Title
Groups Acting on Products of Trees

Author
Brody, Nicolas

Publication Date
2022

Peer reviewed|Thesis/dissertation

Groups Acting on Products of Trees

by

Nicolas Ryan Brody

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Ian Agol, Chair
Professor Mark Haiman
Professor Michael Hutchings

Summer 2022

Groups Acting on Products of Trees

Abstract

Groups Acting on Products of Trees

by

Nicolas Ryan Brody

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Ian Agol, Chair

In this thesis, we explore many aspects of groups acting on trees and on products of trees. These ideas are central to the field of geometric group theory, the study of infinite groups by their large-scale behavior. Many of our techniques are algebraic and arithmetic in nature. Most of this work is motivated by the following question:

Question 1. If G is the fundamental group of a closed surface, can G act freely on a locally compact product of trees?

In fact, the more general question of whether a hyperbolic group which is not virtually free can act properly on a locally compact Euclidean building is open. Of course such an action gives a cubulation of the group in perhaps the simplest combinatorial type of cube complex, so this is a necessary condition on the group. For instance, which right-angled Coxeter or Artin groups admit proper actions on products of trees?

In a landmark paper of 1987, Gromov defines the class of hyperbolic groups, and nearly every branch of modern geometric group theory is represented in Gromov's original paper. However, Dehn's work in 1910 and 1911 may be viewed as the earliest work in geometric group theory. For several decades, Dehn's work was viewed with a more combinatorial lens, until Gromov emphasized the intimate connection with hyperbolic geometry.

Thurston's work in 3-dimensional topology together with Gromov's hyperbolic groups aligned topology and group theory. One might ask *to what extent is a 3-manifold determined by its fundamental group?* The Poincaré conjecture is a precise version of this question, famously resolved by Perelman. In a sense, the fundamental group of a 3-manifold is an excellent replacement for the manifold itself, and in the finite-volume hyperbolic case, the fundamental group miraculously contains all of the metric information!

For closed hyperbolic surfaces, the fundamental group knows only the genus, and there is a high-dimensional space of representations a given surface can support. For a 4-manifold,

the fundamental group says relatively little about the topology or the geometry. Every countable group can occur as the fundamental group of a 4-manifold, and there is a vast land of simply connected four-manifolds. However, it is interesting to note that, at least up to finite index, closed hyperbolic 3-manifold fundamental groups are determined by the data of an automorphism of a surface group. Upon replacing "hyperbolic 3-manifolds" with "discrete subgroups of $\mathsf{PSL}_2(\mathbb{C})$", it is tempting to consider "subgroups of $\mathsf{PSL}_2(\mathbb{C})$". This is the approach we take in this thesis. We observe that $\mathsf{PSL}_2(\overline{\mathbb{Q}})$ acts properly discontinuously and cocompactly on a "restricted" product of infinitely many hyperbolic planes, 3-spaces, and finite valence trees. By allowing transcendental entries, we get a similar statement, but not all of the trees will have finite-valence. But in fact, subgroups of $\mathsf{PSL}_2(\mathbb{C})$ with cofinite volume are actually already conjugate into $\mathsf{PSL}_2(\overline{\mathbb{Q}})$ anyway. That is to say, every hyperbolic 3-manifold of finite volume comes equipped with a canonical action on a product of infinitely many finite-valence trees.

One can ask which properties of discrete subgroups of $\mathsf{PSL}_2(\mathbb{C})$ might carry over to these more general subgroups of $\mathsf{PSL}_2(\mathbb{C})$. Some celebrated properties of Kleinian groups include coherence, tameness, LERFness. It is unknown to what extent each of these properties might hold in this more general setting.

For Julie and Jeff

Contents

List of Figures

Chapter 1

Background

1.1 Introduction

In this thesis, we explore many aspects of groups acting on trees and on products of trees. These ideas are central to the field of geometric group theory, the study of infinite groups by their large-scale behavior. Many of our techniques are algebraic and arithmetic in nature. Most of this work is motivated by the following question:

Question 2. If G is the fundamental group of a closed surface, can G act freely on a locally compact product of trees?

In fact, the more general question of whether a hyperbolic group which is not virtually free can act properly on a locally compact Euclidean building is open. Of course such an action gives a cubulation of the group in perhaps the simplest combinatorial type of cube complex, so this is a necessary condition on the group. For instance, which right-angled Coxeter or Artin groups admit proper actions on products of trees?

In a landmark paper of 1987, Gromov defines the class of hyperbolic groups, and nearly every branch of modern geometric group theory is represented in Gromov's original paper. However, Dehn's work in 1910 and 1911 may be viewed as the earliest work in geometric group theory. For several decades, Dehn's work was viewed with a more combinatorial lens, until Gromov emphasized the intimate connection with hyperbolic geometry.

Thurston's work in 3-dimensional topology together with Gromov's hyperbolic groups aligned topology and group theory. One might ask *to what extent is a 3-manifold determined by its fundamental group?* The Poincaré conjecture is a precise version of this question, famously resolved by Perelman. In a sense, the fundamental group of a 3-manifold is an excellent replacement for the manifold itself, and in the finite-volume hyperbolic case, the fundamental group miraculously contains all of the metric information!

For closed hyperbolic surfaces, the fundamental group knows only the genus, and there is a high-dimensional space of representations a given surface can support. For a 4-manifold, the fundamental group says relatively little about the topology or the geometry. Every countable group can occur as the fundamental group of a 4-manifold, and there is a vast land of simply connected four-manifolds. However, it is interesting to note that, at least up to finite index, closed hyperbolic 3-manifold fundamental groups are determined by the data of an automorphism of a surface group. Upon replacing "hyperbolic 3-manifolds" with "discrete subgroups of $\mathsf{PSL}_2(\mathbb{C})$", it is tempting to consider "subgroups of $\mathsf{PSL}_2(\mathbb{C})$". This is the approach we take in this thesis. We observe that $\mathsf{PSL}_2(\overline{\mathbb{Q}})$ acts properly discontinuously and cocompactly on a "restricted" product of infinitely many hyperbolic planes, 3-spaces, and finite valence trees. By allowing transcendental entries, we get a similar statement, but not all of the trees will have finite-valence. But in fact, subgroups of $\mathsf{PSL}_2(\mathbb{C})$ with cofinite volume are actually already conjugate into $\mathsf{PSL}_2(\overline{\mathbb{Q}})$ anyway. That is to say, every hyperbolic 3-manifold of finite volume comes equipped with a canonical action on a product of infinitely many finite-valence trees.

One can ask which properties of discrete subgroups of $\mathsf{PSL}_2(\mathbb{C})$ might carry over to these more general subgroups of $\mathsf{PSL}_2(\mathbb{C})$. Some celebrated properties of Kleinian groups include coherence, tameness, LERFness. It is unknown to what extent each of these properties might hold in this more general setting.

1.2 Hyperbolic Groups

Suppose G is a group, and S is a subset. We can construct a graph whose vertex set is G, and the edge set corresponds to $G \times S$: for each element $s \in S$, we have an edge (g, gs). This graph is called the *Cayley graph* $\mathsf{Cay}(G, S)$ of G with respect to S. If S is a generating set for G, then every element of G is a (finite) product of elements of S and their inverses. Upon writing $g = s_1 \dots s_n$, we observe that there is a corresponding path in the Cayley graph from the identity of G to g along the partial products $s_1 \dots s_k$. Consequently, $\mathsf{Cay}(G, S)$ is connected exactly when S generates G. A generating set determines a metric d_S on G, which measures the length of a minimal path in the Cayley graph between two elements of G. When S is a finite generating set, this metric space is proper (metric balls are compact) and geodesic (every pair of points is connected by a path whose length is the distance between the two points).

Suppose X is a geodesic metric space, and let $x, y, z \in X$. A *triangle* in X is the union of three geodesics connecting these three points. A triangle is δ-thin ($\delta \geq 0$) if every point in the interior of one of the geodesics is contained in the δ-neighborhood of the other two sides. A metric space is called *δ-hyperbolic* if every triangle is δ-thin, and called *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. Finally, a finitely generated group is called hyperbolic if it admits a finite generating set S for which $\mathsf{Cay}(G, S)$ is hyperbolic.

A space is 0-hyperbolic exactly when every triangle has one side contained in the union of the other two sides. Such a space is called an $\mathbb{R}$-tree. The Cayley graph of a free group with respect to a free generating set S is a $2|S|$-regular tree.

Proposition 1. *If a group G acts freely on a tree, G is free.*

Proof. A free action on a tree is properly discontinuous, because the 1/2-neighborhood of a vertex cannot intersect a translate of itself. The quotient of the tree by G is a graph, and the fundamental group of a graph is free. □

Definition 1. A *surface group* is the fundamental group of a closed orientable surface of genus at least two.

Upon defining hyperbolic groups, Gromov asks whether every hyperbolic group which is not virtually free might contain a surface group. This question has motivated many major works in topology and geometric group theory. It is especially natural in light of the following fact:

Proposition 2. *If a group G acts properly discontinuously and cocompactly on the hyperbolic plane by orientation-preserving isometries, G is a surface group*

A still stronger result due to Tukia is the fact that a finitely generated group is quasi-isometric to the hyperbolic plane if and only if it has a finite index subgroup which is a surface group.

Many cases of the "surface subgroup conjecture" have been resolved, perhaps most notably in the work of Kahn and Markovic [40], who find quasiconvex surface groups in fundamental groups of hyperbolic 3-manifolds of finite volume. This work was a key ingredient in Agol's resolution of the virtual Haken and virtual fibering questions of Thurston.

1.3 Bass-Serre Theory

The fundamental theorem of covering space theory in algebraic topology asserts that there is a Galois correspondence between the collection of connected covering spaces of a topological space X and its fundamental group $\pi_1(X)$. Under relatively mild assumptions, the universal cover $\widetilde{X}$ of X exists, and $\pi_1(X)$ acts on it freely by deck transformations. In this case, we have $X = \widetilde{X}/\pi_1(X)$. In other words, the process of constructing the universal cover of X is reversed by taking the quotient by the deck group.

Suppose we want to study this problem in the other direction. If a group G acts on a space X, when do we have $\pi_1(X/G) = G$? This holds under the assumptions that G acts freely and properly discontinuously on a connected and simply connected space.

It is very easy to construct a great number of examples of group actions that violate these assumptions, in different sorts of ways.

- If S_n acts on $\{1, \ldots, n\}$ by permutations, the quotient space is a point, which is simply connected. In this action, the stabilizer of a point is a copy of S_{n-1}. The space is not connected, and the action is not free.
- If $\mathbb{R}$ acts on $\mathbb{R}$ by translations, the quotient space is a point, which is contractible. This action fails to be properly discontinuous.
- If $\mathbb{Z}/n\mathbb{Z}$ acts on S^1 by rotation, the quotient space is a circle. The space is not simply connected.

- If $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation, the quotient space is topologically $\mathbb{R}^2$. The action is not free.
- If S_3 acts on the edges of the tree which looks like the letter "Y" by permuting the three edges, the quotient is an edge.

Studying the failure of "$\pi_1(X/G) = G''$ in each of these cases actually instructs us to various approaches to resolving the issue. Bass-Serre theory provides a solution to this problem by (i) restricting the type of space X is allowed to be, (ii) changing what we mean by X/G, and (iii) changing what we mean by "π_1". As a remark, each of the above actions can be "fixed" by a suitable adjustment.

Theorem 1. *Suppose G acts on a tree T. Then the fundamental group of the graph of groups T/G is G.*

Theorem 2. *Let $A \subseteq X$ be connected topological spaces with inclusion map ι and $a \in A$, and let $p\colon (Y, b) \to (X, a)$ be a connected covering space. Let $B = p^{-1}(A)$. Then we have*
(i) $\pi_0(B, b) = \pi_1(Y, b) \setminus \pi_1(X, a)/\iota_*\pi_1(A, a)$
(ii) $\pi_1(B, b) = (\iota_A)_*^{-1}(p_*(\pi_1(Y, b)))$.

$$\begin{tikzcd}
(B,b) \arrow[r, dotted, "\iota_B"] \arrow[d, "p"] & (Y,b) \arrow[d, "p"] \\
(A,a) \arrow[r, hook, "\iota_A"] & (X,a)
\end{tikzcd}$$

Proof. First, we note that the set of preimages of the basepoint $p^{-1}(a)$ is naturally identified with the coset space $\pi_1(Y, b) \setminus \pi_1(X, a)$. A (homotopy class of a) path from b to another preimage c of a maps to a loop in X based at a, and precomposing such a path with a loop at b in Y corresponds to left-multiplying with an element of $\pi_1(Y, b)$. If we postcompose a path from b to c with a path from c to c', where the path is contained in a component of the preimage B of A, this path projects to a loop in the subset A of X, or in other words, an element of $\iota_*\pi_1(A, a)$, hence determines the same component of $\pi_0(B, b)$. This proves the first statement.

Note that B may be disconnected, so its fundamental group may depend somewhat dramatically on the choice of basepoint b. The inclusion $\iota_A\colon A \to X$ determines a homomorphism $(\iota_A)_*\colon \pi_1(A, a) \to \pi_1(X, a)$, which need not be injective (or surjective), although the homomorphism induced by the covering map p is injective. The claim is that the subgroup of $\pi_1(A, a)$ which the based covering space (B, b) corresponds to is $(\iota_A)_*^{-1}p_*(\pi_1(Y, b))$.

To see this, we suppose α is a loop in A based at a, and lift this to a path in B. Since $A \subseteq X$, α may be regarded as a loop in X, and lifts to a path based at b in Y. The path it lifts to is a loop in Y if and only if it lives in the p_*-image of $\pi_1(Y, b)$. But if it does lift to a loop based at b in Y, it lifts to something contained in the preimage B of A, since the path is contained in A. If it does not lift to a loop in Y, it is not an element of $\pi_1(B, b)$. ∎

Definition 2. For the particular formulation of Bass-Serre theory we carry out here, we will consider a graph as a small category Γ satisfying some additional properties. So Γ consists of the data of a collection of a set of *objects*, and for every pair of objects, a collection of *morphisms* between them. The objects of Γ consist of the disjoint union of a vertex set V and an edge set E, together with maps $\partial_-, \partial_+ \colon E \to V$ and an inversion map $\iota \colon E \to E$, satisfying $\partial_- \iota = \partial_+$, and $\iota^2 = 1$. We denote $\iota(e) = \overline{e}$, and we will often just describe the edge set of a graph by one of its oriented edges, with the understanding that the opposite orientation is also present. This can be summarized with the diagram below.

$$\begin{tikzcd}
E \arrow[loop above, "1_E"] \arrow[rr, bend left, "\iota"] \arrow[dr, "\partial_+"'] & & E \arrow[loop above, "1_E"] \arrow[ll, bend left, "\iota"] \arrow[dl, "\partial_-"] \\
& V &
\end{tikzcd}$$

Now a *graph of groups over* Γ is a functor $\mathbb{G} \colon \Gamma \to \mathsf{Grp}$, where Grp is the category of groups with *injective* homomorphisms. We think of labeling the vertices of Γ with groups, and the edges with common subgroups of the initial and terminal vertex groups.

Let Path_n be the (finite) category in which the vertex objects consist of $\{v_0, v_1, \ldots, v_n\}$ and the edge objects are $\{e_1, \ldots, e_n\}$, where $\partial_-(e_i) = v_{i-1}$ and $\partial_+(e_i) = v_i$. A path of length n in Γ is a functor $\gamma \colon \mathsf{Path}_n \to \Gamma$. We say that γ *starts* at $\gamma(v_0)$ and *ends* at $\gamma(v_n)$. It is a *loop* if these are equal. If γ_1 ends at v and γ_2 starts at v, we obtain a path $\gamma_1 \circ \gamma_2$ in Γ, of length $\ell(\gamma_1) + \ell(\gamma_2)$.

We also define the *augmented path category* $\widehat{\mathsf{Path}_n}$ to be the category that is obtained by adding "type 2 morphisms," a_i from v_i to itself, and b_i from e_i to itself. The *augmented group category*, $\widehat{\mathsf{Grp}}$, is the category of groups with type 1 morphisms, which correspond to injections between groups, and type 2 morphisms, which can only go from one object of Grp to itself, and these morphisms are in bijection with the elements of that group. If f_g and f_h are type 2 morphisms in a group G, we have $f_h \circ f_g = f_{gh}$.

If $\gamma \colon \mathsf{Path}_n \to \Gamma$ is a path, a *group path above* γ is a commutative diagram

$$\begin{tikzcd}
\widehat{\mathsf{Path}_n} \arrow[r, dotted, "\widetilde{\gamma}"] & \widehat{\mathsf{Grp}} \\
\mathsf{Path}_n \arrow[u, hook] \arrow[r, "\gamma"] & \Gamma \arrow[u, "\mathbb{G}"]
\end{tikzcd}$$

which thereby labels each vertex and each edge of the path γ with an element of the corresponding group $\mathbb{G}(v)$ or $\mathbb{G}(e)$.

We are at last in a position to define the *path groupoid* $\Pi_1(\mathbb{G}_\Gamma)$. The objects of the path groupoid are the collection of all $\mathbb{G}$-group paths above all paths in Γ, up to *homotopy*, which is the equivalence relation generated by the following three types of relations.

A relation of type 0 declares that if γ_1 and γ_2 are paths of length 0 based at the same vertex, we have $\gamma_1 \circ \gamma_2(a_0) = \gamma_1(a_0)\gamma_2(a_0)$.

A type 1 relation declares that $\gamma_0 \sim \gamma_e \sim \gamma_1$, where γ_0 is the group path $\gamma_0(v_0) = \partial_-(g_e)$, $\gamma_0(e_1) = 1$ and $\gamma_0(v_1) = 1$, γ_e is the group path $\gamma_0(v_0) = 1$, $\gamma_e(e_1) = g_e$, and $\gamma_1(v_1) = 1$, and γ_1 is the group path $\gamma_0(v_0) = 1$, $\gamma_e(e_1) = 1$, and $\gamma_1(v_1) = \partial_+(g_e)$. This relation declares that a group element on an edge can be pushed onto an adjacent vertex, and that if a group element on a vertex is in the image of an edge map, it can be pulled all the way across to the other vertex.

A type 2 relation declares that $e\overline{e} = 1_{\partial_- e}$. If there is a path of length two which begins at a vertex v, travels along an edge e to an adjacent vertex w, and then backtracks along $\overline{e}$ back to v, this is homotopic to the constant map at v when each vertex and edge is labeled with the identity of the corresponding group. Note that by using relations of type 0 and type 1, we can sometimes remove backtracking when some group elements are nontrivial (specifically, if the group element at the vertex w is in the image of the map from e).

There is a (partial) multiplication on the path groupoid, which is the composition of two group paths when the first one ends where the second begins.

Finally, if $v_0 \in \Gamma$, the *fundamental group of the graph of groups* $\pi_1(\mathbb{G}_\Gamma, v_0)$ is the collection of group paths lying above loops based at v_0, considered up to homotopy, where the multiplication is concatenation.

The Bass-Serre tree for $\mathbb{G}_\Gamma$ is the tree whose vertices consist of $\sim$-classes of paths based at v_0, with an edge from a reduced path of length n to a reduced path of length $n+1$ when the first one is a prefix of the second.

Example 1. Consider the graph of groups on the graph Γ with vertices $\{v_0, v_1\}$ and edges $\{e_1, e_2, e_3\}$, with $\partial_0(e_i) = v_0$ and $\partial_1(e_i) = v_1$. Then the data of a functor $\mathbb{G}\colon \Gamma \to \mathsf{Grp}$ requires us to select three edge groups with six injections into the two vertex groups. We will take $\mathbb{G}(v_1) = \langle l_1, l_2, l_3 \mid l_1 l_2 l_3 \rangle$ and $\mathbb{G}(v_2) = \langle r_1, r_2, r_3 \mid r_1 r_2 r_3 \rangle$, and three edge groups $\mathbb{G}(e_i) = \mathbb{Z}$.

$$
\begin{tikzcd}
v_0 \arrow[r, bend left, "e_1"] \arrow[r, "e_2"] \arrow[r, bend right, "e_3"'] & v_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[dl, bend right, "l_1"'] \arrow[dr, bend left, "r_1"] & \\
F_2 & \mathbb{Z} \arrow[l, "l_2"] \arrow[r, "r_2"] & F_2 \\
& \mathbb{Z} \arrow[ul, bend left, "l_3"] \arrow[ur, bend right, "r_3"'] &
\end{tikzcd}
$$

Note that F_2 happens to be the fundamental group of the topological realization of the underlying graph in this example, which we will refer to as a θ-graph. We will now construct a space whose fundamental group is the same as the fundamental group of this graph of groups.

Begin with a θ-graph situated at each vertex of this graph, and replace the midpoints of the edges with circles. We now have topological spaces whose fundamental group is the corresponding vertex or edge group. Then, for the edge injections, we glue on mapping cylinders corresponding to the homomorphisms. The resulting space is homeomorphic to a genus two surface, which coincides with the fundamental group of this graph of groups.

Example 2. When each vertex and each edge group is trivial, the fundamental group of the graph of groups is simply the usual fundamental group of the underlying graph, hence a free group whose rank is one minus the Euler characteristic of the graph. As a special case, when the graph consists of a single vertex and a loop at that vertex, labeled by the trivial group, the fundamental group is $\mathbb{Z}$.

If the graph consists of a single edge labeled by the trivial group, the fundamental group is the free product of the vertex groups. If the two vertex groups are cyclic groups of order 2 and 3, the Bass-Serre tree is a $(2,3)$-regular tree (or the first subdivision of a 3-regular tree). Note that the modular surface deformation retracts to the geodesic segment between its two singular points, and the preimage of this is a 3-regular tree embedded in the hyperbolic plane.

If the graph is a loop with the vertex and edge groups $\mathbb{Z}$, and the edge group embeds along ∂_- as $x \mapsto mx$ and along ∂_+ as $x \mapsto nx$, the fundamental group of the graph of groups is the Baumslag-Solitar group $BS(m,n)$.

If φ is a homotopy equivalence $X \to X$, there is a graph of groups on a loop with vertex and edge groups $\pi_1(X)$, and the edge group embeds in the vertex group along ∂_- as the identity, and it embeds in the vertex group along ∂_+ as φ_*. The fundamental group of this graph of groups is the fundamental group of the mapping torus.

As a special case, if φ is a pseudo-Anosov map from a closed hyperbolic surface Σ to itself, the fundamental group of this graph of groups is the fundamental group of the closed hyperbolic manifold which fibers over Σ with monodromy φ.

We remark that a topological realization of the graph of groups is always possible by replacing each vertex group with a $K(G_v,1)$, and edge groups with $K(G_e,1)\times[-1,1]$, where the two ends are glued to the adjacent vertex groups by continuous maps realizing the π_1-injections. Then the fundamental group of the graph of groups is just the usual fundamental group of this topological space.

Suppose a group G acts on a tree T without inverting an edge (although by subdividing such edges, this assumption is merely cosmetic). Then the *quotient graph of groups* is $T//G$, where the underlying graph is the usual quotient graph, and the vertices and edges are labeled by their stabilizers. This depends on a choice of lift to the tree, but any two vertices in the same orbit have conjugate stabilizers. Since the group acts without inversions, if g fixes an edge, it necessarily fixes the two vertices adjacent to it, and so $\mathbb{G}(e)$ naturally injects into both $\mathbb{G}(v)$ and $\mathbb{G}(w)$.

Property (FA)

A group is said to have "Property (FA)" if any action on a tree has a global fixed point (a point in the tree that is fixed by every element of the group). This implies that the group has no nontrivial graph of groups decomposition. Serre proves that a cocompact triangle group $G_{p,q,r} = \langle a, b, c \mid a^p, b^q, c^r, abc \rangle$ has property FA. A torsion element must act on any CAT(0) space with a global fixed point, and so each of a, b, c must fix a vertex. One can show that the fixed point of c must lie on the path between the fixed points of a and b, and since this is symmetric in a, b, c, all three elements must fix the same point.

If we are interested in groups which act on locally finite trees, or even regular trees, there is another type of restriction. Since a torsion element has a global fixed point, it must permute the adjacent vertices, and permute the sphere of radius two in a "blocked" way. This can be formalized in understanding the automorphism group of a rooted tree in terms of an iterated wreath product. If the tree is k-regular and p is a prime larger than k, there is no nontrivial action of $\mathbb{Z}/p\mathbb{Z}$ on the tree whatsoever.

Say a group has Property (FnA) if any action on a product of n trees has a global fixed point. Let H_1, H_2 be two cocompact triangle groups. By Selberg's lemma, they have surface subgroups of finite index, and since all surface groups are commensurable, there is a subgroup G_0 of finite index in each. Then the group $G = H_1 *_{G_0} H_2$ acts on its locally finite Bass-Serre tree, but the vertex stabilizers have Property (FA). Any action of G on a tree restricts to an action of H_1 on a tree, which must have a global fixed point. So G has property (F2A) but not property (FA).

1.4 Number Theory

Many of our techniques use arithmetic notions, which are defined here. We aim to be fairly self-contained.

Number fields

Any field k of characteristic zero contains the rational numbers as a subfield, and the field axioms provide k with the structure of a vector space over $\mathbb{Q}$. The field k is called a *number field* when the dimension of this vector space is finite. Its *degree* is the dimension of $k/\mathbb{Q}$.

An element $x \in k$ is said to be *integral* if it satisfies a *monic* polynomial with integer coefficients, or equivalently, if $\mathbb{Z}[x]$ is a finitely generated $\mathbb{Z}$-module. The collection of integral elements comprise a subring of k, called the *ring of integers*, typically denoted $\mathcal{O}_k$. We have $\mathcal{O}_k \otimes_{\mathbb{Z}} \mathbb{Q} \cong k$, and so the rank of $\mathcal{O}_k$ as a $\mathbb{Z}$-module is the dimension of k as a $\mathbb{Q}$-vector space.

If k is a number field of degree n over $\mathbb{Q}$, k is a *simple* extension of $\mathbb{Q}$, and so there is some $\alpha \in k$ which generates k as a $\mathbb{Q}$-algebra. The minimal polynomial of α over $\mathbb{Q}$ has degree n, and the roots correspond to n possible embeddings $\{\sigma_i\}_{i=1}^n$ of k in $\mathbb{C}$. If an embedding $\sigma_i(k)$ is contained in the real numbers, σ_i is called a *real place*, and is called a complex place otherwise. The embeddings which are not contained in the real numbers come in complex conjugate pairs, and so there are r real embeddings and s pairs of complex embeddings,

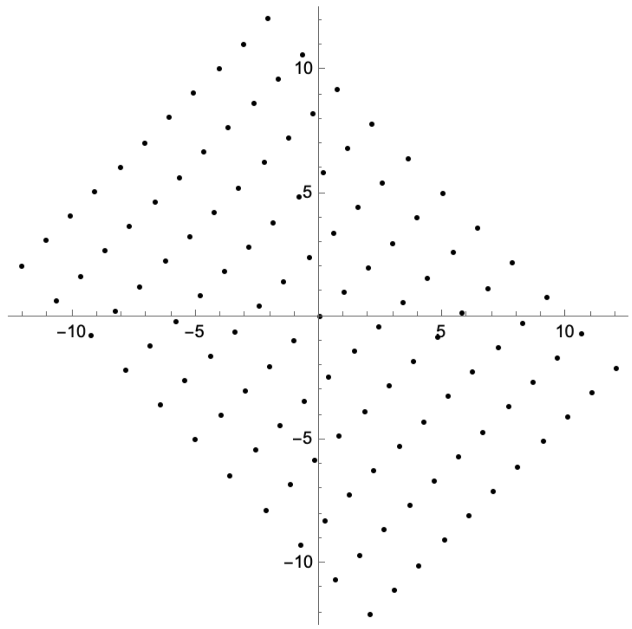


Figure 1.1: Some points from the embedding $\mathbb{Z}[\sqrt{2}] \to \mathbb{R}^2$

where $r + 2s = n$. We make the convention that the first r embeddings σ_i are real, and $\sigma_{j+s} = \overline{\sigma_j}$ for $j \in \{r+1, \ldots, r+s\}$.

Proposition 3. *The image of the map $\mathcal{O}_k \to \mathbb{R}^r \times \mathbb{C}^s$ defined by $x \mapsto (\sigma_i(x))_{i=1}^{r+s}$ is a discrete and cocompact subgroup.*

However, for every proper subset of the complex embeddings, the embedding into $\mathbb{R}^{r_0} \times \mathbb{C}^{s_0}$ is *dense*. This can be seen by the fact that the projection of the lattice onto a subproduct is injective, and if a free abelian subgroup $\mathbb{Z}^m \leq \mathbb{R}^n$ is discrete, we must have $m \leq n$. This is a special case of the strong approximation theorem we will see later, and is also related to the *Oppenheim conjecture* (a theorem of Margulis).

Valuations and Completions

A (real) valuation on a field k is a function $v\colon k \to \mathbb{R} \cup \{\infty\}$ so that $v(x) = \infty$ iff $x = 0$, $v(xy) = v(x) + v(y)$, and $v(x+y) \geq \min\{v(x), v(y)\}$. The valuation is called *discrete* if it has discrete image in $\mathbb{R}$.

The valuation determines a metric on k, where we define $d_v(x,y) = C^{-v(x-y)}$, for an appropriately chosen constant C. Given a valuation, a v-completion of k is a normed field k_v which is complete with respect to v, together with an embedding $k \hookrightarrow k_v$ with dense image.

Theorem 3 (Ostrowski's Theorem)**.** *Every completion $\mathbb{F}$ of $\mathbb{Q}$ is isomorphic to $\mathbb{R}$ or $\mathbb{Q}_p$ for some p.*

Sketch of proof. We can't have $|x| = 1$ for all $0 \neq x \in \mathbb{Z}$, or else the metric is trivial. If some $|x| > 1$, then we can show that the metric is $|x|_\infty^\alpha$ with $\alpha \in (0, 1]$.

Otherwise, $\mathbb{Z}$ is bounded, and there must be some $|x| < 1$. Letting S be the set primes dividing x, not all of these can have norm less than 1. But if $|p| < 1$ and $|q| < 1$, we can write $1 = ap^m + bq^n$ by Bezout's theorem, where $|ap^m| < 1/2$ and $|bq^n| < 1/2$, contradicting the triangle inequality.

□

$\mathbb{Q}_p$

As a set, the p-adic numbers $\mathbb{Q}_p$ consist of the formal expressions

$$x = \sum_{i=k}^{\infty} x_i p^i$$

where $k \in \mathbb{Z}$ and $x_i \in \{0, 1, \ldots, p-1\}$. To add two p-adic numbers, we add *coordinate-wise with carrying*, so that $(x+y)_i = x_i + y_i + \varepsilon_i$, where $\varepsilon_i = 0$ if $x_{i-1} + y_{i-1} < p$ and $\varepsilon_i = 1$ otherwise, and multiplication respects the grading and distributes over addition.

The *valuation* of x is $\inf\{k \in \mathbb{Z} \mid x_k \neq 0\}$, which is ∞ if and only if $x_k = 0$ for all k, that is, $x = 0$. Observe that if $x \neq 0$, we may write $xp^{-k} = x_k + \sum_{i=1}^{\infty} x_{i+k} p^i$, which is an invertible p-adic integer, since we can find the coefficients of the multiplicative inverse recursively.

$\mathbb{Z}_p$

Let p be a prime number. There is a map $\pi_{m,n} \colon \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ whenever $m \geq n$, simply because $p^m\mathbb{Z} \subseteq p^n\mathbb{Z}$. Thus we obtain an inverse system of rings, and we define the p-adic integers to be the inverse limit

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \mid \pi_{m,n}(a_m) = a_n \text{ for } m \geq n\}.$$

By enlarging the inverse system to include all $\mathbb{Z}/n\mathbb{Z}$, we obtain the profinite integers $\widehat{\mathbb{Z}}$. This is equivalent to the product $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$.

Suppose $a \in \mathbb{Z}$ is coprime to p. Then there is a square root of a in $\mathbb{Z}_p$ if and only if $a \pmod p$ is a square. If $x^2 \equiv a \pmod p$, then of course $(-x)^2 \equiv a \pmod p$, and so we see that there is a unique square root lying in the classes $\{1, \ldots, \frac{p-1}{2}\}$ mod p. So we can use the convention that a square in $\mathbb{Z}_p$ has a "positive" and a "negative" square root, according to whether the integer with minimal absolute value in $a + p\mathbb{Z}$ is positive or negative. Note, however, the positive elements are not closed under addition.

Hensel's lemma

Proposition 4. *Suppose f is a polynomial in $\mathbb{Z}_p[x]$ and $\overline{f} \in \mathbb{F}_p[x]$ has a simple root $a \in \mathbb{F}_p$. Then there is a unique $\alpha \in \mathbb{Z}_p$ with $\alpha \equiv a \pmod p$ with $f(\alpha) = 0$.*

Sketch of proof. The assumption that a is a simple root means that $f'(a) \neq 0$. So we seek $a_1 \in \{0, \ldots, p-1\}$ so that $f(a + pa_1) \equiv 0 \pmod{p^2}$.

Note we can write $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ by Taylor's theorem, and observe that $f(a+pa_1) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(pa_1)^i$, and hence $f(a+pa_1) - f(a) = f'(a)pa_1 + p^2 \sum_{i=2}^{n} \frac{f^{(i)}(a)}{i!}(pa_1)^i p^{-2}$. Since $f'(a) \neq 0$, we can solve this equation modulo p^2. Repeating this argument, we obtain a solution modulo p^k for all k, hence a solution in $\mathbb{Z}_p$. $\square$

Ideals

We give a cursory idea of some of the basic behavior that occurs in the interaction between ideals, completions, and extensions. The first observation is that when $p\mathbb{Z}$ is a prime ideal in the integers, and k is a number field (say of degree n), the ideal $p\mathcal{O}_k \leq \mathcal{O}_k$ in the ring of integers of k may or not be prime. If the ideal remains prime, we say p is inert. Otherwise, the ideal may be a product of other ideals in various ways.

If $k = \mathbb{Q}(i)$, the ring of integers $\mathcal{O}_k$ is $\mathbb{Z}[i]$. The prime 5 factors as $5 = (1+2i)(1-2i)$ in $\mathbb{Z}[i]$, and 7 remains prime. Note more generally that any prime which is 3 modulo 4 cannot be written as a sum of squares as 0 and 1 are the only squares mod 4, and the norm of a Gaussian integer $a + bi$ is $(a+bi)(a-bi) = a^2 + b^2$, a sum of squares. That every prime which is 1 mod 4 factors in $\mathbb{Z}[i]$ is attributed to Fermat, but was not proved until much later. The first proof is due to Euler, and uses an infinite descent argument. The ideal $(2) \leq \mathbb{Z}[i]$ is actually the *square* of the ideal $(1+i)$, and for this reason 2 is said to *ramify* in $\mathbb{Z}[i]$.

It is important to note that $\mathbb{Z}[i]/(p)$ is a ring with p^2 elements in each case. When p splits, this is a product $\mathbb{F}_p^2$, and if p is inert the quotient is a field $\mathbb{F}_{p^2}$. If p is ramified, the quotient ring is neither a product ring nor a field.

The operations of extensions and completions interact according to the splitting behavior of primes. For example, the polynomial x^2+1 is irreducible over $\mathbb{Q}$, with splitting field $\mathbb{Q}(i)$. However, since $x^2 + 1 = 0$ has a solution modulo 5 (or any prime which is 1 mod 4), $\mathbb{Z}_5$ already contains a solution to this. So the splitting field for $x^2 + 1$ over $\mathbb{Q}_5$ is $\mathbb{Q}_5$. However, $\mathbb{Q}_7$ does not contain a square root of -1, and so $\mathbb{Q}_7(i)$ is a degree two extension of $\mathbb{Q}_7$.

Adeles

Recall that $\mathbb{Z} \subseteq \mathbb{R}$ has the particularly nice interaction between topology and group theory, in which the action of $\mathbb{Z}$ on $\mathbb{R}$ is discrete and cocompact. The adeles can be profitably considered as a space which is built to mimic this situation for the rational numbers replacing the integers. That is, the additive group of rational numbers $\mathbb{Q}$ is discrete in the ring of adeles, and the quotient by the action $\mathbb{A}/\mathbb{Q}$ is compact.

The adeles are the *restricted direct product* of all of the completions of $\mathbb{Q}$. The p-adic numbers $\mathbb{Q}_p$ have a *maximal compact* subring $\mathbb{Z}_p$, which consists of those p-adic numbers with $v_p \geq 0$.

Suppose $\{(G_i, A_i)\}_{i\in I}$ is a family of pairs where A_i is a compact open subgroup of a topological group X_i. We define the *restricted direct product*

$$\mathbb{X} = \Pi'_{i\in I}(X_i, A_i) = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \in A_i \text{ for almost every } i \right\}$$

Proposition 5. *$\mathbb{X}$ is locally compact.*

Proof. By Tychonoff's theorem, $\mathbb{U} = \prod_{i\in I} A_i$ is a compact open subgroup of $\mathbb{X}$. Note also that any $g \in \mathbb{X}$ is contained in the compact open neighborhood $g\mathbb{U}$. □

The *finite adeles* $\mathbb{A}_{fin}$ are the restricted direct product with respect to the family indexed by the prime numbers $(\mathbb{Q}_p, \mathbb{Z}_p)_{p \text{ prime}}$, and the *adeles* are the product $\mathbb{A}_{fin} \times \mathbb{R}$. The rational numbers can be embedded in $\mathbb{A}$ because they naturally embed in each factor, and a given rational number lies in $\mathbb{Z}_p$ for almost every p. Note that $\widehat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$ is a compact open subgroup of $\mathbb{A}_{fin}$. If S is a subset of the collection of all valuations, we let $\mathbb{A}^S$ denote the restricted direct product over this possibly smaller set.

As an aside, if k is a number field, we can define the k-adeles in an analogous way. The archimedean places (of which there are $d = [k : \mathbb{Q}]$) give d embeddings into the complex numbers (some of which will lie in the real numbers). The behavior of the minimal polynomial for k considered modulo p determines the splitting behavior of the ideal $p\mathcal{O}_k \leq \mathcal{O}_k$, which in turn describes the totality of completions of k.

Proposition 6. *$\mathbb{Q}$ is discrete in $\mathbb{A}$*

Proof. Suppose $\{q_n\}$ is a sequence of rational numbers converging to 0. If the denominators of q_n remain bounded, then the sequence must be eventually zero. Thus the denominators go to infinity. But then it is impossible that the sequence eventually lies in any fixed open set around 0. This is because either (i) infinitely many primes appear as denominators, but open sets in the adeles consist of p-integral elements for all but finitely many p, or (ii) arbitrarily large powers of some prime occur, but open neighborhoods of 0 have a bound on the power of a prime which can occur in a denominator. □

Note that if $x = \sum_{i=k}^{\infty} x_i p^i$ is a p-adic number, there is an additive homomorphism to $\mathbb{Z}[1/p]/\mathbb{Z}$ obtained by $\{x\}_p = \sum_{i=k}^{-1} x_i p^i + \mathbb{Z} = \frac{n}{p^k} + \mathbb{Z}$

Lemma 1. *The kernel of $\phi_p \colon \mathbb{Q}_p \to \mathbb{Z}[1/p]/\mathbb{Z}$ is $\mathbb{Z}_p$.*

Proof. This is just the observation that the set of p-adic numbers with valuation ≥ 0 is just $\mathbb{Z}_p$. □

Definition 3. Suppose $x \in \mathbb{A}_{fin}$ is a finite adele. Then there is a finite set S of primes so that x is p-integral for every $p \notin S$. For the primes in S, $x = \sum_{i=k}^{\infty} x_i p^i$ differs from a p-adic integer by an element of $\mathbb{Z}[1/p]$. However, this element of $\mathbb{Z}[1/p]$ is only defined up to $\mathbb{Z}$.

Proposition 7. *We have $\mathbb{A}_{fin}/\widehat{\mathbb{Z}} \cong \mathbb{Q}/\mathbb{Z}$*

Proof. This is a matter of pasting together all of the ϕ_p maps defined above. Since $x \in \mathbb{A}_{fin}$ is a product $(x_p)_p \in \prod_p \mathbb{Q}_p$ satisfying $x_p \in \mathbb{Z}_p$ for almost all p, the sum $\{x\} = \sum_p \phi_p(x_p)$ is a finite sum of elements of $\mathbb{Z}[1/p]$ (as before, only defined up to $\mathbb{Z}$). Hence the result is an element of $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Z}_p$ is the kernel of each ϕ_p, the kernel of the product is the profinite integers $\prod_p \mathbb{Z}_p = \widehat{\mathbb{Z}}$. $\square$

Proposition 8. *The quotient group $\mathbb{A}/\mathbb{Q}$ fibers over the circle with fiber $\widehat{\mathbb{Z}}$.*

Proof. Consider $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{fin}$, where $\mathbb{A}_{fin}$ denotes the finite adeles. Given $(x, y) \in \mathbb{R} \times \mathbb{A}_{fin}$, we consider $\pi(x, y) = x - \{y\} \in \mathbb{R}/\mathbb{Z}$. This is clearly surjective, and the kernel is the set of $(x, y) \in \mathbb{R} \times \mathbb{A}_{fin}$ with $x \equiv \{y\} \pmod{\mathbb{Z}}$. Note that when $\mathbb{Q}$ is diagonally embedded, we have $\pi(q, q) = q - \{q\} = 0 \in \mathbb{R}/\mathbb{Z}$, hence $\mathbb{Q}$ is in the kernel of π and π descends to a map $\mathbb{A}/\mathbb{Q} \to \mathbb{R}/\mathbb{Z}$. Note that for $\pi(x, y) = 0$, x must be rational to begin with. But then (x, y) is equivalent to $(0, y - x)$ up to the action of $\mathbb{Q}$, and the kernel of $\mathbb{A}_{fin} \to \mathbb{Q}/\mathbb{Z}$ is the profinite integers $\widehat{\mathbb{Z}}$ as remarked above. $\square$

Since $\widehat{\mathbb{Z}}$ is compact, the extension $\mathbb{A}/\mathbb{Q}$ is compact. The set $U = [0, 1) \times \widehat{\mathbb{Z}}$ is a fundamental domain for the action of $\mathbb{Q}$ on $\mathbb{A}$. In fact, $\mathbb{A}/\mathbb{Q}$ is naturally identified with the mapping torus of the map $x \mapsto x + 1$ on $\widehat{\mathbb{Z}}$. We have shown that $\mathbb{Q}$ is a discrete and cocompact subgroup of $\mathbb{A}$, thus we have:

Corollary 1. $\mathbb{Q}$ *is a lattice in* $\mathbb{A}$.

More generally, any subring $R \subseteq \mathbb{Q}$ is in fact of the form $\mathbb{Z}[1/S]$ for some set of primes S, and we can consider the R-adeles $\mathbb{A}_R$, which is the subset of the product $\mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$ with only finitely many non-integral coordinates. In all such cases, R is a lattice in $\mathbb{A}_R$. We remark that this resembles a phenomenon of convex cocompactness of the group action.

Quaternion algebras

Definition 4. Let R be a ring. Then the *(standard) quaternion algebra over* R, denoted $R\{i, j\}$, is the quotient of the free algebra over R by the relations $i^2 = j^2 = -1$ and $ij = -ji$. More general quaternion algebras are obtained by scaling the generators by square roots of elements of R. That is, $R\{\sqrt{\alpha} i, \sqrt{\beta} j\}$ satisfies $(\sqrt{\alpha} i)^2 = -\alpha$ and $(\sqrt{\beta} j)^2 = -\beta$, with the generators anti-commuting as before. We often denote $ij = k$, and observe that $\{1, i, j, k\}$ span $R\{i, j\}$ as an R-module.

The quaternion algebra $R\{i, j\}$ possesses an anti-involution, which sends $a+bi+cj+dk \mapsto a - bi - cj - dk$, which we denote by $q \mapsto \overline{q}$. If q is a quaternion, its (reduced) *norm* is the quantity $N(q) = q\overline{q} = a^2 + b^2 + c^2 + d^2 \in R$.

Suppose A is a quaternion algebra over a number field k. We say that A is *ramified* at a place v if $A \otimes k_v$ is a divison algebra. We note that $\mathbb{Q}\{i, j\}$ is ramified at $\{2, \infty\}$ because $N(q) = a^2 + b^2 + c^2 + d^2 = 0$ has a nonzero solution in a completion of $\mathbb{Q}$ (and hence a nonzero quaternion of norm zero) if and only if it is $\mathbb{Q}_p$ for an odd prime p. Hilbert's reciprocity law insists that the ramification set always has even finite cardinality.

The quaternion algebra we will take the most interest in is $\mathbb{Q}\{i,j\}$, which we call the *rational quaternions*. For the moment, we will use $\mathbb{H}$ to denote the *Hurwitz integers*, which consist of those rational quaternions whose coordinates are either all integers or all half-integers. That is, $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in (\mathbb{Z} + \frac{1}{2})\}$. Note the norm of every Hurwitz integer q is a nonnegative integer, and is zero if and only if $q = 0$. There are 24 units in $\mathbb{H}$.

Lemma 2. *The Euclidean distance from $x \in \mathbb{R}^4$ to $\mathbb{Z}^4$ is at most 1, with equality achieved if and only if $x \in (\mathbb{Z} + \frac{1}{2})^4$.*

Proof. Each real number is within $1/2$ of an integer, with equality if and only if it is in $\mathbb{Z}+\frac{1}{2}$. Thus, $d(x, \mathbb{Z}^4) \leq (1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2 = 1$, with equality achieved if and only if equality is achieved in each coordinate. □

Proposition 9. *$\mathbb{H}$ is a Euclidean domain. In particular, $\mathbb{H}$ is a principal ideal domain.*

Proof. We wish to show that if $a, b \in \mathbb{H}$ with $b \neq 0$, there exist $q, r \in \mathbb{H}$ with $a = bq + r$, and $N(r) < N(b)$.

The rational quaternion $a\bar{b}/N(b)$ can be viewed as an element of $\mathbb{R}^4$, hence the preceding lemma implies that either there is an integer quaternion $q \in \mathbb{Z}\{i,j\} \subseteq \mathbb{H}$ with $N(a\bar{b}/N(b) - q) < 1$, or $a\bar{b}/N(b) = q \in (\mathbb{Z} + \frac{1}{2})^4$. In the first case, for $r = a - qb$, we have $a = qb + r$, where

$$N(r) = N(a - qb) = N((a\bar{b}/N(b) - q)b) = N(a\bar{b}/N(b) - q) \cdot N(b) < N(b),$$

or in the second case we have $a = qb + 0$, with $q \in \mathbb{H}$. □

Definition 5 (Residue Quaternion Algebra)**.** If $I \leq R$ is an ideal, we obtain a *residue quaternion algebra* $(R/I)\{i,j\}$.

Proposition 10. *$\mathbb{F}_p\{i,j\}$ is not a division ring.*

Note this follows easily from Wedderburn's theorem:

Theorem 4 (Wedderburn)**.** *A finite division ring is a field.*

Since $A = \mathbb{F}_p\{i,j\}$ is not commutative when p is odd, it must have zero divisors. For a nonzero zero divisor z, the left multiplication map $A \to Az$ is not injective, since $y \neq 0$ is in the kernel for $yz = 0$. Since $Az \cdot Az = (AzA)z \subseteq Az$ is closed under multiplication, Az is a proper subalgebra, and it contains $z \neq 0$, so it is nontrivial. In fact, Az is 2-dimensional over $\mathbb{F}_p$, and the map $A \to \mathsf{End}(Az)$ determines an isomorphism $\mathbb{F}_p\{i,j\} \cong M_2(\mathbb{F}_p)$. Of course since $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$, we have a projection map $\mathbb{Z}\{i,j\} \to \mathbb{F}_p\{i,j\}$, whose kernel is the index p^4 ideal $p\mathbb{Z}\{i,j\}$.

Let $z = a+bi+cj+dk$ be a zero divisor in $\mathbb{F}_p\{i,j\}$, which exists by Wedderburn's theorem. Then since $z\bar{z}$ is in the center, it is either 0 or a unit. Thus $a^2 + b^2 + c^2 + d^2 = 0 \in \mathbb{F}_p$.

The number of units in $\mathbb{F}_p\{i,j\}$ is the size of $\mathsf{GL}_2(\mathbb{F}_p)$, which is $(p^2-1)(p^2-p)$, and there is one zero element. So the number of zero divisors must be $(p^2-1)(p+1)$, or $(p+1)^2$ up to multiplication by a unit.

Up to the action of the units of $\mathbb{F}_p\{i,j\}$, the left ideals correspond to a copy of $\mathbb{F}_p$ as $\begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$ or a point at infinity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, or in other words, $\mathbb{P}^1\mathbb{F}_p$. Since there is a correspondence between the left ideals of $\mathbb{F}_p\{i,j\}$ and the norm p elements up to multiplication by a unit, we have obtained:

Lemma 3. *If p is an odd prime, there is a correspondence between the primes of norm p in $\mathbb{Z}\{i,j\}$ up to associates and the two-dimensional ideals in $\mathbb{F}_p\{i,j\} \cong M_2(\mathbb{F}_p)$.*

Proof. If $N(x) = p$, then $[x]\mathbb{F}_p\{i,j\}$ is a nontrivial proper ideal, since the norm of every element in the ideal must be 0, and since p^2 does not divide the norm of x, x cannot be 0 in $\mathbb{F}_p\{i,j\}$. If x and y determine the same ideal, then they differ by multiplication by a unit. $\square$

Theorem 5 (Jacobi)**.** *The number of ways to write an odd prime p as a sum of 4 squares is $8(p+1)$.*

Proof. The quaternions of norm p up to the action of the 8 units $\{\pm1, \pm i, \pm j, \pm k\}$ correspond to the projective line $\mathbb{P}^1\mathbb{F}_p$, which has $p+1$ elements. $\square$

Using modular forms, or studying ideals in $\mathbb{Z}/p^m\mathbb{Z}\{i,j\}$, one can obtain the more general:

Theorem 6 (Jacobi [48])**.** *Let p be an odd prime, and n a natural number. The number of ways to write p^n as a sum of four squares is $8(p+1)p^{n-1}$.*

Definition 6. Let p be an odd prime. Then there are $p+1$ quaternions of norm p which are equivalent to $1 \pmod 2$ if $p \equiv 1 \pmod 4$ or equivalent to $1+i+j \pmod 2$ if $p \equiv 3 \pmod 4$. Let A_p denote the set of integral quaternions of norm p.

Corollary 2. *For p an odd prime, we have a bijection $A_p \leftrightarrow \mathbb{P}^1\mathbb{F}_p$.*

We can describe the zero divisors a bit more precisely. There are $(p^2-1)(p+1)$ total, hence $(p+1)^2$ projective classes. Thus a zero divisor is determined by the pair of a right ideal and a left ideal that it generates, and we have a correspondence between projective zero divisors and $(\mathbb{P}^1\mathbb{F}_p)^2$. Since the quaternion algebra comes equipped with the anti-involution of conjugation, it sends left ideals to right ideals, and hence gives an involution on $(\mathbb{P}^1\mathbb{F}_p)^2$ exchanging the two factors.

We summarize this discussion in the following table:

Quaternion algebra	**Matrix algebra**	**Cardinality**
$\mathbb{F}_p\{i,j\}$	$M_2(\mathbb{F}_p)$	p^4
Real quaternions	Scalar matrices	p
Pure quaternions	Traceless matrices	p^3
$\mathbb{PF}_p\{i,j\}$	$\mathbb{P}M_2(\mathbb{F}_p)$	$(p^2+1)(p+1)$
$\mathbb{F}_p\{i,j\}^*$	$\mathsf{GL}_2(\mathbb{F}_p)$	$(p^2-1)(p^2-p)$
A_p	$\mathbb{P}^1\mathbb{F}_p$	$p+1$
	rank 1	$(p-1)(p+1)^2$
$(\mathbb{P}^1\mathbb{F}_p)^2$	projective rank 1	$(p+1)^2$
idempotents	idempotents	$p(p+1)$
nilpotents	nilpotents	$p+1$
Quaternion conjugate $q \to \overline{q}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$	
Conjugation $q(-)q^{-1}$	Adjoint representation	
$N(q) = q\overline{q}$	*det*	
Re(q)	$\frac{1}{2}tr$	

1.5 Arithmetic Groups

A (linear) algebraic group G defined over k is a subgroup of $\mathsf{GL}_n(k)$ which is also a k-algebraic variety in $\mathsf{GL}_n(k)$. The group G inherits the subspace topology from its realization as a subset of k^{n^2}.

Proposition 11. *If R is a discrete subring of k, then $G(R) \leq G(k)$ is discrete.*

Proof. It suffices to observe that if $\{g_n\}$ is a sequence converging to 1 in $\mathbb{G}(R)$, then in particular the matrix entries of g_n converge to those of the identity. But since R is discrete, this means each of the finitely many matrix entries is eventually constant, hence the sequence is eventually the identity. $\square$

For a subring R of a number field k, let $\mathbb{A}_R$ denote the R-adeles, consisting of an open subring of $\mathbb{A}_k$ in which R is discrete and cocompact. For example, $\mathbb{A}_{\mathbb{Z}[1/m]} = \mathbb{R} \times \prod_{p|m} \mathbb{Q}_p \times \prod_{q\nmid m} \mathbb{Z}_q$.

Theorem 7 (Borel–Harish-Chandra)**.** *Suppose G is a connected semisimple k-algebraic group, and $R \leq k$ a subring. Then $\mathbb{G}(R)$ is a lattice in $\mathbb{G}(\mathbb{A}_R)$.*

It is perhaps worth noting that for a ring R, (i) matrix multiplication in $M_n(R)$ is given by polynomial maps in terms of the entries, and (ii) if $A \in M_n(R)$ is invertible, the entries of the inverse of A are given by a polynomial in terms of the entries, divided by the determinant, which if A is invertible to begin with, means that $\det(A) \in R^\times$. Thus if A is merely a *subset* of $\mathsf{GL}_n(k)$, the group generated by A is contained in $\mathsf{GL}_n(R)$, where R is the ring generated by all of the matrix entries of elements of A.

For example, if $A \leq \mathsf{GL}_n(\mathbb{Q})$ is a finitely generated subgroup, there is a finite set $\{a_1, \dots, a_k\}$ of matrices with kn^2 rational numbers, in which only finitely many primes can appear as denominators. Thus $A \leq \mathsf{GL}_n(\mathbb{Z}[1/m])$, for some m. This means that there are infinitely many prime ideals $(p) \leq \mathbb{Z}[1/m]$ available for us to consider reduction maps $A \to \mathsf{GL}_n(\mathbb{F}_p)$.

Theorem 8 (Matthews-Vaserstein-Weisfeiler [56])**.** *Suppose G is a connected, simply connected, and absolutely almost simple algebraic group defined over $\mathbb{Q}$, and suppose $\Gamma \leq G(\mathbb{Q})$ is a finitely generated Zariski dense subgroup. Then Γ surjects $G(\mathbb{F}_p)$ for almost every p.*

In the following special case, (see [49]), we have an even stronger conclusion.

Theorem 9. *For any pair of odd primes p, q, the map $\langle A_p \rangle \to \mathsf{PGL}_2(\mathbb{F}_q)$ has image $\mathsf{PSL}_2(\mathbb{F}_q)$ if p is a square modulo q, and is surjective otherwise.*

An Arithmetic Fuchsian Group

In this section we carry out an explicit and detailed computation of a cocompact arithmetic Fuchsian group.

Let $k = \mathbb{Q}(\sqrt{2})$, and $\mathbb{G} = \mathsf{SL}_2$. Thus $\mathcal{O}_k = \mathbb{Z}[\sqrt{2}]$ and $G(\mathcal{O}_k)$ acts on the hyperbolic plane. However, $\mathsf{SL}_2(\mathbb{Z}[\sqrt{2}])$ acts indiscretely on the hyperbolic plane. By taking a sequence $\{\frac{p_n}{q_n}\}$ of rational numbers satisfying $|\sqrt{2} - \frac{p_n}{q_n}| < \frac{1}{nq_n}$, (e.g., the continued fraction convergents) we note that $t_n = \sqrt{2}q_n - p_n$ converges to zero, and hence the sequence $\begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix}$ converges to the identity.

On the other hand, the ring $\mathbb{Z}[\sqrt{2}]$ embeds in $\mathbb{R} \times \mathbb{R}$ as a discrete subring under the map $a + b\sqrt{2} \mapsto (a + b\sqrt{2}, a - b\sqrt{2})$. We perhaps consider $\mathbb{Z}[\sqrt{2}]$ as a subring of $\mathbb{R}$ under the convention that $\sqrt{2}$ is positive, and then the embedding is the identity in the first factor, and in the second factor the embedding is precomposed with the nontrivial Galois automorphism $\tau \in k/\mathbb{Q}$. It is now apparent that the image is a discrete subring because the image consists of pairs in $\mathbb{R}^2$ whose product is $a^2 - 2b^2$. This is visibly an integer when $a, b \in \mathbb{Z}$, and can be zero only when $a = b = 0$, or else $\frac{a}{b}$ is a rational square root of 2. Now if we have $(x, y) \in \mathbb{R}^2$ with $|xy| \geq 1$, x and y cannot both be small at once.

For any ring A, there is an involution R on the algebra $M_2(A)$ obtained by conjugating by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, inducing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

There is an embedding $M_2(\mathbb{Z}[\sqrt{2}]) \to M_2(\mathbb{R}) \times M_2(\mathbb{R})$ defined by $X \mapsto (X, \tau(R(X)))$ with discrete image, and the restriction to the subvariety $\mathsf{SL}_2(\mathbb{Z}[\sqrt{2}])$ has image contained in $\mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R})$.

Now the natural componentwise action of $\mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R})$ on a product of hyperbolic planes restricts to a proper action of $\mathsf{SL}_2(\mathbb{Z}[\sqrt{2}])$, and the quotient is a 4-orbifold with model geometry $\mathbb{H}^2 \times \mathbb{H}^2$.

Any point in $\mathbb{H}^2 \times \mathbb{H}^2$ has many 2-dimensional totally geodesic planes passing through it. Some of these planes are Euclidean, obtained by taking the product of a geodesic line in each hyperbolic plane, but many have negatively curved metrics.

For any isometry $A \in \mathsf{PGL}_2(\mathbb{R})$, there is a corresponding quasi-isometrically embedded plane P_A obtained as the image of the map $\mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2$ which is defined by $x \mapsto (x, Ax)$. For such a subplane, we might ask which elements $g \in \mathsf{SL}_2(\mathbb{Z}[\sqrt{2}])$ preserve P_A. This will happen precisely when $g \cdot (x, Ax) = (gx, \tau(R(g))Ax) = (y, Ay)$ for some y, or in other words, when $\tau(R(g))A = Ag$, or $\tau(R(g)) = AgA^{-1}$.

The isometry A, being a matrix in $\mathsf{PGL}_2(\mathbb{R})$, also acts on $\mathsf{PSL}_2(\mathbb{R})$ by conjugation, and we can compare the group $\Gamma = \mathsf{PSL}_2(\mathbb{Z}[\sqrt{2}])$ with its conjugate $A\Gamma A^{-1}$. If A has entries in $\mathbb{Q}(\sqrt{2})$, then $\Gamma \cap A\Gamma A^{-1}$ has finite index in Γ (and we say that A *commensurates* Γ). If $A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, this is of course satisfied. Then the condition that $g \in \mathsf{PSL}_2(\mathbb{Z}[\sqrt{2}])$ preserves the plane defined by A in $\mathbb{H}^2 \times \mathbb{H}^2$ amounts to

$$\begin{pmatrix} 3(d_0 - d_1\sqrt{2}) & (c_0 - c_1\sqrt{2}) \\ -3(b_0 - b_1\sqrt{2}) & -(a_0 - a_1\sqrt{2}) \end{pmatrix} = \begin{pmatrix} 3(a_0 + a_1\sqrt{2}) & 3(b_0 + b_1\sqrt{2}) \\ -c_0 - c_1\sqrt{2} & -d_0 - d_1\sqrt{2} \end{pmatrix}$$

and comparing coefficients we see that the matrix should take the form

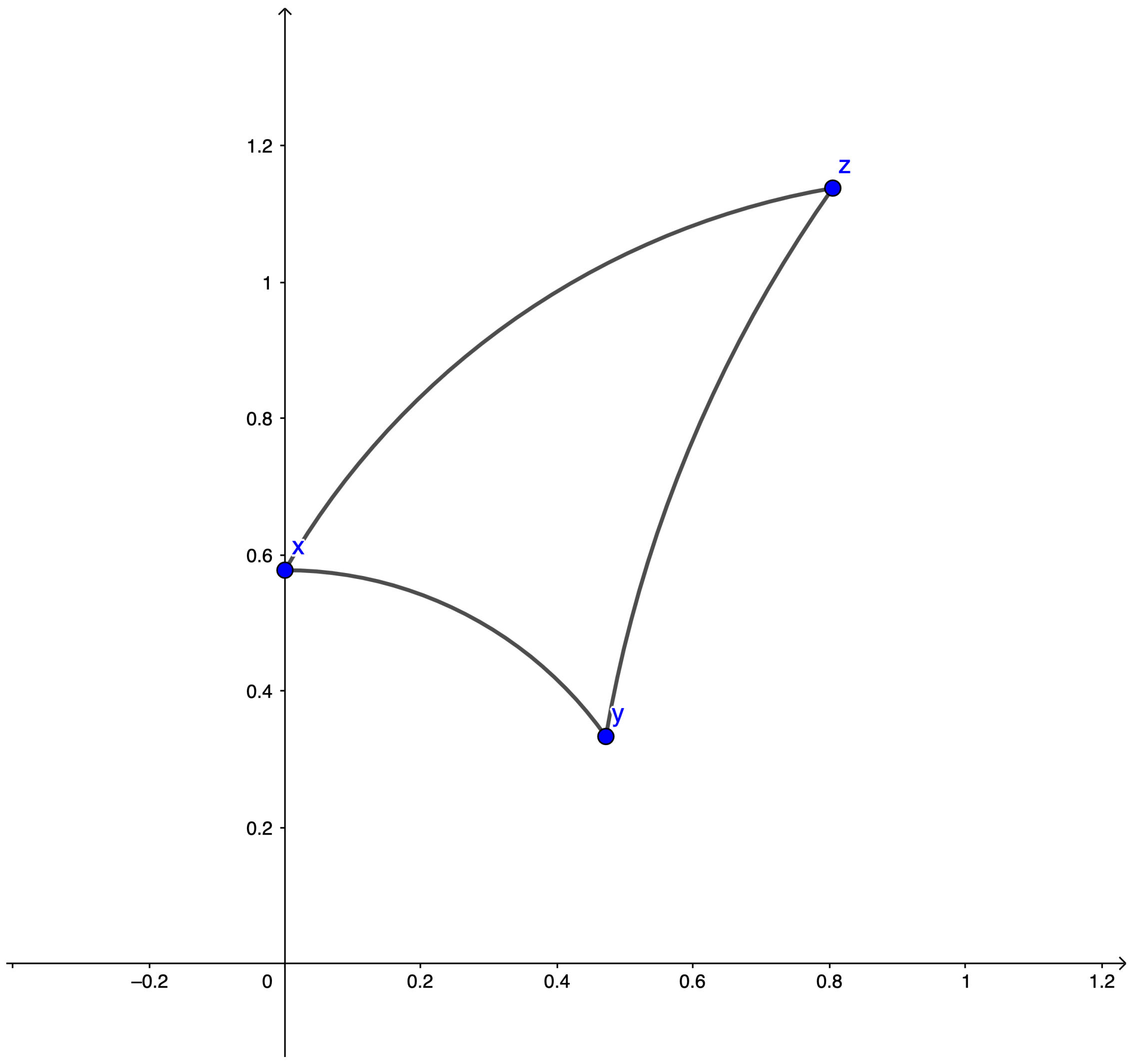
$$\begin{pmatrix} a_0 + a_1\sqrt{2} & b_0 + b_1\sqrt{2} \\ -3b_0 + 3b_1\sqrt{2} & a_0 - a_1\sqrt{2} \end{pmatrix} = \begin{pmatrix} a & b \\ -3\tau(b) & \tau(a) \end{pmatrix}$$

The condition that the determinant is 1 is now asking that $N(a_0, a_1, b_0, b_1) = a_0^2 - 2a_1^2 + 3b_0^2 - 6b_1^2 = 1$. We observe that $N(1, 0, 1, 0) = 4$ and $N(-1, 1, -1, 0) = 2$, and so

$$a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \text{ and } b = \frac{1}{\sqrt{2}}\begin{pmatrix} -1+\sqrt{2} & -1 \\ 3 & -1-\sqrt{2} \end{pmatrix}$$

constitute two basic solutions in $\mathsf{SL}_2(\mathbb{Q}(\sqrt{2}))$ (and as it turns out, every solution can be generated from these). We compute that a fixes $x = \sqrt{3}i/3$ and b fixes $y = (\sqrt{2} + i)/3$ in their action on the hyperbolic plane. Since $a^3 = b^4 = -I$, these are rotations of order 3 and 4 respectively, and we can check that they are both counter-clockwise. Moreover, their product $ab = \frac{1}{2}\begin{pmatrix} 1+\sqrt{2} & -(1+\sqrt{2}) \\ 3(-1+\sqrt{2}) & -1+\sqrt{2} \end{pmatrix}$ is an order 4 rotation about $z = \frac{1+\sqrt{2}i}{3(\sqrt{2}-1)}$. Indeed, the traces of a, b, and ab, are 1, $-\sqrt{2}$ and $\sqrt{2}$ respectively.

If we reduce the coefficients modulo 3 and consider the image in $\mathsf{PSL}_2(\mathbb{F}_3(\sqrt{2})) \cong \mathsf{PSL}_2(\mathbb{F}_9)$, we have $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 1+x & x \\ 0 & 2+x \end{pmatrix}$, and $ab \mapsto \begin{pmatrix} 1+x & 2+2x \\ 0 & 2+x \end{pmatrix}$, where x denotes $\sqrt{2} \in \mathbb{F}_9$. These matrices generate a group of order 36 in $\mathsf{PSL}_2(\mathbb{F}_9)$, as it surjects the upper triangular subgroup. Since every torsion element of the triangle group is conjugate to



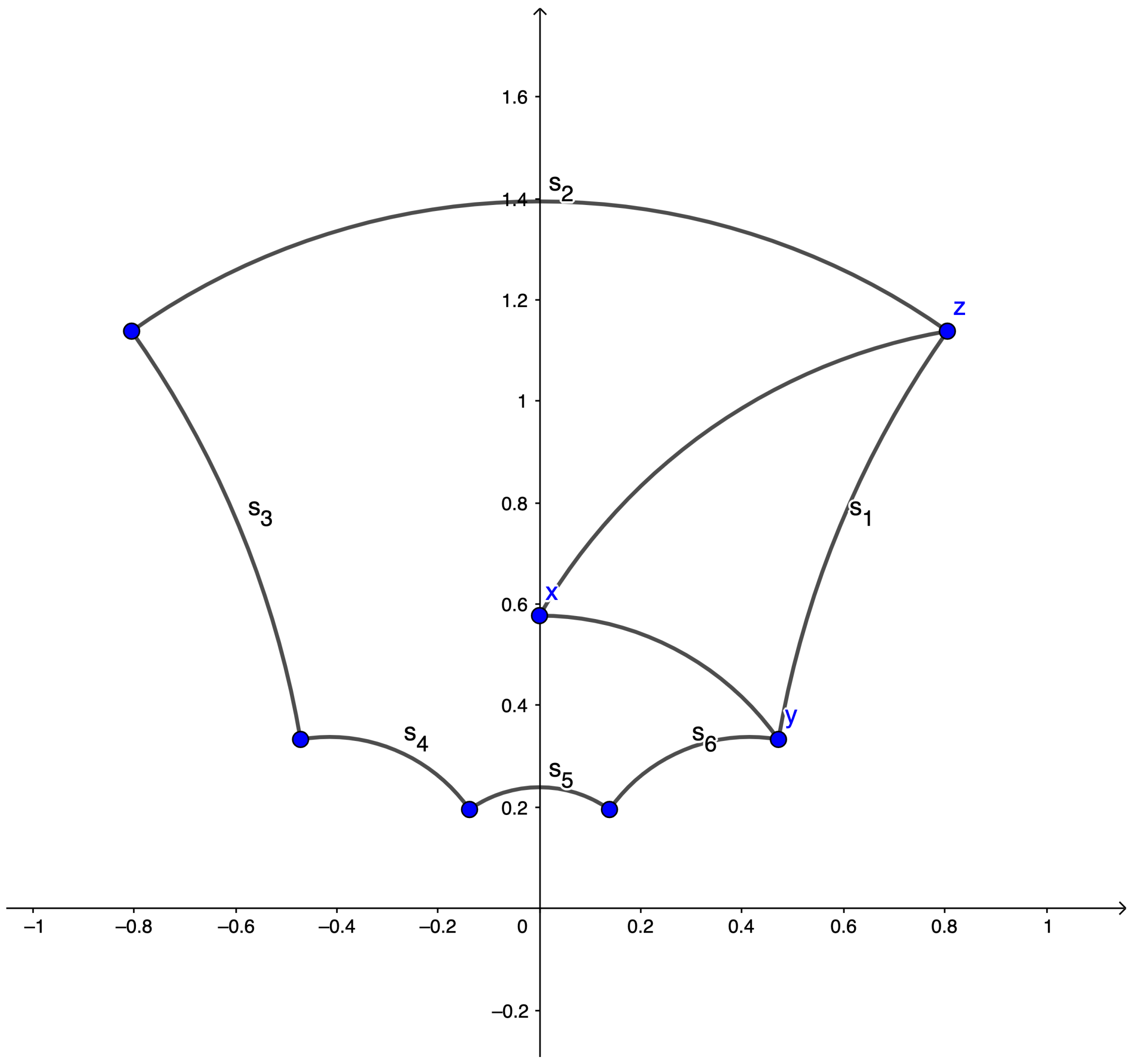
a power of a, b, or ab, every torsion element has nontrivial image in $\mathsf{PSL}_2(\mathbb{F}_9)$, and thus the kernel of this reduction map is a torsionfree subgroup, hence a surface group in $\mathsf{PSL}_2(\mathbb{Z}[\sqrt{2}])$.

Alternatively, the group generated by a, b is the points in a maximal order in a quaternion algebra, and upon intersecting with an integral order we obtain a surface group in $\mathsf{PSL}_2(\mathbb{Z}[\sqrt{2}])$.

Surface groups in $\mathsf{PSL}_2(\mathbb{Z}[\sqrt{n}])$

Now let $n > 2$ be squarefree. Then there is some odd prime p which divides n, and we observe first:

Lemma 4. *There is a positive integer r so that $x^2 + ry^2$ does not represent $0 \pmod p$ nontrivially.*



Proof. Choose r so that $-r$ is not a square modulo p. Observe that if $x^2 + ry^2 = 0$, then $(\lambda x)^2 + r(\lambda y)^2 = 0$ for any λ, and also that it is impossible to have a nonzero solution in which either x or y is zero. Thus, there is solution in which $y = 1$, and we have $x^2 \equiv -r \pmod p$, contradicting our choice of r. $\square$

Note when $p \equiv 3 \pmod 4$, we can choose $r = 1$. Now as before, consider the group of matrices of the form

$$\begin{pmatrix} a + b\sqrt{n} & c + d\sqrt{n} \\ r(-c + d\sqrt{n}) & a - b\sqrt{n} \end{pmatrix}$$

in which $a^2 - nb^2 + rc^2 - nrd^2 = 1$. Since this quadratic form has no solutions over $\mathbb{F}_p$, it has none in $\mathbb{Q}_p$, hence no rational solutions. This implies that the group is a cocompact lattice in $\mathbb{H}^2$, and thus contains a surface group of finite index. Note that the diagonal matrices correspond to solutions of Pell's equation $a^2 - nb^2 = 1$.

$\mathsf{PSL}_2(\mathbb{Q})$

Proposition 12. *The space* $\mathsf{PSL}_2(\mathbb{A})/\left(\mathsf{SO}_2(\mathbb{R}) \times \mathsf{PSL}_2(\widehat{\mathbb{Z}})\right)$ *is a restricted direct product of* $\mathbb{H}^2$ *with a* $(p+1)$*-regular tree for each prime* p*, and* $\mathsf{PSL}_2(\mathbb{Q})$ *is a lattice in this space. The quotient is an adelic modular curve.*

Note that if $\Gamma \leq \mathsf{PSL}_2(\mathbb{Q})$ has a global fixed point in its action on $\mathbb{H}^2$, it is conjugate into $\mathsf{SO}_2(\mathbb{Q})$, which is abelian. So the stabilizer of a point is *not* a lattice in the remaining factors, and is not even Zariski dense. On the other hand, if v is a vertex in one of the trees and we consider the subgroup of $\mathsf{PSL}_2(\mathbb{Q})$ which fixes v, this is a lattice in the remaining factors.

However, we do not immediately obtain a particularly interesting proper action on a product of trees via $\mathsf{PSL}_2(\mathbb{Q})$ or its subgroups.

By fixing a single prime p, it is not hard to construct subgroups of $\mathsf{PSL}_2(\mathbb{Q})$ which act properly, and are even lattices in T_{p+1}. This can be done by finding "ping pong" sets for the action. For example, if g_1 and g_2 are translations in a tree, so that their translation lengths are greater than the overlap of their axes, they will generate a discrete group. It is perhaps interesting to consider the space $\mathcal{L}_p$ of lattices in the p-adic tree. For each isomorphism type of a $(p+1)$-regular graph, there is a space of p-adic structures on this graph, similar to the Teichmüller space of a surface. For example, the space of 2-adic structures on a θ-graph is a 1-dimensional $\mathbb{Q}_2$-manifold.

$\mathsf{SU}_2(\mathbb{Q}(i))$

In analogy with the case of $\mathsf{PSL}_2(\mathbb{Q})$, the group $\mathsf{PSL}_2(\mathbb{Q}(i))$ is a lattice in a product $\mathbb{H}^3 \times T_{\mathcal{P}}$ where $\mathcal{P}$ ranges over the collection of non-dyadic prime ideals in $\mathbb{Z}[i]$ (the group $\mathsf{SU}_2(\mathbb{Z}[i, 1/2])$ is finite). Recall that if p is 1 mod 4, p splits in $\mathbb{Z}[i]$, and is inert otherwise. So we get pairs of $p+1$-regular trees when p is 1 mod 4, and we get trees of valence p^2+1 when p is 3 mod 4.

Proposition 13. *Suppose* $\Gamma \leq G$ *is a discrete subgroup of a topological group, and* $G \to H$ *is a continuous homomorphism to a locally compact group with compact kernel. Then the image of* Γ *is discrete.*

Proof. Let U be a compact neighborhood of 1 in H, and observe its preimage in G is again compact. Thus it intersects Γ in a finite set, and hence only finitely many points of Γ map to U. That this is true for all compact neighborhoods in H implies that the image of Γ is discrete. $\square$

Now we observe that although $\mathbb{R}$ admits a discrete cocompact subring $\mathbb{Z}$, $\mathbb{Q}_p$ has none. In fact, the only discrete subring of $\mathbb{Q}_p$ is the trivial subring! Indeed, if $0 \neq x \in R \leq \mathbb{Q}_p$, $p^n x \in R$ for every n, and this sequence converges to 0. Often $G(\mathbb{Z}) \leq G(\mathbb{R})$ provides a lattice, but there is not an obvious way in general to obtain a lattice or even a discrete subgroup in $G(\mathbb{Q}_p)$.

However, $\mathbb{Z}[1/p]$ is a lattice in $\mathbb{R} \times \mathbb{Q}_p$, and so if the group of real points of an algebraic group $G(\mathbb{R})$ is compact, the group $G(\mathbb{Z}[1/p])$ is a discrete subgroup of $G(\mathbb{R}) \times \mathbb{G}(\mathbb{Q}_p)$, and the preceding proposition implies that $G(\mathbb{Z}[1/p])$ is a discrete subgroup of $G(\mathbb{Q}_p)$. The group $\mathsf{SU}_2(\mathbb{Q}(i))$ is closely related to the group of rational quaternions, which are a primary focus of this thesis. We will see that these quaternions act properly on a restricted direct product of trees of degree $p+1$ over all odd primes p.

1.6 Integer Quaternions

Let $\mathcal{P}$ denote the set of prime numbers. The fundamental theorem of arithmetic is an elementary observation about how the integers behave with respect to multiplication. This motivates the definition and study of *unique factorization domains* in general, which are essentially rings that satisfy the "fundamental theorem of arithmetic" property.

However, we can actually interpret the fundamental theorem of arithmetic through the lens of geometric group theory. In most measures of complexity, the integers form the most basic infinite group. They can be approximated very well by a geodesic metric space: $\mathbb{R}$. The action of $\mathbb{Z}$ on $\mathbb{R}$ has quotient S^1, whose fundamental group is $\mathbb{Z}$. This circle of ideas in which one can understand and study algebraic properties of groups via their actions on geodesic metric spaces, is the very foundation of geometric group theory.

Theorem 10 (The fundamental theorem of arithmetic)**.** *Every positive integer can be expressed uniquely as a product of primes.*

Equivalently, the map $\mathbb{Q}^*_{>0} \to \bigoplus_{p\in\mathcal{P}} \mathbb{Z}$ via $q \mapsto (v_p(q))_{p\in\mathcal{P}}$ is an isomorphism.

Corollary 3. *The group $\mathbb{Q}_{>0}$ acts properly and cocompactly on a restricted product of 2-regular trees.*

Theorem 11 (The fundamental theorem of quaternion arithmetic)**.** *If q is a primitive integer quaternion and p divides the norm of q, there is a unique left-divisor x of q with norm p.*

Proof. We recall that the Hurwitz integers $\mathbb{H}$ form a principal ideal domain by Proposition 9. So the right ideal $p\mathbb{H} + q\mathbb{H}$ is a principal ideal, say $x\mathbb{H}$ for some $x \in \mathbb{H}$. Since $x \mid p$, we have $N(x) \mid p^2$. We claim that the norm of every element of $x\mathbb{H}$ is divisible by p, hence that $x\mathbb{H}$ is a proper ideal and $N(x) \neq 1$. To see this, note that $N(py+qz) = N(py)+2(py\cdot qz)+N(qz) = p^2N(y) + 2p(y \cdot qz) + N(q)N(z)$, which is divisible by p. If $N(x) = p^2$, then since $x \mid p$ and they have the same norm, $xu = p$ for some unit u. But this would imply that $p\mathbb{H}+q\mathbb{H} = p\mathbb{H}$, hence $p \mid q$. But this contradicts the primitive assumption on q. Thus, $N(x) = p$, and $x \mid q$. The choice of x is unique up to right-multiplication by a unit in $\mathbb{H}$, and if x does not have integer coordinates, it has an associate with integer coordinates. ∎

Corollary 4. *The group $\mathcal{H}_{\mathbb{Q}}$ acts properly and cocompactly on a restricted product of $(p+1)$-regular trees, one for each odd prime p.*

Let $\mathbb{P}$ denote the set of primitive integral quaternions.

Proposition 14. *A word $q_1 \dots q_n$ in $\mathbb{P}$ is a geodesic if and only if the coefficients of the product quaternion are relatively prime.*

Proof. If the product is divisible by a prime p, then dividing through by p we obtain an equivalent quaternion which is a product of $n-2$ primes, and hence the path was not a geodesic. If the coefficients are relatively prime, then the fundamental theorem of quaternion arithmetic implies that any reduced expression for the product must have length n. □

Definition 7. Let R be a commutative ring, and $\alpha, \beta \in R$. Then $\mathcal{U}(R\{\sqrt{\alpha}i, \sqrt{\beta}j)$ is the group of units in the quaternion algebra, up to the scaling action of the units of R, that is $\mathcal{U}(R\{\sqrt{\alpha}i, \sqrt{\beta}j\})/\mathcal{U}(R)$. In the case $R = \mathbb{Z}[1/n]$ with n odd, $\mathcal{H}_{\mathbb{Z}[1/n]}$ is the group of units whose primitive integral representative is equivalent to 1 or to $1+i+j$ modulo 2. For example, $\mathcal{U}(\mathbb{Z}\{i,j\}) = \{\pm 1, \pm i, \pm j, \pm k\}/\{\pm 1\}$.

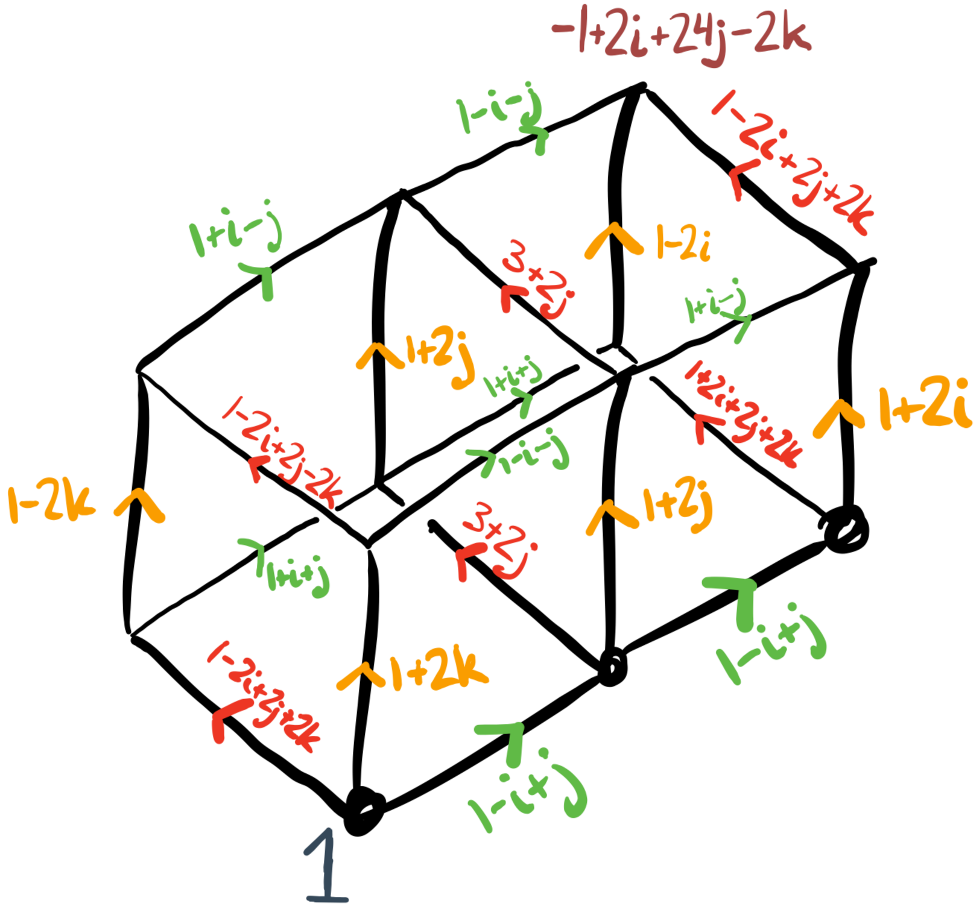
Theorem 12. *If p is an odd prime, the set A_p is a free symmetric generating set of the free group $\mathcal{H}_{\mathbb{Z}[1/p]} \leq \mathcal{H}_{\mathbb{Q}}$.*

Proof. By Theorem 6, the number of primitive integral quaternions of norm p^n is $8(p+1)p^{n-1}$. Up to ± 1, there is exactly one choice of associate which is equivalent to 1 or $1+i+j$ modulo 2, leaving $(p+1)p^{n-1}$ primitive integral quaternions in $\mathcal{H}_{\mathbb{Z}[1/p]}$. Repeatedly applying Theorem 11, we obtain a unique factorization of a quaternion of norm p^n as a product of quaternions of norm p in the set A_p, furnishing a bijection between the set of words in A_p which do not contain $x\overline{x}$ subwords, and the elements of $\mathcal{H}_{\mathbb{Z}[1/p]}$. □

Example 3. Consider the integer quaternion $q = 46+8i+50j$ of norm 4680. Upon factoring $4680 = 2^3 \cdot 3^2 \cdot 5 \cdot 13$, we consider the reductions of q modulo these primes to find its prime factors. Each coefficient is even, so the reduction modulo 2 is zero. Hence q is not primitive, and we can replace $q = 2q_1$, where $q_1 = 23 + 4i + 25j$, whose reduction modulo 2 is $1+j$. The fact that 2 is ramified implies that $1+j$ is both a left and right factor of q_1, and we obtain $q_1 = (1-j)q_2$ where $q_2 = -1 + 2i + 24j - 2k$.

Note reducing modulo 8 at once gives $q \equiv 6 + 2j \pmod 8$, so $(1+j)q \equiv 4 \pmod 8$, yielding $q = 2(1-j)q_2$, where $q_2 \equiv 1 \pmod 2$.

All of this information can be summarized in the following diagram:



For example, reading along the bottom path in this cube complex we see that

$$(1-i+j)(1-i+j)(1+2i)(1-2i+2j+2k) = -1+2i+24j-2k$$

and reading along the top path we see

$$(1-2i+2j+2k)(1-2k)(1+i-j)(1-i-j) = -1+2i+24j-2k.$$

Focusing on a single square in the complex, the path traversing *four* sides demonstrates the equality $(1-2i+2j+2k)(1-2k) = (1+2k)(1-2i+2j-2k)$. The cubes are assembled from *six* of these relation squares, and have *eight* vertices corresponding to the eight divisors of the quaternion of largest norm. The central square with orange and red sides in this diagram reflects the fact that the quaternions $1+2j$ and $3+2j$ commute, hence this square projects to a torus in the quotient complex, in the shape of a *zero*.

A way to understand the relations in this complex is by studying the action of a quaternion of norm p on the projective line for $\mathbb{F}_q$. Choosing the positive imaginary $i \equiv 2 \pmod 5$, we have the correspondence

$$1+2i \mapsto \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix} \equiv \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$1-2i \mapsto \begin{pmatrix} 1-2i & 0 \\ 0 & 1+2i \end{pmatrix} \equiv \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} = \infty$$

$$1+2j \mapsto \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \equiv \begin{pmatrix} 5 & 2 \\ 0 & 1 \end{pmatrix} = 2$$

$$1-2j \mapsto \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \equiv \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix} = 3$$

$$1+2k \mapsto \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \equiv \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix} = 4$$

$$1-2k \mapsto \begin{pmatrix} 1 & -2i \\ -2i & 1 \end{pmatrix} \equiv \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

Consider the quaternion $1-2i+2j+2k \mapsto \begin{pmatrix} 1-2i & 2+2i \\ -2+2i & 1+2i \end{pmatrix}$. Upon choosing $i \equiv 2 \pmod 5$, we obtain $\begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix} \in \mathsf{PGL}_2(\mathbb{F}_5)$. The action on the projective line gives the permutation $(0, 1, 4, 3, 2, \infty)$, which under the above correspondence is the permutation $(1+2i, 1-2k, 1+2k, 1-2j, 1+2j, 1-2i)$.

This permutation is a convenient way to organize the information that the left 5-divisor of $(1-2i+2j+2k)(1+2i)$ is $(1-2i)$, which means that

$$(1-2i+2j+2k)(1+2i) = (1-2i)q$$

for some q with norm 5. To quickly find this q, we can left-multiply both sides by $\overline{1-2i}$ and see that $5+10i+10j+10k = 5q$, or in other words, obtain the relation $(1-2i+2j+2k)(1+2i) = (1-2i)(1+2i+2j+2k)$.

Chapter 2

Actions on Trees

2.1 Models of p-adic trees

The upper half plane model

The group $\mathsf{PGL}_2(\mathbb{Q}_p)$ carries the structure of a topological group, via the quotient of the subspace topology obtained from $\mathsf{GL}_2(\mathbb{Q}_p) \subseteq M_2(\mathbb{Q}_p)$. Following Klein's *Erlangen program*, topological groups present a natural setting to study geometry. By considering K a maximal compact subgroup of a topological group G, the coset space G/K admits a left G-action. For the group $\mathsf{PGL}_2(\mathbb{Q}_p)$, the integral subgroup $\mathsf{PGL}_2(\mathbb{Z}_p)$ is in fact a maximal compact subgroup (indeed this is a maximal proper subgroup as well, an observation of Tits [66]).

There are in fact two maximal compact subgroups of $\mathsf{PGL}_2(\mathbb{Q}_p)$ up to conjugacy, but they are commensurable. Upon constructing the tree, one conjugacy class of maximal compact subgroup is the stabilizer of a vertex, and the other corresponds to the stabilizer of the midpoint of an edge. Much of this section will be dedicated to understanding the structure of the coset space $T_{p+1} = \mathsf{PGL}_2(\mathbb{Q}_p)/\mathsf{PGL}_2(\mathbb{Z}_p)$, and we shall give coordinates to this space (i.e., coset representatives).

The disk model

Here we develop coordinates for a disk model of the p-adic tree, for p odd, and contend that the quaternions give perhaps the most natural coset representatives for $\mathsf{SO}_3(\mathbb{Q}_p)/\mathsf{SO}_3(\mathbb{Z}_p)$. Let $Q\colon \mathbb{Q}_p^3 \to \mathbb{Q}_p$ denote the quadratic form $Q(x, y, z) = x^2+y^2+z^2$ (although any equivalent quadratic form can be used here). Let $\mathsf{SO}_3(\mathbb{Q}_p) = \{g \in \mathsf{SL}_3(\mathbb{Q}_p) \mid Q(gv) = Q(v)\ \forall v\}$. Then the subgroup $\mathsf{SO}_3(\mathbb{Z}_p)$ is a maximal compact subgroup.

Recall that for an odd prime p, A_p denotes the $p+1$ projective integer quaternions with reduced norm p which are equivalent to 1 or $1+i+j$ modulo 2. The adjoint representation identifies A_p with a subset of $\mathsf{SO}_3(\mathbb{Q}_p)$, and $\langle A_p\rangle$ has index 4 in $\mathsf{SO}_3(\mathbb{Z}[1/p])$.

Proposition 15. *The coset space $\mathsf{SO}_3(\mathbb{Q}_p)/\mathsf{SO}_3(\mathbb{Z}_p)$ carries a natural structure of a $(p+1)$-regular tree, and the elements of $\langle A_p\rangle$ form a natural coordinate system for the tree.*

Proof. The group $\mathsf{GL}_2(\mathbb{Q}_p)$ acts on the space $M_2(\mathbb{Q}_p)$ by conjugation. Since trace is a conjugation invariant, this action preserves the 3-dimensional $\mathbb{Q}_p$-vector space $\mathfrak{g}$ of traceless 2-by-2 matrices over $\mathbb{Q}_p$. For $X, Y \in \mathfrak{g}$, we can consider the Lie bracket $[X, Y] = XY - YX$, and obtain a map $\mathsf{ad}\colon \mathfrak{g} \to \mathsf{End}(\mathfrak{g})$ which sends X to the linear map $[X, -]$. The Killing form on $\mathfrak{g}$ is defined by $B(X, Y) = \mathsf{tr}(\mathsf{ad}(X) \circ \mathsf{ad}(Y))$. Note that $\mathsf{ad}(X)$ and $\mathsf{ad}(Y)$ are traceless, but their composition is not necessarily traceless.

The $\mathsf{GL}_2(\mathbb{Q}_p)$ action preserves the Killing form, which is equivalent to the form $Q(x, y, z) = x^2 + y^2 + z^2$ over $\mathbb{Q}_p$. The center acts trivially, and we obtain an injective homomorphism $\rho\colon \mathsf{PGL}_2(\mathbb{Q}_p) \to \mathsf{SO}_3(\mathbb{Q}_p)$, under which $\mathsf{PGL}_2(\mathbb{Z}_p)$ maps to $\mathsf{SO}_3(\mathbb{Z}_p)$. So the coset space is the same tree.

Since $\langle A_p \rangle$ acts simply transitively on the vertices of the tree, it provides a coordinate system. Thus $\rho(g)$, where g ranges over $\langle A_p \rangle$, forms a system of coset representatives for $\mathsf{SO}_3(\mathbb{Q}_p)/\mathsf{SO}_3(\mathbb{Z}_p)$. □

A key takeaway of this discussion is that, with respect to the basis $\{H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$, the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} ad + bc & -ac & bd \\ -2ad & a^2 & -b^2 \\ 2bc & -c^2 & d^2 \end{pmatrix}$$

is an isomorphism to a p-adic orthogonal group for an odd prime p.

Boundary bundles for trees

If T is a tree and $v \in T$, a *geodesic ray at* v is a sequence $r = (v = r_0, r_1, r_2, \dots)$ in the tree, where r_i is connected to r_{i+1} and $r_i \neq r_{i+2}$. This is equivalent to an isometric embedding $\mathbb{N} \to T$. The space of geodesic rays at v carries a topology, where the basic open sets consist of rays that begin with the same finite sequence of vertices. In this way, v determines a boundary for T, denoted $\partial_v T$. However, if w is another vertex in T, there is a unique geodesic path from w to v, and there is a natural homeomorphism $\partial v T \to \partial_w T$, obtained by precomposing with this path and then canceling backtracking.

If T is k-regular for $k \geq 3$, its boundary is a Cantor set. More generally, the boundary may be a closed subset of a Cantor set, but if T admits a lattice, the boundary is empty, two points, or a Cantor set. An *oriented geodesic* in T is a isometric embedding $\mathbb{Z} \to T$, considered up to a change of parametrization.

We can define the *boundary bundle* T^∂ of T to be a directed graph, whose vertices consist of pairs (v, r) where $v \in T$ and $r \in \partial_v T$, and $(v, r) \sim (w, s)$ if $s = (r_1, r_2, r_3, \dots)$. That is, the ray s is obtained from the ray r by forgetting the first coordinate. Note if T is k-regular, so is T^∂. Each vertex has a unique outgoing edge and $k - 1$ incoming edges.

The boundary bundle admits a geodesic flow, which is the map $\varphi\colon T^\partial \to T^\partial$ which sends a vertex (v, r) to the unique vertex to which it has a directed edge. The flow is a $(k-1)$-to-1 surjective map.

Note also that T^∂ comes equipped with a topology and a continuous map $T^\partial \to T$ with compact fibers. Moreover, if T is a regular tree, the automorphism group of T is naturally identified with that of T^∂.

We remark that T^∂ also has a map to the (unbasepointed) boundary of the tree determined by $(v, r) \to r$, where the fibers of the map are copies of the tree T.

Boundary bundle for p-adic trees

Since $\mathsf{PGL}_2(\mathbb{Q}_p)$ acts on the p-adic tree by isometries, it acts on the boundary $\mathbb{Q}_p\mathbb{P}^1$. The point ∞ is stabilized by the Borel subgroup $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Consider the geodesic ray $r_0 = \lfloor 0 \rfloor_0 \to \lfloor 0 \rfloor_{-1} \to \lfloor 0 \rfloor_{-2} \to \dots$. We observe that it is stabilized by $B_{\mathbb{Z}_p} = B \cap \mathsf{PGL}_2(\mathbb{Z}_p)$, and so the boundary bundle may be identified with the coset space $\mathsf{PGL}_2(\mathbb{Q}_p)/B_{\mathbb{Z}_p}$.

If Γ is a lattice in $\mathsf{PGL}_2(\mathbb{Q}_p)$, it determines a finite orbi-graph quotient, but it acts *freely* and cocompactly on the boundary bundle. Another benefit of the boundary bundle space is that it provides a natural space in which to study geodesic laminations in graphs, because infinite paths in a finite graph can embed in this space.

2.2 $\mathsf{PGL}_2(\mathbb{Q}_p)$

Before understanding the role that p-adics play in the theory of algebraic groups, it may appear to be a surprising coincidence or perhaps even a pathology that free groups can be found in the group of rotations of the sphere. Since $\mathsf{SO}_3(\mathbb{R})$ is a compact group, the only finitely generated subgroups which have some obvious geometric meaning are the finite, discrete groups.

However, restricting one's focus to the rational points $\mathsf{SO}_3(\mathbb{Q})$ illuminates a much broader landscape to consider geometrically defined groups. At this point, we can consider $\mathsf{SO}_3(\mathbb{Q})$ as a discrete subgroup of the adelic points $\mathsf{SO}_3(\mathbb{A})$, and obtain a perspective in which the fact that free groups appear in $\mathsf{SO}_3(\mathbb{Q})$ entirely natural, and indeed almost tautological rather than pathological.

In this section, we define the tree for $\mathsf{PGL}_2(\mathbb{Q}_p)$, and classify its isometries.

Proposition 16. *$\mathsf{PGL}_2(\mathbb{Z}_p)$ is a compact subgroup of $\mathsf{PGL}_2(\mathbb{Q}_p)$.*

Proof. Note that $\mathsf{PGL}_2(\mathbb{Z}_p)$ contains $\mathsf{PSL}_2(\mathbb{Z}_p)$ as an index two subgroup, so it is compact if and only if $\mathsf{PSL}_2(\mathbb{Z}_p)$ is. But $\mathsf{SL}_2(\mathbb{Z}_p)$ is a closed subset of the compact set $M_2(\mathbb{Z}_p)$, hence compact. The center of $\mathsf{SL}_2(\mathbb{Z}_p)$ is finite, and so $\mathsf{PSL}_2(\mathbb{Z}_p)$ is compact. $\square$

Once we construct the tree, it will be clear that it is in fact a maximal compact subgroup, because we will be able to prove that it is indeed a maximal subgroup.

Proposition 17. *The collection $V_p = \left\{ \begin{pmatrix} p^n & q \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}, q \in \mathbb{Z}[1/p]/p^n\mathbb{Z} \right\}$ forms a system of representatives for the coset space $\mathsf{PGL}_2(\mathbb{Q}_p)/\mathsf{PGL}_2(\mathbb{Z}_p)$.*

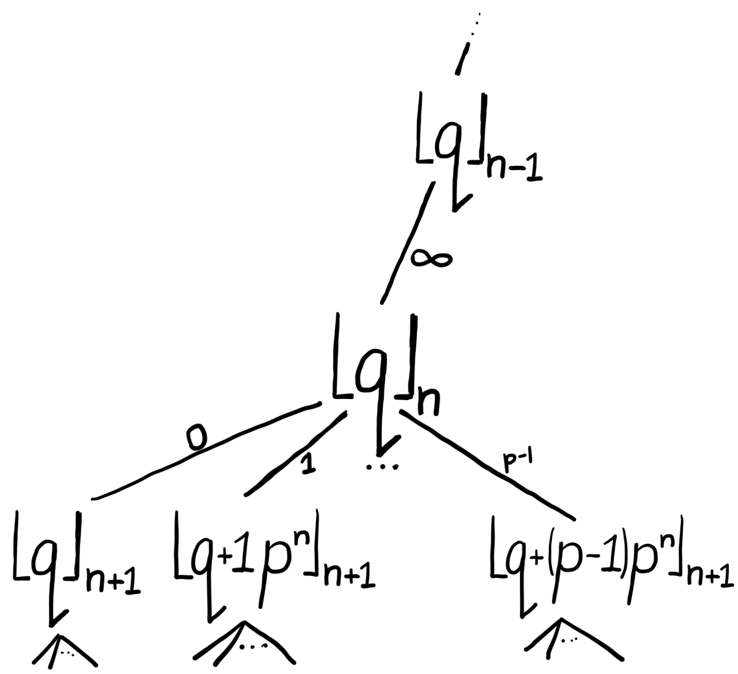


Figure 2.1: The local structure of a tree. The vertices are labeled by $n \in \mathbb{Z}$ and $q \in \mathbb{Z}[1/p]/(p^n)$, and its neighbors are as described.

Proof. First, we describe an algorithm that converts a given p-adic matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with nonzero determinant to an element of V_p. We have a right action by $\mathsf{PGL}_2(\mathbb{Z}_p)$, so we may apply $\mathbb{Z}_p$-linear column operations to the matrix and scale arbitrarily (since the matrix is only determined up to scalars).

Step 1: Possibly exchanging the columns, we assume that $v_p(c) \geq v_p(d)$, so that $\frac{c}{d} \in \mathbb{Z}_p$.

Step 2: This allows us to subtract $\frac{c}{d}$ times the second column from the first column, to obtain $\begin{pmatrix} a - \frac{c}{d}b & b \\ 0 & d \end{pmatrix}$.

Step 3: By scaling the entire matrix, this is equivalent to $\begin{pmatrix} \frac{ad-bc}{d^2} & \frac{b}{d} \\ 0 & 1 \end{pmatrix}$.

Step 4: We can multiply the first column by an element of $\mathbb{Z}_p^\times$ to obtain $\begin{pmatrix} p^n & \frac{b}{d} \\ 0 & 1 \end{pmatrix}$, where $n = v_p(ad - bc) - 2v_p(d)$.

Step 5: We can reduce $\lfloor \frac{b}{d} \rfloor_n$ and obtain an element of $\mathbb{Z}[\frac{1}{p}]/p^n\mathbb{Z}$. $\square$

We introduce the notation $\lfloor q \rfloor_n = \begin{pmatrix} p^n & q \\ 0 & 1 \end{pmatrix}$. We call n the *level* of $\lfloor q \rfloor_n$. We consider q as an element of $\mathbb{Z}[1/p]/p^n\mathbb{Z} \leftrightarrow \mathbb{Q}_p/p^n\mathbb{Z}_p$.

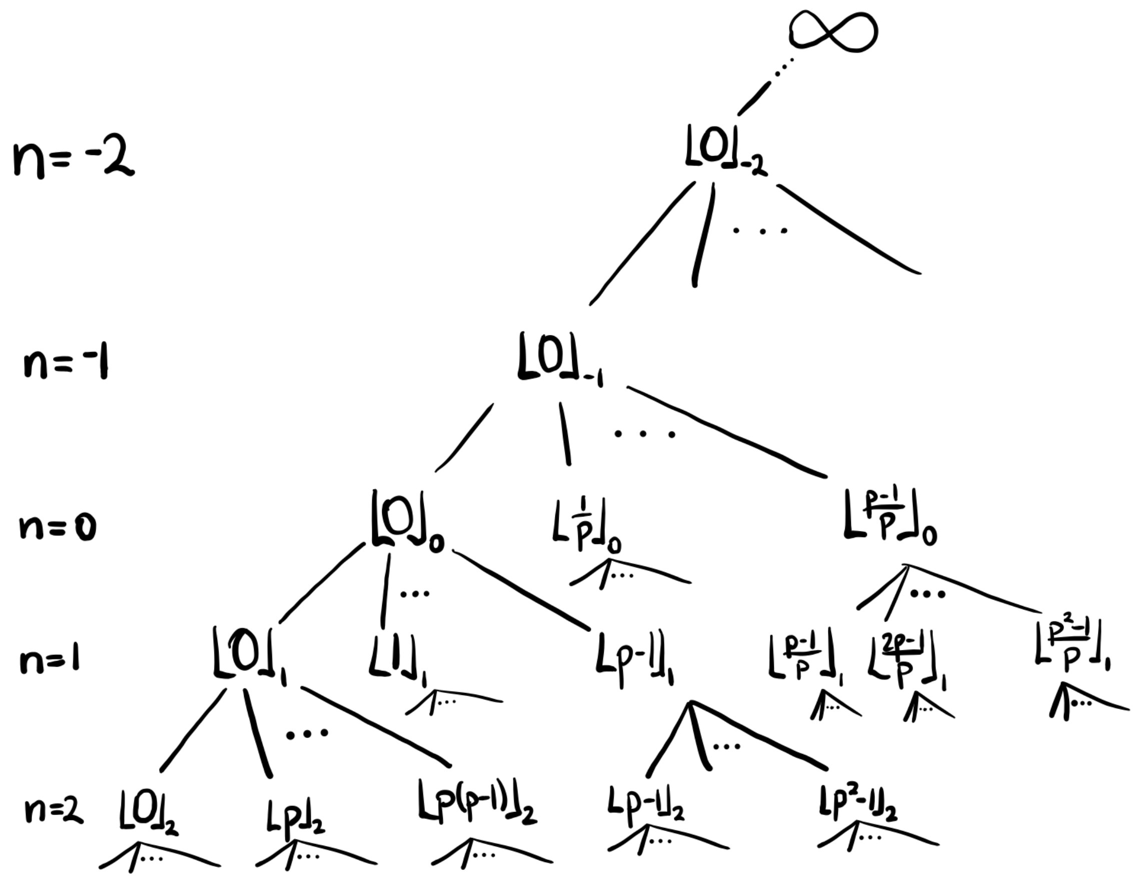


Figure 2.2: The p-adic tree

Proposition 18. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \lfloor q \rfloor_n = \begin{cases} \lfloor \frac{aq+b}{cq+d} \rfloor_{n-2v_p(cq+d)+v_p(ad-bc)} & \textit{if } v_p(cq+d) - v_p(c) \leq n \\ \lfloor \frac{a}{c} \rfloor_{-n-2v_p(c)+v_p(ad-bc)} & \textit{if } v_p(cq+d) - v_p(c) > n \end{cases}$

Proof. We apply the algorithm above to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p^n & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ap^n & aq+b \\ cp^n & cq+d \end{pmatrix}$.

Step 1 tells us to first compare $v_p(cp^n)$ and $v_p(cq+d)$.

Case 1: $v_p(cp^n) \geq v_p(cq+d)$. Then we have $\begin{pmatrix} ap^n & aq+b \\ cp^n & cq+d \end{pmatrix} \sim \begin{pmatrix} p^m & \frac{aq+b}{cq+d} \\ 0 & 1 \end{pmatrix}$, where $m = v_p(ap^n(cq+d) - cp^n(aq+d)) - 2v_p(cq+d) = v_p(p^n(ad-bc)) - 2v_p(cq+d)$.

Case 2: $v_p(cp^n) < v_p(cq+d)$. Then we have $\begin{pmatrix} aq+b & ap^n \\ cq+d & cp^n \end{pmatrix} \sim \begin{pmatrix} p^m & \frac{a}{c} \\ 0 & 1 \end{pmatrix}$, where $m = v_p((aq+b)cp^n - (cq+d)ap^n) - 2v_p(cp^n) = v_p(-(ad-bc)p^n) - 2v_p(cp^n)$.

□

We make some observations about this formula for computing the action of $\mathsf{PGL}_2(\mathbb{Q}_p)$ on the $(p+1)$-regular tree T_p.

First, if the matrix g has determinant 1, $v_p(ad-bc)=0$, which slightly simplifies the formula. But we can also observe that the difference between the height of a vertex and its image under g is even, which implies that the distance between v and gv is even. This reflects the fact that $\mathsf{PSL}_2(\mathbb{Q}_p)$ preserves the bipartite structure of the tree. More generally, $g \in \mathsf{PGL}_2(\mathbb{Q}_p)$ preserves the bipartition if and only if $v_p(\det(g))$ is even for some (hence any) representative matrix.

The formula also simplifies considerably when $c=0$, in which case $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \lfloor q \rfloor_n = \lfloor \frac{aq+b}{d} \rfloor_{n-2v_p(d)+v_p(ad)}$. Since upper triangular matrices fix $\infty \in \mathbb{Q}_p\mathbb{P}^1$, the level of $g.v$ should be independent of v, which is clear from the formula.

We can classify the elements of $\mathsf{PGL}_2(\mathbb{Q}_p)$ by their action on T_p. More general automorphisms of trees are most broadly classified by whether they fix a point in the tree or not. However, for these projective transformations we get a more detailed description of what can occur.

Theorem 13. *There are four types of infinite order elements in $\mathsf{PSL}_2(\mathbb{Q}_p)$ for p an odd prime.*

(i) Hyperbolic elements: $v_p(tr(g)) < 0$

(ii) Loxodrom-ish elliptics: $tr(g)^2 - 4$ is a nonzero square in $\mathbb{Z}_p^\times$

(iii) Strongly elliptics: $tr(g)^2 - 4$ is not a square in $\mathbb{Z}_p^\times$

(iv) Parabolike elliptics: $tr(g)^2 = 4$

Proof. It is clear that these four cases exhaust the possibilities for the trace of g and are thus pairwise nonconjugate. □

Proposition 19 (Hyperbolic elements)**.** *In case (i), g has both an attracting and a repelling fixed point in the boundary of the tree, and there is a unique axis which is invariant. The axis is translated by a length of $2v_p(tr(g))$, and in general if $w \in T$, the distance between w and gw is twice the distance from w to the axis plus the translation length.*

Proof. The characteristic polynomial of g has roots $\lambda_\pm = \frac{tr(g) \pm \sqrt{tr(g)^2-4}}{2}$. Since $tr(g)^2$ is visibly a square in $\mathbb{Q}_p$ and it has negative valuation, its first nonzero coefficient is a square in $\mathbb{F}_p$, which is not changed when adding the integer -4. Thus $tr(g)^2-4$ is also a square in $\mathbb{Q}_p$, and so the two roots lie in $\mathbb{Q}_p$. □

Proposition 20 (Loxodrom-ish elliptics)**.** *In case (ii), the fixed set of g is a neighborhood of a geodesic in the tree. If the fixed set is not precisely an axis, it permutes the points distance r from the fixed set in a p^r-cycle.*

Proof. The matrix g is conjugate to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, for some $\lambda \in \mathbb{Z}_p^\times$. If $\lambda \equiv 1 \pmod p$, then so is λ^{-1}, and g fixes the $v_p(\lambda-1)$-neighborhood of the axis while permuting the p neighbors of each leaf of the fixed subtree in a cycle. Then the p^{th} power of g fixes one additional layer and acts on the next level as a p-cycle.

$\square$

Proposition 21 (Strongly elliptics). *In case (iii), the fixed set of g is a compact set in the tree. By adjoining a square root of $tr(g)^2 - 4$ to $\mathbb{Q}_p$, g has roots in this quadratic extension, and hence fixes these points in the boundary of the tree associated to the quadratic extension.*

Proof. Since the characteristic polynomial of g has no roots in $\mathbb{Q}_p$, it acts on the boundary without fixed points. But this implies that the fixed set is compact. However, the quadratic extension of $\mathbb{Q}_p$ obtained by adjoining the square root of the discriminant will have two fixed points in the boundary, hence a fixed axis. $\square$

Proposition 22 (Parabolike elliptics). *In case (iv), the fixed set of g is a horocycle in the tree. The matrix is conjugate to one of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \neq 0$.*

Proof. The assumption on the trace immediately implies the conjugacy statement. Directly from the formula for the action, it is clear that such a matrix fixes vertices $\lfloor q \rfloor_n$, for $n \leq v_p(x)$, and that g^m fixes vertices at level $v_p(x) + 1$ if and only if $p \mid m$. $\square$

The Cross-ratio

Definition 8. Let $w, x, y, z \in \mathbb{Q}_p\mathbb{P}^1$. The *cross-ratio* $[w, x; y, z]$ is defined by the formula $\frac{(w-y)(x-z)}{(x-y)(w-z)}$, where we extend arithmetic to ∞ in the obvious way. The cross-ratio encodes much of the geometric information in the p-adic tree.

Proposition 23. *A bijection $g \colon \mathbb{Q}_p\mathbb{P}^1 \to \mathbb{Q}_p\mathbb{P}^1$ preserves the cross-ratio of every quadruple if and only if $g \in \mathsf{PGL}_2(\mathbb{Q}_p)$.*

Proof. Supposing $g \in \mathsf{PGL}_2(\mathbb{Q}_p)$, it is a straightforward computation that $g[w, x; y, z] = [gw, gx; gy, gz] = [w, x; y, z]$ for any quadruple. Conversely, if g preserves the cross-ratio, its matrix can be constructed by considering the action on four points, which determines the image of any other point. $\square$

In particular, the preceding proposition characterizes the group $\mathsf{PGL}_2(\mathbb{Q}_p)$ as a subgroup of $\mathrm{Aut}(T)$, once the boundary of the tree has an identification with $\mathbb{Q}_p\mathbb{P}^1$, because any homeomorphism of the boundary which extends over the tree has a unique such extension.

The unit tangent bundle of the tree

When we describe the points of a geometry in terms of a coset space G/K, there is typically a geometric interpretation of G/H for closed subgroups H of K. In the case of the tree $T_p = \mathsf{PGL}_2(\mathbb{Q}_p)/\mathsf{PGL}_2(\mathbb{Z}_p)$, each closed subgroup $H \leq \mathsf{PGL}_2(\mathbb{Z}_p)$ admits an interpretation as a geometric configuration. Analogously, $\mathbb{H}^2 = \mathsf{PSL}_2(\mathbb{R})/\mathsf{SO}_2$ is the hyperbolic plane, and $\mathsf{PSL}_2(\mathbb{R})/1$ is the unit tangent bundle.

$\mathbb{G} = \mathsf{PGL}_2(\mathbb{Q}_p)$ acts simply transitively on the space of *labeled ideal triangles* in T_p. An ideal triangle in the tree is a metric triangle whose vertices lie on the boundary. A triangle in a tree always has a unique center, which is the common intersection of the three sides. From this perspective, an ideal triangle is the same as a vertex in the tree, together with three geodesics emanating from it in different directions.

We can identify a labeled ideal triangle with an ordered triple of distinct points in $\mathbb{Q}_p\mathbb{P}^1$. A basepoint for this is the triple $\{0, 1, \infty\}$, and the center of this triangle is the basepoint of the tree.

Note that $\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \rangle \cong S_3$ acts on the base triangle by permuting its vertices (since the first matrix fixes 1 and exchanges $0, \infty$, and the second matrix fixes ∞ and exchanges $0, 1$), and so $\mathbb{G}/S_3$ is the space of ideal triangles in T_p.

2.3 $\mathsf{PGL}_2(\mathbb{Q})$

Local models

Recall that an n-manifold M is a topological space which is *locally modeled* on $\mathbb{R}^n$, in the sense that each point in M has a neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$. If Δ is a topological space, a Δ-complex is a topological space which is locally modeled on Δ.

Consider the space $S_n = ([0, 1] \times \{1, \dots, n\})/((0, i) \sim (0, j))$ with the quotient topology, which we call the *n-star*. An S_n-complex is a topological space which is locally modeled on S_n. Note an S_n complex is the same as an open subset of an n-regular graph.

Just as an orbifold is a space which is locally modeled on $\mathbb{R}^n$ modulo a finite group, a Δ-orbicomplex is a space which is locally modeled on Δ modulo a finite group. In this section, we construct an object which is best viewed as an $\mathbb{R}^2 \times S_3 \times S_4$-orbicomplex.

Using the theory we have built up so far, we can obtain an explicit description of $\Gamma_n = \mathsf{PGL}_2(\mathbb{Z}[1/n])$ as a finite complex of groups. For each prime p dividing n, there is an action of Γ_n on the p-adic tree without a global fixed point. The action is in fact transitive on the vertices and has a single orbit of edges. Thus we obtain a splitting of Γ_n as an HNN extension with vertex group $\Gamma_{n/p}$ and edge group a subgroup of index $p+1$. We carry this process out explicitly in the case $\Gamma_6 = \mathsf{PGL}_2(\mathbb{Z}[1/6])$. This can also be viewed as a *complex of groups*. Here the complex is CAT(0) and hence developable.

According to the Borel–Harish-Chandra theorem, Γ_6 is a lattice in $\mathbb{H}^2 \times T_{2+1} \times T_{3+1}$. In the present case, we can compute this lattice explicitly. If we consider the action on $T_{2+1} \times T_{3+1}$, the stabilizer of the base vertex is

$$\mathsf{PGL}_2(\mathbb{Z}[1/6]) \cap (\mathsf{PGL}_2(\mathbb{Z}_2) \cap \mathsf{PGL}_2(\mathbb{Z}_3)) = \mathsf{PGL}_2(\mathbb{Z}).$$

We set

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ t_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

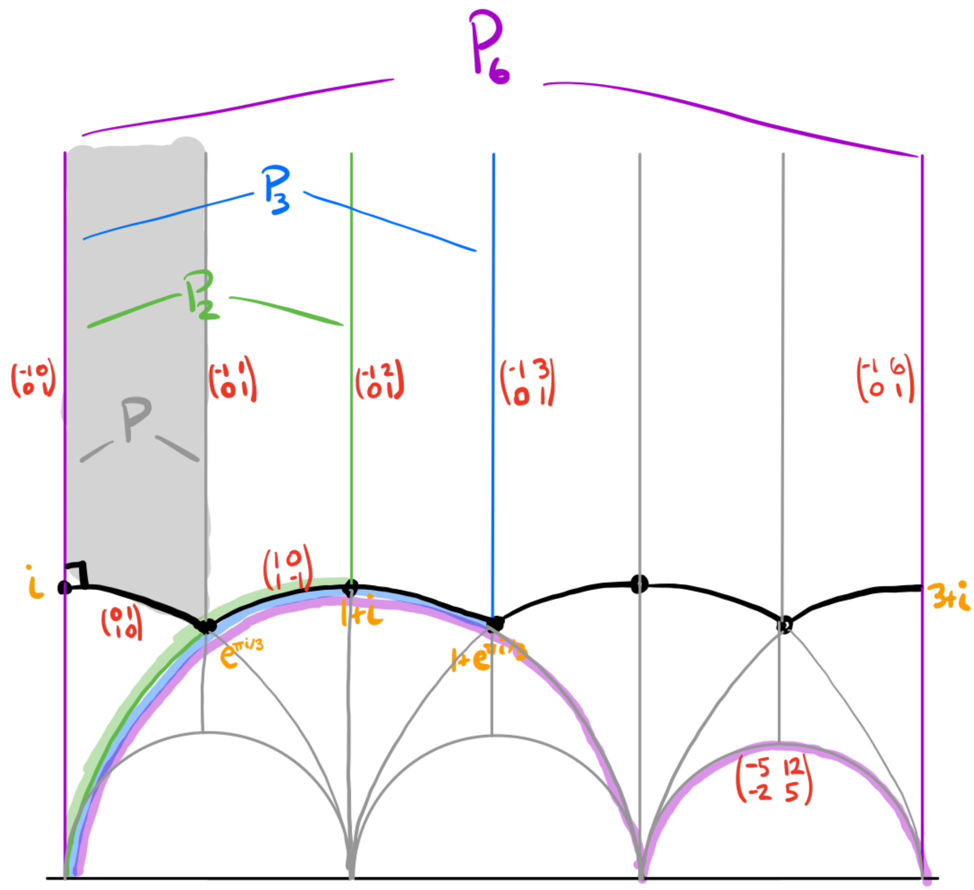


Figure 2.3: Congruence subgroups of the modular group

It is well-known that $\{R, S, T\}$ are generators of $\Lambda = \mathsf{PGL}_2(\mathbb{Z})$. For $N \in \mathbb{Z}$, let $\Lambda_0(N) = \{\begin{pmatrix} a & Nb \\ c & d \end{pmatrix} \in \mathsf{PGL}_2(\mathbb{Z}) \mid ad - Nbc = \pm 1\}$.

We observe that $t_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} t_p^{-1} = \begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}$. Thus it is clear that $\Lambda \cap t_p \Lambda t_p^{-1} = \Lambda_0(p)$. We denote $\Lambda^{t_p} = t_p \Lambda t_p^{-1}$.

Observe that in Figure 2.3, the polygon shaded in gray is a triangle with one ideal vertex, and two vertices with angles of $\pi/2$ and $\pi/3$. The reflections in the sides are labeled with the red matrices, and these reflections generate $\Lambda = \mathsf{PGL}_2(\mathbb{Z})$. The polygon P_2 bound by the green lines and the left vertical axis is a triangle with two ideal vertices and a right angle, and the group generated by reflections in the sides of P_2 is $\Lambda_0(2)$, an index 3 subgroup of Λ. The polygon P_3 bound by the blue lines and the left vertical axis is a triangle with two ideal vertices and a vertex with a $\pi/3$ angle, and the group generated by reflections in the sides of P_3 is $\Lambda_0(3)$, an index 4 subgroup of Λ. The polygon P_6 is an ideal square bound by the purple lines, and reflections in the sides of this square generate $\Lambda_0(6)$, which is an index 12 subgroup of Λ. Note $\Lambda_0(6)$ also covers $\Lambda_0(2)$ and $\Lambda_0(3)$.

Figure 2.4 displays the complex of groups decomposition of $\mathsf{PGL}_2(\mathbb{Z}[1/6])$, which also provides a decomposition of the space $\mathbb{H}^2 \times T_3 \times T_4/\mathsf{PGL}_2(\mathbb{Z}[1/6])$.

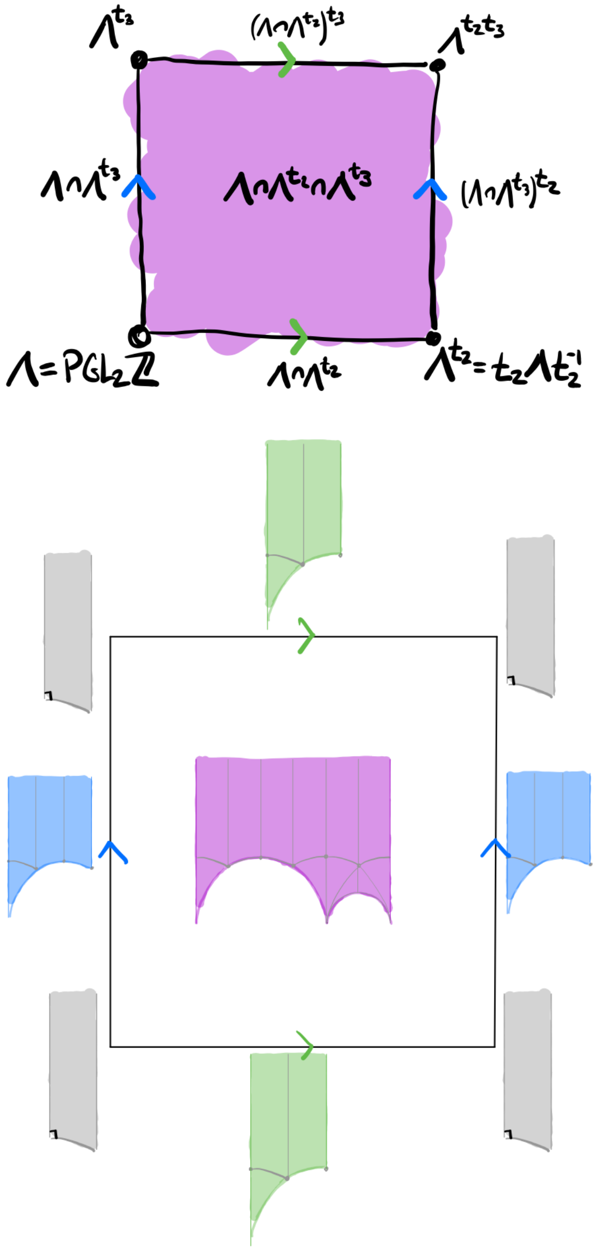


Figure 2.4: The complex of groups and spaces associated to $\mathsf{PGL}_2(\mathbb{Z}[1/6])$

Chapter 3

Surface Group Actions on Products of Trees

3.1 Surface Groups acting on Products of Trees

Recall our motivating question: Suppose X is a locally compact jungle, and G is the fundamental group of a closed surface of genus 2. Can G act freely on X?

Locally Infinite Trees

We first describe explicitly how this can be done without the restriction that X be locally compact.

Let Σ be a genus two surface, and let $R = \{r_1, r_2, r_3\}$ denote the red multicurve and $G = \{g_1, g_2, g_3\}$ the green multicurve in Figure 3.1. Note that these are the fixed sets of the two reflectional symmetries of the surface as it is embedded in $\mathbb{R}^3$, and for each of R and G, the complement $\Sigma \setminus R$ and $\Sigma \setminus G$ is a union of two pairs of pants.

Since $\Sigma \setminus (R \cup G)$ is a union of disks (in fact, four hexagons), a closed curve based at x is fully determined by the sequence of curves it crosses.

We can analyze this picture a bit more. The lift of R to the universal cover is a collection of nonintersecting lines that cut $\mathbb{H}^2$ into half-planes. The action of $\pi_1(\Sigma)$ preserves these lines, and hence permutes the complementary regions. Each region has infinitely many neighbors. If we define a graph whose vertices consist of the complementary regions, and two vertices are connected by an edge if they share a border, we obtain a locally infinite tree.

This tree is in fact precisely the Bass-Serre tree described in Example 1 of the Bass-Serre theory section. The edges adjacent to the base vertex correspond to the cosets of the three $\langle r_i \rangle$ in P, where $P = \pi_1(\Sigma \setminus R, x)$ is the pants group.

Theorem 14. *The action of $\pi_1(\Sigma)$ on $T_\infty \times T_\infty$ is free and determines a quasi-isometric embedding $\mathbb{H}^2 \to T_\infty \times T_\infty$.*

In [20], the authors construct a map $T_\infty \to T$ to a locally finite tree so that the composition $\mathbb{H}^2 \to T_\infty \times T_\infty \to T \times T$ is quasi-isometric, thereby producing a copy of the hyperbolic

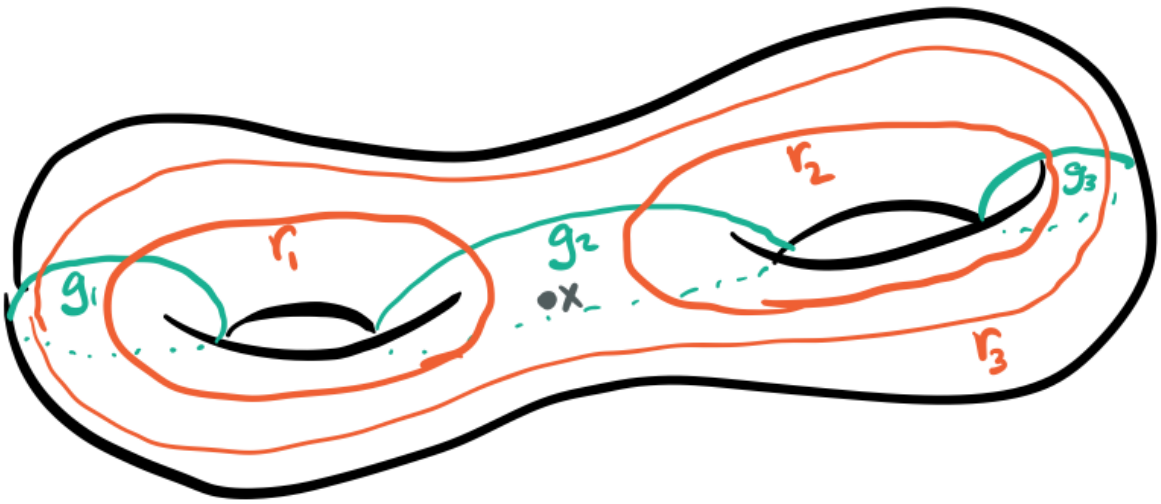


Figure 3.1: Filling multicurves

plane embedded in a product of locally finite trees. However, there is not a clear way to make the resulting plane "periodic". The space of quasi-isometric embeddings of a hyperbolic plane into a product of trees should be open in the space of combinatorial maps, and it is nonempty by this theorem. Perhaps finding a way to "approximate" arbitrarily large portions of a quasi-isometrically embedded hyperbolic plane by a surface group action would allow one to prove the existence of a surface group action on a product of locally finite trees. In the following section, we will construct an alternate quasi-isometric embedding of $\mathbb{H}^2$ in a locally finite product of trees.

Introduction

In section 3.1, G will denote a fixed group, namely $G = \langle a, b \mid [a,b]^2 \rangle$. Note that G is the fundamental group of an orbifold which is a torus with a cone point of order 2.

Proposition 24 (Long-Reid). *The representation* $a \mapsto \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$ *and* $b \mapsto \begin{pmatrix} 1/8 & 3/8 \\ 6/8 & 82/8 \end{pmatrix}$ *determines a faithful action of* G *on* $T_3 \times T_4$.

Proposition 25. *The image of the representation of* G *determined by* $a \mapsto \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$ *and* $b \mapsto \begin{pmatrix} 1+2i & 2+3i \\ -2+3i & 1-2i \end{pmatrix}$ *acts properly on* $T_4 \times T_{14}$.

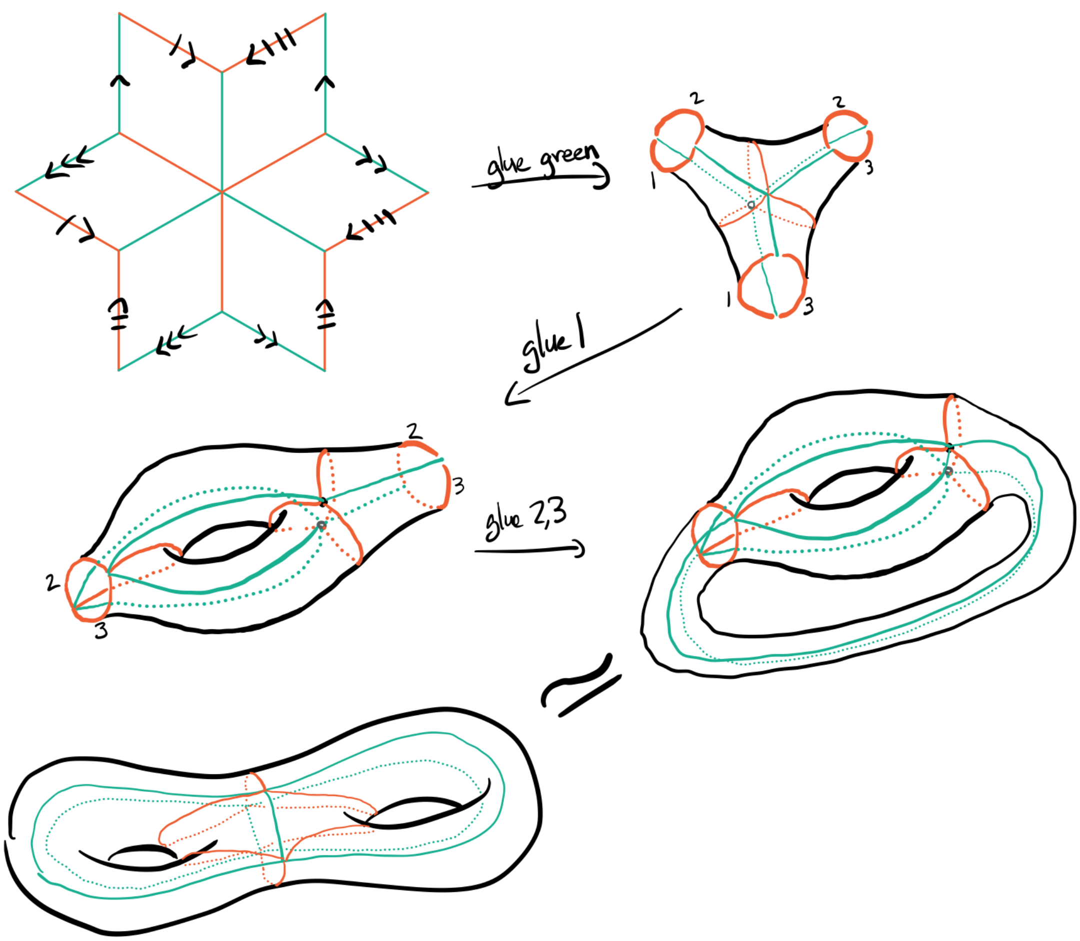


Figure 3.2: How to obtain the filling multicurves from a square complex structure on a surface

The difficulty is that in Proposition 24, there may be an element of G of infinite order which fixes a point in the product of trees, and the representation in Proposition 25 may not be faithful.

Proposition 26. *Unbounded and indiscrete subgroups of* $\mathsf{PSL}_2(\mathbb{Q}_p)$ *are dense or solvable.*

Proof. Suppose $G \leq \mathsf{PSL}_2(\mathbb{Q}_p)$ is unbounded and indiscrete, and let $\mathfrak{g}$ denote the Lie algebra of the closure of G. Then $\mathfrak{g}$ is a subalgebra of $\mathfrak{sl}_2$, and every proper subalgebra is solvable. If it is a proper subalgebra, then G is solvable, otherwise $\overline{G}$ is open in $\mathsf{PSL}_2(\mathbb{Q}_p)$. Theorem (T) in [66] asserts that if G is semi-simple and almost k-simple, then any proper open subgroup of $G(k)^+$ is bounded. But since $\overline{G}$ is an unbounded open subgroup of the simple group $\mathsf{PSL}_2(\mathbb{Q}_p)$, we conclude that G is dense in $\mathsf{PSL}_2(\mathbb{Q}_p)$. $\square$

Proposition 27. *Suppose $G \leq \mathsf{PSL}_2(\mathbb{Q})$ is a surface group. Then there is a nonempty finite set of primes S so that $G \leq \mathsf{PSL}_2(\mathbb{Q}_p)$ is dense for $p \in S$. For every other prime, G is dense in a finite index subgroup of a conjugate of $\mathsf{PSL}_2(\mathbb{Z}_p)$, and this index is 1 and conjugate is trivial for all but finitely many primes.*

Proof. First, if the trace of every element of G is an integer, then G has a finite index subgroup which is contained in $\mathsf{PSL}_2(\mathbb{Z})$, hence G would be virtually free. Thus there must be $g \in G$ with $tr(g)$ not an integer, and there is some nonempty set S of primes which occur as denominators of traces. This set must also be finite, since G is finitely generated and the trace ring of G is therefore finitely generated.

Note that for any prime $p \in S$, G is unbounded in $\mathsf{PSL}_2(\mathbb{Q}_p)$, since $tr(g) \notin \mathbb{Z}_p$ implies that g is a translation in the p-adic tree. If G were unbounded and discrete in $\mathsf{PSL}_2(\mathbb{Q}_p)$, then G would act properly on a tree and hence be virtually free. Surface groups are not virtually free, so G is indiscrete in $\mathsf{PSL}_2(\mathbb{Q}_p)$. But G does contain a free subgroup, hence G is not solvable. Hence, by Proposition 26, G is dense in $\mathsf{PSL}_2(\mathbb{Q}_p)$.

For the primes $q \notin S$, G is bounded in $\mathsf{PSL}_2(\mathbb{Q}_q)$, hence G acts on the q-adic tree with a global fixed point. Then some finite index subgroup of G fixes the base vertex, or in other words, some finite index subgroup of G is contained in $\mathsf{PSL}_2(\mathbb{Z}_q)$. It may be that not all elements of G are q-integral, although they all have q-integral traces. However, there can only be finitely many primes occurring as denominators of entries of G, again since G is finitely generated. In these finitely many additional primes, it may be necessary to conjugate G into $\mathsf{PSL}_2(\mathbb{Z}_q)$, but for the remaining primes, $G \leq \mathsf{PSL}_2(\mathbb{Z}_q)$ and Theorem 8 implies that G is dense for almost every prime, and almost dense for every prime $q \notin S$. □

The Long-Reid group

The orbifold $T_{(2)}$ is a convenient choice of a hyperbolic orbifold because its fundamental group has a very simple presentation with two generators and one relation. Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$ is the identity in $\mathsf{PSL}_2(\mathbb{R})$. Either $b = c = 0$, and we must have $a^2 = d^2 = \pm 1$, and we conclude that $a, d \in \{\pm 1\}$. But this has determinant 1 only if $a = d$, but these are both the identity in $\mathsf{PSL}_2(\mathbb{R})$. So if an element has order 2, it must have b or c nonzero, but this implies that the trace $a + d = 0$. Conversely, any matrix g with trace zero and determinant 1 has characteristic polynomial $x^2 + 1 = 0$, so must satisfy $g^2 = -I$.

If we assume that the generator $\rho(a) = A$ is diagonal (which, as we are interested primarily in rational representations, is a nontrivial restriction - A may be diagonalizable over $\mathbb{R}$ but not over $\mathbb{Q}$!), it is easy to compute which matrices B will satisfy the defining relation.

$$\left[\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} ad - bct^2 & ab(t^2-1) \\ cd(t^{-2}-1) & ad - bct^{-2} \end{pmatrix}$$

So the trace of the commutator is $2ad - bc(t^2 + t^{-2})$. The commutator will have order 2 in $\mathsf{PSL}_2(\mathbb{R})$ exactly when its trace is zero, so we must solve $2ad = bc(t^2 + t^{-2})$ and $ad - bc = 1$. Substituting $ad = bc + 1$ in the first equation gives $2 = bc(t^2 - 2 + t^{-2}) = bc(t - t^{-1})^2$. Thus

$bc = \frac{2}{(t-t^{-1})^2}$, and we see that $c = \frac{2t^2}{b(t^2-1)^2}$ parametrizes solutions in b. We are really just interested in representations up to conjugacy, and conjugating by a diagonal matrix allows us to replace b with $\lambda^2 b$ and c with λ^{-2}. Since not all rational numbers are squares, we lose a bit of generality by fixing a particular solution $b = \frac{1}{t-t^{-1}}$ and $c = \frac{2}{t-t^{-1}}$, but we obtain the explicit solutions in two parameters:

$$\left[\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \begin{pmatrix} s & t \\ 2t & (t^4+1)/s \end{pmatrix}/(t^2-1)\right] = \begin{pmatrix} 1-t^4 & st(t^2-1) \\ \frac{-2(t^4+1)(t^2-1)}{st} & t^4-1 \end{pmatrix}/(t^2-1)^4$$

By setting $t = 3$ and $s = 1$, we recover the representation of Long and Reid. However, we have also obtained the following:

Proposition 28. *For any $t \in \mathbb{Z}$, $\mathsf{PSL}_2(\mathbb{Z}[1/(t^3-t)])$ contains a surface subgroup.*

Note that when $|t| > 3$ is an integer, the set of primes dividing $t^3 - t = (t-1)t(t+1)$ is strictly larger than 2, and so this is the only parameter which gives an action on a product of two trees!

One may ask more generally: *for which rings A does $\mathsf{PSL}_2(A)$ contain a surface subgroup?* For quadratic irrationals α, there is an arithmetic Fuchsian group in $\mathsf{PSL}_2(\mathbb{Z}[\alpha])$. But it is not clear whether $\mathsf{PSL}_2(\mathbb{Z}[1/p])$ or $\mathsf{PSL}_2(\mathbb{Z}[\beta])$ ($\beta^3 - 3\beta - 1 = 0$), for example, contains a surface group.

3.2 The Markoff Equation

If $A \in \mathsf{SL}_2(\mathbb{C})$, its characteristic polynomial takes the form $\lambda^2 - tr(A)\lambda + 1 = 0$, and the Cayley-Hamilton theorem implies that $A^2 - tr(A)A + I = 0$, hence $tr(A^2) - tr(A)^2 + 2 = 0$, or $tr(A^2) = tr(A)^2 - 2$. Now for any other matrix B, we have $BA^{-1}(A^2 - tr(A)A + I) = BA^{-1}0 = BA - tr(A)B + BA^{-1}$, and taking traces yields $tr(BA) - tr(A)tr(B) + tr(BA^{-1}) = 0$, or $tr(A)tr(B) = tr(BA) + tr(BA^{-1})$. We also have the identities $tr(AB) = tr(BA)$ and $tr(A) = tr(A^{-1})$, which are perhaps observed most simply by looking at the formulas for multiplication and inversion.

It follows that

$$tr([A,B]) = tr(AB(BA)^{-1}) = tr(AB)tr(BA) - tr(ABBA)$$

Now $tr(A(BBA)) = tr(A)tr(BBA) - tr(B^2) = tr(A)\left(tr(B)tr(BA) - tr(A)\right) - tr(B^2) = tr(A)tr(B)tr(AB) - tr(A)^2 - tr(B^2)$. Since $tr(B^2) = tr(B)^2 - 2$, we have obtained *the Markoff equation*

$$tr([A,B]) = tr(A)^2 + tr(B)^2 + tr(AB)^2 - tr(A)tr(B)tr(AB) - 2$$

Suppose $\rho\colon \langle a, b \mid [a,b]^2 \rangle \to \mathsf{PSL}_2(\mathbb{Q})$ is a faithful representation. Note the trace is well-defined up to a choice of sign in PSL_2, and $tr(\rho([a,b]))$ must be equal to 0. Set $x = tr(\rho(a))$, $y = tr(\rho(b))$, and $z = tr(\rho(ab))$. Then the trace identity is equivalent to *the Markoff equation*

$$x^2 + y^2 + z^2 = xyz + 2.$$

We set $\delta = tr(\rho(a^2))$, and from the relation $tr(A^2) = tr(A)^2 - 2$, we observe that $\delta = x^2 - 2$. Upon setting $s = y - \frac{x}{2}z$, we thus obtain an equivalent form of the Markoff equation, namely

$$s^2 + \frac{1}{2}z^2 = \delta((\frac{z}{2})^2 - 1).$$

We obtain a handful of integer solutions to this equation by setting $(x, y, z) = (\pm 1, \pm 1, 0)$, $(x, y, z) = (1, 1, 1)$, and $(x, y, z) = (1, -1, -1)$ together with permutations of these solutions. Note that all of these correspond to unfaithful representations, because the generators (which have infinite order in the orbifold group) are mapped to rotations of order 2 or 3.

We now homogenize the Markoff equation, setting $x = \frac{X}{W}$, $y = \frac{Y}{W}$, and $z = \frac{Z}{W}$, and scale by a factor of $4W^4$ to obtain the form:

$$(2WY - XZ)^2 + 2W^2Z^2 = (X^2 - 2W^2)(Z^2 - 4W^2) \tag{3.1}$$

We consider whether there is a faithful representation of this torus orbifold group in which one of the generators is integral. Without loss of generality, suppose x is an integer. If the representation is faithful, we must have $|x| \geq 2$. The condition that x is an integer is equivalent to the condition that $W \mid X$ in the homogeneous form. In this case, the value $\delta = \frac{X^2 - 2W^2}{W^2}$ is an integer, which is 7 mod 8 if x is odd and 2 mod 4 if x is even, since $\{0, 1, 4\}$ are the squares mod 8.

Supposing $x = \frac{X}{W}$ is an integer and canceling factors of W^2, we obtain the integral equation

$$(2Y - xZ)^2 + 2Z^2 = (x^2 - 2)(Z^2 - 4W^2). \tag{3.2}$$

Theorem 15. *If (x, y, z) is a rational solution to $x^2 + y^2 + z^2 = xyz + 2$ and x is an integer, then $|x| \geq 6$, $v_2(x) = 1$ and $v_2(y) = v_2(z) = -1$.*

Proof. Supposing there were such a solution, we obtain an integral solution to Equation 3.2 by writing $(x, y, z) = (x, Y/W, Z/W)$, with W minimal.

Case 1: x is odd. Then $\delta = x^2 - 2 \equiv 7 \pmod 8$. Since also $\delta \geq 2$, there must be a prime $p \equiv 3 \pmod 4$ dividing δ to an odd power, and this prime cannot divide x, or else it divides $x^2 - \delta = 2$. Reducing Equation 3.2 mod p, we obtain $(2Y - xZ)^2 + 2Z^2 \equiv 0 \pmod p$. Noting that $x^2 \equiv 2 \pmod \delta$, and hence $x^2 \equiv 2 \pmod p$, this is equivalent to $(2Y - xZ)^2 \equiv -(xZ)^2 \pmod p$, which can only have a solution if p divides both $(2Y - xZ)$ and xZ, since -1 is not a square mod p when $p \equiv 3 \pmod 4$. Since p does not divide x, it must divide Z. But since p divides $2Y - xZ$ and xZ, it must also divide $2Y$ and hence Y. So p must divide both Y and Z, and therefore cannot divide W since W is minimal. But this implies that p does not divide $Z^2 - 4W^2$, since it is equal to $-4W^2 \pmod p$, which is nonzero because W is not divisible by p.

Now let $k_i = v_p(a_i)$, for $a_1 = 2Y - xZ$ and $a_2 = xZ$. We claim that $v_p((2Y - xZ)^2 + 2Z^2) = 2k$, where $k = \min\{k_1, k_2\}$. To this end, we observe that $A = (a_1/p^k)^2 + (a_2/p^k)^2$ is a sum of integer squares, not both of which are divisible by p. But since there is no nonzero solution to $x^2+y^2=0 \pmod p$ in $\mathbb{F}_p$, A must not be divisible by p. Thus, $v_p((2Y-xZ)^2+2Z^2)=2k$, as claimed.

Thus the left hand side is divisible by an even power of p, and the right hand side is divisible by an odd power of p. This is a contradiction.

Case 2: $4 \mid x$. This implies $\delta = x^2-2 = 2\delta'$, with $\delta' \equiv 7 \pmod 8$. Hence δ' is divisible by some prime $p \equiv 3 \pmod 4$ to an odd power, and we can argue as before. Explicitly, this implies that p does not divide x, but considering Equation 3.2 modulo p can be written $(2Y-xZ)^2+(xZ)^2 \equiv 0 \pmod p$, and we deduce as before that p divides Z and Y. Then the left hand side has even p-adic valuation, and since the right hand side has odd p-adic valuation, we obtain a contradiction.

Thus the only possibility is that $x \equiv 2 \pmod 4$, or equivalently, $v_2(x)=1$. Note that if $x=\pm 2$, the equation becomes $4+y^2+z^2=\pm 2yz+2$, or $(y\mp z)^2=-2$, which clearly has no rational solutions. So $|x|\geq 6$.

Thus, we have shown that if (x,y,z) is a rational solution with x an integer, then $x\equiv 2 \pmod 4$ and $v_2(y)=v_2(z)=-1$. ☐

Setting $t_2^2 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $t_2^2U^{-1} = \begin{pmatrix} 2 & -2 \\ 0 & 1/2 \end{pmatrix}$ and $UL^2U = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

Theorem 16. *The representation* $\rho_2(a) = \begin{pmatrix} 2 & -2 \\ 0 & 1/2 \end{pmatrix}$, $\rho_2(b) = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ *is a faithful representation of the compact orbifold group G into $\mathsf{PSL}_2(\mathbb{Z}[1/2])$, corresponding to the triple $(5/2, 6, 7/2)$.*

Proof. Note that the commutator $\rho_2([a^{-1},b^{-1}]) = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = UL^{-2}$ has trace zero, hence is a rotation of order 2, about $p=\frac{1+i}{2}$. The four points $p, ap=-2+2i$, $bp=(47+i)/34$, and $abp = bap = \frac{2}{17}(13+i)$ form the vertices of a quadrilateral with geodesic sides, and the sum of the interior angles is π. Since the isometries a and b induce side-pairings of this quadrilateral, they generate a Fuchsian group in $\mathsf{PSL}_2(\mathbb{R})$. ☐

Proposition 29. *The group $\rho_2^{-1}(\rho_2(G)\cap \mathsf{PSL}_2(\mathbb{Z}))$ is a maximal subgroup of G, and $H = \rho_2(G)\cap \mathsf{PSL}_2(\mathbb{Z})$ is a thin subgroup of $\mathsf{PSL}_2(\mathbb{R})$ with infinite index in its commensurator, which is a discrete subgroup of $\mathsf{PSL}_2(\mathbb{R})$.*

Proof. For the first statement, we observe that $\rho_2(G)$ is dense in $\mathsf{PSL}_2(\mathbb{Q}_2)$ by Proposition 27, and that since $\rho_2(G)\leq \mathsf{PSL}_2(\mathbb{Z}[1/2])$, and $\mathbb{Z}_2\cap\mathbb{Z}[1/2]=\mathbb{Z}$, we have that $\rho_2(G)\cap\mathsf{PSL}_2(\mathbb{Z}) = \rho_2(G)\cap\mathsf{PSL}_2(\mathbb{Z}_2)$. If there were an intermediate subgroup between H and G, its closure would be an intermediate subgroup between $\mathsf{PSL}_2(\mathbb{Z}_2)\leq\mathsf{PSL}_2(\mathbb{Q}_2)$, which is impossible.

Since $\rho_2(G)$ is a nonarithmetic lattice in $\mathsf{PSL}_2(\mathbb{R})$, it has finite index in its commensurator by Margulis's theorem, which says that a lattice in a simple Lie group is arithmetic if and only if it has indiscrete commensurator [54]. Since the commensurator of $\rho_2(G)$ is therefore discrete, and $\rho_2(G)$ is a lattice, its commensurator must contain it with finite index. Since H is a subgroup of the discrete group G, it is discrete, and since it is not virtually abelian, it must be Zariski dense in $\mathsf{PSL}_2(\mathbb{R})$. $\square$

Theorem 17. *There is no faithful representation* $G = \langle a, b \mid [a,b]^2 \rangle \to \mathsf{PSL}_2(\mathbb{Z}_2)$, *hence* G *is not a subgroup of* $\mathsf{PSL}_2(\mathbb{Z}[1/n])$ *for* n *odd.*

Proof. Note that the only elements which are commutators in $S_3 \cong \mathsf{PSL}_2(\mathbb{F}_2)$ are the 3-cycles and the identity. So any representation must have $\rho([a,b]) \equiv I \pmod 2$. But this implies that $tr(\rho([a,b])) \equiv 2 \pmod 4$, but a faithful representation must have $tr(\rho([a,b])) = 0$. $\square$

Putting together the previous two theorems, we obtain the following corollary.

Corollary 5. $G = \langle a, b \mid [a,b]^2 \rangle$ *is a subgroup of* $\mathsf{PSL}_2(\mathbb{Z}[1/n])$ *if and only if* n *is even.*

Right-Angled Pentagon Groups

We consider an analogous representation of the Coxeter group generated by the reflections in the sides of a right-angled pentagon. For $\lambda, \mu > 0$ satisfying $\tanh(\lambda)^2 + \tanh(\mu)^2 > 1$, there is a unique right-angled pentagon up to isometry with two adjacent sides of length λ, μ. We can take the intermediate vertex at i, with a side of length λ going up to $e^{\lambda} i$. The reflection in the imaginary axis is given by $a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathsf{PGL}_2(\mathbb{R})$. The side of length μ can be chosen to move to the right along the unit circle to the point $(\sinh(\mu) + i)/\cosh(\mu)$ on the unit circle. The inversion in the unit circle is given by $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The edge coming from $e^{\lambda} i$ to the right is a scaling of the unit circle, and reflection in this side is given by $c = \begin{pmatrix} 0 & e^{\lambda} \\ e^{-\lambda} & 0 \end{pmatrix}$. The reflection in the edge orthogonal to the unit circle and passing through $(\sinh(\mu) + i)/\cosh(\mu)$ is given by $d = \begin{pmatrix} -\cosh(\mu) & \sinh(\mu) \\ -\sinh(\mu) & \cosh(\mu) \end{pmatrix}$.

The reflection in the final side must commute with these last two reflections. The condition that $e = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ satisfies $ce = -ec$ implies that $z = -w, y = x/e^{2\lambda}$. Using this information, the condition that $de = -ed$ implies that $e = \begin{pmatrix} -\tanh(\mu) & \tanh(\lambda) + 1 \\ \tanh(\lambda) - 1 & \tanh(\mu) \end{pmatrix}$. The assumption we made that $\tanh(\lambda)^2 + \tanh(\mu)^2 > 1$ is equivalent to the condition that the determinant of e is negative, and since the trace is 0 this is a reflection in $\mathsf{PGL}_2(\mathbb{R})$. Set $\delta^2 = \tanh(\lambda)^2 - \tanh(\mu)^2 - 1$, so that e/δ has determinant -1.

Thus the group

$$\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{\lambda} \\ e^{-\lambda} & 0 \end{pmatrix}, \begin{pmatrix} -\cosh(\mu) & \sinh(\mu) \\ -\sinh(\mu) & \cosh(\mu) \end{pmatrix}, \begin{pmatrix} -\tanh(\mu) & \tanh(\lambda) + 1 \\ \tanh(\lambda) - 1 & \tanh(\mu) \end{pmatrix} /\delta \rangle$$

determines a Fuchsian representation of the right-angled pentagon reflection group into $\mathsf{PGL}_2(\mathbb{R})$. Note that each edge in the pentagon has two orthogonal reflections, and if we multiply them in either order, the translation length is twice the length of the side. This allows us to easily compute the lengths of the sides of the pentagon to be $a = \cosh(\lambda)$, $b = \cosh(\mu)$, $c = \sinh(\lambda)\sinh(\mu)$, $d = \tanh(\lambda)/\delta$, and $e = \tanh(\mu)/\delta$.

Here a and b are the lengths of adjacent sides, and c is the length of the side opposing them. We have the relation $(a^2-1)(b^2-1) = \sinh(\lambda)^2\sinh(\mu)^2$, or equivalently,

$$a^2 + b^2 + c^2 = a^2b^2 + 1.$$

This formula somewhat resembles the Markoff equation, although since this equation has degree four it has somewhat different behavior. We also remark that a solution with a, b, c rational does not guarantee that the other two sides are rational, since the quantity $\delta^2 = 1 - a^{-2} - b^{-2}$ may not be a rational square. The side lengths d and e satisfy the relations $a^2d^2(c^2-1) = b^2-1$ and $b^2e^2(c^2-1) = a^2-1$. Thus any real numbers (a, b, c, d, e) satisfying the relations $r_1 : a^2+b^2+c^2 = a^2b^2+1$, $r_2 : a^2d^2(c^2-1) = b^2-1$ and $r_3 : b^2e^2(c^2-1) = a^2-1$ determines a hyperbolic right-angled pentagon.

We consider whether there are choices of λ and μ so that the image of the representation is contained in $\mathsf{PGL}_2(\mathbb{Q})$. We remark that although this computation essentially determines the $\mathsf{PGL}_2(\mathbb{R})$-character variety, the assumption we made that there is a vertex at i with a vertical side is a nontrivial assumption when only considering representations up to conjugacy in $\mathsf{PGL}_2(\mathbb{Q})$.

The curve $(\cosh(x), \sinh(x))$ admits a rational parametrization equivalent to that of the circle, since any pair of coprime integers (m, n) determines a primitive Pythagorean triple $(m^2-n^2, 2mn, m^2+n^2)$, and thusly a rational point $(\frac{m^2+n^2}{2mn}, \frac{m^2-n^2}{2mn})$. The supposition $e^x = \frac{m}{n}$ determines $\cosh(x) = \frac{m^2+n^2}{2mn}$, $\sinh(x) = \frac{m^2-n^2}{2mn}$, and $\tanh(x) = \frac{m^2-n^2}{m^2+n^2}$.

The substitution $e^\lambda = \frac{r+s}{r-s}$ and $e^\mu = \frac{u}{v}$ yields $a = \frac{r^2+s^2}{r^2-s^2}$, $b = \frac{u^2+v^2}{2uv}$, $c = \frac{rs(u-v)(u+v)}{uv(r-s)(r+s)}$, $d = \frac{rs(u^2+v^2)}{\eta}$, and $e = \frac{(r^2+s^2)(u^2-v^2)}{2\eta}$, where $\eta = \sqrt{(su-rv)(su+rv)(ru-sv)(ru+sv)}$. So if we can choose integers r, s, u, v so that η is an integer ≥ 2, this provides a rational right-angled pentagon group.

A particular solution with entries in $\mathbb{Z}[\sqrt{5}, 1/2, 1/3]$ is obtained by choosing $e^\lambda = e^\mu = \frac{3+\sqrt{5}}{2}$:

$$\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{3+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & 0 \end{pmatrix}, \begin{pmatrix} -3 & \sqrt{5} \\ -\sqrt{5} & 3 \end{pmatrix}/2, \begin{pmatrix} -\sqrt{5} & 3+\sqrt{5} \\ -3+\sqrt{5} & \sqrt{5} \end{pmatrix}/3 \rangle.$$

We remark that a regular right-angled pentagon is obtained by choosing $e^\lambda = e^\mu = \sqrt{\frac{3+\sqrt{5}}{2}}$, and a right-angled hexagon has all edges satisfying $e^\lambda = \sqrt{2}$, and note that a right-angled hexagon is built from six copies of the $(3, 4, 4)$ triangle.

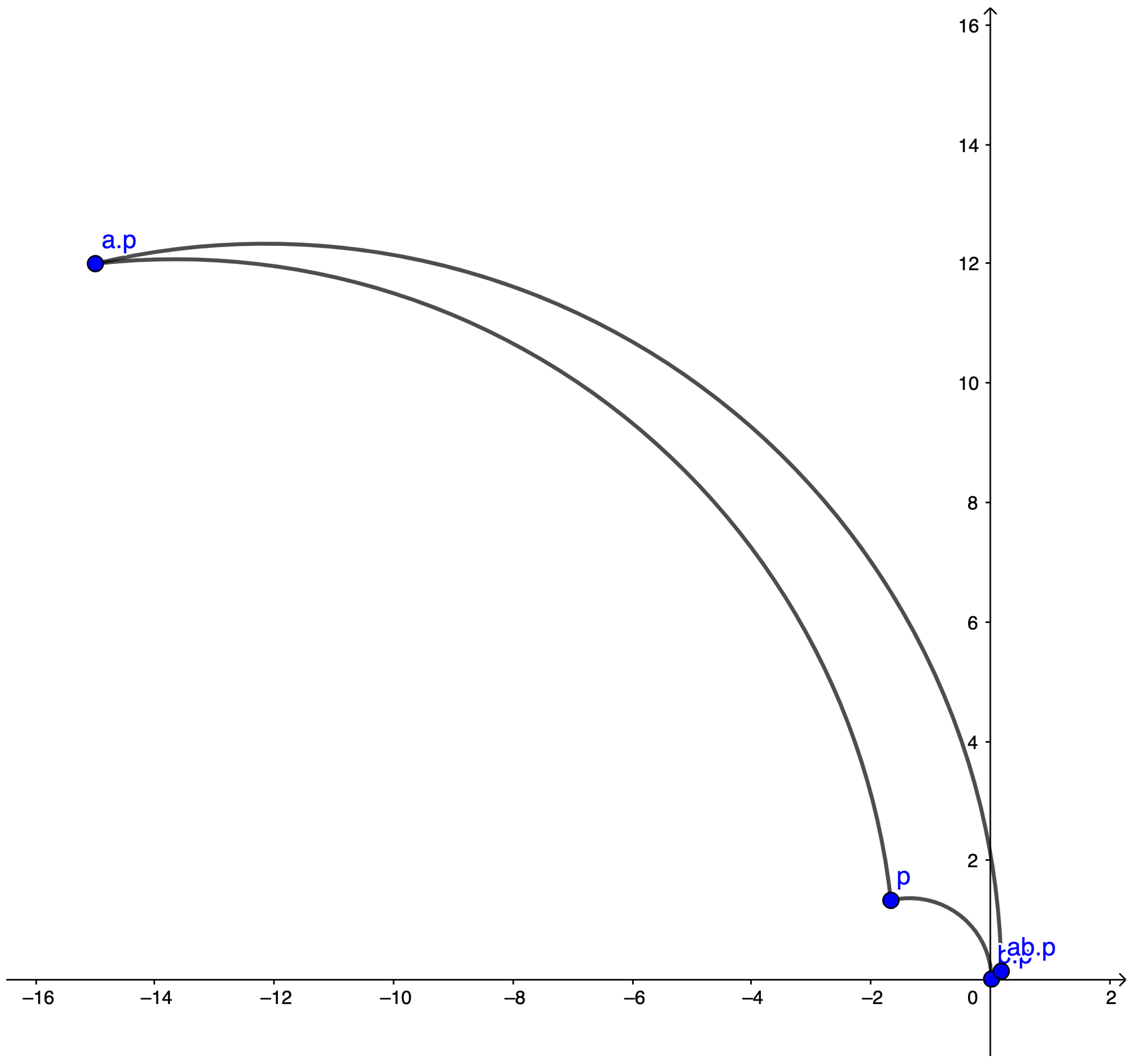


Figure 3.3: The Long-Reid group

Long-Reid representation

We return to the orbifold which is a torus with a single cone point of order 2. We computed some points on the character variety $\rho(a) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$, $\rho(b) = \begin{pmatrix} s & t \\ 2t & (t^4+1)/s \end{pmatrix} /(t^2-1)$, and we now fix $t = 3$, $s = 1$. Set $p = (-5+4i)/3$, so that $ap = -15+12i$, $bp = (5+4i)/(3 \cdot 82)$, and $abp = bap = (15+12i)/82$. Then the quadrilateral with these four vertices and geodesic sides is a fundamental domain for the action of G on $\mathbb{H}^2$. More generally, the point p can be taken to be $\frac{-(t^2+1)+(t^2-1)i}{2st}$ to obtain a fundamental domain. The sum of the interior angles of this quadrilateral can be calculated to be π.

Note that if s and t are both chosen to be rational, we obtain a representation $G \to$

$\mathsf{PSL}_2(\mathbb{Q})$, which is Fuchsian when considered as a representation into $\mathsf{PSL}_2(\mathbb{R})$. This implies that the representation is faithful, and we consequently get a faithful action of a surface group on a tree by considering the composition with $\mathsf{PSL}_2(\mathbb{Q}) \to \mathsf{PSL}_2(\mathbb{Q}_p)$ for any prime p.

The representation into $\mathsf{PSL}_2(\mathbb{Q})$ is actually contained in $\mathsf{PSL}_2(\mathbb{Z}[1/(2st(t^2-1))])$ when $s, t \in \mathbb{Z}$. Let S denote the prime divisors of $2st(t^2-1)$ (in the case of $t = 3, s = 1$, $S = \{2, 3\}$: note that $t^2 - 1 = (t-1)(t+1)$, is a power of a single prime only when $t = 3$). If p does not divide $st(t^2-1)$, the p-adic representation lies in $\mathsf{PSL}_2(\mathbb{Z}_p)$, and acts on the tree with a global fixed point. Thus $G \leq \mathsf{PSL}_2(\mathbb{A}_{fin})$ is discrete if and only if $G \leq \mathsf{PSL}_2(\mathbb{Q}_2) \times \mathsf{PSL}_2(\mathbb{Q}_3)$ is discrete.

The element a is a translation of length 2 in the 3-adic tree, with the endpoints of its axis being $0, \infty$. In the 2-adic tree, it fixes the 1-neighborhood of the axis from 0 to ∞. The element b is a translation of length 6 in the 2-adic tree, with the endpoints of its axis being $\frac{-27\pm\sqrt{737}}{4}$. Since $737 \equiv 1 \pmod 8$, it has a square root in $\mathbb{Q}_2$ by Hensel's lemma. However, $737 \equiv 2 \pmod 3$, so it does not have a square root in $\mathbb{Q}_3$, and thus b has a compact fixed set in its action on the 3-adic tree.

Fixing the base vertex $v_0 = \lfloor 0 \rfloor_0$ in the 2-adic tree, let $H = \{g \in G \mid gv_0 = v_0\} = G \cap \mathsf{PSL}_2(\mathbb{Z}_2)$. We remark that in terms of the generators given in Section 2.3, we have $a = t_3^2$ and $b = ST^{-6}St_2^{-6}T^3$. There is in fact a corresponding continuous π_1-injective map from $T_{(2)}$ into the orbicomplex $(\mathbb{H}^2 \times T_3 \times T_4)\,/\mathsf{PGL}_2(\mathbb{Z}[1/6])$ sending one generator to t_3^2 and the other to $ST^{-6}St_2^{-6}T^3$. This lifts to a map of universal covers $f\colon \mathbb{H}^2 \to \mathbb{H}^2 \times T_3 \times T_4$, and the projection from the product space to the first factor is quasi-isometric on the image of f.

The main question in this case would be answered positively if, for the projection $p\colon \mathbb{H}^2 \times T_3 \times T_4 \to T_3 \times T_4$, the composition $p \circ f$ is quasi-isometric.

Cutting the group

Observe that the group $\pi_1(T_{(2)})$ splits as an HNN extension $G_0 *_{a^b = ca}$, where $c = [b, a] = \begin{pmatrix} 5/4 & -3/8 \\ 41/6 & -5/4 \end{pmatrix}$ and $G_0 = \langle a, c\rangle$. Geometrically, this corresponds to cutting the torus along a simple closed curve to obtain an annulus with a cone point of order 2. This has a virtually free fundamental group G_0, and the two boundary components correspond to the two infinite cyclic groups $\langle a \rangle$ and $\langle bab^{-1}\rangle$.

Under our representation, $\langle a, c\rangle$ acts properly on the 3-adic tree. One can see this by calculating the action of c. We have that c fixes the vertex $\lfloor 0 \rfloor_1$, and permutes its neighbors $\lfloor 0 \rfloor_0 \leftrightarrow \lfloor 6 \rfloor_2$ and $\lfloor 0 \rfloor_2 \leftrightarrow \lfloor 3 \rfloor_2$. This means that the axis for cac passes through $\cdots \to \lfloor 6 \rfloor_2 \to \lfloor 0 \rfloor_1 \to \lfloor 3 \rfloor_2 \to \dots$, and in particular, intersects the axis $\cdots \to \lfloor 0 \rfloor_0 \to \lfloor 0 \rfloor_1 \to \lfloor 0 \rfloor_2 \to \dots$ for a in a single point. Since the index two subgroup $\langle a, bab^{-1}\rangle$ acts properly on T_4, so does G_0.

The action of c on the 2-adic tree fixes the edge $e_0 = \{\lfloor \frac{3}{2} \rfloor_1, \lfloor \frac{3}{2} \rfloor_2\}$, and permutes the two neighbors of each vertex. In particular, we see that the fixed set of c on the product is $e_0 \times \lfloor 0 \rfloor_1$. The element a also fixes the edge e_0. Let $v_0 = \lfloor \frac{3}{2} \rfloor_2$, and $H = \mathrm{Stab}_G(v_0)$. Note G_0 is contained in H.

Proposition 30. *If H acts properly on T_4, the Long-Reid group acts properly on a product of trees.*

Proof. Recall that G acts on $T_3 \times T_4$, and let (v_0, w_0) be the basepoint in the product. If H acts properly on T_4, there can only be finitely many elements of H which fix $w_0 \in T_4$. But this means that only finitely many elements of G fix $(v_0, w_0) \in T_3 \times T_4$, hence the action of G on $T_3 \times T_4$ is proper. □

Since G splits as an extension $G_0 *_{a^b = ca}$, there is a Bass-Serre tree $\mathcal{T}$ associated to this splitting that every subgroup of G acts on. In particular, H acts on this (infinite valence) tree without a global fixed point, and we obtain a splitting of H. Since G_0 is the stabilizer of the base vertex in $\mathcal{T}$ and G_0 is contained in H, the quotient graph of groups $\mathcal{T}//H$ has G_0 as the stabilizer of a base vertex. It appears that this quotient graph of groups has infinitely generated fundamental group, which makes the problem of understanding H recursively quite challenging.

A Free Group

Consider the group

$$F = \langle \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 13/5 & 12/5 \\ 12/5 & 13/5 \end{pmatrix} \rangle.$$

Since the products ab and ab^{-1} of the generators a, b, have trace $26/3$, this is a discrete free subgroup of $\mathsf{PSL}_2(\mathbb{R})$, and it is contained in $\mathsf{PSL}_2(\mathbb{Z}[1/15])$, which is a lattice in a product $\mathbb{H}^2 \times T_4 \times T_6$.

The generator a lies in $\mathsf{PSL}_2(\mathbb{Z}_5)$, and any conjugate of it has a power in $\mathsf{PSL}_2(\mathbb{Z}_5)$. It turns out that $b^k a^l b^{-k}$ is in $\mathsf{PSL}_2(\mathbb{Z}_5)$ only when $2 \cdot 5^{2k-1}$ divides l. Determining whether the action on $T_4 \times T_6$ is proper is closely related to the other questions we consider. Since the group is perhaps simpler than the orbifold group, this might be easier to analyze.

3.3 Totally Unitary Groups

Definition 9. Let k be a number field. A subgroup Γ of $\mathsf{PSL}_2(k)$ is called *totally unitary* if it lies in a compact subgroup at every infinite place.

Our reason for introducing and studying this class of groups is the following proposition.

Theorem 18. *A finitely generated totally unitary group acts properly on a locally compact product of trees.*

Proof. Since Γ is finitely generated, it is contained in $\mathsf{PSL}_2(R)$ for some finitely generated subring $R \subseteq k$. It is discrete in a product $\mathsf{PSL}_2(\mathbb{R})^r \times \mathsf{PSL}_2(\mathbb{C})^s \times \prod_{v \in V_S} \mathsf{PSL}_2(k_v)$ for a finite set S of nonarchimedean valuations.

But since it is contained in a compact subgroup of each real and complex place, the projection to the subproduct $\prod_{v \in S} \mathsf{PSL}_2(k_v)$ remains discrete, and this latter group acts properly on a locally compact product of trees. □

If k is an imaginary quadratic extension of $\mathbb{Q}$, then unitarity is preserved under the Galois automorphism of k, and thus any subgroup of $\mathsf{PSU}_2(k)$ is totally unitary.

For any field, one can choose a subgroup which only uses rational entries to obtain a totally unitary subgroup of $\mathsf{PSL}_2(k)$. For this reason, we are more interested in groups which are Zariski dense in $\mathsf{PSU}_2(k)$.

Theorem 19. *For any totally imaginary number field k, there is a Zariski dense totally unitary group $\Gamma \leq \mathsf{PSU}_2(k)$.*

Proof. Since there are only finitely many primes which are ramified in $\mathcal{O}_k$, we can choose an unramified prime p. Then $\mathsf{PSU}_2(\mathcal{O}[1/p])$ is a Zariski dense totally unitary group. $\square$

Example 4. Consider the group $G = \langle 1+2i, 1+2k, (1+2j)^2, (1+j+\sqrt{3}k)^{1+2j}, (1+j-\sqrt{3}k)^{1+2j}\rangle$. Each $g \in G$ has left and right 5-factor in the set $\{1 \pm 2i, 1 \pm 2j, 1 \pm 2k\}$. However, the conjugate group G^{1-2j} has left and right 5-factors in the set $\{1 \pm 2j, 1 \pm j \pm \sqrt{3}k\}$.

G is a Zariski dense totally unitary group in $\mathsf{PSU}_2(\mathbb{Q}(\sqrt{3}, i))$.

A Totally Unitary Group

We construct surface group representations into $\mathcal{H}_{\mathbb{Q}}$. We suppose the first quaternion $g = x + yj$ and the second is $h = a + bi + cj + dk$. Then $ghg^{-1}h^{-1}$ has real part $(a^2+c^2)(x^2+y^2) + (b^2+d^2)(x^2-y^2)$. This is zero precisely when $\frac{a^2+b^2+c^2+d^2}{b^2+d^2}\frac{x^2+y^2}{y^2} = 2$.

We can find several solutions to this by inspection. For example, if $a^2+b^2+c^2+d^2 = 2y^2$ and $b^2+d^2 = x^2+y^2$, we find the solution $g = 2+3j$, $h = 1+2i+2j+3k$.

If instead $a^2+b^2+c^2+d^2 = 4y^2$ and $b^2+d^2 = 2(x^2+y^2)$, these imply $a^2+c^2 = 2(-x^2+y^2)$, and we find solutions $g = 4+5j$, $h = 3+i+3j+9k$, and $g = 36+77j$, $h = 55+71i+79j+97k$.

For the moment, let us focus on the representation $\langle a, b \mid [a,b]^2\rangle \to \mathsf{PGL}_2(\mathbb{Q}_3)\times\mathsf{PGL}_2(\mathbb{Q}_{13})$ which sends $a \mapsto j(3-2j)$ and $b \mapsto -(1-k)(1+i-j)(1-i-j)$. There is a map to D_8 which sends a to a reflection and b to a rotation of order 4, the kernel of which is an index 8 subgroup which is the fundamental group of a genus 3 surface. This is the 2-congruence subgroup of this representation.

The commutator $c = [a,b] = -2i+2j-k$ is a purely imaginary quaternion, hence is an involution. It fixes the base vertex of the 13-adic tree and reflects in the midpoint of an edge of the 3-adic tree.

The group generated by b and c is isomorphic to $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, because it acts properly on the 3-adic tree. Note that $cbc = 3(1-k)(1+i-j)(1+i+j)^3$, and so c conjugates b to this element. If the representation remains faithful on the HNN extension, we have constructed a proper action of a surface group on a product of trees.

3.4 Quaternion Lattices

In this section, before embarking on a general exploration of quaternion lattices, we will study the group $\mathcal{H}_{\mathbb{Q}}$ and its subgroups.

Recall that $\overline{\mathcal{H}_{\mathbb{Q}}}$ is the projective group of invertible rational quaternions. Note that if q is nonzero, its conjugate $\overline{q}$ satisfies $N(q) = q\overline{q} \in \mathbb{Q}^*$, which is the identity up to scaling. Every nonzero rational quaternion has a unique rational scaling which is an integer quaternion with relatively prime coefficients.

The rational quaternions are ramified at precisely $\{2, \infty\}$, which is reflected in the fact that the group of $\mathbb{Z}[1/2]$-points is finite. In fact, this group is the *binary octahedral group*, consisting of the 48 quaternions $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ and the $\frac{1+i}{\sqrt{2}}$ coset of this set. Let us denote $\varepsilon = \frac{1+i}{\sqrt{2}}$ and $\omega = \frac{-1+i+j+k}{2}$, so that $\varepsilon^2 = i$ and $\omega^3 = 1$. We compute that conjugation by ω sends $i \to k \to j \to i$, and conjugation by ε sends $j \to k \to -j \to -k \to j$. Thus the group generated by ε and ω acts on the set $\{\pm i, \pm j, \pm k\}$ by linear permutations, and thus we have an action on the vertices of an octahedron (or dually, a cube). These two rotations generate the orientation preserving symmetry group, and the kernel of the action is $\{\pm 1\}$. Thus we have a central extension $1 \to \mathbb{Z}/2\mathbb{Z} \to \langle \varepsilon, \omega \rangle \to Isom^+(Oct) \to 1$.

Residue quaternion algebras

The embedding $M_2(\mathbb{Z}_p) \to M_2(\mathbb{Q}_p)$ actually induces an isomorphism $\mathbb{P}M_2(\mathbb{Z}_p) \to \mathbb{P}M_2(\mathbb{Q}_p)$, and so the inverse provides a reduction map $\lfloor - \rfloor_{res} \colon \mathbb{P}M_2(\mathbb{Q}_p) \to \mathbb{P}M_2(\mathbb{F}_p)$. Specifically, given $0 \neq g \in M_2(\mathbb{Q}_p)$, there exists $\lambda \in \mathbb{Q}_p^\times$ so that $\lambda g \in M_2(\mathbb{Z}_p) \setminus pM_2(\mathbb{Z}_p)$, and this λ is defined up to $\mathbb{Z}_p^\times$.

Note that $\lfloor g \rfloor_{res}$ lies in $\mathsf{PGL}_2(\mathbb{F}_p)$ exactly when $g \in \mathsf{PGL}_2(\mathbb{Z}_p)$, and otherwise it determines the *initial* and *terminal directions* of the geodesic between v_0 and gv_0, where $v_0 = \mathsf{PGL}_2(\mathbb{Q}_p)/\mathsf{PGL}_2(\mathbb{Z}_p)$ is the base vertex in the tree. The terminal direction is the initial direction of the inverse g^{-1}.

Subgroups of $\mathcal{H}_{\mathbb{Q}}$

First, we recall some notation. $\mathcal{H}_{\mathbb{Q}}$ is the group of projective rational quaternions. Given a nonzero rational quaternion q, a *primitive integral representative* is a $\mathbb{Q}^*$ multiple of q whose coefficients are integers which are relatively prime. A nonzero rational quaternion has exactly two primitive integral representatives, which differ by ± 1. Thus we may speak unambiguously about the norm of an element of $\mathcal{H}_{\mathbb{Q}}$ as the norm of a primitive integral representative.

There is a finite-index (congruence) subgroup of $\mathcal{H}_{\mathbb{Q}}$ consisting of those rational quaternions whose primitive integral representatives are equivalent to 1 or $1+i+j$ modulo 2, and we will typically try to understand this group rather than the slightly larger $\mathcal{H}_{\mathbb{Q}}$.

For an odd prime p, A_p is the set of $p+1$ norm p elements of $\mathcal{H}_{\mathbb{Q}}$ whose primitive integral representatives are equivalent to 1 or $1+i+j \pmod 2$. Upon choosing a solution $x_p^2 + y_p^2 = -1$ in $\mathbb{Q}_p$, we have an isomorphism $M_{x,y} \colon \mathbb{Q}_p\{i,j\} \to M_2(\mathbb{Q}_p)$ in which $i \mapsto \begin{pmatrix} x_p & y_p \\ y_p & -x_p \end{pmatrix}$, $j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the collection of such isomorphisms $\mathbb{Q}_p\{i,j\} \to M_2(\mathbb{Q}_p)$ up to conjugacy in $M_2(\mathbb{Q}_p)$ are in one-to-one correspondence with the solutions to $x_p^2 + y_p^2 = -1$. Note when $p \equiv 1 \pmod 4$, we can choose $x_p = i$, $y_p = 0$. In general, for p odd, the collection of nonzero solutions to $x_p^2 + y_p^2 + z_p^2 = 0$ in $\mathbb{Q}_p$ up to scaling is in bijection with $\mathbb{P}^1\mathbb{Q}_p$.

Proposition 31. *Under the isomorphism $M_{x,y}$, we obtain a bijection $\pi\colon A_p \to \mathbb{P}^1\mathbb{F}_p$.*

Proof. The elements $q \in A_p$ map to matrices of determinant p in $M_2(\mathbb{Z}_p)$. This implies that the column space of $M_{x,y}(q)$ is a 1-dimensional subspace of $\mathbb{F}_p^2$, which is naturally identified with $\mathbb{P}^1\mathbb{F}_p$. $\square$

Note that $\mathsf{GL}_2(\mathbb{Z}_p)$ acts on $\mathbb{P}^1\mathbb{F}_p$, and thus the elements of $M_2(\mathbb{Q}_p)$ which lie in $\mathsf{GL}_2(\mathbb{Z}_p)$ act on this set. Under the correspondence $M_{x,y}$, we obtain an action α of the quaternions coprime to p on the quaternions of norm p, defined by $\alpha_g(x) = \pi^{-1}(M_{x,y}(g)\pi(x))$. Thus the "metacommutation problem" formulated by Conway-Smith is governed by the action of $\mathsf{PGL}_2(\mathbb{F}_p)$ on $\mathbb{P}^1\mathbb{F}_p$. There are really two such actions, coming from the left and right actions of the matrices, and the conjugation anti-involution on the quaternions swaps the left and right actions.

Given an integer quaternion q, we write $x \mid_L q$ if x is a left-divisor of q.

Lemma 5. *Suppose q, g, and x are primitive integer quaternions, with $N(x) = p$ and $p \nmid N(g)$. If $x \mid_L q$, then $\alpha_g(x) \mid_L gq$.*

Proof. Since $p \nmid N(g)$ and q is primitive, the quaternion gq is not divisible by p. However, since $p \mid N(q)$, p divides the determinant of $M_{x,y}(gq) \in M_2(\mathbb{Z}_p)$, and hence $M_{x,y}(gq)$ reduces to a rank one matrix in $M_2(\mathbb{F}_p)$. The column space of $M_{x,y}(gq)$ coincides with that of $M_{x,y}(g)M_{x,y}(x)$, and hence the point $\pi(\alpha_g(x))$. Since the column spaces of $M_{x,y}(gq)$ and $\pi(\alpha_g(x))$ agree, they are both annihilated by $\overline{M_{x,y}\alpha_g(x)}$, hence $\alpha_g(x)$ left-divides gq. $\square$

Proposition 32. *Suppose x is a primitive integer quaternion of norm $q \neq p$ in A_q. Then $\langle A_p, x\rangle = \langle A_p, A_q\rangle = \mathcal{H}_{\mathbb{Z}[1/pq]}$.*

Proof. Let y be any element of A_q. By Theorem 9, A_p generates a subgroup whose closure contains $\mathsf{PSL}_2(\mathbb{Z}_q)$, so $\langle A_p\rangle \pmod q$ acts transitively on $\mathbb{F}_q\mathbb{P}^1$. So we can find $g \in \langle A_p\rangle$ with $M_{x,y}(g)\pi(x) = \pi(y)$, i.e., $\alpha_g(x) = y$. Thus since $x \mid_L x$ and $N(g) = p^k$ is coprime to $N(x) = q$, by Lemma 5 we have $\alpha_g(x) = y \mid_L gx$. So $gx = yh$ for some integer quaternion h. Since gx has norm p^kq and y has norm q, we must have $N(h) = p^k$. But this implies that h is a product of k quaternions of norm p, hence $h \in \langle A_p\rangle$. Since $g, h \in \langle A_p\rangle$, we have $gxh^{-1} \in \langle A_p, x\rangle$, but this means $y \in \langle A_p, x\rangle$.

This shows that $A_q \subseteq \langle A_p, x\rangle$, hence we must have $\langle A_p, x\rangle = \langle A_p, A_q\rangle$, which is precisely $\mathcal{H}_{\mathbb{Z}[1/pq]}$. $\square$

Proposition 33. *Suppose $x \neq 1$ is a primitive integer quaternion of norm q^k. Then $\langle A_p, x\rangle = \mathcal{H}_{\mathbb{Z}[1/pq]}$ if k is odd, and is an index two subgroup when k is even.*

Proof. Write $x = x_1 \dots x_k$. The element x acts on the q-adic tree by moving the origin to a vertex at distance k. The left action of $\mathsf{PSL}_2(\mathbb{Z}_q)$ is transitive on the vertices at distance k, and so for any other word $y_1 \dots y_k$ of length k in A_q, there is an element $g \in \mathsf{PSL}_2(\mathbb{Z}_q)$ with $gy\mathsf{PSL}_2(\mathbb{Z}_q) = x\mathsf{PSL}_2(\mathbb{Z}_q)$. Since $\langle A_p \rangle$ is dense in $\mathsf{PSL}_2(\mathbb{Z}_q)$, we can choose g from this group.

So in particular, there exists some $g \in \langle A_p \rangle$ so that $gx_1 \dots x_k = x_1 \dots x_{n-1} x'_k g'$ for some $x'_k \neq x_k \in A_q$ and $g' \in \langle A_p \rangle$. Then $x^{-1} g x (g')^{-1} = x_k^{-1} x'_k$. We have shown that $\langle A_p, x \rangle$ contains an element of q-length 2, and indeed all elements of length 2. If k is odd, this group contains $(x_{k-2}x_{k-1})^{-2} \dots (x_1 x_2)^{-1} x = x_{k+1}$, an element of norm q. Hence by Proposition 32, $\langle A_p, x \rangle = \mathcal{H}_{\mathbb{Z}[1/pq]}$ when k is odd. □

Theorem 20. *If $G \leq \mathcal{H}_{\mathbb{Q}}$ is finitely generated and contains A_p for any p, G has finite index in $\mathcal{H}_{\mathbb{Z}[1/p,S^{-1}]}$ for some finite set of primes S. In fact, G is a normal subgroup with quotient $\mathbb{Z}/2\mathbb{Z}^{|T|}$, for some $T \subseteq S$.*

Proof. Let $x \in G$, and abusing notation, identify x with a primitive integral representative. We will show that for any prime q dividing the norm of x, there is a quaternion of norm q^k in G. Then the proof of Proposition 33 implies there is a quaternion of norm q or q^2 in G, and hence that G contains an index 2 subgroup of $\mathcal{H}_{\mathbb{Z}[1/pq]}$. Since G possesses a finite generating set, there is a finite set S of primes which occur as divisors of elements in a generating set.

Write $x = x_1 \dots x_n$, where $x_1, \dots, x_k$ have norm p, and the remaining do not have norm p. By left-multiplying by elements of A_p, $\overline{(x_1 \dots x_k)} x \in G$ and scaling by p^{-k}, we obtain the primitive integral representative of this element, $x_{k+1} \dots x_n \in G$. So we can assume that p does not divide the norm of x.

Write $x = yz$, where the norm of y is a power of q and z is coprime to both p and q. By strong approximation (Theorem 9), there is some $g \in \langle A_p \rangle$ so that $gyz = yz'g'$ with $z \neq z'$, $N(z) = N(z')$ and $g' \in \langle A_p \rangle$. Consequently, $x^{-1} g x g'^{-1} = z^{-1} z' \in G$, and $z^{-1} z'$ has strictly fewer primes dividing its norm. So by repeating this argument, we may assume that the norm of x is the power of a single prime.

In particular, if we let S denote the finite set of primes which divide the norms of primitive integral representatives for generators of G, we have shown that G contains an element of norm q^k for each $q \in S$. Then Proposition 33 implies that G contains an element of norm q or q^2, for each $q \in S$. Setting T to be the set of primes for which G only contains elements of norm q^2, we have shown that G has index $2^{|T|}$ in $\mathcal{H}_{\mathbb{Z}[1/p,S^{-1}]}$. □

Thus, $\langle 1+i+j, 1+i-j, 6749+4573i+5569j+742k \rangle = \mathcal{H}_{\mathbb{Z}[1/3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29]}$, a lattice in a product of 8 trees.

Remark. In fact, even if G is infinitely generated, it will contain the kernel of $\det\colon \mathcal{H}_{\mathbb{Z}[1/S]} \to \mathbb{Z}[1/S]^*/\mathbb{Z}[1/S]^2 \cong ((\mathbb{Z}/2\mathbb{Z})^S$. However, if S is infinite it is possible that this kernel is infinite index in $\mathcal{H}_{\mathbb{Z}[1/S]}$.

We remark that these results are similar to a result of Venkataramana. In [51], it is assumed that the $\mathbb{R}$-rank of G is nonzero, whereas in the present case $G(\mathbb{R})$ is compact.

The preceding proposition implies that the poset of subgroups $H \leq \mathcal{H}_{\mathbb{Q}}$ which contain $\mathcal{H}_{\mathbb{Z}[1/p]}$ is essentially the poset of sets of primes containing p.

By assuming merely that G contains a finite index subgroup of $\mathcal{H}_{\mathbb{Z}[1/p]}$ one can still prove that it must have finite index in one of the lattice groups. In light of the following proposition, we speculate that an arbitrary group contains a lattice in one factor might be satisfied very easily.

Definition 10. Suppose a group G acts on a tree T containing a translation, and let $x \in T$. The *limit set* $\Lambda(G)$ is the closure of the orbit Gx in $\overline{T} = T \cup \partial T$, intersected with ∂T.

Proposition 34. *If $|\Lambda(G)| \geq 3$, the limit set is the smallest nonempty G-invariant closed subset of ∂T.*

Proof. Let B denote the smallest nonempty G-invariant closed subset of ∂T. Since G contains a translation g, the set $\{g^n x \mid n \in \mathbb{Z}\}$ is an infinite discrete subset of T. But since $\overline{T}$ is compact, it must have a limit point which is necessarily in the boundary. So $\Lambda(G)$ is nonempty. It is closed since it is an intersection of closed sets, and it is G-invariant because the orbit Gx is G-invariant. Thus, $B \subseteq \Lambda(G)$.

Suppose $g \in G$ acts on T as a translation. Since every point in ∂T which is not the repelling fixed point of g is eventually mapped arbitrarily close to the attracting fixed point $\partial_+ g$ of g, $\partial_+ g$ lies in any nonempty G-invariant closed subset of the boundary. Since $|\Lambda(G)| \geq 3$, there is a point y which is not $\partial_- g$, and hence $g^n y \to \partial_+ g$. This implies that every point in $\Lambda(G)$ must lie in any nonempty G-invariant closed subset of ∂T. $\square$

Lemma 6. *If $H \leq G$, the limit set of H is contained in the limit set of G. They coincide if $[G : H] < \infty$.*

Proof. Since the limit set of G is closed H-invariant set, the smallest H-invariant closed subset must be contained in $\Lambda(G)$.

If H has finite index in G, there is a bounded set K in T so that the H orbit of K contains any G orbit. This implies that the accumulation set of H in the boundary agrees with the accumulation set of G, and hence $\Lambda(H) = \Lambda(H)$. $\square$

The following proposition generalizes a well-known result about limit sets of normal subgroups.

Proposition 35. *If H is an infinite subgroup of G which is commensurated by G, $\Lambda(H) = \Lambda(G)$.*

Proof. Note that $\Lambda(H)$ is G-invariant, since $g\Lambda(H) = \Lambda(gHg^{-1})$. Since $gHg^{-1} \cap H$ has finite index in each of gHg^{-1} and H, we obtain that $\Lambda(gHg^{-1}) = \Lambda(H)$, and so $\Lambda(H)$ is G-invariant. Thus $\Lambda(G) \subseteq \Lambda(H)$, but the other containment is clear. $\square$

Proposition 36. *Suppose $\Gamma \leq \mathcal{H}_{\mathbb{Q}}$ is nonabelian and nonprimary. Let p be a prime for which $\Gamma_{(p)} = \Gamma \cap \mathcal{H}_{\mathbb{Z}[1/p]}$ is infinite. Then the limit set $\Lambda(\Gamma_{(p)})$ is the entire boundary ∂T_p.*

Proof. Since Γ is nonabelian and nonprimary, Γ embeds in $\mathsf{PSL}_2(\mathbb{Q}_p)$ as a dense subgroup. Since $\Gamma_{(p)}$ is infinite, it has a nonempty limit set in T_p. The limit set of Γ is ∂T_p because it is dense. But by Proposition 35, the limit set of $\Gamma_{(p)}$ is ∂T_p.

By strong approximation, Γ embeds in $\mathsf{PSL}_2(\mathbb{Q}_p)$ as a dense subgroup. The group $\Gamma_{(p)}$ is noncompact and thus contains a translation g and a nontrivial limit set. But since Γ commensurates $\Gamma_{(p)}$, there is an element $\gamma \in \Gamma$ which moves an endpoint of g arbitrarily close to any desired point y in the boundary. But then $\gamma g \gamma^{-1}$ has a boundary point arbitrarily close to y. Thus the limit set is dense in ∂T_p. $\square$

Remark. So it is possible that there are exotic (that is, non-arithmetic, but higher rank and Zariski dense) subgroups of $\mathsf{SO}_3(\mathbb{Q})$, but such groups must have $\Gamma_{(p)}$ an infinite covolume subgroup which still manages to move toward every end. In particular, the action on the product of trees admits no proper convex invariant subset.

Theorem 21. *If Γ is as above, Γ is convex cocompact if and only if Γ is S-arithmetic.*

Proof. The only thing to observe is that the limit set of Γ is the full boundary, and thus the convex hull of an orbit of Γ is the full space. So Γ can only act on its convex hull cocompactly if it acts on the entire space cocompactly, or in other words, is commensurable with a lattice. $\square$

When p, q are two odd primes, the group $\mathcal{H}_{\mathbb{Z}[1/pq]}$ acts on a product of a $(p+1)$ and a $(q+1)$-regular tree, and the quotient is a 1-vertex square complex. This complex has two connected hyperplanes, a vertical and a horizontal one. Under an identification of an oriented edge with a point in $\mathbb{P}^1\mathbb{F}_q$, the vertical hyperplane is a $(p+1)$-regular graph whose vertex set corresponds to $\mathbb{P}^1\mathbb{F}_q$. Each element of A_p is invertible modulo q, and hence determines an element of $\mathsf{PGL}_2(\mathbb{F}_q)$. But since this gives a permutation of $\mathbb{P}^1\mathbb{F}_q$, we can draw the cycle graph of the permutation for each of the $p+1$ generators (which come in conjugate pairs). The symmetric argument in p, q determines the horizontal hyperplane. The graph $Y^{p,q}$ is the quotient of the $p+1$-regular tree by the index $q+1$ subgroup of $\langle A_p \rangle$ which is upper triangular mod q.

This is precisely the graph $Y^{p,q}$ of Theorem 4.4 in [49], in which it is proved that this graph is connected and Ramanujan. Since $Y^{p,q}$ is $(q+1)$-regular, it has $q+1$ as an eigenvalue (and $-(q+1)$ if $Y^{p,q}$ is bipartite). Denoting $\lambda(Y^{p,q})$ as its second largest eigenvalue in absolute value, $Y^{p,q}$ is called *Ramanaujan* if $\lambda(Y^{p,q}) \leq 2\sqrt{q}$.

That these are Ramanujan implies that they can be "navigated efficiently." If p is large compared to q, it must have fairly large diameter, because there are at most $(q+1)q^{k-1}$ vertices in a ball of radius k. But the Ramanujan property implies that the diameter is not too large, and is in fact at most $2\log_p(q) + 3$.

The subgroup conjecture

We thank Richard Schwartz for suggesting the following terminology and Andrei Rapinchuk for helpful comments on the conjecture.

Definition 11. Say $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ is *primary* if Γ is discrete in $\mathsf{SO}_3(\mathbb{Q}_p)$ for some p. Call a subgroup $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ *full* if it is not primary or abelian. A subgroup is full if and only if its Zariski closure is $\prod_{p\in S} \mathsf{PGL}_2(\mathbb{Q}_p)$ for some set S of at least two odd primes.

Note that $\langle 1+2i, 1+2j, 1+2k\rangle$ is primary, and so is any subgroup of it. However, one can check that $\langle (3+2i)(1+2i), 1+2j, 1+2k\rangle$ also has ping-pong sets in its action on the 5-adic tree, hence is a 5-primary subgroup not contained in $\mathsf{SO}_3(\mathbb{Z}[1/5])$. Note also that $\langle 1+8i, 1+8j, 1+8k\rangle$ is both 5-primary and 13-primary (and is in fact a thin subgroup of $\mathsf{PGL}_2(\mathbb{Q}_5) \times \mathsf{PGL}_2(\mathbb{Q}_{13})$). Other examples of this sort may be obtained by taking the group generated by $\langle a+bi, a+bj\rangle$, where a^2+b^2 is composite.

Note that by choosing a nonzero rational axis α, the group of rotations about α gives an abelian subgroup of $\mathsf{SO}_3(\mathbb{Q})$, conjugate in $\mathsf{SO}_3(\mathbb{R})$ to $\mathsf{SO}_2(\mathbb{Q})$.

Conjecture 1. Suppose $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ is a finitely generated subgroup. Then either

(i) Γ is abelian

(ii) Γ is primary

(iii) Γ is conjugate to a finite index subgroup of $\mathsf{SO}_3(A)$, for A a subring of $\mathbb{Q}$.

Stated another way, *there are no higher rank thin groups in* $\mathsf{SO}_3(\mathbb{Q})$.

Note that in each case, Γ is a lattice in a product of trees. In the first case, the trees are 2-regular, and in the second Γ is virtually free and a lattice in a single tree. In particular, the finitely generated subgroups are finitely presented, which is to say that $\mathsf{SO}_3(\mathbb{Q})$ is coherent. We also remark that the finite generation assumption is somewhat superficial, and one might expect arbitrary full subgroups of $\mathsf{SO}_3(\mathbb{Q})$ to be *compact index* rather than finite index.

Proposition 37. *A full subgroup* $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ *is almost dense in* $\mathsf{SO}_3(\mathbb{Q}_p) \cong \mathsf{PGL}_2(\mathbb{Q}_p)$ *for some* p.

Proof. It suffices to show that there is some p so that Γ is not solvable, and is unbounded and indiscrete in $\mathsf{PSL}_2(\mathbb{Q}_p)$. Then Proposition 26 implies that the closure of Γ contains $\mathsf{PSL}_2(\mathbb{Q}_p)$.

Let p be any prime in which $\Gamma \leq \mathsf{PGL}_2(\mathbb{Q}_p)$ is unbounded (if none exists, Γ is finite hence primary for every p). Since Γ is not primary, it is not discrete in $\mathsf{PGL}_2(\mathbb{Q}_p)$. Finally, the only solvable subgroups of $\mathsf{SO}_3(\mathbb{Q})$ are abelian, and since Γ is full, it is not abelian.

□

Recall that a property of groups is called *geometric* if it is preserved under quasi-isometries.

Proposition 38. *The subgroup conjecture implies that* $\mathsf{SO}_3(\mathbb{Q})$ *is coherent, which implies that coherence is not a geometric property.*

Proof. Note that for a finite set S of at least two odd primes, $\mathsf{SO}_3(\mathbb{Z}[1/S])$ is a lattice in a product of trees, and the subgroup conjecture implies that any finitely generated subgroup of it is finitely presented.

Since $\mathsf{SO}_3(\mathbb{Z}[1/S])$ is quasi-isometric to a product of free groups, and direct products of free groups are not coherent, this implies that coherence is not preserved under quasi-isometry. $\square$

We also remark that the subgroup conjecture holds if $\mathsf{SO}_3(\mathbb{Q})$ has the *finitely generated intersection* or *Howson* property, that the intersection of finitely generated subgroups is again finitely generated. This is because, if Γ is a finitely generated full subgroup of $\mathsf{SO}_3(\mathbb{Z}[1/n])$, then $\Gamma_p = \Gamma \cap \mathsf{SO}_3(\mathbb{Z}[1/p])$ is an intersection of finitely generated subgroups which has full limit set. The Howson property would imply that Γ_p is finitely generated, which would necessarily be a lattice because it has full limit set. But then Proposition 49 implies that Γ has finite index in $\mathsf{SO}_3(\mathbb{Z}[1/n])$, which would establish the subgroup conjecture.

Conjecture 2 (Serre [60] p. 734)**.** $\mathsf{SO}_3(\mathbb{Z}[1/n])$ has the congruence subgroup property if n is divisible by at least two odd primes.

If both of these conjectures hold, we not only would classify subgroups of $\mathsf{SO}_3(\mathbb{Q})$ up to commensurability, but we would obtain very strong restrictions on the possible finite index subgroups of $\mathsf{SO}_3(A)$.

Geometric Rigidity

Fix a set S of at least two odd primes, and let $\mathbb{G} = \prod_{p \in S} \mathsf{PGL}_2(\mathbb{Q}_p)$. For an element $g \in \mathbb{G}$, let $\ell(g) = (\ell_p(g_p))_{p \in S}$, where $\ell_p(g_p)$ is the translation length of g_p in the p-adic tree. Here ℓ is the relevant Cartan projection, which takes values in a Weyl chamber $\mathbb{N}^{(S)}$.

Suppose $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ is a full subgroup of $\mathbb{G} = \prod_{p \in S} \mathsf{PGL}_2(\mathbb{Q}_p)$. We say that Γ is *geometrically rigid* if any representation $\rho \colon \Gamma \to \mathbb{G}$ with $\ell(\rho(\gamma)) = \ell(\gamma)$ for every $\gamma \in \Gamma$ is conjugate to the identity. We ask whether full subgroups of $\mathsf{SO}_3(\mathbb{Q})$ may be geometrically rigid.

Definition 12. An action of a group G on a tree T is *minimal* if T admits no proper G-invariant subtree. If $x \in T$, we define the *translation distance function (based at x)* as $\tau_x \colon G \to \mathbb{N}$, with $\tau_x(g) = d(x, gx)$. The *translation length function* of the action $\ell \colon G \to \mathbb{N}$ is defined by $\ell(g) = \inf_{x \in T^{(0)}} \tau_x(g)$.

If $G \leq \mathrm{Aut}(T)$, a *geometric representation* of G is $\rho \colon G \to \mathrm{Aut}(T)$ so that $\ell(g) = \ell(\rho(g))$ for all $g \in G$. An action of a group G on a product of trees X is minimal if X admits no proper G-invariant convex subcomplex, and a geometric representation of G on a product of trees is an action with the same length function for each tree.

Proposition 39. *Suppose $\ell \colon G \to \mathbb{N}$ is a translation distance function on a group.*

Theorem 22. *Suppose $G \leq Aut(T)$ is a minimal action on a tree T, and suppose $\rho \colon G \to Aut(T)$ is a geometric representation. Then there is a unique $\alpha \in Aut(T)$ so that ρ is induced by conjugation by α.*

Proof. Chiswell (Theorem 1, [23]) proves that if $\|\cdot\|\colon G \to \mathbb{N}$ is a translation distance function, there exists a tree X and an action of G on X whose translation distance function is $\|\cdot\|$. Moreover, if $\|\cdot\|$ came from an action on a tree T, there exists a G-equivariant map $X \to T$. Parry (Main Theorem, [63]) proves the same statement for a translation length function $\ell\colon G \to \mathbb{N}$. Thus, given a translation length function on a group, one can construct an action on a tree T with the given translation lengths, but it is easy to obtain the translation distance function with respect to any vertex in that tree T. This implies that translation length and translation distance functions are equivalent.

Because the map is G-equivariant, the image of X in T must be a nonempty invariant subtree of T. But since the G-action on T is minimal, the map is surjective, hence an isomorphism. Thus we have shown that any two minimal actions with the same length function are isomorphic, and hence if the actions are on the same tree, they are conjugate in $\mathrm{Aut}(T)$.

That α is unique follows from the fact that $\mathrm{Aut}(T)$ has trivial center and the action is minimal, for any other α' must commute with the action of G. □

Proposition 40. *Suppose $G \leq Aut(X)$ is a minimal action on a product of non-isomorphic trees X, and $\rho\colon G \to Aut(X)$ is a geometric representation. Then ρ is induced by conjugation by a unique element $\alpha \in Aut(X)$.*

Proof. For each factor tree T of X, the projection to T determines an action G on T and a geometric representation of G to T. Hence by the preceding theorem, the two actions are conjugate for each factor.

The assumption that the trees involved are non-isomorphic is only necessary to conclude that α is unique, or else it is possible that exchanging factors gives additional conjugators. □

Note that since $\mathsf{PGL}_2(\mathbb{Q}_p)$ acts on the $(p+1)$-regular tree T_{p+1}, it embeds in $\mathrm{Aut}(T_{p+1})$ in a natural way.

Lemma 7. *The outer automorphism group of $\mathsf{PGL}_2(\mathbb{Q}_p)$ is trivial.*

Proof. Loo-Keng Hua proves (Theorems 1 and 3 in an appendix to Dieudonné's paper [29]) that for a skew field k, every outer automorphism of $\mathsf{PGL}_2(k)$ is induced by an automorphism of k, and so it suffices to show there are no nontrivial automorphisms of $\mathbb{Q}_p$.

Since a field automorphism must send 1 to 1, the automorphism must be the identity on the dense subfield $\mathbb{Q}$. Say that $x \in \mathbb{Q}_p$ is *infinitely divisible* if there are infinitely many natural numbers n so that $y^n = x$ has a solution in $\mathbb{Q}_p$. It follows directly from Hensel's lemma (see Lemma 6.8 in [70]) that x^{p-1} is infinitely divisible if and only if x is a p-adic unit. Since an automorphism of $\mathbb{Q}_p$ must preserve infinite divisibility, we note that an automorphism of $\mathbb{Q}_p$ must map $\mathbb{Z}_p^\times$ precisely to $\mathbb{Z}_p^\times$. Now writing any $y \in \mathbb{Q}_p$ as $p^n x$ for $x \in \mathbb{Z}_p$, we observe that $\sigma(p^n x) = \sigma(p^n)\sigma(x) = p^n\sigma(x)$, and so σ preserves the p-adic valuation. It follows that σ is continuous, and since it is the identity on a dense subset, σ is the identity.

Thus it follows that $\mathsf{PGL}_2(\mathbb{Q}_p)$ has trivial outer automorphism group. □

Proposition 41. *The normalizer of $\mathsf{PGL}_2(\mathbb{Q}_p)$ in $Aut(T_{p+1})$ is $\mathsf{PGL}_2(\mathbb{Q}_p)$.*

Proof. Note that if $h \in \mathrm{Aut}(T_{p+1})$ normalizes $\mathsf{PGL}_2(\mathbb{Q}_p)$, h induces an automorphism of $\mathsf{PGL}_2(\mathbb{Q}_p)$. By Lemma 7, this automorphism must be inner, and replacing h with hg for some $g \in \mathsf{PGL}_2(\mathbb{Q}_p)$, we can assume h commutes with $\mathsf{PGL}_2(\mathbb{Q}_p)$. But since $\mathsf{PGL}_2(\mathbb{Q}_p)$ acts minimally on the tree, it has trivial centralizer. $\square$

Suppose $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ is full in $\mathsf{SO}_3(\mathbb{Z}[1/n])$. Let $\mathbb{G} = \prod_{p|n} \mathsf{PGL}_2(\mathbb{Q}_p)$, and X the product of the p-adic trees, for p dividing n.

Theorem 23. *If $\rho\colon \Gamma \to \mathbb{G}$ is a geometric representation of a full group, then ρ is conjugate to the identity in $\mathbb{G}$.*

Proof. By Proposition 40, there is a unique $\alpha \in \mathrm{Aut}(X)$ with $\rho = c_\alpha|_\Gamma$, where c_α denotes conjugation by α. Since the factors of X are pairwise non-isometric, α preserves the factors and decomposes as $(\alpha_p)_{p|n}$. Let $\pi_p\colon \mathbb{G} \to \mathsf{PGL}_2(\mathbb{Q}_p)$ be the projection, and $\rho_p = \pi_p \circ \rho$. Under the inclusion, Γ is dense in $\mathsf{PGL}_2(\mathbb{Q}_p)$ by Proposition 37, and ρ_p is another dense embedding of Γ in $\mathsf{PGL}_2(\mathbb{Q}_p)$, which is conjugate by α_p. Since c_{α_p} is continuous, we have

$$c_{\alpha_p}(\mathsf{PGL}_2(\mathbb{Q}_p)) = c_{\alpha_p}(\overline{\Gamma}) = \overline{c_{\alpha_p}(\Gamma)} = \overline{\rho_p(\Gamma)} = \mathsf{PGL}_2(\mathbb{Q}_p).$$

Thus α_p normalizes $\mathsf{PGL}_2(\mathbb{Q}_p)$, hence by Proposition 41, $\alpha_p \in \mathsf{PGL}_2(\mathbb{Q}_p)$. Since this holds for each α_p, we conclude that $\alpha \in \mathbb{G}$. $\square$

Thus, we have shown that full subgroups of $\mathsf{SO}_3(\mathbb{Q})$ are geometrically rigid. As a brief comparison, not even lattices in $\mathsf{PGL}_2(\mathbb{Q}_p)$ satisfy this form of rigidity. There is typically a high-dimensional space of p-adic structures on a given $(p+1)$-regular graph, while any two such structures are conjugate in $\mathrm{Aut}(T)$.

Quaternions over a number field

The quaternion algebra over a field can be thought of as a non-abelian analog of a field extension. As such, we can consider its Galois theory. The primary distinction in this context is that a polynomial may have many more roots over a noncommutative field.

For example, if $\mathbb{H}$ is the algebra of real quaternions, the solution set to x^2+1 is an entire 2-sphere! Note that many automorphisms of $\mathbb{H}$ can be produced by conjugating by a nonzero quaternion q. Note that the plane spanned by $\{1, q\}$ commutes with q, and so conjugation by q will fix this plane, and rotate an orthogonal plane. For example, conjugating $a+bi+cj+dk$ by j yields

$$j(a+bi+cj+dk)(-j) = a - bi + cj - dk$$

effectively negating i and fixing j.

If A is an algebra over a field k, every unit $a \in A$ induces an *inner automorphism* $c_a\colon A \to A$ defined by $c_a(b) = aba^{-1}$. The center $Z(A)$ will be fixed by each inner automorphism. The outer automorphism group of A/k (namely, $\mathrm{Aut}(A/k)/\mathrm{Inn}(A/k)$) serves as a reasonable notion of a Galois group for the algebra A. Note that the Galois group of $Z(A)/k$ will embed in $\mathrm{Out}(A/k)$, because a Galois automorphism of the field $Z(A)$ cannot be induced by conjugation. Galois cohomology allows one to compute this in general. In the case of a quaternion algebra, the outer automorphism group will also contain the kernel of the adjoint representation.

Suppose $\mathbb{Q} \subseteq k \subseteq \mathbb{Q}_p$ is a number field equipped with a p-adic embedding, and suppose $\alpha \in k$ is p-nonintegral but all of its Galois conjugates are p-integral. For example, fixing the embedding $\mathbb{Q}(\sqrt{3}) \to \mathbb{Q}_{13}$ where $\sqrt{3} \equiv 4 \pmod{13}$, the number $\alpha = \frac{5-2\sqrt{3}}{13}$ is not integral, but its Galois conjugate is. These are roots of the polynomial $13\lambda^2 - 10\lambda + 1$.

Commensurability for lattices in trees

We emphasize a connection to the non-archimedean case. Arithmetic Fuchsian groups are commensurable in $\mathsf{PSL}_2(\mathbb{R})$ if and only if they have the same invariant quaternion algebra, but they are commensurable as abstract groups if and only if they are both uniform or both nonuniform. However, arithmetic Kleinian groups are determined up to commensurability by their invariant quaternion algebra.

In the p-adic case, commensurability for arithmetic lattices in $\mathsf{PSL}_2(\mathbb{Q}_p)$ is equivalent to having isomorphic invariant quaternion algebras [53], but any two lattices in any two locally finite trees with at least 3 ends are commensurable as abstract groups! The finer question of whether they are commensurable in the automorphism group of their universal cover is then determined if and only if they have isomorphic universal covers, according to Leighton's theorem:

Theorem 24. *[45] Suppose G and G' are finite graphs. Then G and G' have a common finite cover if and only if they have the same universal cover.*

We might say that $\Gamma_1, \Gamma_2 \leq \mathsf{PSL}_2(\mathbb{Q}_p)$ are *algebraically commensurable* if they are commensurable in $\mathsf{PSL}_2(\mathbb{Q}_p)$, *geometrically commensurable* if they are commensurable in $Aut(T_{p+1})$, and *abstractly commensurable* if they have finite index subgroups which are isomorphic. In the case of lattices in $\mathsf{PSL}_2(\mathbb{Q}_p) \times \mathsf{PSL}_2(\mathbb{Q}_q)$, we are interested in when it is possible for lattices to be algebraically, geometrically, and abstractly commensurable.

3.5 Homology of Quaternion Lattices

The quaternion algebra $\mathbb{Q}\{i,j\}$ is *ramified* at $\{2,\infty\}$; this means that $\mathbb{Q}\{i,j\} \otimes \mathbb{Q}_2$ and $\mathbb{Q}\{i,j\} \otimes \mathbb{R}$ are division algebras, because there are no nonzero solutions to $a^2+b^2+c^2+d^2=0$ in these fields.

Transfinite nilpotence

We show that pro-p groups are transfinitely nilpotent (hence transfinitely solvable).

Definition 13. Let G be a group. Let $G_0 = G$, and inductively define $G_{i+1} = [G, G_i]$ for successor ordinals and $G_\lambda = \cap_{\alpha<\lambda} G_\alpha$ for limit ordinals. G is α-transfinitely nilpotent if $G_\alpha = 1$ for some α.

We show that principal congruence subgroups of linear groups are transfinitely nilpotent.

Proposition 42. *Let k be a number field, and suppose $\Gamma \leq \mathsf{GL}_n(k)$ is finitely generated. Let R be the ring generated by its matrix entries, and $P \trianglelefteq R$ a proper ideal. Suppose the reduction $\Gamma \to \mathsf{GL}_n(R/P)$ has trivial image. Then Γ is transfinitely nilpotent.*

Proof. By assumption, $\Gamma \leq I + PM_n(R)$. We will show that $\Gamma_i \leq I + P^{i+1}M_n(R)$ by induction. Note that if $x = I + P^iX + O(P^{i+1})$, then $x^{-1} = I - P^iX + O(P^{i+1})$. So we have

$$[x,y] = (I + PX + O(P^2))(I + P^iY + O(P^{i+1}))(I - PX + O(P^2))(I - P^iY + O(P^{i+1})$$

and hence $[x,y] \in I + P^{i+1}M_n(R)$.

Now since $\cap_{n\geq 0}P^n = 0$, we must have $\cap_{n\geq 0}\Gamma_n = 1$, and Γ is transfinitely nilpotent. ☐

Corollary 6. *A finitely generated linear group is virtually transfinitely nilpotent.*

Proof. Let $\Gamma \leq \mathsf{GL}_n(k)$ be a finitely generated subgroup, and let R be the ring generated by the entries of elements of entries of Γ. Since this ring is finitely generated, it has a nontrivial proper ideal $I \trianglelefteq R$.

A prime with good reduction must exist as before, and we can take the finite index subgroup which is the kernel of reduction mod p to obtain a finite index subgroup in which we can apply the preceding proposition. ☐

Corollary 7. *If Γ is a finitely generated linear group which admits a solvable congruence quotient, Γ is transfinitely solvable.*

Proof. Since Γ has a solvable congruence quotient, the derived series is contained in a principle congruence subgroup at some finite stage. Then this subgroup is transfinitely nilpotent, hence transfinitely solvable. Since a (finite) term in the derived series is transfinitely solvable, the full group must be. ☐

Homology of $\mathsf{SO}_3(\mathbb{Q})$

The fact that $\mathsf{SO}_3(\mathbb{R})$ is compact means this group is ramified at the infinite place, and Hilbert reciprocity implies that there is an odd number of finite places for which $\mathsf{SO}_3(\mathbb{Q}_p)$ ramifies. It turns out that $\mathsf{SO}_3(\mathbb{Q}_2)$ is the unique ramified place. In particular, there is a reduction map $\mathsf{SO}_3(\mathbb{Q}) \to \mathsf{SO}_3(\mathbb{Z}_2/2^n\mathbb{Z}_2)$.

Let S be the preimage of $\{I_3, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}\}$ under the mod 4 reduction map. This corresponds to the image of the projective odd primitive quaternions which are $1+i+j \pmod 2$.

Proposition 43. $\mathsf{SO}_3(\mathbb{R})$ *is perfect.*

Proof. If α is a nontrivial rotation of the two-sphere, there is a rotation a that goes through half of the angle. Let b be the π-rotation that exchanges the fixed points of a. Then b conjugates a to its inverse, and so $[a,b] = a(ba^{-1}b^{-1}) = aa = \alpha$, and so every element of $\mathsf{SO}_3(\mathbb{R})$ is a commutator. ☐

In fact, $\mathsf{SO}_3(\mathbb{R})$ is even simple, but we would like to highlight the fact that $\mathsf{SO}_3(\mathbb{Q})$ is not even perfect, since we do not always have square roots in $\mathsf{SO}_3(\mathbb{Q})$.

Proposition 44. *There is a surjection $\mathcal{H}_{\mathbb{Q}} \to \bigoplus_{p \text{ odd prime}} \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Q}^*/(\mathbb{Q}^*)^2$.*

Proof. Let $\mathcal{O}$ denote the set of odd primes, and consider the function $\sigma\colon \mathbb{N} \to \mathbb{Z}/2\mathbb{Z}^{(\mathcal{O})}$ defined by $\sigma(n)_p = v_p(n) \pmod 2$. Then $\sigma \circ N\colon \mathcal{H}_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}^{(\mathcal{O})}$ is the desired homomorphism, where N is the norm on $\mathcal{H}_{\mathbb{Q}}$. ☐

Abelian subgroups of quaternion lattices

Lemma 8. *Two non-identity rotations of the 2-sphere commute if and only if they share an axis, or are π rotations between orthogonal axes.*

Proof. It is clear that the rotations described commute, so we suppose that α and β are commuting rotations. Since α is a rotation, it has exactly two fixed points on the sphere, and if $\alpha = \beta\alpha\beta^{-1}$, β must preserve the fixed point set of α. So either β has the same axis, or it interchanges the two fixed points, which implies that β is a π-rotation orthogonal to α. ☐

Corollary 8. *Every abelian subgroup of $\mathsf{SO}_3(\mathbb{R})$ has a fixed point in $\mathbb{RP}^2$.*

Definition 14. A *pure imaginary axis* in $\mathbb{H}$ is a 1-dimensional subspace orthogonal to $\mathbb{R}$.

Proposition 45. *Maximal commutative subalgebras of $\mathbb{H}$ are in 1-to-1 correspondence with pure imaginary axes.*

Proof. If A is a 1-dimensional commutative subalgebra, $\operatorname{span}\{A, 1\}$ is still commutative, and is 2-dimensional unless $A = \mathbb{R}$. Otherwise, $\mathbb{C} = \operatorname{span}\{1, i\}$ properly contains A. If S is a maximal commutative subalgebra of dimension at least 3, it contains two pure imaginary quaternions α, β which are linearly independent. But then $\alpha\beta - \beta\alpha \in S$ is a pure imaginary quaternion orthogonal to both α and β, so $S = \mathbb{H}$. ☐

Definition 15. If $\alpha = [a : b : c] \in \mathbb{QP}^2$ is an axis, the *height* $h(\alpha)$ of α is the norm of a primitive integral representative. For example, $h(3/5, 4/5, 0) = 25$, $h(1/6, -1/3, 1/3) = 9$, and $h(1, 1, 2) = 6$. The height determines the covolume of $\mathsf{Fix}(\alpha)$ on the cube complex it acts on.

In more general arithmetic groups, the number of maximal abelian groups up to commensurability will depend on class field theory invariants.

Bounded Generation and the Congruence Subgroup Property

We turn to the question of the congruence subgroup property.

If k is a field and $\mathcal{O}_S$ the ring of S-integers, any ideal I in $\mathcal{O}_S$ determines a homomorphism $\mathcal{O}_S \to \mathcal{O}_S/I$. For G an algebraic group defined over k, we obtain a map $G(\mathcal{O}_S) \to G(\mathcal{O}_S/I)$, a finite group when I has finite index. So the kernel of this map is a finite index subgroup of $G(\mathcal{O}_S)$, denoted $G(\mathcal{O}_S)[I]$. A *congruence subgroup* is one that contains the kernel of some congruence quotient map.

There are thus two natural profinite groups that contain $G(\mathcal{O}_S)$, and a map between them: $\widehat{G(\mathcal{O}_S)} \to G(\widehat{\mathcal{O}_S})$. The first is the profinite completion of $G(\mathcal{O}_S)$ (with respect to all of its finite index subgroupsx), and the second is called the *congruence completion*, which is the completion with respect to its congruence subgroups. The kernel of this map is called the congruence kernel.

$G(\mathcal{O}_S)$ is said to have the *congruence subgroup property* if the kernel of this map is finite. $\mathsf{PSL}_2(\mathbb{Z})$ fails to have the congruence subgroup property. In fact, every alternating group $A_n, n \geq 5$ is generated by an element of order two and an element of order 3, thus there is a surjective map $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \to A_n$. Simple congruence quotients are all $\mathsf{PSL}_2(\mathbb{F}_p)$, which are not isomorphic to alternating groups for $p \geq 7$. However, $\mathsf{SL}_2(\mathbb{Z}[S^{-1}])$ has the congruence subgroup property [73, Theoreme 1]. It is unknown if $\mathcal{H}_S$ has the congruence subgroup property for $|S| > 1$.

Example 5. Let $a = 1 + i + j$ and $b = 1 + i - j$, and $x = 1 + 2i$, $y = 1 + 2j$, $z = 1 + 2k$. We have

$$\mathcal{H}_{3,5} = \langle a, b, x, y, z \mid ax = zb^{-1}, ay = z^{-1}b, az = x^{-1}b^{-1}, bx^{-1} = y^{-1}b^{-1}, bz = ya, a^{-1}x = y^{-1}a\rangle$$

Let h_n denote the number of conjugacy classes of subgroups of $\mathcal{H}_{3,5}$ of index n. According to GAP (using "LowIndexSubgroupsFpGroup"), we have

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h_n	1	7	0	19	0	0	0	40	0	0	2	1	0	2	0

For a prime not equal to 2, 3, or 5, $\mathcal{H}_{3,5}$ acts transitively on $\mathbb{F}_p\mathbb{P}^1$, and the stabilizer of a point is an index $p + 1$ subgroup. The subgroups of index 11 are perhaps a bit mysterious, as 10 is not prime. However, there is an exceptional transitive action of $\mathsf{PSL}_2(\mathbb{F}_{11})$ on a set of 11 objects, which was observed by Galois in his letter to A. Chavallier [34], considered his final mathematical work. Typically, $\mathsf{PSL}_2(\mathbb{F}_p)$ only admits transitive actions on sets of size $p + 1$, but there happens to be an embedding $A_5 \leq \mathsf{PSL}_2(\mathbb{F}_{11})$, which has index 11, since $|\mathsf{PSL}_2(\mathbb{F}_{11})|/|A_5| = 11$. The preimage of A_5 under the surjection of $\mathcal{H}_{3,5} \to \mathsf{PSL}_2(\mathbb{F}_{11})$ is then an index 11 subgroup. These actions are not conjugate, but are related by an outer automorphism of $\mathsf{PSL}_2(\mathbb{F}_{11})$. The remaining subgroups in this table are indeed congruence subgroups.

Bounded Generation

A group G is said to be *boundedly generated* if there is a finite set $\{g_1, \ldots, g_n\}$ so that every element of G is of the form $g_1^{k_1} \ldots g_n^{k_n}$ for $k_i \in \mathbb{Z}$. In [25, Theorem 1.1], Cooke and Weinberger prove bounded generation for certain groups assuming the generalized Riemann Hypothesis. In [58, Theorem 1.1], the authors show that $\mathsf{SL}_2(\mathcal{O})$ has bounded generation when the group of units of $\mathcal{O}$ is infinite (including all algebraic subrings of $\mathbb{C}$ except for $\mathbb{Z}$ or

$\mathbb{Z}[\sqrt{-d}]$ for positive d). In [26, Theorem 1.3], the authors show that higher rank S-arithmetic subgroups of anisotropic algebraic groups are *not* boundedly generated, including the groups $\mathsf{SO}_3(\mathbb{Z}[1/n])$.

Following [25], we speculate whether GRH might be useful in obtaining other quantitative or finiteness statements regarding $\mathsf{SO}_3(\mathbb{Q})$. For example, assuming $\Gamma \leq \mathsf{SO}_3(\mathbb{Z}[1/n])$ is Zariski dense, for n composite, is Γ a lattice? We have shown that for each prime divisor of n, $\Gamma \cap \mathsf{SO}_3(\mathbb{Z}[1/p])$ has full limit set in the p-adic tree. If in fact this is because it is a lattice, then we have also shown that Γ has finite index in $\mathsf{SO}_3(\mathbb{Z}[1/n])$.

Reflection extensions of quaternion lattices

Recall the embedding $\mathbb{Q}\{i, j\} \to M_2(\mathbb{Q}(i))$ determined by $i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that the matrix $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has order 2. One can also check that conjugation by r fixes i (in other words, r and i commute) and sends j to $-j$. Thus conjugation by the "reflection" r induces the automorphism of $M_2(\mathbb{Q}(i))$ which is the Galois automorphism of $\mathbb{Q}(i)$.

Question 3. Does Scott's argument for right-angled reflection groups have an analog in products of trees? Are geometrically finite subgroups of rational quaternions separable?

Note that products of free groups are incoherent, and thus have finitely generated subgroups which are not convex cocompact. However, irreducible lattices in products of trees may have the property that finitely generated subgroups are convex cocompact, hence finitely presented. This would imply that $\mathsf{SO}_3(\mathbb{Q})$ is coherent.

3.6 Lattices in Products of Trees

Definition 16. A *tree* is a connected and simply connected graph. A *forest* is a union of trees, and a *jungle* is a product of trees.

We have the following important lemma, which we call the *stabilizer principle*:

Lemma 9. *Suppose G acts on a locally finite tree T, and let $v, w \in T$. Then $Stab_G(v)$ and $Stab_G(w)$ are commensurable.*

Proof. The stabilizer $\mathrm{Stab}_G(v)$ acts on the finite set of vertices at distance $d(v, w)$ from v, and so the subgroup which fixes w has finite index in $\mathrm{Stab}_G(v)$. But this is precisely the group $\mathrm{Stab}_G(v) \cap \mathrm{Stab}_G(w)$. ☐

This greatly contrasts with the analogous statement for, say, $\mathbb{H}^3$. If x, y are distinct points in $\mathbb{H}^3$, $\mathrm{Stab}(x)$ and $\mathrm{Stab}(y)$ are both copies of SU_2, but their intersection is SO_2, which is a 1-dimensional subgroup of a 3-dimensional group.

In a much more general context, we can prove a weaker statement. If G acts on a metric space X, the *ε-almost stabilizer* of x is $U_\varepsilon(x) = \{g \in G \mid d(gx, x) < \varepsilon\}$.

Lemma 10. *Suppose a topological group G acts properly on a proper metric space X by isometries, and let x, y be distinct points in X. Then for any $\varepsilon > 0$, $U_\varepsilon(x)$ is covered by finitely many translates of $U_\varepsilon(x) \cap U_\varepsilon(y)$*

Proof. Since X is proper, the $d(x, y)$-sphere about x is compact, so $\mathrm{Stab}(x)$ is a compact subgroup of G. The intersection $U^x_\varepsilon(y) = \mathrm{Stab}(x) \cap U_\varepsilon(y)$ is an open subset of $\mathrm{Stab}(x)$, and so $\cup_{k\in\mathrm{Stab}(x)} kU^x_\varepsilon(y)$ is an open cover of $\mathrm{Stab}(x)$, which must have a finite subcover. $\square$

The stabilizer principle follows from this more general fact because for $\varepsilon < 1$, a vertex in a tree is ε-stabilized if and only if it is a genuine fixed point. The stabilizer principle has the following corollary, which can also be deduced from the fact that $\mathsf{PGL}_2(\mathbb{Z}_p)$ is a compact open subgroup, and any two compact open subgroups are commensurable.

Corollary 9. $\mathsf{PGL}_2(\mathbb{Q}_p)$ *commensurates* $\mathsf{PGL}_2(\mathbb{Z}_p)$*.*

Chapter 4

Quaternions and Lattices in Products of Trees

4.1 The Margulis-Zimmer Conjecture

The Margulis-Zimmer conjecture was first stated in [75] as follows. Let $\mathbb{G}$ be a simply connected and absolutely simple algebraic group defined over a number field k. Fix a set of places S containing all of the Galois embeddings, and suppose that $\Gamma = \mathbb{G}(\mathcal{O}_S)$ has rank at least 2.

Conjecture 3 (Margulis-Zimmer)**.** If Λ is a subgroup of Γ which is commensurated by Γ, Λ is S'-arithmetic for some set of places S' containing all of the Galois embeddings.

Theorem 25 (van Dantzig)**.** *Every totally disconnected locally compact group G has a compact open subgroup K.*

Lemma 11. *If K is a compact open subgroup of a totally disconnected locally compact group G, K is commensurated.*

Proof. First, note that for any $g \in G$, K^g is also a compact open subgroup, since conjugation is a homeomorphism. Then also $K \cap K^g$ is a compact open subgroup. Of course K is the disjoint union of the cosets $C = K/(K \cap K^g)$. Since each $c(K \cap K^g)$ is open, this forms an open cover of K. But the cosets are disjoint, so no proper subcover can still cover K. Thus the set C must be finite to begin with. So K and K^g are commensurable. $\square$

Recall that a symmetric subset $A \subseteq G$ is called a *k-approximate subgroup* if $A \cdot A \subseteq K \cdot A$ for some set $|K| = k < \infty$.

Proposition 46. *If $H \leq G$ is commensurated, then for every $g \in G$, the set $A_g = gH \cup Hg^{-1} \cup \{1\}$ is an approximate subgroup.*

Proof. Since H is commensurated, we know that $HgH \subseteq K_l gH$ and $HgH \subseteq HgK_r$, for some finite sets K_l and K_r depending on g. Of course $1A_g = A_g 1 \subseteq A_g$. We check that pairwise products of the other four types can be finitely covered.

Now

$$(gH)(gH) = g(HgH) \subseteq g(K_l gH) = (gK_l)gH$$

$$(Hg^{-1})(Hg^{-1}) = (Hg^{-1}H)g^{-1} \subseteq (K_r^{-1}g^{-1}H)g^{-1} = (K_r^{-1}g^{-1})(Hg^{-1})$$

$$(Hg^{-1})(gH) = H = g^{-1}(gH)$$

$$(gH)(Hg^{-1}) = g(Hg^{-1})$$

In total, we have that $A_g \cdot A_g \subseteq K \cdot A_g$, where $K = \{1, g, g^{-1}\} \cup gK_l \cup (gK_r)^{-1}$. □

4.2 Hybrid lattices in Jungles

We compute explicitly two lattices in products of trees. For the quaternion algebras $\mathbb{Q}\{i,j\}$ and $\mathbb{Q}\{\sqrt{3}i, j\}$, we consider the projective classes of units of their $\mathbb{Z}[1/5, 1/13]$-points. Let $\mathcal{H}_{\mathbb{Z}[1/65]}$ denote the torsionfree subgroup of the projective units of $\mathbb{Z}[1/65]\{i,j\}$, and $\mathcal{H}'_{\mathbb{Z}[1/65]}$ those of $\mathbb{Z}[1/65]\{\sqrt{3}i, j\}$. We will exhibit explicit generating sets for particular congruence subgroups, chosen so that the groups act simply transitively on the vertices of the product of trees. It is not always possible to do so (try finding norm 3 elements in $\mathbb{Z}\{i, \sqrt{101}j\}$!), but we will exhibit such sets for these groups.

For $\mathcal{H}_{\mathbb{Z}[1/65]}$, we have that $A_5 = \{1 \pm 2i, 1 \pm 2j, 1 \pm 2k\}$ generates $\mathcal{H}_{\mathbb{Z}[1/5]}$ and $A_{13} = \{3 \pm 2i, 3 \pm 2j, 3 \pm 2k, 1 \pm 2i \pm 2j \pm 2k\}$ generates $\mathcal{H}_{\mathbb{Z}[1/13]}$.

For $\mathcal{H}'_{\mathbb{Z}[1/65]}$, $A'_5 = \{1 \pm 2j, 1 \pm j \pm \sqrt{3}k\}$ generates $\mathcal{H}'_{\mathbb{Z}[1/5]}$ and $A'_{13} = \{3 \pm 2j, 3 \pm j \pm \sqrt{3}k, 1 \pm 2\sqrt{3}i, 1 \pm 2\sqrt{3}k, 1 \pm 3j \pm \sqrt{3}k\}$ generates $\mathcal{H}'_{\mathbb{Z}[1/13]}$.

Figures 4.1 and 4.2 exhibit presentations for these groups, obtained with the help of Mathematica, by the following process. For each m, n, range over k and calculate $a_m b_n a_k^{\pm}/5$. For whichever k this result is integral, it will be a prime element b_l of $\mathcal{H}_{\mathbb{Z}[1/13]}$, and thus $a_m b_n a_k^{\pm} b_l^{-1}$ is a relator.

The 21 squares in Figure 4.3 are glued together according to the color and orientation of the arrows on their edges. The gluing respects the vertical or horizontal orientation of the edges. There are 3 horizontal edges and 7 vertical edges. Identifying each of the 21 squares with a unit square $[0,1] \times [0,1]$, the *vertical hyperplane* of the complex is the union of $\{1/2\} \times [0,1]$, ranging over all 21 squares, and the *horizontal hyperplane* is the union of the $[0,1] \times \{1/2\}$. The vertical hyperplane is drawn below and to the left of the complex itself, and the horizontal hyperplane to the right. The vertical hyperplane has three vertices, each of which is (13+1)-valent, and the horizontal hyperplane is (5+1)-valent.

The complex drawn in Figure 4.2 is described similarly, with its hyperplanes beneath. Note that the vertical hyperplanes of the two complexes are the same. The hyperplanes are *bicollared*; for $\varepsilon < 1/2$, the strip $[1/2 - \varepsilon, 1/2 + \varepsilon] \times [0,1]$ embeds in the complex. Since the orientations on edges are not all consistent, it is a nontrivial bundle. Thus, upon cutting the complex along the vertical hyperplane, we obtain a connected complex whose boundary

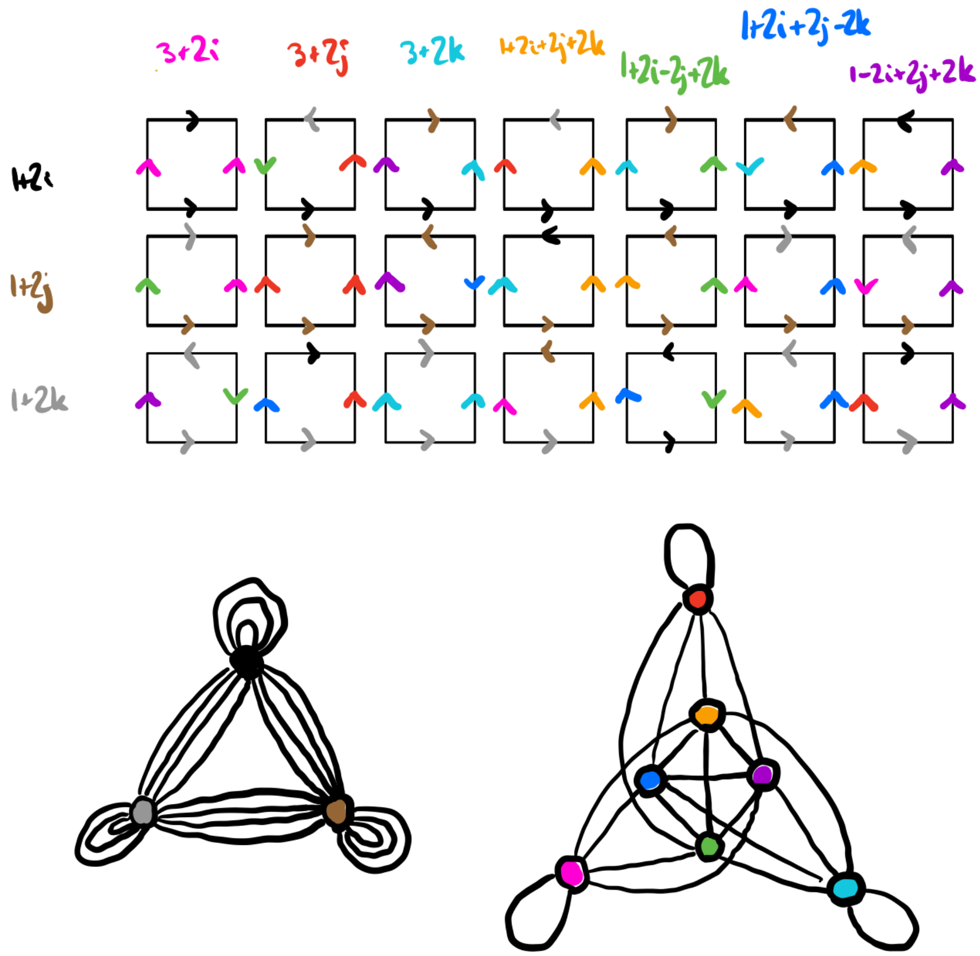


Figure 4.1: A presentation for $\mathcal{H}_{\mathbb{Z}[1/5,1/13]}$.

is a double cover of the hyperplane. We remark that the graphs obtained as the boundaries (double covering the hyperplanes) are isomorphic to the graphs $Y^{5,13}$ and $Y^{13,5}$ described in Lubotzky-Phillips-Sarnak [49, Theorem 4.4], which are explicit Ramanujan graphs.

Since these two "cut" complexes have isomorphic graphs as their boundaries, choosing any isomorphism between their boundaries allows us to glue the two complexes together to get a new square complex, hence a new lattice in a product of trees which is built out of arithmetic pieces.

Note that both of these groups embed in the group $\mathcal{H}_{\mathbb{Z}[\sqrt{3},1/65]}$, which is a lattice in a product $T_{26} \times T_{14} \times T_{14}$. To understand the unit group of $\mathbb{Z}[\sqrt{3}, \frac{1}{5}]\{i, j\}$, a maximal order is given by $\mathbb{Z}[\sqrt{3}]\{1, \zeta, j, \zeta j\}$, where $\zeta = \frac{\sqrt{3}+i}{2}$ is a primitive 12^{th} root of unity. We obtain a free group of rank 13 (since the projective line of $\mathbb{F}_{25}$ corresponds to the generators and their

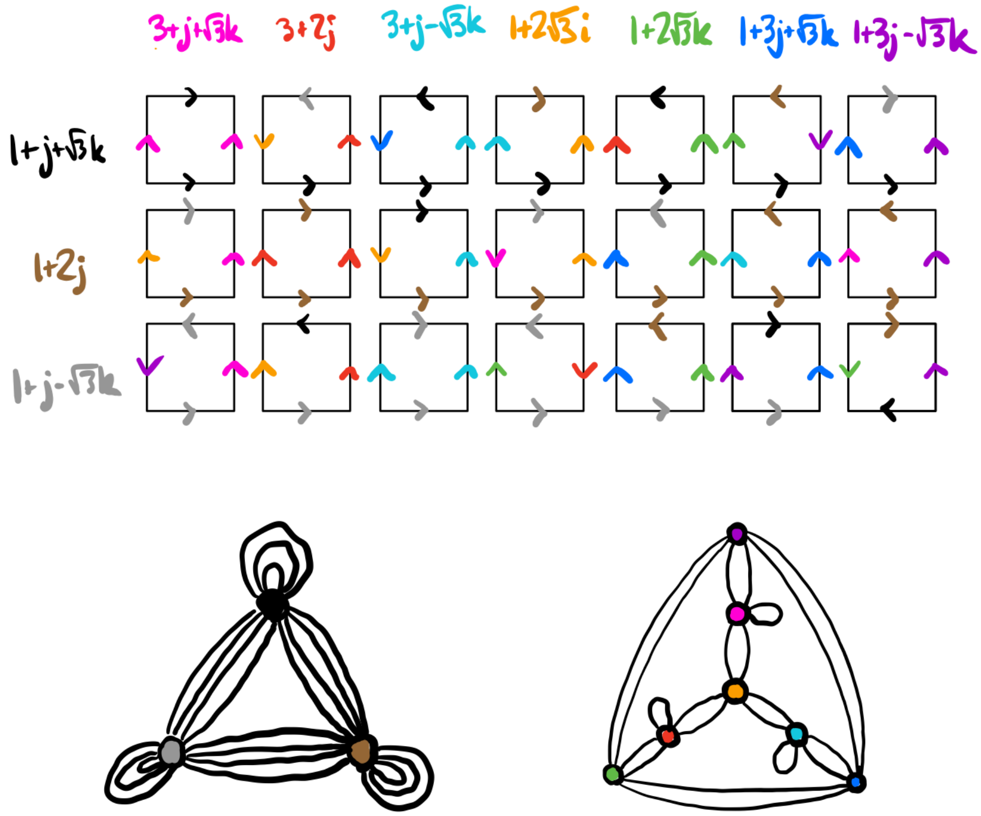


Figure 4.2: A presentation for $\mathcal{H}'_{\mathbb{Z}[1/5,1/13]}$.

inverses). An explicit symmetric generating set is given by the set $A_{25} = \{1 \pm 2i, 1 \pm 2\zeta^m j\}$, where $m \in [12] = \{0, \ldots, 11\}$. Since $13 = (5 + 2\sqrt{3})(5 - 2\sqrt{3})$ factors, and $5 + 2\sqrt{3} = 1^2 + (1 + \sqrt{3})$, we can find generators for $\mathbb{Z}[\sqrt{3}, 1/5 \pm 2\sqrt{3}]\{i, j\}$

The automorphism σ_1 (which is the Galois automorphism on $\mathbb{Q}(\sqrt{3})$ and fixes i, j) fixes $\{1 \pm 2i, 1 \pm 2\zeta^{3m} j\} = A_5$, and σ_2 (which also negates i) fixes $\{1 \pm 2\zeta^{2n} j\} = A'_5$.

4.3 Superrigidity in higher-rank p-adic groups

Definition 17. We say that an algebraic group G *satisfies superrigidity* if for any irreducible lattice $\Gamma \leq G$ and any homomorphism $f \colon \Gamma \to H$ to an algebraic group with Zariski dense image, there is a unique continuous homomorphism $F \colon G \to H$ extending f.

Many modern treatments of Margulis' superrigidity theorem appear to make the assumption that $\mathbb{R}$-rank of G is at least two, leading us to suspect that this may be a necessary assumption. However, Margulis's book ([54], VII.7.1) proves the following:

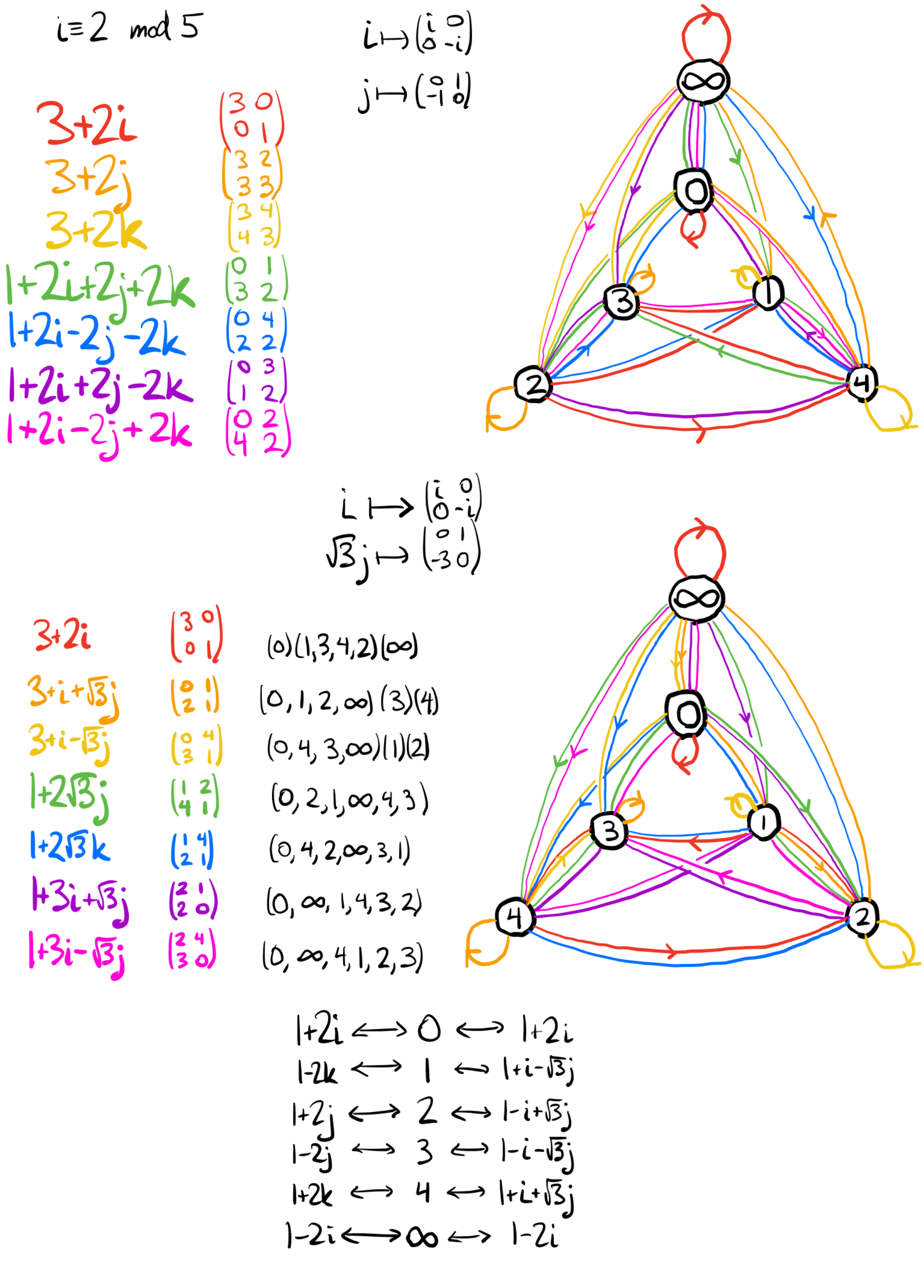


Figure 4.3: The hyperplanes after cutting.

Theorem 26. *Suppose that $\{k_1, \ldots, k_n\}$ is a finite set of local fields, and for each i, G_i is a connected semisimple adjoint algebraic group which is defined, isotropic, and simple over k_i. If $\sum_{i=1}^n rank(G_i(k_i)) \geq 2$, then $G = \prod_{i \in I} G_i(k_i)$ satisfies superrigidity.*

Corollary 10. $\prod_{p \in S} \mathsf{PSL}_2(\mathbb{Q}_p)$ *satisfies superrigidity if $|S| \geq 2$ is a finite set of primes.*

Proposition 47. *Suppose G satisfies superrigidity, and $Out(G) = 1$. Then for any lattice $\Gamma \leq G$, any Zariski dense representation $f \colon \Gamma \to G$ is conjugate to the identity.*

Proof. Suppose $F \colon G \to G$ extends f. Then F is an automorphism of G, which must be induced by conjugation since $\mathrm{Out}(G) = 1$. Thus $f = F|_\Gamma$ is conjugation. □

Proposition 48. *Suppose G satisfies superrigidity, $Out(G) = 1$, and Γ_1, Γ_2 are two lattices in G. Then Γ_1 and Γ_2 are abstractly commmensurable if and only if they are commensurable in G.*

Proof. It is clear that if Γ_1 and Γ_2 are commensurable in G, they are abstractly commensurable.

Now supposing Γ_1 and Γ_2 are abstractly commensurable, then we may replace each with a finite index subgroup $\Lambda_i \leq \Gamma_i$ with $f \colon \Lambda_1 \to \Lambda_2$ an isomorphism of abstract groups. Since Γ_i has finite covolume, $Vol(G/\Lambda_i) = [\Gamma_i : \Lambda_i] Vol(G/\Gamma_i)$ is also finite, and so Λ_i is a lattice. Now since $\Lambda_2 \leq G$, it is natural to consider $f \colon \Lambda_1 \to \Lambda_2 \leq G$ as a representation to G. Since G satisfies superrigidity, f extends to a representation $F \colon G \to G$, which must be surjective by the Borel density theorem. The previous proposition implies that f is conjugate to the identity, which in the present case means that Λ_1 and Λ_2 are conjugate. But this is precisely what it means for Γ_1 and Γ_2 to be commensurable in G. □

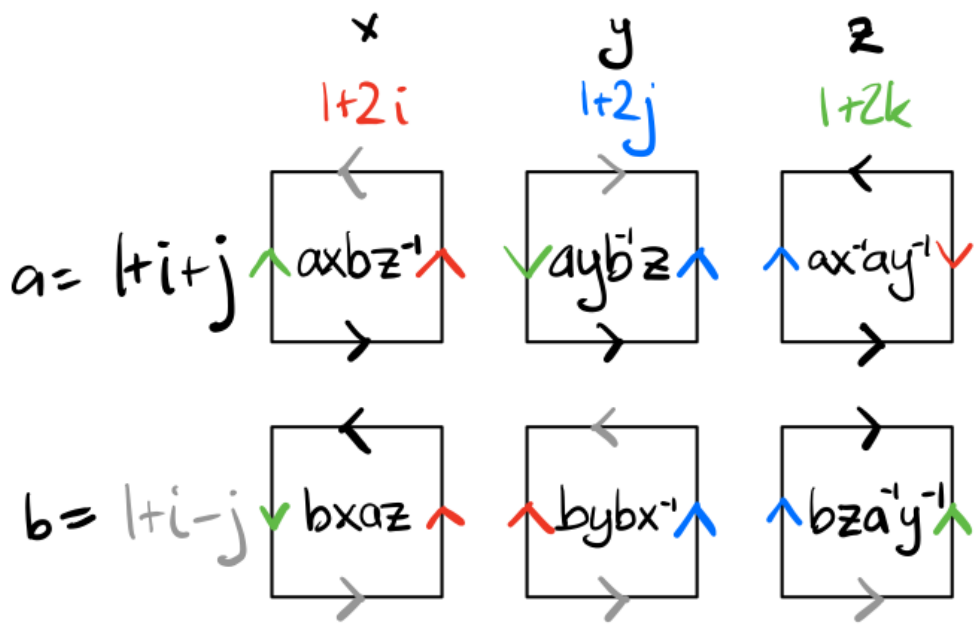
Definition 18. We say that a rational quadratic form f is *ramified at a prime p* if f does not represent zero nontrivially over $\mathbb{Q}_p$

Corollary 11. *Suppose n is composite, and f and g are positive definite ternary quadratic forms which are not ramified with respect to any prime factors of n. Then $\mathsf{SO}_f(\mathbb{Z}[1/n])$ is abstractly commensurable to $\mathsf{SO}_g(\mathbb{Z}[1/n])$ if and only if f and g are equivalent over $\mathbb{Q}$.*

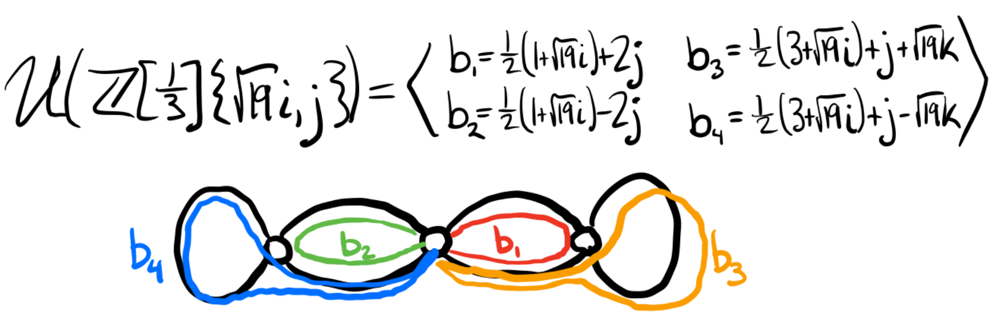
Proof. The assumption on ramification implies that $\mathsf{SO}_f(\mathbb{Q}_p) \cong \mathsf{SO}_g(\mathbb{Q}_p) \cong \mathsf{PGL}_2(\mathbb{Q}_p)$ for each $p \mid n$, and the Borel–Harish-Chandra theorem implies that $\mathsf{SO}_f(\mathbb{Z}[1/n])$ and $\mathsf{SO}_g(\mathbb{Z}[1/n])$ are lattices in the group $\mathbb{G} = \prod_{p|n} \mathsf{PGL}_2(\mathbb{Q}_p)$. Since n is composite, $\mathbb{G}$ has rank at least two, satisfies superrigidity, and $\mathrm{Out}(\mathbb{G}) = 1$ by Lemma 7. If f and g are equivalent over $\mathbb{Q}$, then an equivalence induces a commensuration of the lattices. Otherwise, f and g are not equivalent, and so their orthogonal groups are not commensurable in $\mathbb{G}$. Thus these groups are not abstractly commensurable. □

Example 6. Let $f = x^2 + y^2 + z^2$ and $g = 19x^2 + y^2 + 19z^2$, and consider $\Gamma_1 = \mathsf{SO}_f(\mathbb{Z}[1/15])$ and $\Gamma_2 = \mathsf{SO}_g(\mathbb{Z}[1/15])$. Each of Γ_i is an irreducible lattice in $G = \mathsf{PGL}_2(\mathbb{Q}_3) \times \mathsf{PGL}_2(\mathbb{Q}_5)$. These are abstractly commensurable if and only if they are commensurable in G, but since f and g are not equivalent forms, no finite index subgroup of Γ_1 is isomorphic to a finite index subgroup of Γ_2. Note that 19 is a square mod 3 and a square mod 5, and so $f \sim g$ over $\mathbb{Q}_3$ and $f \sim g$ over $\mathbb{Q}_5$.

We recall the presentation for $\mathsf{SO}_3(\mathbb{Z}[1/15])$. The set $A_3 = \{1+i+j, 1+i-j, 1-i+j, 1-i-j\}$ is a symmetric generating set for the lattice in T_4, and $A_5 = \{1+2i, 1+2j, 1+2k, 1-2i, 1-2j, 1-2k\}$ for the lattice in T_6. These each have covolume 1, as they act simply transitively on the vertices of the trees.

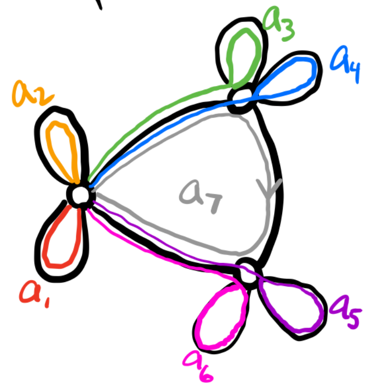


For $\mathsf{SO}_g(\mathbb{Z}[1/15])$, we find generating sets for $\mathsf{SO}_g(\mathbb{Z}[1/3])$ and $\mathsf{SO}_g(\mathbb{Z}[1/5])$ as the elements of small norm in $\mathbb{Z}[x, j]$, where $x = \frac{-1+\sqrt{19}i}{2}$, and x satisfies $x^2 + x + 5 = 0$, or $x(x+1) = -5$, so that x and $1+x$ are conjugates. Note that $1 + x + 2j$ and $1 + x - 2j$ have norm 9 and trace 1, so they act on the 3-adic tree as a translation of length two. The elements $\frac{1}{2}(2+x) + j + \sqrt{19}k$ and $\frac{1}{2}(2+x) + j - \sqrt{19}k$ have norm 27 and trace 3, and so they act on the 3-adic tree as a translation of length 1, about an axis distance 1 from the origin. These four elements generate a discrete group which acts on the tree cocompactly, and we obtain the following quotient:



For norm 5^k elements, we find that $1+2j$ and $1+x$ have norm 5, thus are translations of length 1 with an axis passing through the origin of the tree. The elements $6+2x+9j$ and $6+2x-9j$ each have norm 125 and trace 10, and so act on the 5-adic tree as a translation of length 1 about an axis distance 1 from the origin. The elements $8+x-8j$ and $8+x+8j$ have norm 125 and trace 15, and act similarly to the previous two generators. Finally, $7+2\sqrt{19}k = 7+j+2xj$ has norm 125 and trace 14, hence acts as a translation of length 3 through the origin. By calculating the axes for each of these generators, we find the following quotient:

$$\mathcal{U}(\mathbb{Z}[\tfrac{1}{5}]\{\sqrt{19}i, j\}) = \left\langle \begin{array}{lll} a_1 = 1+2j, & a_3 = 5+\sqrt{19}i+9j, & a_5 = 5+\sqrt{19}i-9j \\ a_2 = \frac{1}{2}(1+\sqrt{19}i), & a_4 = \frac{1}{2}(15+\sqrt{19}i)-8j, & a_6 = \frac{1}{2}(15+\sqrt{19}i)+8j \\ & & a_7 = 7+2\sqrt{19}k \end{array} \right\rangle$$



Thus we have found generators for a torsionfree subgroup of finite index in $\mathsf{SO}_g(\mathbb{Z}[1/15])$, which is a lattice in $T_4 \times T_6$.

Theorem 27. *The groups* $\mathsf{SO}_f(\mathbb{Z}[1/15])$ *and* $\mathsf{SO}_g(\mathbb{Z}[1/15])$ *are lattices in* $\mathsf{PGL}_2(\mathbb{Q}_3)\times\mathsf{PGL}_2(\mathbb{Q}_5)$ *which are not abstractly commensurable. Hence for any prime* $p > 19$*, the hyperplanes in* $\mathsf{SO}_f(\mathbb{Z}[1/(15p)])$ *and* $\mathsf{SO}_g(\mathbb{Z}[1/(15p)])$ *dual to the* p*-adic tree are not isomorphic for any finite index subgroup.*

Proof. The first statement follows from Corollary 11. The hyperplanes dual to the p-adic tree will be finite index congruence subgroups of $\mathsf{SO}_f(\mathbb{Z}[1/15])$ and $\mathsf{SO}_g(\mathbb{Z}[1/15])$. If these had finite index subgroups which were isomorphic, then these groups would be commensurable, a contradiction. □

Question 4. Suppose $\Gamma \leq \mathsf{SO}_3(\mathbb{Z}[1/n])$ is full, and $\mathbb{G} = \prod_{p|n} \mathsf{SO}_3(\mathbb{Q}_p)$. Are there representations $\Gamma \to \mathbb{G}$ which do not extend to $\mathsf{SO}_3(\mathbb{Z}[1/n])$?

If the answer is no, then Γ satisfies a strong enough form of superrigidity to make several deductions. For example, this would imply that Γ cannot be free, because free groups have many more representations to $\mathbb{G}$. We know that any geometric representation must extend.

4.4 The Greenberg-Shalom Question

It is perhaps quite surprising how many simple-to-state open questions one can raise that are concerned with *two-by-two matrices with rational entries* (or entries in a number field). It seems that many of these questions are intimately connected. For example, many of them are direct consequences of an affirmative answer to following question, which should be compared with [31, Question 7.3], [35, p. 231-232], [75, Question 1.2]. The particular statement we will use is slightly less general than what has appeared in the literature, but is a convenient way to state the question for our purposes and use. When we say *Question 5 implies*, we mean of course that the stated consequence would follow from an affirmative answer to the question.

Question 5 (Greenberg-Shalom, Question 1.2 of [33])**.** Let k be a number field, and $G(k)$ a semisimple k-algebraic group. Suppose $\Gamma \leq G(k)$ is Zariski dense and discrete in $G(\mathbb{A}_k^S)$ for a collection of places S containing the infinite places, and its commensurator is indiscrete in this latter group. Then Γ is commensurable with $G(\mathcal{O}_S)$.

It is an observation due to D. Fisher, T. Koberda, M. Mj, and W. van Limbeek that Question 5 implies that the Long-Reid group (Section 3.1) is not discrete in $\mathsf{PSL}_2(\mathbb{Q}_2) \times \mathsf{PSL}_2(\mathbb{Q}_3)$. In this section, we observe other consequences of this conjecture. We thank these authors for helpful conversations.

Applications

It will be useful to cite a generalization of Theorem 20, which we learned after proving our version. However, the lemma makes the assumption that $G(\mathcal{O})$ is infinite, which is not the case in $\mathbb{R}$-rank 0 groups we consider here, so we prove a more general version, closely following the proof of Lemma 2.8 in [51].

Proposition 49 (cf. Theorem 6.9 [75])**.** *Suppose G is an absolutely simple and simply connected k-algebraic group, and S is a finite set of places containing the archimedean places, and such that $G(\mathcal{O}_S)$ is infinite. Suppose $\Gamma \leq G(k)$ is finitely generated and $[G(\mathcal{O}_S) : \Delta] < \infty$ for $\Delta = \Gamma \cap G(\mathcal{O}_S)$. Then there is a finite set of places $S' \supseteq S$ so that $G(\mathcal{O}_{S'})$ and Γ are directly commensurable.*

Proof. Since Γ is finitely generated, its ring of definition is finitely generated over $\mathcal{O}$, and so there is a finite set of places $T \supseteq S$ so that $\Gamma \leq G(\mathcal{O}_T)$. Denote by S' the set of places $v \in T$ for which $\Gamma \leq G(k_v)$ is dense. Since Δ has finite index in $G(\mathcal{O}_S)$, we have that $S' \supseteq S$, since Γ must be unbounded in all of the places v for which $G(\mathcal{O}_{\{v\}})$ is unbounded. By strong approximation [64, Theorem 7.12], the closure of $G(\mathcal{O}_S)$ in $G(k_v)$ is open for any $v \notin S$, and since G is simply connected, any proper open subgroup of $G(k_v)$ is compact [66, Theorem (T)].

So we see that for each $v \notin S'$, since the closure of Γ in $G(k_v)$ is a proper subgroup, it is contained in a compact subgroup. But two compact open subgroups of $G(k_v)$ are directly commensurable, and so $\Gamma \cap G(\mathcal{O}_{k_v})$ has finite index in Γ. Since S' is finite, the intersection $\Gamma \cap G(\mathcal{O}_{S'})$ has finite index in Γ, and we shall assume henceforth that $\Gamma \leq G(\mathcal{O}_{S'})$.

Now let $\overline{(-)}$ denote the closure in $\prod_{v\in S'\setminus S} G(k_v)$. By strong approximation, Δ is dense in $K = \prod_{v\in S'\setminus S} G(\mathcal{O}_v)$, and $\overline{\Gamma}$ contains the compact open subgroup K. But since Γ is unbounded in each $G(k_v)$, its closure must contain $G(k_v)$ for each $v \in S' \setminus S$, hence by [75, Lemma 6.8] Γ is dense in $G(k_{S'\setminus S})$.

Since $[G(\mathcal{O}_S) : \Delta]$ is finite, there is a finite set $F \subseteq G(\mathcal{O}_S)$ of coset representatives for Δ which also serves as a collection of coset representatives for $\overline{\Delta} \subseteq \overline{G(\mathcal{O}_S)}$. We will show that F also serves as a collection of coset representatives for Γ in $G(\mathcal{O}_{S'})$, hence $G(\mathcal{O}_{S'}) = \Gamma F$.

The closure $\overline{G(\mathcal{O}_S)}$ is open in $\prod_{v\in S'\setminus S} G(k_v)$ and Γ is dense, we know that for every $g \in G(\mathcal{O}_{S'})$, there is some $\gamma \in \Gamma \cap g\overline{G(\mathcal{O}_S)}$. But also $(g^{-1}\gamma)^{-1} \in G(\mathcal{O}_{S'}) \cap \overline{G(\mathcal{O}_S)} = G(\mathcal{O}_S) = \Delta F$. But this means that $g \in \gamma\Delta F \subseteq \Gamma F$, and hence $G(\mathcal{O}_{S'}) \subseteq \Gamma F$, as desired. Hence, Γ has finite index in $G(\mathcal{O}_{S'})$.

In the course of the proof, we may have replaced Γ with a finite index subgroup, but note that if Γ had been contained in $G(\mathcal{O}_{S'})$, we have shown that the index of Γ in $G(\mathcal{O}_{S'})$ is at most the index of Δ in $G(\mathcal{O}_S)$. However, the other bound on the index is automatic, because if there were $f_1, f_2 \in F$ so that $f_1\Gamma = f_2\Gamma$, we would have that $f_1^{-1}f_2 \in \Gamma$. But we chose $f_i \in G(\mathcal{O}_S)$, and so $f_1^{-1}f_2 \in \Gamma \cap G(\mathcal{O}_S) = \Delta$, contradicting that these give distinct Δ cosets. ☐

Proposition 50. *If G acts on a locally finite complex, then G commensurates $K = Stab(v)$ for any vertex v.*

Proof. For any $g \in G$, gKg^{-1} is the stabilizer of the vertex gv. Since K fixes v, it permutes the finitely many vertices at distance $d(v, gv)$, and so the subgroup of K which fixes gv has finite index in K. But this is precisely the intersection $K \cap gKg^{-1}$. Thus K and gKg^{-1} intersect with finite index in each, so G commensurates K. ☐

Coherence and the Subgroup Conjecture

In ([60], p. 734), Serre asks most broadly whether $\mathsf{GL}_n(\mathbb{Q})$ is coherent, and remarks afterwards that Baumslag points out that the image of $BS(p,q) = \langle a, t \mid ta^pt^{-1} = a^q\rangle$ under the representation $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $t \mapsto \begin{pmatrix} p/q & 0 \\ 0 & q/p \end{pmatrix}$ is not a finitely presented group, so not even $\mathsf{SL}_2(\mathbb{Z}[1/pq])$ is coherent. Also, since $\mathsf{SL}_2(\mathbb{Z}) \times \mathsf{SL}_2(\mathbb{Z}) \leq \mathsf{SL}_4(\mathbb{Z})$ has a finite index subgroup which is a direct product of free groups (hence proved to be incoherent by Stallings [76, Example 1]), $\mathsf{SL}_4(\mathbb{Z})$ is not coherent.

However, it seems quite possible that $\mathsf{SO}_3(\mathbb{Q})$ is coherent. Recall *the subgroup conjecture* (Conjecture 1), that subgroups of $\mathsf{SO}_3(\mathbb{Q})$ are abelian, primary, or commensurable with $\mathsf{SO}_3(A)$ for A a subring of $\mathbb{Q}$.

Proposition 51. *An affirmative answer to Question 5 implies the subgroup conjecture.*

Proof. Suppose a finitely generated group $\Gamma \leq \mathsf{SO}_3(\mathbb{Q})$ is not abelian or primary. Let S denote the (finite) set of primes p for which Γ acts without a global fixed point on the p-adic tree, so that $\Gamma \leq \prod_{p \in S} \mathsf{PGL}_2(\mathbb{Q}_p)$ is discrete. Since Γ is not primary, Proposition 37 implies that there is some p for which Γ is almost dense in $\mathsf{PGL}_2(\mathbb{Q}_p)$. But then the subgroup $\Gamma \cap \mathsf{PGL}_2(\mathbb{Z}_p)$ is discrete in $\mathsf{PGL}_2(\mathbb{Q}_{S \setminus \{p\}})$, and is commensurated by Γ which is indiscrete in $\mathsf{PGL}_2(\mathbb{Q}_{S \setminus \{p\}})$. Thus Question 5 implies that the group $\Gamma \cap \mathsf{PGL}_2(\mathbb{Z}_p)$ is an $S \setminus \{p\}$-arithmetic lattice, and then Proposition 49 implies that Γ is S-arithmetic. □

Remark. The infinitely generated version of Question 5 implies the analogous version of the subgroup conjecture.

Corollary 12. *Question 5 implies that* $\mathsf{SO}_3(\mathbb{Q})$ *is coherent, and that coherence is not a geometric property.*

Proof. We showed that the subgroup conjecture implies these two properties in Proposition 38, and that Question 5 implies the subgroup conjecture in Proposition 51. □

The Strong Tits Alternative

The *Tits alternative* asserts that any linear group in characteristic zero is either virtually solvable or contains a free group. Tits accomplishes this by ensuring that a group which is not virtually solvable has a pair of independent rank one elements with disjoint attracting and repelling fixed points on some projective spaces, and proves that high enough powers of these elements play "ping-pong".

This dynamical observation has an important application in the theory of hyperbolic groups. Any two elements of infinite order in a hyperbolic group must either lie in a common cyclic subgroup, or else they have powers that play ping-pong.

This property fails for an obvious reason in CAT(0) groups, since torsionfree abelian groups need not be cyclic. For example, $\mathbb{Z}^2$ is a CAT(0) group. However, one might ask if this is the only way that ping-pong can fail. Motivated by this, Wise ([9, Question 2.7]) asks:

Question 6 (Wise)**.** Suppose G is a CAT(0) group and $a, b \in G$. Is there some n so that $\langle a^n, b^n \rangle$ is abelian or free?

One can view this as a strong version of the Tits alternative, asserting that if a, b do not virtually commute, they behave like independent rank one elements. We show that an affirmative answer to Question 5 would produce a counterexample to this question.

Proposition 52. *Let* $a = 1 + 2i$ *and* $b = 3 + 2j$ *be (projective) integer quaternions. Then the group* $H_n = \langle a^n, b^n \rangle$ *is never abelian, and assuming Question 5,* H_n *is never free.*

Proof. For the unconditional statement, it suffices to note that a and b both act on the sphere as rotation by an irrational multiple of π, hence have infinite order. Thus, no power can have order 2, and as they have distinct axes, no nonzero power of a commutes with a nonzero power of b.

Question 5 implies the subgroup conjecture by Proposition 51, and the subgroup conjecture implies that the image of H_n in $\mathsf{SO}_3(\mathbb{Q})$ has finite index in $\mathsf{SO}_3(\mathbb{Z}[1/65])$, a lattice in $T_6 \times T_{14}$. Thus H_n is a lattice in a product of two trees, and therefore cannot be a free group. $\square$

Lyndon-Ullman

For q a rational number, let $P_q = \langle \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \rangle = \langle a, b \rangle$. If $q \geq 2$, these matrices have ping-pong sets in the hyperbolic plane, and thus generate a discrete and free group. When $|q| = 2$, the group is an arithmetic lattice, as it is a finite index subgroup of $\mathsf{PSL}_2(\mathbb{Z})$.

Conjecture 4. (Lyndon-Ullman, see [52],[44]) If $|q| < 2$ is rational, P_q is not free.

Note that the element ab^{-1} has trace $2 - q^2$, which lies in $(-2, 2)$ when $|q| < 2$. Hence ab^{-1} is elliptic. Recall that if r is an elliptic of finite order n, its trace is $\pm 2\cos(\pi/n)$, which is rational only when it is $\pm 2, \pm 1$, and 0. This corresponds to $q = 0$ or $q = \pm 1$, and in any other case for $|q| < 2$ this element is an infinite order elliptic.

So for $q \neq \pm 1$ rational with $|q| < 2$, P_q is indiscrete (and in fact dense) in $\mathsf{PSL}_2(\mathbb{R})$. However, if S is the (nonempty!) collection of primes dividing the denominator of q, P_q acts nontrivially on the p-adic tree for each p in S.

Proposition 53. *Question 5 implies that P_q has finite index in $\mathsf{PSL}_2(\mathbb{Z}[q])$, hence that Conjecture 4 holds.*

Proof. Let $H = P_q \cap \mathsf{PSL}_2(\mathbb{Z})$. Since suitable powers of the generators are integral, H is Zariski dense in $\mathsf{PSL}_2(\mathbb{R})$, and is discrete since it is contained in $\mathsf{PSL}_2(\mathbb{Z})$. However, P_q is indiscrete, since ab^{-1} is an infinite order elliptic. Let S denote the finite set of primes dividing the denominator of q; then P_q acts on $X = \prod_{p \in S} T_{p+1}$ without a global fixed point, with H the stabilizer of the base vertex. In particular, the $\mathbb{R}$-elliptic element ab^{-1} is a translation in each tree, so H is unbounded in every $\mathsf{PSL}_2(\mathbb{Q}_p)$, for p dividing the denominator of q.

By Proposition 50, P_q commensurates H, and P_q is indiscrete in $\mathsf{PSL}_2(\mathbb{R})$ since ab^{-1} is an infinite order elliptic. So Question 5 implies that H is a lattice in $\mathsf{PSL}_2(\mathbb{R})$, and since $H \subseteq \mathsf{PSL}_2(\mathbb{Z})$, it must have finite index in $\mathsf{PSL}_2(\mathbb{Z})$.

Now since P_q contains the arithmetic lattice H, Proposition 49 implies that P_q is commensurable with $\mathsf{PSL}_2(A)$, for some subring $A \leq \mathbb{Q}$. Since $P_q \leq \mathsf{PSL}_2(\mathbb{Z}[q])$, and P_q is unbounded in the archimedean place and the p-adic places for each p dividing the denominator of q, we must have that P_q is a finite index subgroup of $\mathsf{PSL}_2(\mathbb{Z}[q])$, which is not virtually free. Hence P_q cannot be free. $\square$

Bibliography

[1] I. Agol. Generalizations of the four-color theorem. MathOverflow. URL:https://mathoverflow.net/q/249576.

[2] I. Agol. Tameness of hyperbolic 3-manifolds. *arXiv preprint math/0405568*, 2004.

[3] I. Agol. Systoles of hyperbolic 4-manifolds. *arXiv preprint math/0612290*, 2006.

[4] I. Agol. Ideal triangulations of pseudo-anosov mapping tori. 2011.

[5] I. Agol, D. Groves, and J. Manning. The virtual haken conjecture. *Doc. Math*, 18(1):1045–1087, 2013.

[6] I. Agol, D. Groves, and J. F. Manning. Residual finiteness, qcerf and fillings of hyperbolic groups. *Geometry & Topology*, 13(2):1043–1073, 2009.

[7] I. Agol, D. D. Long, and A. W. Reid. The bianchi groups are separable on geometrically finite subgroups. *Annals of Mathematics*, pages 599–621, 2001.

[8] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for sln and sp2n. *Publications mathematiques de l'IHES*, 33:59–137, 1967.

[9] M. Bestvina. Questions in geometric group theory, 2004.

[10] A. Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Bulletin of the American Mathematical Society*, 67(6):579–583, 1961.

[11] L. Bowen. Free groups in lattices. *Geometry & Topology*, 13(5):3021–3054, 2009.

[12] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[13] M. R. Bridson, J. Howie, C. F. Miller, and H. Short. The subgroups of direct products of surface groups. *Geometriae Dedicata*, 92(1):95–103, 2002.

[14] M. R. Bridson and A. W. Reid. Profinite rigidity, fibering, and the figure-eight knot. *arXiv preprint arXiv:1505.07886*, 1505, 2015.

[15] N. Brody and M. R. Klug. Building groups from restricted diagrams of groups. *arXiv preprint arXiv:2109.01939*, 2021.

[16] N. Brody and J. Schettler. Rational hyperbolic triangles and a quartic model of elliptic curves. *Journal of Number Theory*, 164:359–374, 2016.

[17] N. R. Brody. *Limit Roots of Lorentzian Coxeter Systems*. University of California, Santa Barbara, 2016.

[18] M. Burger and S. Mozes. Lattices in product of trees. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 92(1):151–194, 2000.

[19] J. Button. Groups acting faithfully on trees and properly on products of trees. *arXiv preprint arXiv:1910.04614*, 2019.

[20] S. Buyalo, A. Dranishnikov, and V. Schroeder. Embedding of hyperbolic groups into products of binary trees. *Inventiones mathematicae.*, 169(1):153–192, 2007-5-23.

[21] S. Buyalo and V. Schroeder. Embedding of hyperbolic spaces in the product of trees. *Geometriae dedicata*, 113(1):75–93, 2005-06.

[22] D. Calegari and A. Walker. Random groups contain surface subgroups. *Journal of the American Mathematical Society*, 28(2):383–419, 2015.

[23] I. M. Chiswell. Abstract length functions in groups. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 80, pages 451–463. Cambridge University Press, 1976.

[24] J. H. Conway and D. A. Smith. *On quaternions and octonions: their geometry, arithmetic, and symmetry.* AK Peters/CRC Press, 2003.

[25] G. Cooke and P. J. Weinberger. On the construction of division chains in algebraic number rings,with applications to sl2. *Communications in Algebra*, 3(6):481–524, 1975.

[26] P. Corvaja, J. Demeio, A. Rapinchuk, J. Ren, and U. Zannier. Bounded generation by semi-simple elements: quantitative results. *arXiv preprint arXiv:2203.00755*, 2022.

[27] P. Corvaja, A. S. Rapinchuk, J. Ren, and U. M. Zannier. Non-virtually abelian anisotropic linear groups are not boundedly generated. *Inventiones mathematicae*, 227(1):1–26, 2022.

[28] M. Dehn. Uber die topologie des dreidimensionalen raumes. *Mathematische Annalen*, 69(1):137–168, 1910.

[29] J. Dieudonné and L. Hua. *On the Automorphisms of the Classical Groups: On the automorphisms of the classical groups; J. Dieudonné Supplement by LK Hua*. Number 2. American Mathematical Soc., 1951.

[30] B. Farb and L. Mosher. The geometry of surface-by-free groups. *Geometric & Functional Analysis GAFA*, 12(5):915–963, 2002.

[31] D. Fisher. Superrigidity, arithmeticity, normal subgroups: results, ramifications and directions. *arXiv preprint arXiv:2003.02956*, 2020.

[32] D. Fisher, M. Larsen, R. Spatzier, and M. Stover. Character varieties and actions on products of trees. *Israel Journal of Mathematics*, 225(2):889–907, 2018.

[33] D. Fisher, M. Mj, and W. Van Limbeek. Commensurators of normal subgroups of lattices. *arXiv preprint arXiv:2202.04027*, 2022.

[34] E. Galois. Letter to a. chevallier. *translated by AR Singh and CS Yogonanda, http://link.springer.com/content/pdf/10.1007$\tilde{2}$FBF02834265.pdf*, 1846.

[35] L. Greenberg. Commensurable groups of moebius transformations. *Discontinuous groups and Riemann surfaces Ann. of Math. Studies*, 79:227–237, 1974.

[36] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, 1987.

[37] M. Gromov and I. Piatetski-Shapiro. Non-arithmetic groups in lobachevsky spaces. *Publications Mathématiques de l'IHÉS*, 66:93–103, 1987.

[38] A. Guilloux. Equidistribution in s-arithmetic and adelic spaces. *Annales de la faculté des sciences de Toulouse Mathématiques*, 23(5):1023–1048, 2014.

[39] F. Haglund and D. T. Wise. Special cube complexes. *Geometric and Functional Analysis*, 17(5):1551–1620, 2008.

[40] J. Kahn and V. Markovic. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *Annals of Mathematics*, pages 1127–1190, 2012.

[41] J. Kahn and V. Markovic. The good pants homology and the ehrenpreis conjecture. *Annals of Mathematics*, pages 1–72, 2015.

[42] D. Kerr and P. W. Nowak. Residually finite actions and crossed products. *Ergodic Theory and Dynamical Systems*, 32(5):1585–1614, 2012.

[43] D. Kielak. Residually finite rationally solvable groups and virtual fibring. *Journal of the American Mathematical Society*, 33(2):451–486, 2020.

[44] S.-h. Kim and T. Koberda. Non-freeness of groups generated by two parabolic elements with small rational parameters. *Michigan Mathematical Journal*, 1(1):1–25, 2021.

[45] F. T. Leighton. Finite common coverings of graphs. *Journal of Combinatorial Theory, Series B*, 33(3):231–238, 1982.

[46] D. Long and A. Reid. Constructing thin groups. *Thin groups and superstrong approximation*, pages 151–166, 2012.

[47] D. D. Long, A. W. Reid, and M. Thistlethwaite. Zariski dense surface subgroups in sl (3, z). *Geometry & Topology*, 15(1):1–9, 2011.

[48] A. Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125. Springer Science & Business Media, 1994.

[49] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[50] A. Lubotzky and D. Segal. *Subgroup growth*, volume 212. Springer, 2003.

[51] A. Lubotzky and R. J. Zimmer. Arithmetic structure of fundamental groups and actions of semisimple lie groups. *Topology*, 40(4):851–869, 2001.

[52] R. Lyndon and J. Ullman. Groups generated by two parabolic linear fractional transformations. *Canadian Journal of Mathematics*, 21:1388–1403, 1969.

[53] C. Maclachlan and A. W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219. Springer, 2003.

[54] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17. Springer Science & Business Media, 1991.

[55] B. Martelli. Can all n-manifolds be obtained by gluing finitely many blocks? MathOverflow. URL:https://mathoverflow.net/q/42629 (version: 2010-10-22).

[56] C. R. Matthews, L. N. Vaserstein, and B. Weisfeiler. Congruence properties of zariski-dense subgroups i. *Proceedings of the London Mathematical Society*, 48(3):514–532, 1984-05.

[57] N. Monod. Arithmeticity vs. nonlinearity for irreducible lattices. *Geometriae Dedicata*, 112(1):225–237, 2005.

[58] A. Morgan, A. Rapinchuk, and B. Sury. Bounded generation of sl2 over rings of s-integers with infinitely many units. *Algebra & Number Theory*, 12(8):1949–1974, 2018.

[59] G. D. Mostow. Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *Publications Mathématiques de l'IHÉS*, 34:53–104, 1968.

[60] M. F. Newman. *Proceedings of the Second International Conference on the Theory of Groups: Australian National University, August 13-24, 1973*, volume 372. Springer, 2013.

[61] N. Nikolov, D. Segal, and N. Nikolav. On finitely generated profinite groups, i: strong completeness and uniform bounds. *Annals of mathematics*, pages 171–238, 2007.

[62] H. Oh. Adelic version of margulis arithmeticity theorem. *Mathematische Annalen*, 321(4):789–815, 2001.

[63] W. Parry. Axioms for translation length functions. In *Arboreal group theory*, pages 295–330. Springer, 1991.

[64] V. Platonov, A. Rapinchuk, and R. Rowen. *Algebraic groups and number theory*. Academic press, 1993.

[65] H. Poincaré. *Analysis situs*. Gauthier-Villars Paris, France, 1895.

[66] G. Prasad. Elementary proof of a theorem of bruhat-tits-rousseau and of a theorem of tits. *Bulletin de la Société mathématique de France*, 110:197–202, 1982.

[67] C. Radin and L. Sadun. On 2-generator subgroups of so (3). *Transactions of the American Mathematical Society*, 351(11):4469–4480, 1999.

[68] D. Rattaggi. Abelianization conjectures for some arithmetic square complex groups. *arXiv preprint math/0508144*, 2005.

[69] D. A. Rattaggi. *Computations in groups acting on a product of trees: normal subgroup structures and quaternion lattices*. PhD thesis, ETH Zurich, 2004.

[70] A. M. Robert. *A course in p-adic analysis*, volume 198. Springer Science & Business Media, 2013.

[71] P. Sarnak. Notes on the generalized ramanujan conjectures. *Harmonic analysis, the trace formula, and Shimura varieties*, 4:659–685, 2005.

[72] P. Sarnak. Notes on thin matrix groups. *Thin groups and superstrong approximation*, 61:343–362, 2014.

[73] J.-P. Serre. Le probleme des groupes de congruence pour sl2. *Annals of Mathematics*, pages 489–527, 1970.

[74] J.-P. Serre. Trees and amalgams. In *Trees*, pages 1–68. Springer, 1980.

[75] Y. Shalom and G. A. Willis. Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity. *Geometric and Functional Analysis*, 23(5):1631–1683, 2013.

[76] J. Stallings. Coherence of 3-manifold fundamental groups. In *Séminaire Bourbaki vol. 1975/76 Exposés 471–488*, pages 167–173. Springer, 1977.

[77] J. Stallings. How not to prove the poincaré conjecture. In *Topology Seminar Wisconsin, 1965.(AM-60), Volume 60*, pages 83–88. Princeton University Press, 2016.

[78] W. P. Thurston. Three-dimensional geometry and topology, volume 1. In *Three-Dimensional Geometry and Topology, Volume 1*. Princeton university press, 2014.

[79] M.-F. Vignéras. Arithmetic of quaternion algebras. *Springer Lecture Notes in Mathematics*, 800, 1980.

[80] D. T. Wise. Complete square complexes. *Commentarii Mathematici Helvetici*, 82(4):683–724, 2007.

[81] D. T. Wise. *An Invitation to Coherent Groups*, pages 326–414. Princeton University Press, 2020.