause you have  $\frac{1}{3}$ ;

rb: which one is bigger,  $\frac{1}{3}$  or  $\frac{1}{2}$ ?

n:  $\frac{1}{2}$ .

rb: how do you know?

n shows it using the am-fm rep (sets partitions for x axis at 3 moves marker to  $\frac{1}{3}$  and then sets partitions at 2 at which point marker moves automatically to  $\frac{1}{2}$ )

10:40

n shows whole

sets partitions at 3 and 2

moves x marker to  $\frac{1}{3}$  and also correctly names  $\frac{1}{3}$ ,  $\frac{2}{3}$ ,  $\frac{3}{3}$  on the number line but couldn't name  $1\frac{1}{3}$  as  $\frac{4}{3}$  so we moved to mini-lesson

12:10

[mini lesson on naming improper fractions on number line]

rb: so look at the pattern, and how many thirds would this be (pointing to zero).

n: 0 thirds. oh, it would be 4 thirds. that would be 5 thirds and 6 thirds.

rb: and what's 6 divided by 3.

n:  $\frac{1}{2}$

n: 2

rb: and this ( $\frac{5}{3}$ ) would be?

n:  $1\frac{2}{3}$

n states that what makes a fraction improper is when you have a bigger number over a smaller number.

rb: and what's the opposite of an improper fraction?

n: a proper fraction

rb: and this is called a what ( $1\frac{1}{3}$ )?

n: a mixed number

return to am-fm

n moves z marker down to  $\frac{1}{2}$

rb: and what is this piece (shaded) called? what's the name of that piece. remember you called the other piece  $\frac{1}{4}$ . what is the name of that piece?

n: it's  $\frac{1}{2}$  of  $\frac{1}{3}$ .

rb: is that

n: it's half of  $\frac{1}{3}$ .

rb: is it more than a third or less than a third?

n: less than. it's half of it.

n: a sixth. if you had 6 of them it could be thirds. if you had 6 of them you could make (moves cursor around one whole slice).

rb: you could make a whole slice?

n: yeah.

rb: ok, so it this right. the way to find a whole slice. the way to find a name is to figure out how many of these would make up the whole slice?

n: yeah.

rb asks to tile

rb: does that help?

n: yeah.

rb: so it's  $1/6$ .

17:45

$8b^* = 1/3$  of  $1/2$  slice/rat

prediction = equal as 8b (equal was not given as option)

rb: why?

n: because on the other record sheet you said it didn't matter what the order was it would be the same if you were multiplying.

rb presents blue record sheet in which n recognizes the in case 3 (3 rates  $1/2$  slice/rat) adding and multiplying is the same thing.

rb: what are we multiplying here (returning to green record sheet)?

n: 1 into  $1/3$  of a  $1/2$ .  $1/2$  of  $1/3$ . no, how do you multiply that? calculator.

rb: if you were multiple these two numbers ( $1/3$  and  $1/2$ ) what would you get?

n:  $1/6$

rb: how do you know that

n says it's the same as case 8b and the rule for multiplying is to multiple across top and across bottom.

21:00

using am-fm n partitions into halves and thirds. then moves the y maker up to  $1/3$  first and them moved x marker to the left to  $1/2$  (didn't start with whole).

rb points out that n didn't start with the unit.

rb: what's the name of that piece?

n:  $1/6$

n tiles

n: it's one of 6 pieces of 1 whole.

rb: do those 6 pieces always have to be equal?

22:30

[mini lesson on equality of parts within an area model with counter suggestion  $1/4$  verses  $1/6$ ]

rb: what would the name of that fraction be?

n:  $1/6$

rb redraws

rb: how much cheese did you give out?

n:  $\frac{1}{4}$ .

rb: why is it  $\frac{1}{4}$ ?

n: because these two are half (pointing to the two  $\frac{1}{4}$  pieces) and this is  $\frac{1}{4}$  (two of the  $\frac{1}{8}$  pieces) and this is  $\frac{1}{4}$  (two of the other  $\frac{1}{8}$  pieces) and that's a half so you get 1 whole.

rb manipulates drawing by removing lines to produces a whole partitioned into 4ths

rb: so it's not just a matter of counting boxes

rb shows that  $\frac{1}{6}$  would be less than  $\frac{1}{4}$  if using the same whole

rb: so you're pieces do have to be equal in order to just do the count.

25:45

8c =  $\frac{2}{3}$  of  $\frac{1}{3}$  slice/rat

prediction: more or less than  $\frac{1}{6}$ ? in these two cause remember we got  $\frac{1}{6}$ .

n: more.

rb: why?

n: because instead of  $\frac{1}{3}$  of  $\frac{1}{2}$  it's  $\frac{2}{3}$  of  $\frac{1}{3}$ .

rb: I'm not sure I understand why that makes it bigger?

n: can I show you on the (am-fm rep)

rb: well are you saying that  $\frac{2}{3}$  is bigger than  $\frac{1}{3}$ ?

n: yeah.

rb: and  $\frac{1}{3}$  is bigger than  $\frac{1}{2}$ ?

n: no.

rb: so this number is smaller than that one and this number is bigger than that one so why doesn't it just even out?

n: because they're not equal.

rb: like it's not  $\frac{1}{3}$  of  $\frac{1}{2}$ .

n: oh, this is still  $\frac{1}{2}$ .

rb:  $\frac{2}{3}$  counts as  $\frac{1}{2}$

n: it's  $\frac{2}{3}$  half of  $\frac{3}{3}$ ?

rb:  $\frac{2}{3}$  is half of  $\frac{3}{3}$ ?

rb: show me  $\frac{3}{3}$

n using am-fm to come to conclusion that  $\frac{1}{2}$  of  $\frac{3}{3}$  is less than  $\frac{2}{3}$  of  $\frac{3}{3}$ .

rb: for case 8c are we gonna use more or less than 8b and you said

n: more.

rb: ok. what about in comparison to 8b?

n:  $\frac{1}{2}$  of  $\frac{1}{3}$

rb: which one are we gonna end up using more cheese?

n: 8c because  $\frac{2}{3}$  is more than  $\frac{1}{2}$

set partitions at 3 and 3. moves x marker to  $\frac{1}{3}$  and then moves y marker down to  $\frac{2}{3}$ .

rb: how much is that? how much cheese did you end up suing?

n:  $\frac{2}{6}$ . no.  $\frac{1}{9}$ . I'm  $\frac{2}{9}$ .

rb: how are you getting  $2/9$ .

n counts out the 9 squares that make up the whole then tiles

rb: so was that more or less cheese than  $1/6$ ?

n: more.

rb: how do you know?

n: because half would be like there (shows it as an imaginary line between  $1/3$  and  $2/3$ )

32:15

8d =  $3/5$  of  $3/4$  slice/rat

predictions (compared to  $2/3$  of  $1/3$ ): more than 8c because 5ths are more, there's more slices in 5ths.

rb: do these number have anything to do with it?

n:  $3/4$  is bigger than  $1/3$ . no,  $3/4$  is equal to  $1/3$ .

rb: can you show me?

n: actually no they're not.

n uses am-fm to show that  $3/4$  is bigger than  $2/3$  and  $3/4$  is bigger than  $1/3$ .

reset

36:20

set partitions at 5 and 4 and moved y maker up to  $3/4$  and then decided he did it backwards so changed partitions to 4 and 5 and then moved y maker up to  $3/5$ .

rb: so what is that piece called? how much of a slice?

n:  $2/3$ . I think.

rb:  $2/3$ . why is it called  $2/3$ ?

[12 second pause]

rb: so this axis represents slices. so if it went all the way up here (pointing out  $y=1$  on the y axis) it would be one.

n: one.

rb: how much is this? what is this pointed called (pointing to  $y=3/5$ )?

n: the point is called  $3/5$  right now.

rb: so how much of a slice did you take?

n:  $3/5$ ?

rb:  $3/5$ .

n: oh.

rb: remember we stopped doing the 1 by 1. so let's do that.

38:30

reset

n took  $3/4$  of  $3/5$  and rb points out that problem asks for  $3/5$  of  $3/4$ .

rb: does it matter? are you gonna get different answer if you did it differently?

n: no. no.

rb: what's our final output? how much cheese do we end up using?

n:  $6/20$ ?

rb:  $6/20$ . how did you get that.

n: that's just my guess. but yeah.

rb:  $6/20$ , how did you guess that. that's an interesting number to just randomly guess.

n: well, because 5 times 4 is 20 oh, no, it's  $9/20$ . it should be  $9/20$ .

tiles to confirm

43:00

$8e = 5/6$  of  $2/5$  slice/rat

predictions (more or less than  $d = 3/5$  of  $3/4$ ): more because  $5/6$  is bigger than  $3/5$ .

rb: why?

n: your only 1 slice away from having 1 whole.

rb: how are  $3/4$  and  $2/5$  related? what's bigger?

n:  $3/4$  is bigger.

rb: why?

n: because a fifth is smaller is than a fourth.

rb asks then how n shows  $8e$  is gonna be more.

n: I don't know.

rb: you do know that  $5/6$  is bigger than  $3/5$  and that  $2/5$  is smaller than  $3/4$ .

n suggests doing the actually multiplication to figure it out. rb asks n to use am-fm.

47:20

partitions into 5 and 6 and says he likes using the first number on the x axis and goes on to say that you shouldn't get a different answer if you flipped it. shows 1 whole. moves y marker down to zero and then moves up to  $5/6$

rb checks if n can give fraction makes (improper and mixed on y axis and he does correctly for  $6/6$ ,  $8/6 = 1 \frac{2}{6}$ )

rb asks n to go back to  $5/6$

rb: how much cheese do you have there?

n: you have  $5/6$ .

rb explains commutative prop relative to  $8e$ .

rb: so how much is that?

n: that's  $5/6$  of  $2/5$ . yeah.

rb: so how much cheese did we use.? what's our output?

n:  $10/30$ .

rb points out that she notices n looking at chart to get 10 and then double checked it using the am-fm rep and first says no and then says yes. n says when he said no it was because he failed to count for the partition that would arise from the  $1/5$  marker on the x axis but then he noticed it and could see that there would be 10 yellow boxes and the 30 he got from 6 times 5.

52:30

rb: is there another name for  $10/30$ ?

rb recaps the different names for mixed numbers and improper fractions.

rb: feel free to move these around in the box (whole) if that helps. I want to know if we can call that area something else?

n moves files down to get  $2/3$  and then corrects himself while providing a justification to  $1/3$ . then rb takes n to chart and asks what would you do with the number to go from  $10/30$  to  $1/3$ . n says you can just take away the zero. rb then tells n that there's another name for it. how else can you view then. you viewed them as groups of 10. rb suggests using a group of size different than 10.

rb: what other size group could you use?

n moves tiles to see if he can do make groups of 3 works and says there's 1 left over and then quickly follows up with well we know groups of 2 works and moves tiles to original position at which point rb says we don't have an answer for groups of 2.

rb: how much would groups of 2 give you if we counted in groups of 2? how many colored groups of 2 do we have and how many total groups of 2 do we have.

n: we have 5 groups of 2 so  $2/5$ ? no.

rb: how many groups of 2 do we have colored?

n: 5.

rb: and what should we compare that to?

n: 2.

rb recaps what was counting when coming up with  $10/30$  and when coming up with  $1/3$ . and then works with n to come up with  $5/15$

1:04:45

$8f = 4/3$  of  $2/5$  slices/rat

prediction (more or less than  $5/6$  of  $2/5$ ): less.  $4/3$  is equal to  $1 \frac{1}{3}$ .

n:  $8f$  should be more than cause  $4/3$  should be more than  $5/6$ .

rb: you don't seem very convinced of that. why don't you check it out.

n sets partitions at 3 and 4 "because it's  $4/3$ " makes a whole and then moves and then gets stuck. "it would be 1 and  $1/3$ " and then moves the x marker to  $1 \frac{1}{3}$ .

rb: now what are you going to do?

n:  $2/5$

n asks to reset and sets partitions at 5 and 3 so he can have first fraction represented on the the y axis. n show whole (rb asks) and correctly moves y marker up to  $4/3$  (recognizes that so far he's giving out  $1 \frac{1}{3}$  slice of cheese) and then correctly moves x marker to  $2/5$ .

01:11:00

rb asks n to name points on x axis which he correctly does for  $2/5$ ,  $0/5$ ,  $8/5$  (be counting up from  $5/5$ , i.e., 6, 7, 8),  $10/5$ ,  $15/5$ ,  $4/3 = 1 \frac{1}{3}$ )

rb recaps the procedure used by n to construct  $4/3$  of  $2/5$ .

rb: ok, so how much cheese did you end up using.



n counts shaded squares

n: 8 something.  $8/15$ .

rb: how did you get 15?

n: 5 times 3.

rb: what were you pointing to when you said 5.

n scrolls over x and y axis for whole slice.

tiles

n: without having the top part there.

rb: ok, go ahead and move your stuff however you need to move it to show me it's  $8/15$ .

rb asks n to highlight grid.

rb: does  $8/15$  reduce?

n: no.

rb: how do you know?

n begins to move around tiles and concludes that it does not reduce after moving 6 of the 15ths to right two columns and leaves 2 of the 15ths in the upper left corner. rb and n discuss not being able to count by 2s, 3s, 4s, and 5s. rb then moves to chart to show the link to the numbers. is there any number that divides both of these numbers. when 10 divides both of these numbers and also 2 divides both of these numbers. when 10 divides both of these numbers we treated 10 as a whole set. when 2 divided both of these numbers it became  $5/15$  and we were treating it like pairs.

1:18:15

8g =  $2 \frac{2}{3}$  of  $2/5$  slice/rate (compared to  $4/3$  of  $2/5$ )

prediction = more cause you have  $2 \frac{3}{5}$

n: that's equal to  $13/5$ .

rb comments on n looking at the computer screen when coming up with  $13/5$ .

rb again asks why more?

n: if  $4/3$  is only 1 and  $1/3$  and  $13/5$  has 2 so it's more. it has 2 and  $3/5$ .

1:21:45

rb asks n to use am-fm to do 8g.

n shows unit and splits both axes into 5ths, moves the y marker up to  $13/5$  and then moves x marker to the left to  $2/5$ .

rb: so how much is that? how much cheese are we giving to the rat in this case?

n: 26 pieces?

rb: how did you get 26?

n: since you have 13 pieces going up and you half it and 13 plus 13 should equals 26.

rb: I want a fraction. how much of a slice did you give or how many slices did you give. the scientist wants to know not pieces cause he doesn't know what sizes your pieces are. so what are we gonna put for our outputs. 26 what? what are the names of those little pieces, 26 pieces are they fourths?

n: 25ths. yeah, because it's five times five (pointing to unit of one whole).

rb: so we're giving out  $25/26$ ?

n: yeah

tiles

n then moves pieces into unit whole and interrupts results as 1 whole and 1 left over.

rb: so how much did you give?

n: 1 and  $1/25$ .

rb: cause how many times does 25 go into 26. 1 time with 1 left over.

rb: this is what you were doing this is 1 and  $1/4$  and you were saying that can also be called...

n: 1 and  $1/4$  can also be called  $5/4$ .

rb: same thing here.  $26/25$  can also be called  $1 \frac{1}{25}$ .

1:26:50

8h =  $1 \frac{2}{5}$  of  $6/4$  slice/rat

n laughs and when asked he comments on the problems getting harder predictions (compared to  $2 \frac{2}{3}$  of  $2/5$ ) = not asked

rb: what's our unit?

n: the unit is one whole.

rb: okay start there.

n sets partitions at 4 and 5 and then show unit. he then moves y marker up to  $1 \frac{2}{5}$  and then moves x marker to  $6/4$ .

rb: so how much cheese did we give out.

n: that's a lot. I don't know unless I count all the boxes.

rb: it is more than one?

n: yeah, it's more than one.

rb: do you think it's gonna be more than two?

n: no.

rb: okay, why don't you tile.

tiles

n then begins to move tiles around

n: maybe it is more than 2.

n: yeah, it's more than 2

n is moving tiles up above the unit whole to complete the second whole

rb: so how much is that?

n: so it's 2 and [8 second pause]  $6/4$ . 2 and  $6/4$ .

rb: 2 and  $6/4$ . ok, how are you getting the 2 and  $6/4$ ?

n: because you have 2 (scrolling over two tiled wholes), right and then 1, 2, 3, 4, 5, 6 (counting the fourth marks on the x axis).

rb: so that's how much cheese you gave out?

n: 2 and  $6/4$ .

[n highlights grid]

rb: how much cheese did you give out?

n: oh, 2 and  $2/4$ .

rb: what are each of these pieces called. what's the name of these pieces? are they a fourth?  
n: no, they're not a fourth. 20.

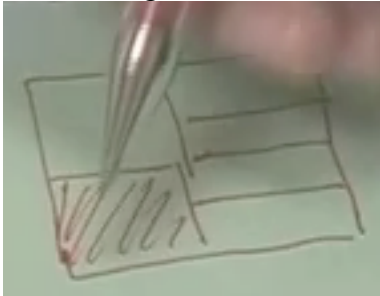
[tape ends]

## Appendix D: Neato's Content Log Summary Table

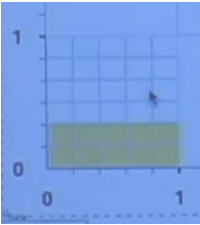
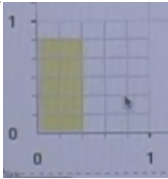
Neato: Day 2 of Clinical Interview

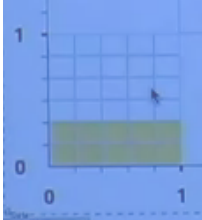
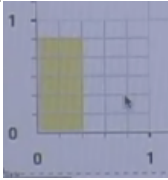
KEY: “\_”=rough quote, L=RB asked leading questions, A=assessment session, T=teaching session, ODM=order doesn't matter, C=correct, I=incorrect, L=less than previous case, M=more than previous case, E=equal to previous case, J=justification, NL=number line, EQ=equivalence, N/A=not applicable

Problem	Content Log	Notes	Other
8A. 1/2x1/2	no prediction asked; gave correct answer (1/4) w/o using am-fm; “ $\frac{1}{4}$ because you start w/ whole, cut in half, and then half of half is one fourth” represents whole; sets X slider at 4 and Y slider at 4, moves X-marker from 1 to $\frac{1}{2}$ and then from $\frac{1}{2}$ to $\frac{1}{4}$ ; “that point is $\frac{1}{4}$ because you cut whole into 4 and you can turn one slice into 4 pieces” “split it into 4ths because it would make it easier than halves” – halves were suggested by RB; (T session)	T = RB gives am-fm demo at the end and asks if N set sliders at 4 because he already knew the answer was 4 and he confirmed this.	
8B. 1/2x1/3	correct prediction (less); “less because you have $\frac{1}{3}$ instead of $\frac{1}{2}$ and $\frac{1}{3}$ is bigger than $\frac{1}{2}$ ” = L; uses x-axis of am-fm to show $\frac{1}{3} < \frac{1}{2}$ ; represents whole; sets X slider at 3 and Y slider at 2, moves X-marker from 1 to $\frac{1}{3}$ , (A session), (T1 session), moves Y-marker line from 1 to $\frac{1}{2}$ ; “name of black area is half of $\frac{1}{3}$ ”; “it's less than $\frac{1}{3}$ , it's half of $\frac{1}{3}$ ” = L; “it's a sixth” = L	A = could correctly identify $\frac{1}{3}$ , $\frac{2}{3}$ , $\frac{3}{3}$ or 1 whole, and $1\frac{1}{3}$ but did not come up with another name for $1\frac{1}{3}$ ( $\frac{4}{3}$ ). T1 = mini lesson on locating pts on number line; after RB mentions that zero can be called $\frac{0}{3}$ Neato had “oh” moment and followed with $\frac{4}{3} = 1\frac{1}{3}$ , $\frac{5}{3}$ , $\frac{6}{3} = 2$ ; RB and N also discussed terminology (improper and proper fractions, mixed number, numerator, and denominator)	

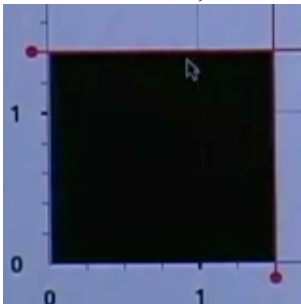
	<p>“because 6 of them will make one” (T2 session)</p>	<p>T2 = RB recaps how to name final black area</p>	
<p>8B*. <math>1/3 \times 1/2</math></p>	<p>correct prediction (equal); “equal because on record sheet we said order doesn’t matter if you’re multiplying” RB: “so are we multiplying here”, N: “we’re adding but adding can be like multiplying”, RB: “what are we multiplying”, N: “1 into <math>1/3</math> of <math>1/2</math>”, RB: “if you where to multiply <math>1/3</math> and <math>1/2</math> what would you get”, N: “<math>1/6</math> because it’s equal to 8B, N: “to multiply fractions you multiply tops and bottoms” did not represent whole; [axes were appropriately partitioned from doing 8B], moves Y-marker from zero to <math>1/3</math> and X-marker from 1 to <math>1/2</math>; “name of black area is <math>1/6</math> because when we tile it it’s <math>1/6</math> [tiles] because it’s <math>1/6</math> of the whole” (T session)</p>	<p>T = mini lesson on part-whole using area model;</p>  <p>N: “the name of that shaded piece is <math>1/6</math> maybe, no I don’t think so, the amount of cheese is <math>1/4</math> because these 2 (<math>1/4</math> pieces) are half and this is half and this is half (the 2 partitioned <math>1/4</math> pieces), you just cut the halves (the 2 partitioned <math>1/4</math> pieces)”;</p> <p>RB recaps how to compare part to whole in order to arrive at a name for the shaded area (pieces have to be equal size if you just want to count boxes)</p>	<p>commutative; operation;</p>
<p>8C. <math>2/3 \times 1/3</math></p>	<p>correct prediction (more) = L (RB: “more or less than <math>1/6</math>”); “more because instead of <math>1/2</math> of <math>1/3</math> it’s <math>2/3</math> of <math>1/3</math>” and (after A session) “<math>2/3</math> is more than <math>1/2</math>” (A session); did not represent whole; sets X slider at 3 and Y slider at 3, moves X-marker from zero to <math>1/3</math> (N: “this is <math>1/3</math>” but not sure if referring to initial area or pt on line), moves Y-marker from 1 to <math>2/3</math>; “the amount of cheese I ended up with is <math>2/6</math>, no, <math>1/9</math>, I mean <math>2/9</math> because there are two</p>	<p>A = RB: “how do you know they’re not going to balance out (<math>8C=8B*=8B</math>)?”, N: “because they are not equal”; N then goes on to correctly state that <math>2/3 &gt; 1/3</math>, incorrectly states that <math>2/3</math> is half of <math>3/3</math> but self-corrected quickly; RB has Neato uses am-fm to show <math>3/3</math> on x-axis and Neato moves marker line from <math>3/3</math> to <math>2/3</math> to where <math>1/2</math> would be and concluded <math>1/2 &lt; 2/3</math>;</p>	<p>commutative applied in justification; initial area vs. pt on line;</p>

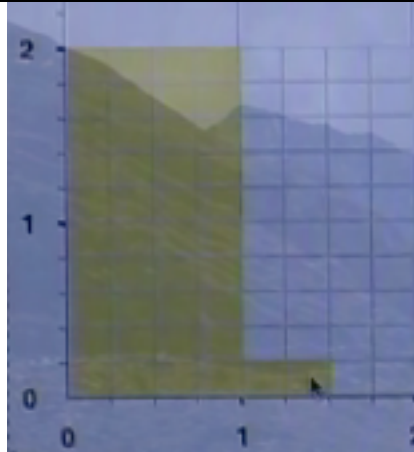
	<p>black tiles and (counts on) 9 total tiles (uses Y-marker line to show 1 tile and then 2 tile by moving down to <math>1/3</math> and then back to <math>2/3</math>)”</p> <p>“that was more cheese than <math>1/6</math> because it would be here (uses cursor to show imaginary horizontal line at <math>Y=1/2</math> to show what <math>1/2 \times 1/3</math> would look like);</p>		
8D. $3/5 \times 3/4$	<p>correct prediction (more) = L (RB: “remember in case C we had <math>2/3</math> of <math>1/3</math> which gave us <math>2/9</math>”);</p> <p>“more because 5ths are not bigger but there are like more slices”, “than in 3rds” = L; (A session);</p> <p>did not represent whole; sets X slider at 5 and Y slider at 4, moves Y-marker from zero to <math>3/4</math>, N: “oh, I did it backwards”, sets Y slider at 5 and X slider at 4, moves Y-marker from zero to <math>3/5</math>; N incorrectly names initial area as <math>2/3</math> (not <math>3/5</math>); (T1 session);</p> <p>represents whole = L; moves Y-marker from 1 to <math>3/5</math>, moves X-marker from 1 to <math>3/4</math>;</p> <p>RB: “you took <math>3/4</math> of <math>3/5</math>, would you get a different answer if you did it the other way?”, N: “no.”;</p> <p>“I used <math>6/20</math> cheese”, “it’s just a guess”, “because 5 times 4 is 20, oh, no it’s <math>9/20</math>” [tiles=L] (T2 session)</p>	<p>A = RB: “do second set of fractions have anything to do with making a prediction?” N correctly states that <math>3/4 &gt; 1/3</math>, then <math>3/4 = 1/3</math>, then uses am-fm to show that that <math>3/4 &gt; 1/3</math> and <math>3/4 &gt; 2/3</math> by partitioning x-axis into thirds and positions X-marker where <math>1/2</math> would be and then where <math>3/4</math> would be which shows it’s greater than <math>1/3</math> and <math>2/3</math>;</p> <p>T1 = RB: “what is that piece (initial area) called (<math>3/5</math>)?”, N: “<math>2/3</math>”, RB shows correspondence between <math>Y=1</math> and unit whole, RB: “what is that point (<math>3/5</math>) called?”, N: “<math>3/5</math>”, RB: “so how many slices did you take?”, N: “<math>3/5</math>?”, RB: “<math>3/5</math>.”, N: “oh.”;</p> <p>T2 = RB points out ways to look at the am-fm representation that instantiate the fractions and the operation of multiplication and repeated addition</p>	<p>T = A blur; initial area = “oh”;</p> <p>commutative; operations;</p>
8E. $5/6 \times 2/5$	<p>incorrect prediction (more); “more because <math>5/6 &gt; 3/5</math></p>	<p>A1 = RB: “what about the second set of fractions?”, N:</p>	<p>operations; commutative;</p>

	<p>because in <math>5/6</math> you're only one slice away from whole" (A1 session) ; represents whole sets Y slider at 6 and X slider at 5;</p> <p>RB: "why do you did it that way?" N: "it's easier for me to look at the y-axis is the first number on this side" and "no, it doesn't matter if you flip it." moves Y-marker down from <math>6/6</math> to <math>1/6</math> and then counts up to <math>5/6</math>;</p> <p>(A2 session);</p> <p>N correctly names initial area (<math>5/6</math>);</p> <p>(T1 session) – brief moves X-marker from <math>5/5</math> to <math>2/5</math>;</p> <p>RB: "how much is that?", N: "<math>5/6</math> of <math>2/5</math>", RB: "how much cheese did you use, what's the output?", N: "<math>10/30</math>";</p> <p>N confirms he got 10 from looking at record sheet and multiplying across and then from looking at the imaginary vertical line at <math>x=1/5</math> that cut the shading in half so you have 10, N gets 30 by multiplying 6 times 5;</p> <p>(T2 session)</p>	<p>"<math>3/4 &gt; 2/5</math> because a <math>5^{\text{th}}</math> is smaller then a <math>4^{\text{th}}</math>", RB: then how do you know they are not going to balance out (<math>8D=8E</math>)?" N: "I don't know", RB: "is there any way you can figure it out?", N: "multiply <math>5/6</math> by <math>2/5</math>";</p> <p>A2 = naming points on y-axis, correctly names <math>6/6</math>, <math>8/6 = 1 \frac{2}{6}</math>;</p> <p>T1 = RB recaps commutative property;</p> <p>T2 = RB introduces equivalence first w/ am-fm and then w/ record sheet, RB: "can you call this area something else?", N: [moves tiles horizontally across within the unit whole]</p>  <p>"<math>2/3</math> because each pair of rows counts as one so it's <math>1/3</math>", RB then directs N's attention to record sheet, RB: "how did we go from this <math>10/30</math> to <math>1/3</math>?", N: "times 10", RB: "can you tell me another name for it by using a different size group?", N tries 3, N" we already know groups of 2 work [returns to original</p>  <p>tiling] , RB asks series of leading questions about colored groups of 2 and total groups of 2 and N</p>	<p>initial area; equivalence;</p>
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<p>8E. 5/6x2/5</p>	<p>incorrect prediction (more); “more because <math>5/6 &gt; 3/5</math> because in <math>5/6</math> you’re only one slice away from whole” (A1 session) ; represents whole sets Y slider at 6 and X slider at 5; RB: “why do you did it that way?” N: “it’s easier for me to look at the y-axis is the first number on this side” and “no, it doesn’t matter if you flip it.” moves Y-marker down from <math>6/6</math> to <math>1/6</math> and then counts up to <math>5/6</math>; (A2 session); N correctly names initial area (<math>5/6</math>); (T1 session) – brief moves X-marker from <math>5/5</math> to <math>2/5</math>; RB: “how much is that?”, N: “<math>5/6</math> of <math>2/5</math>”, RB: “how much cheese did you use, what’s the output?”, N: “<math>10/30</math>”; N confirms he got 10 from looking at record sheet and multiplying across and then from looking at the imaginary vertical line at <math>x=1/5</math> that cut the shading in half so you have 10, N gets 30 by multiplying 6 times 5; (T2 session)</p>	<p>A1 = RB: “what about the second set of fractions?”, N: “<math>3/4 &gt; 2/5</math> because a <math>5^{\text{th}}</math> is smaller than a <math>4^{\text{th}}</math>”, RB: then how do you know they are not going to balance out (<math>8D=8E</math>)?” N: “I don’t know”, RB: “is there any way you can figure it out?”, N: “multiply <math>5/6</math> by <math>2/5</math>”; A2 = naming points on y- axis, correctly names <math>6/6</math>, <math>8/6</math> <math>= 1 \frac{2}{6}</math>; T1 = RB recaps commutative property; T2 = RB introduces equivalence first w/ am-fm and then w/ record sheet, RB: “can you call this area something else?”, N: [moves tiles horizontally across within the unit whole]</p>  <p>“<math>2/3</math> because each pair of rows counts as one so it’s <math>1/3</math>”, RB then directs N’s attention to record sheet, RB: “how did we go from this <math>10/30</math> to <math>1/3</math>?”, N: “times 10”, RB: “can you tell me another name for it by using a different size group?”, N tries 3, N” we already know groups of 2 work [returns to original</p>  <p>tiling] , RB asks series of leading questions</p>	<p>operations; commutative; initial area; equivalence;</p>
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<p>8G.  <math>2\frac{3}{5} \times 2\frac{5}{5}</math>  (mixed #)</p>	<p>correct prediction (more);  “more because you have <math>2\frac{3}{5}</math> which equals <math>1\frac{3}{5}</math> and <math>\frac{4}{3}</math> is only <math>1\frac{1}{3}</math>”;  sets X and Y slider to 1;  represents whole;  moves Y slider to 5 and X slider to 5, moves Y-marker from <math>\frac{5}{5}</math> to <math>2\frac{3}{5}</math>, moves X-marker from <math>\frac{5}{5}</math> to <math>\frac{2}{5}</math>;  “I’m giving out 26 pieces because you have 13 pieces going up and 2 going across (you half it), and <math>13 + 13 = 26</math>” and “the name of the pieces is 25 because 5 times 5” so “you have 1 whole and 1 left over, <math>1\frac{1}{25}</math>”  (T session);</p>	<p>T = RB explores equivalence using the record sheet and the notion of division “how many times does 25 go into 26”</p>	<p>equivalence;  operation</p>
<p>8H.  <math>1\frac{2}{5} \times 6\frac{4}{4}</math>  (mixed #)  (improper)</p>	<p>no prediction asked;  “ok, so <math>1\frac{2}{5}</math> times <math>6\frac{4}{4}</math>”  the X slider is at 5 and Y slider is at 5, moves X slider from 5 to 4;  represents whole = L;  moves Y-marker from <math>\frac{5}{5}</math> to <math>1\frac{2}{5}</math> and moves X-marker from <math>\frac{4}{4}</math> to <math>\frac{6}{4}</math>;</p>  <p>“that’s a lot, I won’t know unless I count all the boxes, it’s more than 1 and less than 2”;  [N tiles and starts moving tiles to fill the whole above the original unit whole] “it’s more than two”;</p>		<p>operations</p>



“It’s  $2\frac{6}{4}$  because you have 2 wholes and 1, 2, 3, 4, 5, 6 [counts partitions on the x-axis up to last tile]”;  
[N highlights grid]  
RB: “how much cheese did you give out?”, N: “oh,  $2\frac{2}{4}$ ”, RB: “what are these tile pieces called?” N: “20ths”, RB: “how many 20ths do we have?”, N: “ $2\frac{2}{20}$ ”;  
RB: “does that reduce”, [N looks to record sheet] “they all have 2 in them”

### Appendix E: Neato's Preliminary Analysis Table

KEY: C=correct, I=incorrect, L=less than previous case, M=more than previous case, E=equal to previous case, ODM=order does matter, T=teaching session, J=justification, NL=number line, EQ=equivalence, N/A=not applicable

Case	Input	Predictions	Prediction Justifications	Construction	Interpretation (initial/pre-tile & final)		Order/Equivalence	Unit/Operation
8a	1/2 of 1/2 slice/rat	N/A	N/A	whole x-slider=4 y-slider=4 x-maker=1, $\frac{1}{2}$ , $\frac{1}{4}$	N/A	C: $\frac{1}{4}$ J: because you cut whole into 4 and you can turn one slice into 4 pieces	N/A	N/A
8b	1/2 of 1/3 slice/rat	C: L	C: $\frac{1}{3} < \frac{1}{2}$ using x-axis of amfm to show it	whole x-slider=3 y-slider=2 x-marker=1, $\frac{1}{3}$ y-marker=1, $\frac{1}{2}$	N/A	Patical to C: $\frac{1}{2}$ of $\frac{1}{3}$ to $\frac{1}{6}$ J: because 6 of them will make 1	NL: $\frac{1}{3}$ , $\frac{2}{3}$ , $\frac{3}{3}=1$ , $1-\frac{1}{3}$ , can't name $1-\frac{1}{3}$ as $\frac{4}{3}$ ; T Session = "Oh"; NL: $\frac{4}{3}=1-\frac{1}{3}$ , $\frac{5}{3}$ , $\frac{6}{3}=2$ ;	
8b*	1/3 of 1/2 slice/rat	C: E	C: from record sheet we know ODM if mult	no whole [from previous case x-slider=2, y-slider=3] y-marker=0, $\frac{1}{3}$ x-marker=1, $\frac{1}{2}$	N/A	C: $\frac{1}{6}$ J: because it's equal to 8b, when you tile it's $\frac{1}{6}$ , [tiles]-it's $\frac{1}{6}$ of whole T Session (1/4 AM of unequal parts) = shaded piece is $\frac{1}{6}$ , no, $\frac{1}{4}$		commutative prop during prediction justification; we're adding but that can be like mult; mult 1 into $\frac{1}{3}$ of $\frac{1}{2}$ ;

						because the $2\frac{1}{4}$ pieces make $\frac{1}{2}$		$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
8c	$\frac{2}{3}$ of $\frac{1}{3}$ slice/rat	C: M	C: instead of $\frac{1}{2}$ of $\frac{1}{3}$ it's $\frac{2}{3}$ of $\frac{1}{3}$ and $\frac{2}{3} > \frac{1}{2}$	no whole x-slider=3 y-slider=3 x-marker=0, $\frac{1}{3}$ (initial area?) y-marker=1, $\frac{2}{3}$	“this is $\frac{1}{3}$ ” (not sure if referring to pt on #line or area)	I to C: $\frac{2}{6}$ to $\frac{2}{9}$ J: because there are 2 black tiles and 9 total tiles	C: $\frac{2}{3} > \frac{1}{3}$ , $\frac{1}{2} > \frac{1}{3}$ , I: $\frac{2}{3} =$ half of $\frac{3}{3}$ , C: no; NL: RB has N use amfm to show $\frac{2}{3} > \frac{1}{2}$ ; Area: N uses amfm to show $\frac{2}{9} > \frac{1}{6}$ by indexing horizontal imaginary line where $y = \frac{1}{2}$	commutative prop implicit in prediction justification;
8d	$\frac{3}{5}$ of $\frac{3}{4}$ slice/rat	C: M	I: 5ths are not bigger than 3rds but there are more slices in a whole	no whole x-slider=5 y-slider=4 y-marker=0, $\frac{3}{4}$ “backwards” y-slider=5 x-slider=4 y-marker=0, $\frac{3}{5}$ ; I: initial area; whole y-marker=1, $\frac{3}{5}$ x-marker=1,	I: $\frac{2}{3}$ (not $\frac{3}{5}$ ) T Session = “Oh” C: $\frac{3}{5}$	I to C: $\frac{6}{20}$ to $\frac{9}{20}$ J: it's a guess, because 5 times 4 is 20, oh, no it's $\frac{9}{20}$	C: $\frac{3}{4} > \frac{1}{3}$ , I: $\frac{3}{4} = \frac{1}{3}$ ; NL: N uses amfm to show that $\frac{3}{4} > \frac{1}{3}$ , and $\frac{3}{4} > \frac{2}{3}$	commutative proper during construction; mult to arrive at output

				$\frac{3}{4}$ N: ODM				
8e	5/6 of 2/5 slice/rat	I: L	I: $5/6 > 3/5$ because only 1 piece from whole & $3/4 > 2/5$ because 5ths are smaller than 4ths “you can mult to know for sure”	whole y-slider=6 x-slider=5; N: ODM; y-marker=1, 1/6, 5/6; C: initial area; x-marker=1, 2/5	C: 5/6 J: N/A	partial to C: 5/6 of 2/5, 10/30; J: because 5 times 2 is 10, also because you would imaginary vertical line at $x=1/5$ would cut the shading in half to give you 10 black tiles, and 6 times 5 is 30	NL: 6/6, 8/6=1-2/6; EQ: N moves tiles to arrive at $10/30=2/3$ , RB has N look at record sheet and N see “times 10”; RB has N uses amfm to arrive at 5/15 as output	mult during prediction justification; commutative prop during construction; mult to justify output; mult for EQ,
8f	4/3 of 2/5 slice/rat ( <i>improper</i> )	C: M	C: $4/3 = 1 - 1/3 > 5/6$ ; RB has N uses amfm to prove it; x-slider=1, y-slider=1, y-slider=4, x-slider=3, x-marker=1, 4/3, realize he didn't need y-slider=4	y-slider=3 x-slider=5 whole y-marker=1, 4/3 C: initial area; x-marker=1, 2/5	C: 1-1/3 J: N/A	partial to C: 8 something because 4 vertical partitions and 2 horizontal partitions; 8/15 because 5 times 3 where 5=horizontal partitions & 3=vertical partitions	see amfm use during prediction justification; NL: C: 1/5, 0/5, 8/5, 6/5, 2=10/5, 15/5, 4/3=1-1/3; EQ: N correctly states 8/15 can't be reduced and uses am-fm to shows it	operation;
8g	2-3/5 of 2/5 slice/rat ( <i>mixed #</i> )	C: M	C: $2-3/5 = 13/5$ and 4/3 is only $1-1/3 (< 2-3/5)$	x-slider=1 y-slider=1 whole y-slider=5	N/A	partial to C: 26 pieces because 13 pieces going up and 2 across,	EQ: $26/25 = 1$ whole w/ 1 left over so 1-1/25	operation;

				x-slider=5 y-marker=1, 2-3/5 x-marker=1, 2/5		13+13=26; 26/25 because 5 times 5		
8h	1-2/5 of 6/4 slice/rat (mixed #) (improper)	N/A	N/A	x-slider=5 y-slider=5 fix x-slider=4 whole y-marker=1, 1-2/5 x-marker=1, 6/4	N/A	I to I to C: don't know unless I tile and count, >1 but <2; tiles and moves tiles and corrects to >2; 2-6/4 because 2 wholes 6 (counts partitions on the x-axis up to position of last tile); highlights grid to arrive at 2-2/4; RB asks for names of tile piece and N correctly answer w/ 20ths and arrives at 2 2/20	N/A	operation;

## Appendix F: Neato's Narrative Summary

Narrative ummary of Neato's growth and change in knowledge

Neato: Day 2 of the Clinical Interview

Objective:

1. What's is he understanding?
  - a. feel more comfortable w/ this then w/ talking about what he doesn't understand
2. What's changing in his understanding?
  - a. for now I'm not going to address why it's change but what is changing

### I. Prediction/Justification

Correct in 6 out of 7 cases, Neato's predictions were correct. In 5 of the 7 cases, Neato's prediction justifications were correct. Neato incorrectly predicted that case  $8E=5/6 \times 2/5$  is less than case  $8D=3/5 \times 3/4$ . The justification was that  $8E$  is less than  $8D$  because  $5/6 > 3/5$ . When prompted to consider the second fraction Neato correctly concluded that  $3/4 > 2/4$ . The statements Neato made regarding fraction order were correct but did not prove that his incorrect prediction. Notice that both fractions changed as you moved from case  $8D$  to case  $8E$ . There was one other instance during which both fractions changed, case  $8C=2/3 \times 1/3$  to  $8D=3/5 \times 3/4$ . Here Neato made a correct prediction ( $8D$  is more than  $8C$ ) but Neato gave an incorrect justification based again on correct statements. This time the correct statements were about the denominators of the first fraction ( $5^{\text{th}}$  are not bigger than  $3\text{rds}$  but there are more slices in whole). Neato's correct justifications were grounded in using the AM-FM representation to show fraction order (1 out of 7 cases), multiplication (1 out of 7 cases), and fraction order (3 out of 7 cases). Also, while discussing case  $8E$ , Neato did say that one way you can know for sure which output is greatest is to multiply the pair of fractions and compare the results.

### II. AM-FM Construction

Neato used the AM-FM representation to compare fractions and to arrive at an output value.

When comparing fractions Neato would use the denominator of one fraction to partition the x-axis, use the x-axis marker line to locate that fraction, and then moving the x-axis marker line to where the second fraction would be. Neato had no problems representing proper fractions on either axes or comparing two fractions using the axes of the am-fm representation. But when first asked to represent the improper fraction  $4/3$ , Neato partitioned the y-axis into fourth and the x-axis into thirds, correctly located  $4/3$  on the x-axis by using the x-axis marker line, and concluded he didn't need to partition the y-axis into fourths.

In terms of constructing the AM-FM representation to arrive at outputs, Neato started by representing the 1 by 1 unit whole in the first 2 cases, stopped doing so for the next 2 cases, and then returned to representing the whole in the remaining 5 cases. Next, I will discuss the sequence in which Neato constructed the output area. In the first case ( $8A=1/2 \times 1/2$ ) Neato set both the x-slider and y-slider at 4, and moved the x-axis marker line from  $4/4$  to  $1/2$  to  $1/4$ . This was very different construction process then the one demonstrated by

RB prior to his beginning work on the first case. RB provided a second demonstration using case 8A. In cases 8B, 8B\*, and 8C, Neato set the x-slider equal to the denominator of the second fraction and the y-slide equal to the denominator of the first fraction. In cases 8B and 8C, Neato represented the second fraction on the x-axis first and then the first fraction on the y-axis. In case 8B\*, he represented the first fraction on the y-axis first and then the second fraction on the x-axis. Case 8D is an interesting turning point. Here Neato set the x-slider equal to the denominator of the first fraction and the y-slider equal to the denominator of the second fraction, moved the y-axis marker line to represent the second fraction, and concluded he had done it backwards. From that point on, Neato would always represent the first fraction on the y-axis followed by the second fraction on the x-axis. In other words, given  $a/b$  of  $c/d$ , Neato chose to construct  $c/d$  of  $a/b$  by first representing  $a/b$  on the y-axis and then taking  $c/d$  of  $a/b$  by representing  $c/d$  on the x-axis.

### III. AM-FM Interpretation

Neato was asked to interpret initial area output and final area output. This of course also involved determining points on the axes, recognizing how those points related to area, and understanding part-whole relationships.

First I shall discuss Neato's initial area interpretation. There were 3 instances during which Neato was asked to interpret initial area (cases 8D, 8E, and 8F). By initial area I mean the area produced after representing one of the two given fractions. In case 8D, the y-axis marker line was positioned at  $y=3/5$  and Neato incorrectly interpreted the initial area as  $2/3$ . Through a series of leading questions Neato arrives at the correct interpretation of  $3/5$ . During the next two cases, (8E and 8F) Neato was again asked to interpret initial area and showed no difficulty in understanding the one-to-one correspondence between the location of the marker line he had moved and the amount of initial area produced.

Now I shall discuss Neato's interpretation of final pre-tiled area. This is the area produced after both of the given fractions have been represented on the axes of the AM-FM representation. In the first four cases Neato produced the correct output, correctly names the output values, and gave justification based on counting the single black tile piece to total number of black tile pieces that make the whole. In case 8D= $3/5 \times 3/4$ , first guessed  $6/20$  and while providing his justification self-corrected to  $9/20$ . This time his justification was based on multiplication ("because 5 times 4 is 20, oh, no it's  $9/20$ "). [SEE VIDEO-not sure what he's attending to when he says 5 times 4 and not sure where the 9 came from]. Similarly in the next case (8E= $5/6 \times 2/5$ ) Neato's multiplies to arrive at a denominator. He counts imaginary tile pieces to arrive at a numerator. In case 8F= $4/3 \times 2/5$  multiplies the number of  $1/3$  intervals and  $1/5$  intervals to that make up the whole to arrive at a denominator. Similarly he multiplies the  $1/3$  intervals and  $1/5$  intervals that make the black shaded area to arrive at a numerator. Neato does the same for case 8G= $2-2/5 \times 2/5$ . In case 8H= $1-2/5 \times 6/4$ , Neato experiences difficulty. He first wanted to tile and count. He knew the area was  $>1$  but also thought it was  $<2$ . After tiling and while moving tiles, he self-corrected and stated the area would actually be  $>2$ . Neato's first answer for final tiled area was  $2-6/4$  "because of the 2 wholes and the 6 came from counting the  $1/4$  intervals across the x-axis from  $0/4$  up to  $6/4$  where the last 2 tile pieces fell. When asked by RB to confirm his answer Neato highlighted the grid and changed his answer to  $2-2/4$ . RB asks names of the tile pieces Neato finally arrived at  $2-2/20$  where 20 came from multiplying the number of  $1/4$  and  $1/5$  intervals that make up the unit whole.



#### IV: Order & Equivalence

Neato had no problem locating proper fractions and mixed number on the axes of the AM-FM representation. However, Neato could not give a fraction equivalent to  $1\frac{1}{3}$ . This resulted in a mini-lesson on locating points on a number line. During this lesson Neato had an “oh” moment. While working with a number line partitioned into thirds (drawn using paper and pencil) RB went through the pattern for naming  $\frac{1}{3}$ ,  $\frac{2}{3}$ ,  $\frac{3}{3}$ , and  $\frac{0}{3}$  at which point Neato responded with “oh,  $\frac{4}{4}$ ” and went on to name  $\frac{5}{3}$  and  $\frac{6}{3}=2$ . From that point Neato had no problem naming any fractions on the number line. Neato also used the axes of the AM-FM representation to compare fractions. This was briefly discussed in the AM-FM construction section above.

In addition to using the number line feature of the AM-FM representation in order to compare fractions, Neato also used the area model feature to compare fractions. When asked to compare the output value of case 8D ( $\frac{1}{3}$  of  $\frac{1}{2}$ ) to the output value of case 8E ( $\frac{2}{3}$  of  $\frac{1}{3}$ ) Neato used area to determine that a final output of  $\frac{2}{9}$  is greater than a final output of  $\frac{1}{6}$ . Neato worked with the AM-FM representation to generate an area model of  $\frac{2}{9}$  with the y-axis marker line set at  $\frac{2}{3}$  and the x-axis marker line set at  $\frac{1}{3}$ . He then indexed an imaginary y-axis marker line at  $y=\frac{1}{2}$  to show that  $\frac{1}{6}$  (the output for  $\frac{1}{2}$  of  $\frac{1}{3}$ ) would result in less shaded area than  $\frac{2}{9}$ .

Finally, Neato also used the area model feature of the AM-FM representation to generate equivalent fractions. It was often the case that a number of the output values could be reduced. To arrive at an equivalent fraction Neato would move tiles either within the  $1 \times 1$  unit whole (as in case 8E:  $\frac{5}{6}$  of  $\frac{2}{5}$ ) or across two  $1 \times 1$  unit wholes (as in case 8G:  $2\frac{3}{5}$  of  $\frac{2}{5}$ ). In case 8E, Neato uses the move function of the AM-FM representation to reduce  $\frac{10}{30}$  to  $\frac{1}{3}$  by counting tiles in groups of 10 instead of single tiles. In case 8G, Neato uses the move function again to reduce  $\frac{26}{25}$  to  $1\frac{1}{25}$  by filling in one  $1 \times 1$  unit whole and recognizing that since there was one tile left over the output was  $1\frac{1}{25}$ . In case 8F ( $\frac{4}{3}$  of  $\frac{2}{3}$ ) Neato correctly stated that the output value of  $\frac{8}{15}$  cannot be reduced and used the move feature of the AM-FM representation to show that the tiles could not be chunked into groups such that the size of the group could cover both 8 tiles and 15 total spaces that constituted the  $1 \times 1$  unit whole.

Of course change in understanding is context sensitive and seldom fixed. Neato's demonstration of “slippage” [can't think of better word right now] when asked to compare case 8C ( $\frac{2}{3}$  of  $\frac{1}{3}$ ) to case 8B\* ( $\frac{1}{3}$  of  $\frac{1}{2}$ ), and then again when asked to compare case 8D ( $\frac{3}{5}$  of  $\frac{3}{4}$ ) to case 8C ( $\frac{2}{3}$  of  $\frac{1}{3}$ ). In the first instance, Neato incorrectly states that  $\frac{2}{3}$  (the first fraction from case 8C) is half of  $\frac{3}{3}$  but quickly self corrects. In the second instance, Neato correctly states that  $\frac{3}{4} > \frac{1}{3}$  then incorrectly states that  $\frac{3}{4} = \frac{1}{3}$ , and then uses the AM-FM representation to show that  $\frac{3}{4} > \frac{1}{3}$ .

#### V: Commutativity & Multiplication

Commutativity and multiplication first came up when Neato was asked to predict whether the outcome in case 8B\* ( $\frac{1}{3}$  of  $\frac{1}{2}$ ) was less than or greater than the output in the previous case, case 8B ( $\frac{1}{2}$  of  $\frac{1}{3}$ ). Neato's predicted that the outputs would be equal because “order doesn't matter if you're multiplying.” Neato appears to recognize multiplication as the operation being performed and also recognizes the commutative property. When Neato was

asked to compare the output values of case 8C ( $2/3$  of  $1/3$ ) to case 8B\* ( $1/3$  of  $1/2$ ) he correctly predicted that the output of 8C would be “more because instead of  $1/2$  of  $1/3$  it’s  $2/3$  of  $1/3$ ” and “ $2/3$  is more than  $1/2$ .” As discussed in the subsection AM-FM construction, in cases 8D ( $3/5$  of  $3/4$ ) and 8E ( $5/6$  of  $2/5$ ) Neato began by representing the first fraction (i.e.,  $3/5$  and  $5/6$  respectively) on the y-axis and then the second fraction (i.e.,  $3/4$  and  $2/5$  respectively). In other words, Neato took  $3/4$  of  $3/5$  (instead of  $3/5$  of  $3/4$ ) and  $2/5$  of  $5/6$  (instead of  $5/6$  of  $2/5$ ). When asked if the output values would differ if he reversed the order of the fractions during the construction process he said, “no, it doesn’t matter if you flip it.”

Neato made the operation of multiplication explicit in a number of ways. In cases 8D ( $3/5$  of  $3/4$ ), 8E ( $5/6$  of  $2/5$ ), 8F ( $4/3$  of  $2/5$ ), and 8G ( $2\frac{3}{5}$  of  $2/5$ ) Neato used the AM-FM representation to construct the output value and then justified the final output by referencing the operation of multiplication. More specifically Neato did this when justifying the denominator value of the final output. For example, in case 8G Neato states, “the name of the pieces is 25 because 5 times 5.”

VI: Unit

See all of the above.

## Appendix G: Transcript of Neato's Clinical Interview

1 Student Name: Neato  
2 Day 2 of Clinical Interview  
3 Transcriber: XXX  
4 Verifier: YYY  
5  
6 KEY (the case numbers in the transcript are numbered differently then in the  
7 proposal):  
8 Case 8 = Case 8, Case 8a = Case 9, Case 8b = Case 10, Case 8b\* = Case 11, Case 8c  
9 = Case 12, Case 8d = Case 13, Case 8e = Case 14, Case 8f = Case 15, Case 8g =  
10 Case 16, and Case 8h = Case 17  
11  
12 RB: To case eight. Just going to write in what you wrote for case eight. So you were  
13 at one rat, two thirds of three fourths slice per rat. And you had one half slice.  
14 Right? <writes "1 rat", "2/3 of 3/4 slice/rat", "1/2 slice" on number chart>.  
15  
16 NP: Right.  
17  
18 RB: Okay, so last time <draws line under first line of data>. So now um, what I will  
19 do is make this more convenient so you don't have to move between this <points to  
20 screen> and this <points to paper>. Ask what you want me to write, instead of you  
21 having to, us having to move things around. The video is going to zoom in on this  
22 part. <points to screen> Okay?  
23  
24 NP: Okay.  
25  
26 RB: Um, so, okay, do you still remember the kind of problems we were doing  
27 before? Just sort of a recap of this <brings cases 0-11 to front>. So we had different  
28 numbers of rats,  
29  
30 NP: Right.  
31  
32 RB: And they were being given different...  
33  
34 NP: Numbers of cheese... amounts of cheese.  
35  
36 RB: Yeah, amounts of cheese. With cutouts we were laying it out how much cheese  
37 that...and we were figuring out outputs like how much cheese we end up using for  
38 each case. And throughout the process I was asking you to make various kinds of  
39 predictions, right?  
40  
41 NP: Right.  
42

43 RB: Um, so I'm going to show [unclear] to you that, I'm going to be asking you for  
44 predictions as we go, but this time instead of using the pads we are going to use the  
45 laptop. Okay?  
46  
47 NP: Okay.  
48  
49 RB: We are going to sort of shift focus, you know how a lot of times we had threes  
50 here <points to first input column>?  
51  
52 NP: Yeah.  
53  
54 RB: Now we are going to be mostly working with one rat. And we are going to be  
55 mostly working with two fractions in this column <points to input columns> Kind  
56 of like what we did to two thirds of three fourths. Do you remember how we did  
57 two thirds of three fourths?  
58  
59 NP: Um, no.  
60  
61 RB: Okay, why don't we, um, okay, so why don't we do the next problem and it will  
62 probably come back to you. So I'm going to call this problem eight A, because it's  
63 kind of related to 8 and it's gonna have one rat, and it's gonna, and the information  
64 the scientist says is, wants us to give out one half of one half slice per rat <writes  
65 "1/2 of 1/2 slice/rat" in input column>. In this case it's only one rat but, we are  
66 going to give that rat one half of one half, right?  
67  
68 NP: Right.  
69  
70 RB: We need to figure out the output.  
71  
72 NP: It will be one, one fourth.  
73  
74 RB: How do you know that?  
75  
76 NP: Because you start with a whole, and then you cut it in half, and then half of the  
77 half is going to be one fourth <hand motions two cuts to a whole slice>.  
78  
79 RB: Good. So you did that right now kind of with your hands gesturing. I want you  
80 to use this <points to the computer screen>, show me, starting off with the whole.  
81  
82 NP: The whole...  
83  
84 RB: Yeah, starting with the whole. How would you start with the whole?  
85  
86 NP: <attempts to move the X-axis marker line but stops >  
87

88 RB: Well you could actually do that, what you were doing. So you can move that red  
89 marker line.  
90  
91 NP: I was going to move this one <moves X-axis marker line to 1>...  
92  
93 RB: Right there. So that's a whole?  
94  
95 NP: That's a whole. And then <moves X\_Division to 4> four.  
96  
97 RB: Um hmm.  
98  
99 NP: I guess I'll put, yeah <moves Y\_division to 4>.  
100  
101 RB: Four and four, so you split both axes into fourths? Can I ask you what, what this  
102 axis <points to X-axis> represents?  
103  
104 NP: Um, slices.  
105  
106 RB: Slices?  
107  
108 NP: Slices.  
109  
110 RB: Slice of cheese, yes. Okay.  
111  
112 NP: So, yeah.  
113  
114 RB: So how much...  
115  
116 NP: So like <moves X-marker line from 1 to  $\frac{1}{2}$ >.  
117  
118 RB: Moved to a half, okay.  
119  
120 NP: Move to a half, right? You have that.  
121  
122 RB: So that's half a slice of cheese...  
123  
124 NP: Half a slice of cheese.  
125  
126 RB: Uh huh.  
127  
128 NP: Then <moves X-axis marker line from  $\frac{1}{2}$  to  $\frac{1}{4}$ > you have, then you go one  
129 fourth, of the half. Yeah.  
130  
131 RB: I would give him another...you cut that half...  
132

133 NP: Because like you have half,  
134  
135 RB: Uh huh.  
136  
137 NP: Right? <moves line to  $\frac{1}{2}$  on X-axis> And then you give them half of one half  
138 <moves line to  $\frac{1}{4}$  on X-axis>, which is one fourth.  
139  
140 RB: I see. I see, very cool. And how do you know that's one fourth <pointing to  
141 position of X-axis marker line at  $X=1/4$ >?  
142  
143 NP: Because you have four, like...I guess no, but, like, let's see, because you have,  
144 because you cut it up into four, I guess.  
145  
146 RB: What do you mean you cut it up into four?  
147  
148 NP: Because you like, you can turn one slice into four pieces.  
149  
150 RB: Four of them would fill the whole, you mean? Four of these pieces <points to  
151 the dark  $\frac{1}{4}$  of the grid>  
152  
153 NP: Four pieces would fill the whole.  
154  
155 RB: I see. Okay, so it's one fourth.  
156  
157 NP: Yeah.  
158  
159 RB: Okay, And now why did you partition, split, divide <points to the X-axis> the X  
160 into fours and the Y into fours? Why did you?  
161  
162 NP: Why? Um...  
163  
164 RB: How did you know to do that?  
165  
166 NP: How did I know like, why did I do that?  
167  
168 RB: Um hmm. Why did you split it into fours? Because the problem says one rat gets  
169 one half of one half, right?  
170  
171 NP: Right.  
172  
173 RB: So I don't see a fourth anywhere here.  
174  
175 NP: Well you had to start out with one whole.  
176  
177 RB: Right. So you had one whole, so let's reset. Just reset it.

178  
179 NP: <hits Reset button>  
180  
181 RB: Move the sliders back to one, one.  
182  
183 NP: <moves X and Y-division to 1>  
184  
185 RB: Okay, now show me what one is.  
186  
187 NP: <moves X-marker line to 1 which causes unit whole to turn black>  
188  
189 RB: So we have a whole slice there, right?  
190  
191 NP: Right  
192  
193 RB: That's our unit. Remember that second meaning of unit, okay? So the unit we  
194 are working, one slice.  
195  
196 NP: Right  
197  
198 RB: And then it says to take half of it. Right? A then half of that.  
199  
200 NP: Half of... <moves X-division to 4>  
201  
202 RB: So why did you split it into fourths?  
203  
204 NP: I split it into fourths because...  
205  
206 RB: Why didn't you split it into two, halves?  
207  
208 NP: Why didn't I split into halves?  
209  
210 RB: Um hmm.  
211  
212 NP: I actually don't know why I hadn't split it into halves but fourths would make it  
213 easier.  
214  
215 RB: Fourth would make it easier?  
216  
217 NP: Yeah.  
218  
219 RB: Based on the way you did it.  
220  
221 NP: Yeah, because it would like, it would be already cut up.  
222

223 RB: Um hmm. Okay what if someone did it like this. I want to tell me what you  
224 think of this method. Okay? So reset. <clicks Reset button and moves X slider back  
225 to 1> Um, so again we start with one. <moves X-axis marker line to 1> Our unit is  
226 one, one slice. And then the problem says to take one half of one half. So I'm  
227 going to say, okay I'm splitting this in two <moves X slider to 2>, right? So I'm  
228 going to move it here <moves X-axis marker line to  $\frac{1}{2}$ > to one half right?  
229  
230 NP: Right.  
231  
232 RB: So if I have take a half, and now I need to take half of that, right?  
233  
234 NP: Right  
235  
236 RB: So then I'm going to split this side <points to the Y-axis w/ cursor> ...  
237  
238 NP: Into half.  
239  
240 RB: Into half <moves Y-division to 2> like that. So because I really just need to take  
241 half of the shaded part <pointing to shaded  $\frac{1}{2}$ >, right?  
242  
243 NP: Right.  
244  
245 RB: So I could do it this way <moves Y-axis marker line down from 1 to  $\frac{1}{2}$ >.  
246  
247 NP: Hmm.  
248  
249 RB: See?  
250  
251 NP: Yeah.  
252  
253 RB: And how much is that of the whole <presses Highlight Grid buttons>?  
254  
255 NP: It's one fourth.  
256  
257 RB: One fourth. Right?  
258  
259 NP: Right.  
260  
261 RB: One fourth. So what do you think of that method?  
262  
263 NP: It's a good method.  
264  
265 RB: So there I didn't have to split to fourths.  
266  
267 NP: Right.



268  
269 RB: Right. You kind of knew the answer was going to one fourth, is that the reason  
270 you split it into fourths?  
271  
272 NP: Yeah.  
273  
274 RB: Because you were looking for that? Okay. So the answer was one fourth slice  
275 <writing in the number chart which is off camera>. Okay, so this piece is called  
276 a... <points to the dark region w/ cursor>  
277  
278 NP: Fourth.  
279  
280 RB: Fourth. And let me tile, tile this really quickly <hits tile button>. If I move this  
281 piece <moves tile of size  $\frac{1}{4}$  to the upper right of the 1x1 square> here, it's still  
282 called...  
283  
284 NP: A fourth.  
285  
286 **TIME=00:08:39**  
287  
288 RB: A fourth. Right, um, okay <moves the tile back>. So now we are going to reset,  
289 <hits reset button> okay. <move X and Y-division back to 1> So the next problem,  
290 8B(8:43), um, 8B is again one rat. And this time the scientist wants us to give that  
291 rat one half of one third of a slice per rat. And before we do anything, I'm going to  
292 ask for a prediction, do you think the amount of cheese we are going to use up in  
293 case 8B is more or less than amount of cheese we used in 8A?  
294  
295 NP: I think it's going to be less.  
296  
297 RB: Less, and why?  
298  
299 NP: Because you have one third.  
300  
301 RB: Instead of ...  
302  
303 NP: Instead of a half.  
304  
305 RB: And so which one is bigger, one third or one half?  
306  
307 NP: One half.  
308  
309 RB: One half is bigger.  
310  
311 NP: Yeah  
312

313 RB: And how do you know that one half is bigger than one third?  
314  
315 NP: Because, can I show you on here <referencing computer screen>.  
316  
317 RB: Yeah.  
318  
319 NP: So...  
320  
321 RB: Remember to always start with a whole slice. I want you to -  
322  
323 NP: Oh, start with a whole slice.  
324  
325 RB: Yeah.  
326  
327 NP: <moves X-axis marker line to 1 to reveal whole>.  
328  
329 RB: Okay now show me one third.  
330  
331 NP: <moves X-division to 3>. This is one third <moves X-axis marker line from 1 to  
332 1/3>.  
333  
334 RB: Um hmm.  
335  
336 NP: Right there, so, two <moves X-Division to 2> and that's one half <the X-axis  
337 marker line automatically jumps from 1/3 to 1/2>.  
338  
339 RB: That's one half.  
340  
341 NP: Yeah.  
342  
343 RB: Okay.  
344  
345 NP: Just a little bit bigger than one third  
346  
347 RB: Okay, let's reset.  
348  
349 NP: <clicks Reset button, moves X-division back to one>  
350  
351 RB: Um, I want you to go ahead and do this problem. So show me this problem's  
352 units. Let's find out what our output is going to be.  
353  
354 NP: So one half of one third. <moves X-axis marker from zero to 1, moves X-  
355 division to 3, moves Y-division to 2>.  
356  
357 RB: Okay, so you are splitting, you are dividing your X-axis into threes, okay.

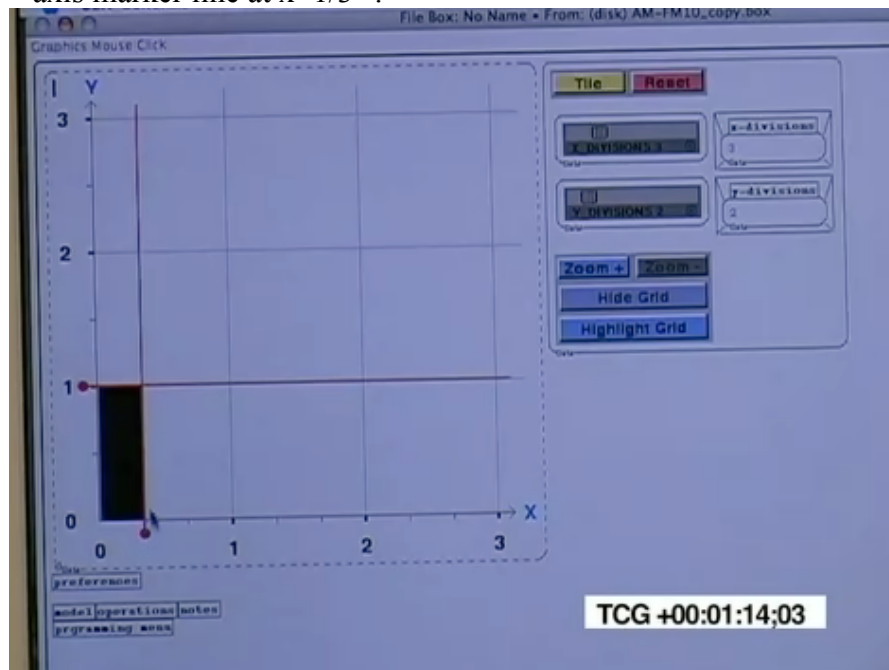
358

359 NP: Okay, so <moves X-axis marker line from 1 to  $1/3$ >

360

361 RB: Okay, and where is that place you positioned it <references the position of the x-axis marker line at  $x=1/3$ >?

362



363

364 NP: In...it's one third.

365

366 RB: It's one third, and what is this point called <points to  $2/3$  on X-axis> on the  
367 number line?

368

369 NP: Two thirds.

370

371 RB: Two thirds, and what would this point be called if we were using thirds <point  
372 to one on the X-axis>?

373

374 NP: One whole but or three thirds.

375

376 RB: Three thirds. And this point would be called &lt;points to one and one third&gt;.

377

378 NP: One and one third.

379

380 RB: And if I wanted it as an improper fraction? If I wanted it...so this is <points to  
381 one third> one third, <points to two thirds> two thirds, <points to one> three  
382 thirds, <points to one and one fourth>...

383

384 NP: One and one third.

385

386 RB: Which could be called, what's another name for one and one third?

387

388 NP: Um.

389

390 RB: Is there another name for that fraction?

391

392 NP: Um, not that I know of.

393

394 [Intervention]

395

396 RB: So one and one third <takes a piece of paper and writes "1\_1/4" instead of  
397 1\_1/3>, right? So we have a number line, <draws a number line on the paper> we  
398 have zero, we have one, two, three <locates "0", "1", "2", and "3" on the number  
399 line>.

400

401 NP: You put four instead of three, you put one and one fourth <points to a "1\_1/4"  
402 that was written on the paper>?

403

404 RB: What do you mean I put... <circles the "1\_1/4" that was written on the paper>  
405 oh no, this is just a fraction. So we have, so you have it split this into thirds, right  
406 <references the partitioning on the x-axis of the AM-FM representation>?

407

408 NP: Right.

409

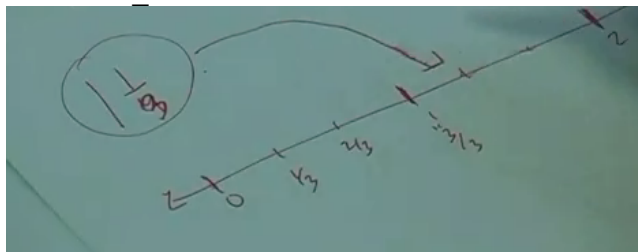
410 RB: This is one third, this is two thirds, this you said is equal to three thirds  
411 <partitions the line segment between zero and one into thirds and marks them as  
412 "1/3", "2/3", and 1 "=3/3">. Right?

413

414 NP: Right.

415

416 RB: So, this <points to "1\_1/4"> oh I see what you are saying. Thank you. One and  
417 one thirds, right <changes "1 1/4" to "1 1/3" and draws an arrow from "1\_1/3" to  
418 where 1\_1/3 would be located on the number line>?



419

420 NP: Right.

421

422 RB: Okay, but if I wanted it like in this form <points to "3/3"> instead of having a  
423 mixed...this is called a mixed number, right <points to "1\_1/3">?

424

425 NP: Right.

426  
 427 RB: Because you have a whole number <points to the “1” in “1\_1/3”> and you have  
 428 a fraction, a proper fraction <points to “1/3” in “1\_1/3”>. And these are just  
 429 <circles “1/3”, “2/3”, “3/3” on the number line> called proper fractions, right?

430  
 431 NP: Right.

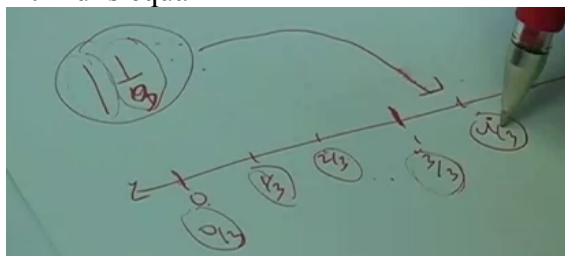
432  
 433 RB: But it’s just two numbers, one over the other, If I wanted that kind of number  
 434 here <points to “1\_1/3” on number line> what would it be? So look at the pattern.  
 435 And this would be...how many thirds would this be <points to zero>?

436  
 437 NP: It would be zero.

438  
 439 RB: Zero thirds <writes “0/3” under “0”>

440  
 441 NP: Oh, it would be, um, four thirds.

442  
 443 RB: It would be four thirds <writes “4/3” on the number line> See that one and one  
 444 third is equal-



445  
 446 NP: To four-

447  
 448 RB: To four thirds. And this would be how many thirds <points to where 1\_2/3  
 449 would be on the number line>?

450  
 451 NP: That would be five thirds.

452  
 453 RB: Five thirds <writes “5/3” on the number line>.

454  
 455 NP: And six thirds.

456  
 457 RB: <writes “6/3” under “2” on the number line> And what’s six divided by three?

458  
 459 NP: Divided by...a half? Right?

460  
 461 RB: Yeah, so what’s six divided by three?

462  
 463 NP: One half, it would be, one, six divi....oh, six divided by, you can <taps table  
 464 twice>, two. It would be two.

465

466 RB: Would equal two, right? <writes “=” between the 2 and the  $\frac{6}{3}$  that are already  
467 written on the paper>. Equals two. Right?

468

469 NP: Right.

470

471 RB: See that <circles  $\frac{5}{3}$ > And This is one and one third <points to  $\frac{4}{3}$ > and this  
472 would be one and... <points to  $\frac{5}{3}$ >

473

474 NP: One and...one...oh, one, one and two thirds.

475

476 RB: One and two thirds <writes  $1\frac{2}{3}$  above the number line over  $\frac{5}{3}$ >. So these  
477 things are equal. There's different ways to write these numbers once you get pass  
478 one, even at one <points to one> one, right? We can either write it as one or we  
479 could write it as three thirds. And zero, we can either write it as zero <points to  
480 “0”> or as zero thirds. So if I want to know the answer in terms of thirds you would  
481 write, you would say zero thirds. Right?

482

483 NP: Yup.

484

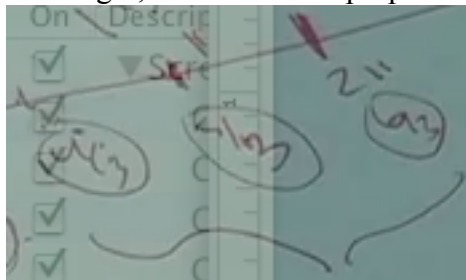
485 RB: So, see how, what do you notice about after we get, after we get pass one,  
486 <points to  $\frac{4}{3}$ ,  $\frac{5}{3}$ > what's the difference between these numbers and these  
487 numbers? <points to  $\frac{2}{3}$ ,  $\frac{1}{3}$ ,  $\frac{0}{3}$ > You notice something about these numbers?

488

489 NP: They are improper

490

491 RB: Right, so these are improper. So what makes something improper?



492

493

494 NP: You have a bigger, like a bigger number over a smaller number.

495

496 RB: Okay, so when your, um, denominator, I mean, denominator is bottom or top?

497

498 NP: Top, wait,

499

500 RB: D down

501

502 NP: D...yeah,

503

504 RB: Denominator is....

505

506 NP: Denominator is....

507

508 RB: Is bottom.

509

510 NP: Yeah, it's bottom.

511

512 RB: So when you are, um, numerator &lt;writes "when n &gt;"&gt;

513

514 NP: Numerator....

515

516 RB: Is greater than...

517

518 NP: Yeah, greater than.

519

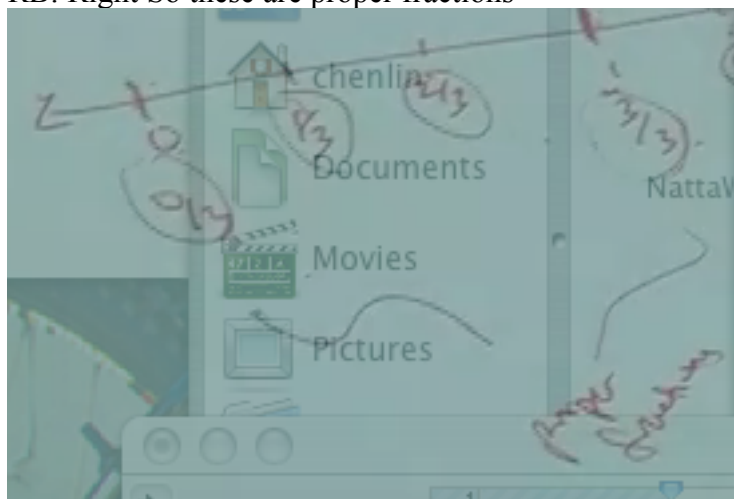
520 RB: <writes 'D' next to "when n >"> Your denominator, D. Okay, so yes, so that's  
521 called an improper fraction. And what's an opposite of an improper fraction?

522

523 NP: A proper fraction?

524

525 RB: Right So these are proper fractions



526

527 And this is called a what? &lt;points to 1 1/3 on top of page&gt;

528

529 NP: A mixed number?

530

531

532 RB: A mixed number <writes "1 1/3 = mixed number"> Okay. So good. So far  
533 we've got one third. But you were asked to take one half of one third and give it to  
534 the rat. So...

535

536 NP: You have one third and... &lt;moves Y-axis marker line down from 1 to 1/2&gt;.

537

538 RB: You moved it down there?

539

540 NP: Yeah.

541

542 RB: Okay. And what is this point called? <points to  $\frac{1}{2}$  on Y-axis> on the number  
543 line? What is that point called?

544

545 NP: A half?

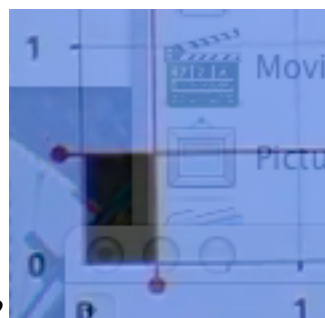
546

547 RB: One...

548

549 NP: Yeah, one half.

550



551 RB: One half, okay. And what is this piece called? How  
552 much area is that?

553

554 NP: That's...

555

556 RB: What's the name of that piece? Remember you called the other piece one fourth,  
557

558 NP: Right.

559

560 RB: What is the name of that piece?

561

562 NP: It would be yeah um, it's half of one third but.

563

564 RB: Is that a fourth?

565

566 NP: Not it's not a fourth.

567

568 RB: Is it a third?

569

570 NP: It is...it's half of one third.

571

572 RB: It's half, of one third.

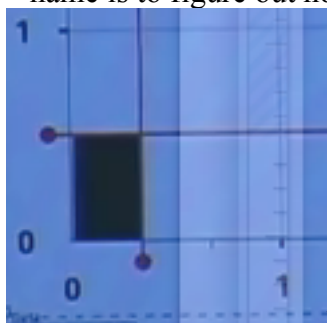
573

574 NP: Yeah.

575



576 RB: So it's not a third.  
 577  
 578 NP: It's not a third.  
 579  
 580 RB: Is it more than a third or less than a third?  
 581  
 582 NP: Less than a third. It's half of it.  
 583  
 584 RB: It's a half of it. So what's the name of this piece? A fifth? A sixth? A seventh?  
 585  
 586 NP: A sixth.  
 587  
 588 RB: A sixth.  
 589  
 590 NP: If you have six of them, you could be, you could be like, you could be thirds.  
 591  
 592 RB: If you have six of them.  
 593  
 594 NP: If you have six of these <pointing to shaded 1/6 piece w/ cursor>.  
 595  
 596 RB: Uh huh.  
 597  
 598 NP: You can make one.  
 599  
 600 RB: A whole slice?  
 601  
 602 NP: Yeah.  
 603  
 604 RB: A whole slice. So the way to find the name, is this right? The way to find the  
 605 name is to figure out how much of these <pointing to the shaded 1/6 piece>



606 would make up the whole slice.  
 607  
 608

609 NP: Right

610

611 RB: So you are saying six of these would make up the whole slice. Right?

612

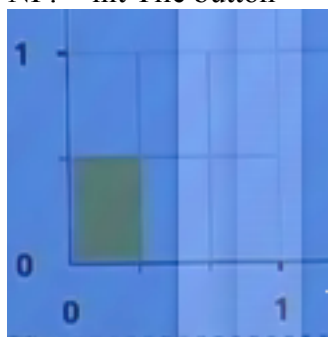
613 NP: Wait, I guess.

614  
 615 RB: So if you moved it here <points to the rest of the square bordered by the lines  
 616  $X=1$ ,  $Y=1$ , and the axis> and up here, here and here. Then you would get six of  
 617 them that complete a whole slice. And so therefore this is called a sixth. And so the  
 618 answer for the amount of cheese you use is...

619  
 620 NP: I used one sixth.

621  
 622 RB: One sixth, slice <writes in number chart which is off camera>. Okay, I want you  
 623 to hit tile for me.

624  
 625 NP: <hit Tile button>



626  
 627

628 RB: Does that help?

629

630 NP: Yeah

631

632 RB: That's one sixth.

633

634 NP: Yeah.

635

636 **TIME=00:17:49**

637

638 RB: Good. Um, let's move to the next problem. So 8C, for 8C we are going to do,  
 639 well actually, before we go to 8C, I'm going to do 8B star. One rat, this time I want  
 640 to take one third of one half, slice per rat <writing in number chart off camera>.  
 641 One third of one half slice per rat.

642

643 NP: One third of one half slice per rat.

644

645 RB: And I want a prediction. Are we going to end up using more or less cheese than  
 646 we used in case 8B?

647

648 NP: Um, I think it will be, equal? I think.

649

650 RB: Equal?

651

652 NP: Um hmm.

653

654 RB: So you're going to. we're going to use exactly one sixth. Why?

655

656 NP: Because, um, on the other sheet, the blue one, the other sheet, um, it had like no  
657 matter what order, you said no matter what order numbers are, it's going to be like,  
658 the same if multiplying.

659

660 RB: Um hmm.

661

662 NP: So, yeah, I think it'll...

663

664 RB: On the blue sheet here you mean, this blue sheet <pulls out the blue number  
665 chart from day 1>?

666

667 NP: Yeah.

668

669 RB: Okay. So here we are multiplying four and three <points to case 0> and got  
670 twelve. Three, four and twelve <points to case 4>

Case #	INPUT (units)	
0	4 rats	12 slices
1	3 rats	6 slices
2	2 rats	3 slices
3	3 rats	2 1/2 slices
4	3 rats	4 1/2 slices

671

672 Multiplying here too? &lt;points to case 4&gt;

673

674 NP: Three and a half equals one and one half.

675

676 RB: So multiplying?

677

678 NP: Well adding, but adding and multiplying can be the same thing.

679

680 RB: So you are adding half to itself three times so that's like multiplying.

681

682 NP: Yeah.

683

684 RB: Okay. So now this was case eight, right? So we repeated case eight up here  
685 <points to the top of the green number chart for day 2>. So now we are  
686 multiplying, what are we multiplying?  
687

688 NP: One into one third of a half <RB pointing to second input column which reads  
689 “1/3 of 1/2 slice/rat”>, I guess.  
690

691 RB: So we are multiplying one <pointing to first input column which reads “1 rat”>  
692 times, what?  
693

694 NP: One, No, one half of one third, I think. How do you multiply...calculate? Um.  
695

696 RB: If you were multiplying these two numbers, what would you get <points to “1/3  
697 of 1/2 slice/rat”>, together?  
698

699 NP: You get, um, one sixth.  
700

701 RB: How do you know that?  
702

703 NP: Because they are equal to, um, case EIGHT B.  
704

705 RB: Eight B. And how do you multiply two fractions <points to 1/3 of 1/2 in case 8-  
706 B\*> together, is there a rule that you follow?  
707

708 NP: You multiply the tops together and the bottoms together?  
709

710 RB: Okay, so that’s how you would multiply fractions?  
711

712 NP: Yeah.  
713

714 RB: Well we will see, we are going to do a couple of these and we will see if it’s  
715 always multiplying that we are doing. Okay?  
716

717 NP: Okay.  
718

719 RB: So you thinking we are going to get the same amount as this <pointing to output  
720 column for case eight B>, and why do you think we are going to get the same  
721 amount, because you said order doesn’t matter?  
722

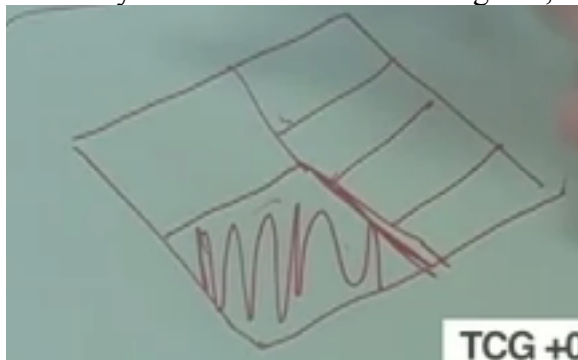
723 NP: Yeah.  
724

725 RB: And we are multiplying? Okay. S show me this problem.  
726

727 NP: One third, <set Y-axis marker line at 1/3 and moves X-axis marker line from 1  
728 to 1/2>. Yeah.

729  
730 RB: Wait, so one third.  
731  
732 NP: Yeah.  
733  
734 RB: Okay can you just hit reset and repeat that for me? Okay.  
735  
736 NP: <hits Reset button>  
737  
738 RB: So you split, hang on a second, so you split the Y-axis into thirds, and you are  
739 moving it to the first point <NP moved Y-axis marker line up from zero to  $1/3$ >.  
740  
741 NP: And now I move this to a half <moves X-axis marker line from 1 to  $1/2$ >.  
742  
743 RB: Okay, okay, so here you didn't start with the unit. One by one. Right?  
744  
745 NP: Right.  
746  
747 RB: But we know the unit's one, right? So I guess it's kind of redundant to have to  
748 always start, right? It still gives you the same thing.  
749  
750 NP: Right.  
751  
752 RB: Okay. And now how much is that? What is the name of that piece?  
753  
754 NP: One sixth.  
755  
756 RB: One sixth. And why is?  
757  
758 NP: Because when we tile it <hits Tile button> it's equal one sixth.  
759  
760 RB: And why is that one sixth?  
761  
762 NP: Because it's one of six pieces of one whole.  
763  
764 RB: Of one whole slice.  
765  
766 NP: Yeah.  
767  
768 RB: Do those six pieces always have to be equal?  
769  
770 NP: Yeah.  
771  
772 MINI LESSON #2  
773

774 RB: They do? So if I drew something like, um. That.



775 . Right? Here is what's shaded,

776

777 NP: Right

778

779 RB: What would that be? What fraction would that be?

780

781 NP: It would be

782

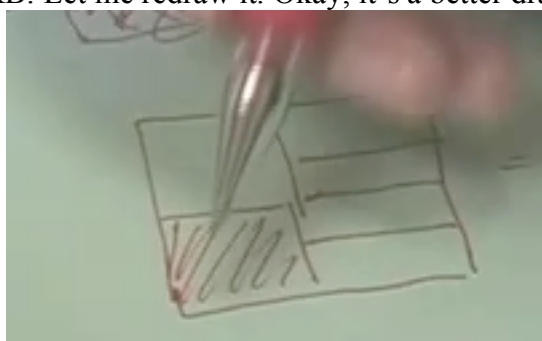
783 RB: What would the name of this piece be <points to shaded region>?

784

785 NP: One sixth maybe? I don't think it will be one sixth.

786

787 RB: Let me redraw it. Okay, it's a better drawing.



788

789 this given to you and someone says okay, so, and you gave out this much cheese  
790 <pointing to the shaded region> let's say this is a cheese, right?

791

792 NP: Right.

793

794 RB: How much cheese did you give out?

795

796 NP: One fourth.

797

798 RB: One fourth. Why is it one fourth?

799

800 NP: Because like, these two are half <points to the two  $\frac{1}{4}$  pieces>. And then these,  
801 this is half and this is half so then it would equal one fourth <points to the right half

802 of the drawing>. You just cut the halves <referring to the two  $\frac{1}{4}$ s that are slit in  
803 half>.

804

805 RB: Oh so if we get rid of these pieces, <crosses the two lines that split the two  $\frac{1}{4}$ s  
806 into half>.

807

808 NP: Yeah

809

810 RB: So you are comparing this area <pointing to the shaded  $\frac{1}{4}$ > to the whole area.

811 And you are trying to find out how many times this area <pointing to the shaded

812  $\frac{1}{4}$ > fits into the whole area?

813

814 NP: Yeah.

815

816 RB: Okay, so it's not just a matter of counting boxes, right? Because if we just  
817 counted boxes, one two three four, five, six <counting all the pieces that make up  
818 the whole> one could say it's one sixth <writes down  $\frac{1}{6}$ >. Right?

819

820 NP: Right.

821

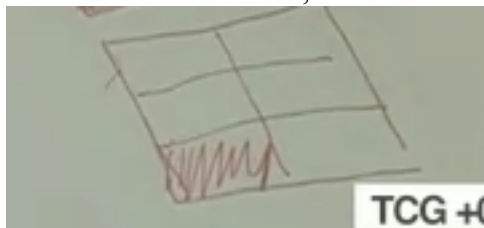
822 RB: But we know that one sixth is smallish, looks smaller than this <points to the  
823 shaded region>, I mean if a whole is the same size. Right?

824

825 NP: Right.

826

827 RB: If I start with a same sized hole, same sized cheese, then one sixth would be like



828 that much.

829

830 NP: Yeah.

831

832 RB: That's <pointing to shaded  $\frac{1}{6}$  of whole> less than that <pointing to the shaded

833  $\frac{1}{4}$  in the previous whole>, right?

834

835 NP: Right.

836

837 RB: So if we told the scientist we used one sixth, when really we used one fourth, it

838 wouldn't be right, right? He would get s sense of exactly how much cheese we

839 used.

840


841 NP: Yeah.

842  
843 RB: Yeah. Okay, your pieces do, like you said, have to be equal, in order to be able  
844 to do the count. If you wanted just to be able to count, one to six, the pieces have to  
845 be equal.  
846  
847 NP: Yeah.  
848  
849 RB: Right? Okay. Good. So we got one sixth again. You were right on about your  
850 prediction, excellent job, Um,  
851  
852 NP: Want me to reset it?  
853  
854 RB: Yeah, reset it, and set it to one, one.  
855  
856 NP: <hits Reset button and moves X-division to one and Y-division to one>  
857  
858 TIME=  
859  
860 RB: So 8C. We are going to do one rat again. And we are going to do two thirds of  
861 one third, slice per rat <writing in the number chart which is not captured on  
862 camera>. Now two thirds of one third. So now I want you to make a prediction  
863 here. In case C [inaudible] are we going to end up using more or less cheese than  
864 one sixth?  
865  
866 NP: Um...  
867  
868 RB: So in these two cases, remember we got, one sixth?  
869  
870 NP: Right.  
871  
872 RB: And you were right, order didn't matter.  
873  
874 NP: More.  
875  
876 RB: More. And why are we going to use more?  
877  
878 NP: Because instead of having one third of, like, one half is one third, it's two thirds  
879 of one third.  
880  
881 RB: Okay. So, okay, so instead of having one third of one half it's two thirds of one  
882 half.  
883  
884 NP: Of one third.  
885  
886 RB: Of one third, okay, so I'm not sure if I understand why that makes it bigger.

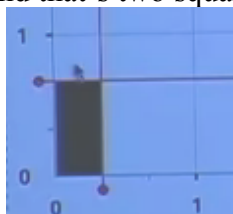


887  
888 NP: Um, can I show you on the <referencing the laptop>....  
889  
890 RB: Well so, is, are you saying that two thirds is bigger than one third?  
891  
892 NP: Yeah.  
893  
894 RB: And one third is bigger than one half?  
895  
896 NP: No.  
897  
898 RB: No. One third is smaller than a half.  
899  
900 NP: Yeah.  
901  
902 RB: So. This numbers is smaller than this one <pointing somewhere on the number  
903 chart which is not on camera but likely to  $2/3 > 1/3$  where case C is  $2/3$  of  $1/3$  and  
904 the previous case is  $1/3$  of  $1/2$  so RB might be asking NP to compare the first two  
905 numbers and the second two numbers>.  
906  
907 RB: And this number is bigger than that one <again pointing somewhere on the  
908 number chart which is not on camera but likely to  $1/3 < 1/2$  where case C is  $2/3$  of  
909  $1/3$  and the previous case is  $1/3$  of  $1/2$  so RB might be asking NP to compare the  
910 second two numbers after having him compare the first two>. So one don't they  
911 just even out?  
912  
913 NP: Because they are not equal.  
914  
915 RB: Oh you mean they are not, like it's not one third of one half?  
916  
917 NP: Oh well, This <points somewhere on the number chart but not captured on  
918 camera> is still one half, it still counts as one half, right?  
919  
920 RB: Two thirds counts as one half?  
921  
922 NP: Because isn't two thirds half of three thirds?  
923  
924 RB: Two thirds is half of three thirds...  
925  
926 NP: Oh wait that's not half of three thirds.  
927  
928 RB: Show me three thirds.  
929  
930 NP: <moves X-division to three and starts moving X-axis marker line towards  $3/3$ >  
931 oh no that's not half, yeah.

932  
933 RB: Okay.  
934  
935 NP: That's three thirds <puts X-marker line at 3/3> and then <moves marker line  
936 from 3/3 to 2/3>  
937  
938 RB: Two thirds would be right there.  
939  
940 NP: Yeah. Oh half would be like... <moves X-axis marker line to about where X  
941 would be equal to 1/2>  
942  
943 RB: Right there, yeah <NP returns X-axis marker line to zero>. So okay, so, then  
944 I'm asking you, for case 8C, are we going to end up using more or less cheese than  
945 case 8B\*? And you said, more, right?  
946  
947 NP: Right.  
948  
949 RB: Okay, what about in comparison to 8B?  
950  
951 NP: One half of one third. Hmm.  
952  
953 RB: Here at least we have the same one third, right?  
954  
955 NP: Right  
956  
957 RB: The same spot. And then this one we did half of it, this one we did two thirds of  
958 it. Which one are we going to end up using more cheese for?  
959  
960 NP: 8C.  
961  
962 RB: 8C, why?  
963  
964 NP: Because it's more than one half. Two thirds is more than one half.  
965  
966 RB: So we are taking more of the one third here <referring to case 8C>, than we took  
967 up here <referring to previous cases>.  
968  
969 NP: Yeah.  
970  
971 RB: Okay, so let's, um, okay so let's see you tile, go ahead and tile so rest it.  
972  
973 NP: Oh, reset <presses reset button and moves X slider back to 1, Y slider already at  
974 1>.  
975  
976 RB: Okay.

- 977  
 978 NP: Okay so. Make this three... <moves X-division to 3> and make this three  
 979 <moves Y-division to 3> because it's two thirds of three thirds.  
 980  
 981 RB: Okay so made both of them three.  
 982  
 983 NP: Move this right there. <moves X-axis marker line to 1/3>. So this right there.  
 984  
 985 RB: So that right there <referring to 1/3 area shaded in black>, tell me what -  
 986  
 987 NP: This is one thirds.  
 988  
 989 RB: So you've got one third of a slice there.  
 990  
 991 NP: Yeah.  
 992  
 993 RB: Okay.  
 994  
 995 NP: And two thirds <moves Y-axis marker line from 1 to 2/3>  
 996  
 997 RB: Two thirds. And then you took two thirds of that one third?  
 998  
 999 NP: Yeah  
 1000  
 1001 RB: Okay. So now um, how much, what is, how much is that? How much cheese did  
 1002 you end up using <references final area of 2/9 produced by Neato>?  
 1003  
 1004 NP: Um, two sixths. Wait. No. One ninth.  
 1005  
 1006 RB: One ninth. How are you getting one ninth?  
 1007  
 1008 NP: I mean two ninths.  
 1009  
 1010 RB: Two ninths. How are you getting two ninths?  
 1011  
 1012 NP: Because that's one square <moves y-axis marker line down from  $y=2/3$  to  
 1013  $y=1/3$ > 

1014 and that's two squares <moves y-axis marker line from  $y=1/3$  to  $y=2/3$ >.



1015

1016

1017 RB: Um Hmm

1018

1019 NP: Three square, four squares, five squares <counts on from the 2 tile pieces that  
1020 make up the shaded region to the remaining tile pieces that make up the 1x1 unit  
1021 whole>, six squares, seven squares, eight squares....

1022

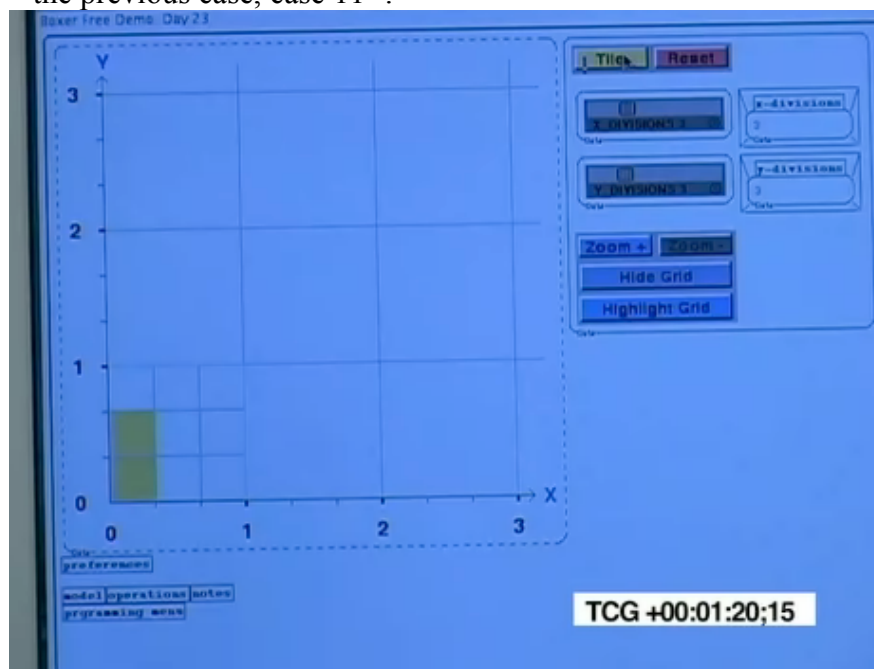
1023 RB: Nine squares. Okay wanna tile?

1024

1025 NP: <hits tile button, shaded region turn to yellow>

1026

1027 RB: Pretty nice, excellent. Okay, I'll write that <2/9> down? So was that more or  
1028 less cheese than one sixth <RB references final area of 1/6 produced by Neato in  
1029 the previous case, case 11>?



1030

1031 NP: One...wait. More.

1032

1033 RB: It was more?

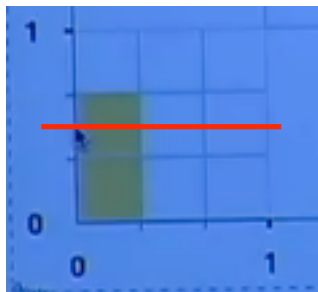
1034

1035 NP: Yeah.

1036

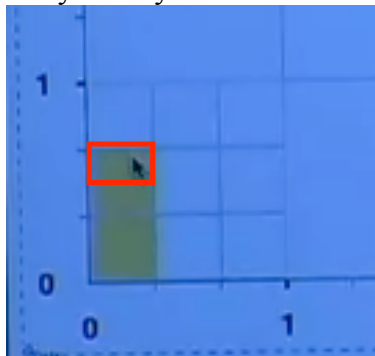
1037 RB: How do you know <that 2/9 is greater than 1/6>?

1038  
 1039 NP: Because it's...because half would be like there <using the cursor to draw an  
 1040 imaginary horizontal line at  $Y=1/2$  through the  $1 \times 1$  unit whole in order to illustrate  
 1041  $1/2$  of  $1/3$ >.



1042  
 1043 RB: Um hmm.

1044  
 1045 NP: And you have this little... <uses cursor to point out the tile half that is left over  
 1046 if you only took  $1/2$  of  $1/3$  instead of  $2/3$  of  $1/3$ >.



1047  
 1048 RB: Little strip left over.

1049  
 1050 NP: Little strip left over.

1051  
 1052 RB: Oh I see so you are saying half should be a line right in here <uses cursor to  
 1053 draw in an imaginary horizontal line at  $Y=1/2$  just as NP had done> that goes  
 1054 across. So there is a little bit over <points out the tile half that is left over just like  
 1055 NP had done>. And why are you pointing out the half?

1056  
 1057 NP: Because a half would be where one third would be.

1058  
 1059 RB: The half....

1060  
 1061 NP: It would be where one sixth would be, like one sixth. So.

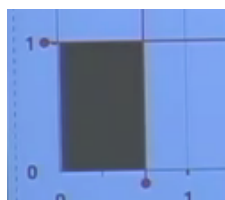
1062  
 1063 RB: Of the third?

1064  
 1065 NP: Yeah.

1066  
 1067 RB: Okay, excellent, good. Next point. 8D

1068  
1069 **TIME=32:16**  
1070  
1071 NP: <hits reset button and moves both sliders back to one>  
1072  
1073 RB: So again we have one rat.  
1074  
1075 NP: Right.  
1076  
1077 RB: And we are going to do three fifths of three fourths slice per rat. Okay? So now  
1078 I'm going to ask for some predictions again. So three fifths of three fourths. Are we  
1079 going to end up using more or less cheese than in case C? So remember what C  
1080 was. Two thirds of one third, which gave us two ninths. So, but in D we are going  
1081 to do three fifths of three fourths.  
1082  
1083 NP: Hmm.  
1084  
1085 RB: What do you think?  
1086  
1087 NP: I think it's going to be more.  
1088  
1089 RB: You think it's going to be more in D than C?  
1090  
1091 NP: Yeah.  
1092  
1093 RB: And why?  
1094  
1095 NP: Because fifths are more than, they are not bigger but there are like more slices.  
1096  
1097 RB: There is more slices in fifths.  
1098  
1099 NP: Yeah.  
1100  
1101 RB: Okay, than thirds?  
1102  
1103 NP: Yeah.  
1104  
1105 RB: Okay, so that's why it's going to be more?  
1106  
1107 NP: Yeah.  
1108  
1109 RB: Okay. Um, so we are just looking at this number and this number <pointing  
1110 somewhere on the number chart off camera, probably the first two fractions in each  
1111 case,  $2/3$  and  $3/5$ >. That's all that matters to be able to find it?  
1112

- 1113 NP: Yeah.  
 1114  
 1115 RB: To find out what's going to be more? What about these two numbers <pointing  
 1116 somewhere on the number chart off camera, probably the second two fractions in  
 1117 each case,  $1/3$  and  $3/4$ >? Do they have anything to do with it?  
 1118  
 1119 NP: Yeah. Um. Three fourths, that's um, bigger than one third.  
 1120  
 1121 RB: Three fourths.  
 1122  
 1123 NP: Are bigger than one third. No, no, three fourths is equal to one third.  
 1124  
 1125 RB: Three fourths are equal to one third? Can you show me?  
 1126  
 1127 NP: Actually, no. No they are not. Never mind. They're not equal cause they had...  
 1128 <moves X slider from 1 to 3 and moves the X-axis marker line to where  $1/2$  would  
 1129 be and Y slider is still at 1> half is right here.  
 1130  
 1131 RB: Uh huh. And you want.  
 1132  
 1133 NP: So, one, two <moves X-axis marker line to imaginary  $1/4$  lines and settles at  
 1134 where  $3/4$  would be to right of  $x=2/3$ >.  
 1135  
 1136 RB: Three fourths.  
 1137  
 1138 NP: I guess that will be three fourths.  
 1139



- 1140 RB: And that's two thirds right there <as NP lets go of the X-axis  
 1141 marker line it gets positioned at  $2/3$ >.  
 1142  
 1143 NP: Two thirds.  
 1144  
 1145 RB: So what's bigger, three fourths or two thirds?  
 1146  
 1147 NP: Hmm, I still think it's three fourths.  
 1148  
 1149 RB: Three fourths is bigger?  
 1150  
 1151 NP: Yeah.  
 1152  
 1153 RB: Okay. And what's bigger, one third or three fourths?

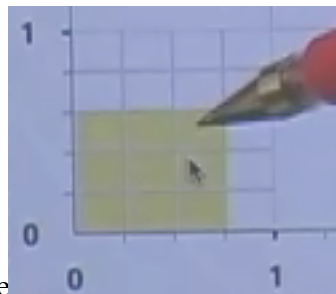
1154  
1155 NP: Three fourths.  
1156  
1157 RB: Three fourths. Okay. So this number is bigger than that number <pointing  
1158 somewhere on the number chart off camera>.  
1159  
1160 NP: Yeah.  
1161  
1162 RB: Okay. Um, and then, we are taking two thirds of on third here <pointing to case  
1163 8C on the number chart>, and here we are taking three fourths, three fifths of three  
1164 fourths <pointing to case 8D on the number chart>. Right? So why don't we check  
1165 it out, test it out. Reset it.  
1166  
1167 NP: So I would have to make this fourths, <moves X-division to 5 and Y-division to  
1168 4> So, three, one, two, three <moves Y-axis maker line up from zero to 3/4>.  
1169  
1170 RB: Okay now tell me what that point is you stopped the red marker line at?  
1171  
1172 NP: That's three fifths.  
1173  
1174 RB: That's three fifths?  
1175  
1176 NP: No, one, two, three, four <counting the fifth marks on the X-axis> oh, I did it  
1177 backwards. Hold on <hits reset button>.  
1178  
1179 RB: What do mean you did it backwards?  
1180  
1181 NP: This one is supposed to be fifths <moves Y-division to 5> and this one was  
1182 supposed to be fourths <moves X-division to 4>.  
1183  
1184 RB: Okay.  
1185  
1186 NP: So, one, two three, oh, three <moves Y-axis marker line from zero to 3/5>  
1187  
1188 RB: Okay what is that point that you....  
1189  
1190 NP: Now it's three fifths.  
1191  
1192 RB: Three fifths. So far how much cheese do we have there?  
1193  
1194 NP: We have...  
1195  
1196 RB: What is that piece called <references initial area of 3/5 produced by Neato>?  
1197  
1198 NP: It would be, you have.



1199  
1200 RB: How much of a slice?  
1201  
1202 NP: Two thirds, I think.  
1203  
1204 RB: Two thirds? Why is it called two thirds? So this axis <points to Y-axis>  
1205 represents slices right?  
1206  
1207 NP: Right.  
1208  
1209 RB: So if it went all the way up here <points to Y=1> it would be one,  
1210  
1211 NP: One.  
1212  
1213 RB: How much is this? <points to Y=3/5> What is this point called?  
1214  
1215 NP: The point is called three fifths right now.  
1216  
1217 RB: Three fifths? So how many slices did you take?  
1218  
1219 NP: Three fifths?  
1220  
1221 RB: Three fifths.  
1222  
1223 NP: Oh <chuckles>.  
1224  
1225 RB: Right? So like when you moved to a half, when you do your first move right, it  
1226 always starts, I know we sort of stopped doing the one by one. Remember when I  
1227 first had you start I said I want see the unit and I want you to move from there. So  
1228 you always started with the one by one, you were shrinking it every single time,  
1229 right?  
1230  
1231 NP: Right.  
1232  
1233 RB: So let's do that. Erase that in this case <NP hits reset button but the X and Y  
1234 sliders are still set at 4 and 5 respectively>. So right now start with the one by one.  
1235  
1236 NP: <moves X-axis marker line from zero to 1 to show 1x1 unit>.  
1237  
1238 RB: Okay, so far now how much cheese do you have?  
1239  
1240 NP: One whole.  
1241  
1242 RB: One whole. And that's what we kind of. So that's our unit. We were told to give  
1243 three fourths of three fifths, okay?

1244  
1245 NP: Yeah.  
1246  
1247 RB: So how do we want to do this? How do you want to start?  
1248  
1249 NP: Well, go back to the three fifths <moves Y-axis marker line down from 1 to  
1250  $3/5$ >.  
1251  
1252 RB: So you took three fifths of one.  
1253  
1254 NP: Yeah.  
1255  
1256 RB: Um hmm.  
1257  
1258 NP: And then three fourths <moves X-axis marker line from 1 to  $3/4$ >.  
1259  
1260 RB: Now you took three fourths of how much?  
1261  
1262 NP: Of three fifths.  
1263  
1264 RB: Of three fifths. So you took three fourths of three fifths. What's the problem I  
1265 asked you to do?  
1266  
1267 NP: Three fifths of three fourths.  
1268  
1269 RB: Does it matter? Are you going to get different answers if you did it differently?  
1270  
1271 NP: Um, No.  
1272  
1273 RB: No, doesn't matter.  
1274  
1275 NP: Yeah.  
1276  
1277 RB: Okay, okay. Um, so what's our final output? How much cheese do we end up  
1278 using <references final area of  $9/20$  produced by Neato>?  
1279  
1280 NP: <looks up into space> Six. Twentieths?  
1281  
1282 RB: Six twentieths. How did you get that?  
1283  
1284 NP: That's just a guess.  
1285  
1286 RB: Six twentieth, how did you guess that? That's an interesting number to just  
1287 randomly guess.  
1288

1289 NP: Well because <looks at number chart> five times four is twenty, Oh no, it's nine  
1290 twentieths.  
1291  
1292 RB: Nine twentieths.  
1293  
1294 NP: It should be nine twentieths.  
1295  
1296 RB: Okay. You want to tile?  
1297  
1298 NP: Sure <hits tile button>  
1299  
1300 RB: And what is that?  
1301  
1302 NP: It's four, so five times four <scrolls over the tiles running across the x-axis and  
1303 running down the y-axis that make up the 1x1 unit whole> yeah, nine twentieths.  
1304  
1305 RB: Nine twentieths. Okay. So wait, so you were about to start counting each  
1306 individual piece, and then you said, oh, five times four. So what were you counting  
1307 when you were doing five times four. What were -  
1308  
1309 NP: I just trying to make sure it was twenty, and then.  
1310  
1311 RB: Okay. But when you said five times four, What were the five and four referring  
1312 to in the picture?  
1313  
1314 NP: How many squares, How many squares inside the whole slice.  
1315  
1316 RB: You only had five and four squares in the whole slice? You said five and four,  
1317 right? Five times four...  
1318  
1319 NP: Five times four.  
1320  
1321 RB: Okay, so when you say five times four what should I be looking at in that  
1322 picture? Is there like...  
1323  
1324 NP: What they equal when you multiply them together.  
1325  
1326 RB: Um hmm. Okay, and then you said three times three.  
1327  
1328 NP: Which is nine. We have nine yellow.  
1329



- 1330 RB: Okay so, I see when you say nine I can count them  
 1331 and say one, two, three, four, five, six, seven eight, nine.  
 1332  
 1333 NP: Right.  
 1334  
 1335 RB: And I know that three times three is nine. Right? Is there a way I can look at  
 1336 this and see the three times three?  
 1337  
 1338 NP: Um. Three times three.  
 1339  
 1340 RB: You kinda said that multiplication is like adding, right? Repeated addition.  
 1341  
 1342 NP: Yeah.  
 1343  
 1344 RB: So what would you be adding three times to get nine?  
 1345  
 1346 NP: Three.  
 1347  
 1348 RB: So if you counted these by triples, like you are counting this <point the first  
 1349 column of shaded region which consists of 3 tiles>  
 1350  
 1351 NP: Yeah, so counted this three times, count three three times.  
 1352  
 1353 RB: Right. So it will be three plus three, plus three <pointing out each column of  
 1354 three tiles> Right?  
 1355  
 1356 NP: That equals nine.  
 1357  
 1358 RB: Nine.  
 1359  
 1360 **TIME=42:35**  
 1361  
 1362 RB: Okay. And then, alright. So let's go on to 8, um, okay, let's go to 8E. So again  
 1363 we have one rat. And we are going to do five sixths of two fifths of slice per rat.  
 1364  
 1365 NP: Five sixths of two fifths slice per rat.  
 1366

1367 RB: So now I want to see a prediction. Um, do you think that for E, we are going to  
1368 end up using more, or less cheese than we used in D?  
1369  
1370 NP: Um, more.  
1371  
1372 RB: More, and why?  
1373  
1374 NP: Because five sixths is bigger than three fifths?  
1375  
1376 RB: Okay, five sixths is bigger than three fifths. Why is five sixths bigger than three  
1377 fifths?  
1378  
1379 NP: Because, you like, you saw one whole, you only like one slice away from having  
1380 one whole. Like one quarter of it.  
1381  
1382 RB: Um hmm. I see. Um, what if it said, five sixths, okay, so five sixths is bigger  
1383 than three fourths.  
1384  
1385 NP: Yeah.  
1386  
1387 RB: Um, okay. What about, so here you are taking five sixths of two fifths.  
1388  
1389 NP: Right  
1390  
1391 RB: Here you are taking three fifths of three fourths,  
1392  
1393 NP: Right.  
1394  
1395 RB: How are these two numbers related, three fourths and two fifths? What's bigger,  
1396 three fourths or two fifths?  
1397  
1398 NP: Hmm, three fourths.  
1399  
1400 RB: And why?  
1401  
1402 NP: Because a fifth is smaller than a fourths,  
1403  
1404 RB: A fifths is smaller than a fourth? So a piece that's called a fifth?  
1405  
1406 NP: Yeah. a piece that is called a fifth, is smaller than one fourth.  
1407  
1408 RB: And how many of those pieces do we have here?  
1409  
1410 NP: Two.  
1411

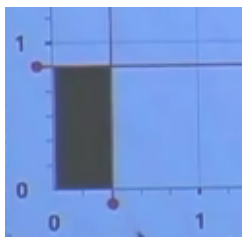
1412 RB: And here we have three of the four pieces, which are bigger. So three fourths is  
1413 bigger than this one <referring to  $2/5$  in case 8E>.  
1414  
1415 NP: Right.  
1416  
1417 RB: And this <referring to  $5/6$  in case 8E> is bigger than that one <referring to  $3/5$  in  
1418 case 8D>.  
1419  
1420 NP: Right.  
1421  
1422 RB: Okay, so then how do you know, ok so like if both of them are bigger, I mean so  
1423 this is bigger than this one, um, and here this is smaller than this one <pointing  
1424 somewhere on the number chart off camera>, how do you know then this is going  
1425 to be more? Right? Because here you are being asked to do smaller amount of  
1426 bigger piece.  
1427  
1428 NP: Right.  
1429  
1430 RB: And here you are asked to take a big amount of the smaller piece. How do you  
1431 know they are not going to balance out or that they are...  
1432  
1433 NP: I don't know.  
1434  
1435 RB: You are not sure.  
1436  
1437 NP: Yeah.  
1438  
1439 RB: Because you do know that this is bigger than this, and this is bigger than this  
1440 <pointing somewhere on the number chart off camera>.  
1441  
1442 NP: Right.  
1443  
1444 RB: So is there any way we can figure that out without going through all that?  
1445  
1446 NP: Well you can, um, multiply five sixths by two fifths.  
1447  
1448 RB: Oh you mean actually do the multiplication and then find the answer and then  
1449 compare the answers to each other?  
1450  
1451 NP: Yeah.  
1452  
1453 RB: I see. Okay. Alright, so I'm going to have you go ahead and do problem E on  
1454 this and talk me through as you are doing it.  
1455

1456 NP: Start with the one unit <moves X-axis marker line from zero to one to reveal  
1457 unit whole>?  
1458  
1459 RB: Uh huh.  
1460  
1461 NP: And then you turn the Y-axis into the, into sixths <moves Y slider to 6>.  
1462  
1463 RB: Okay.  
1464  
1465 NP: And X into fifths <moves X slider to 5>.  
1466  
1467 RB: Okay. Is there a reason why you chose fifths for the X and sixths for the Y? Or  
1468 does it matter?  
1469  
1470 NP: Um, well I just figure like it's easier for me to look at the Y-axis as the first  
1471 number on this side.  
1472  
1473 RB: Okay you like using first number for the Y-axis.  
1474  
1475 NP: Yeah.  
1476  
1477 RB: As the, for the deno... the denominator the first fraction.  
1478  
1479 NP: Yeah.  
1480  
1481 RB: With Y-axis. Okay, okay, and then you like using the second. But, does it matter  
1482 like if you flipped it? Would it matter? Would you get a different answer?  
1483  
1484 NP: No, you wouldn't get a different answer answer.  
1485  
1486 RB: You wouldn't get a different answer.  
1487  
1488 NP: Well you shouldn't get a different answer.  
1489  
1490 RB: Okay, okay, um, okay so far so good. I see what you did.  
1491  
1492 NP: Right.  
1493  
1494 RB: Keep going, now what?  
1495  
1496 NP: So, then you would do, then you would do, five sixths, one two three four five  
1497 <moves Y marker line down to 1/6 and then counts sixth up to 5/6>.  
1498  
1499 RB: Okay, now what did you just do, you sort of counted from bottom up.  
1500

1501 NP: Yeah. I don't know why.  
1502  
1503 RB: No, so that's good. So why did you do that?  
1504  
1505 NP: To make sure that I have five and not, no, not like less than five sixth.  
1506  
1507 RB: Now if you move that red marker line up, can you move it up for me, the one  
1508 you were moving, move it up <NP moves Y-axis marker line from  $5/6$  to  $6/6$ . Um,  
1509 so how many sixths is that?  
1510  
1511 NP: Six sixths.  
1512  
1513 RB: Can you move it further up <NP moves Y-axis maker line to  $7/6$ >? Move it up  
1514 further than that <NP moves Y-axis marker line to  $8/6$ >. How many sixths is that?  
1515  
1516 NP: Eight sixths.  
1517  
1518 RB: Eight sixths. And if I wanted it as a mixed number what would it be?  
1519  
1520 NP: One and two sixths.  
1521  
1522 RB: One and two sixths, okay. Very good. Um, excellent. So let's go back to five  
1523 sixths where you had it <NP moves Y-axis marker line to  $5/6$ >.  
1524  
1525 RB: Okay, um, so now you have, how much cheese do you have there so far?  
1526  
1527 NP: You have...  
1528  
1529 RB: You started off with a whole slice, right?  
1530  
1531 NP: Right.  
1532  
1533 RB: How much do you have...  
1534  
1535 NP: Five sixths.  
1536  
1537 RB: You have five sixths. And you need to give...  
1538  
1539 NP: Two...five sixths of two fifths.  
1540  
1541 RB: Or, two fifths of two sixths right? You can read it anyway you said, remember  
1542 up here we did this, one half of one third or one third of one half. You get the  
1543 same...  
1544  
1545 NP: Yeah.



- 1546  
 1547 RB: Thing because you said we were multiplying.  
 1548  
 1549 NP: Right.  
 1550  
 1551 RB: So, if you, since you started with a five sixths, right?  
 1552  
 1553 NP: Right.  
 1554  
 1555 RB: You can just say, okay so now I need to take two fifths of that. Even though the  
 1556 scientist told you you need to take five sixths of two fifths. You know that, well  
 1557 that's the same thing, it's taking two fifths of five sixths. So in your case you just  
 1558 based on the way you started, like if you had instead of, can you move it back to  
 1559 one, one <NP moves Y-axis marker line from 5/6 to 6/6>? If you were to start by  
 1560 moving this axis <points to X-axis marker line> first.  
 1561  
 1562 NP: Right.  
 1563  
 1564 RB: To what point would you move that to?  
 1565  
 1566 NP: Um, you move this to two <moves X-axis marker line from 5/5 to 2/5>.  
 1567  
 1568 RB: Two fifths?  
 1569  
 1570 NP: Yeah.  
 1571  
 1572 RB: It would be taking five sixths of two fifths. Right? By moving that one  
 1573 <referring to the Y-axis marker line> down. But that's not how you started. So  
 1574 move this back to your original <NP moves X-axis marker line to 5/5 so unit whole  
 1575 is shaded>. You started here so you decided to go five sixths first <NP moves Y-  
 1576 axis marker line from 6/6 to 5/6> which is fine. And now you are going to take two  
 1577 fifths of that <NP moves X-axis marker line from 5/5 to 2/5>.  
 1578



- 1579 NP: Two fifths of that.  
 1580  
 1581 RB: Okay, excellent. Um, so now, how much is that <references final area of 10/30  
 1582 produced by Neato>?  
 1583  
 1584 NP: That's um, <17 second pause> five sixths of two fifths.  
 1585

1586 RB: Um hmm. So how much cheese did we use? What's our output?  
1587  
1588 NP: Um, <10 second pause, looks to number chart> ten, it wouldn't be ten because  
1589 <looks to AM-FM representation> yeah, no, yeah, yeah, ten thirtieths.  
1590  
1591 RB: So I noticed that you looked over here first <points to number chart> and you  
1592 were looking at these numbers and you said ten. So were you multiplying across?  
1593  
1594 NP: Yeah, multiplying across.  
1595  
1596 RB: And so then you went back <points at shaded region> and said, it can't be ten,  
1597 but then...  
1598  
1599 NP: But then I looked at this line <uses cursor to point to  $X=1/5$  and the imaginary  
1600 vertical line that would result from that point> because I forgot that line was there  
1601 and I was like yeah, it's going to be ten. Because you have five going down, cut it  
1602 in half and so you have ten.  
1603  
1604 RB: So ten. Out of how many?  
1605  
1606 NP: <looks to number chart> Thirty.  
1607  
1608 RB: Thirty. And now again you looked at these numbers <points to number chart>  
1609 when you said thirty.  
1610  
1611 NP: Yeah.  
1612  
1613 RB: So how did you know that? How did you get thirty?  
1614  
1615 NP: Because <looks up into space> six times five is thirty.  
1616  
1617 RB: Six times five is thirty, okay. Can you hit tile for me?  
1618  
1619 NP: Sure <hits tile button>.  
1620  
1621 RB: Okay. Very good. So now this is an interesting problem. So your answer is, for  
1622 output, ten over....  
1623  
1624 NP: Ten over thirty.  
1625  
1626 RB: Ten over thirty, okay, is there another name for ten over thirty? You know how  
1627 we talked about mixed numbers and improper fractions, and there's two names for  
1628 that? Right?  
1629  
1630 NP: Right.

1631

1632 RB: You can call it one and, for example when you moved that red marker line



1633 up here, you called it one and two sixths.

1634

1635 NP: Sixths.

1636

1637 RB: Right? These are all sixths. And then I said well, okay what's another name for  
1638 it? It's eight sixths.

1639

1640 NP: Eight sixths.

1641

1642 RB: Right? So those two things mean the same exact thing, they refer to the same  
1643 point on the number line, right?

1644

1645 NP: Right.

1646

1647 RB: But they have two different names for that point, right? Is there another name  
1648 for ten thirtieths?

1649

1650 NP: Ten thirtieths.

1651

1652 RB: And feel free to move these around within the box if that helps. If you can think  
1653 of another way to arrange them,

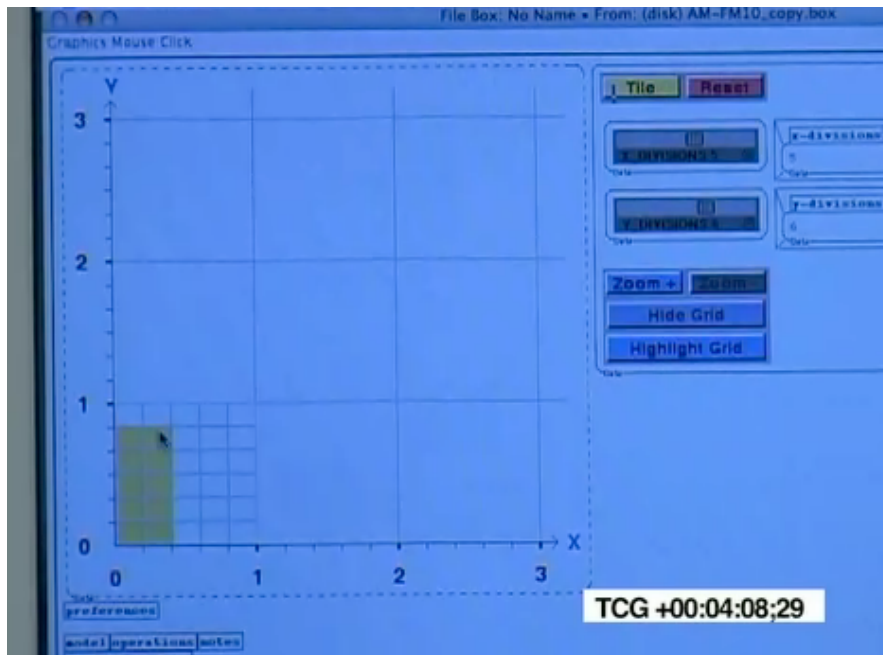
1654

1655 NP: Ten thirtieths. Hmm.

1656

1657 RB: I want to know if you can call that area something else, other than ten thirtieths.

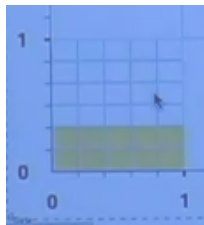
1658 Can you call it something else &lt;references tiled area of 10/30 produced by Neato&gt;?



1659

1660

NP: &lt;moves tiles down&gt;



1661

1662

NP: Two thirds

1663

1664

RB: Two thirds.

1665

1666

NP: Yeah.

1667

1668

RB: How did you get that?

1669

1670

NP: Because you have, this is one &lt;uses cursor to point to the two tiled rows that run from X=0 to X=1&gt;.

1671



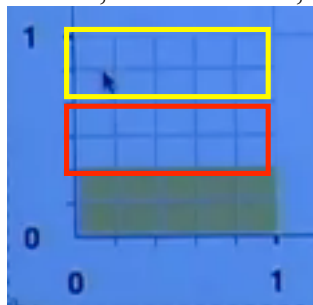
1672

1673

RB: Um hmm.

1674

1675 NP: And then this is two, like these two rows would be like another one <uses cursor  
 1676 to point the two rows above the two tiled rows within the 1x1 unit>, these two  
 1677 count as another one <uses cursor to point to the top two rows within in the 1x1  
 1678 unit>, so it would be, one, one, one third. Yeah, one third.



1679

1680 RB: One third.

1681

1682 NP: Yeah, one third.

1683

1684 RB: Okay, very good. So one third, now if we are just looking at the numbers here.

1685

1686 NP: Yeah.

1687

1688 RB: Right? Ten thirtieths equals one third <written in the output column for case 8E  
 1689 is " $10/30 = 1/3$ ">. How did you think we went from here to here <points to  $10/30$   
 1690 and  $1/3$ > just working with the numbers?

1691

1692 NP: Well times ten, cause like, if you times ten you just add a zero.

1693

1694 RB: Uh huh.

1695

1696 NP: So yeah I guess you took away two zeroes. Why didn't I do that?

1697

1698 RB: No it's fine, no, this is good. Um, okay so one third. Now I'm going to tell you  
 1699 that there is one other way to look at this.

1700

1701 NP: Okay.

1702

1703 RB: So there is another sort of name for this. So there is one third.

1704

1705 NP: Right.

1706

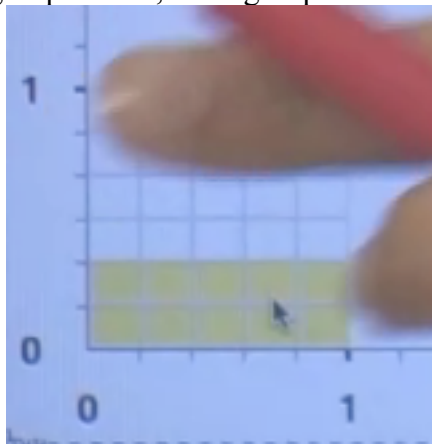
1707 RB: But there is another name for it. Can you see another name for it? So to come up  
 1708 with one third you sort of treated your ten <referring to the 10 yellow tiles>  
 1709 together, as one right? You viewed the ten as one, well I'm telling you you can  
 1710 view this in a different way, and come up with another name for this. How else can  
 1711 you view them? So you viewed them as groups of ten.

1712

1713 NP: Right.

1714

1715 RB: So you took groups of ten and said, okay, so here I have a colored group ten,  
1716 and I have total one group of ten, two groups of ten, three groups of ten <pointing



1717 to two rows made of 10 subunits each> So it's  
1718 one group of ten out of three groups of ten, So it's one third.

1719

1720 NP: Right.

1721

1722 RB: Right? Well, I'm going to suggest using a group of size different than ten. Don't  
1723 use ten, Use a different size group. What other size group could you use?

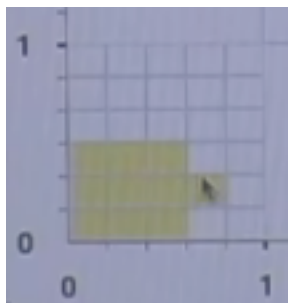
1724

1725 NP: Um <moving cursor around the subunits>. Hmm. Oh no you said... Hmm.

1726

1727 RB: <NP moves tiles into three columns of three with one left over>. So you are  
1728 trying to see if you can get it in groups of three?

1729



1730 NP: Yeah, I got one left over.

1731

1732 RB: Um hmm. So groups of three won't work.

1733

1734 NP: Well we already know groups of two would work, so...

1735

1736 RB: Oh so groups of two work?

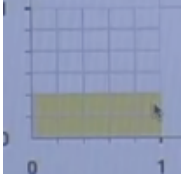
1737



- 1738 NP: Yeah. Because in the beginning it was like
- 1739
- 1740 RB: Well we don't have an answer for groups of two. How much would groups of
- 1741 two give you? If you counted everything as groups of two. How many colored
- 1742 groups of two do we have and how many total groups of two do we have?
- 1743
- 1744 NP: Um, we have, five groups of two.
- 1745
- 1746 RB: Um hmm.
- 1747
- 1748 NP: So two fifths? Or five, no, no, yeah, two fifths, no no, it wouldn't be two fifths.
- 1749
- 1750 RB: How many groups of two do we have colored?
- 1751
- 1752 NP: <counts pairs of yellow tiles> Five.
- 1753
- 1754 RB: Five. And what should we compare that to?
- 1755
- 1756 NP: Two?
- 1757
- 1758 RB: Well so, when we are doing groups of ten,
- 1759
- 1760 NP: Right.
- 1761
- 1762 RB: Right, when you, you were doing the single, you were treating them individuals,
- 1763 right? You weren't pairing them. You weren't doing groups of ten, you weren't
- 1764 doing groups of one, we were just counting boxes, right?
- 1765
- 1766 NP: Right.
- 1767
- 1768 RB: Single boxes, no pairing. So you counted the ten, and you compared it to the
- 1769 whole <points to the 1x1 area in tiles> thirty boxes, right?
- 1770
- 1771 NP: Yeah.
- 1772
- 1773 RB: And then we said, oh wait, you could treat, you could treat it as groups of ten.
- 1774 So you counted this whole thing as a group of ten and you again compare it to the
- 1775 thirty boxes.
- 1776
- 1777 NP: Right.
- 1778

1779 RB: But you found how many groups of ten were in the thirty boxes, and there is  
1780 three, Right? That's how you got the one third. So now you are doing groups of  
1781 two, So okay, you could see these in terms of groups of two <points to the two  
1782 columns of yellow tiles with five in each> and there is one, two, three, four, five,  
1783 how many groups of two are there total?  
1784  
1785 NP: Five. Well how many two groups would be...  
1786  
1787 RB: In the one by one, in the unit.  
1788  
1789 NP: Well, there's five groups. So,  
1790  
1791 RB: Wait, there is five colored groups,  
1792  
1793 NP: Five colored groups  
1794  
1795 RB: How many groups, if two, are there total? Colored including clear.  
1796  
1797 NP: Thirty.  
1798  
1799 RB: Thirty?  
1800  
1801 NP: Well thirty if you...  
1802  
1803 RB: Singles.  
1804  
1805 NP: Oh singles.  
1806  
1807 RB: There are thirty singles.  
1808  
1809 NP: Yeah, thirty singles.  
1810  
1811 RB: And ten single colored one, right?  
1812  
1813 NP: Right.  
1814  
1815 RB: So see look at this one, so there are ten singles <referencing number chart>  
1816  
1817 NP: Right.  
1818  
1819 RB: To thirty singles<referencing number chart>.  
1820  
1821 NP: Ten singles to thirty singles.  
1822  
1823 RB: Right?



- 1824  
 1825 NP: Right.  
 1826  
 1827 RB: Singles when I refer to singles I mean the single square, the little tiny piece  
 1828 <pointing to a single yellow tile>.  
 1829  
 1830 NP: Right.  
 1831  
 1832 RB: Right? So ten tiny pieces to thirty tiny pieces.  
 1833  
 1834 NP: Right.  
 1835  
 1836 RB: When you did one third,  
 1837  
 1838 NP: Right.  
 1839  
 1840 RB: You chunked all the tiny pieces together into ten pieces, right? So there is ten  
 1841 colored pieces.  
 1842  
 1843 NP: Right.  
 1844  
 1845 RB: Right. So those ten pieces together made one.  
 1846  
 1847 NP: Right.  
 1848  
 1849 RB: Right? You treat them when move them all over here <pointing from left to  
 1850 right across X-axis> you made them one. And then you said okay, so how many of  
 1851 those pieces are there in the whole. You said, we moved this down here <moving
- 
- 1852 tiles>. And you said, okay, that's my piece right there, . It consists of  
 1853 ten singles.  
 1854  
 1855 NP: Right.  
 1856  
 1857 RB: Right. So you had one set of ten singles.  
 1858  
 1859 NP: Right.  
 1860  
 1861 RB: Compared to how many sets of ten singles?  
 1862  
 1863 NP: Three.  
 1864

1865 RB: Three, right? So now you had ten singles compared to thirty singles. One pair of  
1866 ten singles to three, thirty singles. Now I'm telling you I want you to do it in pairs  
1867 of two. So you just said five pairs, five sets of twos, right? Colored.  
1868  
1869 NP: Yeah.  
1870  
1871 RB: To how many sets total? Is what I'm asking you. How many sets of two total?  
1872 So, if we are counting, so you said five, right? One, two, three, four, five <uses  
1873 cursor to show pair of five colored tiles>. Is that going to be numerator or  
1874 denominator?  
1875  
1876 NP: Numerator.  
1877  
1878 RB: Numerator. The colored parts are the numerator.  
1879  
1880 NP: Right.  
1881  
1882 RB: Now I want to know how many sets of two there are total. One, two, three, four,  
1883 five, six <uses cursor to point to each pair of two starting with the colored tiles and  
1884 then moving to clear tiles> –  
1885  
1886 NP: Oh.  
1887  
1888 RB: Seven,  
1889  
1890 NP: Eight, nine,  
1891  
1892 RB/NP: Ten, eleven, twelve, thirteen fourteen, fifteen.  
1893  
1894 NP: Fifteen.  
1895  
1896 RB: So another way, another name for this  
1897  
1898 NP: Is five fifteenths.  
1899  
1900 RB: Is five fifteenths. Right?  
1901  
1902 NP: Right.  
1903  
1904 RB: So we can count by twos. Does that make sense?  
1905  
1906 NP: Yeah.  
1907  
1908 RB: Okay. So you said we couldn't count by threes, because when you tried to pair  
1909 by threes you did this, which is good. You said, alright, I'm going to move this

1910 here, gonna move this here, move this here, and I have one left over, right  
 1911 <arranges colored into 3x3 square plus one extra>? If this one wasn't here then  
 1912 you could do it by threes. You would have one, two, three threes, right?

1913

1914 NP: Right.

1915

1916 RB: Compared to three, four, five, six, seven, eight, nine, ten <uses cursor to point  
 1917 out each set of 3 tiles starting with color and moving to clear>, right?

1918

1919 NP: Right.

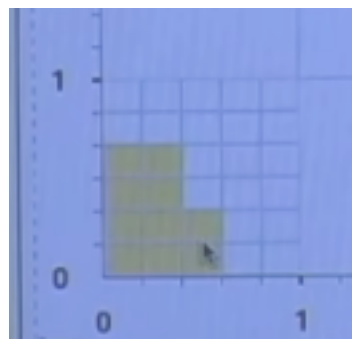
1920

1921 RB: But we have this extra one so you can't really do that. You can't count by  
 1922 threes. You can't count by fours either, right?

1923

1924 NP: Right.

1925



1926 RB: If we did that we would have two left over.  
 1927 here.

1928

1929 NP: Right.

1930

1931 **TIME=1:04:52**

1932

1933 RB: And we did count by fives that worked.

1934

1935 NP: Yeah.

1936

1937 RB: Um, so good, next one. Okay, here I'm going to do 8F, we have one rat. And we  
 1938 are going to have four thirds of two fifths slice per rat. Four thirds of two -

1939

1940 NP: Four

1941

1942 RB/NP: Four Thirds of two fifths.

1943

1944 RB: Okay, um, so some predictions, in comparison to E, are we going to end up use  
 1945 more or less cheese?

1946

1947 NP: Um, more or less cheese. Less?  
1948  
1949 RB: Less. So, why less? So tell me what we did in eight E, How much, what do we  
1950 do? We took...  
1951  
1952 NP: We took five sixths of two fifths.  
1953  
1954 RB: Um hmm. And here we are being asked to do what?  
1955  
1956 NP: Four thirds of two fifths.  
1957  
1958 RB: Okay.  
1959  
1960 NP: Four thirds is equal to one and one third.  
1961  
1962 RB: Um hmm. And this is five sixths <referencing the input for case 8E>.  
1963  
1964 NP: Right.  
1965  
1966 RB: So where are we being asked to take more cheese? Are we being asked to take  
1967 more here two fifths, or are we asked to take more here? Here we are asked to take  
1968 five sixths of two fifths.  
1969  
1970 NP: Right.  
1971  
1972 RB: Here we ask to take four thirds of two fifths.  
1973  
1974 NP: Well yeah, um, 8F should be more than...  
1975  
1976 RB: Than E?  
1977  
1978 NP: Than E.  
1979  
1980 RB: Why?  
1981  
1982 NP: Because four thirds should be more than five sixths?  
1983  
1984 RB: Should be more? You don't seem very convinced of that. Why don't you check  
1985 it out.  
1986  
1987 NP: <hits reset button> So move all these back to one <moves both sliders to 1>. So,  
1988 um, four and <moves Y-division to 4 and then X-division to 3> three.  
1989  
1990 RB: Why did we set it at four and three?  
1991

1992 NP: Because it's four thirds.  
1993  
1994 RB: Okay.  
1995  
1996 NP: So start off with one, <moves X-axis marker line from zero to 1>.  
1997  
1998 RB: Um hmm.  
1999  
2000 NP: And for um, how would I do this? It will be, you could say it would be one, one  
2001 third, so <moves X-axis marker line from 3/3 to 4/3>.  
2002  
2003 RB: Four thirds?  
2004  
2005 NP: Yeah.  
2006  
2007 RB: Is that four thirds right there? That point? What's the name of that ...  
2008  
2009 NP: Yeah, four thirds.  
2010  
2011 RB: Four thirds.  
2012  
2013 NP: Because you have one hole and then you have <uses cursor to point to 1/3  
2014 shaded that falls outside the first 1x1 unit whole>  
2015  
2016 RB: Okay, four thirds. And now you are being asked to take, now what are you  
2017 going to do?  
2018  
2019 NP: Two fifths?  
2020  
2021 RB: Um hmm,  
2022  
2023 NP: Four thirds of two fifths.  
2024  
2025 RB: What are those, those pieces you split up into? On the Y-axis?  
2026  
2027 NP: Oh the Y-axis, it split it up into fourths.  
2028  
2029 RB: Why did you do that?  
2030  
2031 NP: Well because you asked me to show four thirds.  
2032  
2033 RB: Okay, so you showed me four thirds here, what did that have to do with this part  
2034 <pointing to the Y-axis>?  
2035  
2036 NP: Oh well, I guess that had nothing to do with that part. I was just thinking weird.

2037  
2038 RB: No, what were you thinking?  
2039  
2040 NP: Hmm?  
2041  
2042 RB: What were you thinking?  
2043  
2044 NP: I thought that it would have like the four would have to do something of four  
2045 thirds.  
2046  
2047 RB: Oh, so you.  
2048  
2049 NP: Yeah.  
2050  
2051 RB: I see. I see. You thought you had make that red marker line move somewhere  
2052 over here <points to the Y-axis> too. Okay, so now I want you to finish up this  
2053 problem.  
2054  
2055 NP: Four, two fifths, two fifths, so can I restart this problem?  
2056  
2057 RB: I think you can move this slider. It should let you.  
2058  
2059 NP: Oh <has cursor positioned on the X slider>.  
2060  
2061 RB: So which slider do we want to move? Y?  
2062  
2063 NP: Yeah.  
2064  
2065 RB: So that's the X?  
2066  
2067 NP: Oh yeah. <moves Y-division to 3>.  
2068  
2069 RB: And why are we moving it to thirds?  
2070  
2071 NP: And then move this one to fifths <moves X-division to 5 causes the X-axis  
2072 marker line to move from  $4/3$  to  $7/5$ >. Because four, um, four thirds, remember  
2073 how I said the first number, I like it to be on the Y?  
2074  
2075 RB: Okay so let's reset.  
2076  
2077 NP: <hits reset button>  
2078  
2079 RB: Okay, so I want you to start with the one by one, unit.  
2080  
2081 NP: <moves X-axis marker line from zero to  $5/5$ >.

2082  
2083 RB: Okay, so, okay. so now, we have the first number on Y?  
2084  
2085 NP: Yeah.  
2086  
2087 RB: Wait, we do? Yeah. The thirds.  
2088  
2089 NP: So four thirds<moves Y-axis marker line from  $3/3$  to  $4/3$ >  
2090  
2091 RB: Okay so we got bigger than one slice now, so we are giving out, okay how  
2092 much, how much cheese are we giving out so far?  
2093  
2094 NP: Um, one and one third.  
2095  
2096 RB: One and one third, slice. Okay, and then what?  
2097  
2098 NP: Two <moves X-axis marker line from  $5/5$  to  $2/5$ >.  
2099  
2100 RB: Two fifths.  
2101  
2102 NP: Yeah.  
2103  
2104 RB: Okay, um, what is this point here<points to  $1/5$  on X-axis>?  
2105  
2106 NP: That's one fifth.  
2107  
2108 RB: And this point here <points to 0>?  
2109  
2110 NP: Zero fifths.  
2111  
2112 RB: What's, this point here <points to  $8/5$ >?  
2113  
2114 NP: Um. Eight fifths, yeah, eight fifths.  
2115  
2116 RB: Eight fifths?  
2117  
2118 NP: Wait, one two three four, five, six seven <miscounts the marks on the X-axis>.  
2119 Oh, wait, so this is six <points to  $6/5$ >, seven, eight.  
2120  
2121 RB: How did you know so quickly that this was six? <pointing to  $6/5$ >  
2122  
2123 NP: Six. because this is already five <points to 1>.  
2124  
2125 RB: Oh, so five fifths will be one?  
2126

2127 NP: Yeah.  
2128  
2129 RB: Okay, how many fifths is two?  
2130  
2131 NP: Ten fifths.  
2132  
2133 RB: And here <points to  $x=3$ >?  
2134  
2135 NP: Fifteen fifths.  
2136  
2137 RB: Fifteen fifths, Are you seeing the patterns?  
2138  
2139 NP: Yeah.  
2140  
2141 RB: Okay good. Okay, so far what we have done is taking, we started with a whole  
2142 unit, one by one unit, and you said okay, we need one, we need four thirds, right?  
2143 That's what you did. So this is four thirds <points to  $4/3$  on Y-axis>.  
2144  
2145 NP: Right.  
2146  
2147 RB: Which is equivalent to?  
2148  
2149 NP: One and one third.  
2150  
2151 RB: One and one third. And then you said okay, and now we are going to take two  
2152 fifths of that, right?  
2153  
2154 NP: Right  
2155  
2156 RB: So you moved, you stretched it, I mean you shrunk it down to  $2/5$ .  
2157  
2158 NP: Right.  
2159  
2160 RB: Right? Um, okay, so how much cheese did we end up using <references final  
2161 area of  $8/15$  produced by Neato>?  
2162  
2163 NP: <10 second pause> Let's see. So... One two three four <uses cursor to point out  
2164 where the yellow shaded tiles would be moving up the Y-axis> and this is cut in  
2165 half <referencing the  $1/5$  mark that would split the tiles in half>, It would equal  
2166 eight. Eight, eight something.  
2167  
2168 RB: Of what?  
2169  
2170 NP: This is, um, eight fifteenths. Yeah.  
2171



2172 RB: Eight fifteenths. How did you get fifteen?  
2173  
2174 NP: Because five <moves curser across x-axis which is partitioned into fifths> times,  
2175 well <moves curser up and down y-axis which is partitioned into thirds from  $y=0$   
2176 to  $y=1$ >...  
2177  
2178 RB: Five times three?  
2179  
2180 NP: Three is fifteen.  
2181  
2182 RB: What were you pointing to when you said five?  
2183  
2184 NP: Five, how many in the whole <points the marks on the X-axis that make up the  
2185 1x unit>, squares that you have in one whole slice.  
2186  
2187 RB: Like how much you split it into?  
2188  
2189 NP: Yeah.  
2190  
2191 RB: Five, because they are fifths?  
2192  
2193 NP: Yeah.  
2194  
2195 RB: And then three because they are thirds?  
2196  
2197 NP: Yeah.  
2198  
2199 RB: I see. Five times three will tell you how piece there are in here <pointing to the  
2200 1x1 unit whole>.  
2201  
2202 NP: Yeah.  
2203  
2204 RB: Okay, so it's fifteen.  
2205  
2206 NP: Fifteen.  
2207  
2208 RB: And the eight you got how?  
2209  
2210 NP: Eight?  
2211  
2212 RB: You said eight fifteenths.  
2213  
2214 NP: Eight fifteenths because you have one, two, three, four, four thirds <moving  
2215 cursor up the Y-axis>.  
2216

2217 RB: Um hmm.

2218

2219 NP: And this half line <pointing to line that would extend vertical from  $X=1/5$ >.

2220

2221 RB: So there is...

2222

2223 NP: Fifths, so then it would equal eight fifteenths.

2224

2225 RB: Okay, let's hit tile.

2226

2227 NP: <hits tile button>

2228

2229 RB: Okay.

2230



2231 NP: Without having the top part there. <uses cursor to point to the  
2232 gray partition in the second 1x1 unit whole>.

2233

2234 RB: Okay so go ahead and move your stuff however you need to show me why it's  
2235 eight fifteenths.

2236

2237 NP: One two, five, six, seven eight <counts as he moves tiles into the original 1x1  
2238 unit whole and within the original 1x1 unit whole>.

2239

2240 RB: You want to high light grid, so it highlights the units for us?

2241



2242 NP: <hits highlight grid button>

2243

2244 RB: So we see that it's eight out of fifteen for our one by one unit, right?

2245

2246 NP: Yeah.

2247

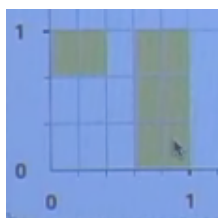
2248 RB: Okay excellent. And does eight out of fifteen reduced?

2249  
 2250 NP: Eight out of fifteen, eight out of fifteen is reduced...can I reduce <moves cursor  
 2251 over the yellow tiles>? No?

2252

2253 RB: How do you know?

2254



2255 NP: So if I moved...hold on. Yeah. No it doesn't reduce.

2256

2257 RB: It doesn't reduce, We can't count by twos?

2258

2259 NP: Two, three, four, no.

2260

2261 RB: Why can't we count by twos? Because I can count the yellows by two, right?

2262

2263 NP: Right.

2264

2265 RB: So two,

2266

2267 NP: Two, four

2268

2269 RB: Six, eight <uses cursor to point out pairs of yellow tiles>. So how many pairs of  
 2270 twos do we have? Four of them.

2271

2272 NP: Four of them. And...

2273

2274 RB: Can we count the totals by twos? Two...

2275

2276 NP: Two...

2277

2278 RB: So there is one.

2279

2280 NP: Right.

2281

2282 RB: There's two, three, four, five, six, seven. oh and one <uses cursor to point out  
 2283 pairs of twos that constitute the 1x1 unit whole>.

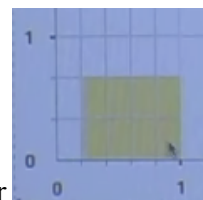
2284

2285 NP: Yeah. One.

2286

2287 RB: This one <points to the corner tile> doesn't have a pair right? Because there's  
 2288 fifteen and fifteen is an odd number.

2289  
 2290 NP: Yeah.  
 2291  
 2292 RB: So we can't count by twos. So that won't work. Can we count by threes? No,  
 2293 because we have eight colored, right?  
 2294  
 2295 NP: Right.  
 2296  
 2297 RB: So that won't work.  
 2298  
 2299 NP: Can't count by fives because...  
 2300  
 2301 RB: We have eight.  
 2302  
 2303 NP: Yeah.  
 2304  
 2305 RB: Can we count by fours? So we count four and four for the yellow. But can we  
 2306 count four and four for the fifteen?  
 2307



2308 NP: Oh, yeah <begins to move yellow tiles>. Four and four So  
 2309 yeah, that is equal. So you have four, four.  
 2310  
 2311 RB: Four, so you have two there. Two sets of the four, to how many total?  
 2312  
 2313 NP: That still uneven though.  
 2314  
 2315 RB: It's still uneven, right? We are missing a box.  
 2316  
 2317 NP: Yeah.  
 2318  
 2319 RB: So we can't count by fours either.  
 2320  
 2321 NP: No.  
 2322  
 2323 RB: So it doesn't reduce, yeah.  
 2324  
 2325 NP: It doesn't reduce.  
 2326  
 2327 RB: Excellent. And the way you check when you look back at the number is you say,  
 2328 is there any number that divides both of these numbers <pointing to "8/5" written  
 2329 in the output column of case 8F>? Right?

2330  
2331 NP: No.  
2332  
2333 RB: Nothing So here we knew that <points to the output column for case 8E  
2334 “10/30=1/3=5/15”> ten divides both these numbers <referring to the 10 and 30 in  
2335 “10/30”> and two divides both these numbers.  
2336  
2337 NP: Right  
2338  
2339 RB: Right? When ten divided both these numbers <10 and 30 in “10/30”>, we  
2340 treated the ten as a whole set. And then when five, two divided both these numbers  
2341 it became five fifteenths, so we treating, we were counting by pairs.  
2342  
2343 NP: Right.  
2344  
2345 **TIME=1:18:09**  
2346  
2347 RB: Okay, um, okay, next problem. 8G. So 8G, case 8G, one rat again. And now you  
2348 are going to do two and three fifths of two fifths slice per rat <fills in the input  
2349 columns for case 8G>. So again we have our, I’m going to ask for a prediction, Are  
2350 we going to end up using more or less cheese than we used here <points to case  
2351 8F>?  
2352  
2353 NP: Hmm. Um. More.  
2354  
2355 RB: Why more?  
2356  
2357 NP: Cause you already have...  
2358  
2359 RB: Mike, what time am I at?  
2360  
2361 Mike: Um, ten minutes.  
2362  
2363 RB: Okay, because?  
2364  
2365 NP: Because you have more like, because two, you have two and three fifths.  
2366  
2367 RB: As opposed to...  
2368  
2369 NP: I have to find out what that is. Um.  
2370  
2371 RB: So in F you were going to take four thirds of two fifths.  
2372  
2373 NP: Right.  
2374

2375 RB: And G you are being asked to take two and three fifths of two fifths. And you  
2376 are saying that in G we are going to get more cheese, we are going to end up using  
2377 more cheese. Right?  
2378  
2379 NP: Right.  
2380  
2381 RB: Okay. So why are we going to end up using more cheese? Now you are looking  
2382 over  
2383  
2384 NP: It should be, that's equal to thirteen fifths. right?  
2385  
2386 RB: Thirteen fifths?  
2387  
2388 NP: Yeah.  
2389  
2390 RB: What is equal to thirteen fifths?  
2391  
2392 NP: Two and three fifths.  
2393  
2394 RB: Okay. how did you get that, because I saw you looking over here at the  
2395 computer screen when you were doing that, you were counting something.  
2396  
2397 NP: Yeah I was counting how many...  
2398  
2399 RB: The fifths?  
2400  
2401 NP: Yeah.  
2402  
2403 RB: Okay, because it's already split into fifths here <pointing to the computer screen  
2404 off camera that has not been reset from previous case>.  
2405  
2406 NP: Yeah.  
2407  
2408 RB: Mike are you getting this one or this one? <points to the computer screen and  
2409 then the paper>. Both of them? Okay, so this one was split into fifths.  
2410  
2411 NP: Right.  
2412  
2413 RB: Um, okay, so you said it's thirteen fifteenths.  
2414  
2415 NP: Um hmm.  
2416  
2417 RB: So again my question is are we going to end up using more cheese for G or F,  
2418 you said G, right?  
2419

2420 NP: Right.  
2421  
2422 RB: Why?  
2423  
2424 NP: Because you have more cause like, if four thirds is only one in one third.  
2425  
2426 RB: Um hmm,  
2427  
2428 NP: And thirteen fifths is two so it's more, because that's two.  
2429  
2430 RB: Um hmm.  
2431  
2432 NP: Two and three fifths.  
2433  
2434 RB: And you are working with the same number, right?  
2435  
2436 NP: Yeah. Two fifths.  
2437  
2438 RB: Right. Okay, so here the prediction here is a little easier to make than when your  
2439 numbers are all so different from each other, right?  
2440  
2441 NP: Right.  
2442  
2443 RB: So here, you can start, you can either say you are starting with the same number,  
2444 or you can say you are starting with two numbers, one is different. But you are  
2445 taking the same amount of them, right?  
2446  
2447 NP: Yeah.  
2448  
2449 RB: Okay, so let's have you go ahead and reset, and do this problem.  
2450  
2451 NP: <hits reset button> reset. Okay so, moves these back <moves X and Y divisions  
2452 back to one> So it will be...so start <move X-axis marker line from zero to 1>.  
2453  
2454 RB: With the unit.  
2455  
2456 NP: My unit.  
2457  
2458 RB: Okay.  
2459  
2460 NP: And then since they both are fifths, then I guess they both would have five  
2461 <moves Y and then X divisions to 5>  
2462  
2463 RB: Um hmm. Okay.  
2464

2465 NP: And then so since this is five, it has to be ten, one, two, three <moves Y-axis  
2466 marker line up to  $2\frac{3}{5}$ >. So there's thirteen fifths.  
2467  
2468 RB: Um hmm.  
2469  
2470 NP: You have to have two fifths <moves X-axis marker line from 1 to  $2/5$ >.  
2471  
2472 RB: Two fifths, okay. Alright so, how much is that? How much cheese are we giving  
2473 out in this case, for this rat?  
2474  
2475 NP: Um <9 second pause, looks to the AM-FM representation> twenty-six, twenty-  
2476 six pieces?  
2477  
2478 RB: Twenty-six pieces? How did you get twenty-six?  
2479  
2480 NP: Okay since you have thirteen pieces going up <moves cursor up the Y-axis>,  
2481 and you half it <points  $X=1/5$  on to X-axis>, thirteen plus thirteen should equal  
2482 twenty-six.  
2483  
2484 RB: Thirteen plus thirteen equals twenty-six, okay. So you get twenty-six pieces but  
2485 how is that going to help the scientist, if you wrote, I want to have a fraction, how  
2486 much of a slice did you give? Or how many slices did you give, right? The scientist  
2487 wants to know not pieces, because he doesn't know what size your pieces are.  
2488  
2489 NP: Right  
2490  
2491 RB: So, what are we going to put for the output? How much cheese, so you give  
2492 twenty-six, what?  
2493  
2494 NP: Twenty-six...  
2495  
2496 RB: What is the name of those little pieces, those twenty-six little pieces? Are they a  
2497 fourth, each? Are each of these little pieces <pointing to what would be one tile if  
2498 shaded region was tiled> -  
2499  
2500 NP: Twenty-fifths.  
2501  
2502 RB: Twenty-fifths.  
2503  
2504 NP: Yeah, because five times five <makes vertical and horizontal motion with arm>.  
2505  
2506 RB: Five times five, so they are going to be a fifth each,  
2507  
2508 NP: Yeah.  
2509



2510 RB: So we are giving out twenty-six twenty-fifths?

2511

2512 NP: Uh, yeah.

2513

2514 RB: Okay, I'll write that down.

2515

2516 NP: And, yeah.

2517

2518 RB: And what?

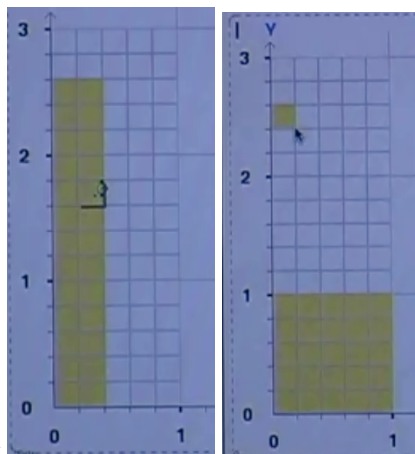
2519

2520 NP: And can I tile it?

2521

2522 RB: Yeah. Go for it.

2523



2524 NP: <hits tile button and moves tiles around>

2525

2526 RB: Five minutes? Okay, thanks Mike.

2527

2528 NP: So you have one whole and one left over.

2529

2530 RB: Okay, so how much did you give?

2531

2532 NP: One and one twenty-fifths.

2533

2534 RB: Okay, so that's the same as...

2535

2536 NP: As...

2537

2538 RB: Twenty-six twenty-fifths.

2539

2540 NP: Yeah, twenty-six twenty-fifths.

2541

2542 RB: Right? Because how many times does twenty-five go into twenty-six? One time  
2543 and one left over.

2544  
2545 NP: Right.  
2546  
2547 RB: Right. So, this fraction can sometimes be thought of as divisions. Remember  
2548 when I said what's six divided by two, and you said three. Right?  
2549  
2550 NP: Right.  
2551  
2552 RB: Um, it's the same thing here. And this is what you were doing with the, when  
2553 you said this <points to  $5/4$  on Y-axis> is one and one fourth. And you said, oh that  
2554 can also be called  
2555  
2556 NP: What? Oh, one and one fourth can be called five, five fourths?  
2557  
2558 RB: Five fourths, right. So same thing here. This is twenty-six fifths, you can  
2559 visualize a number line if that helps, right?  
2560  
2561 NP: Right.  
2562  
2563 RB: Twenty-six fifths can also be called, one, and one, twenty-fifths.  
2564  
2565 NP: Right.  
2566  
2567 **TIME=1:26:40**  
2568  
2569 RB: Right? Okay. Um, good. Next problem? Okay. So now what I'm going to do is  
2570 8H. One rat again, and we are going to do one and two fifths times six fourths slice  
2571 per rat. Why did you laugh?  
2572  
2573 NP: This is going to get hard.  
2574  
2575 RB: That's the idea. You are getting smarter so the problems have to get harder.  
2576  
2577 NP: Okay, so one and two fifths times six fourths.  
2578  
2579 RB: I don't like writing times here, I should write "of".  
2580  
2581 NP: Oh.  
2582  
2583 RB: But even though you know you said we were multiplying, I think that's why I  
2584 keep writing times.  
2585  
2586 NP: Yeah. So reset that <presses reset button but divisions are still at fifths for the x-  
2587 axis and the y-axis>. So...  
2588

2589 RB: So what's our unit?

2590

2591 NP: The unit is one whole.

2592

2593 RB: Okay, so start there.

2594

2595 NP: Okay, so <moves x division from 5 to 4> unit is one whole <moves X-axis  
2596 marker line from zero to 4/4>.

2597

2598 RB: So you always like to set you sliders first before you go to your unit, huh?

2599

2600 NP: Yeah.

2601

2602 RB: That's good. And you always pick the Y-slider for the first fraction.

2603

2604 NP: Yeah.

2605

2606 RB: Yeah, okay.

2607

2608 NP: So one and two fifths <moves Y-axis marker line from 5/5 to 1 2/5>.

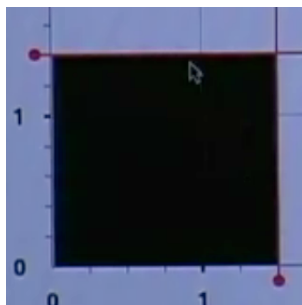
2609

2610 RB: One and two fifths, okay.

2611

2612 NP: Okay. And six fourths <moves X-axis marker line from 4/4 to 6/4>.

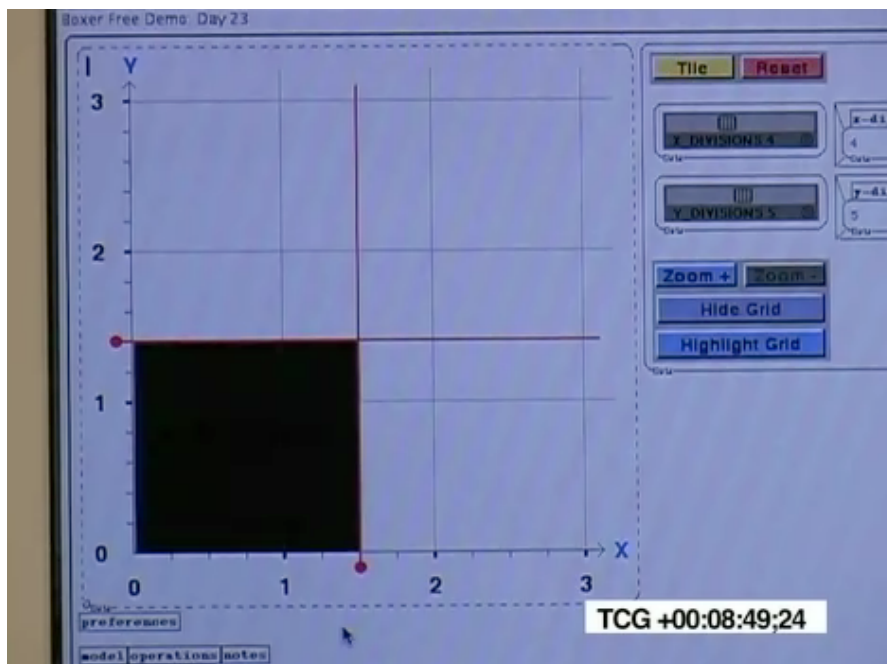
2613



2614

2615

2616 RB: Okay, so how much cheese we give out <references final area of 42/20 produced  
2617 by Neato>?



2618

2619 NP: <chuckles> Um, that's a lot <12 second pause>.

2620

2621 RB: Mike, can you stop that camera, over there, yeah. And um, in case we run out of  
2622 this camera, I can have you play that camera for a while. So just stop it.

2623

2624 NP: <chuckles> I don't know, unless I count all the boxes.

2625

2626 RB: Unless you count all the boxes?

2627

2628 NP: Yeah.

2629

2630 RB: Is it more than one?

2631

2632 NP: Yeah, it's more than one.

2633

2634 RB: Do you think it's going to be more than two?

2635

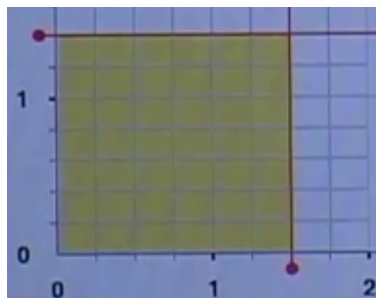
2636 NP: No.

2637

2638 RB: No? No. Okay, why don't you tile?

2639

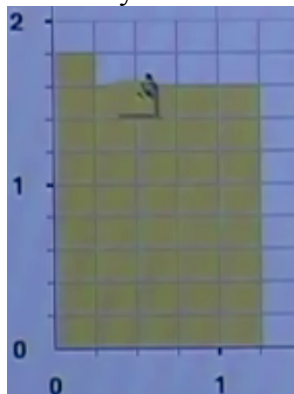
2640 NP: <hits tile button>



2641

2642

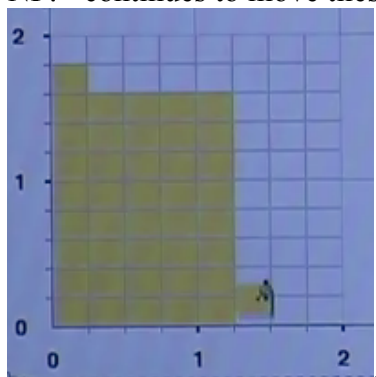
NP: Okay <starts moving a few tiles>. Maybe it is more than two.



2643

2644

NP: <continues to move tiles> Yeah, it's more than two.



2645

2646

RB: It is?

2647

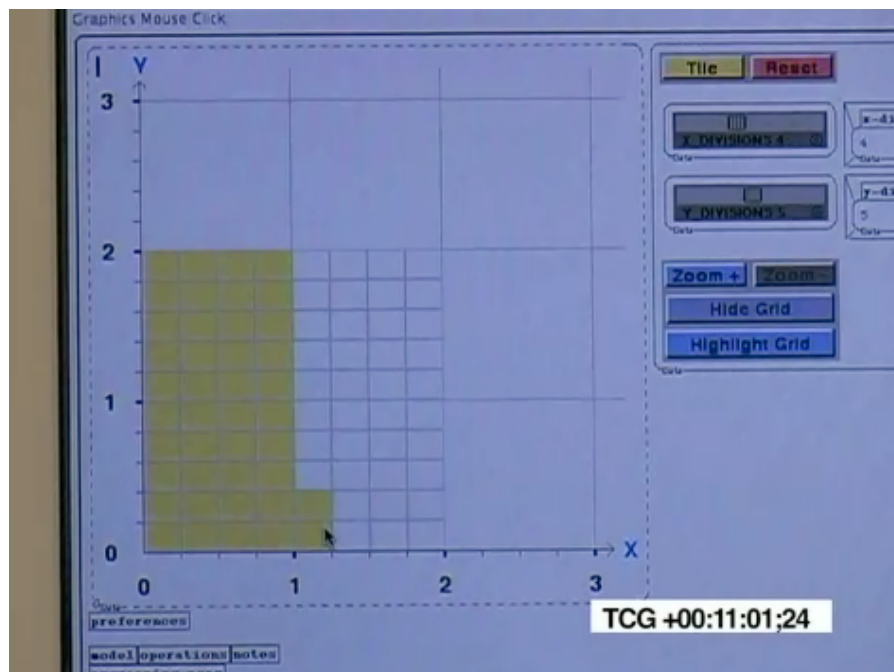
2648

NP: <finishes moving tiles> Yeah.

2649

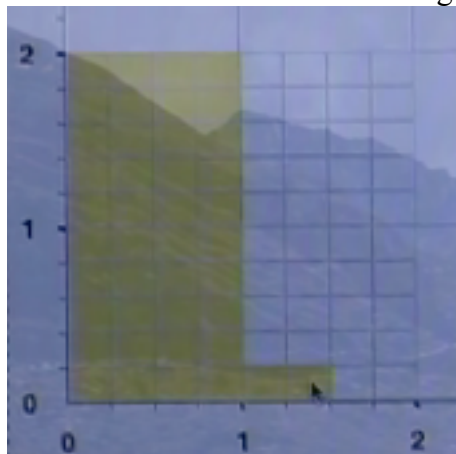
2650

RB: So how much is that?



2651  
2652

NP: So it's two and <moves a single tile>.



2653  
2654  
2655  
2656  
2657  
2658  
2659  
2660  
2661  
2662  
2663  
2664

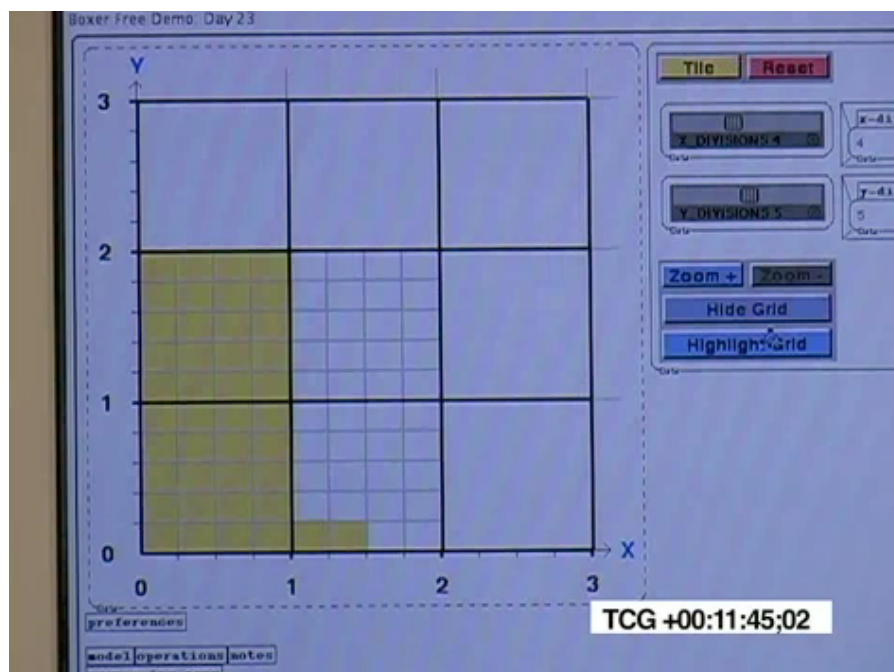
N: And <8 second pause> six fourths. Two and six fourths.

RB: Two and six fourths. Okay, how are you getting the two and six fourths

NP: Because you have two <uses cursor to point out the two tiled wholes>, right  
and then one, two, three, four, five, six < counts the six  $\frac{1}{4}$  line segments across the  
x-axis from first position of first tile to position of last tile>.

RB: So that's how much cheese you gave out?

NP: Two and six fourths <hits highlight grid button>.



2665

2666 RB: How much cheese did you give out?

2667

2668 NP: Oh, two and two fourths.

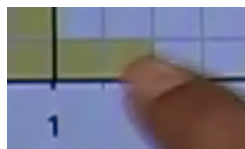
2669

2670 RB: Two fourths.

2671

2672 NP: Yeah.

2673

2674 RB: So these two, what are each of these pieces called, what's the  
2675 name of these pieces? Are they a fourth?

2676

2677 NP: Oh no they are not a fourth. They are...

2678

2679 [end of video]

2680

2681 [fieldnotes: via a series of guided questions by RB, NP arrives at the correct final  
2682 area output of  $2\frac{2}{20}$ ]

## Appendix H: Transcript of Oscar's Clinical Interview

1 Student: Oscar  
2 Day 2 of Clinical Interview  
3 Transcribed by: XXX  
4 Verifier: YYY  
5

6 RB: So today like I said, we are gonna be working mostly with the laptop. And, what we  
7 are gonna do is, we are gonna kinda continue with the cases; remember how we went  
8 through the cases? So I have...the sheet from last time of yours and what we are gonna  
9 do is, do the same stuff but using the laptop. So I have another record sheet that I'm  
10 gonna fill out, just so it's easier in terms of us not having to move the laptop around,  
11 I'll do the filling it out for you. I'll ask you what you want me to write and I'll write it  
12 out, okay? In terms of the output. Do you have any questions before we get started?  
13 Okay, great. This is done and... Okay, so quick recap we are gonna start with case  
14 eight.  
15

16 **Time: 01:09**  
17

18 RB: Um, I'm just gonna write what you have written for case eight, right? We did case  
19 eight with one rat. And we had two thirds of three fourths slices per rat and your output  
20 was one half slices, right? So I'm just gonna rewrite that in. Now, most of the cases we  
21 are gonna be doing today are gonna be related to case eight. So, notice how in the first  
22 sheet we had-how we were working with whole numbers, four and three, three four,  
23 three two, three one and then we moved to whole number and a fraction, three and a  
24 half, three and four fifths and then three and three halves. And then case eight we did  
25 something interesting, in that we had one rat but then we had two fractions is this  
26 column, right? So the kinds of problems we are gonna be doing, um, today are mostly  
27 these kinds of problems. We are gonna have one rat and two fractions in this column,  
28 and you are gonna use the laptop to answer these, um, these question of how much  
29 cheese you are using, okay?  
30

31 **Time: 02:31**  
32

33 RB: So, the first case, I'm gonna call it case 8A. And, it would be one half-or actually  
34 one rat, and then one half of one half slice per rat. So the scientist is telling you, you  
35 have one rat and the amount of cheese you have to give this rat is one half of half a  
36 slice. Um, so...why don't-we started this, right?  
37

38 OA: A little bit.  
39

40 RB: A little bit, okay. So, I want you to go ahead by showing me what the unit is. What  
41 is our unit in this case?  
42

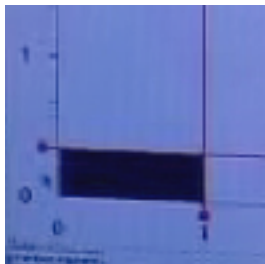
43 OA: Rats.  
44

45 RB: Um, unit, remember our cheeses.  
46



- 47 OA: Oh, cheeses.  
48  
49 RB: Yeah, so a single slice, two slices, what is our unit?  
50  
51 OA: Two slices?  
52  
53 RB: Our unit is two slices? We always treat two slices as our unit?  
54  
55 OA: Um...one slice.  
56  
57 RB: Okay, so why did you say two slices?  
58  
59 OA: Um... I don't know.  
60  
61 RB: So this for example-remember we were talking about the second meaning of unit, so  
62 there is the unit that refers to this word, right? What word-what the numbers are  
63 referring to-this word is rats, this is was slices per rat. And then we were talking about  
64 unit with respect to cheese, right <RB holds up a single paper cutout>? And how if  
65 your unit was bigger, if it was this size <RB holds up a larger paper cutout> it would  
66 change the problem, right? Taking a half of this <RB shows a large cutout> is different  
67 than taking a half of this <RB shows original paper cutout>. Or, if your unit was two  
68 slices <RB holds up two single paper cutouts>, a half of two slices would be one slice,  
69 right? And so, if we were talking about two slices, the scientist wouldn't understand if  
70 we said one half, right? Because he would be thinking a half of a single slice < RB  
71 shows original paper cutout>, right? So for the scientist, the unit is always a single  
72 slice, so when you write an output-when you write an output in your output, if you  
73 write three fourths, he is always gonna think three fourths of a single slice. He is never  
74 gonna think you mean three fourths of two slices, right? Okay, so, unit is a single slice.  
75 So I want you to show me that here <RB points to computer screen>.  
76  
77 OA: Um, that? <OA moves x-axis marker from zero to one>.  
78  
79 RB: Yep. So that right there is what a single slice of cheese looks like, right? So,  
80 equivalent to one of these <RB holds up a single paper cutout>. Okay. So, now you are  
81 being told to give the rat one half of one half, so how are you gonna do that?  
82  
83 OA: Um... <moves the x-division slider> missed up <return x-division slider to default  
84 and moves y-division slider to three>.  
85  
86 RB: So you set the y division at three. Why three?  
87  
88 OA: Because half has to be around there somewhere <points to where  $y=1/2$ > and half of  
89 that would be like right there somewhere <points to where  $y=1/4$ >.  
90  
91 RB: Wait, wait, say that again before you move that a half.  
92  
93 OA: Like half of it, if I made it two <OA moves the y-division slider to two>, it would  
94 be right there <points to where  $y=1/2$ >. If I made it three <OA moves the y-division

95 slider back to three>, it would be half of that, half of half. So to make it that much you  
 96 have to move it here <OA moves y-axis marker from one to one third>.



97

98 RB: To make half of half you put it there. Okay, so, this is half of half? Okay, so, well,

99 I'm still-so this is where you had it, right? <RB move the y-axis marker back to point

100 1>, and then you set this <points to the y-division slider> at three, how did you know to

101 set this at three?

102

103 OA: Um, cause I know that two would be right here <OA references imaginary

104 horizontal line at  $y=1/2$ >. So I made it into three, and it would give me that thing.

105

106 **Time: 07:00**

107

108 RB: I see, I see, so I'm gonna draw something really quick, and tell me what you think,

109 okay? So that, and you said that a half would be right there, right? Is that right <draw

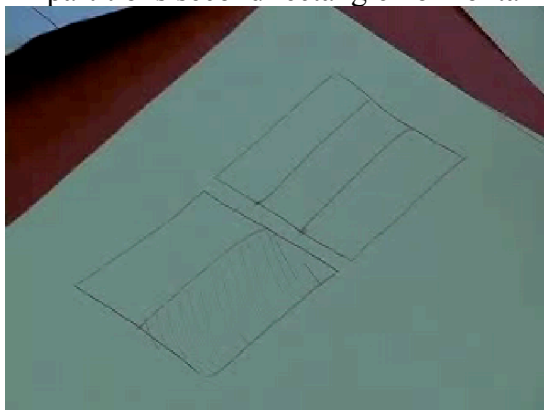
110 rectangle partitions horizontal into two halves>? Okay, so half <RB shades bottom half

111 of rectangle>. Now I'm gonna draw the same rectangle, the same size, like this, right

112 <draws another rectangle adjacent to the first>? So I'm gonna split into three equal

113 parts, that's what you did, right? Is that right? So like right there and right there

114 <partitions second rectangle horizontal into three equal parts>.



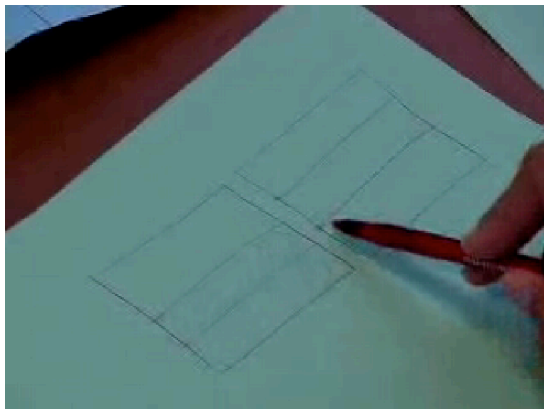
115

116 RB: Right? Okay, now, a half of this is about here <draws another horizontal line

117 partitioning shaded half into halves>? Does that match up exactly?

118

119 **Time: 07:53**



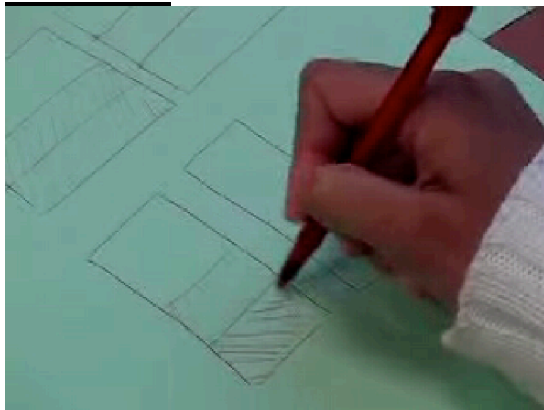
120  
121 OA: This one <OA points to rectangle on the right> is a little higher.

122  
123 RB: That one is a little higher. Do you think is just because I drew the pictures wrong or  
124 do you think that there's something wrong here?

125  
126 OA: Um...um... there is something wrong.

127  
128 RB: You think there is something wrong, okay. So, lets see. How about if we exaggerate  
129 the length for a second? Lets say our whole look like this. <RB draws two "tall"  
130 rectangles>. So half would be right here, right? <RB horizontally partitions the  
131 rectangle on the left in two equal parts>. And, so that's a half. <RB shades the bottom  
132 half part of the first rectangle>. And you said we are gonna take half of that, right. So  
133 that would be about there <RB horizontally partitions the bottom half into two equal  
134 parts>. So it would be this much that we are taking <RB shades 1/4 of the rectangle on  
135 the left>. Okay.

136 **Time: 09:06**



137  
138 RB: Now, now I'm gonna do this part <RB horizontally partitions the second rectangle  
139 in three and shades the bottom third>. A third.



140

141 OA: That one looks higher &lt;OA points to second rectangle&gt;.

142

143 RB: That one looks higher, yeah. Okay. So, lets go back here<referring to the AM-FM  
144 representation with unit whole shaded and y-division slider set at three>. So right now  
145 we have it at thirds, right? If we move it there, right? This is the middle? <RB moves y-  
146 axis marker from one to a half and hold it there>.

147

148 OA: Uhum.

149

150 RB: Um, then we want to take half of that, right? What would half of that be?

151

152 OA: Like right here <OA point to a point half way between zero and one half below the  
153  $1/3$  marker>. Below that.

154

155 RB: Below that line, right <referencing  $y=1/3$  marker>? So, splitting it into thirds is not  
156 quite giving us a half of a half. Is there something else we can split it into?

157

158 OA: Fourths.

159

160 RB: Fourths. Why fourths?

161

162 OA: Because then there would be more lines and this one <points to  $y=1/3$  marker>  
163 would be right there <points where  $y=1/4$  would be>.

164

165 RB: Okay, you wanna try it?

166

167 OA: &lt;OA moves the y-division slider to four&gt;

168

169 RB: Okay, so first show me a half.

170

171 OA: <OA moves y-axis marker from four fourths to two fourths>

172

173 RB: Okay, so, so far you've taken a half of your single slice, right? And-but you were  
174 told to take a half of a half, so show me half of a half.

175

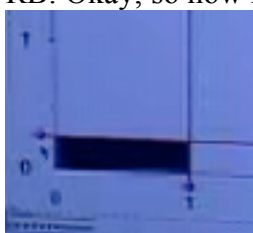
176 OA: <OA moves y-axis marker from one half to one fourth>.

177

178 **Time: 10:54**

179

180 RB: Okay, so how much cheese is that? How much cheese did you give out?



181

182 OA: One fourth.

183

184 RB: How do you know that?

185

186 OA: Because there were four slices in all and then-like you split them into four, and you  
187 only give them like one of those.

188

189 RB: Okay, do you wanna hit "Tile"?

190

191 OA: <OA hits "Tile">

192

193 RB: So that's one fourth, okay. Good, I'm gonna write that as my input, so um, my next  
194 question. Do these pieces, these smaller pieces <RB points to  $\frac{1}{4}$  tile pieces>, these four  
195 pieces, do they all have to be equal sized?

196

197 OA: Yeah.

198

199 RB: Why?

200

201 OA: Because when its bigger its gonna give a different fraction.

202

203 **Time: 11:58**

204

205 RB: It's gonna give a different fraction. Okay, so I'm gonna draw so you tell me what  
206 you think. If I gave you something like this, and I said that this is equal to one sixth.



207

208 RB: What would you say to that?

209

210 OA: That it's wrong.

211

212 RB: That it's wrong, so-could you find for me-what is it equal to, how much cheese did I  
213 give out in this case?

214

215 OA: Its equal to-to two eighths.

216

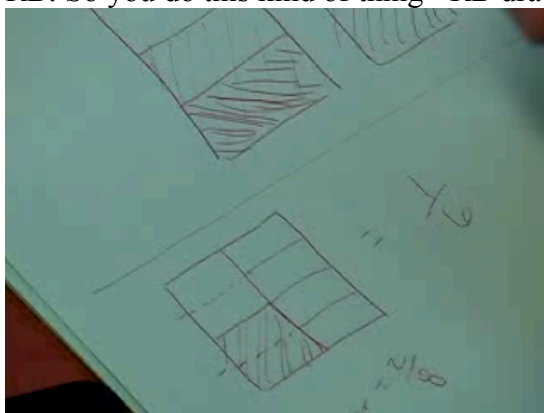
217 RB: Two eights. So OA says two eights, why two eights? <RB writes " $OA = 2/8$ ">

218

219 OA: Cause right- I just split it right there <OA points to half of the shaded part> and  
220 right there <OA points to half of top-left part> so that's four and four is eight <four  
221 parts in the left half plus four parts in the right half>.

222

223 RB: So you do this kind of thing &lt;RB draws in the partitions referenced by OA&gt;.



224

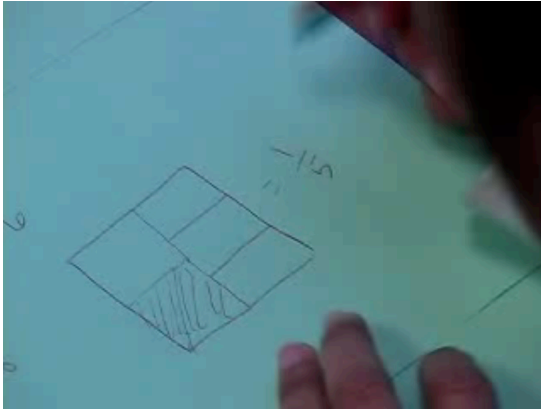
225 OA: Yeah.

226

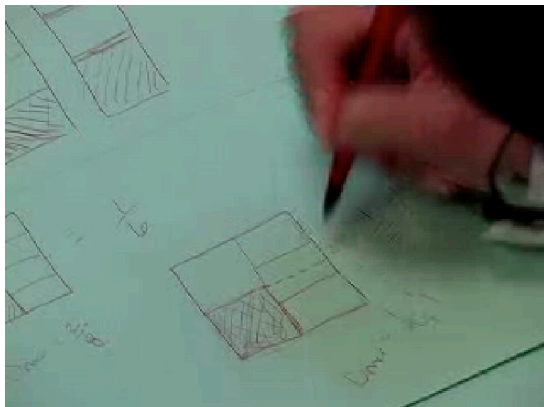
227 **Time: 13:02**

228

229 RB: Okay, alright, so what if I gave you this, and I said that this is equal to one fifth.



- 230  
 231 RB: What would you say to that?  
 232  
 233 OA: Um... <18 second pause>.  
 234  
 235 RB: Would you say yes or no? Yes it's one fifth or no?  
 236  
 237 OA: No.  
 238  
 239 RB: No, okay. What would you say it is?  
 240  
 241 OA: Um...You could say... I don't know.  
 242  
 243 RB: How much cheese did I give out <RB darkens the shaded area>?  
 244  
 245 OA: One...  
 246  
 247 RB: One... What is this piece called <RB points to shaded area>?  
 248  
 249 OA: One slice.  
 250  
 251 RB: One slice, I gave out the whole-this is my slice <RB points to the whole rectangle>.  
 252 Did I give out one slice?  
 253  
 254 OA: No...you gave one fourth.  
 255  
 256 RB: Gave one fourth, how did you figure that one out <RB writes "OA=1/4">?  
 257  
 258 OA: I split it right here <references imaginary line that would partition the left half of the  
 259 area model into half>.  
 260  
 261 RB: I see <RB draws in imaginary OA's imaginary line>



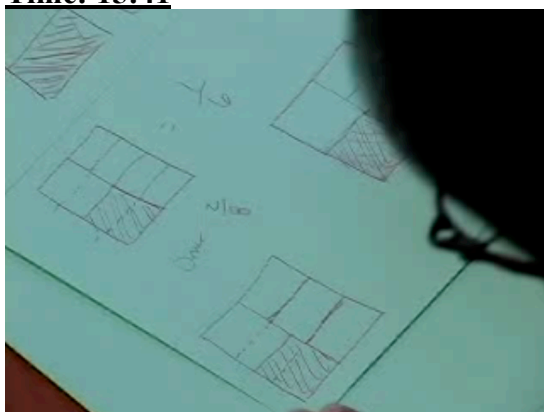
262

263 RB: Very cool.

264

265 OA: And these lines <points to both horizontal lines that partitions the right half of the  
266 rectangle into thirds> I <inaudible>

267

268 RB: This you just imagine they weren't there <RB points to both horizontal lines that  
269 partitions the right half of the rectangle into thirds>. And so you got one fourth. Very  
270 cool. Now what if I did this. And I added those lines in.271 **Time: 15:41**

272

273 RB: What would you say that is?

274

275 OA: Um...

276

277 RB: How much cheese did I give out?

278

279 OA: ...one fourth.

280

281 RB: Still one fourth &lt;RB writes "OA=1/4"&gt;, and how do you know that?

282

283 OA: Same thing, I just added a line there <OA uses the same approach as the previous  
284 example> and imagined that those weren't there.

285

286 RB: Okay, so you added this line again <RB draws in the imaginary line partitioning the  
287 left half of the rectangle into half>.

288

289 OA: Uhum.

290



291 RB: Now if I said I want the answer in terms of sixths. These are sixths, right? <RB  
 292 points to smaller parts>. One, two, <RB counts the total number of  $\frac{1}{6}$  parts that  
 293 constitute the whole>, three, four, five, six, right? And they are all equal, so I want the  
 294 answer in terms of a sixth. What would you say my answer is?

295

296 **Time: 16:39**



297

298 RB: How many sixths do I have? Is there a way to do that, or is that impossible?

299

300 OA: A half of three sixth?...two sixth.

301

302 RB: Half of three sixths.

303

304 OA: No. Yeah, three sixths.

305

306 RB: So what are the three sixths you are counting?

307

308 OA: These three <points to the three parts that make up the left half of the rectangle>.

309

310 RB: Okay. Half of three sixth <RB writes "half of 3-sixths"> Okay, but I wanted as one  
 311 fraction, you know how this is written as one over four, how would I write that as a  
 312 fraction?

313

314 OA: Um... I don't know.

315

316 RB: So, you said half of three sixths and then you said and then you said two sixths, how  
 317 much would two sixth be?

318

319 OA: These two <points to the shaded  $\frac{1}{6}$  and the  $\frac{1}{6}$  part above which is only half  
 320 shaded>.

321

322 RB: These two, but we don't have two, right? We have less than two. Do we have one?

323

324 OA: Yeah.

325

326 RB: Do we have exactly one?

327

328 OA: Yeah-oh no, more than one.

329

330 RB: We have more than one, right? So, how many sixths do we have?

331

332 OA: One and a half.

333

334 RB: One and a half? Okay, so how can I write a fraction?

335

336 OA: <8 second pause>.

337

338 RB: Any ideas in terms of sixths, not sure? Okay, okay. Good job. Okay, so the next  
339 case, and now, again, I'm gonna start asking you for predictions kinda the way I was  
340 doing with the other cases.

341

342 **Time: 18:47**

343

344 RB: Um, so we are gonna do 8B. And is one third, one rat again, one third of one half  
345 slice per rat. <RB writes "1 rat" and "1/3 of 1/2 slice/rat"> Okay? So before you do  
346 anything, I'm gonna ask you for a prediction. Are we gonna end up using more or less  
347 cheese in B than we used in A. So in A we used-we had one rat and we were asked to  
348 give one half of one half slice per rat, and the output was one fourth slice, right? So in  
349 B we have one rat, but we are giving one third of one half a slice per rat, so are we  
350 gonna use more or less cheese here? <RB points to case 8B>

351

352 OA: Less.

353

354 RB: Less, why do say less?

355

356 OA: Because I pictured in my mind like the square, and, um, I split it into half and then I  
357 got the half of that-

358

359 RB: -Uhum-

360

361 OA: -And then another one, split into three and then a half of that.

362

363 RB: Wait-you split the original-the square-

364

365 OA: -The one third and then half-and then one half of that.

366

367 RB: Okay, so the second, you had a square and then you imagine splitting it up into  
368 three, the whole square?

369

370 OA: Uhum.

371

372 RB: And then taking one of those.

373

374 OA: And three into half.

375

376 R: And splitting that one into half? Okay, or splitting all three into half?

377

378 OA: No, no. I have the half and split that half into three.

379

380 RB: Oh, okay. You took a square, you take half of it and you split that half into three.  
381 And then what do you do?

382

383 OA: Then I get the answer.

384

385 RB: Well, how many of those-so you have a half that is split into three. How many of  
386 those do you take?

387

388 OA: One

389

390 RB: Just one of them, okay. So then this one <RB points to case 8B> is gonna be less  
391 than this one <RB points to case 8A>. Okay, okay, so lets have you go ahead and do it.

392

393 OA: So reset it?

394

395 RB: Yeah, reset it. And usually when you reset it doesn't it doesn't-so you are gonna  
396 have to move the slider back.

397

398 OA: So one. <OA moves x-axis marker from zero to one>

399

400 RB: One unit, okay. Good.

401

402 OA: <OA mumbles something and then moves y-division slider from three to six>

403

404 RB: Oh, why did you set it at six?

405

406 OA: Because-I thought of that because it was one half and then times-it was the answer  
407 times two, like that times two is four <referencing case 8A> so I did three times two.

408

409 RB: Uhum

410

411 OA: That six so I'll try it maybe it is six.

412

413 RB: Okay.

414

415 OA: So I'll try it right now. So half of that is right here <OA moves the y-axis marker  
416 from 6/6 to 3/6>

417

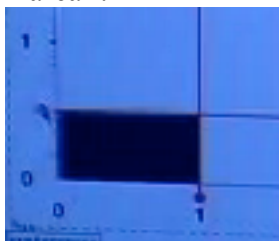
418 RB: What's that-sorry, go ahead.

419

420 OA: <inaudible>

421

422 RB: Okay. So that's half so far, so much cheese is this right here? <RB points to shaded  
423 area>.



424

425 OA: That's half.

426

427 RB: And how much were you asked to give?

428

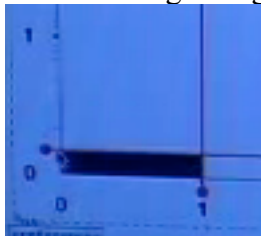
429 OA: One third of half.

430

431 RB: Uhum

432

433 OA: So this goes right there <OA moves y-axis marker from  $\frac{3}{6}$  to  $\frac{1}{6}$ >.



434

435 RB: Its right there, okay. Okay, I see.

436

437 **Time: 22:30**

438

439 RB: So I have a couple of questions. So first of all, so this x-axis, so they are these  
440 numbers , so zero, one, two, and three <RB points to the numbers on the x-axis>. So  
441 what do these numbers refer to?

442

443 OA: To rats.

444

445 RB: To rats, okay, so you have rats here <RB points to x-axis>, so you have one rat  
446 <points to location of x-axis marker line at  $x=1$ >. So these numbers here, zero, and the  
447 fractions in between and one and the fraction in between two and the fraction in  
448 between <RB points to the partitions along the y-axis> what do those things-those  
449 numbers refer to?

450

451 OA: Slices.

452

453 RB: Slices?

454

455 OA: Yeah.

456

457 RB: Slices, okay. Just slices, or slices per rat?

458

459 OA: Slices...per-no...like if I did one <OA points to number one on x-axis> it would give  
460 me this <OA points to the  $1 \times 1$  unit whole>. But if it's two <points to the number two  
461 on the x-axis> it'll give me these two <points to two of the  $1 \times 1$  unit wholes>. This is  
462 for the one <points to one of the  $1 \times 1$  unit whole> and this is for the other <points to the  
463 other  $1 \times 1$  unit whole>.

464

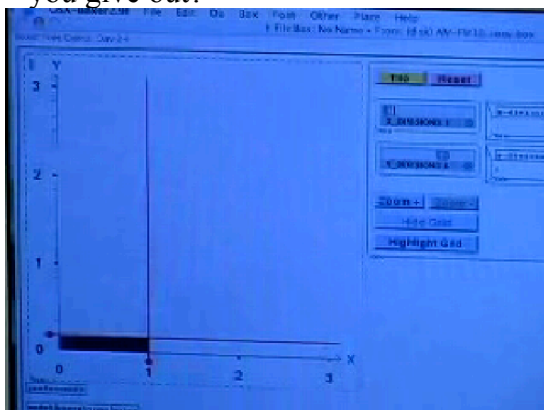
465 RB: I see. Okay, so it is just slices, this one here <RB points to y-axis> and this one is  
466 rats <RB points to x-axis> Okay, good, next question.

467

468 **Time: 23:37**

469

470 RB: When you, okay I'm gonna go back here for a second-oh, lets have you tile first, so-  
 471 oh before you tile, how much is that? <RB points to grey area> How much cheese did  
 472 you give out?



473  
 474 OA: One sixth.

475  
 476 RB: How do you know that?

477  
 478 OA: Cause there is six of them, six like little boxes right here <OA points to each sixth  
 479 on the y-axis>.

480  
 481 RB: Uhum.

482  
 483 OA: And there is one of these <points to shaded 1/6 piece>, so its one sixth.

484  
 485 RB: Okay now hit "tile," Lets see. <OA hits "tile">. Okay, one sixth, so that's one sixth.  
 486 Okay, so do you want me to write one sixth for the output? Okay, so was that more or  
 487 less what we got in case A?

488  
 489 OA: Umm, less.

490  
 491 RB: Less, so you were right about your prediction. Good job. So next question.

492  
 493 **Time: 24:30**

494  
 495 RB: I'm gonna reset <hits "Reset">, so you did this right? So the way you started this is  
 496 you moved-you showed me your unit, a single slice <RB moves x-axis marker to one>,  
 497 then you said, I'm gonna take half of that, so you moved this here <RB moves y-axis  
 498 marker from 6/6 to 3/6>.

499  
 500 OA: Yeah.

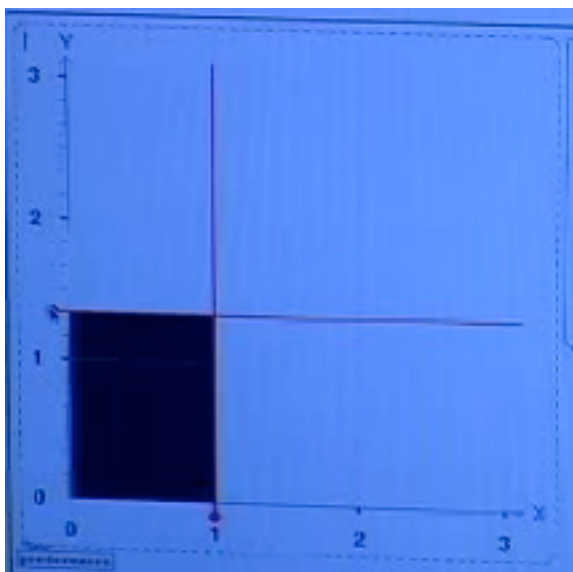
501  
 502 RB: Is that right?

503  
 504 OA: Yeah.

505  
 506 RB: What is this point called right here? <RB points to 3/6 > What is the fraction name  
 507 for this point?

508

- 509 OA: One half.  
510  
511 RB: One half. Okay, what is the fraction name for this point <RB moves y-axis marker  
512 from  $\frac{3}{6}$  to  $\frac{4}{6}$ >?  
513  
514 OA: Four sixth.  
515  
516 RB: Four sixth. What is the fraction name for that point <RB moves y-axis marker from  
517  $\frac{4}{6}$  to  $\frac{2}{6}$ >?  
518  
519 OA: Two sixth.  
520  
521 RB: Two sixth. And if I want the fraction name for this point in terms of sixths <RB  
522 moves y-axis marker from  $\frac{2}{6}$  to  $\frac{3}{6}$ >.  
523  
524 OA: Three sixth.  
525  
526 RB: Three sixth, which is equal to one half?  
527  
528 OA: Yeah.  
529  
530 RB: One half. Okay. What is the fraction name for this point in terms of sixths <RB  
531 moves y-axis marker from  $\frac{3}{6}$  to  $\frac{0}{6}$ >?  
532  
533 OA: Zero.  
534  
535 RB: Zero sixth?  
536  
537 OA: Yeah.  
538  
539 RB: And this point <RB moves y-axis marker from  $\frac{0}{6}$  to  $\frac{6}{6}$ >?  
540  
541 OA: Um, six six.  
542  
543 RB: Six sixth. Okay. And this point <RB moves y-axis marker from  $\frac{6}{6}$  to  $\frac{8}{6}$ >?



544

OA: Um, one and two sixth.

546

RB: One and two sixth. Now, that's called a mixed number, right? One and two sixths, cause you have the one which is whole number and you have a fraction with it. If I wanted this just as a regular fraction, no mixed number, what would I call this?

550

OA: Um...eight twelfths.

552

RB: Eight twelfths? Why eight twelfths?

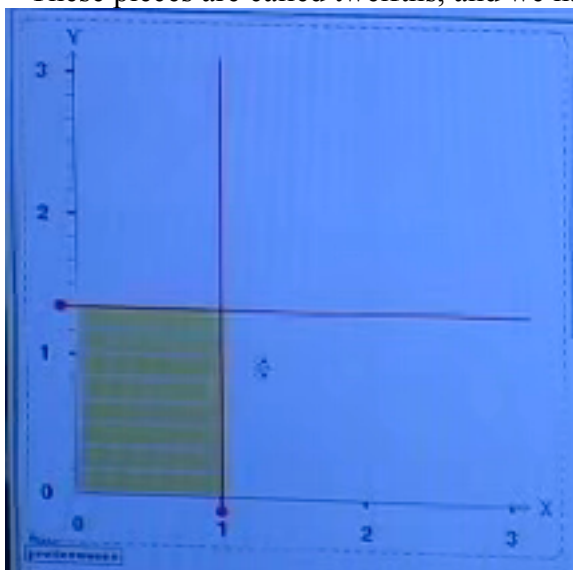
554

OA: Cause I counted this six <points to the six  $1/6$  pieces that make up the shaded  $1 \times 1$  unit whole> and then six <points to the six  $1/6$  pieces that make the second  $1 \times 1$  unit whole which has  $2/6$  shaded> so that's twelve and then six <the shaded  $1/6$  pieces from the first unit whole> plus two <the shaded  $1/6$  pieces from the second unit whole> that's eight. So that's eight twelfths.

560

RB: Eight twelfths, so then these little pieces, like when I hit tile <RB hits "tile">, right? These pieces are called twelfths, and we have eight of them?

562



563

564 OA: Uhum.

565

566 RB: They are twelfths of-what's our unit? Are they twelfths of one slice or twelfths of  
567 two slices <RB lifts up a single  $1/6$  tile piece>?



568

569 OA: Twelfths of two slices.

570

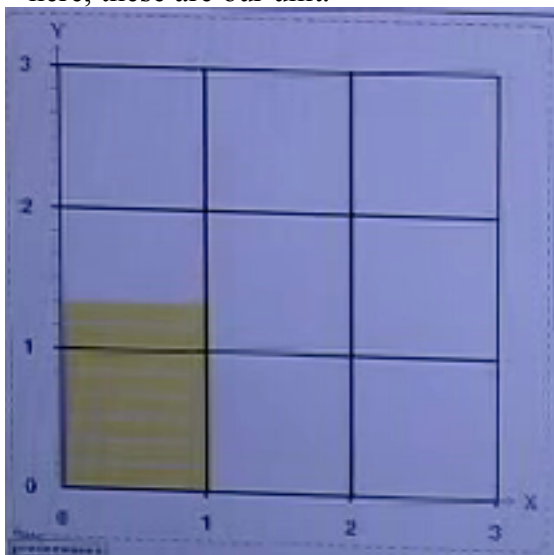
571 RB: So our unit is two when you call them twelfths. Now, if you told the scientist that  
572 your giving out twelfths, he is not gonna think of twelfths for two units, he is gonna  
573 think of twelfths for one unit.

574

575 OA: Yeah.

576

577 RB: So our unit is always one, right? Remember I said that? So let me highlight the units  
578 here, these are our unit.



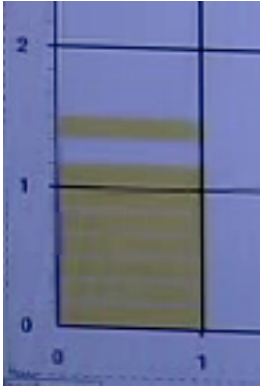
579

580 RB: So what should this <RB lifts up a single  $1/6$  tile piece> be called if our unit is one?

581 So if our unit is two it should be called twelfths, right? But if our unit is one, what

582 should this piece be called?





583

584 OA: Umm.

585

586 RB: What's the name of that piece?

587

588 OA: This one &lt;points in the middle of the tiled 1x1 unit whole&gt;?

589

590 RB: Uhum.

591

592 OA: One-one.

593

594 RB: The whole piece-what's one?

595

596 OA: This whole box &lt;points to 1x1 unit whole&gt;.

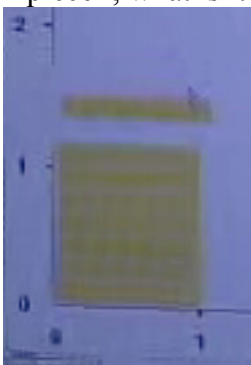
597

598 RB: That whole box is one.

599

600 OA: Yeah.

601

602 RB: What if we pick one single piece like this right here <RB lifts a single 1/6 tile  
603 piece>, what is it called?

604

605 OA: Um...

606

607 RB: Is it a fourth? Is it a twelfth?

608

609 OA: Yeah.

610

611 RB: It's a twelfth of two slices.

612

613 OA: Yeah.

- 614  
615 RB: I wanna know what it is of one slice.  
616  
617 OA: One sixth.  
618  
619 RB: One sixth. Okay, one sixth. How many sixths do we have here?  
620  
621 OA: How many sixths? Two.  
622  
623 RB: Total.  
624  
625 OA: Oh total, twelve.  
626  
627 RB: The colored, these <points to tiled  $1/6$  piece>.  
628  
629 OA: Oh, eight.  
630  
631 RB: Eight. Okay. Um, so now if we have eight sixths here <pointed to total tiled area>,  
632 what is this point <points to  $y=8/6$ > gonna be called?  
633  
634 OA: Um... eight twelfths.  
635  
636 **Time: 28:40**  
637  
638 RB: Okay. I'm gonna go to that piece of paper again, and I'm gonna show you  
639 something. So we have a number line, right? That's what these are <points to the x-axis  
640 and y-axis of the AM-FM representation> number lines. <RB draws a number line  
641 from zero to three> And it's going from zero to three, one, two, three. Good so far?  
642 And we have it split into sixths, right? You set the divisions at six. So I'm gonna go  
643 and do that. <RB partitions the number line between zero and one into sixths>. Did I do  
644 that right?  
645  
646 OA: Uhum  
647  
648 RB: Okay its not even, but <RB partitions the number line between one and two into  
649 sixths and two and three into sixths>. Okay. So I'm gonna ask you to tell me what the  
650 names of these points are, okay?



651

652 OA: Uhum

653

654 RB: So we have this number line. What's this point right here called <points to  $4/6$ >?

655

656 OA: Four sixths.

657

658 RB: Four sixths. And how do you know that?

659

660 OA: Cause the fourth line right there.

661

662 RB: So you are counting the line?

663

664 OA: Yeah

665

666 RB: So this is the first line, this is the second, now why don't you count this line &lt;RB points to zero marker&gt;.

667

668 OA: Cause it would be zero sixth.

669

670 RB: Okay so this is zero sixth <RB labels  $0/6$ >. What is this point here <points to  $3/6$ >?

671

672 OA: A half-one half.

673

674 RB: One half, is there another name for one half <RB labels  $1/2$ >?

675

676 OA: Three sixths.

677

678 RB: Three sixths, okay <RB labels  $3/6$ >. How do you go from three sixths to one half?

679

680 OA: If you reduce it.

681

682 RB: If you reduce it, and how do you reduced it?

683

684 OA: You do three divided by three is one and six divided by three is two.

685

686 RB: Okay, good, what is this point right here &lt;points to 1&gt;? In terms of sixths?

687

688

689 OA: Six sixths.

690

691 RB: Okay, six sixth, good. Now, what is this point right here <points to  $8/6$ >?

692

693 OA: Eight sixths.

694

695 RB: Eight sixths. Good, and what if I wanted as a mixed number?

696

697 OA: It would be one and two-one and two sixth.

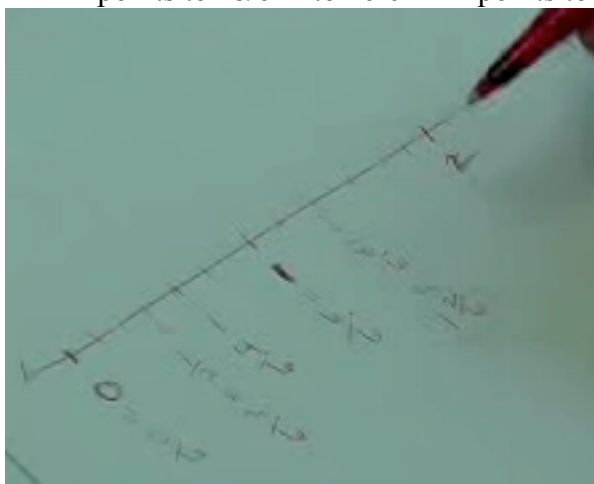
698

699 RB: <RB labels  $1\frac{2}{6}$ >. How did you get that?

700

701 OA: Cause its one <point to marker labeled 1> and then two <points to the 2 line  
702 segments following the point labeled 1> sixths.

703

704 RB: Two sixth, now if I was just working with the numbers, how could I go from here  
705 <RB points to " $8/6$ "> to here <RB points to " $1\frac{2}{6}$ ">?

706

707 OA: Um...

708

709 RB: How would I go from here <RB points to " $8/6$ "> to here <RB points to " $1\frac{2}{6}$ "> if I  
710 was just working with the numbers?

711

712 OA: This bottom number has to be the same-

713

714 RB: -The same, uhum.

715

716 OA: ...then eight divided by two.

717

718 RB: Eight divided by two?

719

720 OA: No, four.

721

722 RB: Eight divided by four is what?

723

724 OA: Two.

- 725  
726 RB: And that's how you get that <the "2" in  $1\frac{2}{6}$  from  $\frac{8}{6}$ >? And how do you get this  
727 one? You don't know, and why do you divide eight by four?  
728  
729 OA: It's half of eight.  
730  
731 RB: It's half of eight, and why did you choose half of eight?  
732  
733 OA: Um... I don't know.  
734  
735 RB: Okay, okay. Um, and what about from here <RB points to " $1\frac{2}{6}$ "> to here <RB  
736 points to " $\frac{8}{6}$ ">, is there a way you can work with the numbers? If you are given this  
737 and you wanted to find this kind of fraction, is there something you can do with these  
738 numbers-  
739  
740 OA: -Six plus two.  
741  
742 RB: Six plus two is what?  
743  
744 OA: Eight.  
745  
746 RB: Eight, and why do you do six plus two?  
747  
748 OA: Um...  
749  
750 RB: You're not sure?  
751  
752 OA: No.  
753  
754 RB: Okay, okay, good, so what is-so this is six sixths, this is eight sixths, what would  
755 this point be <points to  $\frac{10}{6}$ >?  
756  
757 OA: Ten sixth.  
758  
759 RB: Ten six <labels  $\frac{10}{6}$ >, or?  
760  
761 OA: Um... one, one and...four sixth.  
762  
763 RB: One and four sixth <labels  $1\frac{4}{6}$ >, how did you get that?  
764  
765 OA: Well, same. It's one right here <points to marker labeled 1>, and then four sixths  
766 <points to remaining line segment from 1 to  $1\frac{4}{6}$ >.  
767  
768 RB: Okay, so one and four sixths, and this point here would be <points to  $\frac{11}{6}$ >?  
769  
770 OA: One and fifth sixths.  
771  
772 RB: Five sixths, or?  
773

- 774 OA: One and five sixth-or eleven sixth.  
775  
776 RB: Eleven sixths, and this one <points to 2>?  
777  
778 OA: Two.  
779  
780 RB: Two, what's the other name for two?  
781  
782 OA: Six sixths  
783  
784 RB: I though you said six sixths-  
785  
786 OA: -that's one over six sixths-no, one and six sixths.  
787  
788 RB: One and six sixths, okay, another name?  
789  
790 OA: Umm....  
791  
792 RB: You got two  
793  
794 OA: Six sixths.  
795  
796 RB: Six sixths is one <RB points to 6/6 label>. Seven sixths, eight sixths, nine, ten,  
797 eleven <RB points out each marker following 6/6 up to 11/6>-  
798  
799 OA: -twelve sixths.  
800  
801 RB: Twelve sixths, okay. How do you go from twelve sixths to two?  
802  
803 **Time: 34:15**  
804  
805 OA: Um... what the mixed number, it's six sixths which equals one, so the one and the  
806 one is two.  
807  
808 RB: Is two.  
809  
810 OA: Yeah.  
811  
812 RB: And what about from twelve sixths to two? Why is twelve sixths the same as two?  
813  
814 OA: <9 second pause> Twelve divided by six is two.  
815  
816 RB: Twelve divided by six is two. So what is eight divided by six <points to 8/6 on the  
817 number line>?  
818  
819 OA: Um...um... I don't know.  
820  
821 RB: Is it less than one? More than one?  
822

- 823 OA: Less than one.  
824  
825 RB: So eight divided by six <RB writes 8 divided by 6 using long division notation>.  
826 How many times will six go into eight?  
827  
828 OA: Zero.  
829  
830 RB: Six doesn't go into eight? Why not?  
831  
832 OA: No one-once.  
833  
834 RB: Once. Okay once and then I get six, what do I get left over?  
835  
836 OA: Two.  
837  
838 RB: Okay. <RB does the long division to arrive at "1 2/6">.  
839  
840 OA: Oh, it's two sixths.  
841  
842 RB: Uhum. So you get one-  
843  
844 OA: -one and two sixths.  
845  
846 RB: Do you see that?  
847  
848 OA: Yeah.  
849  
850 RB: Okay. So what's ten divided by six <point to 10/6 on the number line>?  
851  
852 OA: One and <RB writes out the long division for 10 divided by 6>- one and-six-it  
853 equals four <OA subtracts 6 from 10 as RB is writing out the long division>.  
854  
855 RB: Is that what we got? Okay. So that's how you go from improper-do you know that  
856 these are called improper fractions <points to improper fractions on the number line>?  
857 Okay, so these fractions <points to the proper fraction>, when the top number is what?  
858 Less than the bottom number?  
859  
860 OA: Yeah.  
861  
862 RB: They are called proper fractions.  
863  
864 OA: Yeah.  
865  
866 RB: Uhum, and then when you start getting these type of fractions, seven sixths, eight  
867 sixths, nine sixths, ten sixths, eleven sixth, twelve sixths, thirteen sixths, these are all  
868 improper fractions. And with improper fraction, you can convert them to, what? What  
869 are these called <points to a mixed number on the number line>?  
870  
871 OA: Mixed numbers.

872

873 RB: Mixed numbers, and mixed numbers you can convert to improper fractions, right?

874 The way you go about converting them is you, actually you did it, you- when you are

875 given an improper fraction like this, it's the top number divided by the bottom number,

876 right? So ten divided by six gives you one and four sixths. Eight divided by six gives

877 you one and two sixths. Now, to go backwards, there is a formula for this and your

878 teacher should've-you probably learned this like, I don't know when you probably

879 learned this.

880

881 OA: I learned that sometime but I don't remember when.

882

883 RB: Do you remember the formula-how do go-

884

885 OA: -&lt;inaudible&gt;

886

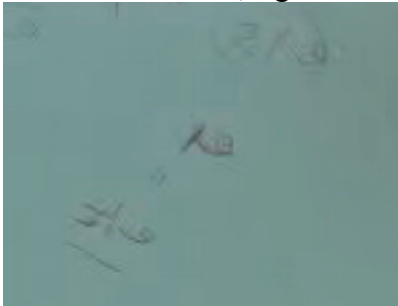
887 **Time: 37:33**

888

889 RB: So let's say you have one and four sixths, right? And we know the answer should be

890 ten six, right? And this is where we wanna get. And you said-you did say we keep the

891 bottom the same, right?



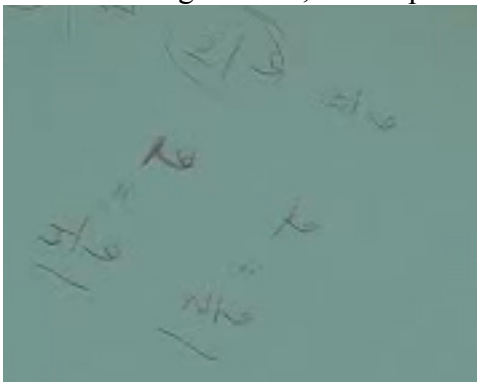
892

893 OA: Uhum.

894

895 RB: Okay, let's look at some more, so we have one and two sixths, and we know the

896 answer is eight sixths, we keep the bottom the same.



897

898 RB: What would this point be called right here <points to  $2\frac{3}{6}$ >?

899

900 OA: What point?

901

902 RB: This one <points again to  $2\frac{3}{6}$ >.

903



904 OA: Three six-no, two and three sixths.

905

906 RB: Two and three sixths, or?

907

908 OA: Or five-sixths.

909

910 RB: Okay so this is <points to  $12/6$ >-

911

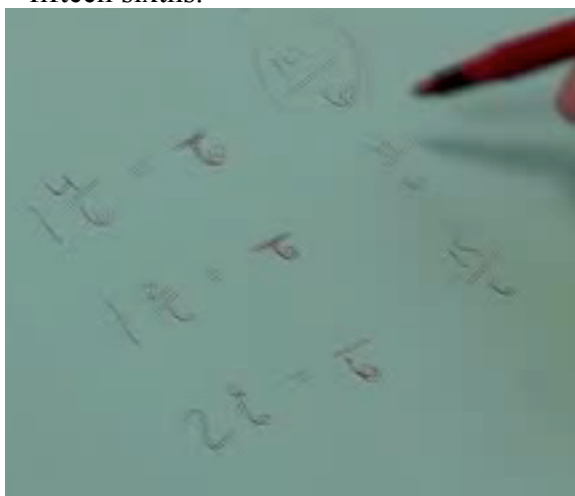
912 OA: -Twelve-

913

914 RB: -Thirteen, fourteen, fifteen-five sixths, okay. Good. So we have two and three

915 sixths or five sixths, right? So we know what we should get here, we should get

916 five sixths.



917

918 RB: Let's see if we can figure out how to do this. So we know that the six has to stay in  
919 the bottom, right? So look, it checks out so far. Here we have to get ten, here we have  
920 to get eight, here we have to get fifteen <points to the numerators of the improper  
921 fraction>. What can-what are we doing here? So, with these numbers <points to  $1_4/6$ >  
922 to get ten <the numerator in  $10/6$ >? Or, what are we doing with these numbers <points  
923 to  $1_2/6$ > to get eight <the numerator in  $8/6$ >? Or, what are we doing with these  
924 numbers <points to  $2_3/6$ > to get fifteen <the numerator in  $15/6$ >? Help me find  
925 patterns here.

926

927 OA: You do six plus four equals ten <points to the 4 and 6 in  $1_4/6$ >.

928

929 RB: Okay. Does that work here <points to  $1_2/6$ >?

930

931 OA: Yeah.

932

933 RB: Does that work here <points to  $2_3/6$ >?

934

935 OA: No.

936

937 RB: No, so that's not right. Well, let's work with this one then <points to  $2_3/6$ >, maybe  
938 there's something-what can we do with these numbers? You wanna get fifteen.

939

940 OA: Two times six is twelve plus three.

- 941  
942 RB: Okay, that works. Does that work here <points to  $1\frac{2}{6}$ >?  
943  
944 OA: Um..  
945  
946 RB: One times six is...  
947  
948 OA: Six.  
949  
950 RB: Plus two is...  
951  
952 OA: Yeah.  
953  
954 RB: That works. One times six is <points to  $1\frac{4}{6}$ >...  
955  
956 OA: Six.  
957  
958 RB: Plus four...  
959  
960 OA: Yeah. It does work.  
961  
962 RB: That works, right? So you figure it out the rule all by yourself Omar, okay, so what  
963 was it again <pointing to  $2\frac{3}{6}$ >?  
964  
965 OA: Two times six.  
966  
967 RB: So we are timesing these two <writes a multiplication symbol between the 2 and the  
968 6 in  $2\frac{3}{6}$ >.  
969  
970 OA: And then plus three.  
971  
972 RB: And then adding this one <writes an addition symbol between 2 and 3 in  $2\frac{3}{6}$ >.  
973  
974 OA: Yeah.  
975  
976 RB: So that's twelve plus three is fifteen <writes  $2\frac{3}{6} = "15"/6$ >. Good, so now you  
977 know how to go back and fourth between these two, right? Good job. Okay, let's go  
978 back to our cases.  
979  
980 **Time: 40:22**  
981  
982 RB: 8B\*. So here we have one rat. And here we are gonna go one half of one third slice  
983 per rat. So now...I want you to-let's reset this <hits "Resit">. Okay, so now, prediction.  
984 Here, are we gonna end up using more or less cheese than in 8B?  
985  
986 OA: The same.  
987  
988 RB: Why the same?  
989

- 990 OA: Because this one <OA points to case 8B> right here and this <points to 8B\*>-it's  
991 just switched, the order is just switched.  
992
- 993 RB: And why doesn't order matter?  
994
- 995 OA: Um...  
996
- 997 RB: What are we doing with these two fractions? Are we adding them, subtracting them,  
998 multiplying them, are we dividing them, what are we doing?  
999
- 1000 OA: We are dividing them.  
1001
- 1002 RB: Dividing them, so we are dividing <points to case 8B:  $1/3$  of  $1/2$ >-  
1003
- 1004 OA: -One third-it's one half divided by-no one third divided by one half.  
1005
- 1006 RB: This one here <case 8B:  $1/3$  of  $1/2$ > is one third divided by one half?  
1007
- 1008 OA: Yeah.  
1009
- 1010 RB: 8B?  
1011
- 1012 OA: Uhum.  
1013
- 1014 RB: So this is one third divided by one half? And this one is <RB points to case 8B\*:  $1/2$   
1015 of  $1/3$ >  
1016
- 1017 OA: One half divided by one third.  
1018
- 1019 RB: So 8B\* is one half divided by one third?  
1020
- 1021 OA: Yeah.  
1022
- 1023 RB: Okay, so when we divide one third by one half, we get one sixth? Okay, so last time  
1024 you were saying you were multiplying. You said you did, one times one is one and  
1025 three times two is six <points to case 8B:  $1/3$  of  $1/2$ >. One times one is one, two times  
1026 two is four <points to case 8A:  $1/2$  of  $1/2$ >. But now we are dividing <points to case  
1027 8B\*:  $1/2$  of  $1/3$ >?  
1028
- 1029 OA: No wait, we are not gonna get the same-well, two times three is six <points to  
1030 denominators in case 8B\*:  $1/2$  of  $1/3$ >.  
1031
- 1032 RB: Uhum.  
1033
- 1034 OA: So yeah, it's the same.  
1035
- 1036 RB: So we are getting the same, but now we are multiplying?  
1037
- 1038 OA: Yeah.

1039  
 1040 RB: So are we multiplying or dividing?  
 1041  
 1042 OA: Multiplying.  
 1043  
 1044 RB: Okay, so now you think we were multiplying. Why did you think we were dividing  
 1045 before?  
 1046  
 1047 OA: Because they were switched so I thought that-that-that-if you divide that it's gonna  
 1048 be the same as if you divide that.

1049  
 1050 RB: Um, okay. So let's have you do this one.

1051

1052 **Time: 42:49**

1053

1054 OA: So one rat <OA moves the x-axis marker to one, the y-division slider is set at 6 from  
 1055 previous case>.

1056

1057 RB: Okay, so how much cheese is that so far?

1058

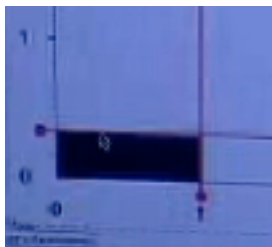
1059 OA: One whole thing.

1060

1061 RB: One slice? One slice. Okay, now tell me what you are doing.

1062

1063 OA: It's on half of one third, so one third, wait, split it up in two, so one third would  
 1064 be...Right <Moves y-axis marker from  $6/6$  to  $5/6$ > there <moves y-axis marker from  
 1065  $5/6$  to  $2/6$ >.



1066

1067 RB: Right there, how do you know that's one third?

1068

1069 OA: Cause what I did is, I did two of these <top  $1/6$  marks on the y-axis> equals one and  
 1070 two of these <middle  $1/6$  marks on the y-axis> is another one and two of these <bottom  
 1071  $1/6$  marks on the y-axis> is another one. So, that's three and that's one <points to  
 1072 bottom  $1/6$  marks that make up the shaded area> third.

1073

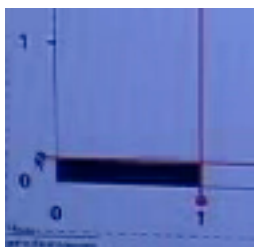
1074 RB: That's one third, okay.

1075

1076 OA: And, then half of that is right here <moves y-axis marker from  $2/6$  to  $1/6$ >..

1077

1078 RB: Uhum. So how much cheese did you give out?



- 1079  
 1080 OA: One sixth.  
 1081  
 1082 RB: One sixth a slice, okay. Do you wanna tile <OA hits "Tile">? Okay, so what's the  
 1083 name of this piece right here <RB lifts up the 1/6 tile piece>?  
 1084  
 1085 OA: Um.  
 1086  
 1087 RB: What do we call this?  
 1088  
 1089 OA: One six.  
 1090  
 1091 RB: It's a sixth?  
 1092  
 1093 OA: Sixth. Yeah.  
 1094  
 1095 RB: And we have one of them so it's one sixth, it's not a twelfth?  
 1096  
 1097 OA: No.  
 1098  
 1099 RB: No, not a twelfth. One sixth. Okay, good job.  
 1100  
 1101 **Time: 44:30**  
 1102  
 1103 RB: Let's do 8C. Okay, we have again one rat, and this time you are gonna give the rat  
 1104 two thirds of one third slice per rat. Okay, so prediction. Two thirds of one third. Are  
 1105 we gonna end up using more or less cheese than in here, in 8B\* <1/2 of 1/3>?  
 1106  
 1107 OA: Um...um...less  
 1108  
 1109 RB: Why less?  
 1110  
 1111 OA: This fraction is smaller.  
 1112  
 1113 RB: Which fraction is smaller?  
 1114  
 1115 OA: This one <points to 2/3 in case 8C>.  
 1116  
 1117 RB: Smaller than what?  
 1118  
 1119 OA: One half <points to 1/2 in case 8B\*>.  
 1120  
 1121 RB: Smaller than a half. How do you know two thirds is smaller than one half?

- 1122  
1123 OA: I pictured in my mind that three-three-no, it's more. This will have more <OA  
1124 points to case 8C:  $2/3$  of  $1/3$ >  
1125  
1126 RB: 8C is gonna have more?  
1127  
1128 OA: Yeah.  
1129  
1130 RB: Why is 8C gonna have more?  
1131  
1132 OA: Because this is bigger <OA points to " $2/3$ " in 8C>.  
1133  
1134 RB: Okay, and how do you know that?  
1135  
1136 OA: I picture in my mind three-  
1137  
1138 RB: -Uhum-  
1139  
1140 OA: -And then in the same box I put little line in the middle of... Can I show you right  
1141 here <points to the AM-FM representation> what I did?  
1142  
1143 RB: Yeah.  
1144  
1145 OA: <hits "Resit"> I split it in three <sets y-division slider from 6 to 3>, and then I  
1146 pictured, that's one half <OA moves y-axis marker to about half>, and two thirds is  
1147 right there <moves y-axis marker to  $1/3$  instead of  $2/3$ >.  
1148  
1149 RB: Two thirds is where?  
1150  
1151 OA: Right there <OA keep the y-axis marker at  $1/3$ >-no right there <OA moves the y-  
1152 axis marker up to  $2/3$ >?  
1153  
1154 RB: Right there. Why is two thirds right there <RB points to  $2/3$  on y-axis> and not right  
1155 there <RB points to  $1/3$  on y-axis>.  
1156  
1157 OA: Cause that's  $1/3$  <lets go of the y-axis marker line at  $y=2/3$ >. Right here is one third  
1158 <points to  $y=1/3$  mark>.  
1159  
1160 RB: How do you know this one is one third and this one is two thirds?  
1161  
1162 OA: Because one third is smaller than two thirds <moves y-axis marker from  $2/3$  to  $1/3$ >  
1163 and if I make it like that it's less cheese <moves y-axis marker form  $1/3$  back to  $2/3$ >.  
1164  
1165 RB: Okay, okay, so this one is gonna give us more. Good, so go ahead and do the  
1166 problem for me <case 8C:  $2/3$  of  $1/3$ >, so hit reset and start all over <OA hits "Resit">.  
1167 So what are you doing there?  
1168  
1169 **Time: 46:46**  
1170

1171 OA: Splitting it into six <sets y-division slider at 6>.

1172

1173 RB: Why six?

1174

1175 OA: No it's nine-wait <moves y-division slider to limit of 8>...

1176

1177 RB: Why is it nine?

1178

1179 OA: It doesn't go to nine.

1180

1181 RB: Yep, it only goes to eight.

1182

1183 OA: Um...um...um...um...um...

1184

1185 RB: So what are we gonna do? Is there a problem here? Why do you wanna set it at  
1186 nine?

1187

1188 OA: Because here <points to previous cases on the number chart> there was two times  
1189 two equals four, then three times two is six, three times two is six, and then three times  
1190 three is nine but it doesn't go to nine.

1191

1192 RB: But it doesn't go to nine, so what can we do? Is there some other way we can do  
1193 this?

1194

1195 OA: Um...three <sets y-division slider at three and moves x-axis marker from zero to  
1196 1>.

1197

1198 RB: So that's-how much cheese-

1199

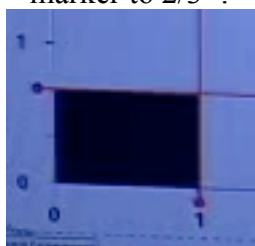
1200 OA: -That's one rat, no that's one slice.

1201

1202 RB: One slice, okay.

1203

1204 OA: And then one third, one third of two thirds, that's two thirds <OA moves y-axis  
1205 marker to  $\frac{2}{3}$ >.

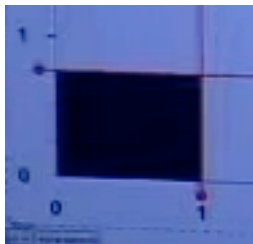


1206

1207 RB: Uhum.

1208

1209 OA: And it says one third of two thirds. So if I split it <moves y-division slider from 3 to  
1210 4 which cases y-axis marker to jump from  $\frac{2}{3}$  to  $\frac{3}{4}$ >. Hmm.



1211

1212 RB: How much is that?

1213

1214 OA: That's three-three fourths

1215

1216 RB: Three fourths, did you want that?

1217

1218 OA: No <moves y-division slider to 5 then to 4 and then to 3 and shading shifts  
1219 accordingly to  $4/5$ ,  $3/4$ , and  $2/3$ >.

1220

1221 RB: So what's that right now, how much cheese?

1222

1223 OA: That's two thirds.

1224

1225 RB: Uhum. You want two thirds of one third.

1226

1227 OA: Um...oh, two thirds, so that's one third <moves y-axis marker from  $2/3$  to  $1/3$ >.

1228

1229 RB: Now you want two thirds of that, how are you gonna get two thirds of that?

1230

1231 OA: Split it into six <moves y-division slider from 3 to 6 so that  $2/6$  area is shaded>-no,  
1232 to eight <moves y-division slider from 6 to 8 which cases  $3/8$  area to be shaded>.

1233

1234 RB: That moved up &lt;the y-axis marker line&gt;.

1235

1236 OA: Right there <moves y-axis marker line from  $3/8$  to  $2/8$ >.

1237

1238 RB: How much is that cheese right there? How much cheese?



1239

1240 OA: Two eights.

1241

1242 RB: Two eights. Was that two thirds of one third?

1243

1244 OA: Uhum.

1245

1246 RB: Okay I'm gonna draw something. So you did this <draw rectangle area model  
1247 partitioned horizontally into thirds>, and you took the bottom third <shaded bottom  $1/3$   
1248 area> like that.



1249

1250 **Time: 50:00**

1251

1252 RB: Right? And you wanna take two thirds of that. So how many pieces should I split  
1253 this into <RB points to bottom third>?

1254

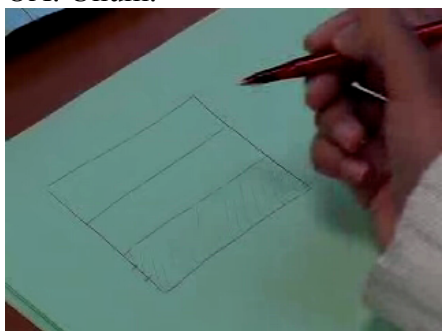
1255 OA: Three.

1256

1257 RB: Three, like that &lt;RB partitions the bottom third in three&gt;.

1258

1259 OA: Uhum.



1260

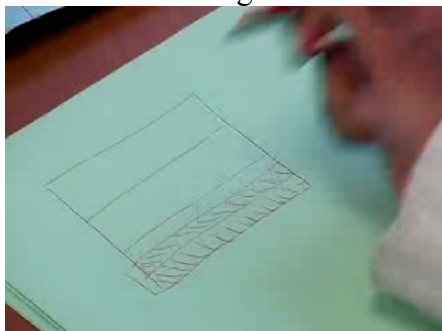
1261 RB: Okay. So then, how many of them do you want me to take?

1262

1263 OA: Two.

1264

1265 RB: Two. So right here. Right? And this one right there <shaded 2/3 of bottom 1/3 area>.  
1266 So we are talking about this much, right?



1267

1268 RB: Now how much is this of the whole thing?

1269

1270 OA: Ummm...it is...

1271

1272 RB: What fraction am I giving out? How can we figure this out? You've done this before  
1273 with the other examples, right?

1274

1275 OA: It's...

1276

1277 RB: Do you wanna draw stuff in?

1278

1279 OA: <OA partitions the top two 1/3 parts also in three by adding marks across the  
1280 vertical length>. Split all of them in three.

1281

- 1282 RB: Uhum.  
1283  
1284 OA: So, <OA counts the total number of parts> It's two ninths.  
1285  
1286 RB: It's two ninths. So that's two eights <pointing to the AM-FM construction>, so  
1287 that's not exactly what we are looking for here. But you can't set this at nine-it won't  
1288 let you go up to ninths. So is there another way we can do this problem?  
1289  
1290  
1291 OA: Um...  
1292  
1293 RB: Does it always have to be split up this way <gestures horizontally>, can't it be split  
1294 up this way <gestures vertically>?  
1295  
1296 **Time: 51:52**  
1297  
1298 OA: Um, yeah. Oh wait. <OA sets y-division slider at 1, which results in a shading of the  
1299 1x1 unit whole and then sets the x-division slider at 4>.  
1300  
1301 RB: What did you set it at four?  
1302  
1303 OA: <OA mumbles something and then changes the y-division slider from 4 to 3> So,  
1304 one third is right there <OA moves the x-axis marker from 3/3 to 1/3>  
1305  
1306 RB: Uhum.  
1307  
1308 OA: And then two thirds of that <OA moves x-axis division from 3 to 6 and then to 7  
1309 which shifts shading from 1/3 to 2/6 to 2/7>. It got lower <moves x-axis division from  
1310 7 to 8 which shifts shading from 2/7 to 2/8>.  
1311  
1312 RB: So you are doing what you did before. Two eights, right?  
1313  
1314 OA: Yeah, two eights.  
1315  
1316 RB: But we know that it's two ninths <pointing to the area model construction drawn on  
1317 paper>.  
1318  
1319 OA: Um...divided <inaudible>.  
1320  
1321 RB: Okay, what if you use both of these, what if you split them? Both the x and the y,  
1322 will that help you?  
1323  
1324 OA: Yeah.  
1325  
1326 RB: What if we do that, so why don't we do that, so start here <RB sets both x-division  
1327 slider and y-division slider at 1 and moves the x-axis marker from zero to one>. Now,  
1328 that's my hint to you, to split both of them  
1329  
1330 OA: <OA moves y-division slider to 3 and x-division slider to 3>.

1331

1332 RB: Okay, so split them both into three, so why threes?

1333

1334 OA: Because it goes like that <gestures horizontally three times and vertically three  
1335 times>, then it splits into nine. So, can I press tile?

1336

1337 RB: No, no, no, I want you to show me where-I want you to do this problem before you-

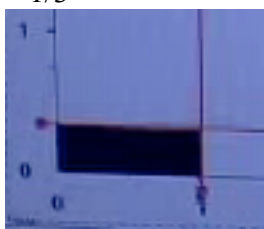
1338

1339 OA: -Oh, okay <OA moves y-axis marker from  $3/3$  to  $2/3$ >.

1340

1341 RB: So tell what you just did there.

1342

1343 OA: Just two thirds-no, one third-it's right there <OA moves y-axis marker from  $2/3$  to  
1344  $1/3$ >

1345

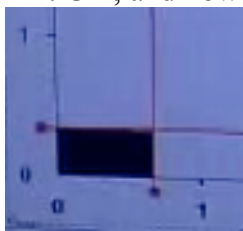
1346 RB: So you moved it to one third.

1347

1348 OA: And then two thirds of it is right there <OA moves x-axis marker from  $3/3$  to  $2/3$ >

1349

1350 RB: Um, and how much is that?



1351

1352 OA: That's two ninths.

1353

1354 RB: How do you know?

1355

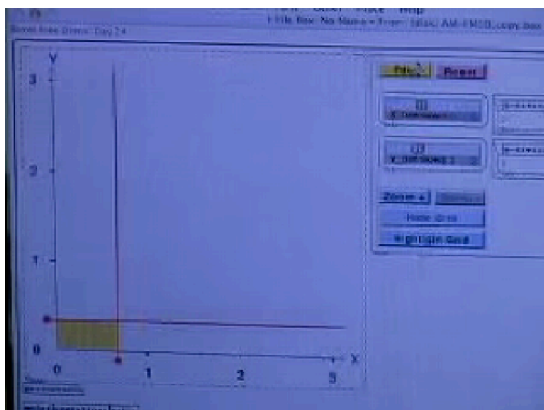
1356 OA: Because if I split it up like that <gestures the vertical and horizontal partitions that  
1357 would result from tiling>, there will be nine, so it's two <points to the shaded area>  
1358 ninths.

1359

1360 RB: Okay, you wanna hit tile?

1361

1362 OA: &lt;OA hits "Tile"&gt;



1363  
 1364 RB: Okay, so then my question, what is this piece called? <RB lifts up one of the  $\frac{1}{9}$   
 1365 tiles>.

1366  
 1367 OA: A ninth.

1368  
 1369 RB: A ninth. Okay, of how many slices? What's our unit? What's a ninth of our unit?

1370  
 1371 OA: Two.

1372  
 1373 RB: Two slices? This is a ninth of two slices?

1374  
 1375 OA: One.

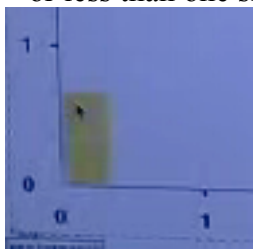
1376  
 1377 RB: Of one slice. Okay. So it's a ninth of one slice, our unit is one, one slice. Okay.  
 1378 Now, does it matter if I move it to some place else, does it change-

1379  
 1380 OA: -No.

1381  
 1382 RB: No, okay, final answer was two ninths.

1383  
 1384 OA: Uhum.

1385  
 1386 RB: Okay, let me put this back <places  $\frac{1}{9}$  tile on top of the  $\frac{1}{9}$  tile> Now, was that more  
 1387 or less than one sixth?



1388  
 1389 OA: Umm... less.

1390  
 1391 RB: Two ninths is less than one sixth? Why?

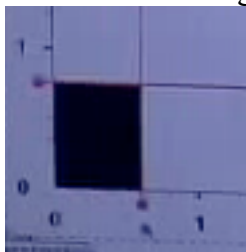
1392  
 1393 **Time: 55:33**

1394

- 1395 OA: Because, one sixth was like right here <OA gestures to a horizontal area across the  
1396 x-axis from zero to 1> where this line is at <looks to be pointing at  $y=1/3$ >.  
1397
- 1398 RB: One sixth was right, where?  
1399
- 1400 OA: Like right here <OA points to what looks to be  $y=1/3$ >.  
1401
- 1402 RB: That's a third, right? This line is one third <RB points to  $y=1/3$ >. How did we get  
1403 one sixth? What were the two fractions we were working with to get the one sixth?  
1404 What were the two fractions we were working?  
1405
- 1406 OA: One half of one third.  
1407
- 1408 RB: Uhum. So.  
1409
- 1410 OA: We got more.  
1411
- 1412 RB: So we got more where? With two ninths or-  
1413
- 1414 OA: -Right here.  
1415
- 1416 RB: We have more with two ninths.  
1417
- 1418 OA: Uhum.  
1419
- 1420 RB: Okay. Where would the line be for one sixth.  
1421
- 1422 OA: Right there-half of it <OA points to the y-axis between zero and  $1/3$ >.  
1423
- 1424 RB: Right here <points to  $y=1/6$ >.  
1425
- 1426 OA: Half of one third. It's like right here <points to  $y=1/6$ >.  
1427
- 1428 RB: It would be right there <points to  $y=1/6$ >.  
1429
- 1430 OA: Yeah.  
1431
- 1432 RB: And your shading would go how far?  
1433
- 1434 OA: Here. <OA points to  $x=1/3$ >.  
1435
- 1436 RB: It would.  
1437
- 1438 OA: No.... It would go all of this <OA points from  $x=0$  to  $x=1$ >.  
1439
- 1440 RB: It would go all of that, right? Cause you didn't split this part <the x-axis> up. So it  
1441 would go all of that.  
1442
- 1443 OA: Yeah.

- 1444  
1445 RB: So, then how many things would it fill up? How many boxes would it fill up? If  
1446 there was a line right here <RB references an imaginary horizontal line at  $y=1/6$ > The  
1447 bottom three <tiles> would be shaded <half way>, right?  
1448  
1449 OA: There is-this one has more.  
1450  
1451 RB: This one has more. How much more of a box?  
1452  
1453 OA: One sixth more.  
1454  
1455 RB: So does it have a complete more extra box shaded?  
1456  
1457 OA: No, wait. Well, what I did was, I split this in half <points to bottom  $1/9$  tile>, and  
1458 then I put, I split this box in half <points to bottom  $1/9$  tile again> so I left one half  
1459 there <points to tile area from  $x=0$  to  $x=1/3$ > and I put the other half there <points to  
1460 tile area from  $x=1/3$  to  $x=2/3$ > and then I split this half <points to the top  $1/9$  tile>, put  
1461 one half there <points to tile area from  $x=2/3$  to  $x=3/3$ >, put the other one up here  
1462 <points to tile area from  $x=1/3$  to  $x=2/3$  and  $y=1/3$  to  $y=2/3$ >. So it would be, it would  
1463 be...um...  
1464  
1465 RB: So how much yellow do we have here, in two ninths?  
1466  
1467 OA: Half of that <points to top  $1/9$  tile>.  
1468  
1469 RB: Half a box?  
1470  
1471 OA: Yeah.  
1472  
1473 **Time: 58:12**  
1474  
1475 RB: Good. Okay. Eight 8D, we again have one rat, this time we are gonna take three  
1476 fifths of three fourths. Okay, so prediction. Three fifths of three fourths, are we gonna  
1477 end up using more or less cheese than 8C < $2/3$  of  $1/3$ >?  
1478  
1479 OA: Um...um...less.  
1480  
1481 RB: Less, why less?  
1482  
1483 OA: Because, right here we split it right here and here <OA points to y-axis and x-axis>.  
1484  
1485 RB: Uhum.  
1486  
1487 OA: So maybe right here we're gonna split three fifths here <OA points to y-axis> and  
1488 three fourths there <OA points to x-axis>.  
1489  
1490 RB: Uhum.  
1491  
1492 OA: So there is gonna be more like, boxes.

- 1493  
 1494 RB: There is gonna be more boxes?  
 1495  
 1496 OA: So it's gonna be less.  
 1497  
 1498 RB: Cause there are more boxes? Okay. Okay, good. Let's have you go ahead and do this  
 1499 problem. So hit resear <OA hits "Reset" but the both sliders are set positioned at 3>,  
 1500 yeah. So show me the unit-start with the unit. What's the unit?  
 1501  
 1502 OA: So <OA moves y-axis marker from zero to  $3/3$ >.  
 1503  
 1504 RB: Singles slice, right? Good.  
 1505  
 1506 OA: So divide this into five <OA moves the x-division slider to 5> and this into four  
 1507 <OA moves the y-division slider to 4>.  
 1508  
 1509 RB: Okay.  
 1510  
 1511 OA: And three fourths is...right there <OA moves y-axis marker from  $4/4$  to  $3/4$ >.  
 1512  
 1513 RB: Why did you start with three fourths first?  
 1514  
 1515 OA: Because I am gonna get three fourths and then I'm gonna get three fifths of that.  
 1516  
 1517 RB: Okay.  
 1518  
 1519 OA: And that is right there. <OA moves x-axis marker from  $5/5$  to  $3/5$ >.



- 1520  
 1521 RB: That's right there, that's three fifths. What is this point called right here <RB points  
 1522 to point  $6/5$  on x-axis>?  
 1523  
 1524 **Time: 1:00:05**  
 1525  
 1526 OA: That's six fifths.  
 1527  
 1528 RB: Six fifths, or?  
 1529  
 1530 OA: Or, one and one fifth.  
 1531  
 1532 RB: One and one fifth. What is this point called right here <RB points to point  $7/4$  on y-  
 1533 axis>?  
 1534  
 1535 OA: That's one-three fourths.

1536

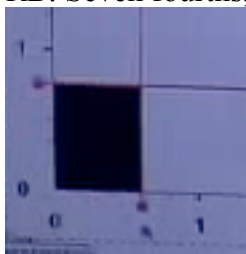
1537 RB: Or?

1538

1539 OA: Or...seven fourths.

1540

1541 RB: Seven fourths, good. Okay. How much is shaded here? How much did you give out?



1542

1543 OA: That is...umm...that's...nine twentieths.

1544

1545 RB: Nine twentieths. How did you get that?

1546

1547 OA: Well, I did-there's four boxes here <OA points to the four y-axis line segments from  
 1548  $y=0$  to  $y=4/4$ >, and then five here <points to the five x-axis line segments from  $x=0$  to  
 1549  $x=5/5$ > so then four times five, and that's twenty, so that's my bottom number.

1550

1551 RB: Uhum.

1552

1553 OA: And then I did-there's three spaces here <points to the three x-axis line segments  
 1554 from  $x=0$  to  $x=3/5$ >, so that's two lines <gestures two vertical lines at  $x=1/5$  and  
 1555  $x=2/5$ > and then two <lines> right there <gestures two horizontal lines at  $y=1/4$  and  
 1556  $y=2/4$ >, so it's three times three, that's nine.

1557

1558 RB: Uhum. So you did three time three and four times five?

1559

1560 OA: Yeah.

1561

1562 RB: Okay, and what was your answer again?

1563

1564 OA: Nine twentieths.

1565

1566 RB: Nine twentieths. So was that more or less than two ninths?

1567

1568 OA: It's more.

1569

1570 RB: How do you know it's more?

1571

1572 OA: Because the other one, a third was like right there <OA points to where  $x=1/3$ >.

1573

1574 RB: Uhum.

1575

1576 OA: And there were two boxes. Like one right here and another one right here <points  
 1577 out where the two  $1/9$  tiles fell across the x-axis in the previous case>.

1578



1579 RB: Uhum.

1580

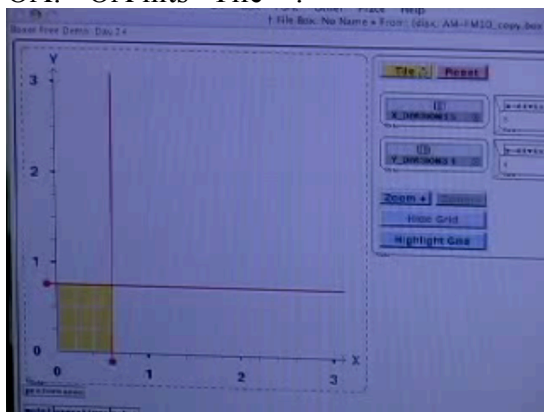
1581 OA: So I pictured it was less.

1582

1583 RB: Okay. Do you wanna hit tile?

1584

1585 OA: <OA hits "Tile">.



1586

1587 RB: Good, nine twentieths, excellent. Um...is nine twentieths more or less than a half?

1588

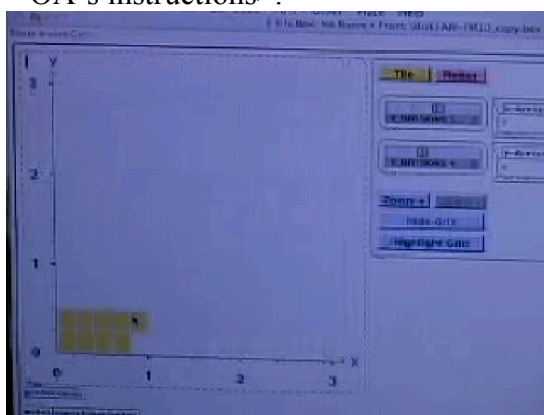
1589 OA: Nine twentieths is <OA mumbles something> that is...less.

1590

1591 RB: How do you know?

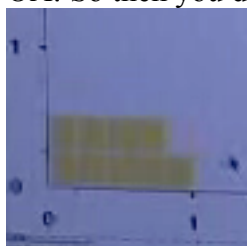
1592

1593 OA: Because these three <OA points to the top three 1/20 tile pieces>, I put one right  
1594 here, another one right here, and another one right here <RB moves tiles according to  
1595 OA's instructions>.



1596

1597 OA: So then you do this <OA moves a tile pieces>.



1598

1599 RB: Uhum.

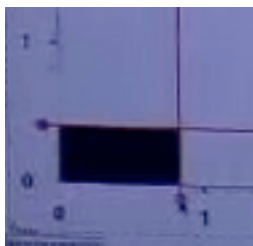
1600

1601 OA: And you need one more cube.

1602  
1603 RB: You need one more cube. So you need how many more twentieths?  
1604  
1605 OA: One.  
1606  
1607 RB: One more twentieth, and that would give you how many twentieths? If you had one  
1608 more?  
1609  
1610 OA: Half-If I put one little more there <point to blank 1/20 slot in configures> it would  
1611 give me half of the whole <points to the 1x1 unit whole> slice of cheese.  
1612  
1613 RB: Of a whole slice of cheese, okay. And what is the name of this again? <RB lifts up  
1614 one of the 1/20 tile piece>.  
1615  
1616 OA: Twentieths.  
1617  
1618 RB: Twentieths, so if I added one more twentieth here, how many twentieths would we  
1619 have?  
1620  
1621 OA: Twelve.  
1622  
1623 RB: Twelve twentieths?  
1624  
1625 OA: No, ten.  
1626  
1627 RB: Ten twentieths. Does ten twentieths reduce?  
1628  
1629 OA: Yeah.  
1630  
1631 RB: To what?  
1632  
1633 OA: One half.  
1634  
1635 RB: How do you know that?  
1636  
1637 OA: Cause ten times two is twenty, so that means it is half of it. And that's one half.  
1638  
1639 **Time: 01:03:47**  
1640  
1641 RB: Okay, good. Next one, 8E. One rat again, and we are gonna do five sixth of two  
1642 fifths. So prediction. Five sixth of two fifths, are we gonna end up using more or less  
1643 cheese than in 8D <8D: 3/5 of 3/4>?  
1644  
1645 OA: Um...more.  
1646  
1647 RB: Why more?  
1648  
1649 OA: Because this numbers is like-that's smaller than three fifths.  
1650

- 1651 RB: What's smaller than three fifths?  
1652  
1653 OA: Five sixths.  
1654  
1655 RB: Five sixths is smaller than three fifths? Okay, why?  
1656  
1657 OA: Um...because if you do like a square, like, divide it into five and shade three, you  
1658 would get more than if you were to divide into six- no wait is that a five <referring the  
1659 "5" in  $5/6$ >?  
1660  
1661 RB: Uhum.  
1662  
1663 OA: Oh, never mind, I thought it was a one.  
1664  
1665 RB: It's a five, and this is a two fifths.  
1666  
1667 OA: Okay. You are gonna get...you are gonna use more cheese.  
1668  
1669 RB: In which one, 8D or 8E?  
1670  
1671 OA: 8E.  
1672  
1673 RB: We are gonna use more in 8E, why?  
1674  
1675 OA: Because the fraction is bigger.  
1676  
1677 RB: Five sixths is bigger than what?  
1678  
1679 OA: Three fifths.  
1680  
1681 RB: Three fifths. Okay, and what about these two fractions <points to  $3/4$  in case 8D and  
1682  $2/5$  in case 8E>, do they have anything to do with it?  
1683  
1684 OA: Umm...yeah.  
1685  
1686 RB: Okay, so, five sixths of two fifths is gonna be more than three fifths of three fourths,  
1687 because five sixths is bigger. Okay, let's see you do it.  
1688  
1689 **Time: 1:05:50**  
1690  
1691 OA: <OA resets and moves x-axis marker from zero to  $5/5$ >.  
1692  
1693 RB: So tell me-walk me through what you are doing.  
1694  
1695 OA: I did-so that gives me one rat, for one rat.  
1696  
1697 RB: Okay, so if it is for one rat-what does your x-axis represent?  
1698  
1699 OA: For the rats.

- 1700  
 1701 RB: For the rats, and your y-axis?  
 1702  
 1703 OA: Number of slices, like how many slices.  
 1704  
 1705 RB: Slices.  
 1706  
 1707 OA: Yeah.  
 1708  
 1709 RB: Okay, good.  
 1710  
 1711 OA: Okay...first two fifths <OA moves the y-division slider from 4 to 5 and then moves  
 1712 the y-axis marker to 5/5 to 2/5>.  
 1713  
 1714 RB: Okay, so what are you doing?  
 1715  
 1716 OA: That's two fifths, so I gotta get 5/6 of it so I divide the x-axis into six <moves x-  
 1717 division slider from 5 to 6>, and then I put right there <OA moves x-axis marker from  
 1718 6/6 to 5/6>. So that's five sixths of two fifths.  
 1719  
 1720 RB: Okay. So that's five sixths right there. So this point is called five sixths right here  
 1721 <RB points to point five-sixth on x-axis>?  
 1722  
 1723 OA: Yeah.  
 1724  
 1725 RB: What is that five sixth referring to? Rats?  
 1726  
 1727 OA: Um, no. Five sixth of the two fifths. That's two fifths <OA points to y-axis> and  
 1728 that's five sixths right there <OA points to x-axis>  
 1729  
 1730 RB: So the x-axis now has changed meaning, right?  
 1731  
 1732 OA: Yeah.  
 1733  
 1734 RB: To what? Not rats anymore, but what?  
 1735  
 1736 OA: Slices.  
 1737  
 1738 RB: Slices, slices, okay, good. So, how much is that? How much cheese did you give out  
 1739 <references final area of 10/30 produced by Oscar>?



- 1740  
 1741 OA: Ten...ten thirtieths.  
 1742

1743 RB: How did you get that so quick?

1744

1745 OA: I did, one, two, three, four, five, six, seven, eight, nine, ten <OA points out and  
1746 counts aloud the total number of shaded boxes>.

1747

1748 RB: Uhum.

1749

1750 OA: And then six times five <points to the sliders> is thirtieths-thirty.

1751

1752 RB: So six times five you pointed to the sliders when you did that?

1753

1754 OA: Yeah. It's the same thing as right here <points to the x-axis and y-axis>.

1755

1756 RB: Same thing is right where?

1757

1758 OA: Right with these numbers right here <OA points to the y-axis from 1 to zero and  
1759 points to the x-axis from zero to 1>.

1760

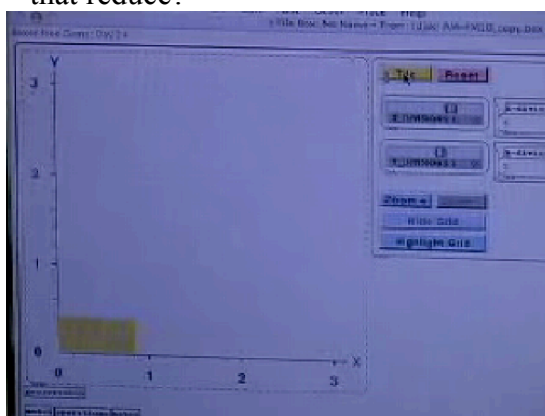
1761 RB: So the number of little-

1762

1763 OA: -Cubes.

1764

1765 RB: Cubes. I see. Okay, let's see you tile. <OA hits "Tile"> Ten thirtieths, okay. Does  
1766 that reduce?



1767

1768 OA: Um...yeah.

1769

1770 **Time: 1:08:12**

1771

1772 RB: To what?

1773

1774 OA: Three fifteenths-no, one-one fifth.

1775

1776 RB: One fifth, how did you get that?

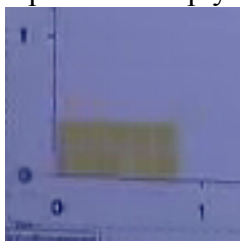
1777

1778 OA: So first I did three fifteenths, so three divided by three is one, fifteen divided by  
1779 three is five. So it's one fifth.

1780

1781 RB: How did you get three fifteenths?

- 1782  
 1783 OA: Um, three-no, um...that's five fifteenths-five fifteenths.  
 1784  
 1785 RB: Five fifteenths. Okay, so five fifteenths and then you got what?  
 1786  
 1787 OA: Then...no.  
 1788  
 1789 RB: Does that reduce further?  
 1790  
 1791 OA: No.  
 1792  
 1793 RB: Five fifteenths, that's it, it doesn't reduce any further than this, are you sure?  
 1794  
 1795 OA: Yeah.  
 1796  
 1797 RB: There is no number that goes into five and fifteen evenly?  
 1798  
 1799 OA: No.  
 1800  
 1801 RB: One, two, three-  
 1802  
 1803 OA: -Oh five divided by five-  
 1804  
 1805 RB: -Hum-  
 1806  
 1807 OA: -Is one and then fifteen divided by five is three. One third.  
 1808  
 1809 RB: One third. Okay, so it's equal to all these things, right <RB referring fractions  
 1810 equivalent to  $10/30$  written on the number chart not captured on video>? So I want you  
 1811 to help me see al these things in this picture here, using this program. Is there a way we  
 1812 can move these tiles around to see one third?  
 1813  
 1814 OA: Um, change it to one third?  
 1815  
 1816 RB: Change what to one third?  
 1817  
 1818 OA: The divisions.  
 1819  
 1820 RB: But if you change the divisions to one third, this will still be empty space <RB  
 1821 points to empty tile space between  $x=5/6$  and  $x=6/6$ >.



- 1822  
 1823 OA: Um.  
 1824

1825 RB: Right? The divisions would just get rid of some of these lines, right? If we change  
1826 the divisions, but there is still empty space right here.

1827

1828 OA: To make one third.

1829

1830 RB: Can we move the tiles? Somehow?

1831

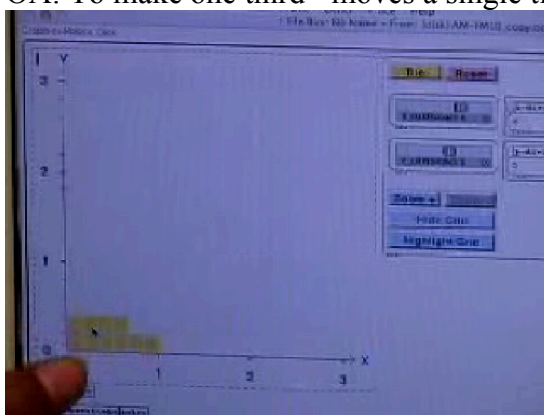
1832 OA: Um...So...to make one third?

1833

1834 RB: Uhum.

1835

1836 OA: To make one third <moves a single tile>. It's already one third.



1837

1838 RB: What do you mean it's already one third?

1839

1840 OA: Well it's more than one third.

1841

1842 RB: What's more than one third?

1843

1844 OA: This-like-...wait, wait, wait, wait, wait. Well because when I did it I got rid of these  
1845 things <OA points to  $1/5$  markers from  $y=0$  to  $y=5/5$ >.

1846

1847 RB: Uhum.

1848

1849 OA: And got rid of this one, this one and this one <OA points the  $x=1/6$ ,  $x=3/6$ ,  $x=5/6$   
1850 markers>. So that's thirds.

1851

1852 RB: Uhum.

1853

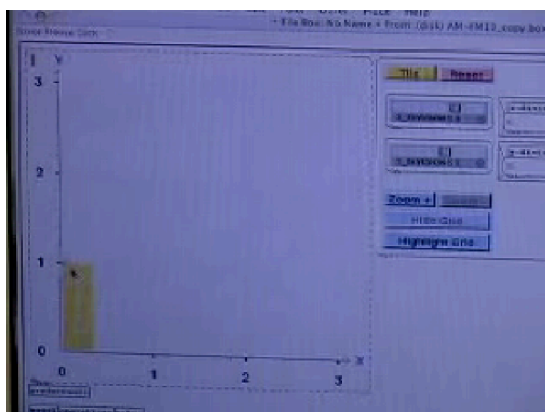
1854 OA: And these two and this one that's one <point to two tile pieces laying across x-axis  
1855 from  $x=0$  to  $x=2/6=1/3$ > that's one third.

1856

1857 RB: But would it be the whole column, or would it be just the-

1858

1859 OA: -No it would be the whole column. So <OA moves tile pieces up to fill first  $1/3$   
1860 column from  $x=0$  to  $x=2/6=1/3$ >.



- 1861
- 1862 RB: I see what you did. Okay, and why is that one third?
- 1863
- 1864 OA: Um, because that's one, that's two and that's three <OA points to the 1/6 line
- 1865 segments from  $x=0$  to  $x=6/6$  in pairs of two>.
- 1866
- 1867 RB: Uhum.
- 1868
- 1869 OA: So that's thirds, and these aren't here <points to the five 1/5 line segments from  $y=0$
- 1870 to  $y=5/5$ >.
- 1871
- 1872 RB: -Uhum-
- 1873
- 1874 OA: -So this would be like-this is one third.
- 1875
- 1876 RB: Okay. So, you wanna take-what is this axis, x-division, you want to set the x-
- 1877 divisions at three?
- 1878
- 1879 OA: Um, I'm not sure. Should I change them.
- 1880
- 1881 RB: Yeah go for it. See what happens. <OA changes the x-division slider from 6 to 3, the
- 1882 y-division slider is still set at 5>. Okay. Um, now what is that fraction-what is this
- 1883 piece called <RB lifts up a 1/15 tile piece>?
- 1884
- 1885 OA: That's one-no that's...
- 1886
- 1887 RB: What's the name of that piece?
- 1888
- 1889 OA: Fifteenths.
- 1890
- 1891 RB: Fifteenths. Okay, and how many fifteenths do we have here?
- 1892
- 1893 OA: Five.
- 1894
- 1895 RB: Five fifteenths, is that what the other fraction was?
- 1896
- 1897 OA: Yeah.
- 1898



1899 RB: So how do we get from ten to five, what happened? From ten thirtieths to five  
1900 fifteenths?

1901

1902 OA: Um...

1903

1904 **Time: 1:13:18**

1905

1906 RB: So remember before we had, reset <RB hits "Reset">. We had this at six, right  
1907 <moves x-division slider from 3 to 6, leaving y-division slider at 5>?

1908

1909 OA: Well, if I make that it's six.

1910

1911 RB: And then you did that <RB moves x-axis marker from zero to 6/6>, right? And then  
1912 you took two fifths of a slice of cheese <RB moves y-axis marker to from 5/5 to 2/5>.  
1913 And then you took, five sixths of the two fifths <RB moves x-axis marker from 6/6 to  
1914 5/6 >, right here, right?

1915

1916 OA: Uhum.

1917

1918 RB: And then we hit tile <hits "Tile">, so that was ten thirtieths, how do we get five  
1919 fifteenths?



1920

1921 OA: Five fifteenths?

1922

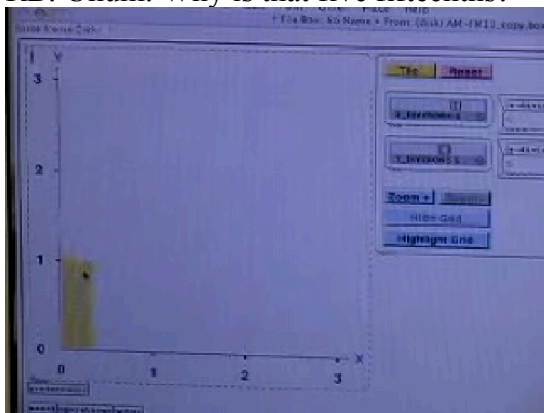
1923 RB: Uhum.

1924

1925 OA: We move them like <OA moves the tiles the same way he did it before>

1926

1927 RB: Uhum. Why is that five fifteenths?



1928

1929 OA: Well, because if we move <moves cursor towards y-division slider>-

1930

1931 RB: -I don't want you to change this <RB points to the x-division and y-division  
1932 sliders>.

- 1933  
1934 OA: Oh no?  
1935  
1936 RB: Why is that five fifteenths?  
1937  
1938 OA: Because if you divide <points to tiles moving up the y-axis>.  
1939  
1940 RB: So you just counted, one, two, three, four, five, what were you counting?  
1941  
1942 OA: The yellow boxes.  
1943  
1944 RB: You weren't counting each of them, right? You were counting them together, in  
1945 pairs.  
1946  
1947 OA: No, I was counting each one.  
1948  
1949 RB: Well, what if you counted them in pairs?  
1950  
1951 OA: There is five pairs, five pairs of two.  
1952  
1953 RB: So there is five pairs. There is five pairs of yellow boxes.  
1954  
1955 OA: Uhum.  
1956  
1957 RB: How many pairs of total boxes do we have?  
1958  
1959 OA: Fifteen.  
1960  
1961 RB: So if we count by twos, we get five fifteenths, right?  
1962  
1963 OA: Uhum.  
1964  
1965 **Time: 01:15:20**  
1966  
1967 RB: Okay. Let's go to the next one, 8F. Good job. We have one rat, four thirds of two  
1968 fifths. Okay, prediction time. Four thirds of two fifths, here we were taking five sixths  
1969 of two fifths and now we are gonna take four thirds. Are we gonna end up using more  
1970 or less cheese?  
1971  
1972 OA: Um-  
1973  
1974 RB: -In 8F <4/3 of 2/5> than we use in 8E <5/6 of 2/5>?  
1975  
1976 OA: That's two fifths?  
1977  
1978 RB: Yeah these are both-this is 2/5 <points to 2/5 in case 8F>, yeah.  
1979  
1980 OA: We use less.  
1981

- 1982 RB: And then we're gonna take  $\frac{4}{3}$  of it. Why less?
- 1983
- 1984 OA: Because the fraction is smaller than that one.
- 1985
- 1986 RB: Which fraction-
- 1987
- 1988 OA: -No wait, wait, wait, no more.
- 1989
- 1990 RB: Which fraction-okay, we are gonna use more in 8E or 8F?
- 1991
- 1992 OA: 8F.
- 1993
- 1994 RB: 8F, why?
- 1995
- 1996 OA: This is a mixed number <points to  $\frac{4}{3}$ >.
- 1997
- 1998 RB: That's a mixed number? So what does that mean?
- 1999
- 2000 OA: That it is bigger than one.
- 2001
- 2002 RB: It's bigger than one, okay.
- 2003
- 2004 OA: And this is smaller than one <points to  $\frac{5}{6}$  in case 8E>.
- 2005
- 2006 RB: That's smaller than one, cause its what kind of fraction?
- 2007
- 2008 OA: An improper.
- 2009
- 2010 RB: Improper?
- 2011
- 2012 OA: Proper.
- 2013
- 2014 RB: Proper. Improper is the one you can change to a mixed number. Okay, okay, so let's
- 2015 see you do it.
- 2016
- 2017 OA: So, <OA hits "Reset">.
- 2018
- 2019 RB: Talk me through what you are doing.
- 2020
- 2021 OA: Change it to one rat <OA moves x-axis marker from zero to  $\frac{6}{6}$ >.
- 2022
- 2023 RB: Uhum
- 2024
- 2025 OA: And then, um...no, wait <moves y-division slider from 5 to 3>.
- 2026
- 2027 RB: Okay, so what did you just do there?
- 2028
- 2029 OA: I changed it into thirds, cause it's gonna be the bottom number.
- 2030

2031 RB: Uhum.

2032

2033 OA: That's a fifth so change this one into fifths <moves x-division slider from 6 to 5>.

2034 Then, two fifths <OA moves x-axis marker from  $5/5$  to  $2/5$ >.

2035

2036 RB: Uhum, so how much-before you do anything-how much cheese is that right there?



2037

2038 OA: That is...

2039

2040 RB: How much did you just take of a slice?

2041

2042 OA: Nine. Nine little cube things.

2043

2044 RB: Nine little cub things?

2045

2046 OA: The total <appears to be gesturing the act of counting tiles that make up the 1x1 unit  
2047 whole> there was like... there was fifteen total, in like the whole there was fifteen.

2048

2049 RB: Uhum. And how much is in there?

2050

2051 OA: Then there's twelve now. No, wait, there is... six.

2052

2053 RB: Six, okay. Okay, and what is the name of those little cube things?

2054

2055 OA: Fifteenths.

2056

2057 RB: So six fifteenths? Do six fifteenths reduce? To what?

2058

2059 OA: No-wait-yeah.

2060

2061 RB: To what?

2062

2063 OA: To...um...two-no-two fifths.

2064

2065 RB: Two fifths? How much-this problem asks you to take how much of a slice of  
2066 cheese?

2067

2068 OA: Four thirds.

2069

2070 RB: Of?

2071

2072 OA: Two fifths.

2073

2074 RB: And how much have you taken so far? Two fifths, right? So you didn't have to  
 2075 count all those little cubes when I asked you how much of the cheese did you just take,  
 2076 you could've just said two fifths, right? You moved it to two fifths. You took two fifths  
 2077 of the whole slice, right?

2078  
 2079 OA: Uhum.

2080  
 2081 RB: But it's the same thing if you had said six fifteenths, right? There are the same thing,  
 2082 they are equivalent, they are equal fractions.

2083  
 2084 OA: Uhum

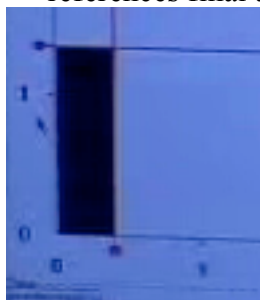
2085  
 2086 RB: Alright, so you have two fifths, now what?

2087  
 2088 OA: It's four thirds of two fifths <OA moves y-axis marker from  $3/3$  up to  $4/3$ >. That's  
 2089 four thirds of two fifths.

2090  
 2091 RB: That's four thirds?

2092  
 2093 OA: Yeah.

2094  
 2095 RB: So how much cheese did you just use? How much cheese did you give out  
 2096 <references final area of  $8/15$  produced by Oscar>?



2097  
 2098 OA: Four, four fifteenths.

2099  
 2100 RB: Four fifteenths?

2101  
 2102 OA: Yeah.

2103  
 2104 RB: How are you getting four fifteenths?

2105  
 2106 OA: No wait...oh yeah four fift-no...four fifteenths yeah.

2107  
 2108 RB: And where is the four fifteenths coming from? Where is the number four coming  
 2109 from?

2110  
 2111 OA: Well, one, two, thee, four <points to four  $1/3$  line segments from  $y=0$  to  $y=4/3$ >.

2112  
 2113 RB: -Uhum-

2114  
 2115 OA: -And then out of fifteen <points to  $1 \times 1$  unit whole>.

2116  
 2117 RB: So what are counting, just this little, these little segments <RB points to segments on  
 2118 y-axis> or are you counting boxes?

2119  
 2120 OA: The...boxes.

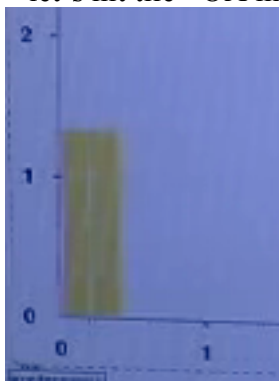
2121  
 2122 RB: And there is-

2123  
 2124 OA: -Four of them.

2125  
 2126 RB: So there is just this box here <makes horizontal line at  $y=1/3$ >-

2127  
 2128 OA: -No, wait, wait, wait <appears to be counting to himself> eight-eight fifteenths.

2129  
 2130 RB: Eight fifteenths. So there is eight boxes and each box is called a fifteenth? Okay,  
 2131 let's hit tile <OA hits "Tile">. Is that eight?



2132  
 2133 OA: Uhum.

2134  
 2135 RB: So it's eight fifteenths, do you wanna write that down?

2136  
 2137 OA: -No wait...eight thirtieths.

2138  
 2139 RB: Okay, so first you said eight fifteenths and now you are saying eight thirtieths. Why  
 2140 eight thirtieths?

2141  
 2142 OA: Because, that's fifteen <OA points to first 1x1 unit whole from  $y=0$  to  $y=3/3$ > and  
 2143 that's fifteen <OA points to second 1x1 unit whole from  $y=3/3$  to  $y=6/3$ >, so that's  
 2144 thirty.

2145  
 2146 RB: So is this piece called a thirtieth <lifts up a single  $1/15$  tile piece>?

2147  
 2148 OA: Yeah.

2149  
 2150 RB: It's not a fifteenth.

2151  
 2152 OA: No.

2153  
 2154 RB: Okay. It's a thirtieth of how many slices?

2155

2156 OA: Um, two.

2157

2158 RB: Okay, I want the unit to be one.

2159

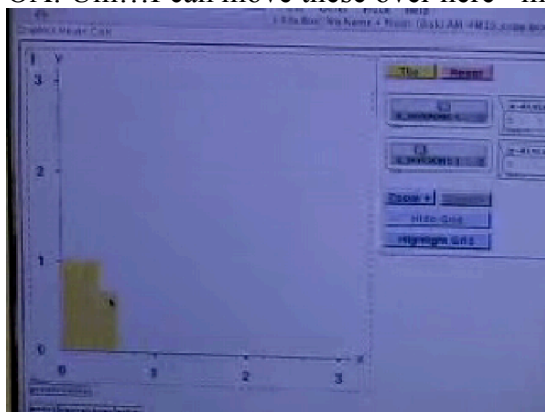
2160 OA: No-one.

2161

2162 RB: It's a thirtieth of one slice? Remember, I always want the unit to be one.

2163

2164 OA: Um...I can move these over here <moves tiles into a single 1x1 unit>.



2165

2166 RB: So how much cheese did you give out?

2167

2168 OA: Eight fifteenths.

2169

2170 RB: Eight fifteenths. So back to eight fifteenths, are you happy with that?

2171

2172 OA: Yeah.

2173

2174 RB: Okay, if my unit was two slices, if I was okay with my unit being two slices, what  
2175 would I call this? Eight what?

2176

2177 OA: Eight thirtieths.

2178

2179 RB: Thirtieths, but my unit is not two slices, my unit is one slice, right? So this pieces are  
2180 called a fifteenths of one slice. They would be thirtieths of two slices, right? Okay.

2181

Right, good job.

2182

2183 **Time: 01:22:51**

2184

2185 RB: 8G, you got one rat and we are gonna take two and three fifths of two fifths slice per  
2186 rat. Okay, so we are gonna take two and three fifths of two fifths slice per rat,  
2187 prediction? Are we gonna end up using more or less cheese than 8F <4/3 of 2/5>?

2188

2189 OA: Um...more.

2190

2191 RB: Why more?

2192

2193 OA: Because, this is like bigger than two <points to  $2\frac{3}{5}$  in case 8G>.

2194

2195 RB: Uhum.

2196

2197 OA: And that one is only bigger than one <points to  $\frac{4}{3}$  in case 8F>.

2198

2199 RB: That one is only bigger than one but less than two.

2200

2201 OA: Yeah.

2202

2203 RB: Okay, good, let's see you do it.

2204

2205 OA: <hits "Reset" the x-division sliders remains at 5 and the y-division slider remains at  
2206 3>.

2207

2208 RB: So talk me through what you are doing as you do it.

2209

2210 OA: So, two fifths, first I'm gonna do the one slice <OA moves the x-axis marker from  
2211 zero to  $\frac{5}{5}$ >.

2212

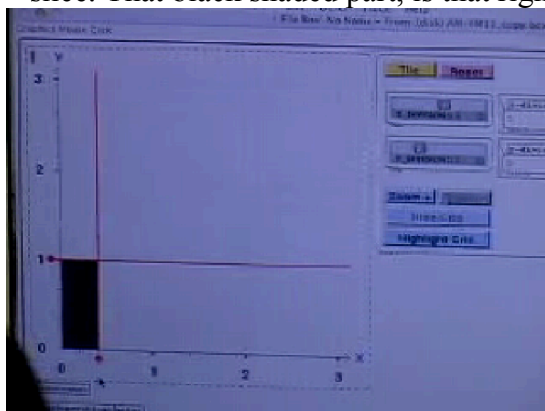
2213 RB: Uhum.

2214

2215 OA: And then two fifths of that, is right there <OA moves x-axis marker from  $\frac{5}{5}$  to  
2216  $\frac{2}{5}$ >.

2217

2218 RB: So, so far you you've taken two fifths-so that amount right there is two fifths of a  
2219 slice. That black shaded part, is that right? Okay.



2220

2221 OA: Yeah.

2222

2223 RB: Now what are you gonna do? <OA moved y-division slider from 3 to 5>. So you  
2224 moved you y-divisions to five, why?

2225

2226 OA: Because, it's three fifths.

2227

2228 RB: Uhum. Okay.

2229

2230 OA: So this is right here <moves y-axis marker from  $\frac{5}{5}$  to  $2\frac{3}{5}$ >.

2231

2232 RB: Right there.

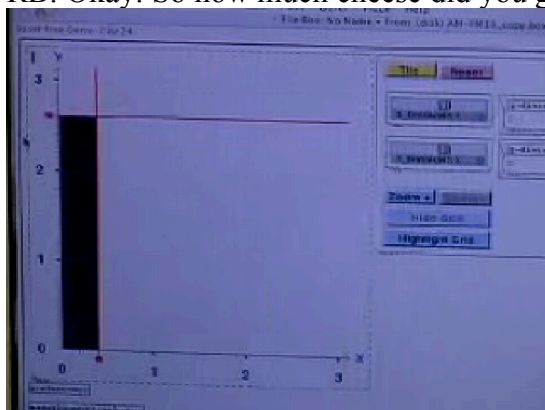
2233



2234 **Time: 01:24:50**

2235

2236 RB: Okay. So how much cheese did you give out?



2237

2238 OA: Um...that's...twenty six-no.

2239

2240 RB: twenty sixths?

2241

2242 OA: No, wait, wait... So, I put this over here.

2243

2244 RB: You put which one over there? What are you trying to-

2245

2246 OA: - I move this block right here <OA points to shaded area between  $y=5/5$  to  $y=10/5$ >, right here <OA points bottom  $1 \times 1$  unit whole>, and then I put the five <OA points to shaded area between  $y=10/5$  to  $y=15/5$ > there <OA points to bottom  $1 \times 1$  unit whole>. So there is gonna be one left.

2250

2251 RB: On left, so how much cheese did you give out?

2252

2253 OA: That's one and one fifth-

2254

2255 RB: -One and-

2256

2257 OA: -One and one...

2258

2259 RB: What are those little pieces called?

2260

2261 OA: One and one twenty fifths.

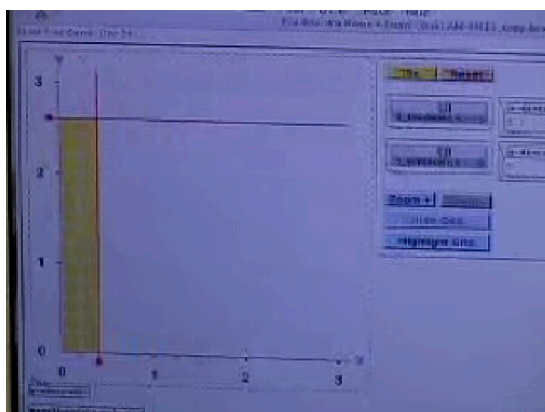
2262

2263 RB: One and one twenty fifths, that's amazing. That's great that you thought of it that way. Okay, let's see you tile.

2264

2265 OA: <hits "Tile">.

2266



- 2267  
 2268 RB: And now you could move all these down, but I'm not gonna have you do that. Can I  
 2269 ask what is the name of this point right here? <RB points to  $y=2\frac{3}{5}$ >  
 2270  
 2271 OA: That's a...um...  
 2272  
 2273 RB: What is that?  
 2274  
 2275 OA: Thirteen...thirteen, thirteen fifth-no.  
 2276  
 2277 RB: Thirteen fifths?  
 2278  
 2279 OA: Yeah.  
 2280  
 2281 RB: Thirteen fifths. So this is called thirteen fifths?  
 2282  
 2283 OA: Yeah.  
 2284  
 2285 RB: Does you rule work <RB looks over to the number chart>?  
 2286  
 2287 OA: Yeah.  
 2288  
 2289 RB: Five times two plus three equals thirteen, yeah.  
 2290  
 2291 OA: Yeah.  
 2292  
 2293 RB: Okay, and then, um, how could this < $1\frac{1}{25}$ > be written if I wanted it an improper  
 2294 fraction?  
 2295  
 2296 OA: Twenty-five.  
 2297  
 2298 RB: Twenty-five on the bottom?  
 2299  
 2300 OA: Yeah.  
 2301  
 2302 RB: What's gonna go on the top?  
 2303  
 2304 OA: Twenty-six.  
 2305

- 2306 RB: Twenty-six. Okay, okay. So what are we doing to these two numbers to get this  
2307 number?  
2308
- 2309 OA: Which two numbers?  
2310
- 2311 RB: These two numbers? Thirteen fifths and two fifths, what are we doing to these two  
2312 numbers to get twenty-six fifths?  
2313
- 2314 OA: Well, five times five is twenty-five and thirteen times two is twenty-six.  
2315
- 2316 RB: So multiplying?  
2317
- 2318 OA: Yeah.  
2319
- 2320 RB: What are we doing here <points to case 8E:  $5/6$  of  $2/5$ >?  
2321
- 2322 OA: Um, six times five is thirty and-  
2323
- 2324 RB: -Uhum.  
2325
- 2326 OA: Five times two is ten.  
2327
- 2328 RB: So multiplying.  
2329
- 2330 OA: Yeah.  
2331
- 2332 RB: So of means to multiply?  
2333
- 2334 OA: Yeah.  
2335
- 2336 **Time: 1:27:55**  
2337
- 2338 RB: Of means multiply, okay. 8H. One rat, and I want you to do one and two fifths of six  
2339 fourths slice per rat. So prediction, are we gonna end up using more or less cheese  
2340 <than case 8G:  $2\frac{3}{5}$  of  $2/5$ >?  
2341
- 2342 OA: More.  
2343
- 2344 RB: Why more? Which one?  
2345
- 2346 OA: This one <points to case 8H> is gonna get more.  
2347
- 2348 RB: Why?  
2349
- 2350 OA: Both of the fractions are mixed numbers.  
2351
- 2352 RB: Both of the fractions are mixed numbers. Okay, okay, let's see you do it.  
2353
- 2354 OA: <hits "Reset" and both sliders remain at 5> six fourths that right, umm.

2355  
2356 RB: So what are you doing in your head right now? What are you trying to do? Convert  
2357 it?  
2358  
2359 OA: Yeah.  
2360  
2361 RB: Okay, how are we gonna convert it?  
2362  
2363 OA: Four divided by six.  
2364  
2365 RB: Is it four divided by six or-  
2366  
2367 OA: -Six divided by four.  
2368  
2369 RB: Six divided by four.  
2370  
2371 OA: That is one...one and two fourths.  
2372  
2373 RB: One and two fourths, okay. Do you want me to write that down somewhere?  
2374  
2375 OA: Yeah.  
2376  
2377 RB: One and two fourths. Okay.  
2378  
2379 OA: Um...so I have to do one and two fourths so, I divide the-first I have to one <OA  
2380 moves x-axis marker from zero to  $5/5$ >.  
2381  
2382 RB: Okay.  
2383  
2384 OA: And then I divide the y-axis into four <OA moves y-division slider from 5 to 4>.  
2385  
2386 RB: Okay.  
2387  
2388 OA: <OA moves x-axis marker from  $5/5$  to  $2/5$ >  
2389  
2390 RB: What did you just do there?  
2391  
2392 OA: I did two-it's two-wait. Oh, no it's the x. Oh, wait, I did the wrong one. <OA moves  
2393 x-axis marker back to  $5/5$  and moves the y-axis marker from  $4/4$  to  $2/4$ >.  
2394  
2395 RB: Wait, what did you just do there?  
2396  
2397 OA: It's two fourths.  
2398  
2399 RB: Wait, but it's one and two fourths.  
2400  
2401 OA: Yeah, I know...so...<OA moves y-axis marker to  $6/4$ >.  
2402  
2403 RB: Okay, so that's one and two fourths?

2404

2405 OA: Yeah.

2406

2407 RB: Okay, so now what?

2408

2409 OA: Now, twelve fifths, make it a mixed number is...two,-

2410

2411 RB: -Wait, what are we doing here?

2412

2413 OA: I was gonna make that <points to  $1\frac{2}{5}$ > into a mixed number.

2414

2415 RB: This is a mixed number, one.

2416

2417 OA: Oh, oh-i thought it was twelve over five.

2418

2419 RB: Oh, no, no, no, no. One.

2420

2421 OA: One and two fifths?

2422

2423 RB: Uhum.

2424

2425 OA: So divided it into fifths, <OA moves cursor towards x-division slider which is  
 2426 already set at 5 and then moves x-axis marker from  $5/5$  to  $1\frac{2}{5}$ > one and two fifths  
 2427 right there.

2428

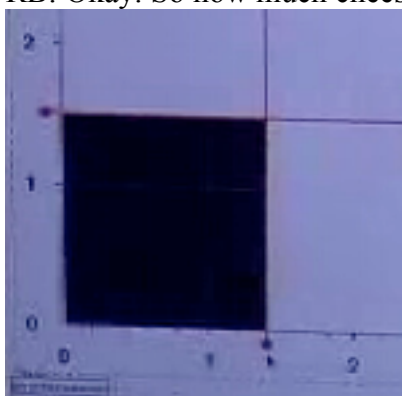
2429 RB: One and two fifths, so you are happy with that?

2430

2431 OA: Yeah.

2432

2433 RB: Okay. So how much cheese did we used then?



2434

2435 OA: Um...two slices.

2436

2437 RB: Two slices? Exactly two slices?

2438

2439 OA: Yeah.

2440

2441 RB: How do you know?

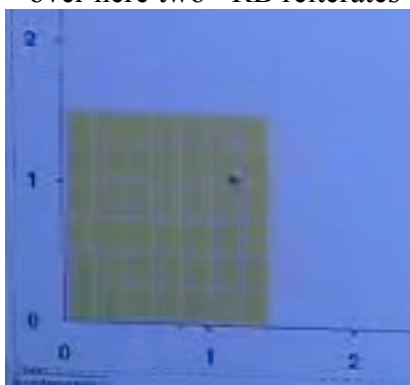
2442

2443 OA: Well, I put this one and it fits right there and there's two spaces here so I divide  
 2444 these two and put it right there <OA visualizes moving black shading to the right of  
 2445  $x=1$  into the top  $1 \times 1$  unit whole>.  
 2446

2447 RB: Wait you fit which one where?  
 2448

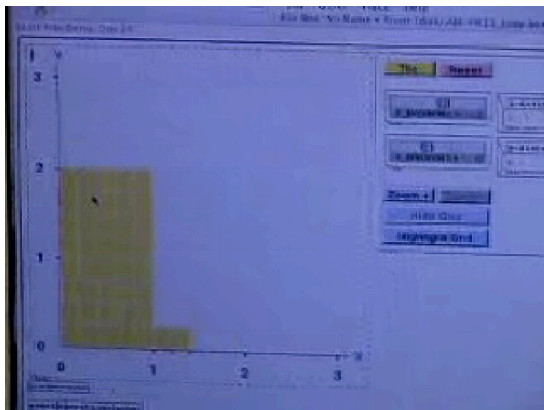
2449 OA: Put this little one right there <points to top right corner shaded area> and then two  
 2450 of these <points again to the top right corner shaded area> fit <points to what would be  
 2451 the remaining unshaded area in the top  $1 \times 1$  unit whole> so I divide this one <points to  
 2452 shaded area in the right most unit whole> and it make exactly that and I put it right  
 2453 there <points again to the remaining unshaded area in the top  $1 \times 1$  unit whole>.  
 2454

2455 RB: So wait, you put this one right there and you divide this by two and two of those fit  
 2456 over here two <RB reiterates what OA demonstrated above>. So let's see you tile.



2457

2458 OA: So I move them <OA moves around tile pieces>. There is two pieces left.



2459

2460 RB: There are two pieces left, so how much cheese did you give out?  
 2461

2462

2463 OA: So, two and two...two and two twentieths?  
 2464

2465

2466 RB: Two and two twentieths, how do you know that?  
 2467

2468

2469 OA: Cause there's two <OA points to the two tiled unit wholes> and then two <points to  
 2470 left over tile pieces> twentieths out of the thing <points to the  $1 \times 1$  unit whole in which  
 2471 the two left over tile pieces fall>.  
 2472

2470

2471 RB: Okay, so what is this piece called right here? <RB lifts up a single  $1/20$  tile pieces>.  
 2472

2471

2472 OA: That's a twentieth.

2473

2474 RB: That's a twentieth, of one slice.

2475

2476 OA: Yeah.

2477

2478 **Time: 1:33:10**

2479

2480 RB: Excellent, good. So now I'm gonna do something tricky. So this is gonna be called  
2481 number 9. It's one rat <RB talks to someone else>, one rat and then I want you to take  
2482 two thirds of three fourths of two slices per rat, my unit is changing. So remember how  
2483 I said my unit is one? So now, I'm gonna switch my unit to two. So now I want you to  
2484 do this using this

2485

2486 OA: Okay, so reset it?

2487

2488 RB: Uhum.

2489

2490 [tape ends]

2491