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Highly structured orientations from equivariant Thom spectra

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Highly structured orientations from equivariant Thom spectra

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by

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ABSTRACT OF THE DISSERTATION

Highly structured orientations of equivariant Thom spectra

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Doctor of Philosophy in Mathematics

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Professor Michael Anthony Hill, Chair

We report significant progress toward establishing an obstruction theory of equivariant Thom spectra with multiplicative structures arising from maps of E_V spaces. Focusing on the Fujii-Landweber Real bordism spectrum, we explain an argument to show that a homotopy ring Real orientation of an E_ρ-spectrum satisfying strong equivariant evenness and mild additional commutativity condition lifts to an E_ρ map.

First, we address the foundational issues. We discuss model categories for equivariant C_2-spectra indexed on Real inner product spaces and comparisons among them. We explain which foundations are needed for our project and describe the parts that would be original even non-equivariantly.

Next, we construct equivalences between several E_V operads, including little disks and Steiner operads. We show that algebras over $E_{V\oplus\mathbb{R}}$ operads can be strictified to monoids in the category of E_V-algebras.

To compute obstruction groups for maps of E_V algebras, we analyze the spectra of the derived indecomposibles of augmented E_V-algebras. We determine that such spectra are V-fold desuspensions of a lift of the May delooping machine to the augmented algebras.

Next, we discuss the induced map on cohomology for a map between spaces whose $RO(C_2)$-graded cohomology carries obstruction classes for homotopy ring and E_ρ ring maps out of the Real bordism spectrum.

Finally, we review the strategies for the proofs of the goal results of our project, explaining how to use the technique of climbing the slice tower of the target to construct highly structured orientations out of Real bordism.

The dissertation of Bar Roytman is approved.

Daniel Dugger

Sucharit Sarkar

Burt Totaro

Michael Anthony Hill, Committee Chair

University of California, Los Angeles

2023

To Elisa, who kept me calm, and my mom, who kept me comfortable

Contents

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Vita

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Chapter 1

Introduction

The Real bordism spectrum $MU_{\mathbb{R}}$ has an essential role in equivariant approaches to chromatic homotopy theory. Namely, Real bordism is the ring spectrum that carries the universal formal group law of a formal group law with the genuine action of the cyclic group C_2 realizing the inverse of the formal group law. Real bordism was first introduced by Landweber in [45] as the bordism theory of manifolds with C_2 actions and Real vector bundles, i.e., complex vector bundles with antilinear C_2-actions. Development of the corresponding Real K-theory of Atiyah only predated this by a couple years [4] with the basics of the cohomological approach established by Fujii [23], [24] and later Araki [3].

However, the collaborations of Hu–Kriz [34] and Hill–Hopkins–Ravenel [30] that have convincingly demonstrated the computational tractability and importance of Real bordism in equivariant chromatic homotopy theory, with the resolution of most cases of the Kervaire invariant problem. Subsequent inspirational computations performed in [31], [28], [9] show that Real orientations make much of the structure of the Real Landweber exact spectra tractable.

While Real orientations, in the sense we use here, are morphisms of homotopy ring genuine C_2-spectra out of $MU_{\mathbb{R}}$, one desirable tool of equivariant homotopy theory has not been available for many of these computations. The inputs of ring operations are parameterized by sets of the form $C_2 \times {\Sigma_n}^{\mathrm{op}}/C_2$, while $MU_{\mathbb{R}}$ and many interesting spectra have GE_∞ or similar structures that have operations with inputs of the form $C_2 \times {\Sigma_n}^{\mathrm{op}}/\Gamma$ for other graph subgroups Γ. See [10] for elaboration on this insight. Therefore, our computational toolbox misses certain connected components, elements of equivariant $\underline{\pi}_0$, of spaces of potential multiplicative structure for morphisms of highly structured ring spectra. The purpose of our project is to rectify this situation.

Circumstantial evidence that this can be done satisfactorily can be found in the work of Chadwick–Mandell, which proves that complex orientations of even E_2-ring spectra with commutative π_0 can always be lifted to the category of E_2-ring spectra [20]. Indeed, the analogue of E_2-ring spectra in C_2-equivariant homotopy theory are the E_ρ ring spectra, with ρ the regular C_2-representation and E_ρ ring spectra admit multiplications of every kind, lacking only in forms of equivariant homotopy coherent commutativity.

This dissertation focuses on the mathematically interesting aspects of the following result. Some of the more lengthy and routine details of the proof will be excluded due to constrained time; however, aspects of the proof with new ideas will be presented in more detail, with the remainder summarized.

Anticipated theorem 1.1. *Let R be an E_ρ ring spectrum $\underline{\pi}_{n\rho-1}R$ are trivial for odd $n > 0$, and $\underline{\pi}_{n\rho}R$ are constant Mackey functors for $n \geq 0$. Suppose also that $\underline{\pi}_0(R)$ is a commutative Green functor. Then, any homotopy class of homotopy ring maps $MU_{\mathbb{R}} \to R$ admits a lift to a homotopy class of E_ρ-ring maps. In particular, the Real Quillen idempotent is an E_ρ map and $BP_{\mathbb{R}}$ admits a construction as an E_ρ algebra.*

The strategy of the proof is largely the same as the analogous one of [20] but following their outline requires equivariant generalizations of the main computation Chadwick–Mandell used and the non-equivariant theory they cited. First, because the task is to prove that there are no obstructions to lifting ring morphisms, an appropriate carrier of obstruction classes is required. Basterra–Blumberg–Hill–Lawson–Mandell have already developed this [6].

We begin with a review of basic notions, including operads, in Chapter 2. Chapter 3 is a discussion of the model categorical foundations of our work and why they matter. These foundations are lengthy, incompletely written, and only definitions are included here.

In Chapter 4, we prove an equivariant generalization of [17, C] which shows that there are homopically well-behaved E_{V+1}-operads of the form $\mathscr{O} \otimes_{\mathsf{BV}} \mathcal{A}ss$ where $\mathscr{O}$ is an E_V-operad and $\mathcal{A}ss$ is the associative operad. In other words, a strictly associative operation can be identified. There are essentially no conditions on the group on which V acts, and the method of proof is entirely new and geometric, with the idea of using the radius parameter of an embedded little cylinder to serve a secondary purpose in determining an ideal projection with the non-disk coordinate. This strictification result is necessary for a convenient theory of Thom spectra as right modules and is a key ingredient of an E_V-Thom isomorphism theorem generalizing [20, 2.1]. Along the way, we will also correct an error from [26] in the comparison of little disk and Steiner operads.

In Chapter , we relate the equivariant delooping machine of May to the spectra of the derived indecomposibles of E_V augmented spectra. This applies well to algebras of the form $\Sigma^\infty_+ X$.

The computational backbone of our work is an equivariant lifting of the one in [20, 7.3] and is our Theorem 6.1, to which Chapter 6 is dedicated. This chapter describes the effect of the canonical map $\Sigma^\rho BU_{\mathbb{R}}(1) \to B^\rho BU_{\mathbb{R}}$ from the ρ-suspension of the space of units required to orient all Real bundles in a homotopy ring genuine C_2-spectrum to the space of units required to orient all Real bundles with an E_ρ-spectrum. The cofiber being even is what ultimately causes the vanishing of the obstruction groups.

We close with Chapter 7 sketching a proof of Anticipated theorem 7.2.

Chapter 2

Spaces and Operads

We begin with a review of basic notions required to study algebraic structures in equivariant homotopy theory. The notions of operads are revisited in later chapters.

2.1 Spaces

Let us disambiguate some basic terminology. Strong familiarity with these notions is assumed of the reader.

Notation 2.1. A *space* refers to a compactly generated weak Hausdorff topological space. The category of spaces and continuous maps is denoted $\mathscr{U}$. A *subspace* Y of a space X is a space with underlying set a subset of the underlying set of X and whose topology is final among compactly generated topologies at least as fine as the subspace topology. A *based space* refers to a space equipped with a point known as the *basepoint*. A *based map* between based spaces is a continuous map sending the basepoint of the source to the basepoint of the target. The category of based spaces and based maps is denoted $\mathscr{T}$.

We do, of course, need corresponding notions for equivariant homotopy theory.

Notation 2.2. A *topological group* refers to a group object in $\mathscr{U}$. If G is a topological group, an object in $\mathscr{U}$ with a continuous left G-action is called a *G-space* and a *G-map* between

G-spaces is a continuous map of the underlying spaces that commutes with the G-action maps. The category of G-spaces and G-maps is denoted $G\mathscr{U}$. A *based G-space* is a G-space equipped with a fixed point of the G action called the *basepoint* and a *based G-map* between based G-spaces is a continuous map that sends the basepoint of the source to the basepoint of the target. The category of based G-spaces and based G-maps is denoted $G\mathscr{T}$. For C_2-spaces X, we write X_+ to denote the based C_2-space that is the coproduct of X and a singleton basepoint as a space.

Note that the singleton space has a unique topological group structure and action on every space, which is compatible with every continuous map. Therefore, the equivariant notions can be regarded as a true generalization of the non-equivariant notions. We are usually concerned with topological groups G with finitely many points, which can be identified with finite groups and especially the cyclic group C_2 of order 2. We often write as though G is fixed, even when a statement applies to every topological group G.

Notation 2.3. The category $G\mathscr{U}$ is cartesian closed. The category $G\mathscr{T}$ has a closed symmetric monoidal product denoted $\wedge$ called the *smash product*. The smash product $X \wedge Y$ is the topological quotient space $X \times Y / X \vee Y$ where $X \vee Y$ is the coproduct of X and Y in $G\mathscr{T}$ with the equivalence class $X \vee Y$ serving as the basepoint. Other points in $X \wedge Y$ will be referred to as (x, y), where $x \in X$, $y \in Y$, and neither x nor y are basepoints of X or Y, respectively. We will use similar notation in other situations where a point in a space is a singleton equivalence class. Therefore G-spaces are the objects of a canonical $G\mathscr{U}$-enriched category $\mathscr{U}_G$ with morphism spaces $\mathscr{U}_G(X, Y)$ obtained as the right adjoint of $X \times (-)$ applied to Y. The $G\mathscr{T}$ enriched category of based G-spaces $\mathscr{T}_G$ is defined similarly.

As usual for equivariant homotopy theory, we make extensive use of representation spheres.

Notation 2.4. If V is a G-space then S^V is the one-point compactification of V. with the basepoint taken to be the point at infinity. We will only use the notation S^V in cases where V is an orthogonal G-representation with real scalars after possibly forgetting complex structure. If V and W are both real orthogonal G-representations or both complex vector spaces with real form arising as the fixed point subspace of a C_2-action, then $W - V$ is the orthogonal complement of V in W with the induced G or C_2-action. We write $\Omega^V X$ for $\mathscr{T}_G(S^V, X)$ and call it the *V-fold loop space* of X. We write $\Sigma^V X$ for $S^V \wedge X$ and call it the *V-fold suspension* of X.

2.2 Operads

Operads are essentially parameterized operations and axioms for algebraic structures in which one cannot refer to the same variable twice in an axiom internal to a symmetric monoidal category. Much of the theory relies on a common extra condition for the symmetric monoidal category.

Definition 2.5. A *cocomplete symmetric monoidal category* is a symmetric monoidal category that is cocomplete with a monoidal product $\otimes$ such that for every object X, the endofunctor $X \otimes (-)$ preserves colimits. An enriched category is said to be a cocomplete symmetric monoidal category if its underlying category is cocomplete symmetric monoidal.

We will also occasionally need to weaken the unit axiom for symmetric monoidal categories.

Definition 2.6. A *weak symmetric monoidal category* is a category $\mathscr{C}$ bifunctor $\otimes$, a weak unit U, associators, unitors, and braiding satisfying the standard axioms of [52, XI.1] (with the weak unit as the unit), except for the axiom demanding that the unitor $\lambda_X : U \otimes X \to X$ is an isomorphism. A *cocomplete weak symmetric monoidal category* is a weak symmetric monoidal category that is cocomplete (in an unenriched sense), and each endofunctor $X \otimes (-)$ for objects X preserves colimits.

For use as mapping spaces, we often need the monoidal product to admit a right adjoint.

Definition 2.7. A symmetric monoidal category or weak symmetric monoidal category is *closed* if for every object X, the endofunctor $X \otimes (-)$ has a right adjoint.

Our notion of an enriched cocomplete symmetric monoidal category is weak and not standard but is sufficient for our purposes. Operads are carriers of the axioms of the algebraic structures of interest throughout this study. Raw collections of operations without composition data are stored in collections.

Definition 2.8. A *collection* C in a category $\mathscr{C}$ is a collection of objects $C(n)$ of $\mathscr{C}$ indexed over $n \in \mathbb{N}$ together with an action of the opposite group of the symmetric group on n elements ${\Sigma_n}^{\mathrm{op}}$ on $C(n)$. A *morphism* $f : C \to D$ *of collections* of $\mathscr{C}$ consists of a sequence indexed over $n \in \mathbb{N}$ of morphisms $f_n : C(n) \to D(n)$ of ${\Sigma_n}^{\mathrm{op}}$-objects. The category of collections in $\mathscr{C}$ is denoted $\mathsf{Coll}(\mathscr{C})$.

Examples of collections abound as underlying collections of operads but for the time being, we have the following fundamental example.

Example 2.9. *The (weak) unit collection U_{Coll} in a (weak) symmetric monoidal category $\mathscr{C}$ is the collection with $U_{\mathsf{Coll}}(1) = U$, U being the unit of $\mathscr{C}$ with the unique action by the trivial group and $U_{\mathsf{Coll}}(n) = \varnothing$ where $\varnothing$ is the initial object of $\mathscr{C}$ and has its unique action by ${\Sigma_n}^{\mathrm{op}}$.*

The elements of collections of free operads on free Σ_n sets of generators in the category of sets are rooted planar trees with vertices labeled by the generators and a composition relation determined by tree grafting. Considerations of symmetric group actions on the sets of rooted planar trees lead one to consider maps such as the following.

Definition 2.10. Let $\mathbf{k} = (k_i)_{i=1}^n$ be a list of elements of $\mathbb{N}$ of length n. The map $\Gamma = \Gamma_{\mathbf{k}} : \Sigma_n \times \prod_{i=1}^k \Sigma_{k_i} \to \Sigma_{\sum_{i=1}^n k_i}$ is the composite of the top row of the following diagram.

$$\begin{tikzcd}
\{1,\dots,k\} \arrow[r, "\bullet^{-1}"] & \coprod_{i=1}^n \{1,\dots,k_i\} \arrow[r] & \coprod_{i=1}^n \{1,\dots,k_{\sigma^{-1}(i)}\} \arrow[r, "\bullet"] & \{1,\dots,k\} \\
& \{1,\dots,k_j\} \arrow[u, "\iota_j"] \arrow[r, "\tau_j"'] & \{1,\dots,k_j\} \arrow[u, "\iota_{\sigma(j)}"'] &
\end{tikzcd}$$

The morphism marked $\bullet$ is a bijection of totally ordered finite sets, where the source is a coproduct of totally ordered finite sets with the concatenation ordering and $\bullet^{-1}$ is the inverse of a similarly defined bijection. The central morphism is produced via the universal property of coproducts and is determined by the square in the diagram commuting for all choices of j.

This definition of an operad is monochromatic, describing algebraic structures on a single object, and describes algebraic operations algebraically. The following definition is standard and abstracted from the original definition in $\mathscr{U}$ given in [57] with minor changes in the presentation of equivariance axioms with one major difference. The original condition has a condition on $\mathscr{O}(0)$ that we do not demand.

Definition 2.11. An *operad* in a weak symmetric monoidal category $\mathscr{C}$ is a collection $\mathscr{O}$, equipped with an *identity* map $1_{\mathscr{O}} : U \to \mathscr{O}(1)$, and *composition* maps

$$\gamma_{\mathbf{k}} : \mathscr{O}(n) \otimes \bigotimes_{i=1}^{n} \mathscr{O}(k_i) \to \mathscr{O}(\sum_{i=1}^{n} k_i)$$

for every $\mathbf{k} = (k_i)_{i=1}^{n} \in \coprod_{N \in \mathbb{N}} \mathbb{N}^n$ satisfying the following axioms.

1. (Left unit) For every $n \in \mathbb{N}$, the following diagram commutes.

$$\begin{tikzcd} U \otimes \mathscr{O}(n) \arrow[r, "1_{\mathscr{O}} \otimes \mathsf{id}"] \arrow[dr, "\lambda"'] & \mathscr{O}(1) \otimes \mathscr{O}(n) \arrow[d, "\gamma_{(n)_{i=1}^{1}}"] \\ & \mathscr{O}(n) \end{tikzcd}$$

2. (Right unit) For every $n \in \mathbb{N}$, the following diagram commutes.

$$\begin{tikzcd} \mathscr{O}(n) \otimes U^{\otimes n} \arrow[r, "\mathsf{id} \otimes 1_{\mathscr{O}}^{\otimes n}"] \arrow[dr, "\rho"'] & \mathscr{O}(n) \otimes \mathscr{O}(1)^{\otimes n} \arrow[d, "\gamma_{(1)_{i=1}^{n}}"] \\ & \mathscr{O}(n) \end{tikzcd}$$

3. (Associativity) For every $\underline{\mathbf{k}} = (\mathbf{k}_i)_{i=1}^m \in \coprod_{M\in\mathbb{N}}(\coprod_{N\in\mathbb{N}} \mathbb{N})^n$ where $\mathbf{k}_i = (k_{i,j})_{j=1}^{n_i}$, the following diagram commutes.

$$\begin{tikzcd}
\mathscr{O}(m)\otimes\bigotimes_{i=1}^m \mathscr{O}(n_i)\otimes\bigotimes_{i=1}^m\bigotimes_{j=1}^{n_i}\mathscr{O}(k_{i,j}) \arrow[r, "\sim"] \arrow[dd, "\gamma\otimes\mathsf{id}"] & \mathscr{O}(m)\otimes\bigotimes_{i=1}^m(\mathscr{O}(n_i)\otimes\bigotimes_{j=1}^{n_i}\mathscr{O}(k_{i,j})) \arrow[d, "\mathsf{id}\otimes\bigotimes\gamma"] \\
 & \mathscr{O}(m)\otimes\bigotimes_{i=1}^m\mathscr{O}(\sum_{j=1}^{n_i}k_{i,j}) \arrow[d, "\gamma"] \\
\mathscr{O}(\sum_{i=1}^m n_i)\otimes\bigotimes_{i=1}^m\bigotimes_{j=1}^{n_i}\mathscr{O}(k_{i,j}) \arrow[r, "\gamma\circ(\mathsf{id}\otimes\alpha)"] & \mathscr{O}(\sum_{i=1}^m\sum_{j=1}^{n_i}k_{i,j})
\end{tikzcd}$$

The top horizontal arrow of the diagram is a rearrangement of factors formed from the symmetric monoidal structure.

4. (Equivariance) If $[\sigma]$ denotes a morphism that permutes the n factors of a product in $\mathscr{C}$, the following diagram commutes.

$$\begin{tikzcd}
\mathscr{O}(n)\otimes\bigotimes_{i=1}^n\mathscr{O}(k_i) \arrow[r, "\sigma^{\mathrm{op}}\otimes\bigotimes\tau_i{}^{\mathrm{op}}"] \arrow[d, "\mathsf{id}\otimes[\sigma]"] & \mathscr{O}(n)\otimes\bigotimes_{i=1}^n\mathscr{O}(k_i) \arrow[dd, "\gamma"] \\
\mathscr{O}(n)\otimes\bigotimes_{i=1}^n\mathscr{O}(k_{\sigma^{-1}(i)}) \arrow[d, "\gamma"] & \\
\mathscr{O}(k) \arrow[r, "\Gamma(\sigma;(\tau_i)_{i=1}^n)^{\mathrm{op}}"'] & \mathscr{O}(k)
\end{tikzcd}$$

A *morphism of operads* is a morphism of collections such that the diagrams induced by it and the operad structure maps are commutative. The category of operads in a symmetric monoidal category $\mathscr{C}$ is denoted $\mathsf{Op}(\mathscr{C})$

Example 2.12. *If $\mathscr{C}$ is a cocomplete symmetric monoidal category, the associative operad $\mathcal{Ass}_{\mathscr{C}}$ has collection defined by $\mathcal{Ass}_{\mathscr{C}}(n) = U^{\amalg\Sigma_n}$ with the action map corresponding to $\tau^{\mathrm{op}} \in {\Sigma_n}^{\mathrm{op}}$ defined through the universal property of coproduct so that for each $\pi\in\Sigma_n$ the diagram* *below commutes.*

$$
\begin{tikzcd}
U^{\amalg \Sigma_n} \arrow[r, "\tau^{\mathrm{op}}"] & U^{\amalg \Sigma_n} \\
U \arrow[u, "\iota_\pi"] \arrow[ur, "\iota_{\pi\tau}"'] &
\end{tikzcd}
$$

The composition maps are defined as the composites

$$U^{\amalg \Sigma_n} \otimes \bigotimes_{i=1}^n U^{\amalg \Sigma_{k_i}} \xrightarrow{\sim} (U \otimes \bigotimes_{i=1}^n U)^{\amalg (\Sigma_n \times \prod_{i=1}^n \Sigma_{k_i})} \xrightarrow{\lambda^{\amalg \Gamma}} U^{\amalg \Sigma_{\Sigma_{i=1}^n k_i}}$$

for all finite lists of naturals $(k_i)_{i=1}^n$.

Example 2.13. *If $\mathscr{C}$ is a cocomplete symmetric monoidal category, the commutative operad $\mathcal{C}omm_{\mathscr{C}}$ is the operad with $\mathcal{C}omm_{\mathscr{C}}(n) = U$ for all $n \in \mathbb{N}$, trivial ${\Sigma_n}^{\mathrm{op}}$ actions, and each composition map is a composite of unitors.*

Another family of fundamental examples of operads is endomorphism operads.

Example 2.14. *Let $\mathscr{C}$ be a closed weak symmetric monoidal category and X be an object of $\mathscr{C}$. The endomorphism operad $\mathcal{E}nd(X)$ is the operad with $\mathcal{E}nd(X)(n) = F(X^{\otimes n}, X)$, where $F(Y, -)$ is the right adjoint of $Y \otimes (-)$ with the action of ${\Sigma_n}^{\mathrm{op}}$ induced by pulling back the factor rearranging action of Σ_n on $X^{\otimes n}$. The operad composition takes the form*

$$
\begin{tikzcd}
F(X^{\otimes k}, X) \otimes \bigotimes_{i=1}^k F(X^{\otimes n_i}, X) \arrow[d, "\mathrm{id} \otimes \otimes"] \\
F(X^{\otimes k}, X) \otimes F(X^{\otimes \Sigma_{i=1}^k n_i}, X^{\otimes k}) \arrow[d, "\circ"] \\
F(X^{\otimes \Sigma_{i=1}^k n_i}, X)
\end{tikzcd}
$$

and the unit map is adjoint to the left unitor $U \otimes X \to X$

Operads are useful insofar as their algebras are. If operads encode axioms for certain types of algebraic structures, their algebras are the algebraic structures themselves.

Definition 2.15. A *algebra A over an operad $\mathscr{O}$*, or $\mathscr{O}$-algebra, in a symmetric monoidal category $\mathscr{C}$ is an object A of $\mathscr{C}$ and maps

$$a_n : \mathscr{O}(n) \otimes A^{\otimes n} \to A$$

satisfying the following axioms.

1. (Unit) The following diagram commutes.

$$\begin{tikzcd} U \otimes A \arrow[r, "1_{\mathscr{O}} \otimes \mathsf{id}_A"] \arrow[dr, "\lambda"'] & \mathscr{O}(1) \otimes A \arrow[d, "a_1"] \\ & A \end{tikzcd}$$

2. (Associativity) The diagram

$$\begin{tikzcd} \mathscr{O}(k) \otimes \bigotimes_{i=1}^{k} (\mathscr{O}(n_i) \otimes A^{\otimes n_i}) \arrow[r, "\mathsf{id} \otimes \bigotimes a_{n_i}"'] \arrow[d, "f"] & \mathscr{O}(k) \otimes A^{\otimes k} \arrow[dd, "a_k"'] \\ (\mathscr{O}(k) \otimes \bigotimes_{i=1}^{k} \mathscr{O}(n_i)) \otimes A^{\Sigma n_i} \arrow[d, "\gamma_{k,\mathbf{n}}"] & \\ \mathscr{O}(\sum_{i=1}^{k} n_i) \otimes A^{\Sigma n_i} \arrow[r, "a_{\Sigma n_i}"] & A \end{tikzcd}$$

   commutes for every $\mathbf{n} = (n_i)_{i=1}^{k} \in \coprod_{N \in \mathbb{N}} \mathbb{N}^N$ where f is the unique isomorphism, which is the composite of symmetric monoidal category structure maps that do not rearrange the order of factors labeled A or factors of the form $\mathscr{O}(-)$.

3. (Equivariance) The diagram

$$\begin{tikzcd} \mathscr{O}(n) \otimes A^{\otimes n} \arrow[r, "\sigma^{\mathrm{op}} \otimes [\sigma^{-1}]"] \arrow[dr, "a_n"'] & \mathscr{O}(n) \otimes A^{\otimes n} \arrow[d, "a_n"] \\ & A \end{tikzcd}$$

   commutes for every $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$. A *morphism of algebras over an operad $\mathscr{O}$*

is a map of underlying objects that induces commutative diagrams with $\mathscr{O}$-algebra structure maps. The category of $\mathscr{O}$ algebras in the symmetric monoidal category $\mathscr{C}$ is denoted $\mathscr{C}[\mathscr{O}]$.

Our first examples of algebras are standard algebraic notions.

Example 2.16. *In a cocomplete weak symmetric monoidal category, algebras over $\mathcal{A}ss$ are equivalent to monoid objects, with the algebra A over $\mathcal{A}ss$ corresponding to a monoid object with operation*

$$A \otimes A \xrightarrow{\lambda^{-1}} U \otimes A^{\otimes 2} \xrightarrow{\iota_e} (\coprod_{\sigma \in \Sigma_2} U) \otimes A^{\otimes 2} \xrightarrow{a_2} A$$

Example 2.17. *Algebras over $\mathcal{C}omm$ in a cocomplete symmetric monoidal category are commutative monoid objects with equivalence realized in a manner similar to Example 2.16.*

The free algebra adjunction for algebras over an operad assembles into a monad in the usual manner of categorical algebra. We review standard notation and introduce the construction next.

Notation 2.18. In a cocomplete weak symmetric monoidal category, if X has a right ${\Sigma_n}^{\mathrm{op}}$ action and Y has a left Σ_n action we write $X \otimes_{\Sigma_n} Y$ for the colimit of the $B{\Sigma_n}^{\mathrm{op}}$-shaped diagram sending the object to $X \otimes Y$ and morphisms sending $\sigma \mapsto (\sigma^{\mathrm{op}})_X \otimes (\sigma^{-1})_Y$ where subscripted group elements denote the corresponding action map on the subscript.

Definition 2.19. Suppose $\mathscr{C}$ is a cocomplete symmetric monoidal category and let $\mathscr{O}$ be an operad. The underlying functor of the free algebra monad is $\mathbb{O}X = \coprod_{k \in \mathbb{N}} \mathscr{O}(k) \otimes_{\Sigma_k} X^{\otimes k}$ with action maps induced by compositions and universal property of colimits. More generally, if $\mathscr{C}$ is a weak symmetric monoidal category, the underlying functor of the free algebra monad fits into the diagram below

$$\begin{tikzcd}
U \otimes X \arrow[r, "1_{\mathscr{O}} \otimes \mathrm{id}"] \arrow[d, "\lambda"] & \mathscr{O}(1) \otimes X \arrow[r, "\iota_1"] & \coprod_{k \in \mathbb{N}} \mathscr{O}(k) \otimes_{\Sigma_k} X^{\otimes k} \arrow[d] \\
X \arrow[rr] & & \mathbb{O}X
\end{tikzcd}$$

defining $\mathbb{O}X$ as a pushout.

Remark 2.20. The monad May originally used in [57] is not a special case of the one of Definition 2.19, but we will need May's construction later.

Proposition 2.21. *Algebras over an operad and algebras over the associated monad are isomorphic categories.*

Proof. This can be checked directly from the definitions, and we omit the details. □

We conclude this chapter with a central example of operads and algebras used in our work.

Example 2.22. *Let V be a finite-dimensional orthogonal G-representation over $\mathbb{R}$. Let $\mathbb{D}(V)$ denote the open unit disk (or ball) in V. A V-little disk $d : \mathbb{D}(V) \to \mathbb{D}(V)$ is a G-map of the form $d(v) = c + rv$ for some fixed $c \in \mathbb{D}(V)$ and real number $r \in (0, 1 - \|c\|]$. We say two little V-disks d and d' are disjoint, written $d \perp d'$ if d and d' have disjoint images. The V-little disks operad $\mathscr{D}_V$ is defined by $\mathscr{D}_V(1)$ is the G-stable subspace of $\mathscr{T}_G(\mathbb{D}(V), \mathbb{D}(V))$ corresponding to the set of little V-disks and $\mathscr{D}_V(n)$ is the subspace of $\mathscr{D}_V(1)^n$ consisting of tuples of pairwise disjoint little V-disks. The group ${\Sigma_n}^{\mathrm{op}}$ acts on the left of $\mathscr{D}_V(n)$ with σ^{op} acting by the restriction of the canonical action of σ^{-1} on $\mathscr{D}_V(1)^n$. The operad unit of $\mathscr{D}_V$ is the inclusion of the point corresponding to the identity map in $\mathscr{D}_V(1)$. The composition maps*

$$\gamma : \mathscr{D}_V(k) \times \prod_{i=1}^{k} \mathscr{D}_V(n_i) \to \mathscr{D}_V(\sum_{i=1}^{k} n_i)$$

are induced by universal property as indicated in the following diagram.

$$
\begin{tikzcd}
\mathscr{D}_V(k) \times \prod_{i=1}^{k} \mathscr{D}_V(n_i) \arrow[r, "\subseteq"] \arrow[dddd, dashed] & \mathscr{D}_V(1)^k \times \prod_{i=1}^{k} \mathscr{D}_V(1)^n \arrow[r, "f_1"] & \prod_{i=1}^{k} \mathscr{D}_V(1) \times \mathscr{D}_V(1)^{n_i} \arrow[d, "\prod(\Delta \times \mathrm{id})"] \\
& & \prod_{i=1}^{k} \mathscr{D}_V(1)^{n_i} \times \mathscr{D}_V(1)^{n_i} \arrow[d, "f_2"] \\
& & \prod_{i=1}^{k} (\mathscr{D}_V(1) \times \mathscr{D}_V(1))^{n_i} \arrow[d, "\prod(\circ)^{n_i}"] \\
& & \\
\mathscr{D}_V(\sum_{i=1}^{k} n_i) \arrow[rr, "\subseteq"] & & \mathscr{D}_V(1)^{\sum n_i}
\end{tikzcd}
$$

The maps f_1 and f_2 are the expected rearrangement of cartesian factors so that every little V-disk coordinate of $\mathscr{D}_V(n_i)$ is composed as the right factor with the i^{th} coordinate of $\mathscr{D}_V(k)$ and the (lexicographic) ordering of the coordinates of $\prod_{i=1}^{k} \mathscr{D}_V(n_i)$ is respected.

Example 2.23. *Spaces of the form $\Omega^V X = \mathscr{T}_G(S^V, X)$ (regarded as objects of $\mathscr{U}_G$ in the exposition below) have canonical $\mathscr{D}_V$-algebra structures. Consider the function $\zeta = \zeta_V : V \to \mathbb{D}(V)$ defined by*

$$\zeta(u) = \frac{1}{\sqrt{1 + \|u\|^2}} u$$

for all $u \in V$. Note that ζ is a G-homeomorphism with inverse

$$\zeta^{-1}(u) = \frac{1}{\sqrt{1 - \|u\|^2}} u$$

There are G-maps $f_n : \mathscr{D}_V(n) \times S^V \to \bigvee_{i=1}^{n} S^V$ described as follows. For points of the form $p = ((d_1, \ldots, d_n), v)$ where there exists a d_i and a $w \in V$ such that $d_i(\zeta(w)) = \zeta(v)$ we have $f_n(p) = (i, w)$, where (i, w) denote the point corresponding to w in the i^{th} wedge summand S^V. All other points are taken to the basepoint by f_n. The continuity of f_n can be checked using the properties of one-point compactification and the gluing lemma along closed sets. The $\mathscr{D}_V$ action on $\Omega^V X$ has structure maps

$$\begin{array}{c}
\mathscr{D}_V(n) \times \mathscr{T}_G(S^V, X)^n \\
\Big\downarrow{\scriptstyle g_1 \times g_2} \\
\mathscr{T}_G(S^V, \bigvee_{i=1}^n S^V) \times \mathscr{T}_G(\bigvee_{i=1}^n S^V, X) \\
\Big\downarrow{\scriptstyle \mathrm{swap}} \\
\mathscr{T}_G(\bigvee_{i=1}^n S^V, X) \times \mathscr{T}_G(S^V, \bigvee_{i=1}^n S^V) \\
\Big\downarrow{\scriptstyle q} \\
\mathscr{T}_G(\bigvee_{i=1}^n S^V, X) \wedge \mathscr{T}_G(S^V, \bigvee_{i=1}^n S^V) \\
\Big\downarrow{\scriptstyle \circ} \\
\mathscr{T}_G(S^V, X)
\end{array}$$

explained as follows. The map q is the canonical quotient map for smash products and $\circ$ is the enriched composition of mapping spaces of based G-spaces. The morphism g_1 is the adjoint of f_n up to restriction of the codomain, and g_2 is induced by the universal property of coproducts in based G-spaces.

Chapter 3

Real Spectra

We continue with an introduction to Real spectra for the study of Real bordism. Due to the subtle technical considerations in the multiplicative theory, we will confine our discussion of Real bordism to the setting of certain categories of Real spectra that are explicitly discussed here for the first time. The general theory of equivariant spectra when the group acts on the real division algebra of scalars for the indexing vector spaces is not discussed below but is worthy of attention for future work. The theory of Real spectra is equivalent to the theory of genuine C_2-spectra in a robust manner; however, the details of some comparisons are left to later work. While it is possible to work with a version of Real bordism in C_2-spectra for our work, our choice not to is informed by the awareness that this would only complicate the required comparison results. The purpose of this chapter is to precisely state a version of the main results as applied to Real spectra.

In some sense, this chapter discusses material necessary to fill an omission in the literature about comparisons of models of multiplicative Thom spectra produced via forgetting complex or quaternionic module structure to those produced geometrically and natively in corresponding categories of spectra. A key comparison result in [30] shows the equivalence between Real spectra and geniune C_2-spectra. In general, for classical equivariant Thom spectra indexed on vector spaces with G-actions twisted by automorphisms on the real division algebra of scalars, no such comparison results are known, and there is potential for fascinating new phenomena. Considering this, reviewing foundational principles in a new context may be worthwhile.

Without this material, it is still possible to prove that a Thom spectrum equivalent to $MU_{\mathbb{R}}$ produced as a Thom spectrum via a map to BO_{C_2} satisfies our result, but a multiplicative equivalence between such a Thom spectrum and the $MU_{\mathbb{R}}$ of [30] would be missing.

Many definitions of this chapter are taken from [22, A] rather than [47] to modernize and simplify the foundations.

3.1 Indexing Categories

The term “Real spectrum”, when used imprecisely, refers to an object in a category modeling equivariant C_2-spectra where complex vector spaces with real form, which we will define shortly, are used as indexing objects. These are not C_2-representations in the usual sense but are modules over a real division algebra with a C_2-action by automorphisms.

Definition 3.1. A real form W of a complex vector space V is a subspace of the underlying real vector space $V_{\mathbb{R}}$ of V such that the induced map $\mathbb{C} \otimes_{\mathbb{R}} W \to V$ is a linear isomorphism. The C_2-structure on a complex vector space with real form V is induced by the complex conjugation action on $\mathbb{C}$ and the trivial actions on V (and $\mathbb{R}$). A real form W of a complex inner product space V with inner product $\langle -, \bullet \rangle$ is a real form of the underlying complex vector space such that the image of the restriction $\langle -, \bullet \rangle|_{W \times W}$ is contained in $\mathbb{R}$.

The next definition describes the objects of the various indexing categories.

Definition 3.2. A Real vector space V is a finite or countable dimensional complex inner product space topologized as the colimit of their finite dimensional subspaces with the Euclidean topology and equipped with a real form and the induced C_2-space structure.

The canonical example of a Real vector space is $\mathbb{C}^\infty$ identified with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^\infty$. Let us introduce the categories with Real vector space objects relevant to this work.

Definition 3.3. The $C_2\mathscr{U}$-enriched category $\mathscr{I}^c_{\mathbb{R}}$ has objects Real vector spaces and morphism spaces $\mathscr{I}^c_{\mathbb{R}}(V, W)$, i.e., the C_2-stable closed subspace of the exponential object W^V consisting of inner product preserving linear maps, also called *Real linear isometries* with composition maps arising from being a subcategory of $\mathscr{U}_{C_2}$. The full subcategory of $\mathscr{I}^c_{\mathbb{R}}$ consisting of finite-dimensional Real vector spaces is denoted $\mathscr{I}_{\mathbb{R}}$.

The categories $\mathscr{I}^c_{\mathbb{R}}$ and $\mathscr{I}_{\mathbb{R}}$ are useful tools for describing and producing algebras over linear isometry operads and are involved in the theory of Thom spectra It is useful to have notation for the following category.

Definition 3.4. If $\mathcal{U}$ is a Real vector space of countable dimension, then it is called a *Real universe* and the poset category of finite-dimensional Real subspaces of $\mathcal{U}$ is denoted $\mathsf{fdSub}^{\mathcal{U}}_{\mathbb{R}}$. We write $\mathsf{fdSub}_{\mathbb{R}}$ when $\mathcal{U} = \mathbb{C}^\infty$.

The next category can be regarded as a Thom space construction on the previous category. It is used to index Real LMS spectra. The initials LMS refer to the authors of [47] where the theory of the corresponding category of equivariant spectra is studied.

Definition 3.5. For every Real universe $\mathcal{U}$, the based $\mathscr{T}_{C_2}$- category $\mathcal{I}^{\mathcal{U}}_{\mathbb{R}}$ is the category with objects the finite dimensional Real subspaces of $\mathcal{U}$ and morphism spaces $\mathcal{I}^{\mathcal{U}}_{\mathbb{R}}(V, W) = S^{W-V}$ when $V \subseteq W$ and $\mathcal{I}^{\mathcal{U}}_{\mathbb{R}}(V, W)$ is the singleton otherwise whose composition maps are the canonical homeomorphisms

$$\mathcal{I}^{\mathcal{U}}_{\mathbb{R}}(V, W) \wedge \mathcal{I}^{\mathcal{U}}_{\mathbb{R}}(U, V) = S^{W-V} \wedge S^{V-U} \to S^{W-U} = \mathcal{I}_{\mathbb{R}}(V, W)$$

obtained from $+ : W - V \times V - U \to W - U$ and the trivial map otherwise. We often write $\mathcal{I}_{\mathbb{R}}$ instead of $\mathcal{I}^{\mathbb{C}^\infty}_{\mathbb{R}}$.

The next category is named after the authors of [53] who characterized orthogonal G-spectra with an analogous category. It was first used in [30] to construct a model of $MU_{\mathbb{R}}$ of what they call "Real spectra" and we call "Real unitary spectra" to help distinguish models.

Definition 3.6. The Real Mandell–May category $\mathscr{J}_{\mathbb{R}}$ is the $\mathscr{T}_{C_2}$-enriched category with objects Real vector spaces and morphism spaces $\mathscr{J}_{\mathbb{R}}(V,W)$ defined as subspaces of $\mathscr{I}_{\mathbb{R}}(V,W)_+ \wedge S^W$ consisting of the basepoint and pairs (T,w) where $w \perp \operatorname{image} T$. In other words, $\mathscr{J}_{\mathbb{R}}(V,W)$ is the Thom space of the vector bundle on $\mathscr{I}_{\mathbb{R}}(V,W)$ that is the orthogonal complement of the canonical image image subbundle of the trivial bundle associated with W. The composition $\mathscr{J}_{\mathbb{R}}(V,W) \wedge \mathscr{J}_{\mathbb{R}}(U,V) \to \mathscr{J}_{\mathbb{R}}(U,W)$ is the based map sending $((T,w),(T',w')) \mapsto (TT', w+Tw')$. The Real Mandell–May category is symmetric monoidal with a symmetric monoidal structure induced by direct sum as an enriched functor and corresponding vector addition.

3.2 Categories of Real Spectra

Let us begin by describing the various categories of Real spectra. The first is not the simplest, but it is the best studied, having been treated in [30], as mentioned above.

Definition 3.7. The $C_2\mathscr{T}$-enriched category $\mathscr{I}_{\mathbb{R}}\mathscr{S}_{\mathbb{R}}$ of Real unitary spectra is the enriched functor category $[\mathscr{J}_{\mathbb{R}}, \mathscr{T}_{C_2}]_{C_2\mathscr{T}}$. The category $\mathscr{U}_{\mathbb{R}}\mathscr{S}$ inherits a symmetric monoidal structure from those on $\mathscr{J}_{\mathbb{R}}$ and $\mathscr{T}_{C_2}$ by Day convolution.

We now turn to Real spectra resembling those of the categories discussed in [47].

Definition 3.8. The category of Real LMS prespectra $\mathscr{P}\mathscr{S}_{\mathbb{R}}$ is the $C_2\mathscr{T}$-enriched functor category $[\mathcal{I}_{\mathbb{R}}, \mathscr{T}_{C_2}]_{C_2\mathscr{T}}$. Analogous categories $\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$ exist where $\mathcal{U}$ is a Real universe replacing the role of $\mathbb{C}^\infty$. as well as similar categories $\mathscr{C}\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$ indexed by full subcategories $\mathscr{C}$ of $\mathcal{I}_{\mathbb{R}}^{\mathcal{U}}$ with set of objects that form cofinal full subcategories of $\mathsf{fdSub}_{\mathbb{R}}^{\mathcal{U}}$.

For our purposes, spectra modeled by presheaf categories are insufficient because we require compatibility with tools from infinite loop space theory and direct access to a highly structured version of the Thom diagonal.

Definition 3.9. The category of Real LMS spectra $\mathscr{S}_{\mathbb{R}}$ is the full subcategory of $\mathscr{P}\mathscr{S}_{\mathbb{R}}$ consisting of X such that for any nested finite-dimensional Real subspaces $V \subseteq W \subseteq \mathbb{C}^{\infty}$, the map $X(V) \to \Omega^{W-V}X(W)$ corresponding by adjunction to the structure map $S^{W-V} \to \mathscr{T}_{C_2}(X(V), X(W))$ of X is a homeomorphism. Similarly, there are categories $\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$ of Real LMS spectra $\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$ indexed on other Real universes as well as categories $\mathscr{C}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$ spectra indexed on full subcategories $\mathscr{C}$ of $\mathscr{I}_{\mathbb{R}}^{\mathcal{U}}$ on objects forming a cofinal full subcategory of $\mathsf{fdSub}_{\mathbb{R}}^{\mathcal{U}}$

The last type of Real LMS spectra described above is essentially canonically identified with those of the form $\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$.

Proposition 3.10. *If $\mathscr{C}$ is as in Definition 3.9 then the Real LMS spectrum category indexed on $\mathscr{C}$ is equivalent to and a retract of $\mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$ via the pullback of the inclusion $\mathscr{C} \to \mathcal{I}_{\mathbb{R}}^{\mathcal{U}}$.*

Proof. An inverse equivalence F can be obtained by choosing for every finite dimensional Real $U \subseteq \mathcal{U}$ an object V_U of $\mathscr{C}$ such that $U \subseteq V_U$ and setting $F(X)(U) = \Omega^{V_U - U}X(V_U)$. $\square$

A fact similar to the one above justifies an equivalence between Real spectra and genuine (LMS) C_2-spectra, as studied in [47]. This equivalence says nothing about smash products or the multiplicative aspects of algebra. One of the intentions behind this chapter of our work is to illustrate the delicate topological structures involved in the multiplicative theory of Real spectra and to provide convincing and technically accurate reasons for its equivalence with the multiplicative theory of C_2-spectra.

One of the weaknesses of $\mathscr{S}_{\mathbb{R}}$ is the lack of a canonical smash product. The development of multiplicative algebra in $\mathscr{S}_{\mathbb{R}}$ and more sophisticated categories to remedy this issue requires the establishment of a few facts.

Notation 3.11. The category $\mathscr{N}$ is the poset category associated to the natural numbers $\mathbb{N}$ with its usual order.

Theorem 3.12 ([47]). *The inclusion $\mathscr{S}_{\mathbb{R}} \to \mathscr{P}\mathscr{S}_{\mathbb{R}}$ admits a left adjoint $L_{\mathbb{R}} : \mathscr{P}\mathscr{S}_{\mathbb{R}} \to \mathscr{S}_{\mathbb{R}}$ and there are similar left adjoints $L_{\mathbb{R}} : \mathscr{C}\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}} \to \mathscr{S}_{\mathbb{R}}^{\mathcal{U}}$.*

Proof. The proof of this result is essentially the same as the one in [47] for LMS G-spectra indexed by G-representatons. The strategy is to factor the inclusion of categories through the full subcategory of Real inclusion prespectra. A *Real inclusion prespectrum* X is a Real prespectrum such that for any nested finite-dimensional Real subspaces $V \subseteq W \subseteq \mathbb{C}^\infty$, the maps $X(V) \to \Omega^{W-V}X(W)$ corresponding by adjunction to the structure map $S^{W-V} \to \mathscr{T}_{C_2}(X(V), X(W))$ of X is, up to isomorphism, the inclusion of a subspace. The proof of the existence of a left adjoint to the inclusion of the category of Real inclusion prespectra in Real prespectra is the same as that in the appendix of [47]. The left adjoint $L'_{\mathbb{R}}$ of inclusion of the category of Real LMS spectra in Real inclusion prespectra is described on objects evaluated on objects by the formula

$$L'_{\mathbb{R}}X(V) = \operatorname{colim}_{W \in V \backslash \mathsf{fdSub}_{\mathbb{R}}} \Omega^{W-V}X(W)$$

where the right-hand side is the colimit of a functor determined by adjunctions from the structure maps of X. The fact $L'_{\mathbb{R}}X$ is actually a Real LMS spectrum is a consequence of two ingredients. First, the functor $\Omega^{(-)-V}X(-)$ is valued in inclusions on morphisms. Second, there is an interchange law between Ω^U and $\mathscr{N}$-indexed colimits (or equivalently, $\mathscr{C}$-indexed colimits for $\mathscr{C}$ admitting a cofinal functor $\mathscr{N} \to \mathscr{C}$,) in the subcategory of $C_2\mathscr{T}$ consisting of inclusions.

The first ingredient is a consequence of the functors Ω^U preserving inclusions. The proof of this uses the compactness of S^U. The second ingredient is a based equivariant version of [48, A.9.5], where Lewis describes a bijection between the underlying sets of both sides of the interchange. The argument that the bijection underlies a homeomorphism is similar to portions of the argument of [65, 3.8]. □

Corollary 3.13. *The category of LMS Real spectra and its variants are tensored and cotensored by* $\mathscr{T}_{C_2}$.

Proof. The tensoring on a reflective subcategory of a presheaf category can be obtained by applying the tensoring on the presheaf level followed by the reflection left adjoint. A similar statement holds for the cotensoring. □

Next, we describe the twisted half-smash products following the approach of [22, A], which are needed to describe the remaining categories of Real spectra we need. The twisted half-smash product depends on the data of a C_2-map $\alpha : A \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U}, \mathcal{U}')$ between two Real universes, which we fix for now.

Definition 3.14. For every finite dimensional $V \subseteq \mathcal{U}$ and $V' \subseteq \mathcal{U}'$ let $A_{V,V'}$ denote the pullback as in the following diagram.

$$\begin{tikzcd}
A_{V,V'} \arrow[rr] \arrow[d, "p_{V,V'}"] & & \mathscr{I}^c_{\mathbb{R}}(V, V') \arrow[d] \\
A \arrow[r, "\alpha"'] & \mathscr{I}^c_{\mathbb{R}}(\mathcal{U}, \mathcal{U}') \arrow[r] & \mathscr{I}^c_{\mathbb{R}}(V, \mathcal{U}')
\end{tikzcd}$$

Let $T\alpha_{V,V'}$ be the based closed subspace of $A_{V,V'+} \wedge S^{V'}$ consisting of the basepoint and pairs (x, v') such that $v' \perp \operatorname{image} \alpha(p_{V,V'}(x))$. In other words, $T\alpha_{V,V'}$ is the Thom space arising from pulling back the vector bundle realizing $\mathscr{J}_{\mathbb{R}}(V, V')$ as a Thom space along the map $A_{V,V'} \to \mathscr{I}^c_{\mathbb{R}}(V, V')$.

If V'' is a finite-dimensional subspace of $\mathcal{U}'$ containing V', then applying $\mathscr{I}^c_{\mathbb{R}}(V, -)$ to the chain of inclusions induces a C_2-map $f : A_{V,V'} \to A_{V,V''}$ and a based C_2-map $T\alpha_{V,V'} \wedge S^{V''-V'} \to T\alpha_{V,V'}$ sending $((x, v), w)) \mapsto (f(x), v + w)$. Up to adjunction, these are the data of a Real prespectrum $\mathscr{T}\alpha_V$. The Real spectrum $\mathscr{M}\alpha_V$ is defined as the spectrification $L_{\mathbb{R}}\mathscr{T}\alpha_V$ of $\mathscr{T}\alpha_V$.

The following result is the Real analogue of [22, A.4.3], proven identically.

Proposition 3.15. *Consider the (variant) Real prespectrum maps* $S^{W-V} \wedge \mathscr{T}\alpha_W \to \mathscr{T}\alpha_V$ *corresponding to pullback-induced maps of spaces* $a_{W,V,V'} : A_{W,V'} \to A_{V,V'}$ *with adjoints to structure based* G*-maps* $S^{W-V} \wedge T\alpha_{W,V'} \to T\alpha_{V,V'}$ *sending*

$$(w,(x,v')) \mapsto (a_{W,V,V'}(x), v' + (p_{W,V'}(x))(w)).$$

Applying $L_{\mathbb{R}}$ *results in an isomorphism of (variant) Real LMS spectra* $S^{W-V} \wedge \mathscr{M}\alpha_W \to \mathscr{M}\alpha_V$ *where* $\wedge$ *denotes the Real LMS spectrum tensoring.*

Proof. Cofinal agreement at the Real pre-spectrum level is leveraged to isomorphism at the Real LMS spectrum level. □

Definition 3.16. The *twisted half-smash product of* A *with Real prespectra* $\alpha \ltimes_{\mathscr{P}} : \mathscr{P}\mathscr{S}^{\mathcal{U}}_{\mathbb{R}} \to \mathscr{P}\mathscr{S}^{\mathcal{U}'}_{\mathbb{R}}$, often written $A \ltimes_{\mathscr{P}} (-)$, sends X to the colimit of the functor $\mathsf{fdSub}^{\mathcal{U}}_{\mathbb{R}} \to \mathcal{S}^{\mathcal{U}'}_{\mathbb{R}}$ sending V to $X(V) \wedge \mathscr{M}\alpha_V$ and nested $V \subseteq W$ to the composites

$$X(V) \wedge \mathscr{M}\alpha_V \to X(V) \wedge S^{W-V} \wedge \mathscr{M}\alpha_W \to X(W) \wedge \mathscr{M}\alpha_W$$

with first map arising from Proposition 3.15 and the second map arising from an adjoint to a structure map of X.

Definition 3.17. The *twisted half-smash product of* A *with Real LMS spectra* $\alpha \ltimes (-) : \mathscr{S}^{\mathcal{U}}_{\mathbb{R}} \to \mathscr{S}^{\mathcal{U}'}_{\mathbb{R}}$, often written $A \ltimes (-)$ is the composite $L_{\mathbb{R}}(\alpha \ltimes_{\mathscr{P}} (-))u$, where u is the forgetful functor to Real prespectra. We usually write $A \ltimes (-)$ for $\alpha \ltimes (-)$. By properties of reflections held by $L_{\mathbb{R}}$, $A \ltimes (-)$ can be equivalently defined in the same way as $A \ltimes_{\mathscr{P}} (-)$, with the role of Real prespectra replaced with the corresponding notions of Real LMS spectra.

Twisted half smash products were originally developed in [47] but we will draw Real analogues of a few basic results from [22, A], which works from definitions corresponding to ours and are proven identically.

Proposition 3.18. ([22, A.6.2]) *Let* $\alpha : A \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U},\mathcal{U}')$ *and* $\beta : B \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U}',\mathcal{U}'')$ *be* C_2*-maps for Real universes* $\mathcal{U}$, $\mathcal{U}'$, *and* $\mathcal{U}''$. *If* $\gamma : B \times A \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U},\mathcal{U}'')$ *is the composite* $\gamma = \circ(\beta,\alpha)$, *then there is a natural isomorphism in all arguments*

$$B \ltimes (A \ltimes X) \to (B \times A) \ltimes X.$$

Proof. Unpacking definitions one finds the above isomorphism is induced by isomorphisms of the form $B \ltimes \mathscr{M}\alpha_V \to \mathscr{M}\gamma_V$ arising from the maps $T\beta_{V',V''} \wedge T\alpha_{V,V'} \to T\gamma_{V,V''}$ defined by $((b,v''),(a,v')) \mapsto ((b,a), v'' + \beta(b)v')$. Checking that the natural transformation is a natural isomorphism is done by reducing to the case of compact A and B and using a cofinality argument. $\square$

Proposition 3.19. ([22, A.5.3]) *For a* C_2*-map* $\mathsf{id}_{\mathcal{U}} : * \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U},\mathcal{U})$ *from the singleton space, there is a natural isomorphism* $* \ltimes X \to X$.

With the twisted half-smash product in hand, we can describe the remaining categories of Real spectra needed.

Definition 3.20. The monad $\mathbb{L}_{\mathbb{R}}$ on $\mathscr{S}_{\mathbb{R}}$ is has underlying functor $\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty) \ltimes (-)$ associated with the identity map of C_2-spaces $\mathsf{id} : \mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty) \to \mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)$ and the monad structure arises by Propositions 3.18 and 3.19 from this being a map corresponding to the inclusion of the full subcategory of a single object of $\mathscr{I}^c_{\mathbb{R}}$, equivalently regarded as a monoid. The *category of EKMM* $\mathbb{L}_{\mathbb{R}}$*-spectra* is the category $\mathscr{S}_{\mathbb{R}}[\mathbb{L}_{\mathbb{R}}]$ of algebras over $\mathbb{L}_{\mathbb{R}}$.

Convention 3.21. When using the map $\mathsf{id} : \mathscr{I}^c_{\mathbb{R}}(\mathcal{U},\mathcal{U}') \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U},\mathcal{U}')$ to form a twisted half-smash product, we will not explicitly mention the use of id, which our notation already suppresses.

Unlike LMS Real spectra, the category of EKMM $\mathbb{L}_{\mathbb{R}}$-spectra has a well-behaved but not symmetric monoidal smash product. As a first step toward the concept, we consider smash products that join differently indexed categories of LMS Real spectra.

Definition 3.22. Let $\mathcal{U}$,$\mathcal{U}'$ be Real universes and let $\mathscr{C}_{\mathcal{U}}$ and $\mathscr{C}_{\mathcal{U}'}$ be full subcategories of $\mathcal{I}_{\mathbb{R}}^{\mathcal{U}}$ and $\mathcal{I}_{\mathbb{R}}^{\mathcal{U}'}$ respectively so that their object sets are cofinal in $\mathsf{fdSub}_{\mathbb{R}}^{\mathcal{U}}$ and $\mathsf{fdSub}_{\mathbb{R}}^{\mathcal{U}'}$ respectively. Let $\mathscr{C}_{\mathcal{U},\mathcal{U}'}$ be the full subcategory of $\mathcal{I}_{\mathbb{R}}^{\mathcal{U}\oplus\mathcal{U}'}$ generated by all $V \oplus V'$ where V and V' are objects of $\mathscr{C}_{\mathcal{U}}$ and $\mathscr{C}_{\mathcal{U}'}$, respectively.

The *external smash products of Real prespectra*

$$\underline{\wedge}_{\mathscr{P}} : \mathscr{C}_{\mathcal{U}}\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}} \times \mathscr{C}_{\mathcal{U}'}\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}'} \to \mathscr{C}_{\mathcal{U},\mathcal{U}'}\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}\oplus\mathcal{U}'}$$

is defined by

$$(X\underline{\wedge}Y)(V \oplus V') = X(V) \wedge Y(V')$$

and on morphisms given by the dotted arrow induced by the universal property of quotients in the commutative diagram below.

$$\begin{tikzcd}
\mathscr{C}_{\mathcal{U}}(V,W) \times \mathscr{C}_{\mathcal{U}'}(V',W') \arrow[d, equal] & \\
S^{W-V} \times S^{W'-V'} \arrow[r] \arrow[d, "q"'] & \mathscr{T}_{C_2}(X(V),X(W)) \times \mathscr{T}_{C_2}(Y(V'),Y(W')) \arrow[d] \\
S^{(W\oplus W')-(V\oplus V')} \arrow[r, dashed] \arrow[d, equal] & \mathscr{T}_{C_2}(X(V)\wedge Y(V'), X(W)\wedge Y(W')) \\
\mathscr{C}_{\mathcal{U},\mathcal{U}'}(V\oplus V', W\oplus W') &
\end{tikzcd}$$

Definition 3.23. Let $\mathcal{U}$ and $\mathcal{U}'$ be Real universes and $\mathscr{C}_{\mathcal{U},\mathcal{U}'}$ is the full subcategory generated by all $V \oplus V'$ where V and V' are finite-dimensional Real subspaces of $\mathcal{U}$ and $\mathcal{U}'$ respectively. The *external smash product of Real LMS spectra* $\underline{\wedge} : \mathscr{S}_{\mathbb{R}}^{\mathcal{U}} \times \mathscr{S}_{\mathbb{R}}^{\mathcal{U}'} \to \mathscr{S}_{\mathbb{R}}^{\mathcal{U}\oplus\mathcal{U}'}$ is defined as the composite

$$\mathscr{S}_{\mathbb{R}}^{\mathcal{U}} \times \mathscr{S}_{\mathbb{R}}^{\mathcal{U}'} \xrightarrow{u \times u'} \mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}} \times \mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}'} \xrightarrow{\wedge_{\mathscr{P}}} \mathscr{C}_{\mathcal{U},\mathcal{U}'}\mathscr{P}\mathscr{S}_{\mathbb{R}}^{\mathcal{U}\oplus\mathcal{U}'} \xrightarrow{L_{\mathbb{R}}} \mathscr{S}_{\mathbb{R}}^{\mathcal{U}\oplus\mathcal{U}'}$$

where u and u' are the right adjoints of the spectrification functors $L_{\mathbb{R}}$.

The external smash product of Real LMS spectra is compatible with twisted half-smash products.

Proposition 3.24. ([22, A.6.3]) *Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1'$, and $\mathcal{U}_2'$ be Real universes and let $\alpha : A \to \mathscr{I}_{\mathbb{R}}^c(\mathcal{U}_1, \mathcal{U}_1')$ and $\beta : B \to \mathscr{I}_{\mathbb{R}}^c(\mathcal{U}_2, \mathcal{U}_2')$ be C_2-maps. There is a natural isomorphism*

$$(A \times B) \ltimes E_1 \underline{\wedge} E_2 \to (A \ltimes E_1) \underline{\wedge} (B \ltimes E_2)$$

in α, β, $E_1 \in \mathscr{S}_{\mathbb{R}}^{\mathcal{U}_1}$ and $E_2 \in \mathscr{S}_{\mathbb{R}}^{\mathcal{U}_2}$.

We are now ready to define the smash product of spectra needed.

Definition 3.25. The *smash product* $X \wedge_{\mathscr{L}_{\mathbb{R}}} Y$ of two EKMM $\mathbb{L}_{\mathbb{R}}$-spectra is defined via the coequalizer diagram

$$\begin{tikzcd} (\mathscr{I}_{\mathbb{R}}^c((\mathbb{C}^\infty)^{\oplus 2}, \mathbb{C}^\infty) \times (\mathscr{I}_{\mathbb{R}}^c(\mathbb{C}^\infty, \mathbb{C}^\infty) \times \mathscr{I}_{\mathbb{R}}^c(\mathbb{C}^\infty, \mathbb{C}^\infty))) \ltimes (X \underline{\wedge} Y) \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ \mathscr{I}_{\mathbb{R}}^c((\mathbb{C}^\infty)^{\oplus 2}, \mathbb{C}^\infty) \ltimes (X \underline{\wedge} Y) \arrow[d] \\ X \wedge_{\mathscr{L}_{\mathbb{R}}} Y \end{tikzcd}$$

where f and g are described as follows. The map f is induced by functoriality applied to the composite map

$$\begin{array}{c} \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty)\times(\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)\times\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)) \\ \Big\downarrow{\scriptstyle \mathsf{id}\times\oplus} \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty)\times\mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},(\mathbb{C}^\infty)^{\oplus 2}) \\ \Big\downarrow{\scriptstyle \circ} \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty) \end{array}$$

, which is also the defining map for which the twisted half smash product in the source of f is formed. The map g is the composite

$$\begin{array}{c} (\mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty)\times(\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)\times\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)))\ltimes(X\underline{\wedge}Y) \\ \Big\downarrow \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty)\ltimes(((\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)\times\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty))\ltimes(X\underline{\wedge}Y)) \\ \Big\downarrow \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty)\ltimes((\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)\ltimes X)\underline{\wedge}(\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty,\mathbb{C}^\infty)\ltimes Y)) \\ \Big\downarrow \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2},\mathbb{C}^\infty)\ltimes(X\underline{\wedge}Y) \end{array}$$

obtained from Propositions 3.18 and 3.24 and the $\mathbb{L}_{\mathbb{R}}$ actions on X and Y.

Definition 3.26. If $X\in\mathscr{T}_{C_2}$ and $V\subseteq\mathbb{C}^\infty$ is a finite-dimensional Real subspace, the Real shift desuspension spectrum $\Sigma^\infty_V X$ is defined by $\Sigma^\infty_V X = L_{\mathbb{R}}\Sigma^\infty_{V,\mathscr{P}}X$ where $\Sigma^\infty_{V,\mathscr{P}}X(W) = S^{W-V}\wedge X$ when $V\subseteq W$ and is the singleton otherwise with structure maps induced by the usual homeomorphisms relating spheres and their smash products. When $V = 0$, we write *the suspension Real spectrum of* X $\Sigma^\infty X$ for $\Sigma^\infty_V X$ and the *sphere spectrum* $S_{\mathbb{R}}$ is $\Sigma^\infty S^0$.

The next proposition gives suspension Real spectra the structure of $\mathbb{L}_{\mathbb{R}}$ algebras.

Proposition 3.27. ([22, A.5.5]) *Let* $\mathcal{U}$ *and* $\mathcal{U}'$ *be Real universes and let* $\alpha: A\to\mathscr{I}^c_{\mathbb{R}}(\mathcal{U},\mathcal{U}')$*.*

Then, there is an isomorphism

$$A \ltimes \Sigma^\infty X \to A_+ \wedge \Sigma^\infty X$$

natural in α.

Proof. The functors $A \ltimes (-)$ and $A \ltimes_{\mathscr{P}} (-)$ are left adjoints with right adjoints $F[\alpha, -)$ and similarly defined $F[\alpha, -)_{\mathscr{P}}$. The former is defined by $F[\alpha, -)(V) = \mathscr{S}_{\mathbb{R}}^{\mathcal{U}'}(\mathscr{M}\alpha_V, -)$ and structure maps induced by the isomorphisms $\mathscr{M}\alpha_V \to S^{W-V} \wedge \mathscr{M}\alpha_W$ and properties of the tensoring and cotensoring. Composing adjoint functors yields a natural isomorphism

$$A \ltimes \Sigma^\infty X = L_{\mathbb{R}}(A \ltimes_{\mathscr{P}} L_{\mathbb{R}}\Sigma^\infty_{\mathscr{P}} X) \to L_{\mathbb{R}}(A \ltimes_{\mathscr{P}} \Sigma^\infty_{\mathscr{P}} X).$$

The defining diagram of $A \ltimes_{\mathscr{P}} \Sigma^\infty_{\mathscr{P}} X$ consists of isomorphisms. By evaluating at the Real vector space 0 we obtain

$$A \ltimes_{\mathscr{P}} \Sigma^\infty_{\mathscr{P}} X \to X \wedge \mathscr{M}\alpha_0$$

and $\mathscr{M}\alpha_0 = \Sigma^\infty A_+$ by direct level-wise comparison of $\mathscr{T}\alpha_0$ and $\Sigma^\infty_{\mathscr{P}} A_+$. These facts together with the observation that suspension spectra are the corresponding tensorings of the sphere spectrum gives the result. $\square$

By virtue of the equivalence of Real LMS spectra and the corresponding C_2-spectra $S_{\mathbb{R}}$ can be identified with the sphere spectrum S_{C_2} in the other context, although we emphasize again that this comparison ignores multiplicative considerations.

Although $S_{\mathbb{R}}$ is not a unit for the smash product $\wedge_{\mathscr{L}_{\mathbb{R}}}$ we do have a weak version of the statement.

Proposition 3.28. ([22, 1.8.3]) *There is a natural transformation*

$$\lambda : S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} X \to X$$

of EKMM $\mathbb{L}_{\mathbb{R}}$-spectra.

Proof. First note that the construction of $\wedge_{\mathscr{L}}$ implies $\wedge_{\mathscr{L}}$ commutes with colimits in either argument. In particular, every EKMM $\mathbb{L}_{\mathbb{R}}$-spectrum being an algebra over a monad has a canonical resolution by free algebras (by a well-known result in [42], that is easier for most English speakers to find in [5]) and it suffices to check naturality on free algebras. By properties of the twisted half-smash product, we can find an isomorphism

$$S_{\mathbb{R}} \wedge_{\mathscr{L}} \mathbb{L}_{\mathbb{R}} X \to A \ltimes (S_{\mathbb{R}} \underline{\wedge} X)$$

for an explicit map $\alpha : A \to \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus 2}, \mathbb{C}^\infty)$. Let $\iota_2 : \mathbb{C}^\infty \to (\mathbb{C}^\infty)^{\oplus 2}$ be the second argument inclusion and observe that $S_{\mathbb{R}} \underline{\wedge} X$ is isomorphic to $\{\iota_2\} \ltimes X$. Consequently $S_{\mathbb{R}} \wedge_{\mathscr{L}} \mathbb{L}_{\mathbb{R}} X$ is isomorphic to $A \ltimes X$ for a map $\alpha' : A \to \mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty, \mathbb{C}^\infty)$. The desired natural map on free algebras is induced by the terminal map of C_2-spaces over $\mathscr{I}^c_{\mathbb{R}}(\mathbb{C}^\infty, \mathbb{C}^\infty)$. $\square$

We can finally define our last category of Real spectra.

Definition 3.29. The *category $\mathscr{M}_{S_{\mathbb{R}}}$ of EKMM $S_{\mathbb{R}}$-modules* is the full subcategory of EKMM $\mathbb{L}_{\mathbb{R}}$-spectra consisting of X such that $\lambda : S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} X \to X$ is an isomorphism. The smash product $\wedge$ of EKMM $S_{\mathbb{R}}$-modules is the restriction of $\wedge_{\mathscr{L}_{\mathbb{R}}}$.

The Real analogue of the remarkable and celebrated work we have reviewed a fraction of to explain our notions yields the following.

Theorem 3.30. [22] *The category of EKMM $S_{\mathbb{R}}$-modules is a $C_2\mathscr{T}$-enriched cocomplete symmetric monoidal category with unit $S_{\mathbb{R}}$, smash product $\wedge$, and left unitor λ. The category of EKMM $\mathbb{L}_{\mathbb{R}}$-spectra is a $C_2\mathscr{T}$-enriched cocomplete weak symmetric monoidal category with product $\wedge_{\mathscr{L}_{\mathbb{R}}}$.*

3.3 Algebraic Structures in Real Spectra

Since Real EKMM $S_{\mathbb{R}}$-modules form a symmetric monoidal category, the usual notion of algebras over operads applies. Moreover, the suspension spectrum functor Σ^∞ and $\Sigma^\infty_+ = \Sigma^\infty(-)_+$ can be used to send operads in $C_2\mathscr{U}$ or $C_2\mathscr{T}$ to operads in $\mathscr{M}_{S_{\mathbb{R}}}$. The following fact justifies this.

Proposition 3.31. *Suspension spectra are $S_{\mathbb{R}}$-modules and $\Sigma^\infty : \mathscr{T}_{C_2} \to \mathscr{M}_{S_{\mathbb{R}}}$ is strong symmetric monoidal.*

The situation is more subtle for Real LMS spectra and EKMM $\mathbb{L}_{\mathbb{R}}$-spectra. For the former, we have an alternative notion of algebra over an operad in $\mathscr{U}_{C_2}$ based on the twisted half smash product. We need the following result to make sense of equivariance axioms for operadic algebras.

Proposition 3.32 ([47])**.** *Let $\mathcal{U}$ be a Real universe and $\alpha : \Sigma_n \to \mathscr{I}^c_{\mathbb{R}}(\mathcal{U}^{\oplus n}, \mathcal{U}^{\oplus n})$ be the permutation map. Note that the permutation map is compatible with composition. Then, for every $X \in \mathscr{S}^{\mathcal{U}}_{\mathbb{R}}$, Σ_n acts on $X^{\triangle n}$ in the sense that $X^{\overline{\wedge} n}$ is an algebra over the monad $\Sigma_n \ltimes (-)$ with multiplication arising from Proposition 3.18 and compatibility from composition and the unit map arising from Proposition 3.19 and the inclusion of the identity of Σ_n.*

Proof. We sketch here an original argument from Cole's definitions, which we have adopted. By computing the defining colimit of the source by restricting to the cofinal subcategory $\mathscr{C}$ of finite dimensional Real subspaces of the form $\bigoplus_{i=1}^n V_i$ in $\mathsf{fdSub}^{\mathcal{U}^{\oplus n}}_{\mathbb{R}}$ and one can show that the canonical maps

$$\mathscr{M}\alpha_{\bigoplus_{1=1}^n V_i} \xrightarrow{\sim} \bigvee_{\sigma \in \Sigma_n} \Sigma^\infty_{\bigoplus_{i=1}^n V_{\sigma^{-1}(i)}} S^0 \xrightarrow{\sim} \bigvee_{\sigma\in\Sigma_n} \overline{\bigwedge}_{i=1}^n \Sigma^\infty_{V_{\sigma^{-1}(i)}} S^0$$

are isomorphisms by comparing defining Real prespectra. Levelwise comparison yields an isomorphism for all $Y \in \mathscr{S}_{\mathbb{R}}^{\mathcal{U}^{\oplus n}}$

$$\operatorname{colim}_{U \in \mathscr{C}} \Sigma_U^\infty S^0 \wedge Y(U) \xrightarrow{\sim} Y$$

whenever U ranges over a cofinal full subcategory of $\mathsf{fdSub}_{\mathbb{R}}^{\mathcal{U}^{\oplus n}}$. Therefore, the natural (in $\mathscr{C}$) isomorphism

$$\begin{tikzcd}
\left(\bigwedge_{i=1}^{n} X(V_i)\right) \wedge \mathscr{M}\alpha_{\bigoplus_{1=1}^{n} V_i} \arrow[r, "\sim"] & \bigvee_{\sigma \in \Sigma_n} \left(\bigwedge_{i=1}^{n} X(V_i)\right) \wedge \overline{\bigwedge}_{i=1}^{n} \Sigma_{V_{\sigma^{-1}(i)}}^\infty S^0 \arrow[d, "\sim"] \\
 & \bigvee_{\sigma \in \Sigma_n} \overline{\bigwedge}_{i=1}^{n} X(V_{\sigma^{-1}(i)}) \wedge \Sigma_{V_{\sigma^{-1}(i)}}^\infty S^0
\end{tikzcd}$$

after passing to colimits yields an isomorphism

$$\Sigma_n \ltimes X^{\otimes n} \to \bigvee_{\sigma \in \Sigma_n} X$$

and the action map is the composite of this with the fold map. The remainder of the proof consists of formal checks given the required facts. □

There is a canonical operad that bears a strong relationship to the twisted half-smash product on which we rely.

Definition 3.33. The *Real linear isometries operad* $\mathscr{L}_{\mathbb{R}}$ has spaces

$$\mathscr{L}_{\mathbb{R}}(n) = \mathscr{I}_{\mathbb{R}}^c((\mathbb{C}^\infty)^{\oplus n}, \mathbb{C}^\infty)$$

with ${\Sigma_n}^{\mathrm{op}}$ acting by precomposition with permutations of the copies of $\mathbb{C}^\infty$. The composition is

$$\begin{array}{c} \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus k}, \mathbb{C}^\infty) \times \prod_{i=1}^k \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus n_i}, \mathbb{C}^\infty) \\ \downarrow \scriptstyle{\mathsf{id}\times\oplus} \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus k}, \mathbb{C}^\infty) \times \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus \Sigma n_i}, (\mathbb{C}^\infty)^{\oplus k}) \\ \downarrow \scriptstyle{\circ} \\ \mathscr{I}^c_{\mathbb{R}}((\mathbb{C}^\infty)^{\oplus \Sigma n_i}, \mathbb{C}^\infty) \end{array}$$

and the unit is the inclusion of the identity $\mathbb{C}^\infty \to \mathbb{C}^\infty$.

We can now define operadic algebras in $\mathscr{S}_{\mathbb{R}}$. This is a notion necessary to develop the theory of Thom spectra.

Definition 3.34. Let $\mathscr{O}$ be an operad in $C_2\mathscr{U}$ equipped with a map $\phi : \mathscr{O} \to \mathscr{L}_{\mathbb{R}}$. An *algebra A over the operad $\mathscr{O}$ with respect to ϕ in $\mathscr{S}_{\mathbb{R}}$* is a Real LMS spectrum A equipped with maps

$$a_n : \mathscr{O}(n) \ltimes A^{\triangle n} \to A$$

satisfying the following axioms.

1. (Unit) The diagram

$$\begin{tikzcd} * \ltimes A \arrow[r, "1_{\mathscr{O}} \ltimes \mathsf{id}"] \arrow[dr, "f"'] & \mathscr{O}(1) \ltimes A \arrow[d, "a_1"] \\ & A \end{tikzcd}$$

   commutes, where f is the map of Proposition 3.19.

2. (Associativity) The diagrams

$$\begin{tikzcd}
\mathscr{O}(k) \ltimes \overline{\bigwedge}_{i=1}^{k} \left(\mathscr{O}(n_i) \ltimes A^{\underline{\wedge} n_i}\right) \arrow[r, "\mathsf{id} \ltimes \overline{\wedge} a_{n_i}"] \arrow[d, "g_1"'] & \mathscr{O}(k) \ltimes A^{\underline{\wedge} k} \arrow[ddd, "a_k"] \\
\mathscr{O}(k) \ltimes \left(\prod_{i=1}^{k} \mathscr{O}(n_i) \ltimes A^{\underline{\wedge} \sum n_i} \right) \arrow[d, "g_2"'] & \\
\left(\mathscr{O}(k) \times \prod_{i=1}^{k} \mathscr{O}(n_i) \right) \ltimes A^{\underline{\wedge} \sum n_i} \arrow[d, "\gamma \ltimes \mathsf{id}"'] & \\
\mathscr{O}\left(\sum_{i=1}^{k} n_i \right) \ltimes A^{\underline{\wedge} \sum n_i} \arrow[r, "a_{\sum n_i}"'] & A
\end{tikzcd}$$

commute for all $(n_i)_{i=1}^{n} \in \coprod_{N \in \mathbb{N}} \mathbb{N}^N$ where g_1 and g_2 are maps induced from Propositions 3.24 and 3.18 respectively.

3. (Equivariance) The diagrams

$$\begin{tikzcd}
\mathscr{O}(n) \ltimes (\Sigma_n \ltimes A^{\underline{\wedge} n}) \arrow[r, "h_2"'] \arrow[dd, "h_1"] & (\mathscr{O}(n) \times \Sigma_n) \ltimes A^{\overline{\wedge} n} \arrow[r, "\sim"] & ({\Sigma_n}^{\mathrm{op}} \times \mathscr{O}(n)) \ltimes A^{\underline{\wedge} n} \arrow[d, "h_3"] \\
 & & \mathscr{O}(n) \ltimes A^{\overline{\wedge} n} \arrow[d, "a_n"'] \\
\mathscr{O}(n) \ltimes A^{\underline{\wedge} n} \arrow[rr, "a_n"] & & A
\end{tikzcd}$$

commute for every $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$ where h_1 and h_3 are action maps (with h_3 well-defined since ϕ is an operad map) and h_2 is the map of Proposition 3.18.

A *morphism of algebras over operads* is a morphism of LMS Real spectra that induces commutative squares with each structure map pair of the source and target. The category of $\mathscr{O}$-algebras (with respect to ϕ) is denoted $\mathscr{S}_{\mathbb{R}}[\mathscr{O}]$.

The following construction is useful for finding a well-behaved replacement of an operad with another over the linear isometries operad.

Definition 3.35. If $\mathscr{C}$ is a symmetric monoidal category, $\mathscr{O}$ and $\mathscr{P}$ are operads in $\mathscr{C}$, then there is an operad $\mathscr{O} \otimes^{\|} \mathscr{P}$ called the *parallel product of* $\mathscr{O}$ *and* $\mathscr{P}$ described as follows. The underlying collection is $\mathscr{O} \otimes^{\|} \mathscr{P}(n) = \mathscr{O}(n) \otimes \mathscr{P}(n)$ with the diagonal ${\Sigma_n}^{\mathrm{op}}$ action. The composition is defined by using the braidings and associators to sort factors of the source and apply the compositions of $\mathscr{O}$ and $\mathscr{P}$.

Convention 3.36. The operad $\mathscr{O} \times^{\|} \mathscr{L}_{\mathbb{R}}$ is regarded as an operad over $\mathscr{L}_{\mathbb{R}}$ via the second projection.

Another way to realize operads in U_{C_2} in the context of spectra is by the suspension spectrum.

Proposition 3.37. *The functors* $\Sigma^\infty(-)_+ : \mathscr{U}_{C_2} \to \mathscr{S}_{\mathbb{R}}[\mathbb{L}_{\mathbb{R}}]$ *and* $\Sigma^\infty(-)_+ : \mathscr{U}_{C_2} \to \mathscr{M}_{S_{\mathbb{R}}}$ *are strong symmetric monoidal and consequently induce functors on corresponding categories of operads and algebras.*

The utility of these constructions is noted in [20, 3.5.i]. It is stated there, and appears to have been regarded as folklore beforehand, that the categories $\mathscr{S}_{\mathbb{R}}[\mathbb{L}_{\mathbb{R}}][\mathscr{O}]$ and $\mathscr{S}_{\mathbb{R}}[\mathscr{O} \times^{\|} \mathscr{L}_{\mathbb{R}}]$ are equivalent because of an argument that compares the monad $\mathbb{O}\mathbb{L}_{\mathbb{R}}$ with the monad for free $\mathscr{O} \times \mathscr{L}_{\mathbb{R}}$-algebras. Unfortunately, the statement is incorrect because of the necessity of the pushout of Definition 2.19 in the weak symmetric monoidal context. Nevertheless, the difference between the two is a mild technical point, and we have the following relationship implied by the relationships between respective free algebras.

Proposition 3.38. *The category* $\mathscr{S}_{\mathbb{R}}[\mathbb{L}_{\mathbb{R}}][\mathscr{O}]$ *is equivalent to a reflective subcategory of* $\mathscr{S}_{\mathbb{R}}[\mathscr{O} \times^{\|} \mathscr{L}_{\mathbb{R}}]$.

With the standard model structures on $\mathscr{S}_{\mathbb{R}}[\mathbb{L}_{\mathbb{R}}][\mathscr{O}]$ and $\mathscr{S}_{\mathbb{R}}[\mathscr{O} \times^{\|} \mathscr{L}_{\mathbb{R}}]$, the inclusion functor of 3.38 is a right Quillen equivalence.

We do not describe any of the comparison functors of our various categories of spectra and operadic algebras in this document, but note that all the ideas and definitions needed parallel material found in the existing work of [20, 30, 53]. In particular, ideas of [30, B] are needed to establish model categorical structures for algebras over equivariant operads. The indexed smash product in that setting has a reformulation that applies to ours. One simply takes the indexed smash product as a non-equivariant object to obtain analogues of Real spectra that are $\mathscr{T}_{C_2 \times C_2}$ valued functors from an enriched category on which the left C_2 factor acts trivially on mapping spaces and then applies the forgetful functor along the diagonal subgroup. We also note that the construction of a model structure on operads $\mathscr{S}_{\mathbb{R}}[\mathscr{O}]$ is needed considering the non-equivalence of categories of Proposition 3.38.

It also appears that we need mixed model structures for Real LMS spectra, $\mathbb{L}_{\mathbb{R}}$-spectra, and EKMM $S_{\mathbb{R}}$ modules in the sense of [18] so that the cofibrant objects in the mixed structure are objects homotopy equivalent to the cofibrant objects of the standard structure. These model structures may be necessary for applying space-level results about the May delooping machine of [26] to algebras over operads in EKMM $S_{\mathbb{R}}$-modules because the free algebra functors for relevant operads only preserve the mixed cofibrancy notion.

3.4 Fujii–Landweber Real Bordism

We now summarize the relevant aspects of the Real analogue of the classical highly structured theory of Thom spectra from [47]. The proofs apply verbatim to our situation after replacing the corresponding categories and objects.

We have a functor $U_{\mathbb{R}}(-) : \mathscr{I}_{\mathbb{R}}^c \to \mathscr{T}_{C_2}$ such that $U_{\mathbb{R}}(V)$ is the unitary group of V. Abusing notation, we also use $U_{\mathbb{R}}$ to denote $U_{\mathbb{R}}(\mathbb{C}^\infty)$. The space BH denotes the geometric realization of the simplicial bar construction $|B(*, H, *)|$ on a topological monoid H, and $E_| H$ denotes $|B(*, H, H)|$ similarly. (We use this unusual notation because $BU_{\mathbb{R}}$ is used later for a C_2-space homotopy equivalent but not homeomorphic to $BU_{\mathbb{R}}$.) The category $\mathscr{I}_{\mathbb{R}}^c$ is symmetric monoidal as a $C_2\mathscr{U}$-enriched category with direct sum acting as the symmetric monoidal product. The functors $U_{\mathbb{R}}(-)$, $BU_{\mathbb{R}}(-)$, and $E_{|}U_{\mathbb{R}}$ are lax symmetric monoidal in the enriched sense and and this structure allows one to define actions of $\mathscr{L}_{\mathbb{R}}$ on their evaluations at $\mathbb{C}^\infty$. Such functors are called $\mathscr{I}^c_{\mathbb{R}}$-FCPs.

Before proceeding with a discussion of Thom spectra, let us introduce definitive Real models of $MU_{\mathbb{R}}$, starting from [30].

Definition 3.39. The Real unitary Real bordism spectrum $\mathcal{MU}_{\mathbb{R}}$ is the functor $\mathscr{J}_{\mathbb{R}} \to \mathscr{T}_{C_2}$ with $\mathcal{MU}_{\mathbb{R}}(V) = B(*, (U_{\mathbb{R}}(V))_+, S^V)$ is a two-sided bar construction for the based monoid object $(U_{\mathbb{R}}(V))_+$ acting on S^V in the canoncial manner. On morphisms, the map is adjoint to the based C_2-map

$$\mathscr{J}_{\mathbb{R}}(V, W) \to \mathscr{T}_{C_2}(B(*, (U_{\mathbb{R}}(V))_+, S^V), B(*, (U_{\mathbb{R}}(W))_+, S^W))$$

that can be described simplicially by sending (T, v) to the map that applies $U_{\mathbb{R}}(T)$ to every $U_{\mathbb{R}}(V)$ coordinate of the source and $v + T(-)$ to the V coordinate.

For a well-behaved Thom spectrum functor, we must look to Real prespectra or Real LMS spectra. The analogue of $\mathcal{MU}_{\mathbb{R}}$ is not difficult to produce.

Definition 3.40. The Real prespectrum $MU^{\mathscr{P}}_{\mathbb{R}}$ is composite of $\mathscr{I}_{\mathbb{R}} \to \mathscr{J}_{\mathbb{R}}$ and $\mathcal{MU}_{\mathbb{R}}$. The Real LMS spectrum $MU^{\mathbf{LMS}}_{\mathbb{R}}$ is $L_{\mathbb{R}} MU^{\mathscr{P}}_{\mathbb{R}}$.

In addition, the EKMM $S_{\mathbb{R}}$-module $MU^{\mathbf{EKMM}}_{\mathbb{R}}$ is $S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} MU^{\mathbf{LMS}}_{\mathbb{R}}$ (although we will not justify this construction here) and another model obtained from $\mathcal{MU}_{\mathbb{R}}$ by a (derived) left Quillen functor $\mathscr{I}_{\mathbb{R}}\mathscr{S}_{\mathbb{R}} \to \mathscr{M}_{S_{\mathbb{R}}}$. With their corresponding algebra over operad structures, over $\mathcal{C}omm$ or $\mathscr{L}_{\mathbb{R}}$ in the various senses depending on the setting, each of these models can and needs to be compared by following the techniques of [53].

Next, we describe a functor that assigns a Thom spectrum to a C_2-map $f : X \to BU_{\mathbb{R}}$. Although the theory has a few more minor technicalities, one can generalize this as in the non-equivariant theory of Thom spectra of [47] to a functor assigning Thom spectrum to a map $f : X \to BF$ where F is the colimit of the subspaces of the $\Omega^V S^V$ consisting of non-equivariant homotopy automorphisms of S^V.

Definition 3.41. Let $f : X \to \phi^* BU_{\mathbb{R}}$ be a map of $\mathscr{O}$-algebras where $\phi : \mathscr{O} \to \mathscr{L}_{\mathbb{R}}$ is an operad map and ϕ^* is the forgetful functor. Then, the *Thom Real pre-spectrum* $Mf^{\mathscr{P}}$ is defined by setting $Mf^{\mathscr{P}}(V)$ to be the coequalizer

$$\begin{tikzcd}
U_{\mathbb{R}}(V)_+^{\mathrm{op}} \times (X \times_{BU_{\mathbb{R}}(V)} E_{|}U_{\mathbb{R}}(V))_+ \wedge S^V \arrow[d, shift right=2ex, "a"'] \arrow[d, shift left=2ex, "\pi"] \\
(X \times_{BU_{\mathbb{R}}(V)} E_{|}U_{\mathbb{R}}(V))_+ \wedge S^V \arrow[d] \\
Mf^{\mathscr{P}}(V)
\end{tikzcd}$$

where π is the projection and a is the action map with each $t \in U_{\mathbb{R}}(V)$ acts on the $(X \times_{BU_{\mathbb{R}}} E_{|}U_{\mathbb{R}})_+$ factor by multiplication by $U_{\mathbb{R}}$ and on S^V via its action on V. When $V \subseteq W$, the structure map $S^{W-V} \to \mathscr{T}_{C_2}(Mf^{\mathscr{P}}(V), Mf^{\mathscr{P}}(W))$ is defined as the based map sending $w \in W - V$ to the map defined as the geometric realization of a simplicial map that results from applying the inclusion $U_{\mathbb{R}}(V) \to U_{\mathbb{R}}(W)$ and adding w to the last coordinate (unless it is the basepoint). The Thom spectrum Mf is $L_{\mathbb{R}} Mf^{\mathscr{P}}$.

The operad action map $\mathscr{O}(n) \ltimes Mf^{\triangle n} \to Mf$ is constructed up to canonical isomorphism by applying $L_{\mathbb{R}}$ to a map $\mathscr{O}(n) \ltimes_{\mathscr{P}} Mf^{\mathscr{P} \triangle n} \to Mf$ constructed using the universal property of colimit for the appropriate diagram with objects ranging over the data of compact subspaces $K \subseteq \mathscr{O}(n)$, finite dimensional subspaces $V_1, \ldots, V_n$ in $\mathbb{C}^\infty$ with cocone maps

$$\left(\bigwedge_{i=1}^{n} Mf^{\mathscr{P}}(V_i) \right) \wedge \mathscr{M}(\phi_n|_K)_{\oplus_{i=1}^n V_i} \to Mf^{\mathscr{P}}$$

induced by maps

$$h_{K;V_1,\dots,V_n} : \left(\bigwedge_{i=1}^{n} Mf^{\mathscr{P}}(V_i)\right) \wedge \Sigma^{\infty}_{\bigoplus_{i=1}^{n} W_i,\mathscr{P}} T(\phi_n|_K)_{\bigoplus_{i=1}^{n} V_i,W} \to Mf^{\mathscr{P}}$$

by way of a canonical isomorphism $L_{\mathbb{R}}\Sigma^{\infty}_{W,\mathscr{P}} T(\phi_n|_K)_{\bigoplus_{i=1}^{n} V_i,W} \to \mathscr{M}(\phi_n|_K)_{\bigoplus_{i=1}^{n} V_i}$ for sufficiently large W (which can be chosen to be minimal to specify a unique map). The map $h_{K,V_1,\dots,V_k}$ is the based map arising from a simplicial map that applies the $\mathscr{O}$-algebra structure map to X and $U_{\mathbb{R}}$ coordinates (after identifying every $U_{\mathbb{R}}(V)$ with a subspace of $U_{\mathbb{R}}(\mathbb{C}^{\infty})$ and if the vector coordinates together are listed in order as $((v_i)_{i=1}^{n}, w)$ for a point with K coordinate k, the vector coordinate of the output is $\phi_n(k)(v_1 \oplus \dots \oplus v_n) + w$.

An additional compatibility of operad action maps in the above definition is that $\mathscr{O}(n) \ltimes Mf^{\triangle n}$ factors through $M(fa_n)$, where $a_n : \mathscr{O}(n) \times X^n \to X$ is the action map on X. The main utility of this notion of Thom spectrum is a highly structured model of the Thom diagonal.

Proposition 3.42. *The commutative triangle*

$$\begin{tikzcd} BU_{\mathbb{R}} \arrow[r, "\Delta"] \arrow[dr, "\mathsf{id}"'] & BU_{\mathbb{R}} \times BU_{\mathbb{R}} \arrow[d, "\pi_2"] \\ & BU_{\mathbb{R}} \end{tikzcd}$$

with π_1 the second projection induces a map $M\mathsf{id} \to M\pi_2$ of Thom spectra which is

$$MU_{\mathbb{R}}^{\mathbf{LMS}} \to BU_{\mathbb{R}+} \wedge MU_{\mathbb{R}}^{\mathbf{LMS}}$$

after composing with isomorphisms.

It is important for this project to transport this Thom diagonal to the Real EKMM $S_{\mathbb{R}}$-module setting. The fact that $BU_{\mathbb{R}+}$ appears in a tensoring rather than as part of a suspension spectrum is a technically significant point and aids in keeping the factors of

$BU_{\mathbb{R}+} \wedge MU_{\mathbb{R}}^{\mathbf{LMS}}$ separate as Quillen equivalences are applied to find highly structured Thom diagonals in other models.

Chapter 4

Strictification

Recall the V-little disks operad $\mathscr{D}_V$ of Example 2.22 and that V-fold loop spaces as examples of its algebras in $\mathscr{U}_G$. The spectrum $MU_{\mathbb{R}}^{\mathbf{EKMM}}$ and other commutative ring spectra in symmetric monoidal categories are $\mathscr{D}_V$-algebras as well. The [20] argument begins with a cohomological Thom isomorphism theorem for $\mathscr{D}_V$-algebras in spectra in which the target is a $\mathscr{D}_{V\oplus\mathbb{R}}$ algebra. It is cumbersome to replicate the argument equivariantly without developing an equivariant analogue to the result of [17] which allows one to replace algebras (in symmetric monoidal categories enriched by $\mathscr{U}$) over little $n+1$-cubes operad with monoid objects in the categories of algebras over little n-cubes with a canonical symmetric monoidal structure that commutes with the forgetful functor to the underlying category.

In this chapter, we demonstrate an equivariant version of the result that applies for all topological groups G using concrete methods. We will also prove equivalences between operads similar to $\mathscr{D}_V$, including the Steiner operads, which have applications in infinite loop space theory. Recent work of Szczesny [66] has compared little disk operads and their perpendicular products, in our lexicon, in the setting of colored operads.

4.1 Geometric Operads

Mostly for exposition, we begin by defining geometric operads and follow that with several examples that will be shown to be E_V operads. In this chapter, G refers to an arbitrary topological group.

Definition 4.1. An operad $\mathscr{O}$ in $\mathscr{U}_G$ is *unital* if $\mathscr{O}(0) \simeq *$.

The following notion can be defined for arbitrary unital operads and implicitly appears as part of a result of Boardman and Vogt [14], which we recall as Theorem 4.32 below.

Definition 4.2. Let $\mathscr{O}$ be a unital operad in $\mathscr{U}_G$. The disjointness relation $\perp = \perp_{\mathscr{O}}$ associated to $\mathscr{O}$ is a relation on $\mathscr{O}(1)$ defined by $x \perp y$ if and only if there exists a $z \in \mathscr{O}(2)$ such that $\gamma(z; 1, *) = x$ and $\gamma(z; *, 1) = y$.

Roughly speaking, a geometric operad is a unital operad that arises from a monoid and a well-behaved binary relation.

Definition 4.3. Let M be a monoid in $\mathscr{U}_G$ and $\perp$ be a binary relation on M that satisfies the following properties for all $a, b, c \in M$.

1. (Symmetry) If $a \perp b$, then $b \perp a$.
2. (Left invariance) If $a \perp b$, then $ca \perp cb$.
3. (Right stability) If $a \perp b$, then $a \perp bc$.
4. (Non-degeneracy) The condition $1 \not\perp a$ holds.

Then the *geometric operad $\mathscr{O}_M$ associated to* $(M, \perp)$ has $\mathscr{M}(n)$ given by the $G \times {\Sigma_n}^{\mathrm{op}}$-invariant subspace of M^n consisting of $(a_1, \ldots, a_n)$ such that for distinct i, j, the relation $a_i \perp a_j$ holds. The unit is the identity element of M regarded as $\mathscr{O}_M(1)$. The composition is defined in the same way as the composition of $\mathscr{D}_V$ was in Example 2.22. A *geometric operad* is an operad $\mathscr{O}$ equipped with data $(M, \perp)$ and an isomorphism $\phi : \mathscr{O} \to \mathscr{O}_M$.

Non-degeneracy is included in the above definition because it guarantees that symmetric group actions on the spaces of the operad are free and because the patterns of reasoning we use in concrete situations working with examples rely on it. Note that if $\mathscr{M}$ is the geometric operad associated to $(M, \perp)$, then $\mathscr{M}(1) = M$ with the operad composition and unit agreeing with those of the monoid.

The fundamental example of operads used in our work are the little disk operads which we recall again here with some additional notation.

Example 4.4. *Let V be an orthogonal G-representation and let $\mathbb{D}(V)$ denote the open unit disk in V. The subspace of $\mathscr{U}_G(\mathbb{D}(V), \mathbb{D}(V))$ consisting of $d[c, r]$ for $c \in \mathbb{D}(V)$ and $r \in (0, 1 - ||c||]$ defined by $d[c, r](x) = c + rx$ forms a submonoid $\mathscr{D}_V(1)$. The relation $\perp$ defined by $f \perp g$ if and only if the image of f and the image of g are disjoint satisfies the four necessary properties, and there is a corresponding geometric operad $\mathscr{D}_V$.*

A related family of operads is the Steiner operads, first studied in [64].

Example 4.5. *Let V be an orthogonal G-representation and let $R(V)$ be the submonoid of $\mathscr{U}_G(V, V)$ consisting of embeddings $f : V \to V$ such that*

$$||f(v) - f(w)|| \leq ||v - w|| \tag{4.1}$$

for every pair of vectors $v, w \in V$, regarded as a based space with the identity as the basepoint. Let $\mathscr{K}_V(1)$, the space of Steiner paths for V, be the space of based maps $I \to R(V)$ where the unit interval $I = [0, 1]$ is given the basepoint 1 with the pointwise multiplication monoid structure inherited from $R(V)$. The relation $\perp$ defined by $f \perp g$ if and only if $f(0)$ and $g(0)$ have disjoint images satisfies the four necessary properties. These give rise to a geometric operad $\mathscr{K}_V$ called the Steiner operad for V.

A map f satisfying inequality 4.1 is also called Lipschitz with constant 1. We will use basic properties of the relationship between differentiable and Lipschitz functions in Section

4.4.

Recall the parallel product of operads of Definition 3.35. Geometric operads are closed under the parallel product, and we will record this fact here.

Proposition 4.6. *Suppose $(\mathscr{O}_\alpha)_{\alpha\in A}$ is a family of geometric operands. Then, the parallel product $\prod^{\parallel}_{\alpha\in A}\mathscr{O}_\alpha$ is the geometric operad associated to $(\prod_{\alpha\in A}\mathscr{O}_\alpha(1), \perp_\parallel)$ where $(x_\alpha)_{\alpha\in A} \perp_\parallel (y_\alpha)_{\alpha\in A}$ if and only if for all $\alpha \in A$, $x_\alpha \perp_{\mathscr{O}_\alpha} y_\alpha$.*

However, there is another well-behaved product for geometric operads that we will define and use.

Definition 4.7. Suppose $(\mathscr{O}_\alpha)_{\alpha\in A}$ is a family of geometric operads. The monoid $\prod_{\alpha\in A}\mathscr{O}_\alpha(1)$ has a binary relation $\perp_\perp$ defined by $(x_\alpha)_{\alpha\in A} \perp_\perp (y_\alpha)_{\alpha\in A}$ if and only if there exists an $\alpha \in A$ such that $x_\alpha \perp_{\mathscr{O}_\alpha} y_\alpha$. Then the *perpendicular product of* $(\mathscr{O}_\alpha)_{\alpha\in A}$ is the geometric operad $\prod^{\perp}_{\alpha\in A}\mathscr{O}_\alpha$.

Another way of producing new geometric operads from old ones is as follows.

Proposition 4.8. *Suppose $\mathscr{O}$ is the geometric operad associated to $(M, \perp)$ and $f : N \to M$ is a map of monoid objects. If $f^*\perp$ is defined by $x f^* \perp y$ if and only if $f(x) \perp f(y)$, then there is a geometric operad $\mathscr{P}$ associated to $(N, \perp)$ and a morphism of operads $\mathscr{P} \to \mathscr{O}$ extending the map of unary operations f.*

The geometric operad structure on the Steiner operads can be understood in this way by mapping to a similarly defined geometric operad of tuples of disjoint embeddings $V \to V$.

4.2 Boardman–Vogt Tensor Products

In this section, we review the Boardman–Vogt tensor product introduced in [13], state the main result of the chapter, and discuss its main application.

We mention the following basic bit of relevant theory to help define the Boardman–Vogt tensor product.

Proposition 4.9. *In a cocomplete symmetric monoidal category $\mathscr{C}$, there is a left adjoint $F_{\mathsf{Op}} : \mathsf{Coll}(\mathscr{C}) \to \mathsf{Op}(\mathscr{C})$ to the forgetful functor and $\mathsf{Op}(\mathscr{C})$ is cocomplete.*

Proof. The proof is routine after using the result of [41] reinterpreting operads as monoids for a certain monoidal product on $\mathsf{Coll}(\mathscr{C})$ with unit U_{Coll}. □

Definition 4.10. The *Boardman–Vogt tensor product* $\mathscr{O} \otimes_{\mathsf{BV}} \mathscr{P}$ of two operads $\mathscr{O}$ and $\mathscr{P}$ in a cocomplete cartesian monoidal category $\mathscr{C}$ is defined as a coequalizer diagram of the form

$$F_{\mathsf{Op}}\left(\Sigma_n \times \coprod_{(m_1,m_2)\in\mathbb{N}^2 : m_1 m_2 = n} \mathscr{O}(m_1) \times \mathscr{P}(m_2)\right) \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{O} \amalg \mathscr{P} \longrightarrow \mathscr{O} \otimes_{\mathsf{BV}} \mathscr{P}$$

where f is induced by the map

$$\begin{array}{c} \mathscr{O}(m_1) \times \mathscr{P}(m_2) \\ \downarrow{\scriptstyle \mathsf{id}\times\Delta} \\ \mathscr{O}(m_1) \times \mathscr{P}(m_2)^{m_1} \\ \downarrow{\scriptstyle \iota_1\times(\iota_2)^{m_1}} \\ \mathscr{O} \amalg \mathscr{P}(m_1) \times \mathscr{O} \amalg \mathscr{P}(m_2)^{m_1} \\ \downarrow{\scriptstyle \gamma} \\ \mathscr{O} \amalg \mathscr{P}(m_1 m_2) \end{array}$$

and g is induced by the map

$$\begin{array}{c} \mathscr{O}(m_1) \times \mathscr{P}(m_2) \\ \downarrow{\scriptstyle \sim} \\ \mathscr{P}(m_1) \times \mathscr{O}(m_2) \\ \downarrow{\scriptstyle \mathsf{id}\times\Delta} \\ \mathscr{P}(m_1) \times \mathscr{O}(m_2)^{m_1} \\ \downarrow{\scriptstyle \iota_2\times(\iota_1)^{m_2}} \\ \mathscr{O} \amalg \mathscr{P}(m_2) \times \mathscr{O} \amalg \mathscr{P}(m_1)^{m_2} \\ \downarrow{\scriptstyle \gamma} \\ \mathscr{O} \amalg \mathscr{P}(m_1 m_2) \end{array}$$

on underlying G-spaces of collections. More generally, if $\mathscr{C}$ is symmetric monoidal and $\mathscr{O}$ and $\mathscr{P}$ are cocommutative comonoid objects in $\mathsf{Op}(\mathscr{C})$ with respect to the parallel product $(\otimes^{\|})$ of operads, then $\mathscr{O} \otimes_{\mathsf{BV}} \mathscr{P}$ is defined in the same way with $\times$ replaced with $\otimes$.

Our interest in the Boardman-Vogt tensor product lies in the following result.

Proposition 4.11. *Let $\mathscr{C}$ be a cocomplete symmetric monoidal category. Then $\mathcal{A}ss$ and $\mathcal{C}omm$ are canonically cocommutative comonoids. For any cocommutative comonoid $\mathscr{O}$ in $\mathsf{Op}(\mathscr{C})$ with respect to $\otimes^{\|}$, $\mathscr{C}[\mathscr{O}]$ is symmetric monoidal with monoidal product $X \otimes Y$ lifting the $\mathscr{C}$ monoidal product with the action maps given as the composites*

$$\begin{tikzcd}
\mathscr{O}(n) \otimes (X \otimes Y)^{\otimes n} \arrow[d, "\Delta \otimes \mathrm{id}"] \\
(\mathscr{O}(n) \otimes \mathscr{O}(n)) \otimes (X \otimes Y)^{\otimes n} \arrow[d, "f"] \\
(\mathscr{O}(n) \otimes X^{\otimes n}) \otimes (\mathscr{O}(n) \otimes Y^{\otimes n}) \arrow[d, "\gamma"] \\
X \otimes Y
\end{tikzcd}$$

with f being a rearrangement of factors. The category of monoids in this symmetric monoidal structure is equivalent to the category $\mathscr{C}[\mathscr{O} \otimes_{\mathsf{BV}} \mathcal{A}ss]$.

Our work here in spaces will be useful for work in symmetric monoidal categories of spectra because of the following elementary categorical fact.

Proposition 4.12. *Colimit -preserving strong symmetric monoidal functors induce functors that preserve cocommutative coalgebras in operads, preserve $\otimes_{\mathsf{BV}}$ between pairs of them, and preserves $\mathcal{A}ss$.*

4.3 E_V Operads

The notion of equivalence of operads in $G\mathscr{U}$ we use is that of what we name here a Strøm weak equivalence, a stronger notion than the standard one developed in [27] obtained by asking for levelwise equivariant weak equivalences on H fixed point sets for graph subgroups (as they are called in [10]) of $G \times {\Sigma_n}^{\mathrm{op}}$. The Strøm weak equivalences are also defined in greater generality.

Definition 4.13. A *Strøm weak equivalence of operads* $\phi : \mathscr{O} \to \mathscr{P}$ in $G\mathscr{U}$ is a morphism of operads such that each $n \neq 1$, $\phi_n : \mathscr{O}(n) \to \mathscr{P}(n)$ is a homotopy equivalence in the category $(G \times {\Sigma_n}^{\mathrm{op}})\mathscr{U}$ and $\mathscr{O}(1) \to \mathscr{P}(1)$ is a homotopy equivalence in $(G \times {\Sigma_1}^{\mathrm{op}})\mathscr{T}_G$ between spaces based by respective operad identities.

The relevance of this definition lies in the fact that change of operads behaves better in the topologically sensitive settings, especially the Real LMS and EKMM $S_{\mathbb{R}}$-module settings in which we are interested. Change of operads arising from a strong symmetric monoidal functor from $\mathscr{U}_G$ by a Strøm weak equivalence will induce a Quillen equivalence in any reasonable setting. Changes of operads that are Quillen equivalences compatible with the Thom diagonal are needed, and the extra strength of our equivalences will be useful to maintain control over the two smash factors of the target.

With our notion of equivalence fixed, we can define the notion of an E_V-operad.

Definition 4.14. An operad $\mathscr{O}$ is an E_V operad (with respect to Strøm weak equivalences) if and only if there exist operads $\mathscr{O}_1, \dots, \mathscr{O}_n$ (with n odd, for convenience in this statement) and a zig-zag of morphisms of operads

$$\mathscr{O} \leftarrow \mathscr{O}_1 \rightarrow \dots \leftarrow \mathscr{O}_n \rightarrow \mathscr{D}_V$$

such that each morphism is a Strøm weak equivalence.

A technical detail that should be considered is whether for an E_V operad $\mathscr{O}$ (in our sense), the operad $\Sigma^\infty \mathscr{O}_+$ has category of algebras equivalent to the category of algebras of a cofibrant replacement for an appropriate model category on operads in $\mathscr{M}_{S_{\mathbb{R}}}$.

The main result of this chapter can now be stated.

Theorem 4.15. *The operad $\mathcal{A}ss \otimes_{\mathsf{BV}} \mathscr{D}_V$ is an $E_{V\oplus\mathbb{R}}$ operad.*

We prove this from first principles in the remainder of the chapter.

4.4 From Little Disks to Steiner Paths

The goal of this section is to prove the following two results.

Theorem 4.16. *Let V be a finite-dimensional orthogonal G-representation. Then, the Steiner operad $\mathscr{K}_V$ is an E_V operad.*

Theorem 4.17. *Let $(V_\ell)_{\ell=1}^m$ be a finite family of orthogonal G-representations. Then, the perpendicular product $\prod_{1\le\ell\le m}^{\perp} \mathscr{D}_{V_\ell}$ is an $E_{\bigoplus_{1\le\ell\le m} V_\ell}$ operad.*

Our first task will be to construct and describe intermediate operads for use in zig-zags. We work with a family $(V_\ell)_{\ell=1}^m$ of orthogonal G-representations, and statements for a single vector space V are provided for a notationally simple and essentially comprehensive case. We must relate $\bigoplus_{\ell=1}^m V_\ell$ to the cartesian product of disks $\prod_{\ell=1}^m \mathbb{D}(V_\ell)$ to relate the operands to which they correspond. Let $\zeta : \bigoplus_{\ell=1}^m V_\ell \to \prod_{\ell=1}^m \mathbb{D}(V_\ell)$ be the diffeomorphism given by

$$\zeta(u) = \left(\frac{1}{\sqrt{1+\|u_1\|^2}} u_1, \dots, \frac{1}{\sqrt{1+\|u_m\|^2}} u_m \right) \tag{4.2}$$

for all direct sum decompositions $u = (u_1 \dots u_m)$ with each $u_\ell \in V_\ell$. This is indeed invertible with inverse

$$\zeta^{-1}(v) = \left(\frac{1}{\sqrt{1-\|v_1\|^2}} v_1, \dots, \frac{1}{\sqrt{1-\|v_m\|^2}} v_m \right)$$

for all direct sum decompositions of $v = (v_1, \dots, v_m) \in \prod_{\ell=1}^m \mathbb{D}(V_\ell)$. This ζ is a generalization of the map used in Example 2.23 to establish V-fold loop spaces are algebras over $\mathscr{D}_V$. For any orthogonal G-representaion V, we will also similarly refer to $V \to \mathbb{D}(V)$ defined by $v \mapsto \frac{1}{\sqrt{1+\|u_1\|^2}} u_1$ as ζ.

The following notion of a sufficiently small product of little disks will be used to form intermediate operads in our zig-zag.

Definition 4.18. For an orthogonal G-representation V, a ζ-little V-disk is an element $d \in \mathscr{D}_V(1)$ such that

$$\|\zeta^{-1} d\zeta(v) - \zeta^{-1} d\zeta(w)\| \le \|v - w\|. \tag{4.3}$$

for all $v, w \in V$. The subspace of $\mathscr{D}_V(1)$ consisting of ζ-little V disks is denoted $\mathscr{D}_V^\zeta(1)$.

We warm up by establishing some of the basic point-set topology of the subspace.

Proposition 4.19. *For any orthogonal G-representation V, the subspace of ζ-little V disks is G-stable and closed. The subspace $\prod^\perp_{1\le i\le m} \mathscr{D}_{V_i}^\zeta(1)$ (which will earn its name before the end of the section) of $\prod_{i=1}^m \mathscr{D}_{V_i}(1) = \prod^\perp_{1\le i\le m} \mathscr{D}_{V_i}(1)$ consisting of $d_1 \times \dots \times d_m$ where d_i is a ζ-little V_i-disk that is G-stable and closed. For every $d \in \prod^\perp_{1\le i\le m} \mathscr{D}_{V_i}^\zeta(1)$, $\zeta^{-1} d\zeta$ satisfies the inequality that appears as 4.3.*

Proof. The group G acts by isometries on V and $\mathbb{D}(V)$. For every pair of points $v, w \in V$, both sides of the inequality 4.3 are continuous in $d \in \mathscr{U}_G(\mathbb{D}(V), \mathbb{D}(V))$. Since $\mathbb{D}(V)$ is locally compact and Hausdorff, $\mathscr{U}_G(\mathbb{D}(V), \mathbb{D}(V))$ has the compact-open topology, and thus $D_V(1)$ can be seen to be a closed subspace by the usual methods. The claims on products of disks follow from properties of products. The last claim follows because distance-reducing maps of m metric spaces commute with the ℓ_2 product metric functor. $\square$

There is an alternative proof of the above proposition replacing the discussion of the topology of $\mathscr{D}_V(1)$ with the following characterization.

Proposition 4.20. *For any G-representation V, there is a homeomorphism*

$$\mathscr{D}_V(1) \to \{(c,r) : c \in \mathbb{D}(V), r \in (0, 1 - ||c||]\} \subseteq \mathbb{D}(V) \times (0,1].$$

such that

$$d \mapsto (d(0), 2||d(\frac{1}{2}v) - d(0)||)$$

for any unit vector $v \in V$. This map is the inverse of $(c,r) \mapsto d[c,r]$. This induces a homeomorphism on products $\prod^{\perp}_{1\leq i\leq m} \mathscr{D}_{V_i}(1) \to \{(c_1, r_1, \ldots, c_n, r_n) : c_i \in \mathbb{D}(V_i)$ and $r_i \in (0, 1 - ||c_i||]\}$

Proof. In the case of a single representation, fixing a choice for the vector v helps to prove the continuity of the map. The continuity of the inverse map follows by universal property. The product of homeomorphisms is a homeomorphism. □

We now turn our attention to characterizing the ζ-little V-disks among all little V-disks. We begin by checking a crucial algebraic property.

Lemma 4.21. *The subspace $\mathscr{D}^{\zeta}_V(1)$ of $\mathscr{D}_V(1)$ is closed under the monoid operation, composition. This operation is described by the equation*

$$d[c_1, r_1]d[c_2, r_2] = d[c_1 + r_1c_2, r_1r_2]$$

for all $c_1, c_2 \in \mathbb{D}(V)$, $r_1 \in (0, 1 - ||c_1||]$, and $r_2 \in (0, 1 - ||c_2||]$. The subspace $\prod^{\perp}_{1\leq i\leq m} \mathscr{D}^{\zeta}_{V_i}(1)$ of $\prod^{\perp}_{1\leq i\leq m} \mathscr{D}_{V_i}(1)$ is closed under the monoid operation.

Proof. If d and d' are ζ-little V-disks then $\zeta^{-1}dd'\zeta = (\zeta^{-1}d\zeta)(\zeta^{-1}d'\zeta)$ and inequality 4.23 can be applied for d and d' to obtain it for dd'. This conclusion passes to products. The explicit formula for products of little disks is checked by

$$\begin{aligned}d[c_1, r_1]d[c_2, r_2](p) = d[c_1, r_1](c_2 + r_2 p)\\ =&c_1 + r_1(c_2 + r_2 p)\\ =&(c_1 + r_1 c_2) + r_1 r_2 p\\ =& d[c_1 + r_1 c_2, r_1 r_2](p)\end{aligned}$$

for every $p \in \mathbb{D}(V)$. □

The next proposition provides some elementary examples and closure properties of ζ-little V-disks. Every little V-disk centered at the origin and every little V-disk smaller than a ζ-little V disk with the same center is a ζ little V-disk.

Lemma 4.22. *For a G-representation V and for every $r \in (0,1]$, $d[0,r] \in \mathscr{D}^{\zeta}_{V}(1)$. If $0 < r' \leq r \leq 1$, and $c \in \mathbb{D}(V)$ is such that $d[c,r] \in \mathscr{D}^{\zeta}_{V}(1)$, then $d[c,r'] \in \mathscr{D}^{\zeta}_{V}(1)$.*

More generally, for every $(r_1, \ldots, r_m) \in (0,1]^m$, $d[0,r_1] \times \ldots \times d[0,r_m] \in \prod^{\perp}_{1 \leq i \leq m} \mathscr{D}^{\zeta}_{V_i}(1)$. If for each i between 1 and m, $0 < r'_i \leq r_i \leq 1$ and $c_i \in \mathbb{D}(V_i)$ is such that $d[c_1,r_1] \times \ldots \times d[c_m,r_m] \in \prod^{\perp}_{1 \leq i \leq m} \mathscr{D}^{\zeta}_{V_i}(1)$, then $d[c_1,r'_1] \times \ldots \times d[c_m,r'_m] \in \prod^{\perp}_{1 \leq i \leq m} \mathscr{D}^{\zeta}_{V_i}(1)$

Proof. For every non-zero $p \in V$, the derivative $D(\zeta^{-1} d_{0,r} \zeta)_p$ is a symmetric linear operator with eigenvalue $r(1 + (1-r^2)||p||^2)^{-3/2}$ on eigenspace $\mathbb{R}p$ and eigenvalue $r(1 + (1-r^2)||p||^2)^{-1/2}$ on the eigenspace $(\mathbb{R}p)^{\perp}$. Because these eigenvalues are positive and bounded above by 1 and V is convex, the first claim of the result follows. The claims on the cartesian product follow similarly by eigenspace decompositions of the derivative on each V_i.

The second conclusion follows from lemma 4.21 and the first because when $d_{c,r}$ is a ζ-little disk and $0 < r' \leq r$, the identity $d[c,r'] = d[c,r]d[0,r'/r]$ holds. In the case of the product one composes on the right with $d[0, r'_1, r_1] \times \ldots \times d[r'_m/r_m]$. □

We employ more analysis to find sufficiently many ζ-little V-disks and begin with a useful inequality. The strategy of the proof repeats the strategy of Lemma 4.22.

Lemma 4.23. *The maps $\zeta : V \to \mathbb{D}(V)$ for an orthogonal G-representation V and $\zeta : \bigoplus_{i=1}^m V_i \to \prod_{i=1}^m \mathbb{D}(V_i)$ decrease distances, i.e.*

$$||\zeta(v) - \zeta(w)|| \le ||v - w||$$

for all v, w in the domain of ζ.

Proof. We prove this for the case of $V_1, \dots, V_m$. The same proof applies to V as a list of length 1. The Fréchet derivative of ζ at a point $p = (p_1, \dots, p_m)$, $D\zeta_p$, is a symmetric linear operator with eigenvalue $(1 + ||p_i||^2)^{-3/2}$ for eigenspace $\mathbb{R}p_i$ (when $p_i \neq 0$) and eigenvalue $(1+||p_i||^2)^{-1/2}$ for eigenspace $(\mathbb{R}p_i)^{\perp_{V_i}}$, with each in the interval $(0,1]$ Since $\bigoplus_{i=1}^m V_i$ is convex and open in itself, ζ is Lipschitz with constant 1. □

Computing minimal Lipschitz constants for little V-disks or finding that there are none is a difficult geometric task. Rather than doing this, we use the next two propositions to argue enough ζ-little V-disks exist for our purposes.

Lemma 4.24. *The evaluations at the origin*

$$\mathscr{D}_V^\zeta(1) \to \mathbb{D}(V) \text{ and more generally } \prod_{1 \le i \le m} \mathscr{D}_{V_i}^\zeta(1) \to \prod_{i=1}^m \mathbb{D}(V)$$

, sending $d \mapsto d(0)$ and $d_1 \times \dots d_m \mapsto d_1(0) \times \dots \times d_m(0)$ have a G-equivariant continuous section.

Proof. It suffices to consider the case of a single V by the universal property of products. For $c \in \mathbb{D}(V)$, we define

$$\underline{\mathrm{r}}(c) = \min \left\{ \frac{1 - ||c||}{2}, \left(1 - \left(\frac{1 + ||c||}{2}\right)^2\right)^{3/2} \right\}.$$

It suffices to show by proposition 4.20 and continiuty of $\underline{\mathrm{r}}(c)$ in the variable $c \in \mathbb{D}(V)$ that $d_{c, \underline{\mathrm{r}}c}$ is a ζ-little V-disk because G acts orthogonally.

The restriction $\widetilde{\zeta^{-1}}$ of ζ^{-1} to the compact ball $\overline{B}_V(c, \frac{1-\|c\|}{2})$ has derivative $D(\widetilde{\zeta^{-1}})_p$ at $p \in \mathbb{D}(V)$ which is symmetric everywhere with maximal eigenvalue $(1 - (\frac{1+\|p\|}{2})^2)^{-3/2}$. This eigenvalue depends only on and increases in $\|p\|$, which is maximized at $\frac{1+\|c\|}{2}c$. Because the domain $\overline{B}_V(c, \frac{1-\|c\|}{2})$ is convex and open in $\mathbb{D}(V)$, it follows that $\widetilde{\zeta^{-1}}$ is Lipschitz with constant $(1 - (\frac{1+\|c\|}{2})^2)^{3/2}$.

We have a codomain restriction $d\widetilde{[c, \underline{\mathsf{r}}(c)]} : \mathbb{D}(V) \to \overline{B}_V(c, \frac{1-\|c\|}{2})$ of $d[c, \underline{\mathsf{r}}(c)]$. By the above analysis and lemma 4.23 the product of Lipschitz constants of the factors of $\widetilde{\zeta^{-1}}d\widetilde{[c, \underline{\mathsf{r}}(c)]}\zeta = \zeta^{-1}d_{w,r_w}\zeta$ gives a Lipschitz constant of at most 1 for the composite, as desired. □

Proposition 4.25. *If $d[c,r] \in \mathscr{D}^{\zeta}_V(1)$ and $c' \in V$ satisfies $\|c'\| \le \|c\|$, then $d[c', r] \in \mathscr{D}^{\zeta}_V(1)$. If $d[c_1, r_1] \times \ldots \times d[c_n, r_n] \in \prod^{\perp}_{1\le i\le n} \mathscr{D}_{V_i}(1)$ and $\|c'_i\| < \|c_i\|$ for all i, then $d[c_1, r'_1] \times \ldots \times d[c_n, r'_n]$.*

Proof. The case of products follows from the case of a single orthogonal G-representation V using the ℓ_2 functoriality trick.

Let $p \in V$ be arbitrary aside from the finite number of exceptions to the claims we make in our argument. Our goal is to compare $D(\zeta^{-1}d_{c',r}\zeta)_p$ to $D(\zeta^{-1}d_{c,r}\zeta)_{p'}$ for a well-chosen $p' \in V$. We assume $d_{c',r}\zeta(p) \ne 0$ and note that at most one element of V is excluded in this way. Let c'' be the intersection of the ray beginning at c' and extending in the direction parallel to the ray from 0 to $\zeta(p)$ with the sphere of radius $\|c\|$ centered at 0. Let $T : V \to V$ be a (not neccessarily G-equivariant) orthogonal linear isomorphism such that $Tc'' = c$ and take $p' = T^{-1}p$.

We will use a few basic relevant facts. First note that $D(d_{c',r})_y = r\mathsf{id} = D(d_{c'',r})_y$ for each $y \in \mathbb{D}(V)$. Let $\tilde{T} : \mathbb{D}(V) \to \mathbb{D}(V)$. Then $\tilde{T}\zeta = \zeta\tilde{T}$, $\tilde{T}d_{c'',r} = d_{c,r}\tilde{T}$, and $T = DT_x$ for any $x \in V$.

The construction of c'' implies $d_{c'',r}\zeta(p) = ad_{c,r}p$ for some $a \ge 1$. This implies that $D(\zeta^{-1})_{d_{c',r}\zeta(p)}$ and $D(\zeta^{-1})_{d_{c'',r}\zeta(p)}$ have the same eigenspaces, with corresponding eigenvalues on the latter greater than those of the former. This yields the crucial inequality

$$\|D\zeta^{-1}_{d_{c',r}\zeta(p)}w\| \le \|D\zeta^{-1}_{d_{c'',r}\zeta(p)}w\|$$

for every $w \in V$.

Putting all these pieces together we have

$$\begin{aligned}
\|D(\zeta^{-1}d_{c',r}\zeta)_p v\| &= \|D(\zeta^{-1})_{d_{c',r}\zeta(p)} D(d_{c'',r})_{\zeta(p)} D\zeta_p v\| \\
&\leq \|D(\zeta^{-1})_{d_{c'',r}\zeta(p)} D(d_{c'',r})_{\zeta(p)} D\zeta_p v\| \\
&= \|D(\zeta^{-1}d_{c'',r}\zeta T)_{p'} T^{-1}v\| \\
&= \|D(T\zeta^{-1}d_{c,r}\zeta)_{p''} T^{-1}v\| \\
&= \|TD(\zeta^{-1}d_{c,r}\zeta)_{p'} T^{-1}v\| \\
&= \|D(\zeta^{-1}d_{c,r}\zeta)_{p'} T^{-1}v\|
\end{aligned}$$

for all $v \in V$ and $\|T^{-1}v\| = \|v\|$. From this and the convexity of V, we can conclude that the minimal Lipschitz constant of $\zeta^{-1}d_{c',r}\zeta$ is no more than the minimal Lipschitz constant of $\zeta^{-1}d_{c,r}\zeta$, which is bounded above by 1, by hypothesis. We can extend these conclusions by continuity to the excluded p (when it exists) or provide a similar argument. The conclusion follows. $\square$

This completes our focused study of $\mathscr{D}_V^{\zeta}(1)$ and more generally $\prod_{1\leq i\leq m}^{\perp} \mathscr{D}_{V_i}^{\zeta}(1)$. The space $D_V^{\zeta}(1)$, as the notation suggests, is the space of unary operations of the operad defined next.

Definition 4.26. For an orthogonal G-representation V, the ζ*-little* V*-disks operad* $\mathscr{D}_V^{\zeta}$ is the geometric operad for the submonoid $\mathscr{D}_V^{\zeta}(1)$ of $\mathscr{D}_V(1)$ with the restriction of the disjointness relation of $\mathscr{D}_V$ as in proposition 4.8. The operad $\prod_{1\leq i\leq m}^{\perp} \mathscr{D}_{V_i}^{\zeta}$ is defined as in 4.7.

Proposition 4.27. *For each n, in the commutative squares of inclusions*

$$\begin{tikzcd}
\mathscr{D}_V^{\zeta}(n) \arrow[r] \arrow[d] & \mathscr{D}_V(n) \arrow[d] \\
\mathscr{D}_V^{\zeta}(1)^n \arrow[r] & \mathscr{D}_V(1)^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_{1\leq i\leq m}^{\perp} \mathscr{D}_V^{\zeta}(n) \arrow[r] \arrow[d] & \prod_{1\leq i\leq m}^{\perp} \mathscr{D}_V(n) \arrow[d] \\
(\prod_{1\leq i\leq m}^{\perp} \mathscr{D}_V^{\zeta}(1))^n \arrow[r] & (\prod_{1\leq i\leq m}^{\perp} \mathscr{D}_V(1))^n
\end{tikzcd}$$

each map is the inclusion of a closed subset.

Proof. The $\perp$ relation for $\mathscr{D}_V$ is closed, as can be checked with Proposition 4.20 since $d_{c_1,r_1} \perp d_{c_2,r_2}$ if and only if $r_1 + r_2 \leq \|c_1 - c_2\|$. Since $\mathscr{D}_V(1)$ is locally compact Hausdorff, $\mathscr{D}_V(1)^n$ has the ordinary product topology identified with a Euclidean topology and $\mathscr{D}_V(n)$ is the intersection of closed sets corresponding to $\perp$ holding for each pair of coordinates. The product of the closed sets $\mathscr{D}_V^{\zeta}(1)^n \subseteq \mathscr{D}_V(1)^n$ is closed and its intersection with the subspace $\mathscr{D}_V(n)$ is also closed. □

Definition 4.28. For a G-space X, $F(X,n)$ denotes the configuration G-space of n distinct points in X. That is, $F(X,n)$ is the G-invariant open subspace of $(x_1,\ldots,x_n) \in X$ such that $x_i \neq x_j$ for all pairs of indices i,j with $i \neq j$.

Proposition 4.29. *The inclusion map*

$$\mathscr{D}_V^{\zeta} \to \mathscr{D}_V$$

or more generally,

$$\prod_{1\leq i\leq m}^{\perp} \mathscr{D}_{V_i}^{\zeta} \to \prod_{1\leq i\leq m}^{\perp} \mathscr{D}_{V_i}$$

is a Strøm weak equivalences of operads. For every $n \in \mathbb{N}$, there is a commutative diagram

$$\begin{tikzcd}
\prod^{\perp}_{1\leq i\leq n} \mathscr{D}_{V_i}^{\zeta}(n) \arrow[rr, "\subseteq"] \arrow[dr, "\phi_1"'] & & \prod^{\perp}_{1\leq i\leq n} \mathscr{D}_{V_i}(n) \arrow[dl, "\phi_2"] \\
& F(\prod_{1\leq i\leq n} \mathbb{D}(V_i), n) &
\end{tikzcd}$$

in $G \times {\Sigma_n}^{\mathrm{op}}$-spaces corresponding to the evaluation at the origins of the little disks, and each map in the diagram is a $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant homotopy equivalence.

Proof. The first step of this proof is that ϕ_1 has a section ψ_1. It follows from this that ϕ_2 has a section ψ_2 such that the restriction of ψ_2 to $\mathscr{D}_V^{\zeta}(n)$ is the inclusion composed with ψ_2.

For $n = 1$, Lemma 4.24 supplies the section. Let $\underline{\mathsf{r}}_j$ be defined as $\underline{\mathsf{r}}$ in the proof of Lemma

4.24 for the choice of $V = V_j$. We define ψ_1 by

$$\psi_1(c^1, \ldots, c^n) = (d^1, \ldots, d^n)$$

where with the notation $c^i = (c^i_1, \ldots, c^i_n)$ and $d^i = (d^i_1, \ldots, d^i_n)$ the coordinates are given by

$$d^i_j = d[c^i_j, \min(\underline{\mathrm{r}}_j(c^i_j), \frac{1}{2} \min_{k \neq i} \max_{\ell} \|c^i_\ell - c^k_\ell\|)]$$

and note that the map is well-targeted. The c and r coordinates realize $\prod^{\perp}_{1 \le i \le n} \mathscr{D}^{\zeta}_{V_i}(n)$ and $\prod^{\perp}_{1 \le i \le n} \mathscr{D}_{V_i}(n)$ as subsets of Euclidean space. Up to this homeomorphism $\psi_1 \phi_1$ and $\psi_2 \phi_2$ are homotopic to their respective identities by a linear homotopy. □

Proposition 4.30. *There is a Strøm weak equivalence of operads $\mathscr{D}^{\zeta}_V \to \mathscr{K}_V$ or more generally $\prod^{\perp}_{1 \le i \le m} \mathscr{D}^{\zeta}_{V_i} \to \mathscr{K}_{\oplus^m_{i=1} V_i}$. For each n, there is a commutative square*

$$\begin{tikzcd}
\prod^{\perp}_{1 \le i \le m} \mathscr{D}^{\zeta}_{V_i}(n) \arrow[r] \arrow[d, "\phi_1"'] & \mathscr{K}_{V_1 \oplus \ldots \oplus V_m}(n) \arrow[d, "\phi_3"] \\
F(\prod^n_{i=1} \mathbb{D}(V_i), n) \arrow[r, "\sim"] & F(\oplus^m_{i=1} V_i, n)
\end{tikzcd}$$

with each map a $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant homotopy equivalence.

Proof. The last claim implies the rest. Take the map ϕ_3 to be evaluation at $0 \in I$ and $0 \in V$. The bottom arrow induced by ζ is an isomorphism, ϕ_1 is an equivariant homotopy equivalence by Proposition 4.29, and ϕ_3 is a $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant homotopy equivalence by the proof of a result of [Steiner]. We need to construct the operad map and in such a way that the resulting diagram commutes.

For this purpose, we take the map $f : \prod^{\perp}_{1 \le i \le m} \mathscr{D}^{\zeta}_{V_i}(1) \to \mathscr{K}_{\oplus_{i=1}^m V_i}(1)$ defined by

$$f((d[c_1, r_1], \dots d[c_n, r_n]))(t) =$$
$$\begin{cases} \zeta^{-1} d[(1-2t)c_1, r_1] \times \dots \times d[(1-2t)c_n, r_n]\zeta, & \text{if } 0 \le t \le \frac{1}{2}, \\ \zeta^{-1} d[0, (2-2t)r_1 + (2t-1)] \times \dots \times d[0, (2-2t)r_n + (2t-1)]\zeta, & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

and we check that f is a monoid homomorphism preserving $\perp$ and inducing a map of geometric operads. Most notably, it is well defined by Lemmas 4.25, 4.22, and 4.21. Continuity and the monoid homomorphism property can be checked using the previous results of this section. Unpacking the definitions also shows the required triangles commute. □

We prove the main results of this section together.

Proof of Theorems 4.16 and 4.17. There is a zig-zag of Strøm weak equivalences

$$\prod^{\perp}_{1 \le i \le m} \mathscr{D}_{V_i} \leftarrow \prod^{\perp}_{1 \le i \le m} \mathscr{D}^{\zeta}_{V_i} \to \mathscr{K}_{\oplus_{i=1}^m V_i} \leftarrow \mathscr{D}^{\zeta}_{\oplus_{i=1}^m V_i} \to \mathscr{D}_{\oplus_{i=1}^m V_i}$$

by Propositions 4.29 and 4.30. □

4.5 Strictification

The main theorem of this section is the following.

Theorem 4.31. *Let V be an orthogonal G-representation. There is a Strøm weak equivalence $\eta : \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V \to \mathcal{A}ss \otimes_{\mathsf{BV}} \mathscr{D}_V$ where $\mathbb{R}$ denotes the trivial representation.*

The case of $V = 0$ is well known; therefore, we assume $V \neq 0$ in this section. We will need to use the following result.

Theorem 4.32 ([14]). *Let $\mathscr{O}$ be a unital operad in $\mathscr{U}_G$ such that the corresponding relation $\perp$ is closed. Consider the relation $\sim$ on $\mathcal{Ass}(n) \times \mathscr{O}(1)^n$ where*

$$(\pi_1, x_1^1, \ldots, x_n^1) \sim (\pi_2, x_1^2, \ldots, x_n^2)$$

if and only if $x_i^1 = x_i^2$ for all i and regarding π_1 and π_2 as elements of Σ_n, if $\pi_1^{-1}(i) < \pi_1^{-1}(j)$ and $\pi_2^{-1}(i) > \pi_2^{-1}(j)$, then $x_i^1 \perp x_j^1$. Then, there is a homeomorphism $\mathcal{Ass}(n) \times \mathscr{O}(1)^n / \sim \to \mathcal{Ass} \otimes_{\mathsf{BV}} \mathscr{O}(n)$ sending $[(\pi, x_1, \ldots, x_n)] \mapsto \gamma(\pi \otimes 1; 1 \otimes x_1, \ldots, 1 \otimes x_n)$.

Proof. The theorem of Boardman and Vogt is as above, except that the category from which $\mathscr{O}$ is taken is that of Bourbaki topological spaces, and there is no condition on $\perp$. The proof for the above statement is identical, except for the use of $\perp$ being closed to guarantee that the quotient spaces constructed in the argument are in $\mathscr{U}_G$. □

We use the notation $[\pi; x_1, \ldots, x_n]$ for $\gamma(\pi \otimes 1; 1 \otimes x_1, \ldots, 1 \otimes x_n)$ in the remainder of this section. It is helpful to have a better understanding of the equivalence relation involved in the above theorem. We relate the equivalence classes to partial orders through the combinatorial fact, which we prove next.

Definition 4.33. For each $pi \in \Sigma_n$, let $<_\pi$ denote the total order on $\{1, \ldots n\}$ such that $i <_\pi j$ if and only if $\pi^{-1}(i) < \pi^{-1}(j)$

Let $\perp$ be a symmetric antireflexive relation on $\{1, \ldots, n\}$. To each $\pi \in \Sigma_n$ we can assign a strict partial order $<_\pi^\perp$ defined by $i <_\pi j$ if and only if there exists a finite sequence $i = k_0, k_1, \ldots, k_m = j$ in $\{1, \ldots, n\}$ such that $\pi^{-1}(k_0) < \pi^{-1}(k_1) < \ldots < \pi^{-1}(k_m)$ and $k_\ell \not\perp k_{\ell+1}$ for integers ℓ such that $0 \leq \ell < m$.

Proposition 4.34. *Suppose $\perp$ is a symmetric antireflexive relation on $\{1, \ldots, n\}$ and $\sim$ is the equivalence relation on Σ_n characterized by $\pi_1 \sim \pi_2$ if and only if for all $i, j \in \{1, \ldots, n\}$, if $\pi_1^{-1}(i) < \pi_1^{-1}(j)$ and $\pi_2^{-1}(i) > \pi_2^{-1}(j)$, then $i \perp j$. Then $\pi_1 \sim \pi_2$ if and only if the strict partial orders $<_{\pi_1}^\perp$ and $<_{\pi_2}^\perp$ are the same relation.*

Proof. First suppose that $\pi_1 \sim \pi_2$ and take i, j arbitrary such that $i <^{\perp}_{\pi_1} j$. We find a sequence $i = k_0, k_1, \ldots, k_m = j$ as in Definition 4.33 and note that the elements of this sequence are necessarily distinct. Therefore, it follows $\pi_2^{-1}(k_\ell) \neq \pi_2^{-1}(k_{\ell+1})$ for each ℓ with $0 \leq \ell < m$. The assumption that $\pi_2^{-1}(k_\ell) > \pi_2^{-1}(k_{\ell+1})$ is contradictory because it implies $k_\ell \perp k_{\ell+1}$ and by the trichomotomy law $\pi_2^{-1}(k_\ell) < \pi_2^{-1}(k_{\ell+1})$. We conclude that $i <^{\perp}_{\pi_2} j$. Because i and j are arbitrary and because $\sim$ is symmetric, it follows $<^{\perp}_{\pi_1}$ and $<^{\perp}_{\pi_2}$ are the same relation.

Conversely, suppose $<^{\perp}_{\pi_1}$ and $<^{\perp}_{\pi_2}$ are the same relation and fix $i, j \in \{1, \ldots, n\}$ such that $\pi^{-1}(i) < \pi_1^{-1}(j)$ and $\pi_2^{-1}(i) < \pi_2^{-1}(j)$. Assume towards a contradiction that $i \not\perp j$. Then $i <^{\perp}_{\pi_1} j$ and $i >^{\perp}_{\pi_2} j$, the latter of which is equivalent to $i >^{\perp}_{\pi_1} j$ by hypothesis, contradicting that $<^{\perp}_{\pi_1}$ is a strict partial order. We conclude $i \perp j$. This proves $\pi_1 \sim \pi_2$. $\square$

We begin by constructing η. We will need some notation.

Definition 4.35. For any $n \in \mathbb{N}$ and strict partial order $\prec$ on $\{1, \ldots, n\}$, let $X(n, \prec)$ be the G-stable subspace of $\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n)$ with elements

$$((d[c_1, r_1], d_1), \ldots, (d[c_n, r_n], d_n))$$

such that for every distinct $i, j \in \{1, \ldots, n\}$,

1. if $i \prec j$, then $c_i \leq c_j$ or $d_i \perp_{\mathscr{D}_V} d_j$, and
2. if $i \nprec j$ and $j \nprec i$, then $d_i \perp_{\mathscr{D}_V} d_j$.

The subspaces $X(n, \prec)$ is G-stable because $\perp$ is a G-stable relation and the action of G on $\mathbb{D}(\mathbb{R})$ is trivial. The additional properties required are described by the following two lemmas.

Lemma 4.36. *Each $X(n, \prec)$ is closed in $\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n)$ and the subspaces $X(n, \prec)$ as $\prec$ varies cover $\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n)$.*

Proof. The first claim is checked using the fact that $\perp_{\mathscr{D}_V}$ is a closed relation. An arbitrary element $((d[c_1, r_1], d_1), \ldots, (d[c_n, r_n], d_n))$ is in $X(n, <_\pi)$ where π is the element of Σ_n such that $\pi^{-1}(i) < \pi^{-1}(j)$ if and only if $c_i < c_j$ or both $c_i = c_j$ and $i < j$. □

Lemma 4.37. *There is an intersection identity $X(n, \prec) \cap X(n, \prec') = X(n, \prec'')$ where $\prec''$ is defined by $a \prec'' b$ if and only if $a \prec b$ and $a \prec' b$, i.e. $\prec'' = \prec \cap \prec'$.*

Proof. ($\subseteq$) Take $((d[c_1, r_1], d_1), \ldots, (d[c_n, r_n], d_n)) \in X(n, \prec) \cap X(n, \prec')$ and fix distinct $i, j \in \{1, \ldots, n\}$.

1. Suppose $i \prec'' j$. Then $i \prec j$, from which $c_i \leq c_j$ or $d_i \perp_{\mathscr{D}_V} d_j$ follow.

2. Suppose $i \nprec'' j$ and $j \nprec'' i$. Suppose toward a contradiction that $d_i \not\perp_{\mathscr{D}_V} d_j$. This hypothesis implies i and j are ordered by both $\prec$ and $\prec'$ by the definition of $X(n, \prec) \cap X(n, \prec')$. The assumptions regarding $\prec''$ imply that the two orders disagree, from which we obtain $c_i = c_j$, which implies $d[c_i, r_i] \not\perp_{\mathscr{D}_{\mathbb{R}}} d[c_j, r_j]$. The definition of $\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V$ as a geometric operad guarantees from the non-disjointness of the first coordinates that $d_i \perp_{\mathscr{D}_V} d_j$, which gives the desired contradiction. We conclude that $d_i \perp_{\mathscr{D}_V} d_j$.

($\supseteq$) By symmetry, it suffices to show $X(n, \prec'') \subseteq X(n, \prec)$. Fix

$$((d[c_1, r_1], d_1), \ldots, (d[c_n, r_n], d_n)) \in X(n, \prec'')$$

and fix distinct $i, j \in \{1, \ldots, n\}$.

1. Suppose $i \prec j$. In the case $i \prec' j$, then $i \prec'' j$, implying that $c_i \leq c_j$ or $d_i \perp_{\mathscr{D}_V} d_j$. In the remaining case of $i \nprec' j$, we have $i \nprec'' j$ and $j \nprec'' i$. Consequently, $d_i \perp_{\mathscr{D}_V} d_j$, which tautologically implies $c_i \leq c_j$ or $d_i \perp_{\mathscr{D}_V} d_j$.

2. Suppose $i \nprec j$ and $j \nprec i$. Therefore, $i \nprec'' j$ and $j \nprec'' i$, which implies $d_i \perp_{\mathscr{D}_V} d_j$.

Hence $((d[c_1, r_1], d_1), \ldots, (d[c_n, r_n], d_n)) \in X(n, \prec)$, as desired. □

Definition 4.38. Suppose $\prec, \prec'$ are two strict partial orders on a set S. Then, for all $s_1, s_2 \in S$, $s_1 \prec s_2$ implies $s_1 \prec' s_2$, then we say $\prec'$ is a stronger order then $\prec$.

Lemma 4.39. *There is a map $\eta : \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V \to \mathcal{A}ss \otimes_{\mathsf{BV}} \mathscr{D}_V$ such that for every $n \in \mathbb{N}$ the restriction of $\eta_n : \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n) \to \mathcal{A}ss \otimes \mathscr{D}_V(n)$ to $X(n, \prec)$ maps*

$$((d_1^1, d_2^1), \ldots, (d_1^n, d_2^n)) \mapsto [\pi; d_2^1, \ldots, d_2^n]$$

for every partial order $\prec$ on $\{1, \ldots, n\}$ and for every $\pi \in \Sigma_n$ such that $<_\pi$ is a strong order than $\prec$.

Proof. The restrictions of η to $X(n, \prec)$ as prescribed are well-defined by Proposition 4.34 as well as the fact that for every partial order $\prec$ on $\{1, \ldots, n\}$, there is a total order stronger than $\prec$, which corresponds to an element of Σ_n. The map η is well-defined and continuous by Lemma 4.36 on the level of G-spaces by gluing the prescribed maps on the closed cover comprised of the subspaces $X(n, \prec)$ of their respective $\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n)$.

The resulting glued map is Σ_k equivariant by a routine argument. The operad unit is preserved, i.e. $\eta_1(1) = \eta_1((d[0,1], d[0,1]) = [1; 1] = 1$. Compatibility with composition is a consequence of the existence of commutative diagrams of the form

$$\begin{tikzcd}
X(k, \prec_0) \times \prod_{i=1}^k X(n_i, \prec_i) \arrow[r] \arrow[d] & X(\sum_{i=1}^k n_i, \prec_0 \bullet(\prec_1, \ldots, \prec_k)) \arrow[d] \\
\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(k) \times \prod_{i=1}^k \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n_i) \arrow[r] & \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(\sum_{i=1}^k n_i)
\end{tikzcd}$$

with vertical maps as inclusions and $\prec_0 \bullet(\prec_1, \ldots, \prec_k)$ defined by $p \prec_0 \bullet(\prec_1, \ldots, \prec_k) q$ if and only if

1. there exist distinct $\ell_1, \ell_2 \in \{1, \ldots, k\}$ such that $\sum_{i=1}^{\ell_1 - 1} n_i < p \leq \sum_{i=1}^{\ell_1} n_i$, $\sum_{i=1}^{\ell_2 - 1} n_i < q \leq \sum_{i=1}^{\ell_2} n_i$, and $\ell_1 \prec_0 \ell_2$, or

2. there exists an $\ell \in \{1, \ldots, k\}$ such that $\sum_{i=1}^{\ell_1 - 1} n_i < p \leq \sum_{i=1}^{\ell_1} n_i$, $\sum_{i=1}^{\ell - 1} n_i < q \leq \sum_{i=1}^{\ell} n_i$, and

$p - \sum_{i=1}^{\ell-1} n_i \prec_\ell q - \sum_{i=1}^{\ell-1} n_i$.

If elements $\pi_0, \ldots, \pi_k$ with $\pi_0 \in \Sigma_k = \mathcal{Ass}(k)$ and $\pi_i \in \Sigma_{n_i} = \mathcal{Ass}(n_i)$ are such that for each i, $<_{\pi_i}$ is stronger than $\prec_i$, then we observe that $<_{\gamma(\pi_0;\pi_1,\ldots,\pi_k)}$ is stronger than $\prec_0 \bullet (\prec_1, \ldots, \prec_k)$. This implies η commutes with composition maps. □

We fix $n \in \mathbb{N}$. The strategy of the proof of Theorem 4.31 below, is to find subspaces X and Y (up to homeomorphism) of the source and target of η_n respectively such that the inclusion maps of X and Y are homotopy equivalences and η_n restricts to a homeomorphism $X \to Y$. For notational simplicity, however, we describe X as a subspace of a Euclidean space.

Construction 4.40. Let $F : \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(1) \to ([-\infty, \infty]^2 \times V \times \mathbb{R})$ be the map defined by

$$F((d[a,s], d[c,r])) = \left(\zeta^{-1}(a-s), \zeta^{-1}(a+s), c, r\right)$$

where $\zeta : [-\infty, \infty] \to [-1, 1]$ is a monotone homeomorphism, such as the one from Section 4.4 continuously extended to the extended reals. Note that F is G-equivariant homeomorphism onto its image. The restriction of the n^{th} cartesian power of F to $\mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n)$, $F^n : \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n) \to (([-\infty, \infty]^2 \times V \times \mathbb{R}))^n$, is also a $G \times {\Sigma_n}^{\text{op}}$-equivariant homeomorphism onto its image X_0. The underlying set of X_0 is the set of $((x_1, y_1, c_1, r_1), \ldots, (x_n, y_n, c_n, r_n))$ such that

1. $x_i < y_i$ and $r_i \le 1 - \|c_i\|$ for all $i \in \{1, \ldots, n\}$, and

2. at least one of $y_i < x_j$, $y_j < x_i$, or $\|c_i - c_j\| \le r_i + r_j$ hold for all pairs of distinct $i, j \in \{1, \ldots, n\}$.

The $G \times {\Sigma_n}^{\text{op}}$-invariant subspace X_1 of X_0 consists of elements that satisfy the additional condition

3. for all $i \in \{1, \ldots, n\}$, $x_i, y_i \in \mathbb{R}$.

The $G \times \Sigma_n{}^{\mathrm{op}}$ invariant subspace X of X_1 consists of elements satisfying the conditions

4. the equation for all $i \in \{1, \ldots n\}$,

$$\zeta(\frac{y_i + x_i}{2}) + 1 = y_i - x_i = 2r_i$$

   holds for all $i \in \{1, \ldots n\}$, and

5. the inequality

$$\min_{i \neq j} \min(||c_i - c_j||, |\frac{y_i + x_i}{2} - \frac{y_j + x_j}{2}|) \geq 2 \max_k r_k$$

   is satisfied, where the minimum is taken over distinct indices $i, j \in \{1, \ldots, n\}$ and the maximum is taken over $k \in \{1, \ldots, n\}$

Our next goal is the following result.

Lemma 4.41. *The inclusion map $X \to X_0$ is a $G \times \Sigma_n{}^{\mathrm{op}}$ equivariant homotopy equivalence.*

Proof. This is a consequence of the following pair of lemmas. □

Lemma 4.42. *The inclusion map $X_1 \to X_0$ is a $G \times \Sigma_n{}^{\mathrm{op}}$-equivariant homotopy equivalence.*

Lemma 4.43. *The inclusion map $X \to X_1$ admits a $G \times \Sigma_n{}^{\mathrm{op}}$-equivariant deformation retraction.*

The next result is at the heart of our proof of Lemma 4.42.

Proposition 4.44. *Consider the subspace A of $[-\infty, \infty]^2$ consisting of (x, y) such that $x < y$ and let $B = A \cap \mathbb{R}^2$. The inclusion $\iota : B \to A$ is a homotopy equivalence via linear homotopies with a homotopy inverse $f = (f_1, f_2) : A \to B$ such that if $h = (h_1, h_2)$ is a homotopy from* id *to ιf,*

$$x \leq h_1((x, y), t) < h_2((x, y), t) \leq y \tag{4.4}$$

for all $(x, y) \in A$ and $t \in I$.

Proof. Conjugating with ζ reduces the proposition to the same question except A is a subspace of $[-1,1]^2$ and $B = A \cap (-1,1)^2$. In this case, we use $f(x,y) = (\frac{2x+y}{3}, \frac{x+2y}{3})$ and linear homotopies to arrive at the conclusion. □

Proof of Lemma 4.42. Apply Proposition 4.44 to each pair of coordinates labeled (x_i, y_i). The homotopies are well-targeted because inequality 4.4 guarantees condition 2 of 4.40 is satisfied for every $t \in I$. Equivariance is a routine check. □

Proof of Lemma 4.43. We first define several auxiliary maps $X_1 \to \mathbb{R}$ as follows. Let

$$\begin{aligned}
d(p) &= \min_{i \neq j} \min(\|c_i - c_j\|, |\frac{y_i + x_i}{2} - \frac{y_j + x_j}{2}|) \\
\upsilon_1(p) &= \max(0, \max_i(\frac{y_i + x_i}{2} - \zeta^{-1}(\min(1, \frac{y_i - x_i}{2} - 1)))), \\
\upsilon_2(p) &= \max_i(\frac{y_i + x_i}{2} - \zeta^{-1}(2r_i - 1)), \\
\upsilon_3(p) &= \max_i(\frac{y_i + x_i}{2}) - \zeta^{-1}(d(p) - 1)),
\end{aligned}$$

and

$$\phi(p) = \max(\upsilon_1(p), \upsilon_2(p), \upsilon_3(p))$$

where $p = ((x_1, y_1, c_1, r_1), \dots, (x_n, y_n, c_n, r_n))$. Now we define $\epsilon : X_1 \to X$ by

$$\epsilon = ((x_1', y_1', c_1', r_1'), \dots, (x_n', y_n', c_n', r_n'))$$

where

$$\begin{aligned}
x_i'(p) &= \frac{y_i + x_i}{2} - \phi(p) - \frac{1}{2}((\zeta(\frac{y_i + x_i}{2} - \phi(p)) + 1), \\
y_i'(p) &= \frac{y_i + x_i}{2} - \phi(p) + \frac{1}{2}((\zeta(\frac{y_i + x_i}{2} - \phi(p)) + 1), \\
c_i'(p) &= c_i, \text{ and} \\
r_i'(p) &= \frac{1}{2}\zeta(\frac{y_i + x_i}{2} - \phi(p))
\end{aligned}$$

with many properties to check, tediously but routinely. We note that ϕ is identically 0 on X and so ϵ restricts to the identity on X. Routine checks also verify that the linear interpolation homotopy between the inclusion $\iota : X_1 \to (\mathbb{R}^2 \times \mathbb{D}(V) \times \mathbb{R})^n$ and $\iota\epsilon$ has image in X_1 and that this homotopy is equivariant. $\square$

This completes our analysis of the source of η_n. We now turn our attention to the target of η_n.

Construction 4.45. We define Y_0 as the subspace of $\mathcal{A}ss \otimes_{\mathsf{BV}} \mathscr{D}_V(n)$ consisting of

$$q = [\pi; d[c_1, r_1], \ldots, d[c_n, r_n]]$$

such that

1. $r_{\pi^{-1}(i)} < r_{\pi^{-1}(j)}$ whenever $\pi^{-1}(i) < \pi^{-1}(j)$ and $d[c_{\pi^{-1}(i)}, r_{\pi^{-1}(i)}] \not\subset d[c_{\pi^{-1}(j)}, r_{\pi^{-1}(j)}]$, and
2. $r_i < 1 - ||c_i||$ for all $i \in \{1, \ldots, n\}$.

Let Y_1 be defined as the subspace of $\mathbb{D}(V) \times (0, 1])^n$ such that a point

$$p = ((c_1, r_1), \ldots, (c_n, r_n)) \in Y_1$$

if and only if

1. the strict inequality for little disks $r_i < 1 - ||c_i||$ holds, and
2. for any pair of distinct $i, j \in \{1, \ldots, n\}$ such that $||c_i - c_j|| < r_i + r_j$ we have $r_i \neq r_j$.

The subspace Y of Y_1 consists of all points p satisfying the addition condition that

3. the inequality

$$\min_{i \neq j} \min(||c_i - c_j||, |\zeta^{-1}(2r_i - 1) - \zeta^{-1}(2r_j - 1)|) \geq 2 \max_k r_k$$

is satisfied.

Our analysis of the target will show that Y is homotopy equivalent to it.

Lemma 4.46. *There is a sequence of maps $Y \to Y_1 \to Y_0 \to \mathcal{Ass} \otimes_{\mathsf{BV}} \mathscr{D}_V(n)$ consists of $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant homotopy equivalences.*

Proof. This will be shown by Lemmas 4.47, 4.48, and 4.49. □

Lemma 4.47. *The inclusion map $Y \to Y_1$ admits a $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant deformation retraction.*

Lemma 4.48. *The map $F : Y_1 \to Y_0$*

$$F((c_1, r_1), \dots, (c_n, r_n)) = [\pi; d[c_1, r_1], \dots, d[c_n, r_n]]$$

where $\pi \in \Sigma_n$ is any element such that $r_{\pi^{-1}(i)} < r_{\pi^{-1}(j)}$ and $\|c_i - c_j\| \le r_i + r_j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$ is a well-defined $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant homeomorphism.

Lemma 4.49. *The inclusion map $Y_0 \to \mathcal{Ass} \otimes_{\mathsf{BV}} \mathscr{D}_V(n)$ is a $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant homotopy equivalence.*

Proof of Lemma 4.47. Let $\psi : Y_1 \to \mathbb{R}$ be the $G \times {\Sigma_n}^{\mathrm{op}}$-equivariant map defined by

$$d(p) = \min_{i \neq j} \max(\|c_i - c_j\|, |\zeta^{-1}(2r_i - 1) - \zeta^{-1}(2r_j - 1)|),$$

let $\phi : Y_1 \to \mathbb{R}$ be the equivariant map defined by

$$\phi(p) = \max(0, \max_i(\zeta^{-1}(r_i)) - \zeta^{-1}(d(p) - 1))$$

for all $p = ((c_1, r_1), \dots, (c_n, r_n)) \in Y_1$. Define $\epsilon : Y_1 \to Y$ by

$$\epsilon = ((c'_1, r'_1), \dots, (c'_n, r'_n))$$

where

$$c_i'(p) = c_i, \text{ and}$$
$$r_i'(p) = \frac{1}{2}(\zeta(\zeta^{-1}(2r_i - 1) - \phi(p)) + 1)$$

This ϵ is a deformation retraction via a linear homotopy. □

Proof of Lemma 4.48. Well-definedness of F requires the existence of π, which follows from the existence of π such that $<_\pi$ extends the transitive closure $\prec$ of the relation R such that iRj if and only if $r_{\pi^{-1}(i)} < r_{\pi^{-1}(j)}$ and $\|c_i - c_j\| \leq r_i + r_j$. Suppose π' is another element of Σ_n satisfying the same conditions and i, j are such that $\pi^{-1}(i) < \pi^{-1}(j)$ and $\pi'^{-1}(i) > \pi'^{-1}(j)$. Then $\pi^{-1}(i) \nsucc \pi^{-1}(j)$ and $\pi'^{-1}(i) \nprec \pi'^{-1}(j)$. Using the conditions on π and π', it becomes a matter of Boolean logic to show that $\|c_i - c_j\| \leq r_i + r_j$.

Up to homeomorphism, the inverse map is well-defined and continuous by the universal property of quotients. □

Proof of Lemma 4.49. Let

$$k_i(q) = \sum \max((r_j + r_k) - \|c_j - c_k\|, 0)$$

where the sum is indexed over pairs (j, k) such that $i <_\pi^\perp j$ and $i <_\pi^\perp k$. The definition of $<_\pi^\perp$ implies the independence of the choice of representative π in the definition of k_i.

We define a homotopy inverse to the inclusion $g : \mathcal{A}ss \otimes_{\mathsf{BV}} \mathscr{D}_V(n) \to Y_0$ by

$$g(q) = [\pi; d[c_1, r_1'(q)], \ldots, d[c_n, r_n'(q)]]$$

where

$$r_i'(q) = \frac{1}{2^{1+k_i(q)}} \min_\ell r_\ell$$

and note that the well-definedness of all k_i implies the well-definedness of r'_i and g as well. Note that for each $\pi \in \Sigma_n$, g lifts to a map on $\mathscr{D}_V(1)^n$, which is homotopic to the identity up to a linear homotopy (after identifying $\mathscr{D}_V(1)^n$ with a subspace of a euclidean space using the c_i, r_i coordinates). The linear homotopies jointly preserve the $\sim$ equivalence relation, so g is homotopic to the identity. This homotopy restricts to a homotopy between $g|_{Y_0}$ and id_{Y_0}. □

When comparing Y with X, we should observe redundancy among the conditions.

Lemma 4.50. *Condition 3 of Construction 4.45 implies Condition 2 of Construction 4.45.*

Proof. Suppose condition 4 holds. Condition 2 is satisfied because otherwise condition 4 gives $0 \geq 2\max_k r_k$, which is impossible. □

Proof of Theorem 4.31. The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathscr{D}_{\mathbb{R}} \times^{\perp} \mathscr{D}_V(n) \arrow[r, "\eta_n"'] & \mathcal{Ass} \otimes_{\mathsf{BV}} \mathscr{D}_V(n) \end{tikzcd}$$

where the top horizontal map is the projection removing the x_i and y_i coordinates (which is well-defined as seen using Lemma 4.50), and the vertical maps are homotopy equivalences by Lemmas 4.41 and 4.46. The map $X \to Y$ is a homeomorphism because it is surjective and has an inverse guaranteed by Condition 4 of Construction 4.40. □

4.6 Mixed cofibrancy of E_V operads

In this final section, we compare our E_V operads to a cofibrant operad in $\mathscr{U}_G$ in the sense of [27] when G is a compact Lie group. We will show that the cofibrant replacement is a Strøm weak equivalence for E_V operads in our sense.

Definition 4.51. The operad $\mathscr{D}'_V$ in $\mathscr{U}_G$ is the geometric operad associated to $\mathscr{D}_V(1)$ and the relation $\perp'$ defined by $d[c_1, r_1] \perp' d[c_2, r_2]$ if and only if $\|c_1 - c_2\| < r_1 + r_2$, i.e. the closure of the images of $d[c_1, r_1]$ and $d[c_2, r_2]$ are disjoint.

Definition 4.52. For a finite-dimensional orthogonal G-representation V, the operad $\mathcal{D}_V$ in $\mathscr{U}_G$ is the non-geometric suboperad of $\mathscr{D}'_V$ such that $\mathcal{D}_V(n) = \mathscr{D}'_V(n)$ for $n \in \{0, 1\}$ and $\mathcal{D}_V(n)$ is the set of $(d[c_1, r_1], \dots, d[c_n, r_n])$ in $\mathscr{D}'_V(n)$ such that $\|c_i\| + r_i < 1$ for each $i \in \{1, \dots, n\}$.

Proposition 4.53. *For every finite dimensional G-representation V, the inclusion map $\mathcal{D}_V \to \mathscr{D}_V$ is a Strøm weak equivalence.*

Proof. The map $\mathscr{D}_V(n) \to \mathcal{D}_V(n)$ sending

$$(d[c_1, r_1], \dots, d[c_n, r_n]) \mapsto (d[c_1, r_1(1 - \sqrt{\|c_1\|})], \dots, d[c_n, r_n(1 - \sqrt{\|c_n\|})])$$

is well-defined, $G \times \Sigma_n{}^{\mathrm{op}}$-equivariant, and based when $n = 1$. Both witnessing homotopies of the desired equivalence take the form

$$\begin{array}{c} ((d[c_1, r_1], \dots, d[c_n, r_n]), t) \\ \downarrow \\ (d[c_1, tr_1(1 - \sqrt{\|c_1\|}) + (1-t)r_1], \dots, tr_n(1 - \sqrt{\|c_n\|}) + (1-t)r_n] \end{array}$$

which completes the proof up to routine verifications. □

Let us consider the case of V being an orthogonal G-representation that is topologized as a colimit of its finite-dimensional subspaces with their euclidean topology. Every construction in this and the previous two sections has been functorial in the finite G-representations involved an under our new hypothesis that G is compact, and every V we are now considering is a colimit of finite dimensional ones. We extend the definition of each of the operad families we have discussed indexed by a vector space by taking colimits along the diagram of subspaces. All our results hold in this case.

The next result is an application of Illman's work. In the case where G is a finite group, the necessary results are found in the 1970s paper [38]. Certainly, the difficulties of the compact Lie case are handled in the sequel paper [36], which proves that G-equivariant triangulations exist for any manifold with smooth G-action and that G-equivariant triangulated spaces are G-CW structures with the same cells. However, it was not until Illman's turn of the century paper [37] that a statement regarding extending G-triangulated structures of G manifolds from closed G-submanifolds appeared under a hypothesis of real analyticity, which is superfluous for our purposes.

Proposition 4.54. *If V is a countable dimensional orthogonal G-representation such that V is the union of its finite dimensional subrepresentations, then $\mathcal{D}_V(n)$ has a $G \times {\Sigma_n}^{\mathrm{op}}$-CW structure for each $n \in \mathbb{N}$.*

Proof. Choose an exhaustive increasing sequence of finite-dimensional subrepresentations $W_1 \subseteq W_2 \subseteq \ldots \subseteq V$ and observe that $\operatorname{colim} \mathcal{D}_{W_m}(n) = \mathcal{D}_V(n)$.

First, consider the case of $n \neq 1$. Note that the $(c_1, r_1, \ldots, c_n, r_n)$ coordinate system witnesses $W_m(n)$ as an open subset of a Euclidean space with a real analytic $G \times {\Sigma_n}^{\mathrm{op}}$-action, closed in $W_{m+1}(n)$. By Illman's theorem [36][37] and by induction, there is a $G \times {\Sigma_n}^{\mathrm{op}}$-CW structure on each $\mathcal{D}_{W_m}(n)$ such that $\mathcal{D}_{W_m}(n)$ is a subcomplex of $\mathcal{D}_{W_{m+1}}(n)$. Passing to the colimit yields a G-CW structure on $\mathcal{D}_V(n)$.

In the $n = 1$ case we have a $G \times {\Sigma_1}^{\mathrm{op}}$-equivariant homeomorphisms $\psi_n : \mathcal{D}_{W_m}(1) \to \mathbb{D}(W_m) \times (0,1]$ given by $d[c,r] \mapsto (c, (1 - \|c\|)^{-1} r)$ and commuting with inclusion maps. Each $\mathbb{D}_{W_m}(1)$ is open in W_m with real analytic G action and is closed in $\mathbb{D}_{W_{m+1}}(1)$. Using Illman's theorem [36, 37] again, we obtain a $G \times {\Sigma_1}^{\mathrm{op}}$-CW structure on $\operatorname{colim} \mathbb{D}(W_m)$. Taking any CW-decomposition of $(0,1]$, we note that the homeomorphisms ψ_n glue together to a homeomorphism $\mathcal{D}_V(1) \to \operatorname{colim}(\mathbb{D}(W_m) \times (0,1]) \simeq (\operatorname{colim} \mathbb{D}(W_m)) \times (0,1]$ and that $\mathcal{D}_V(1)$ inherits a $G \times {\Sigma_1}^{\mathrm{op}}$-CW structure. $\square$

We now turn to the main result of this section.

Proposition 4.55. *If $\mathscr{O}$ is an E_V-operad with respect to Strøm weak equivalences, then the cofibrant replacement $Q\mathscr{O} \to \mathscr{O}$ in the model category of operads in $\mathscr{U}_G$ is a Strøm weak equivalence.*

Proof. Fix a zig-zag of Strøm weak equivalences

$$\mathscr{O} \leftarrow \mathscr{O}_1 \to \ldots \leftarrow \mathscr{O}_n \to \mathscr{D}_V$$

connecting $\mathscr{O}$ and $\mathscr{D}_V$ and note that it extends to a zig-zag of Strøm weak equivalences connecting $\mathscr{O}$ and $\mathscr{D}_V^\circ$ by Proposition 4.53. In particular, this is a zig-zag of weak equivalences of fibrant objects in the category of operads, because all operads are fibrant in Top_G. Hence, by cofibrancy of $W\mathscr{O}$, there exists a commutative diagram in the model categorical homotopy category of operads

$$\begin{tikzcd}
W\mathscr{O} \arrow[d] \arrow[dr, dashed] \arrow[drrr, dotted] \arrow[drrrr, dotted] \arrow[drrrrr, dotted] & & & & & \\
\mathscr{O} & \mathscr{O}_1 \arrow[l] \arrow[r] & \ldots & \mathscr{O}_n \arrow[l] \arrow[r] & \mathscr{D}_V & \mathscr{D}_V^\circ \arrow[l]
\end{tikzcd}$$

extending the zig-zag with maps from $W\mathscr{O}$ to each stage of the zig-zag. By induction on the stages of the zig-zag and the two-out-of-three property of isomorphisms, the maps $W\mathscr{O} \to \mathscr{O}_i$ stages are isomorphisms in the model categorical homotopy category of operads. We choose representatives in the category of operads $W\mathscr{O} \to \mathscr{O}_i$ which must be weak equivalences because they lift an isomorphism. By Proposition 4.54 and since $W\mathscr{O}(n)$ has a $G \times {\Sigma_n}^{\mathrm{op}}$-CW structure, the map $W\mathscr{O} \to \mathscr{D}_V^\circ$ is a Strøm weak equivalence. For each k, we have a commutative diagram

$$\begin{tikzcd}
W\mathscr{O}(k) \arrow[d] \arrow[dr] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr, "\sim"] & & & & & \\
\mathscr{O}(k) & \mathscr{O}_1(k) \arrow[l, "\sim"'] \arrow[r, "\sim"'] & \ldots & \mathscr{O}_n(k) \arrow[l, "\sim"'] \arrow[r, "\sim"'] & \mathscr{D}_V(k) & \mathscr{D}_V^\circ(k) \arrow[l, "\sim"']
\end{tikzcd}$$

in the homotopy category of $\mathrm{Top}_{G\times\Sigma_n{}^{\mathrm{op}}}$ with $G \times \Sigma_k{}^{\mathrm{op}}$ homotopy equivalences marked with $\sim$. Inductively, using the 2-out-of-3 property for homotopy equivalences, we obtain that $W\mathscr{O}(k) \to \mathscr{O}(k)$ is a $G \times \Sigma_k{}^{\mathrm{op}}$-equivariant homotopy equivalence, as desired. $\square$

Chapter 5

Derived Indecomposibles of Augmented Algebras

In the study of obstruction theory for algebras in spectra, the maps from the source to any k invariant of the target factors through the spectrum of derived indecomposibles. This chapter describes these spectra for augmented algebras with the intention of applying them to $\Sigma^\infty B_| U_{\mathbb{R}+}$, which is augmented by the sphere spectrum $S_{\mathbb{R}}$.

In this chapter, we work in the category of EKMM $S_{\mathbb{R}}$-modules or EKMM S_G-modules with a mixed model structure that has the usual weak equivalences and collection of cofibrant objects consisting of all objects that are homotopy equivalent to the cofibrant objects of the standard model structures. The construction and basic properties of this model structure, including facts about geometric realizations of simplicial objects, have not been fully written yet, and the results in this chapter are conditional on their verification. Alternatively and more easily, one can work in orthogonal G-spectra or Real unitary spectra, where the model structures are well understood.

Analogous non-equivariant results of the same type with significant point-set differences are proven in [7]. One difference between our work and the previous work is that our approach only requires the base algebra (that has reserved notation A below) for the augmentation to be $E_{V\oplus 1}$ rather than commutative. This is needed to apply the ideas to operadic algebra analogues of the rigid algebraic Thom spectra of [2].

5.1 Notions of Highly Structured Algebra

We begin by reviewing some essential definitions.

Definition 5.1. Let A be an algebra over an operad $\mathscr{O}$. The universal enveloping operad $U_{\mathscr{O}}\mathsf{A}$ is defined as the collection underlying the coequalizer

$$\bigvee_{k\in\mathbb{N}} \mathscr{O}(n+k) \wedge_{\Sigma_k} (\mathbb{O}\mathsf{A})^{\wedge k} \rightrightarrows \bigvee_{k\in\mathbb{N}} \mathscr{O}(n+k) \wedge_{\Sigma_k} \mathsf{A} \longrightarrow U_{\mathscr{O}}\mathsf{A}(n)$$

formed through the map induced by the operad action and the map induced by the composition map of $\mathscr{O}$ obtained after unpacking the definition of $\mathbb{O}\mathsf{A}$. Considering the summand with $k=0$ in the parallel pair, the unit arises from the map of $\mathscr{O}(1) \to U_{\mathscr{O}}\mathsf{A}(1)$ and composition maps for $U_{\mathscr{O}}\mathsf{A}$ are induced by universal property of colimit from composing the $\mathscr{O}$ and concatenating copies of A.

Definition 5.2. A module M over an algebra A over an operad $\mathscr{O}$ is a left $U_{\mathscr{O}}\mathsf{A}(1)$-module. Equivalently, these are algebras over an operad $U_{\mathscr{O}}\mathsf{A}[1]$ such that $U_{\mathscr{O}}\mathsf{A}[1](1) = U_{\mathscr{O}}\mathsf{A}(1)$ and $U_{\mathscr{O}}\mathsf{A}[1](n)$ is trivial otherwise, with composition induced by multiplication.

Definition 5.3. Let A be an $\mathscr{O} \otimes_{\mathsf{BV}} \mathcal{A}ss$ operad for a unital operad $\mathscr{O}$ with a cocommutative comonoid structure for the parallel product. The operad $\mathscr{P}_{\mathscr{O},\mathsf{A}}$ for *left* A*-modules in the category of* $\mathscr{O}$*-algebras* is given by $\mathscr{P}_{\mathscr{O},\mathsf{A}}(n) = \mathscr{O}(n) \wedge \mathsf{A}$ with unit map arising from the operad unit in $\mathscr{O}(1)$ and the unital operad unit in $\mathscr{O}(0)$ via the composition

$$S_{\mathbb{R}} \longrightarrow S_{\mathbb{R}} \wedge S_{\mathbb{R}} \longrightarrow \mathscr{O}(1) \wedge \mathscr{O}(0) \longrightarrow \mathscr{O}(1) \wedge \mathsf{A}$$

and multiplication of the form

$$\begin{array}{c}
(\mathscr{O}(k) \wedge \mathsf{A}) \wedge \bigwedge_{i=1}^{k} (\mathscr{O}(n_i) \wedge \mathsf{A}) \\
\Big\downarrow{\scriptstyle \Delta} \\
(\mathscr{O}(k) \wedge \mathscr{O}(k) \wedge \mathsf{A}) \wedge \bigwedge_{i=1}^{k} (\mathscr{O}(n_i) \wedge \mathsf{A}) \\
\Big\downarrow{\scriptstyle \sim} \\
\big(\mathscr{O}(k) \wedge \bigwedge_{i=1}^{k} \mathscr{O}(n_i)\big) \wedge (\mathsf{A} \wedge (\mathscr{O}(k) \wedge \mathsf{A}^{\wedge k})) \\
\Big\downarrow \\
\mathscr{O}(\sum_{i=1}^{k} n_i) \wedge \mathsf{A}
\end{array}$$

where the last map is a wedge of an instance of composition in $\mathscr{O}$ and application of the $\mathscr{O} \otimes_{\mathsf{BV}} \mathcal{A}ss(k+1)$ operations that use $\mathscr{O}(k)$ to multiply k arguments and apply the ordinary binary multiplication of $\mathcal{A}ss(2)$ to multiply the result with another copy of A. This multiplication can be seen to be unital and associative, the latter using the interchange law of the Boardman–Vogt tensor product.

Definition 5.4. If A is an $\mathcal{A}ss$-algebra, i.e., a monoid object, then the operad $\mathscr{L}\mathscr{M}od_{\mathsf{A}/}$ is defined by $\mathscr{L}\mathscr{M}od_{\mathsf{A}/}(k) = \mathsf{A}$ when $k = 0$ or $k = 1$ and $\mathscr{L}\mathscr{M}od_{\mathsf{A}/}(k)$ is the initial object initial otherwise. Every composition map is either trivial due to its source being trivial or the monoid multiplication, and the unit is the unit of A. The algebras of $\mathscr{L}\mathscr{M}od_{\mathsf{A}/}$ are left modules of A under A.

Definition 5.5. An operad $\mathscr{O}$ is *without constants* when $\mathscr{O}(0)$ is an initial object. If $\mathscr{O}$ is any operad in a cocomplete symmetric monoidal category, then the *associated operad without constants* $\overline{\mathscr{O}}$ is the operad without constants such that $\overline{\mathscr{O}}(n) = \mathscr{O}(n)$ for $n \neq 0$ with the same unit as $\mathscr{O}$ and composition maps initial when involving $\mathscr{O}(0)$, and induced by $\mathscr{O}$ otherwise.

Definition 5.6. If $\mathscr{O}$ is an operad and I is an $\overline{\mathscr{O}}$-algebra, then QI is defined by the coequalizer diagram in the symmetric monoidal category of G-sepctra

$$\bigvee_{n \geq 2} \overline{\mathscr{O}}(n) \wedge I^{\wedge n} \rightrightarrows I \longrightarrow QI$$

with the upper map arising from the operad action and the lower map trivial.

Definition 5.7. The square-zero extension ZM of a module M over and algebra A over an operad $\mathscr{O}$ has underlying object $\mathsf{A} \vee M$ with operad action maps such that the composite

$$\mathscr{O}(n+k) \wedge \mathsf{A}^{\wedge n} \wedge M^{\wedge k} \longrightarrow \mathscr{O}(n+k) \wedge (\mathsf{A} \vee M)^{n+k} \longrightarrow \mathsf{A} \vee M$$

is the action map for A when $k = 0$, the module structure map when $k = 1$, and the zero map when $k \geq 2$.

Definition 5.8. Let A be a $\mathscr{D}_V \otimes_{\mathsf{BV}} \mathcal{A}ss$-algebra in a symmetric monoidal category of spectra. The monad $\mathbb{P}_V^{\mathsf{A}}$ is the monad on the category of left A modules under A associated to the change of operads $\mathscr{L}\mathscr{M}od_{\mathsf{A}/} \to \mathscr{P}_{\mathsf{A}}$.

The monad above is a structured spectral analogue of the monads used throughout [57].

Definition 5.9. If $\mathscr{O}$ is an operad and X is an $\mathscr{O}$-algebra augmented by the initial object $\mathscr{O}(0)$, then the functor $\mathbb{O}^{\mathsf{aug}}$ defined via pushout squares

$$\begin{tikzcd}
\mathbb{O}\mathscr{O}(0) \arrow[r] \arrow[d] & \mathbb{O}X \arrow[d] \\
\mathscr{O}(0) \arrow[r] & \mathbb{O}^{\mathsf{aug}}X
\end{tikzcd}$$

for each $\mathscr{O}(0)$ augmented object with the induced unit and multiplication maps.

Proposition 5.10. *The monad $\mathbb{P}_V^{\mathsf{A}}$ induces a monad $\mathbb{P}_V^{\mathsf{A},\mathsf{aug}}$ on the category of augmented left A-modules, i.e., the slice category of $\mathscr{L}\mathscr{M}od_{\mathsf{A}/}$-algebras over the initial object A.*

Proof. We first observe that the base change of operads is computed by a coequalizer diagram

$$\mathbb{P}_{\mathcal{D}_V,\mathsf{A}}(\mathsf{A} \vee (\mathsf{A} \wedge X)) \rightrightarrows \mathbb{P}_{\mathcal{D}_V,\mathsf{A}}X \longrightarrow \mathbb{P}_V^{\mathsf{A}}X$$

where the upper morphism is induced by the left module under A structure and the lower morphism is induced by the operad composition combined with the morphism of operads. With this description, when X is over A, the claimed structure map $\mathbb{P}_V^{\mathsf{A}} X \to \mathsf{A}$ is arises from the universal property of coequalizers applied to the composite

$$\mathbb{P}_{\mathcal{D}_V,\mathsf{A}} X \longrightarrow \mathbb{P}_{\mathcal{D}_V,\mathsf{A}} \mathsf{A} \xrightarrow{\alpha} \mathsf{A}$$

where α is the map associated to the initial algebra structure on $\mathscr{P}_{\mathcal{D}_V,\mathsf{A}}(0) = S \wedge \mathsf{A} \simeq \mathsf{A}$. Routine diagrammatic methods are used to show that the resulting endofunctor on augmented left A-modules has multiplication and unit lifted from the category of left A-modules under A. $\square$

Definition 5.11. Suppose X is an augmented A-bimodule and Y is an augmented left A-module. The *augmented smash product* $X \wedge_{\mathsf{A}}^{\mathsf{aug}} Y$ is defined as the pushout

$$\begin{tikzcd} X \vee_{\mathsf{A}} Y \arrow[r] \arrow[d] & \mathsf{A} \arrow[d] \\ X \wedge_{\mathsf{A}} Y \arrow[r] & X \wedge_{\mathsf{A}}^{\mathsf{aug}} Y \end{tikzcd}$$

where $X \vee_{\mathsf{A}} Y$ is the coproduct of X and Y in the category of left A-modules under A regarded as an augmented left A-module.

We will mostly apply the above definition in the case where $X = S^V_{+,\mathsf{A}}$ which is defined as follows.

Definition 5.12. The augmented A bimodule $S^V_{+,\mathsf{A}}$ is defined by a pushout square

$$\begin{tikzcd} \mathsf{A} \wedge (S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} \mathbb{L}_{\mathbb{R}} \Sigma^\infty S^0) \arrow[r] \arrow[d] & \mathsf{A} \wedge \Sigma^\infty S^0 \arrow[d] \\ \mathsf{A} \wedge (S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} \mathbb{L}_{\mathbb{R}} \Sigma^\infty S^V) \arrow[r] & S^V_{+,\mathsf{A}} \end{tikzcd}$$

where the left vertical map is induced by a map of spaces and the top horizontal map is induced by the $\mathbb{L}_{\mathbb{R}}$ action on suspension spectra.

In the model category of augmented A-modules, $S^V_{+,\mathsf{A}}$ is a cofibrant model of $\mathsf{A} \wedge S^V_+$.

Notation 5.13. Let $n \in \mathbb{N}$. We set $U_{n,0}$ to be the subspace of $S^V \times \mathcal{D}_V(n)$ consisting of $(x, (d_1, \ldots, d_n))$ such that x is either the basepoint of S^V or an element of V in the complement of the images of the little disks $d_1, \ldots, d_n$.

For $i_0 = 1, \ldots, n$, we define U_{n,i_0} as the subspace $S^V \times \mathcal{D}_V(n)$ consisting of $(x, (d_1, \ldots, d_n))$ such that either x is in the image of d_i or $(x, (d_1, \ldots, d_n)) \in U_{n,0}$.

For each $\mathbf{n} = (n_i)_{i=1}^k \in \coprod_{\mathsf{k} \in \mathbb{N}} \mathbb{N}^k$, we set $W_{\mathbf{n},0}$ to be the subspace of $S^V \times \mathcal{D}_V(k) \times \prod_{i=1}^k \mathcal{D}_V(n_i)$ that is the preimage of $U_{\sum_{i=1}^k n_i, 0}$ under the composition map $S^V \times \mathcal{D}_V(k) \times \prod_{i=1}^k \mathcal{D}_V(n_i) \to S^V \times \mathcal{D}_V(\sum_{i=1}^k n_i)$. Similarly $W_{\mathbf{n}, i_0, j_0}$ is the preimage of $U_{\sum_{i=1}^k n_i, j_0 + \sum_{i=1}^{i_0 - 1} n_i}$ under the same map $S^V \times \mathcal{D}_V(k) \times \prod_{i=1}^k \mathcal{D}_V(n_i) \to S^V \times \mathcal{D}_V(\sum_{i=1}^k n_i)$.

5.2 The May Delooping Machine

Proposition 5.14. *There exists a natural transformation*

$$S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} \mathbb{P}^{\mathsf{A}}_V X \to S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} X$$

in the category of augmented left A*-modules that gives* $S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} (-)$ *the structure of a right module over the monad* $\mathbb{P}^{\mathsf{A}}_V$.

Proof. (Sketch). We now indicate the structure of the cumbersome proof. First, we express $S^V_{+,\mathsf{A}} \wedge_{\mathsf{A}} \mathbb{P}^{\mathsf{A}}_V X$ as an iterated colimit of simpler pieces. If Y is the coequalizer

$$\begin{array}{c}
\bigvee_{k \in \mathbb{N}} \bigvee_{i=1}^k \mathsf{A} \wedge \Sigma^\infty_+ (* \times_{\mathscr{L}} \mathbb{L} U_{k,0}) \wedge \mathsf{A} \wedge X^{\wedge k} \\
\downarrow \quad \downarrow \\
\bigvee_{k \in \mathbb{N}} \bigvee_{i=0}^k \mathsf{A} \wedge \Sigma^\infty_+ (* \times_{\mathscr{L}} \mathbb{L} U_{k,i}) \wedge \mathsf{A} \wedge X^{\wedge k} \\
\downarrow \\
Y
\end{array}$$

where the left map sends the wedge summand indexed by (k,i) into $(k,0)$ isomorphically, and the right map is induced by the inclusion of the corresponding summands. With this Y, we have a pushout square

$$\begin{tikzcd}
S^V_{+,\mathsf{A}} \wedge_{\mathsf{A}} (\mathsf{A} \wedge \Sigma^\infty_+(* \times_{\mathscr{L}} \mathbb{L}*) \wedge \mathsf{A} \wedge X) \arrow[r] \arrow[d] & Y \arrow[d] \\
S^V_{+,\mathsf{A}} \wedge_{\mathsf{A}} (\mathsf{A} \wedge S_{\mathbb{R}} \wedge \mathsf{A} \wedge X) \arrow[r] & S^V_{\mathsf{A},+} \wedge_{\mathsf{A}} \mathbb{P}_{\mathcal{D}_V,\mathsf{A}} X
\end{tikzcd}$$

corresponding to correcting the operadic unit in the lifting of $\mathcal{D}_V$ to $\mathbb{D}_V$. Then, we have a canonical pushout square

$$S^V_{+,\mathsf{A}} \wedge_{\mathsf{A}} \mathbb{P}_{\mathbb{D}_V,\mathsf{A}}(A \vee (A \wedge X)) \rightrightarrows S^V_{\mathsf{A},+} \wedge_{\mathsf{A}} \mathbb{P}_{\mathcal{D}_V,\mathsf{A}} X \longrightarrow S^V_{+,\mathsf{A}} \wedge_{\mathsf{A}} \mathbb{P}^{\mathsf{A}}_V X$$

and note that the source of the coequalizer pair can be written as an iterated colimit of diagrams in a manner similar to that of the target. The defining pushout square of $S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} \mathbb{P}^{\mathsf{A}}_V X$ completes the description as an iterated colimit.

The maps $\phi_{k,i} : U_{k,i} \to S^V$ mapping $U_{k,0}$ to the basepoint and any other $(x,(d_1,\dots,d_n))$ to the unique $y \in V$ such that $d_i(y) = x$, together with defining pushout squares, assemble to the desired natural transformation. As the most non-trivial example, we use maps

$$\begin{tikzcd}
\mathsf{A} \wedge \Sigma^\infty_+(* \times_{\mathscr{L}} \mathbb{L}U_{k,i}) \wedge \mathsf{A} \wedge X^{\wedge k} \arrow[d, "\alpha_0"'] \\
\mathsf{A} \wedge \Sigma^\infty_+(* \times_{\mathscr{L}} \mathbb{L}(U_{k,i})) \wedge \Sigma^\infty_+(* \times_{\mathscr{L}} \mathbb{L}(U_{k,i})) \wedge \mathsf{A} \wedge X^{\wedge k} \arrow[d, "\alpha_1"'] \\
\mathsf{A} \wedge \Sigma^\infty_+(* \times_{\mathscr{L}} \mathbb{L}S^V) \wedge \Sigma^\infty_+ \mathcal{D}_V(k-1) \wedge \mathsf{A} \wedge \underbrace{\mathsf{A} \wedge \ldots \wedge \mathsf{A} \wedge X \wedge \mathsf{A} \wedge \ldots \wedge \mathsf{A}}_{i} \arrow[d, "\alpha_2"'] \\
S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} X
\end{tikzcd}$$

where the maps are described as follows. The map α_0 is the composite of the map induced by the diagonal map on $U_{k,i}$ with known natural isomorphisms. The map α_1 is constructed using the map $\phi_{k,i}$ on the left factor and the map induced by the projection $U_{k,i} \to \mathcal{D}_V(k-1)$ sending $(x,(d_1,\dots,d_k)) \mapsto (d_1,\dots,\hat{d}_i,\dots,d_k)$. The map α_2 arises from applying the $\mathcal{D}_V(k-1)$ operad multiplication to the last $k-1$ copies of A.

To verify that this is a right module structure, one similarly decomposes $S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} \mathbb{P}^V_{\mathsf{A}}\mathbb{P}^V_{\mathsf{A}}$ using the spaces $W_{\mathbf{n},i,j}$ serving a similar role to the spaces $U_{k,i}$. The proof is a lengthy formal check. $\square$

Proposition 5.15. *The functor Z is the left adjoint of a Quillen adjunction*

$$\mathsf{LMod}^{\mathsf{aug}}_{\mathsf{A}} \underset{I}{\overset{Z}{\rightleftarrows}} \mathsf{LMod}_{\mathsf{A}}$$

between the category of left A-modules and augmented left A-modules. This induces an adjunction between categories of algebras over monads (or equivalently an operad in the case of the source of the left adjoint) pictured below.

$$\mathsf{Alg}(\overline{\mathbb{P}_{\mathbb{D}_V,\mathsf{A}}}) \underset{I}{\overset{Z}{\rightleftarrows}} (\mathsf{LMod}^{\mathsf{aug}}_{\mathsf{A}})^{\mathbb{P}^{\mathsf{A}}_V}$$

One of the omitted model categorical details of our work is that the above adjunction is a Quillen adjunction for the mixed model structure.

5.3 Derived Indecomposibles via Delooping Machine

We now state the main theorem of this chapter.

Theorem 5.16. *There is a natural equivalence $Z^{\mathbf{L}}S^V_{\mathsf{A}} \wedge_{\mathsf{A}} Q^{\mathbf{L}}I^{\mathbf{R}}R \simeq B(S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} (-), \mathbb{P}^{\mathsf{A}}_V, R)$ of augmented left A-modules for $\mathbb{P}^{\mathsf{A}}_V$-algebras R. In other words, the augmentation module of the equivariant E_V topological André–Quillen homology over A of an augmented E_V-algebra R is the V-fold suspension lifting of the May delooping machine to the category of augmented* *left* A*-modules.*

As a first step, we show the following relationship.

Proposition 5.17. *For any augmented cofibrant space X, the 0-cell map*

$$S^V_{+\mathsf{A}} \wedge \Sigma^\infty X_+ \to B(S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{S_\mathbb{R}} (-), \mathbb{P}^{\mathsf{A}}_V, \mathsf{A} \wedge \Sigma^\infty_+ \mathbb{D}_V(X)))$$

is a weak equivalence.

Proof. Note that $\mathbb{P}^{S_\mathbb{R}}_V = (\Sigma^\infty(\mathbb{D}_V)_+)^{\mathsf{aug}} = \Sigma^\infty_+(\mathbb{D}^{\mathsf{aug}})$. The case $\mathsf{A} = S_\mathbb{R}$ follows from the extra degeneracy argument (which goes through even though the augmentations are incompatible with the extra degeneracy, because it is irrelevant to the extra degeneracy argument), and the general case follows from the commutativity of base change with the relevant constructions. $\square$

Disregarding standard methods of resolutions, the main technical problem in passing from the Proposition 5.17 to Theorem is that the suspension is modeled via a smash product with a sphere with basepoint at the augmentation unit in one and at the spectral basepoint in the other. Considering this, we introduce some notation.

Notation 5.18. The spectrum S^V_{A} is defined as the pushout

$$\begin{tikzcd} S_\mathbb{R} \wedge A \arrow[r] \arrow[d] & S^V_{\mathsf{A},+} \arrow[d] \\ * \arrow[r] & S^V_{\mathsf{A}} \end{tikzcd}$$

corresponding to the pushout identifying the augmentation point of S^V_+ (or point at infinity of V) with the basepoint. We set $\overline{\mathbb{P}^{\mathsf{A}}_V}$ to be the monad of the operad $\overline{\mathscr{P}_{\mathbb{D}_V,\mathsf{A}}}$ without constants.

For a G-space X, we define a modified suspension spectrum $F(X)$ by a pushout diagram

$$\begin{tikzcd} S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} \mathbb{L}_{\mathbb{R}} \Sigma^\infty *_+ \arrow[r] \arrow[d] & * \arrow[d] \\ S_{\mathbb{R}} \wedge_{\mathscr{L}_{\mathbb{R}}} \mathbb{L}_{\mathbb{R}} \Sigma^\infty X_+ \arrow[r] & F(X) \end{tikzcd}$$

Lemma 5.19. *The natural map*

$$S^V_{\mathsf{A}} \wedge_{\mathsf{A}} (\mathsf{A} \wedge F(X)) \to B(S^V_{\mathsf{A}} \wedge_{\mathsf{A}} (-), \overline{\mathbb{P}^{\mathsf{A}}_V}, \overline{\mathbb{P}^{\mathsf{A}}_V}(\mathsf{A} \wedge F(X)))$$

is a weak equivalence for augmented cofibrant space X. Here $S^V_{\mathsf{A}} \wedge_A (-)$ has the induced right $\overline{\mathbb{P}^{\mathsf{A}}_V}$-module structure from the action of $\mathbb{P}^{\mathsf{A}}_V$ on $S^V \wedge^{\mathsf{aug}}_{\mathsf{A}} (-)$.

Proof. We have a canonical zig-zag of weak equivalences in the homotopy category of $\mathscr{P}_{\mathbb{D}_V,\mathsf{A}}$-algebras

$$\begin{tikzcd} & \mathbb{P}_{\mathbb{D}_V,\mathsf{A}}(Y) \arrow[d] & \\ \mathbb{P}^{\mathsf{A}}_V(\mathsf{A} \wedge \Sigma^\infty_+ X) \arrow[r, "\iota_0"] \arrow[ur, dashed] & \mathbb{P}_{\mathbb{D}_V,\mathsf{A}}(\mathsf{A} \wedge \Sigma^\infty_+ I \wedge \Sigma^\infty_+ X) & \mathbb{P}_{\mathbb{D}_V,\mathsf{A}}(\mathsf{A} \wedge \Sigma^\infty_+ X) \arrow[l, "\iota_1"'] \arrow[ul, dashed] \arrow[d, "\sim"] \\ & & \mathsf{A} \wedge \Sigma^\infty_+ \mathbb{D}_V(X) \end{tikzcd}$$

and obtain Y via a choice of cofibrant replacement and the dotted morphisms to it using fibrancy of the target and cofibrancy of the sources. For the zig-zag passing through Y, we describe the augmentations for each algebra. The left end has an augmentation arising from the augmentation on X, and the right end has an augmentation arising from the terminal map of G-spaces. The augmentation on $\mathbb{P}_{\mathbb{D}_V,\mathsf{A}}(Y)$ is induced by any map $Y \to \mathsf{A} \vee (\mathsf{A} \wedge \Sigma^\infty_+ X)$ is constructed to represent the difference between the maps induced by augmentation and the identity of X smashed with A (c.f. [7, Proof of 7.7]). This gives us a zig-zag of weak equivalences between $B(S^V_{\mathsf{A},+} \wedge^{\mathsf{aug}}_{\mathsf{A}}, \mathbb{P}^{\mathsf{A}}_V, \mathbb{P}^{\mathsf{A}}_V(\mathsf{A} \wedge \Sigma^\infty_+ X))$ and $B(S^V_{\mathsf{A},+} \wedge^{\mathsf{aug}}_{\mathsf{A}}$

$(-), \mathbb{P}_V^{\mathsf{A}}, \mathsf{A} \wedge \Sigma_+^\infty \mathcal{D}_V^{\mathsf{aug}}(X))$.

The derived fiber of the augmentation $I^{\mathbf{R}}$ can be computed as the cofiber of the structural coagmentation map, and applying this yields an isomorphism in the homotopy category between $B(S_{\mathsf{A}}^V \wedge_{\mathsf{A}} (-), \overline{\mathbb{P}_V^{\mathsf{A}}}, \overline{\mathbb{P}_V^{\mathsf{A}}}(\mathsf{A} \wedge F(X)))$ and $B(S_{\mathsf{A}}^V \wedge_{\mathsf{A}} (-), \overline{\mathbb{P}_V^{\mathsf{A}}}, \mathsf{A} \wedge \Sigma^\infty \mathcal{D}_V^{\mathsf{aug}}(X))$.

We construct a similar zig-zag of equivalences of augmented left A-modules

$$\begin{tikzcd}
 & Z \arrow[d] & \\
S_{+,\mathsf{A}}^V \wedge_{\mathsf{A}}^{\mathsf{aug}} \Sigma_+^\infty X \arrow[r] \arrow[ur, dashed] & S_{+,\mathsf{A}}^V \wedge_{\mathsf{A}}^{\mathsf{aug}} \Sigma_+^\infty X \wedge \Sigma_+^\infty I & \mathsf{A} \vee (S_{\mathsf{A}}^V \wedge_A (A \wedge \Sigma^\infty X)) \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

where the vertical morphism is a cofibrant replacement, the dotted arrows are induced maps from cofibrant objects to fibrant objects, I has basepoint 0, and the right horizontal map has the right wedge summand factor through

$$S_{\mathsf{A}}^V \wedge_{\mathsf{A}} (\mathsf{A} \wedge \Sigma^\infty X) \to S_{+,\mathsf{A}}^V \wedge_{\mathsf{A}}^{\mathsf{aug}} \Sigma_+^\infty X \wedge \Sigma_+^\infty \{1\}$$

so that taking 1 as the basepoint of $\{1\}$, the induced map of augmented A modules $\mathsf{A} \vee (S_{\mathsf{A}}^V \wedge_{\mathsf{A}} (A \wedge \Sigma^\infty X)) \to S_{+,\mathsf{A}}^V \wedge_{\mathsf{A}}^{\mathsf{aug}} \Sigma_+^\infty X \wedge \Sigma_+^\infty \{1\}$ is a homotopy inverse to the canonical map of non-augmented left A modules. This zig-zag is compatible with the zig-zag of bar constructions of augmented algebras, and so the resulting zig-zag of cofibers of coagmentations are compatible zig-zags as well. The result follows. $\square$

Theorem 5.16 is now a formal consequence of our work. We split the remaining work into two lemmas.

Lemma 5.20. *For any cofibrant left A-module M, the natural unit composed with 0-cell map*

$$S_{\mathsf{A}}^V \wedge_{\mathsf{A}} M \to B(S_{\mathsf{A}}^V \wedge_{\mathsf{A}} (-), \overline{\mathbb{P}_V^{\mathsf{A}}}, \overline{\mathbb{P}_V^{\mathsf{A}}} M)$$

is a weak equivalence.

Proof. First suppose that $M = \mathsf{A} \wedge N$ for a cofibrant $S_{\mathbb{R}}$-module N. The simplicial object $B_\bullet(S^V_{\mathsf{A}} \wedge_{\mathsf{A}} (-), \overline{\mathbb{P}^{\mathsf{A}}_V}, \overline{\mathbb{P}^{\mathsf{A}}_V} M)$ has a level-wise filtration arising from the arity of the relevant total composite $\mathbb{D}_V$ operation arity in expressing each level as a wedge sum. Studying the action of $\mathbb{P}^{\mathsf{A}}_V$ on $S^V_{+,\mathsf{A}} \wedge^{\mathsf{aug}}_{\mathsf{A}} (-)$ and because the action of $\overline{\mathbb{P}^{\mathsf{A}}_V}$ on $S^V_{\mathsf{A}} \wedge_{\mathsf{A}}$ is induced by taking cofibers of respective coaugmentations, the action map is level preserving by sending every summand corresponding to an arity greater than 1 identically to the spectral basepoint and sending arity 1 pieces to themselves (up to composing with the unit map to get an endomorphism). After forgetting the left A-module structures we can rewrite the geometric realization in the form $\bigvee_{m>0} \mathscr{B}(m) \wedge_{\Sigma_m} N^{\wedge m}$ as in [7, 7.6]. Take N to be $F(\underline{m}_+)$ where $\underline{m}_+$ is a cofibrant augmented G-space replacing the discrete topological space $\{1, \ldots, m\}_+$ to find a case where $\mathscr{B}(m)$ is a retract of $\mathscr{B}(m) \wedge_{\Sigma_m} N^{\wedge m}$ for all M. Now apply Lemma 5.19 for this X and see that $\mathscr{B}(1) \simeq S^V_{\mathsf{A}}$ and $\mathscr{B}(m)$ is weakly contractible for $m \geq 2$. The result follows for free M. For other values of cofibrant left A-modules M, we note that both the source and target of the map and the filtration behave well in the module argument with respect to filtered colimit and every cofibrant left A-module M is a filtered homotopy colimit of free left A-modules (up to an actual homotopy equivalence of the colimit of the projective model categorical cofibrant replacement, as every object is fibrant). □

We now use Lemma 5.20 to begin studying equivariant topological André–Quillen homology.

Lemma 5.21. *For any cofibrant $\overline{\mathbb{P}^{\mathsf{A}}_V}$-algebra T, the natural map*

$$f : B(S^V_{\mathsf{A}} \wedge_{\mathsf{A}}, \overline{\mathbb{P}^{\mathsf{A}}_V}, T) \to S^V_{\mathsf{A}} \wedge_{\mathsf{A}} QT$$

is a weak equivalence

Proof. Once again, we begin with the free case, which is $T = \overline{\mathbb{P}^{\mathsf{A}}_V} M$ here for some cofibrant left A-module M. The functor $Q\overline{\mathbb{P}^{\mathsf{A}}_V}$ is naturally isomorphic to endofunctor $\Sigma^\infty_+ \mathcal{D}_V(1) \wedge (-)$, which is naturally homotopy equivalent to the identity functor. Now, making this identification Lemma 5.20 states that a morphism that splits f is a weak equivalence. Hence f itself must be a weak equivalence. The general case follows from expressing T as a filtered colimit of free algebras up to weak equivalence. $\square$

Proof of Theorem 5.16. The result is reduced to Lemma 5.21 by application of the weak equivalence creating functor $I^{\mathbf{R}}$ implemented as the cofiber of the coaugmentation. $\square$

Chapter 6

Loop Spaces and Cohomology

This chapter describes a simple computation at the heart of our claim the existence of E_ρ maps out of $MU_\mathbb{R}$. We assume a basic understanding of C_2-Mackey functors. The material in [29] more than suffices. We also assume familiarity with the equivariant Serre spectral sequence of Kronholm [43]. We will begin to assume some standard notations from those sources. For example, ρ being the regular representation for C_2, σ and is the sign representation.

6.1 Equivariant deloopings of Bott periodicity

The main result of this chapter is the Real equivariant analogue of [20, 7.3].

Theorem 6.1. *The canonical map $\Sigma^\rho BU_\mathbb{R}(1) \to B^\rho BU_\mathbb{R}$ induces a quotient map in $RO(G)$-graded Mackey functor-valued cohomology, as indicated in the diagram below.*

$$\begin{tikzcd}
\underline{H}^\star(B^\rho BU_\mathbb{R}, \underline{\mathbb{Z}}) \arrow[r] \arrow[d, "\sim"'] & \underline{H}^\star(\Sigma^\rho BU_\mathbb{R}(1), \underline{\mathbb{Z}}) \arrow[d, "\sim"] \\
\underline{H}^\star(C_2/C_2, \underline{\mathbb{Z}})[c_m : m \geq 2] \arrow[r] & \underline{H}^\star(C_2/C_2, \underline{\mathbb{Z}})[c_m : m \geq 2]/(c_m c_n : m, n \geq 2)
\end{tikzcd}$$

where $|c_i| = i\rho$, $|x_m| = (m+1)\rho$ and the map is the identity on coefficients and sends $c_{m+1} \mapsto$

$(-1)^m x_m$.

In the statement of the above theorem, it is important for our intended application that $B^\rho(-)$ is the functor corresponding to the May delooping machine. This aligns the results of this chapter with Theorem 5.16.

Proposition 6.2. *Let SU_T denote the C_2-space that has SU as its underlying space with C_2 generator g acting by $g \cdot A = A^T$. Then, the map $SU \to (SU_T)^{C_2}$ defined by $A \mapsto AA^T$ descends to a homeomorphism $SU/SO \to (SU_T)^{C_2}$.*

Proof. It suffices to show that the restrictions $SU(n)/SO(n) \to (SU_T(n))^{C_2}$ are bijections for each n because the source is compact and the target is Hausdorff. If $AA^T = BB^T$ for special unitary A and B, using the identity $A^T = \overline{A}^{-1}$ we can rewrite this as $BA^{-1} = \overline{BA^{-1}}$, demonstrating injectivity. Surjectivity is an application of the Autonneâ€"Takagi factorization, where one makes choices to ensure that the determinant of each matrix involved is 1. Explicitly, if $M \in (SU_T)^{C_2}$, then we may write $M = X + iY$ for two real matrices X and Y, and the fact that M is unitary implies $XY = YX$. Therefore, we may choose a special orthogonal V such that VXV^T and VYV^T are both diagonal. The determinant of 1 can be ensured by adjusting the first column of V with a sign if needed. Therefore, VMV^T is a complex diagonal matrix of determinant 1, and there exists a complex diagonal matrix D with $D^2 = VMV^T$. By replacing the first column of D with its negative if needed, D can be chosen so that its determinant is 1. Now $M = (V^T D)(V^T D)^T$, proving surjectivity. □

As we appeal to the classical descriptions of Bott periodicity, we replace the $BU_{\mathbb{R}}$ of Chapter 3 with another equivalent model, as defined in the statement of the following proposition.

Proposition 6.3. *There is a weak equivalence of algebras over $\mathscr{L}_{\mathbb{R}}$ $BU_{\mathbb{R}} \to \Omega^\sigma SU_{\mathbb{R}}$ where $\Omega^\sigma SU_{\mathbb{R}}$ inherits its algebra structure from $SU_{\mathbb{R}}$ and $SU_{\mathbb{R}} = \operatorname{colim}_{V \subseteq \mathbb{C}^\infty} SU_{\mathbb{R}}(V)$ and $BU_{\mathbb{R}} = \operatorname{colim}_{V \subseteq \mathbb{C}^\infty} U_{\mathbb{R}}(V \oplus V)/U_{\mathbb{R}}(V) \times U_{\mathbb{R}}(V)$ have algebra structures induced by $\mathscr{I}_{\mathbb{R}}$-FCPs $SU_{\mathbb{R}}(-)$ and $BU_{\mathbb{R}}(-)$.*

Proof. This is a consequence of Bottâ€™s famous periodicity theorem [15] and for relevant maps treated explicitly see [21]. Note that in [21] ΩX is the component of the trivial map in the based loop space of X. Observe that the Bott map $BU \to \Omega SU$ is equivariant when the source and target are given the C_2-space structures $BU_{\mathbb{R}}$ and $\Omega^{\sigma} SU_{\mathbb{R}}$ respectively. The map $BU_{\mathbb{R}} \to \Omega^{\sigma} SU_{\mathbb{R}}$ also respects the Real linear isometry operad action because it can be assembled from $BU_{\mathbb{R}}(V) \to \Omega^{\sigma} SU_{\mathbb{R}}(V)$. The compactness of S^{σ} and the structure maps of $SU_{\mathbb{R}}(-)$ being inclusions guarantees that $\operatorname{colim}_{V \subseteq \mathbb{C}^{\infty}} \Omega^{\sigma} SU_{\mathbb{R}}(V)$ is canonically isomorphic to $\Omega^{\sigma} SU_{\mathbb{R}}$. The main step is to show that the classical map $BU_{\mathbb{R}} \to \Omega^{\sigma} SU_{\mathbb{R}}$ is a C_2-equivariant weak equivalence. Let SU_T denote the special unitary group with the transpose C_2-action. Note that $A \mapsto AA^T$ defines a map $SU \to (SU_T)^{C_2}$. We have a fiber sequence

$$\Omega^{\sigma} SU_{\mathbb{R}} \longrightarrow SU_{\mathbb{R}} \longrightarrow \mathscr{T}_{C_2}(C_{2+}, SU_{\mathbb{R}}) \longrightarrow SU_T$$

that arises from the cofiber sequence $C_{2+} \to C_2/C_{2+} \to S^{\sigma}$ in the first three terms as a classical homotopy fiber and the C_2-locally trivial fibration defined by $B \mapsto B(e)B(g)^{-1}$ in the last three terms where g is the generator of C_2. Consequently, there is an explicit equivariant weak equivalence $\Omega^{\sigma} SU_{\mathbb{R}} \to \Omega SU_T$ arising as the composite

$$\Omega^{\sigma} SU_{\mathbb{R}} \longrightarrow (\mathscr{T}(C_{2+}, SU_{\mathbb{R}}) \times^h_{SU_T} *) \times^h_{\mathscr{T}(C_{2+}, SU_{\mathbb{R}})} * \longrightarrow \Omega SU_T$$

and the composite $BU_{\mathbb{R}} \to \Omega SU_T$ agrees with the map $BU_{\mathbb{R}} \to \Omega^{\sigma} SU_{\mathbb{R}}$ upon application of the forgetful functor to topological spaces. We show that $BU_{\mathbb{R}} \to \Omega SU_T$ is an equivariant weak equivalence by checking that the induced natural transformation of fixed point functors is a weak equivalence. Indeed, on objects and restriction maps, it is precisely the diagram

From Bott periodicity, we know that the classical commutative diagram

$$\begin{tikzcd} BO \arrow[r] \arrow[d, "\sim"'] & BU \arrow[d, "\sim"'] \\ \Omega SU/SO \arrow[r] & \Omega SU \end{tikzcd}$$

of Bott maps has vertical arrows that represent the desired weak equivalences, completing the proof. □

Proposition 6.4. *There is a commutative diagram in the category of algebras over the Real linear isometries operad $\mathscr{L}_{\mathbb{C}}$*

$$\begin{tikzcd} SU_{\mathbb{R}} \arrow[r] \arrow[d] & U_{\mathbb{R}} \arrow[d] \\ \Omega BSU_{\mathbb{R}} \arrow[r] & \Omega BU_{\mathbb{R}} \end{tikzcd}$$

where the vertical maps are weak equivalences and

$$BSU_{\mathbb{R}} = U_{\mathbb{R}}(\mathbb{C}^{\infty} \oplus \mathbb{C}^{\infty})/SU_{\mathbb{R}}(\mathbb{C}^{\infty}) \times U_{\mathbb{R}}(\mathbb{C}^{\infty})$$

inherits its $\mathscr{L}_{\mathbb{C}}$ structure from a similar colimit diagram as $BU_{\mathbb{R}}$ does.

Proof. We will show the existence of the weak equivalence for $U_{\mathbb{R}}$. The proof for $SU_{\mathbb{R}}$ and the existence of the commutative square induced by inclusion is straightforward. The desired map is defined by

$$A \mapsto \left(t \mapsto \left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \mathsf{id} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \right]_{\sim U_{\mathbb{R}} \times U_{\mathbb{R}}} \right)$$

where $\theta = 0$ when t is the basepoint and $\theta = \frac{\pi}{2} + \frac{\pi t}{2\sqrt{1+t^2}}$ for $t \in \mathbb{R}$. This θ is a reparameterization of the interval $[0, \pi]$. Compatibility with the linear isometries operad structure can be argued with $\mathscr{J}$-FCPs again and equivariance is immediate from the definition. The inclusion $U_{\mathbb{R}} \simeq U_{\mathbb{R}}(\mathbb{C}^{\infty} \oplus 0) \to U_{\mathbb{R}}(\mathbb{C}^{\infty} \oplus \mathbb{C}\infty)/1 \times U_{\mathbb{R}}(\mathbb{C}^{\infty})$ is a filtered colimit of closed embeddings of manifolds with smooth G action along h-cofibrations and is thus an h-cofibration. Therefore, we can choose a morphism that completes the commutative diagram

$$\begin{tikzcd}
U_{\mathbb{R}} \arrow[rr] \arrow[d, hook] & & \Omega BU_{\mathbb{R}} \arrow[d] \\
U_{\mathbb{R}}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)/1 \times U_{\mathbb{R}}(\mathbb{C}^\infty) \arrow[rr, dashed] \arrow[dr] & & PBU_{\mathbb{R}} \arrow[dl, two heads] \\
& BU_{\mathbb{R}} &
\end{tikzcd}$$

relating two fiber sequences of C_2-spaces with contractible total spaces. The contractibility of the source follows from the Puppe sequence associated to the fiber sequence

$$\begin{tikzcd}
U_{\mathbb{R}} \arrow[r] & U_{\mathbb{R}}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) \arrow[d] \\
& U_{\mathbb{R}}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)/1 \times U_{\mathbb{R}}(\mathbb{C}^\infty)
\end{tikzcd}$$

and the map $U_{\mathbb{R}}(n) \to U_{\mathbb{R}}$ and its composite with $U_{\mathbb{R}} \to U_{\mathbb{R}}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)$ induces isomorphisms on all $\pi_n^H(-)$. By the five lemma, $U_{\mathbb{R}} \to \Omega BU_{\mathbb{R}}$ is a weak equivalence. □

Theorem 6.5. *Let $\mathcal{U}$ be a Real universe, $\mathscr{K}_{\mathcal{U}}$ the corresponding Steiner operad, and $\mathbb{E}$ be the infinite loop space machine sending $\mathscr{K}_{\mathcal{U}}$-algebras to LMS spectra of* [26, 2.11]*. Then, $\Omega^\infty\mathbb{E}$ is connected to the identity functor by a zig-zag of natural transformations, which consists of weak equivalences on group-like objects and induces a natural isomorphism g in the homotopy category of spaces. For every G-representation $V \subseteq \mathcal{U}$, there is a natural map $\omega_V : \mathbb{E}\Omega^V Z \to \Omega^V \mathbb{E}Z$ for $\mathscr{K}_{\mathcal{U}}$-spaces Z where $\Omega^V Z$ has the $\mathscr{K}_{\mathcal{U}}$ structure inherited from mapping into Z. Assume that the inclusion of the identity element of Z is an h-cofibration. If either Z is group-like and V contains a copy of the trivial representation (or is 0), or Z*

is connected, then there is a commuting natural triangle

$$\begin{tikzcd} & \Omega^V Z \arrow[dl] \arrow[dr] & \\ \Omega^\infty \mathbb{E}\Omega^V Z \arrow[rr, "\Omega^\infty \omega_V"] & & \Omega^V \Omega^\infty \mathbb{E} Z \end{tikzcd}$$

of weak equivalences. Moreover, under the same conditions, there is a natural weak equivalence $f : B^V Z \to \mathbb{E}Z(V)$ and a functorial commutative square in the (model categorical) homotopy category of $\mathscr{K}_{\mathcal{U}}$-spaces

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"] & \Omega^V B^V Z \arrow[d, "\Omega^V f"'] \\ \Omega^\infty \mathbb{E} Z \arrow[r, "\sim"] & \Omega^V(\mathbb{E}Z(V)) \end{tikzcd}$$

where g is the induced isomorphism.

Proof. The first claim is proven in [26, 2.13]. The second claim follows from the argument of [58, 15.1] with minor changes (e.g. replacing ordinary spheres with representation spheres and using Σ^V and Ω^V in place of Σ and Ω). Before embarking on the proof of the last claim, we recall a characterization of weak equivalences of G-spaces that follows immediately from a result of Waner (and Lewis in the compact Lie case) [46, 1.2]. A map of G-spaces is a weak equivalence if application of the representation sphere homotopy set functors

$$\pi^K_{(\iota_K V - V^K)\oplus \mathbb{R}^n}(-) = [G_+ \wedge_K S^{\iota_K V - V^K \oplus \mathbb{R}^n}, -]^G$$

where ι_K is the forgetful functor to K-representations induce isomorphisms. These homotopy set functors will also be useful for describing the space $\mathbb{E}Z(V)$ as we will see. Towards the final claim, we describe $\mathbb{E}Z(V)$ with a chain of G-homeomorphisms below.

$$\begin{tikzcd}
\mathbb{E}Z(V) \arrow[d, "\alpha_0"] \\
\operatorname{colim}_{V\subseteq W\subseteq\mathcal{U}} \Omega^{W-V} B(\Sigma^\infty(-)(W), K_{\mathcal{U}}, Z) \arrow[d, "\alpha_1"] \\
\operatorname{colim}_{V\subseteq W\subseteq\mathcal{U}} \Omega^{W-V} B(\operatorname{colim}_{W\subseteq U\subseteq\mathcal{U}} \Omega^{U-W}\Sigma^U, \mathscr{K}_{\mathcal{U}}, Z) \arrow[d, "\alpha_2"] \\
B(\operatorname{colim}_{V\subseteq U\subseteq\mathcal{U}} \Omega^{U-V}\Sigma^U, \mathscr{K}_{\mathcal{U}}, Z)
\end{tikzcd}$$

The G-homeomorphism α_0 exists because $\mathbb{E}Z$ is the spectrification of the geometric realization $B(\Sigma^\infty, \mathscr{K}_{\mathcal{U}}, Z)_\bullet$ taken in prespectra and because geometric realization preserves inclusion prespectra. Similarly and more simply, α_1 follows from the suspension prespectra being an inclusion prespectra. The map α_2 is built from a zig-zag of straightforward equivalences induced by the following facts. The loop space functors commute with colimits of inclusions that have a cofinal $\mathscr{N}$-sequence. Second, Ω^{W-V} can be moved inside the two-sided bar construction (as a functor composed on the left of the left argument) homeomorphically when each of the homotopy set functors $\pi^K_{\iota_K(W-V)-(W-V)^K\oplus\mathbb{R}^n}(-)$ indexed by subgroups K of G and $n < \dim(W-V)^K$ vanish level-wise and inclusions of basepoints are h-cofibrations level-wise. The natural transformation $B^V Z \to \mathbb{E}Z(V)$ is now induced by the natural transformation of functors $\Sigma^V \to \operatorname{colim}_{V\subseteq U\subseteq\mathcal{U}} \Omega^{U-V}\Sigma^U$. Suppose now that K is a subgroup of G and $n < \dim V^K$. Then, for every $m \in \mathbb{N}$, the levels $B(\Sigma^V, \mathscr{K}_V, Z)_m$ and $B(\operatorname{colim}_{V\subseteq U\subseteq\mathcal{U}} \Omega^{U-V}\Sigma^U, \mathscr{K}_{\mathcal{U}}, Z)_m$ have vanishing $\pi^K_{\iota_K V - V^K\oplus\mathbb{R}^n}(-)$, the first from being a V-fold suspension and the latter from being a colimit of inclusions with cofinal $\mathscr{N}$-sequence of spaces that also have vanishing $\pi^K_{\iota_K V - V^K\oplus\mathbb{R}^n}(-)$. Indeed, by adjunction, for a given U, we have an isomorphism

$$\pi^K_{\iota_K V - V^K\oplus\mathbb{R}^n}(\Omega^{U-V}\Sigma^U(\mathscr{K}_V)^{\circ m}Z) \longrightarrow \pi^K_{\iota_K U - U^K\oplus\mathbb{R}^n}(\Sigma^U(\mathscr{K}_V)^{\circ m}Z)$$

and the vanishing of the target follows from basic G-connectivity properties of suspensions. We need to verify that the remaining homotopy groups are isomorphisms and take $n \geq \dim V^K$. By applying the change of groups adjunction, we may assume without loss of generality reduce to the case that $G = K$. Note that our task is now to compare homotopy groups of maps out of $S^{(V - V^K \oplus \mathbb{R}^n)} \simeq S^{n - \dim V^K} \wedge S^V$. Therefore, it suffices to show that the induced map $\Omega^V B^V Z \to \Omega^V \mathbb{E}Z(V) \simeq \Omega^\infty \mathbb{E}Z$ is a weak equivalence because it suffices for the map of equivariant homotopy groups $\pi_n^K(\Omega^V B^V Z) \to \pi_n^K(\Omega^\infty B^V Z)$ is an isomorphism. It is not difficult to verify the commutativity of the diagram of $\mathscr{K}_{\mathcal{U}}$-spaces

$$\begin{tikzcd}
Z \arrow[d, equal] & B(\mathscr{K}_V, \mathscr{K}_V, Z) \arrow[l] \arrow[r, "\alpha"'] \arrow[d] & B(\Omega^V \Sigma^V, \mathscr{K}_V, Z) \arrow[r] \arrow[d] & \Omega^V B^V Z \arrow[d] \\
Z & B(\mathscr{K}_{\mathcal{U}}, \mathscr{K}_{\mathcal{U}}, Z) \arrow[l] \arrow[r] & B(\Omega^\infty \Sigma^\infty, \mathscr{K}_{\mathcal{U}}, Z) \arrow[r] & \Omega^\infty \mathbb{E}Z
\end{tikzcd}$$

whose rows are taken from [26, 1.13, 2.12] and consist of weak equivalences. Therefore, the rightmost vertical map is a weak equivalence. The morphism marked α is a weak group completion because V contains a copy of the trivial representation. Therefore, because Z is grouplike, α is a weak equivalence. Therefore, the map $B^V Z \to \mathbb{E}Z(V)$ is a weak equivalence. Moreover, the diagram above also produces the desired triangle in the homotopy category. $\square$

Lemma 6.6. *Let X be a cofibrant grouplike $\mathscr{K}_{\mathcal{U}}$ algebra for a G-universe $\mathcal{U}$ and $U \subseteq W$ be nested finite-dimensional subrepresentations of $\mathcal{U}$. Suppose at least one of the following holds.*

1. *The representation $W - U$ has a trivial summand.*
2. *The representation W contains a trivial summand and X is G-connected.*
3. *Both X and $\Omega^U X$ are G-connected.*

Then, there exists an equivalence $f : B^{W-U} X \to B^W \Omega^U X$ in the homotopy category of G-spaces such that in the homotopy category of K_W-spaces, there is a commutative diagram of *isomorphisms*

$$\begin{tikzcd}
\Omega^{W-U}B^{W-U}X \arrow[rr, "\Omega^W f"] & & \Omega^{W-U}B^W\Omega^U X \\
& X \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where the α and β are induced by the zig-zag from [26, 1.13].

Proof. The map f arises from the zig-zag

$$B^{W-U}X \longrightarrow \mathbb{E}(X)(W-U) \xrightarrow{\sim} \Omega^U\mathbb{E}(X)(W) \xleftarrow{\omega_U} \mathbb{E}(\Omega^U X)(W) \longleftarrow B^W\Omega^U X$$

with the unmarked maps arising from Theorem 6.5. Taking Ω^U (which preserves weak equivalences) and using more of Theorem 6.5 and [26, 1.13,2.12] with forgetful functors. we have a commutative diagram of equivalences in the homotopy category of K_W-spaces

$$\begin{tikzcd}
\Omega^{W-U}\mathbb{E}(X)(W-U) \arrow[rr, "\sim"] \arrow[drr, "\sim"] & & \Omega^{W-U}\Omega^U\mathbb{E}(X)(W) \arrow[d, "\sim"] & & \Omega^{W-U}\mathbb{E}(\Omega^U X)(W) \arrow[ll, "\omega_U"'] \arrow[d, "\sim"'] \\
& & \Omega^\infty\mathbb{E}(X) \arrow[dd] & & \Omega^\infty\mathbb{E}(\Omega^U X) \arrow[ll, "\omega_U"] \arrow[ddll] \\
\Omega^{W-U}B^{W-U}X \arrow[uu] \arrow[urr] \arrow[drr] & & & & \Omega^{W-U}B^W\Omega^U X \arrow[u] \arrow[dll] \arrow[uu, bend right] \\
& & X & &
\end{tikzcd}$$

as desired. ∎

Proposition 6.7. *Let $U, V,$, and W be G-representations (over $\mathbb{R}$) in a G-universe $\mathcal{U}$ such that $U \subseteq W$ and $V \subseteq W$ and let $\phi : \mathscr{O} \to K_{\mathcal{U}}$ be a weak equivalence from a cofibrant G-operad to the Steiner operad of the G-universe $\mathcal{U}$. Suppose X and Y are cofibrant grouplike $\mathscr{O}$-algebras for and $g : Q_{\mathscr{O}}\Omega^U X \to \Omega^V Y$ is a map of $\mathscr{O}$-algebras where $Q_{\mathscr{O}}$ is the cofibrant replacement in $\mathscr{O}$-algebras. Assume that (U, W, X) and (V, W, Y) satisfy one of the numbered hypotheses of*

Lemma 6.6. If ϕ_* is the change of operads functor, then there is a morphism

$$f : B^{W-U}\phi_*X \to B^{W-V}\phi_*Y$$

in the homotopy category of G-spaces via a morphism f such that the diagram

$$\begin{tikzcd}
\Omega^U X \arrow[r, dash, "\sim"] & \Omega^U\Omega^{W-U}B^{W-U}\phi_*X \arrow[r, "\sim"] & \Omega^W B^{W-U}\phi_*X \arrow[dd, "\Omega^{W-U}f"] \\
Q_{\mathscr{O}}\Omega^U X \arrow[u, "\sim"] \arrow[d, "g"] & & \\
\Omega^V Y \arrow[r, dash, "\sim"] & \Omega^V\Omega^{W-V}B^{W-V}\phi_*Y \arrow[r, "\sim"] & \Omega^W B^{W-V}\phi_*Y
\end{tikzcd}$$

commutes in the homotopy category of G-spaces. If g is a weak equivalence, so is f.

Proof. Recall that ϕ_* induces an isomorphism on homotopy categories and that because all G-spaces are fibrant, the forgetful functor ϕ^* (which we will often suppress in the notation below) from $\mathscr{K}_{\mathcal{U}}$-algebras to $\mathscr{O}$-algebras is derived and induces an adjoint to ϕ_* in homotopy categories. Because X and Y are cofibrant, we can find an equivalence of $\mathscr{O}$-algebras lifting the unit in homotopy categories $Y \to \phi^*\phi_*Y$ and $X \to \phi^*\phi_*X$. Applying the homotopical endofunctors Ω^U and Ω^V on $\mathscr{O}$-algebras and composing, we obtain the zig-zag in the top row of the commutative diagram

$$\begin{tikzcd}
\Omega^U\phi^*\phi_*X \arrow[d, equal] & \Omega^U X \arrow[l, "\sim"'] & Q_{\mathscr{O}}\Omega^U X \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & \Omega^V Y \arrow[r, "\sim"] & \Omega^V\phi^*\phi_*Y \arrow[d, equal] \\
\phi^*\Omega^U\phi_*X & & \phi^*\phi_*Q_{\mathscr{O}}\Omega^U X \arrow[ll] \arrow[rr] & & \phi^*\Omega^V\phi_*Y
\end{tikzcd}$$

which induces a zig-zag

$$\Omega^U\phi_*X \longleftarrow \phi_*Q_{\mathscr{O}}\Omega^U X \longrightarrow \Omega^V\phi_*Y$$

by adjunction. It follows $\phi_*Q_{\mathscr{O}}\Omega^U X \to \Omega^U\phi_*X$ is a weak equivalence from the two-out-of-three property after applying the weak equivalence reflecting functor ϕ^* as in the second row of the first diagram. The next diagram defines g.

$$\begin{tikzcd}
B^{W-U}\phi_*X \arrow[r] & B^W\Omega^U\phi_*X \arrow[dl, "\alpha"] \\
B^W\Omega^V\phi_*Y & B^{W-V}\phi_*Y \arrow[l]
\end{tikzcd}$$

The horizontal morphisms are the isomorphisms in the homotopy category of G-spaces of Lemma 6.6. The final claim about the commutative square in the homotopy category of spaces is verified by using the diagram below. Applying Ω^W, and naturality of the zig-zag of [26, 1.13] we have the commutative diagram in the homotopy category of G-spaces

$$\begin{tikzcd}
\Omega^W B^{W-U}\phi_*X \arrow[d] & \Omega^U\Omega^{W-U}B^{W-U}\phi_*X \arrow[l, "\sim"'] \arrow[d] \arrow[dr] & \\
\Omega^W B^W\Omega^U\phi_*X \arrow[d, "\Omega^W B^W\alpha"] & \Omega^U\Omega^{W-U}B^W\Omega^U\phi_*X \arrow[l, "\sim"'] \arrow[r] & \Omega^U\phi_*X \arrow[d, "\alpha"] \\
\Omega^W B^W\Omega^V\phi_*Y & \Omega^V\Omega^{W-V}B^W\Omega^V\phi_*Y \arrow[l, "\sim"'] \arrow[r] & \Omega^V\phi_*Y \\
\Omega^W B^{W-V}\phi_*Y \arrow[u] & \Omega^V\Omega^{W-V}B^{W-V}\phi_*Y \arrow[l, "\sim"'] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Because each morphism save α and $\Omega^W B^W\alpha$ in the diagram is an equivalence, this completes the proof. □

Let $BU_{\mathbb{R}}(1) = U_{\mathbb{R}}(\mathbb{C}^\infty \oplus \mathbb{C})/U_{\mathbb{R}}(\mathbb{C}^\infty) \times U_{\mathbb{R}}(\mathbb{C})$.

Proposition 6.8. *Consider the canonical map $F : \Sigma^\rho BU_{\mathbb{R}}(1) \to B^\rho\phi_*BU_{\mathbb{R}}$ induced by the inclusion map $BU_{\mathbb{R}}(1) \to BU_{\mathbb{R}}$ and various natural transformations. Then, there exists a* *commutative diagram*

$$\begin{tikzcd}
\Sigma^\rho BU_{\mathbb{R}}(1) \arrow[r] \arrow[d, "F"'] & \Sigma^\rho BU_{\mathbb{R}} \arrow[dl] \arrow[dd] \arrow[ddd, bend left, "\beta"] \\
B^\rho \phi_* BU_{\mathbb{R}} \arrow[d, "\lambda"'] & \\
B^1 \phi_* SU_{\mathbb{R}} \arrow[r] \arrow[dr] & BSU_{\mathbb{R}} \arrow[d] \\
 & BU_{\mathbb{R}}
\end{tikzcd}$$

in the homotopy category of G-spaces such that the upper triangle is induced by inclusions, λ is an equivalence, and β is an equivariant lift of the equivariant Bott map.

Proof. We begin with the chain of maps

$$\begin{tikzcd}
 & & \Omega^\rho BSU_{\mathbb{R}} \arrow[d, "\simeq"] \arrow[drr] & & \\
BU_{\mathbb{R}}(1) \arrow[r] & BU_{\mathbb{R}} \arrow[r] \arrow[ur] & \Omega^\sigma SU_{\mathbb{R}} \arrow[r] & \Omega^\sigma U_{\mathbb{R}} \arrow[r] & \Omega^\rho BU_{\mathbb{R}}
\end{tikzcd}$$

of inclusions and the maps of Propositions 6.3 and 6.4 with homotopy commutative factorization through $\Omega^\rho BSU_\rho$. The upward-pointing diagonal map is a factorization given by the same formulas as the composite $BU_{\mathbb{R}} \to \Omega^\rho BU_{\mathbb{R}}$ and it is straightforward to check that it is well defined. The composite $BU_{\mathbb{R}} \to \Omega^\rho BU_{\mathbb{R}}$ is the (adjoint of the) equivariant Bott map. Using Proposition 6.7 we can form a commutative diagram

$$\begin{tikzcd}
\Sigma^\rho BU_{\mathbb{R}}(1) \arrow[r] \arrow[ddr, "F"] & \Sigma^\rho BU_{\mathbb{R}} \arrow[r] \arrow[dd] & BSU_{\mathbb{R}} \arrow[r] \arrow[d] & BU_{\mathbb{R}} \arrow[dd, "\sim"'] \\
 & & \phi_* BSU_{\mathbb{R}} \arrow[d] & \\
 & B^\rho \phi_* BU_{\mathbb{R}} \arrow[r] & B^1 \phi_* SU_{\mathbb{R}} \arrow[r] & \phi_* BU_{\mathbb{R}}
\end{tikzcd}$$

in the homotopy category of spaces. To verify that the induced map $\Sigma^\rho BU_{\mathbb{R}} \to BU_{\mathbb{R}}$ is the equivariant Bott map, we apply more of the statement of Proposition 6.7 to compare the adjoint to the original chain of maps to complete the proof. $\square$

6.2 Cohomology computation

We will need to make some comments on the $RO(G)$ equivariant Serre spectral sequence of [43], building on [44]. The idea of using hypercohomology to understand the $RO(G)$-grading of this sequence is an insight from Mike Hill. Let $f : X \to Y$ be a map of G-spaces. First, the spectral sequence of [43] is not stated to be Mackey functor valued, but by functoriality in the coefficients and because Mackey functors have corresponding Mackey functors of Mackey functors, we can take it to be Mackey functor valued. More precisely, we invoke the fact that any abelian group valued functor F from the category of G-Mackey functors is the evaluation at G/G of a unique G-Mackey functor valued functor F' from G-Mackey functors such that $F'(\underline{M}) = F(\underline{M}(G \times -))$. Then, there is an induced functor of equivariant simplices $\Delta_G(X) \to \Delta_G(Y)$ and an uninteresting version (due to having no easily computable pages in general) of the equivariant Serre spectral sequence that looks like the hypercohomological Grothendieck spectral sequence converging to associated to the composition of hypercohomological functors appearing as the vertical arrows below.

$$\begin{tikzcd}
{[\Delta_G^{\mathrm{op}}(E), \underline{A}-Mod]} \arrow[r, "\iota"] & D_+([\Delta_G(E)^{\mathrm{op}}, \underline{A}-Mod]) \arrow[d, "\lim^R \circ F^R(S^{-V},-)"] & \\
& D_+([\Delta_G(B)^{\mathrm{op}}, \underline{A}-Mod]) \arrow[d, "\lim^R"] & \\
& D_+(\underline{A}-Mod) \arrow[r, "H^*(F^R(-,\underline{M}))"] & (\underline{A}-Mod)^{\mathbb{Z}}
\end{tikzcd}$$

Here, the first functor is the result of applying the Mackey functor analogue of the Hom functor (adjoint to the box product) out of the chain complex associated to the constant functor of the cellular chain complex of a representation sphere S^V (using duality to extend this notion for virtual representations) and then applying a fiber-wise derived limit functor.

The second functor is simply a derived limit functor. When the map $E \to B$ is a G-fibration (and when beginning with a cochain complex of coefficients concentrated in degree 0 as shown), the functor $[\Delta_G^{\text{op}}(E), \underline{A} - Mod] \to D_+([\Delta_G(B)^{\text{op}}, \underline{A} - Mod])$ is well-behaved and we can describe the E_2 page as $\underline{E}_2^{p,q}(V) = \underline{H}^p(B, \underline{\mathscr{H}}^{q+V}(f, M))$ for an $RO(G)$ graded local coefficient system $\underline{\mathscr{H}}^\star(f, M)$, i.e., a collection of functors indexed on the G-representation group from $\Delta_G(B)$ to G-Mackey functors that factors through the G-fundamental groupoid of X in a designated way. In the special case where B is G-simply connected and F is the fiber of $E \to B$, the E_2 page is $\underline{E}_2^{p,q}(V) = \underline{H}^p(B, \underline{H}^{q+V}(f, M))$. However, we can work just as easily with the more general spectral sequence for which the lower $\lim^R$ functor of the base is replaced with $\lim^R \circ F^R(S^W, -)$ for another G-representation W. An essential reason for this is that the functors $F^R(S^{-V}, -)$ commute with the limit functors, and applying them with the first or second composite results in the same total complex but different filtrations. Consequently, following the arguments of [43], we have the following computational tool, which is presented in the special case we use.

Proposition 6.9. *There exists a functorial spectral sequence of $H\underline{\mathbb{Z}}^{*+\star} \square H\underline{\mathbb{Z}}^{*'+\star'}$-algebras of bidegree $(r, 1-r)$ with parameters $(\star, \star')$ from $RO(G) \times RO(G)$ (or strictly speaking, a groupoid with objects in $RO(G) \times RO(G)$ and morphisms of homotopy classes of equivalences of the corresponding representation sphere) for G-fibrations with $E \to B$ with simply connected base B and fiber F of the form*

$$\underline{E}_2^{p,q}(W, V) = \underline{H}^{p+W}(B, \underline{H}^{q+V}(F, \underline{\mathbb{Z}})) \Rightarrow \underline{H}^{p+q+W+V}(E, \underline{\mathbb{Z}})$$

which converges strongly. Moreover, there exists a cup product structure

$$\underline{E}_r^{p,q}(W, V) \square \underline{E}_r^{p',q'}(W', V') \to \underline{E}_r^{p+p',q+q'}(W \oplus W', V \oplus V')$$

(for pairs of equivalences $S^W \wedge S^{W'} \to S^{W \oplus W'}$ and $S^V \wedge S^{V'} \to S^{V \oplus V'}$) converging to the cup

product

$$\underline{H}^{p+q+W+V}(E,\underline{\mathbb{Z}}) \square \underline{H}^{p'+q'+W'+V'}(E,\underline{\mathbb{Z}}) \to \underline{H}^{p+p'+q+q'+W+W'+V+V'}(E,\underline{\mathbb{Z}}).$$

We will now use this spectral sequence to make our computation relatively straightforward.

Proposition 6.10. *The inclusion $BSU_{\mathbb{R}} \to BU_{\mathbb{R}}$ induces on Mackey functor-valued $RO(C_2)$-graded cohomology with coefficients in $\underline{\mathbb{Z}}$ the quotient map*

$$H\underline{\mathbb{Z}}^{\star}[[c_n : n \geq 1]] \to H\underline{\mathbb{Z}}^{\star}[[c_n : n \geq 2]]$$

sending $c_1 \mapsto 0$. Here c_i is in degree $i\rho$ evaluated at the G-set G/G.

Proof. The cohomology of $BU_{\mathbb{R}}$ is computed as stated in [34, 2.25] using the Thom isomorphism and because $H\underline{\mathbb{Z}}$ is Real-oriented by truncation. We compute the cohomology of $BSU_{\mathbb{R}} \to BU_{\mathbb{R}}$ via the spectral sequence associated to the C_2-fibration

$$S^{\sigma} \simeq U_{\mathbb{R}}(1) \to BSU_{\mathbb{R}} \to BU_{\mathbb{R}}$$

and has the one associated to

$$S^{\sigma} \to BSU_{\mathbb{R}}(1) \simeq * \to BU_{\mathbb{R}}(1)$$

as a retract via the inclusion and determinant maps Note that $BU_{\mathbb{R}}$ and $BU_{\mathbb{R}}(1)$ are classifying spaces of G-connected spaces and thus G-simply connected. The Serre spectral sequence as stated thus applies, and by a freeness argument, the result is determined. □

Proposition 6.11. *The cohomology ring of $\Sigma^{\rho} BU_{\mathbb{R}}(1)$ with $\underline{\mathbb{Z}}$ coefficients is abstractly isomorphic to $H\underline{\mathbb{Z}}^{\star}[[c_n : n \geq 2]]/(c_i c_j : i, j \geq 2)$.*

Proof. Suspension of a based space shifts cohomology apart from the basepoint summand and has cup product structure given by the square zero extension of the cohomology of a point by standard arguments. Since $BU_{\mathbb{R}}(1)$ has polynomial $\underline{\mathbb{Z}}$-cohomology ring, the result follows. $\square$

Proof of Theorem 6.1. We identify the map $\Sigma^{\rho} BU_{\mathbb{R}}(1) \to B^{\rho} BU_{\mathbb{R}}$ with $\Sigma^{\rho} BU_{\mathbb{R}}(1) \to BSU_{\mathbb{R}}$ as justified by Proposition 6.8. Evaluation of cohomology Mackey functors at the C_2-set C_2/e admits a natural isomorphism with non-equivariant cohomology; therefore, the lower square of the diagram below commutes.

$$\begin{tikzcd}
\underline{H}^{n\rho}(BSU_{\mathbb{R}}, \underline{\mathbb{Z}})(C_2/C_2) \arrow[r] \arrow[d, "\sim"'] & \underline{H}^{n\rho}(BSU_{\mathbb{R}}, \underline{\mathbb{Z}})(C_2/C_2) \arrow[d, "\sim"] \\
\underline{H}^{n\rho}(BSU_{\mathbb{R}}, \underline{\mathbb{Z}})(C_2/e) \arrow[r] \arrow[d, "\sim"'] & \underline{H}^{n\rho}(\Sigma^{\rho} BU_{\mathbb{R}}(1), \mathbb{Z})(C_2/e) \arrow[d, "\sim"] \\
H^{2n}(BSU, \mathbb{Z}) \arrow[r] & H^{2n}(\Sigma^2 BU(1), \mathbb{Z})
\end{tikzcd}$$

The upper square of the diagram is formed by the restriction structure maps of Mackey functors and consists of isomorphisms. What we have shown reduces our problem to [20, 7.3]. $\square$

Chapter 7

Orientations

We conclude this report with a review of how the main arguments of [20] and results of equivariant stable homotopy theory would complete the proof of 7.2. All results are unoriginal or conditional on the rest of the project, and we will not prove any claims.

Throughout this chapter, we set G as a finite group. An E_V-algebra refers to an algebra over an E_V operad, modulo technicalities combining the categories.

7.1 Obstruction Theory

We begin by summarizing obstruction theory in genuine equivariant stable homotopy theory. One powerful source of computational information since the Kervaire Invariant One work of [30] has been the slice spectral sequence. The genuine equivariant stable homotopy category admits a filtration, called the slice filtration, along which key examples of spectra, including $MU_{\mathbb{R}}$ and Real Landweber exact theories, lend themselves to tractable analysis. We use the regular slice tower and filtration of [67] rather than the original one in [30].

Definition 7.1. A *slice sphere* $\hat{S}$ is a cofibrant spectrum equivalent to one of the form $G_+ \wedge_H S^{n\rho_H}$ for some $n \in \mathbb{Z}$ in which case the *dimension of the slice sphere* $\dim \hat{S}$ is $n|H|$, where H is a subgroup of G and ρ_H is the real regular representation of H.

Proposition 7.2 ([30]). *There is a functorial tower*

$$\dots P^2X \to P^1X \to P^0X \to P^{-1}X \to P^{-2}X \to \dots$$

for G-spectra X with homotopy limit X such that P^nX is the universal (in the homotopical sense) spectrum under X for which, for every slice sphere $\hat{S}$, the set of homotopy classes of maps $[\hat{S}, P^nX]$ is 0 when $\dim \hat{S} > n$. Moreover, the natural map $[\hat{S}, X] \to [\hat{S}, P^nX]$ is an isomorphism when $\dim \hat{S} \leq n$.

Definition 7.3. The n^{th} slice P_n^nX of a genuine G-spectrum X is the fiber of the map $P^nX \to P^{n-1}X$. An n-slice is a spectrum equivalent to its n^{th} slice.

The slice filtration is sufficiently refined.

Proposition 7.4. *If X and Y are n-slices the G-space mapping spectrum $F(X,Y)$ has vanishing homotopy groups outside the coefficient system (or Mackey functor) $\underline{\pi}_0 F(X,Y)$.*

The slice tower can be constructed through Bousfield localization, and an elementary explanation of the construction of P^nX is that one attaches cells of dimension at least n+2 (corresponding to slice spheres of dimension at least n+1 to induce vanishing of the slice sphere-sourced homotopy groups, as in one of the classical constructions of Eilenberg–Mac Lane spaces from Moore Spaces.

One perspective on slice spheres is that they are the result of applying a multiplicative followed by an additive change of group functors to non-equivariant spheres of non-equivariant stable homotopy theory. From this viewpoint, slice spheres, which arise from an externalized multiplicative phenomenon, are well suited for filtering multiplicative objects. In the multiplicative case, connectivity is required of an operad $\mathscr{O}$ and its algebra A for each P^nA to admit a canonical algebraic structure.

The behavior of the slice filtration under the smash product makes the following fact a routine exercise.

Proposition 7.5. *If $\mathscr{O}$ is a connective operad and R is a connective algebra over $\mathscr{O}$, then P^nR has a canonical structure of a module over the algebra P^0R over $\mathscr{O}$ in the homotopy category.*

The following obstruction theory has been identified.

Theorem 7.6 ([6]). *Let $\mathscr{O}$ be a connective operad and R be an algebra over $\mathscr{O}$. Then, the slice tower of R can be constructed to be a tower of $\mathscr{O}$-algebras and there are homotopy pullback squares*

$$\begin{tikzcd} P^{n+1}R \arrow[r] \arrow[d] & P^0R \arrow[d] \\ P^nR \arrow[r] & P^0R \vee \Sigma^1 P^{n+1}_{n+1}R \end{tikzcd}$$

where $P^0R \vee \Sigma^1 P^{n+1}_{n+1}R$ has a $\mathscr{O}$-algebra structure a square-zero extension of P^0R.

Our contribution begins when we turn to Thom spectra. Suppose the $\mathscr{O}$-algebra map $f : X \to B\mathbb{G}$ with X based and connected is an input for $\mathscr{O}$-algebra Thom spectrum that admits an $\mathscr{O}$-algebra Thom diagonal, and $\mathscr{O}$ a well-chosen E_V operad (in a sense appropriate to its category), where V contains a copy of the trivial representation. Further, suppose that R is a connective E_V algebra where the Green functor structure on P^0R induced by the E_V structure is commutative. Suppose there is a given map $Mf \to P^nR$ for some n. To lift this to a map $Mf \to P^{n+1}R$, it is necessary and sufficient to show the composite

$$Mf \to P^nR \to P^0R \vee \Sigma^1 P^{n+1}_{n+1}R$$

is homotopic to the composite

$$Mf \to P^nR \to P^0R \to P^0R \vee \Sigma^1 P^{n+1}_{n+1}R.$$

The conditions we assumed for R and the discreteness of the endomorphism mapping spaces of the slices of R can be used to find an essentially unique algebra over $E_{V\oplus\mathbb{R}}$ operad structure on $P^0R \vee \Sigma^1 P^{n+1}_{n+1}R$ extending the $\mathscr{O}$-structure. Our strictification result Theorem 4.15 allows us to regard $P^0R \vee \Sigma^1 P^{n+1}_{n+1}R$ as a monoid in E_V algebras.

Let B be any $E_{V\oplus\mathbb{R}}$-algebra under Mf as an E_V-algebra; we apply our conclusions to the choice $P^0R \vee P^{n+1}_{n+1}R$. The homology Thom isomorphism theorem states that a homotopy class of maps of G-spectra

$$B \wedge Mf \to B \wedge \Sigma^\infty_+ X,$$

namely the one corresponding to the geometric cap product with the orientation obtained from the map $Mf \to B$, is an isomorphism.

With the assumed highly structured Thom spectrum construction, this map can be obtained as the adjoint to the composite of $\mathscr{O}$-algebra maps

$$Mf \to Mf \wedge \Sigma^\infty_+ X \to B \wedge \Sigma^\infty X$$

under the adjunction between the categories of $\mathscr{O}$-algebras and the category of left B-modules in the category of $\mathscr{O}$-algebra maps. Because the free (only with respect to B-module structure) left B-modules in $\mathscr{O}$-algebras $B \wedge Mf$ and $B \wedge \Sigma^\infty_+ X$ are isomorphic, the set of maps $Mf \to B$ and $\Sigma^\infty_+ X \to B$ of $\mathscr{O}$-algebras are in bijection in the homotopy category.

In the case of interest, the map $\Sigma^\infty_+ X \to P^0_0 R \vee \Sigma^1 P^{n+1}_{n+1}R$ has a target that is a square-zero extension of $P^0_0 R$ and the fact that X is connected and based determines the map $\Sigma^\infty_+ X \to P^0_0 R$ and by general properties of augmented algebras, one can lift essentially uniquely to a map $\ell : \Sigma^\infty_+ X \to S_G \vee P^{n+1}_{n+1} X$ between two $\mathscr{O}$-algebras with S_G augmentation. Moreover, ℓ is trivial if and only if the obstruction we are studying vanishes.

By Theorem 5.16 and some adjunctions between augmented algebras, the obstruction to lifting $Mf \to P^n R$ to a map $Mf \to P^{n+1}R$ lies in the group $(P^{n+1}_{n+1}R)^V(B^V X)$.

7.2 Orientations of Real Bordism

There are several more ingredients to Anticipated Theorem .

First, for orientations of non-connective spectra, we need a mild generalization of [31, 2.7].

Proposition 7.7. *If R is a fibrant $\mathscr{O}$-algebra in a symmetric monoidal category of G-spectra, where $\mathscr{O}$ is a connective operad, then there is an $\mathscr{O}$-algebra map $r : \overline{R} \to R$ that is a connective cover on the underlying spectrum.*

Sketch. The idea behind the proof of the above proposition is to perform the same procedure as one would with CW approximation, but only using connective cells everywhere. One reduces to the case where $\mathscr{O}$ is build from non-negative dimensional slice cells and applies basic properties of the slice tower. □

Let us now consider the question of lifting a homotopy ring map from $MU_{\mathbb{R}}$ to an E_ρ map. For a homotopy ring spectrum B, homotopy ring maps $MU_{\mathbb{R}} \to B$ correspond to maps $\Sigma^\infty_\rho BU_{\mathbb{R}}(1) \to B$ such that the composite

$$S \cong \Sigma^\infty_\rho \Sigma U_{\mathbb{R}}(1) \to \Sigma^\infty_\rho BU_{\mathbb{R}}(1) \to B$$

is the homotopy ring unit. Lifting a homotopy ring map to a E_ρ map is therefore equivalent to the following extension problem.

$$\begin{tikzcd} \mathbb{O}^{\mathsf{aug}}\Sigma^\infty_\rho BU_{\mathbb{R}}(1) \arrow[r] \arrow[d] & R \\ MU_{\mathbb{R}} \arrow[ur, dashed] & \end{tikzcd} \qquad (*)$$

We also recall that $\Sigma^\infty_\rho BU_{\mathbb{R}}(1)$ is the Thom spectrum associated to the inclusion $BU_{\mathbb{R}}(1) \to BU_{\mathbb{R}}$.

Suppose R is an E_ρ connective ring spectrum such that the homotopy groups $\underline{\pi}_{n\rho-1}R$ are trivial for odd $n > 0$, and $\underline{\pi}_{n\rho}R$ are constant Mackey functors for $n \geq 0$. Further, suppose that

$\underline{\pi}_0(R)$ is a commutative Green functor. Let $f : MU_{\mathbb{R}} \to R$ be a homotopy ring map. Then, because the homomorphism induced by the unit $\underline{\pi}_0(S_{\mathbb{R}}) \to \underline{\pi}_0(MU_{\mathbb{R}})$ is an epimorphism, the homotopy ring map $MU_{\mathbb{R}} \to R \to P^0R$ is guaranteed to lift to the E_ρ algebra category.

Now assume that we have an E_ρ lift $MU_{\mathbb{R}} \to P^nR$ of the homotopy ring map $MU_{\mathbb{R}} \to R \to P^nR$ and seek to lift one more stage of the tower. Using the argument of the previous section, the compatibility of the Thom spectrum construction with the free algebra construction, and computing the derived indecomposibles of free augmented algebras, the lifting problem becomes the problem of extending a nullhomotopy of the horizontal arrow to the diagonal arrow in the following commutative triangle.

$$\begin{tikzcd}
\Sigma^\infty BU_{\mathbb{R}}(1) \arrow[r] \arrow[d] & \Sigma^1 P^{n+1}_{n+1} R \\
\Sigma^{-\rho}\Sigma^\infty B^\rho BU_{\mathbb{R}} \arrow[ur] &
\end{tikzcd}$$

Known facts about the slice tower for C_2-spectra, Theorem 6.1 and the conditions we have placed on the homotopy groups of R guarantee that the extension problem is solvable.

One must know more to lift up the entire slice tower to the homotopy limit.

Theorem 7.8 (Milnor sequence for spaces [16, IX.3.1])**.** *There is a short exact sequence of pointed sets*

$$* \longrightarrow \lim^1_{n\in\mathscr{N}^{\mathrm{op}}} \pi_{q+1}(X_n) \longrightarrow \pi_q(\lim_{n\in\mathscr{N}^{\mathrm{op}}} X_n) \longrightarrow \lim_{n\in\mathscr{N}^{\mathrm{op}}} \pi_q(X_n) \longrightarrow *$$

where for any inverse sequence of groups $H : \mathscr{N}^{\mathrm{op}} \to \mathsf{Grp}$, the pointed set $\lim^1_{n\in\mathscr{N}^{\mathrm{op}}} H_n$ is the set of orbits of $\prod_{n\in\mathbb{N}} H_n$ acting on itself via the left action

$$(g_n)_{n\in\mathbb{N}} \cdot (h_n)_{n\in\mathbb{N}} = (g_n h_n H([n \le n+1]^{\mathrm{op}})(g_{n+1})^{-1})_{n\in\mathbb{N}}.$$

For inverse sequences of abelian groups, $\lim^1$ has an abelian group structure, and the short exact sequence above is one in the category of groups for $q \ge 1$ and in the category of abelian *groups for $q \geq 2$.*

The proof of Anticipated Theorem 7.2 goes through the argument described above in a topologically enriched setting and inductively shows each relevant fundamental group in the Milnor sequence (with $q = 0$) with X_n as the space of solutions to the extension problem (*) vanishes, from which the theorem follows.

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