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Abstract

This paper proves that the vehicle trajectories predicted by (i) a simple linear car-following model, CF(L), (ii) the kinematic wave model with a triangular fundamental diagram, KW(T), and (iii) two cellular automata models CA(L) and CA(M) match everywhere to within a tolerance comparable with a single "jam spacing". Thus, CF(L) = KW(T) = CA(L,M).

1 The CF(L) model

One of the oldest and simplest car-following models [1] relates speed and spacing by a linear rule, with a time lag. If we use $x^n(t)$ and $v^n(t)$ for the position and speed of car n at time t, and number cars in the reverse direction of travel, as is customary, the aforementioned rule is:

$$v^n(t+\epsilon_r) = V(x^{n-1}(t) - x^n(t)), \tag{1}$$

where ϵ_r is a reaction time, and V is a (linear) function of the spacing $s^n(t) = x^{n-1}(t) - x^n(t)$ in front of vehicle n. Because speeds are bounded we truncate the linear relation and use:

$$V(s) = \min(\frac{s-\delta}{\tau},\ v_f), \qquad s \geq \delta \tag{2}$$

where v_f is the maximum (or "free-flow") speed, δ is the minimum possible (or "jam") spacing and τ is a sensitivity coefficient with units of time. In the truncated model, speed increases linearly with spacing for $s \leq \delta_o$, where $\delta_o = \delta + \tau v_f$ is the "optimum" spacing, and equals v_f for $s > \delta_o$.

The CF(L) model is a modification of the truncated model for discrete time. Drivers do not continuously track their position and speed. Instead, they sample their spacing at times $t_i = i\epsilon_s$ and adjust their acceleration for the interval (t_i, t_{i+1}) to achieve an average speed in this interval, $v^n_{i,i+1}$, consistent with rule (2); i.e.,

$$v^n_{i,i+1} = V(s^n_i). \tag{3}$$

It is further assumed that the sampling interval is $\epsilon_s = \tau$.

This last assumption endows the model with properties that are both realistic and mathematically useful for the purposes of this paper.[1] In particular, if $s^n_i \leq \delta_o$ then $v^n_{i,i+1} = (s^n_i - \delta)/\tau$ and the position of vehicle n at the end of the interval is: $x^n_{i+1} = x^n_i + \tau v^n_{i,i+1} = x^n_i + s^n_i - \delta = x^{n-1}_i - \delta$. If $s^n_i > \delta_o$, then $v^n_{i,i+1} = v_f < (s^n_i - \delta)/\tau$ and the position is $x^n_{i+1} < x^{n-1}_i - \delta$. In both cases, $x^n_{i+1} \leq x^{n-1}_i - \delta$. Since $x^{n-1}_i \leq x^{n-1}_{i+1}$, this guarantees that $x^n_{i+1} \leq x^{n-1}_{i+1} - \delta$; thus, the jam spacing is preserved at all times. Moreover, the two results for the final vehicle position can be neatly combined in a single formula:

$$x^n_{i+1} = \min\{x^n_i + \tau v_f, x^{n-1}_i - \delta\}. \tag{4}$$

[1] Realism is discussed in Sec. 4.

This is the discrete-time expression of Newell's "lower order" car-following model [2]. The formula implies that a vehicle always takes the most advanced position possible, limited by its free-flow speed and the position of the car in front. After a vehicle catches up with the car in front, the two trajectories become translationally symmetric.

Consider now a standard "lead-vehicle problem" (LVP), which will be our basis for comparing models. Given are: (i) the trajectory of a "lead vehicle", $x^0(t)$ for $t \geq 0$ with speeds in the feasible range $[0, v_f]$, and (ii) initial positions for all following cars $x^n(0)$ satisfying the minimum spacing requirement. We look for the trajectories of all the following cars. To find the CF(L) solution, apply (4) for $n = 1$ and all $i \geq 0$. Then iterate with increasing n to obtain the trajectories, x^n, of all vehicles.[2] We shall use $\mathcal{F}$ for the mapping implied by one of these iterations: $x^n = \mathcal{F}(x^{n-1}, x_0^n)$.

2 The KW(T) model

We show here that the CF(L) model and the kinematic wave model with a triangular fundamental diagram, KW(T), are identical if the parameters of the latter are chosen to be consistent with the equilibrium values of the former. That is, if the jam density is $\kappa = 1/\delta$, the free-flow speed is v_f, and the wave-speed in the congested regime (the slope of the declining branch of the fundamental diagram) is $w = -\delta/\tau$. Some important features of the KW(T) model are now introduced. For additional information see [3, 4, 5, 6].

Consider an LVP. Kinematic wave theory looks for a continuous function, $N(t, x)$, giving the vehicle number at every point in continuous space-time. The streamlines of $N(t, x)$ (curves where $N(t, x) = n$ for fixed n) are the vehicle trajectories. In a well-posed LVP the lead-vehicle trajectory is given by specifying a curve $N(t, x) = 0$ with speeds in the range $[0, v_f]$. The initial vehicle positions are given by specifying a Lipschitz-continuous and non-increasing function, $N(0, x)$, such that $N(0, x) - N(0, x + \Delta x) \leq \delta \Delta x$ for all x and Δx. (The initial positions $x^n(0)$ are the values of x where $N(0, x)$ is an integer.)

Kinematic wave theory contains recipes to obtain the n^{th} vehicle trajectory in terms of the $n - 1^{st}$. These recipes show that if $N(0, x)$ is linear between every pair of consecutive initial

[2] In this paper when an index or a variable is omitted from a symbol, it denotes a vector.

vehicle positions then:

$$x^n(t) = x^n(0) + tV(x^{n-1}(0) - x^n(0)) \qquad \text{if} \qquad t \le \tau, \tag{5}$$

$$x^n(t) = \min\{x^n(0) + tv_f, x^{n-1}(t-\tau) - \delta\} \qquad \text{if} \qquad t \ge \tau. \tag{6}$$

These equations always define a continuous curve, as the reader can easily verify. We shall use $\mathcal{K}$ to abbreviate the mapping based on (5) and (6) that gives the trajectory of vehicle n in terms of its initial position and the trajectory of $n-1$: $x^n = \mathcal{K}(x^{n-1}, x_0^n)$. It is easy to see from (5-6) that $\mathcal{K}$ is a contraction mapping with respect to the l_∞ (maximum absolute difference) norm, $\|\cdot\|$; i.e.,

Contraction property of $\mathcal{K}$: If for some $\epsilon > 0$, $\|x^{n-1} - \overline{x}^{n-1}\| \le \epsilon$ and $|x_0^n - \overline{x}_0^n| \le \epsilon$, then $\|x^n - \overline{x}^n\| \le \epsilon$. □

This in turn implies by iteration that the complete vector of KW(T) trajectories, x, also satisfies a similar property with respect to a problem's data,

Contraction property of KW: If for some $\epsilon > 0$, $\|x^0 - \overline{x}^0\| \le \epsilon$ and $\|x_0 - \overline{x}_0\| \le \epsilon$, then $\|x - \overline{x}\| \le \epsilon$. □

Finally note that if we are only interested in the position of the vehicles at the lattice times $\{t_i = \tau i\}$, Eq.(5) is unnecessary and recursion (5-6) becomes:

$$x_{i+1}^n = \min\{x_0^n + (i+1)\tau v_f, x_i^{n-1} - \delta\}. \tag{7}$$

Obviously, this discrete version of the problem also exhibits the contraction properties.

We now prove that KW(T) = CF(L). We shall use x and $\underline{x}$ to identify the vehicle positions obtained with (4) and (7), respectively.

Equivalence Theorem: If $\{x_i^0 = \underline{x}_i^0; \forall i \ge 0\}$ and $\{x_0^n = \underline{x}_0^n; \forall n \ge 0\}$, then $\{x_i^n = \underline{x}_i^n; \forall n, i \ge 0\}$.

Proof: Consider $n = 1$ and note from (4) that if the second term is a minimum for some i, then it is a minimum from then on. This happens because the second term can never increase in one jump by more than τv_f - since the speeds of the lead vehicle are assumed to be in the range $[0, v_f]$. Thus, (4) predicts that vehicle

1 will advance by τv_f units for as many consecutive intervals as possible until the second term of (4) becomes a minimum; i.e., that vehicle $n = 1$ moves to position $x^n_{i+1} = x^n_0 + (i+1)\tau v_f$ after the $(i+1)$ jump if and only if $x^n_0 + (i+1)\tau v_f \leq x^{n-1}_i - \delta$. Otherwise it advances to position $x^{n-1}_i - \delta$. This, of course, is a restatement of (7). Thus, the theorem is true for $n = 1$.

Note from (4) and (7) that the output speeds produced by $\mathcal{K}$ or $\mathcal{F}$ are in the same range, $[0, v_f]$, as the input speeds. Thus, the argument in the preceding paragraph can be successfully iterated for $n = 2, 3, ...$ Evidently the trajectories must match for arbitrary n. $\square$

3 The CA(L,M) models

A strong connection between between fluid and cellular automata (CA) models of traffic flow has been suspected for some time; see e.g., the links established in [7] and [8]. We add to this body of knowledge by presenting two CA models that match the CF(L) and KW(T) vehicle trajectories precisely everywhere with a tolerance comparable with the jam spacing. The first model - a variant of the original in [9] - achieves the match only for certain parameter combinations. The second model achieves it for all conditions. The correspondence between the parameters of the CA and KW(T) models is established.

In this section space is discretized in increments δ, so that the j^{th} lattice line is at location $X_j = j\delta$. We shall also work with dimensionless distance, $z = x/\delta$, because it conveniently expresses the lattice number: $z_j = j$.

3.1 The CA(L) model

Define now the dimensionless quantity, $\tau v_f/\delta = \omega$, and express (4) in dimensionless form:

$$z^n_{i+1} = \min\{z^n_i + \omega, z^{n-1}_i - 1\} \tag{8}$$

It is found experimentally time after time that $\omega \approx 7$. If ω is a positive integer, (8) is a relationship among integers. Thus, if the data of a dimensionless LVP $\{z^0_i, z^n_0\}$ are integer, its CF(L) solution $\{z^n_i\}$ is also integer. LVPs with integer data will be called “ILVPs”.

Associated with every LVP there is an ILVP obtained by rounding up every data point to the next integer value. We shall use an overbar to denote the data of an associated ILVP; thus,

$\{\overline{z}_i^0, \overline{z}_0^n\} = \{\lceil z_i^0 \rceil, \lceil z_0^n \rceil\}$. The CA(L) model ("cellular automata (linear)") is the application of the CF(L) rule (8) to the associated ILVP of an LVP. A word description of (8) is as follows:

> **The CA(L) rule:** Jump to the most forward point possible without reaching the old position of the followed car or jumping more than ω intervals. $\square$

By construction, CA(L) = CF(L) = KW(T) for the associated ILVP of any LVP. Thus, we write $\overline{z} \doteq \{\overline{z}_i^n\}$ for all three ILVP solutions. We know, however, that the CF(L) and KW(T) solutions of the LVP are the same: $z \doteq \{z_i^n\}$. Since, by construction, all the data points of the LVP and ILVP differ by less than 1 unit, the contraction property of KW(T) guarantees that $\|\overline{z} - z\| \leq 1$.

This is the desired result. It shows that if the position of every vehicle in the initial data is advanced to the nearest lattice point, then the CA(L) model predicts the same vehicular positions as the KW(T) and CF(L) models to within one jam spacing; i.e., to within the degree of accuracy allowed by the CA granularity. The CA(L) model requires integer ω, however.

3.2 The CA(M) model

We now present an alternative that can be readily modified for non-integer ω, and match under this condition the CF(L) and KW(T) predictions with a tolerance comparable with the jam spacing.

3.2.1 Definition and performance for integer ω:

We still discretize space in jumps of δ but use a finer time-lattice: $T_k = k\delta/v_f$. Assume for now that $\omega \geq 1$ is integer. Then, the old time instants $\{t_i\}$ are a subset of the new $\{T_k\}$, such that $t_i = T_{k\omega}$. The intervals of the new lattice are ω times smaller than those of the old.

We shall work with dimensionless vehicular positions on the fine lattice: $Z_k^n = x^n(T_k)/\delta$. And, again, for any LVP we define an associated ILVP by rounding up its fine-lattice data to the nearest integer: $\{\overline{Z}^0, \overline{Z}_k\} = \{\lceil Z^0 \rceil, \lceil Z_k \rceil\}$.

In terms of the new dimensionless positions, the KW(T) predictions (5) and (6) are:

$$Z_k^n = Z_0^n + kJ(Z_0^{n-1} - Z_0^n) \qquad \text{if} \qquad k \leq \omega, \qquad \text{and} \tag{9}$$

$$Z_k^n = \min\{Z_0^n + k, Z_{k-\omega}^{n-1} - 1\} \qquad \text{if} \qquad k > \omega, \tag{10}$$

where J is the version of (2) that returns the dimensionless speed ($J = v/v_f \in [0, 1]$) in terms of the dimensionless spacing $S = s/\delta$: $J(S) = \min\{1, (S-1)/\omega\}$ for $S \geq 1$. These relationships are next put into CA form.

Since (9) is not a relationship among integers, we replace it in the CA formulation by:

$$\overline{Z}_k^n = \overline{Z}_0^n + \lfloor kJ(\overline{Z}_0^{n-1} - \overline{Z}_0^n) \rfloor \qquad \text{if} \qquad k \leq \omega, \tag{11}$$

which produces integer outputs for integer inputs. The CA(M) ("cellular automata (memory)") model is the application of (11-10) to an associated ILVP.

Recall that (5) and therefore (9) arose by assuming that the initial density in each vehicular spacing was constant and equal to the reciprocal of the spacing - but this assumption was made for convenience only. Simple KW theory considerations show that a modified distribution always exists for which the KW solution of the ILVP obeys (11-10).[3] Thus, the output of these two equations, $\overline{Z} \doteq \{\overline{Z}_k^n\}$, is a KW(T) solution of the ILVP on the fine lattice.

Since the maximum absolute difference between the LVP and ILVP data is inferior to 1 unit, the contraction property of the KW(T) model again guarantees that the discrepancy between the CA(M) and KW(T)-CF(L) predictions on the fine lattice is always less than 1 jam spacing.

3.2.2 Simplified expression of the CA(M) rule:

The Equivalence Theorem showed that (7) could be replaced by an expression (4) that did not involve the initial conditions. Since (10) is just a dimensionless version of (7) a similar simplification is now possible. The new expression is:

$$Z_{k+1}^n = \min\{Z_k^n + 1, Z_{k-\omega+1}^{n-1} - 1\} \qquad \text{if} \qquad k > \omega. \tag{12}$$

Equation (12) is slightly different from the dimensionless version of (4) because it pertains to a finer lattice, but the logic behind it is the same.[4] Equations (11) and (12) are an alternative expression of the CA(M) rule.

A further simplification is possible. Since dimensionless speeds are in the range [0,1], the Z_k^n are non-decreasing functions of k. Therefore, if the second argument in the right side of (12) is a strict minimum then $Z_{k+1}^n = Z_k^n$; the vehicle does not move. Otherwise it jumps one position. Thus, with the CA(M) rule vehicles make the following simple binary choices:

[3] This distribution only involves "jam", "optimum" and zero densities

[4] In this case too, the first argument of (12) always increases with each jump at least as much as the second, and therefore the prediction of (12) for Z_{k+1}^n must be either $Z_0^n + (k+1)$ or $Z_{k-\omega}^{n-1} - 1$; i.e., the same as (10).

The CA(M) rule: After initialization with (11) for ω periods, vehicles jump to the next lattice position if this position has been vacant for $\omega - 1$ intervals. Otherwise, they stay where they are. $\square$

3.2.3 Modification and performance for arbitrary ω:

We consider here LVPs with driver-specific lags, τ^n, and restrict our attention to problems with super-optimal initial spacings:

$$x^{n+1}(0) \leq x^n(0) - \tau^{n+1} v_f - \delta. \tag{13}$$

We also pretend that the system has been in a steady state before $t = 0$, so that: $x^n(t) = x^n(0) + tv_f$ for $t < 0$. These stricter assumptions are reasonable since in practice most systems are simulated starting with an empty (or lightly congested) system. For this special case of the LVP it is pointed out in [2] that:[5]

$$x^n(t) = \min\{x^n(0) + tv_f, x^0(t - \Gamma_n) - n\delta\} \qquad \text{where} \qquad \Gamma_n = \sum_{m=1}^{n} \tau^m. \tag{14}$$

Result (14) allows us to bound the discrepancies between LVP problems that only differ in the driver populations. If two populations have cumulative lags Γ^n and $\widehat{\Gamma}^n$, (14) implies:

$$|\widehat{x}^n(t) - x^n(t)| \leq v_f |\Gamma^n - \widehat{\Gamma}^n|, \quad \forall t. \tag{15}$$

This shows that a driver population with any given lag behaves similarly to a heterogeneous population with variable lags, but similar cumulative values. We now use this idea.

If $\omega \geq 1$ is not an integer, approximate the problem by one where drivers are inhomogeneous with integer $\widehat{\omega}_n$. Choose these values according to the rule: $\sum_{m=1}^{n} \widehat{\omega}_m = \lceil n\omega - \frac{1}{2} \rceil$ and solve the ILVP associated with an LVP with the (heterogeneous) CA(M) rule.

We have seen that the output of the CA(M) rule for a fixed n matches that produced by the heterogeneous KW(T) model on the fine lattice if the input data are integer. Thus, a perfect

[5] A proof is by induction. For this problem (5)-(6) can be replaced by (6) with unrestricted t. Equation (14) is obviously true for $n = 1$. Thus, it suffices to show that it is true for vehicle $n + 1$ if it is true for n. If it is true for n, we can insert (14) into the instance of (6) for vehicle $n + 1$ to find its trajectory. The result is: $x^{n+1}(t) = \min\{x^{n+1}(0) + tv_f,\ x^n(0) + tv_f - \tau^{n+1} v_f - \delta,\ x^0(t - \Gamma_n - \tau^{n+1}) - (n+1)\delta\}$. Inequality (13) guarantees that the middle argument of the minimum function can never be smaller than the first. Thus, the result for trajectory $x^{n+1}(t)$ is as predicted by (14).

match is obtained for $n = 1$. Since the CA(M) rule produces integer trajectories, the perfect match continues for $n = 2, 3, \ldots$ Hence, we can use a single vector, $\widehat{\overline{x}}$, to describe both ILVP solutions on the fine lattice. (The "hat" stands for "inhomogeneous" and the "overbar" for integer.)

We can now use (15) to compare the inhomogeneous and homogeneous KW(T) solutions of the ILVP. By construction, $|\Gamma^n - \widehat{\Gamma}^n| \leq \frac{1}{2}\tau$. Thus, an upper bound to $\|\widehat{\overline{x}} - \overline{x}\|$ on the fine lattice is $\frac{1}{2}\tau v_f = \frac{1}{2}\omega\delta$. We have already seen that δ is an upper bound to $\|\overline{x} - x\|$, where x now denotes the homogeneous KW(T) solution vector on the fine lattice. Thus, $\|\widehat{\overline{x}} - x\| \leq (\frac{1}{2}\omega + 1)\delta$, showing that if $\omega \geq 1$ is allowed to be real, then the error in vehicle position achieved by the CA(M) method is still bounded by a quantity comparable with the jam spacing.

In practical applications, where perfectly matching the deterministic KW(T) model is not necessary, it may be easier and just as realistic to randomize the memory parameter, ω, across drivers. This, of course, is equivalent to randomizing τ in the KW(T) and CF(L) models.

4 Discussion

Parallel derivations show that the results of this paper also apply to the "finite highway problem" - where data include the flow of traffic wishing to enter at the upstream end. The CA(M) model is particularly useful because its law of motion involves only one cell at a time. This facilitates the formulation of meaningful boundary conditions for merges, diverges and lane-changes; and the integration of CA methods into "next-generation" hybrid models of traffic flow.

Experimental evidence suggests that parsimonious multi-lane hybrid models can achieve an unprecedented level of realism [10]. These models treat traffic on individual lanes as parallel KW(T) streams intermittently interrupted by lane changing vehicles with bounded acceleration. These vehicles are then treated as lead-vehicles in an LVP. The models have a demand component that triggers the lane changes, and a "constrained-motion model" in continuous time that generates the lead-vehicle trajectories. The models are easy to estimate; they only involve three KW(T) parameters and one parameter for the demand model.

The same "hybrid" idea - combining a detailed model in continuous time and continuous time for the lane changes, with a coarser model for the individual lane streams - can also be applied if one replaces the KW(T) module of the model by a CA(L,M) module. The results of this paper suggest that this hybrid model would yield similar results as [10], and this could be advantageous in some application contexts.

Unlike many car-following models, the CF(L) model keeps all shock fronts sharp. This is in agreement with numerous experiments, as explained in [2]. The CF(L) model, however, assumes that drivers are very precise (e.g., with $\tau = \epsilon_s$) and act synchronously. Since this is not possible, but the model fits reality well, one would expect other forms of driving to produce similar macroscopic results. Analysis and numerical experiments with very long lines of cars [11] show that both, smooth control rules for throttle control, and very crude (but realistic) "bang-bang" rules, generate vehicle trajectories similar to those of the CF(L) model.

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