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Perverse Equivalences in the Dg-Stable Category

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Perverse Equivalences in the Dg-Stable Category

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by

Jeremy Rollin Bundick Brightbill

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ABSTRACT OF THE DISSERTATION

Perverse Equivalences in the Dg-Stable Category

by

Jeremy Rollin Bundick Brightbill

Doctor of Philosophy in Mathematics

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Professor Raphael Alexis Rouquier, Chair

Let A be a finite-dimensional self-injective algebra, graded in non-positive degree. We define $A\text{-dgstab}$, the differential graded stable category of A, to be the quotient of the bounded derived category of dg-modules by the thick subcategory of perfect dg-modules. We prove that $A\text{-dgstab}$ is the triangulated hull of the orbit category $A\text{-grstab}/\Omega(1)$, which allows computations in the dg-stable category to be performed in the graded stable category. We provide a sufficient condition for the orbit category to be equivalent to $A\text{-dgstab}$ and show this condition is satisfied by Nakayama algebras and Brauer tree algebras. When A is a symmetric algebra with socle concentrated in degree $-d < 0$, we show that $A\text{-dgstab}$ has Calabi-Yau dimension $-d-1$.

Chuang and Rouquier [CR17] describe an action by perverse equivalences on the set of bases of a triangulated category of Calabi-Yau dimension -1. We develop an analogue of their theory for Calabi-Yau categories of arbitrary negative dimension and apply this theory to the dg-stable category.

As an example, we analyze the dg-stable category of a Brauer tree algebra, with an arbitrary non-positive grading. We compute the Auslander-Reiten quiver, then develop a combinatorial model for $A\text{-dgstab}$, which we use to describe the action of perverse equivalences. Using our model, we show that perverse equivalences act transitively on the set of bases.

The dissertation of Jeremy Rollin Bundick Brightbill is approved.

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2020

to my parents, Marilyn and Sam, for a childhood full of love and happiness;

to my brother, Tim, for many words and many games;

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and to my teacher, Jed Laderman, for showing me how much there is to learn.

TABLE OF CONTENTS

LIST OF FIGURES

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CHAPTER 1

Introduction

1.1 Perverse Equivalences

Perverse equivalences are equivalences of triangulated categories performed with respect to a stratification. They were first used by Chuang and Rouquier [CR08] in their proof of Broué's abelian defect conjecture for symmetric groups; the theory of perverse equivalences was later formalized by the same authors in [CR17]. In this work, Chuang and Rouquier define an action of perverse equivalences on a collection of t-structures—parametrized by tilting complexes—in the bounded derived category of a symmetric algebra A. The further study of this action is the primary motivation of this dissertation.

The bounded derived category of A has many t-structures, and the action of perverse equivalences exhibits braid-like relations, twisted by some other structure. To investigate this extra structure, one natural approach would be to quotient out the braid relations, which correspond to certain autoequivalences of the bounded derived category. These autoequivalences are given by tensoring with certain complexes of projective bimodules. In the stable category, these complexes become isomorphic to zero and thus the braid relations vanish. However, t-structures need not exist in the stable category, hence the definition of the action must be modified. Chuang and Rouquier successfully adapt their action to the stable module category and, more generally, to Calabi-Yau categories of dimension -1; in this setting, perverse equivalences act on the set of bases of the category.

In this work, we adapt the action of Chuang and Rouquier to a new setting. First, we introduce a grading on the algebra A, which we view as a dg-algebra with zero differential. We then define the dg-stable category of A, $A\text{-}\mathrm{dgstab}$, to be the quotient of the bounded derived category of finite-dimensional dg-modules by the thick subcategory generated by A (i.e., the perfect dg-modules). The dg-stable category is a generalization of the ordinary stable module category; when the algebra is concentrated in degree zero, the two notions coincide. In the dg-stable category, there are nontrivial interactions between the grading data of A and the homological structure of $A\text{-dgstab}$. By viewing the grading data as a parameter, one obtains a family of triangulated categories on which the action of perverse equivalences can be defined.

1.2 The Dg-Stable Category

If A is a self-injective k-algebra, then $A\text{-stab}$, the stable module category of A, admits the structure of a triangulated category. This category has two equivalent descriptions. The original description is as an additive quotient: One begins with the category of A-modules and sets all morphisms factoring through projective modules to zero. More categorically, we define $A\text{-stab}$ to be the quotient of additive categories $A\text{-mod}\,/A\text{-proj}$. The second description, due to Rickard [Ric89a], describes $A\text{-stab}$ as a quotient of triangulated categories. Rickard obtains $A\text{-stab}$ as the quotient of the bounded derived category of A by the thick subcategory of perfect complexes. Once this result is known, the triangulated structure on $A\text{-stab}$ is an immediate consequence of the theory of triangulated categories. When translated back into the additive description, the homological shift functor $[-1]$ inherited from $D^b(A\text{-mod})$ becomes identified with the syzygy functor Ω, which maps each module to the kernel of a projective cover. The triangulated description provides a well-behaved technical framework for transferring information between $A\text{-stab}$ and the derived category, while the additive description allows computations of morphisms to be performed in $A\text{-mod}$ rather than $D^b(A\text{-mod})$. If A is made into a graded algebra, analogous constructions produce two equivalent descriptions of the graded stable category $A\text{-grstab}$.

If A is a dg-algebra, we use the triangulated description to define the differential graded stable category $A\text{-dgstab}$. More precisely, $A\text{-dgstab}$ is defined to be the quotient of the derived category $D^b_{dg}(A)$ of dg-modules by the thick subcategory of perfect dg-modules.

The most immediately interesting feature of the dg-stable category is the presence of non-trivial interactions between the grading data and the triangulated structure. In $D^b_{dg}(A)$, the grading shift functor coincides with the homological shift functor, and so in $A\text{-dgstab}$ the grading shift functor (-1) can be identified with Ω. This phenomenon does not occur in the graded stable category, since the grading shift and homological shift functors in $D^b(A\text{-grmod})$ are distinct.

However, working with dg-modules introduces new complications. Dg-modules need not arise from complexes of graded modules, which poses an obstacle to obtaining a simple additive definition of $A\text{-dgstab}$, without which computation of morphisms becomes much harder, as it must be done in the triangulated setting. We shall consider the problem of finding a simple additive description of $A\text{-dgstab}$.

The dg-stable category has been studied by Keller [Kel05], using the machinery of orbit categories; our approach is motivated by his work. In Chapter 3, we consider the case where A is a non-positively graded, finite-dimensional, self-injective algebra, viewed as a dg-algebra with zero differential. There is a natural functor $A\text{-grstab} \to A\text{-dgstab}$ which is faithful but not full. This is due to the fact that $X \cong \Omega X(1)$ for all $X \in A\text{-dgstab}$; the corresponding isomorphism almost never holds in $A\text{-grstab}$. To recover the missing morphisms, we turn to the orbit category $\mathcal{C}(A) := A\text{-grstab}/\Omega(1)$. The objects of $\mathcal{C}(A)$ are those of $A\text{-grstab}$, and the morphisms $X \to Y$ are finite formal sums of morphisms $X \to \Omega^n Y(n)$ in $A\text{-grstab}$. Orbit categories need not be triangulated, but Keller proves they always fit inside a "triangulated hull". We shall construct a fully faithful functor $F_A : \mathcal{C}(A) \to A\text{-dgstab}$ whose image generates $A\text{-dgstab}$ as a triangulated category and show that $A\text{-dgstab}$ is the triangulated hull of $\mathcal{C}(A)$.

F_A is an equivalence of categories precisely when it identifies $\mathcal{C}(A)$ with a triangulated subcategory of $A\text{-dgstab}$. This is in general not the case, as there is no natural way to take the cone of a formal sum of morphisms with different codomains. We provide a sufficient condition for F_A to be an equivalence and show that this condition is satisfied by self-injective Nakayama algebras. An example for which F_A is not an equivalence is also provided.

A triangulated category $(\mathcal{T}, \Sigma)$ is said have Calabi-Yau dimension w if the Σ^w is a Serre functor for $\mathcal{T}$. The stable category of any finite-dimensional self-injective algebra is (-1)-Calabi-Yau. We show that if A is symmetric and graded with socle in degree $-d$ for some integer $d \geq 0$, then $A\text{-dgstab}$ has Calabi-Yau dimension $-d-1$.

Having defined the dg-stable category, in Chapter 4 we adapt the action of perverse tilts to this new setting. Since projective modules become zero in the dg-stable category, we can no longer use tilting complexes as the basis for the action. Instead, the correct notion is that of a basis. Just as the summands of a tilting complex mimic the behavior of projective generators in the derived category, a basis of a (-1)-Calabi-Yau triangulated category is a collection of objects which behave analogously to the set of simple modules in the stable category. Chuang and Rouquier show that perverse equivalences act on the set of bases of a (-1)-Calabi-Yau category. For any $w < 0$, we define the analogous notion of a $|w|$-basis for a w-Calabi-Yau category and show that the action of perverse equivalences is defined in this context.

1.3 Brauer Tree Algebras

In the second half of this work, we investigate the dg-stable category of non-positively graded Brauer tree algebras. A Brauer tree is the data of a tree, a cyclic ordering of the edges around each vertex, a marked vertex (called the exceptional vertex) and a positive integer multiplicity associated to the exceptional vertex. The data of a Brauer tree determines, up to Morita equivalence, an algebra whose composition factors reflect the combinatorial data of the tree. When the graph in question is a star, the resulting Brauer tree algebra is a Nakayama algebra, i.e. all indecomposable modules are uniserial. We refer to Schroll [Sch18] for a detailed introduction to the theory of Brauer tree algebras and their appearance in group theory, geometry, and homological algebra, but we mention here one application which is of particular relevance. Khovanov and Seidel [KS02] link the category $D^b_{dg}(A)$, where A is a graded Brauer tree algebra on the the line with n vertices, to the triangulated subcategory of the Fukaya category generated by a chain of knotted Lagrangian spheres. The braid group acts on $D^b_{dg}(A)$ by automorphisms, and the category $A\text{-dgstab}$ can be viewed as the quotient of $D^b_{dg}(A)$ by this action.

In Chapter 5, we show that $\mathcal{C}(A)$ is equivalent to $A\text{-dgstab}$ for any Brauer tree algebra. When A corresponds to the star with n edges and multiplicity one, we classify the objects and morphisms of $A\text{-dgstab}$, producing the category's Auslander-Reiten quiver. Since all Brauer tree algebras are derived equivalent to the star, their dg-stable categories are also equivalent, hence this result describes all (basic) Brauer tree algebras of multiplicity one.

Having described the dg-stable category of a Brauer tree algebra, in Chapter 6 we turn our attention to the action of perverse equivalences. We develop a combinatorial model of $A\text{-}\mathrm{dgstab}$, in which objects are represented by interlocking beads of varying lengths on a circular wire. In this model, a basis corresponds to maximal non-overlapping configurations of beads, and perverse equivalences act via physically intuitive transformations of beads. Our main result establishes transitivity of the action: every basis can be obtained by applying successive perverse equivalences to the original collection of simple A-modules.

CHAPTER 2

Notation and Definitions

2.1 Triangulated Categories

A **triangulated category** is the data of an additive category $\mathcal{T}$, an automorphism Σ of $\mathcal{T}$ (called the **suspension** or **shift**), and a family of **distinguished triangles** $X \to Y \to Z \to \Sigma X$ obeying certain axioms. For more details on the definition and theory of triangulated categories, see Neeman [Nee01].

In a triangulated category, any morphism $X \xrightarrow{f} Y$ can be completed to a triangle $X \xrightarrow{f} Y \to Z \to X[1]$. We refer to Z as a **cone** of f; it is unique up to non-canonical isomorphism. Abusing notation, for any morphism $f : X \to Y$ in a triangulated category $\mathcal{T}$, we shall write $C(f)$ to refer to any choice of object completing the triangle $X \xrightarrow{f} Y \to C(f) \to X[1]$. This will cause no confusion.

An additive functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is said to be **exact** or **triangulated** if it commutes with the suspension functor and preserves distinguished triangles. A full additive subcategory $\mathcal{I}$ of $\mathcal{T}$ is called **triangulated** if it is closed under isomorphisms, shifts, and cones. A triangulated subcategory $\mathcal{I}$ of $\mathcal{T}$ is **thick** if $\mathcal{I}$ is closed under direct summands. Given a thick subcategory, one can form the quotient category $\mathcal{T}/\mathcal{I}$ by localizing at the class of morphisms whose cone lies in $\mathcal{I}$. There is a natural functor $\mathcal{T} \to \mathcal{T}/\mathcal{I}$ which is essentially surjective and whose kernel is $\mathcal{I}$.

A **Serre functor** of $\mathcal{T}$ is an autoequivalence $\mathbb{S}$ of $\mathcal{T}$ such that there exists an isomorphism $\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong \mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)^*$ which is natural in X and Y.

$\mathcal{T}$ is said to be w-**Calabi-Yau** for some $w \in \mathbb{Z}$ if Σ^w is a Serre functor for $\mathcal{T}$.

If $\mathcal{S}$ is any subcategory of $\mathcal{T}$, we define $\mathcal{S}^\perp = \{X \in Ob(\mathcal{T}) \mid \mathrm{Hom}(Y, X) = 0 \text{ for all } Y \in \mathcal{S}\}$

and ${}^{\perp}\mathcal{S} = \{X \in Ob(\mathcal{T}) \mid \mathrm{Hom}(X, Y) = 0 \text{ for all } Y \in \mathcal{S}\}$.

2.2 Complexes

If $\mathcal{A}$ is any additive category, we write $Comp(\mathcal{A})$ for the category of (cochain) complexes over $\mathcal{A}$. We shall write our complexes as $(C^\bullet, d_C^\bullet)$, where $d_C^n : C^n \to C^{n+1}$ for all $n \in \mathbb{Z}$. We write $Ho(\mathcal{A})$ for the category of complexes and morphisms taken modulo homotopy. If $\mathcal{A}$ is an abelian category, we let $D(\mathcal{A})$ denote the derived category of $\mathcal{A}$. On any of these subcategories, we shall use the superscript b (resp., $+, -$) to denote the full, replete subcategory generated by the bounded (resp., bounded below, bounded above) complexes.

If $\mathcal{A} = A\text{-mod}$ for some algebra A, we shall write $Ho^{perf}(A\text{-mod})$ (resp., $D^{perf}(A\text{-mod})$) for the thick subcategory of $Ho(A\text{-mod})$ (resp., $D(A\text{-mod})$) generated by A. The objects are those complexes which are homotopy equivalent (resp., quasi-isomorphic) to bounded complexes of projective modules; we refer to them as the **strictly perfect** (resp., **perfect**) complexes.

On any of the above categories, we let $[n]$ denote the n-th shift functor, defined by $(C^\bullet[n])^i = C^{i+n}$ and $d^\bullet_{C[n]} = (-1)^n d^\bullet_C$. We write $H^n(C^\bullet) := ker(d^n_C)/im(d^{n-1}_C)$ for the n-th cohomology group of $C^\bullet$.

Given a morphism of complexes $f : X^\bullet \to Y^\bullet$, we define the **cone** of f to be the complex $C(f)^\bullet = X^\bullet[1] \oplus Y^\bullet$ with differential $\begin{pmatrix} d^\bullet_{X[1]} & 0 \\ f[1] & d^\bullet_Y \end{pmatrix}$. We obtain an exact triangle $X \xrightarrow{f} Y \to C(f) \to X[1]$ in $Ho(\mathcal{A})$.

We write $\tau_{\leq n}, \tau_{\geq n}, \tau_{<n}, \tau_{>n}$ for the truncation functors on $D(\mathcal{A})$ defined by the canonical t-structure. More explicitly, if $X^\bullet$ is a complex, the kth term of $\tau_{\leq n} X^\bullet$ is X^k if $k < n$, 0 if $k > n$, and $ker(d^n_X)$ if $k = n$. $\tau_{>n} X^\bullet$ is defined analogously; here the nth term equal to $im(d^n_X)$. We also denote by $X^{\leq n}$ the complex whose kth term is X^k for $k \leq n$ and 0 for $k > n$. We denote $X^{\geq n}, X^{<n}, X^{>n}$ similarly, and refer to these complexes as the **sharp truncations** of $X^\bullet$.

2.3 Modules and the Stable Category

If A is an algebra over a field k, let $A\text{-mod}$ denote the category of finitely generated right A-modules, and let $A\text{-proj}$ denote the full subcategory of finitely generated projective modules. A-Mod and A-Proj will denote the categories of all modules and projective modules, respectively.

Given an A-module X, we define the **socle** of X, $soc(X)$ to be the sum of all simple submodules X. We define the **radical** of X, $rad(X)$, to be the intersection of all maximal submodules of X, and we define the **head** of X to be the quotient $hd(X) = X/rad(X)$. We note that $rad(A)$, where A is viewed as a right module over itself, is equal to the Jacobson radical of A. If X is finitely generated, then $rad(X) = Xrad(A)$.

An algebra A is said to be **self-injective** if A is injective as a right A-module. In a self-injective algebra, the classes of finitely-generated projective and injective modules coincide. A is said to be **symmetric** if there is a linear map $\lambda : A \to k$ such that $ker(\lambda)$ contains no left or right ideals of A, and $\lambda(ab) = \lambda(ba)$ for all $a, b \in A$. All symmetric algebras are self-injective.

We let $A\text{-stab}$ denote the **stable module category** of A. The objects of $A\text{-stab}$ are the objects of $A\text{-mod}$, and $\mathrm{Hom}_{A\text{-stab}}(X, Y)$ is defined to be the quotient of $\mathrm{Hom}_{A\text{-mod}}(X, Y)$ by the subspace of morphisms factoring through projective modules. There is a full, essentially surjective functor $A\text{-mod} \twoheadrightarrow A\text{-stab}$ which is the identity on objects. If A is self-injective, then $A\text{-stab}$ admits the structure of a triangulated category, and it has been shown by Rickard [Ric89a] that $A\text{-stab}$ is equivalent as a triangulated category to $D^b(A\text{-mod})/D^{perf}(A\text{-mod})$.

For a more detailed introduction to the theory of finite-dimensional algebras, see for instance Benson [Ben91].

We say that an additive category $\mathcal{C}$ is a **Krull-Schmidt category** if every object in $\mathcal{C}$ is isomorphic to a finite direct sum of objects with local endomorphism rings.

If A is finite-dimensional, both $A\text{-mod}$ and $A\text{-stab}$ are Krull-Schmidt. Any indecomposable object of $A\text{-stab}$ is the image of an indecomposable object in $A\text{-mod}$.

2.4 Graded Modules

Let A be a graded algebra over a field k. We denote by $A\text{-grmod}$ (resp., $A\text{-grproj}$) the category of finitely generated graded right modules (resp., finitely generated graded projective right modules). We shall use upper case letters when the modules are not required to be finitely generated, in analogy with Section 2.3.

The **graded stable module category** $A\text{-grstab}$ is defined analogously to $A\text{-stab}$. When A is self-injective, $A\text{-grstab}$ can be given the structure of a triangulated category and which is equivalent to $D^b(A\text{-grmod})/D^{perf}(A\text{-grmod})$.

If X is a graded A-module, we write X^i to denote the homogenous component of X in degree i. (If $X^\bullet$ is a complex of graded modules, we shall denote the degree i component of the nth term of the complex by $(X^n)^i$.) On any category of graded objects, we define the grading shift functor (n) by $X(n)^i = X^{i+n}$. If $x \in X$ is a homogeneous element, we let $|x|$ denote the degree of x.

For a graded module X, we define the **support** of X to be the set $supp(X) = \{n \in \mathbb{Z} | X^n \neq 0\}$. We also define $max(X) = sup(supp(X))$ and $min(X) = inf(supp(X))$. Note that if A is finite-dimensional and X is a finitely generated nonzero A-module, then X is a finite-dimensional k-vector space, therefore $supp(X)$ is a finite, nonempty set and $max(X)$ and $min(X)$ are finite.

Given graded modules X and Y, define $\mathrm{Hom}^\bullet_{A\text{-grmod}}(X, Y)$ to be the graded vector space whose degree n component is the space $\mathrm{Hom}_{A\text{-grmod}}(X, Y(n))$ of degree n morphisms. If X is a graded left B-module for some graded algebra B, then $\mathrm{Hom}^\bullet_{A\text{-grmod}}(X, Y)$ is a graded right B-module.

2.5 Differential Graded Modules

A **differential graded algebra** is a pair (A, d_A), where A is a graded k-algebra and d_A is a degree 1 k-linear differential which satisfies, for all homogenous $a, b \in A$, the equation $d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b)$.

If (A, d_A) is a differential graded k-algebra, a **differential graded right A-module** (or dg-module, for short) is a pair (X, d_X) consisting of a graded right A-module X and a degree 1 k-linear differential $d_X : X \to X$ satisfying $d_X(xa) = d_X(x)a + (-1)^{|x|} x d_A(a)$ for all homogeneous elements $x \in X, a \in A$. A morphism of differential graded modules is defined to be a homomorphism of graded A modules which commutes with the differential. We denote by $A\text{-dgmod}$ the category of finitely-generated right dg-modules. As above, we shall write $A\text{-dgMod}$ for the category of arbitrary dg-modules.

As with complexes, we write $H^i(X)$ for the ith cohomology group of X, i.e. the degree i component of $ker(d_X)/im(d_X)$.

Any graded algebra A can be viewed as a differential graded algebra with zero differential. In this case, for any dg-module (X, d_X), d_X is a degree 1 morphism of graded modules, and so the kernel and image of d_X are dg-submodules of X with zero differential. In this paper, we shall work exclusively with dg-algebras with zero differential.

The grading shift functor (n) can be extended to dg-modules by $(X, d_X)(n) := (X(n), d_{X(n)})$, where $d_{X(n)} = (-1)^n d_X(n)$.

There is a faithful functor $\widehat{\ } : Comp(A\text{-grmod}) \to A\text{-dgMod}$ sending the complex $(X^\bullet, d_X^\bullet)$ to the dg-module $(\widehat{X}, d_{\widehat{X}})$ whose underlying graded module is $\widehat{X} = \bigoplus_{n \in \mathbb{Z}} X^n(-n)$ and whose differential $d_{\widehat{X}}$ restricts to $d_X^n(-n)$ on $X^n(-n)$. If A is finite-dimensional, $\widehat{\ }$ restricts to a functor $Comp^b(A\text{-grmod}) \to A\text{-dgmod}$. Note also that $\widehat{X^\bullet[k]} = \widehat{X}(k)$.

Identifying graded modules with complexes concentrated in degree zero yields a fully faithful functor $A\text{-grmod} \hookrightarrow Comp^b(A\text{-grmod})$. The restriction of $\widehat{\ }$ to $A\text{-grmod}$ is fully faithful. Note that $\widehat{X(k)} = \widehat{X}(k)$.

If $f, g : X \to Y$ are morphisms of dg-modules, we say f and g are **homotopic** if there is a degree -1 graded morphism $h : X \to Y$ such that $f - g = h \circ d_X + d_Y \circ h$. We write $Ho_{dg}(A)$ for the category of right dg-modules over A and homotopy classes of morphisms. By formally inverting the quasi-isomorphisms of $Ho_{dg}(A)$, we obtain $D_{dg}(A)$, the derived category of dg-modules. We again use the superscript b (resp., $+, -$) to denote the full subcategory whose objects are isomorphic to dg-modules with bounded (resp. bounded below, bounded above) support. We write $Ho_{dg}^{perf}(A)$ (resp., $D_{dg}^{perf}(A)$) for the thick subcategories of $Ho_{dg}^b(A)$ (resp., $D_{dg}^b(A)$) generated by the dg-module A. We refer to the objects of $Ho_{dg}^{perf}(A)$ (resp., $D_{dg}^{perf}(A)$) as the **strictly perfect** (resp., **perfect**) dg-modules.

If P is strictly perfect, then $\mathrm{Hom}_{Ho_{dg}(A)}(P,X) \cong \mathrm{Hom}_{D_{dg}(A)}(P,X)$ for any dg-module X. In addition, if A is a finite-dimensional, self-injective graded algebra with zero differential, then $\mathrm{Hom}_{Ho_{dg}(A)}(X,P) \cong \mathrm{Hom}_{D_{dg}(A)}(X,P)$. Any perfect dg-module is quasi-isomorphic to a strictly perfect dg-module.

If $P^\bullet \in Comp^-(A\text{-grproj})$, then $\mathrm{Hom}_{Ho_{dg}(A)}(\widehat{P},X) \cong \mathrm{Hom}_{D_{dg}(A)}(\widehat{P},X)$ for any dg-module X. If A is finite-dimensional, self-injective, and has zero differential, then $\mathrm{Hom}_{Ho_{dg}(A)}(X,\widehat{I}) \cong \mathrm{Hom}_{D_{dg}(A)}(X,\widehat{I})$ for any $I^\bullet \in Comp^+(A\text{-grproj}, X \in A\text{-dgmod}$.

We define the **differential graded stable module category**, or **dg-stable category**, of A to be the quotient $A\text{-dgstab} := D_{dg}^b(A)/D_{dg}^{perf}(A)$.

If A is finite-dimensional, we have essentially surjective functors $A\text{-dgmod} \twoheadrightarrow Ho_{dg}^b(A) \twoheadrightarrow D_{dg}^b(A) \twoheadrightarrow A\text{-dgstab}$, each of which is the identity on objects. By composing with the inclusion $A\text{-grmod} \hookrightarrow A\text{-dgmod}$, we obtain an additive functor $A\text{-grmod} \to A\text{-dgstab}$ whose kernel contains $A\text{-grproj}$. Hence this functor factors through $A\text{-grmod} \twoheadrightarrow A\text{-grstab}$.

Given a morphism of dg-modules $f : X \to Y$, we define the **cone** of f to be the complex $C(f) = X(1) \oplus Y$ with differential $\begin{pmatrix} d_{X(1)} & 0 \\ f(1) & d_Y \end{pmatrix}$. We obtain an exact triangle $X \xrightarrow{f} Y \to C(f) \to X(1)$ in $Ho_{dg}(A)$.

2.6 Functors and Resolutions

Let A be a finite-dimensional, self-injective graded algebra.

Let $m : A^{op} \otimes_k A \twoheadrightarrow A$ denote the multiplication map, viewed as a morphism of graded $(A^{op} \otimes_k A)$-modules, and let $I = ker(m)$. We define the functor $\Omega := - \otimes_A I : A\text{-grmod} \to A\text{-grmod}$. Note that Ω has a right adjoint $\Omega' := \mathrm{Hom}^\bullet_{A\text{-grmod}}(I, -)$.

Since I is projective both as a right and left A-module, Ω is exact and $\Omega(A\text{-grproj}) \subset A\text{-grproj}$. Thus Ω lifts to $D^b(A\text{-grmod})$ and descends to $A\text{-grstab}$. Additionally, Ω preservers the thick subcategory ($D^{perf}(A\text{-grmod})$). The complex $P^\bullet = 0 \to I \hookrightarrow A^{op} \otimes_k A \twoheadrightarrow A \to 0$ is an exact complex of projective right A-modules, hence is homotopy equivalent to zero. Then for any $X \in A\text{-grmod}$, we have that $X \otimes_A P^\bullet$ is homotopy equivalent to zero, hence exact. But $X \otimes_A P^\bullet \cong 0 \to \Omega X \hookrightarrow X \otimes_k A \twoheadrightarrow X \to 0$, hence ΩX is the kernel of a surjection from a projective right A-module onto X. Thus Ω is an autoequivalence of $A\text{-grstab}$ and is isomorphic to the desuspension functor for the triangulated structure. In $A\text{-grstab}$, ΩX is isomorphic to the kernel of a projective cover of X.

Similarly, for any complex $X^\bullet \in Comp^b(A\text{-grmod})$, we have a short exact sequence of complexes $0 \to \Omega(X^\bullet) \hookrightarrow X^\bullet \otimes_k A \twoheadrightarrow X^\bullet \to 0$. From the resulting triangle in $D^b(A\text{-grmod})$, we obtain a natural transformation $[-1] \to \Omega$. Since $X^\bullet \otimes_k A \in D^{perf}(A\text{-grmod})$, this natural transformation descends to a natural isomorphism $[-1] \to \Omega$ in $A\text{-grstab}$.

By a similar argument, Ω defines a functor $A\text{-dgmod} \to A\text{-dgmod}$ which is exact and preserves direct summands of A. Thus Ω lifts to $D^b_{dg}(A)$ and preserves $D^{perf}_{dg}(A)$, and so Ω descends to $A\text{-dgstab}$. We also have a natural transformation $(-1) \to \Omega$ of endofunctors of $D^b_{dg}(A)$ which descends to an isomorphism in $A\text{-dgstab}$.

Similarly, Ω' is exact and preserves projective modules, and so descends to $A\text{-grstab}$ and $A\text{-dgstab}$ and lifts to $D^b(A\text{-grmod})$ and $D^b_{dg}(A)$. Since Ω' is right adjoint to Ω, we have that Ω' is quasi-inverse to Ω in $A\text{-grstab}$ and $A\text{-dgstab}$.

For any $X \in A\text{-grmod}$, we can construct a projective resolution $(P^\bullet_X, d^\bullet_{P_X})$ of X, such that $coker(d^{-n-1}_{P_X}) = \Omega^n(X)$ for any $n \geq 0$. More specifically, for $n \geq 0$, we let $P^{-n}_X = \Omega^n X \otimes_k A$, and for $n \geq 1$ we let $d^{-n}_{P_X}$ be the composition $P^{-n}_X = \Omega^n X \otimes_k A \twoheadrightarrow \Omega^n X \hookrightarrow P^{-n+1}_X$. Likewise, we can construct an injective resolution $(I^\bullet_X, d^\bullet_{I_X})$ such that $ker(d^n_{I_X}) = (\Omega')^n(X)$ for all $n \geq 0$. We let $I^n_X = \mathrm{Hom}^\bullet_{A\text{-grmod}}(A^{op} \otimes_k A, (\Omega')^n(X))$ and define the differential analogously. Joining $P^\bullet_X$ and $I^\bullet_X$ via the map $P^0_X \twoheadrightarrow X \hookrightarrow I^0_X$, we can define an acyclic biresolution $(B^\bullet_X, d^\bullet_{B_X})$ with $B^n_X = I^n_X$ for $n \geq 0$ and $B^n_X = P^{n+1}_X$ for $n < 0$. We refer to these resolutions as the **standard resolutions** of X.

2.7 Dg-categories

We give a brief introduction to the terminology and machinery of dg-categories. For more details, the reader may consult Keller [Kel06].

A **differential graded category** or **dg-category** over a field k is a category enriched over $k\text{-dgMod}$. A **dg-functor** between two dg-categories is a functor of categories enriched over $k\text{-dgMod}$. Thus morphism spaces in a dg-category have a natural structure of complexes of k-vector spaces, and dg-functors induce morphisms of complexes on these Hom spaces. Let $dgcat_k$ denote the category of all small dg-categories over the field k.

Given a dg-category $\mathcal{A}$, we define the **homotopy category** of $\mathcal{A}$ to be the category $H^0(\mathcal{A})$ whose objects are the objects of $\mathcal{A}$ and whose morphsms are given by $\mathrm{Hom}_{H^0(\mathcal{A})}(X, Y) := H^0(\mathrm{Hom}_{\mathcal{A}}(X, Y))$. Similarly, we define $Z^0(\mathcal{A})$ to be the category with the same objects whose morphisms are the closed degree 0 morphisms of $\mathcal{A}$.

A dg-functor $F : \mathcal{A} \to \mathcal{B}$ is called a **quasi-equivalence** if F induces quasi-isomorphisms on all morphism spaces and $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is an equivalence. By inverting the quasi-equivalences in $dgcat_k$, we obtain the **homotopy category of dg-categories**, denoted $Ho(dgcat_k)$.

Define the **dg-category of dg k-modules** $k\text{-dgMod}_{dg}$ as follows: The objects of $k\text{-dgMod}_{dg}$ are the differential graded k-modules. The degree n piece of $\mathrm{Hom}_{k\text{-dgMod}_{dg}}(X, Y)$ is defined to be $\bigoplus_i \mathrm{Hom}_k(X^i, Y^{i+n})$, and the differential is $d_n(f_i) = d_{Y^{n+i}} f_i + (-1)^n f_i d_{X^{i-1}}$.

Given a dg-category $\mathcal{A}$, we define a **right dg $\mathcal{A}$-module** to be a dg-functor $M : \mathcal{A}^{op} \to k\text{-dgMod}_{dg}$. A morphism of dg $\mathcal{A}$-modules is a natural transformation of dg-functors. We let $\mathcal{A}\text{-dgMod}$ denote the category of dg $\mathcal{A}$-modules. Moreover, we can define the **dg-category of dg $\mathcal{A}$-modules**, $\mathcal{A}\text{-dgMod}_{dg}$ in analogy to $k\text{-dgmod}_{dg}$.

Each object X of $\mathcal{A}$ defines a dg $\mathcal{A}$-module $X^\wedge : Y \mapsto \mathrm{Hom}_{\mathcal{A}}(Y, X)$, and $\wedge : X \mapsto X^\wedge$ defines a fully faithful dg-functor from $\mathcal{A}$ to $\mathcal{A}\text{-dgMod}_{dg}$. We say a dg-module M is **representable** if it is isomorphic to a dg-module in the image of $\wedge$. We can define the shift $M[n]$ of a dg-module M by $M[n](X) = M(X)[n]$. Similarly, we define the cone of a morphism of dg-modules $f : M \to N$ to be the dg-module given by $C(f)(X) = C(f_X)$. The homology of a dg-module is also defined object-wise.

Note that when $\mathcal{A}$ has one object, $*$, we can identify $\mathcal{A}$ with the dg-algebra $A = \mathrm{End}_{\mathcal{A}}(*)$; in this case a dg $\mathcal{A}$-module is the same data as a dg A-module.

Localizing $\mathcal{A}\text{-dgMod}$ at the quasi-isomorphisms, we obtain the **derived category** $D_{dg}(\mathcal{A})$ of dg $\mathcal{A}$-modules. Each dg-functor $F : \mathcal{A} \to \mathcal{B}$ defines an $\mathcal{A}^{op} \otimes \mathcal{B}$-module given by $X_F : (A, B) \mapsto \mathrm{Hom}_{\mathcal{B}}(B, FA)$. Denote by $rep(\mathcal{A}, \mathcal{B})$ the set of objects $X \in D_{dg}(\mathcal{A}^{op} \otimes \mathcal{B})$ such that for all $A \in \mathcal{A}$, $X(A, -)$ is isomorphic in $D_{dg}(\mathcal{B})$ to a representable $\mathcal{B}$-module. The correspondence $F \mapsto X_F$ defines a bijection between the morphisms $\mathcal{A} \to \mathcal{B}$ in $Ho(dgcat_k)$ and the isomorphism classes of objects in $rep(\mathcal{A}, \mathcal{B})$. This correspondence respects composition, where composition of bimodules is given by the derived tensor product.

A dg-category $\mathcal{A}$ is **pretriangulated** if for every $X \in \mathcal{A}$ and $n \in \mathbb{Z}$, the dg-module $X^\wedge[n]$ is representable, and for any closed morphism f of degree zero, $C(f^\wedge)$ is representable. If $\mathcal{A}$ is pretriangulated, then $H^0(\mathcal{A})$ is a triangulated category, with shift and cone induced from $\mathcal{A}$. Every dg-category $\mathcal{A}$ embeds into a **pretriangulated hull** $PreTr(\mathcal{A})$. The pretriangulated hull has an explicit construction, due to Bondal and Kapranov [BK91], in terms of one-sided twisted complexes. (Note that [BK91] uses the notation $PreTr^+(\mathcal{A})$ for this construction.) Every morphism in $Ho(dgcat_k)$ from a dg-category $\mathcal{A}$ to a pretriangulated dg-category $\mathcal{B}$ factors uniquely through $PreTr(\mathcal{A})$, and if $\mathcal{A}$ is already pretriangulated then it is quasi-equivalent to $PreTr(\mathcal{A})$. Define the **triangulated hull** of $\mathcal{A}$ to be the triangulated category $Tr(\mathcal{A}) := H^0(PreTr(\mathcal{A}))$. $H^0(\mathcal{A}) \hookrightarrow Tr(\mathcal{A})$ generates $Tr(\mathcal{A})$ as a triangulated category; if $\mathcal{A}$ is pretriangulated, then $H^0(\mathcal{A})$ is equivalent to $Tr(\mathcal{A})$.

Given a dg-category $\mathcal{A}$ and a full dg-subcategory $\mathcal{B}$, there is a dg-category $\mathcal{A}/\mathcal{B}$, called the **dg-quotient of $\mathcal{A}$ by $\mathcal{B}$**. The dg-quotient is obtained by "setting all objects of $\mathcal{B}$ to zero" by formally adjoining a morphism $\mathcal{E}_B$ for each object $B \in \mathcal{B}$ satisfying $d(\mathcal{E}_B) = id_B$. (See Drinfeld [Dri04] for details.) The dg-quotient generalizes the Verdier quotient in the sense that $Tr(\mathcal{A})/Tr(\mathcal{B}) \cong Tr(\mathcal{A}/\mathcal{B})$. It is characterized by the universal property that any morphism $\mathcal{A} \to \mathcal{C}$ in $Ho(dgcat_k)$ sending each object of $\mathcal{B}$ to a contractible object in $\mathcal{C}$ factors uniquely through $\mathcal{A}/\mathcal{B}$ (Tabuada, [Tab10]).

Given a dg-category $\mathcal{A}$ and $F \in rep(\mathcal{A}, \mathcal{A})$, one can define the dg-orbit category $P : \mathcal{A} \to \mathcal{A}/F$. This category is characterized by the universal property that for any dg-category $\mathcal{B}$ and any $G \in rep(\mathcal{A}, \mathcal{B})$ such that $G \circ F \cong G$, G factors through P. For a more precise description, see Keller [Kel05, Section 9.3].

2.8 Autoequivalences and Automorphisms

Given any category $\mathcal{C}$ and an autoequivalence $F : \mathcal{C} \to \mathcal{C}$, there is a category $\widetilde{\mathcal{C}}$, an automorphism $\widetilde{F} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$, and an equivalence of categories $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ such that $\pi \circ \widetilde{F} = F \circ \pi$. Identifying $\mathcal{C}$ with $\widetilde{\mathcal{C}}$, we can assume without loss of generality that the autoequivalence F of $\mathcal{C}$ is an automorphism.

2.9 Left Dg-Modules

In this paper, we work exclusively with right dg-modules. All the results presented are valid for left dg-modules, but minor adjustments must be made to account for numerous unpleasant sign conventions. We describe the necessary adjustments here.

If (A, d_A) is a differential graded k-algebra, we define a **left differential graded A-module** (or dg-module, for short) to be a pair (X, d_X) consisting of a graded left A-module X and a degree 1 k-linear differential $d_X : X \to X$ satisfying $d_X(ax) = d_A(a)x + (-1)^{|a|} a d_X(x)$ for all homogeneous $a \in A, x \in X$. We let $A\text{-dgmod}_l$ denote the category of left dg-modules over A.

If A is a graded algebra, we define the algebra $(\overline{A}, \circ)$ to be the set A with multiplication given by $a \circ b = (-1)^{|a||b|}(ab)$. Similarly, if M is a graded right A-module, we denote by $(\overline{M}, \circ)$ the graded right $\overline{A}$-module with M as the underlying set and the operation given by $m \circ a = (-1)^{|m||a|} ma$. We define $\overline{M}$ similarly for left graded modules. The functor sending M to $\overline{M}$ and acting as the identity on morphisms defines an isomorphism between $A\text{-grmod}$ and $\overline{A}\text{-grmod}$.

Let A^{op} denote the opposite algebra. We call $\overline{A}^{op}$ the graded opposite algebra. If (A, d_A) is a dg-algebra, then $(\overline{A}^{op}, d_A)$ is also a dg-algebra. If $(M, d_M) \in A\text{-dgmod}$, then $(\overline{M}, d_M) \in \overline{A}^{op}\text{-dgmod}_l$, and this defines an isomorphism of categories.

As before, there is a faithful functor $\widehat{\ }: Comp(A\text{-grmod}_l) \to A\text{-dgMod}_l$ sending the complex $(M^\bullet, d_M^\bullet)$ to the dg-module $(\widehat{M}, d_{\widehat{M}})$. The underlying graded module $(\widehat{M}, *)$ is given by $\widehat{M} = \bigoplus_{n\in\mathbb{Z}} M^n(-n)$; the operation $*$ is defined by $a * m = (-1)^{|a|n} am$, where $m \in M^n(-n)$. The differential $d_{\widehat{M}}$ restricts to $d_M^n(-n)$ on $M^n(-n)$. This definition of $\widehat{\ }$ is equivalent to converting to complexes of right $\overline{A}^{op}$-modules, applying the original definition of $\widehat{\ }$, and then converting back to left dg-modules over A.

If M is a left dg-module, define the dg-grading shift functor $\langle n\rangle : (M, d_M) \mapsto (M\langle n\rangle, d_{M\langle n\rangle})$. The underlying set of the left graded module $(M\langle n\rangle, \cdot_n)$ is $M(n)$, and the operation $\cdot_n$ is given by $a \cdot_n m = (-1)^{|a|n} am$. The differential is given by $d_{M\langle n\rangle} = (-1)^n d_M$. Triangles in the homotopy or derived categories take the form $X \to Y \to Z \to X\langle 1\rangle$. For $M^\bullet \in Comp(A\text{-grmod}_l)$ we have that $\widehat{M^\bullet[n]} = \widehat{M}\langle n\rangle$ and $\widehat{M^\bullet(n)} = \widehat{M}(n)$.

If X and Y are graded modules, we say that a function $f : X \to Y$ is a **graded skew-morphism of degree** n if it is a degree n k-linear map such that $f(ax) = (-1)^{n|a|} af(x)$ for all $x \in X$ and all homogeneous $a \in A$. We say two morphisms of left dg-modules $f, g : X \to Y$ are **homotopic** if there is a graded skew-morphism $h : X \to Y$ of degree -1 such that $f - g = h \circ d_X + d_Y \circ h$. We also note that if A has zero differential, then d_A is a graded skew-morphism of degree 1.

2.10 Rooted Plane Trees

A **tree** is a connected graph without cycles. We write V_T and E_T for the sets of vertices and edges, respectively, in T; the subscripts will be omitted when there is no risk of confusion. A **rooted tree** is a pair (T, r) where $r \in V_T$; r is called the **root** of T. Each vertex v admits a unique minimal path γ_v to the root; the **depth**, $d(v)$, of v is the number of edges in this path. If a vertex $u \neq v$ lies on γ_v, we say that u is an **ancestor** of v and that v is a **descendant** of u. Each vertex $v \neq r$ has a unique adjacent ancestor, called the **parent** of v, denoted $p(v)$. For any vertex v, we say u is a **child** of v if v is the parent of u, and we denote by $c(v)$ the set of children of v. We say v is a **leaf** if v has no children.

For a finite rooted tree (T, r), we define the **weight** $W(v)$ of v to be the number of vertices of the subtree consisting of v and its descendants; thus $W(r) = |V_T|$ and $W(v) = 1$ if and only if v is a leaf. It is clear that $W(v)$ is given recursively by $W(v) = 1 + \sum_{u \in c(v)} W(u)$. There is a one-to-one correspondence between the edges of (T, r) and the non-root vertices, with each edge corresponding to the incident vertex of greater depth. Using this bijection, we define the **weight** of an edge $W(e)$ to be the weight of the corresponding vertex.

When we wish to emphasize the dependence on a choice of root, we will write $d_r(v)$, $p_r(v)$, $W_r(v)$, etc.

Let T be a tree with n edges. We say a rooted tree (T, r) is **balanced** if for all $v \in c(r)$ (or, equivalently, for all $v \neq r$), $W(v) \leq \lfloor \frac{n+1}{2} \rfloor$. If v is a child of r in (T, r), we say the rooted tree (T, v) is a **rebalancing of** (T, r) **in the direction of** v.

A **plane tree** is a tree T together with a cyclic ordering of the edges incident to each vertex. One can specify this data by drawing T in the plane such that each vertex is locally embedded.

Let $\mathcal{T}_n$ denote the set of (isomorphism classes of) trees with n edges. Let $\mathcal{PT}_n$ denote the set of (isomorphism classes of) plane trees with n edges.

Example 2.10.1. Let T be a line with n edges. If n is even, then (T, r) is balanced if and only if r is the middle vertex of T. If n is odd, then (T, r) is balanced if and only if r is either of the vertices incident to the middle edge of T.

Proposition 2.10.2. *Let T be a tree with n edges. Then there exists $r \in V_T$ such that (T, r) is balanced. Either T has a unique balancing root, or it has exactly two balancing roots r and r'. In the latter case, n is odd, r and r' are adjacent and the edge joining them has weight $\frac{n+1}{2}$. Conversely, if n is odd and the rooted tree (T, r) has an edge $r - r'$ of weight $\frac{n+1}{2}$, then r and r' are both balancing roots of T.*

Proof. Choose a vertex $r \in V_T$ such that the quantity $max\{W_r(w) \mid w \in c_r(r)\}$ is minimized. Suppose that (T, r) is not balanced. Choose the (necessarily unique) vertex $v \in c_r(r)$ such that $W_r(v) > \lfloor \frac{n+1}{2} \rfloor$, and consider the tree (T, v). We will show that $W_v(w) < W_r(v) = max\{W_r(w) \mid w \in c_r(r)\}$ for all $w \in c_v(v)$, which will contradict the minimality of r.

For all $w \in c_v(v) - \{r\}$, we have that $W_v(w) = W_r(w) < W_r(v)$. Finally,

$$W_v(r) = 1 + \sum_{w \in c_v(r)} W_v(w) = 1 + \sum_{w \in c_r(r) - \{v\}} W_r(w) = n + 1 - W_r(v)$$
$$< n + 1 - \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil$$

hence $W_v(r) \leq \lfloor \frac{n+1}{2} \rfloor < W_r(v)$. We have obtained our contradiction, thus (T, r) is balanced.

Next, suppose (T, r) and (T, r') are balanced, with $r \neq r'$. Let v be the parent of r' with respect to (T, r). Then, since (T, r') is balanced, we have that

$$W_r(r') = 1 + \sum_{w \in c_r(r')} W_r(w) = 1 + \sum_{w \in c_{r'}(r') - \{v\}} W_{r'}(w) = n + 1 - W_{r'}(v)$$
$$\geq n + 1 - \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil$$

But since (T, r) is balanced, we have that $W_r(r') \leq \lfloor \frac{n+1}{2} \rfloor \leq \lceil \frac{n+1}{2} \rceil$. Thus $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil$, hence n is odd. Furthermore, $W_r(v) > W_r(r') = \frac{n+1}{2}$, which implies that $v = r$. Thus r and r' are adjacent, and the edge between them has weight $W_r(r') = \frac{n+1}{2}$. Since the total weight of the children of r is n, there can be no other children of weight $\frac{n+1}{2}$, hence r and r' are the only balancing vertices of T.

For the final statement, suppose r and r' are adjacent vertices in T such that $W_r(r') = \frac{n+1}{2}$. The other children of r have weight at most $n - \frac{n+1}{2} = \frac{n-1}{2}$, hence (T, r) is balanced. We have already seen that $W_{r'}(r) = n + 1 - W_r(r') = \frac{n+1}{2}$; a symmetric argument then shows that (T, r') is also balanced. □

Remark. One can find the balancing root(s) of a tree T via a simple algorithm: Pick an arbitrary vertex r as the root. If the tree is not balanced, rebalance the tree in the (unique) direction of the highest weighted child of r, until the tree is balanced. If the balancing root has an incident edge of weight $\frac{n+1}{2}$, then both vertices incident to this edge are balancing roots.

CHAPTER 3

The Dg-Stable Category

3.1 A Structure Theorem for the Dg-Stable Category

3.1.1 Construction of the Orbit Category

Let A be a finite-dimensional, non-positively graded, self-injective k-algebra, viewed as a dg-algebra with zero differential. In this section, we shall provide a description of the dg-stable category of A in terms of the graded stable category. In Definition 3.1.5, we define the orbit category $\mathcal{C}(A) = A\text{-grstab}/\Omega(1)$. In Definition 3.1.8 we define a functor $F_A : \mathcal{C}(A) \to A\text{-dgstab}$ and in Theorem 3.1.10 we show that F_A is fully faithful with essential image generating $A\text{-dgstab}$ as a triangulated category.

We begin with some simple facts about graded A-modules.

Proposition 3.1.1. *Let* $X, Y \in A\text{-grmod}$. *If* $supp(X) \cap supp(Y) = \emptyset$, *then* $\mathrm{Hom}_{A\text{-grmod}}(X, Y) = 0$ *and* $\mathrm{Hom}_{A\text{-grstab}}(X, Y) = 0$.

Proof. The first part of the statement follows immediately from the definition of morphisms of graded modules. Since $\mathrm{Hom}_{A\text{-grstab}}(X, Y)$ is defined as a quotient of $\mathrm{Hom}_{A\text{-grmod}}(X, Y)$, the second part of the statement follows from the first. □

Proposition 3.1.2. *Let* $X \in A\text{-grmod}$. *Then*

$$max(hd(X)) = max(X)$$
$$min(soc(X)) = min(X)$$

Proof. The radical of A is a graded submodule of A (see Kelarev, [Kel92]), and so $rad(X) =$

$Xrad(A)$ is a graded submodule of X. Thus $hd(X) = X/rad(X)$ is graded with $supp(hd(X)) \subset supp(X)$. Therefore $max(hd(X)) \leq max(X)$.

To establish the reverse inequality, take a nonzero element $x \in X^{max(X)}$. If $x \notin rad(X)$, then the image of x in $hd(X)$ is a nonzero element in degree $max(X)$, and we are done. Suppose $x \in rad(X)$. Note that since X is finitely generated and A is finite-dimensional, X is also finite dimensional. Thus $rad^k(X)$ becomes zero for sufficiently large k. Since x is nonzero, there is a maximum $n > 0$ such that $x \in rad^n(X) = Xrad^n(A)$. Write $x = \sum_{i=1}^{m} x_i a_i$ for some homogeneous $a_i \in rad^n(A), x_i \in X$. Without loss of generality, we may assume that all terms are nonzero and that $deg(x_i a_i) = deg(x)$ for all i. Since $deg(x) = max(X)$ and A is non-positively graded, we must have that $deg(x_i) = max(X)$ and $deg(a_i) = 0$ for all i. Since each $a_i \in rad^n(A)$ and $x \notin rad^{n+1}(X)$, there must be some j such that $x_j \notin rad(X)$. Thus we have obtained a nonzero $x_j \in X^{max(X)} - rad(X)$, and so $max(hd(X)) = max(X)$.

For the second equation, note that $soc(X)$ is a graded submodule of X (see Năstăsescu and Van Oystaeyen, [NV85]). Thus $supp(soc(X)) \subset supp(X)$ and so $min(soc(X)) \geq min(X)$.

For the reverse inequality, it suffices to show that $soc(X) \cap X^{min(X)} \neq 0$. Since A is non-positively graded, $X^{min(X)}A \subset X^{min(X)}$ and so $X^{min(X)}$ is a submodule of X. Since X is finite-dimensional, $X^{min(X)}$ has a simple submodule and thus has nonzero intersection with $soc(X)$. Therefore $min(soc(X)) = min(X)$. □

Proposition 3.1.3. *Let $X \in Ob(A\text{-}grmod)$. Then*

1) $max(\Omega X) \leq max(X)$

2) $min(\Omega' X) \geq min(X)$

3) $max(P_X^{-n}(n)) = max(\Omega^n X(n)) \leq max(X) - n$

4) $min(I_X^n(-n)) = min((\Omega')^n X(-n)) \geq min(X) + n$

(See Section 2.6 for notation.)

Proof. Since $I \subset A^{op} \otimes_k A$, we have that $max(I) \leq max(A^{op} \otimes_k A) = 0$. Thus $max(\Omega X) = max(X \otimes_A I) \leq max(I) + max(X) \leq max(X)$.

Similarly, $min(\Omega' X) = min(\mathrm{Hom}_A^{\bullet}(I, X)) \geq min(X) - max(I) \geq min(X)$.

The last two equations follow from the first two and the definitions of the standard projective and injective resolutions. $\square$

Recall from Section 2.6 that the functor $\Omega(1)$ is an autoequivalence of A-grstab and A-dgstab. By replacing A-grstab and A-dgstab with equivalent categories $\widetilde{A\text{-grstab}}$ and $\widetilde{A\text{-dgstab}}$ (see Section 2.8), we may assume without loss of generality that $\Omega(1)$ is an automorphism of both categories. We let Ω^{-1} denote the inverse of Ω, and we shall identify it with the isomorphic functor Ω'.

Going forward, we shall write Ω^{-n} to mean $(\Omega')^n$ for $n \geq 0$, even on A-grmod and A-dgmod. This is a dangerous abuse of notation as Ω is not invertible in either category. However, adopting this convention allows us to greatly simplify certain expressions and is safe as long as we avoid expressions of the form $\Omega\Omega^{-1}X$ outside the stable category.

We obtain the following corollary of Proposition 3.1.3.

Proposition 3.1.4. *Let* $X, Y \in Ob(A\text{-}grmod)$. *Then* $\mathrm{Hom}_{A\text{-}grstab}(X, \Omega^n Y(n)) = 0$ *for all but finitely many* $n \in \mathbb{Z}$.

Proof. By Proposition 3.1.3, $max(\Omega Y(1)) \leq max(Y(1)) = max(Y) - 1$ and so $max(\Omega^n Y(n)) \leq max(Y) - n$. Thus for $n >> 0$, we have that $max(\Omega^n Y(n)) < min(X)$. We also have that $min(\Omega^{-n} Y(-n)) \geq min(Y) + n$, and so for $n >> 0$ we have that $min(\Omega^{-n} Y(-n)) > max(X)$. Thus $\mathrm{Hom}_{A\text{-grmod}}(X, \Omega^n Y(n)) = 0$ for all but finitely many n. $\square$

We are now ready to state the main definitions.

Definition 3.1.5. Let $\mathcal{C}(A)$ be the category given by:

1) $Ob(\mathcal{C}(A)) = Ob(A\text{-grstab})$

2) For $X, Y \in Ob(\mathcal{C}(A))$, $\mathrm{Hom}_{\mathcal{C}(A)}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n Y(n))$

3) For $(f_n)_{n \in \mathbb{Z}} : X \to Y$ and $(g_m)_{m \in \mathbb{Z}} : Y \to Z$, define composition by

$(g_m) \circ (f_n) = (\sum_{i \in \mathbb{Z}} \Omega^i g_{j-i}(i) \circ f_i)_{j \in \mathbb{Z}}$.

Remark. If we do not wish to assume that $\Omega(1)$ is an automorphism of A-grstab, natural isomorphisms $\varepsilon_{n,m} : \Omega^n \Omega^m \to \Omega^{n+m}$ satisfying the appropriate coherence conditions must be inserted into the composition formula.

We note that the sum in the composition formula is finite by Proposition 3.1.4. It is clear that $\mathcal{C}(A)$ is an additive category. In fact, $\mathcal{C}(A)$ is precisely the orbit category $A\text{-grstab}/\Omega(1)$ as defined by Keller, [Kel05]. Keller shows that while such a category need not be triangulated, it can always be included in a "triangulated hull". We shall see that $A\text{-dgstab}$ is the triangulated hull of $\mathcal{C}(A)$.

Proposition 3.1.6. *The orbit category $\mathcal{C}(A)$ is a Krull-Schmidt category.*

Proof. Since $A\text{-grstab}$ is Krull-Schmidt and the natural map $A\text{-grstab} \to \mathcal{C}(A)$ is additive and essentially surjective, any object of $\mathcal{C}(A)$ can be written as a direct sum of indecomposable objects in $A\text{-grstab}$. Thus, it suffices to show that any indecomposable object X of $A\text{-grstab}$ has local endomorphism ring in $\mathcal{C}(A)$.

First, we claim that for any indecomposable X in $A\text{-grmod}$ and any $i \neq 0$, any map $f : X \to \Omega^i X(i) \to X$ lies in $rad(\mathrm{End}_{A\text{-grmod}}(X))$. By Proposition 3.1.3, $supp(X) \not\subseteq supp(\Omega^i X(i))$, hence f cannot be surjective and thus is not an isomorphism. Since $\mathrm{End}_{A\text{-grmod}}(X)$ is a local finite-dimensional algebra, f lies in the unique maximal two-sided ideal, which is equal to the Jacobson radical. Since $\mathrm{End}_{A\text{-grstab}}(X)$ is a quotient of $\mathrm{End}_{A\text{-grmod}}(X)$, we also have that the image of f lies in $rad(\mathrm{End}_{A\text{-grstab}}(X))$.

Let $X \in A\text{-grstab}$ be indecomposable. We must show that $\mathrm{End}_{\mathcal{C}(A)}(X)$ is local. Write $V_n = \mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n(X)(n))$ for the n-th graded component of $\mathrm{End}_{\mathcal{C}(A)}(X)$. We claim that the subspace $V := rad(V_0) \oplus \bigoplus_{n \neq 0} V_n$ is the unique maximal two-sided ideal of $\mathrm{End}_{\mathcal{C}(A)}(X)$.

To show V is a two-sided ideal, take f_i in the ith graded piece of V and $g_j \in V_j$. If $i + j \neq 0$, then $g_j f_i \in V$. If $i = -j \neq 0$, then $g_j f_i : X \to \Omega^i X(i) \to X$ is an element of $rad(V_0)$. Finally, if $i = j = 0$, then we immediately have that $g_0 f_0 \in rad(V_0)$. Thus V is a left ideal, and a parallel argument shows it is a right ideal.

Clearly, $\mathrm{End}_{\mathcal{C}(A)}(X)/V \cong \mathrm{End}_{A\text{-grstab}}(X)/rad(\mathrm{End}_{A\text{-grstab}}(X))$, which is a division ring since X is indecomposable. Thus V is maximal.

To show V is the unique maximal ideal, it suffices to show that it is equal to $rad(\mathrm{End}_{\mathcal{C}(A)}(X))$. As a maximal two-sided ideal, V contains the radical. For the reverse inclusion, it suffices to show that every element $f = (f_i) \in V$ is nilpotent, since the Jacobson radical contains every nil ideal.

Let N be such that $V_i = 0$ for all $|i| > N$. Then note that the i-th graded piece of f^n is a sum of maps of the form $X \xrightarrow{h_1} \Omega^{i_1}X(i_1) \xrightarrow{h_2} \Omega^{i_2}X(i_2) \cdots \xrightarrow{h_n} \Omega^i X(i)$, where the h_j are translations of the f_k by various powers of $\Omega(1)$. If $|i_j| > N$ for any j, the composite map is zero, so we may assume that $|i_j| \leq N$ for all $1 \leq j \leq n$. If $n > m(2N+1)$ for some $m \geq 1$, then by the pigeonhole principle there exists $-N \leq r \leq N$ such that $\Omega^r X(r)$ appears as the codomain of one of the h_j at least $m+1$ times. Grouping terms, we can then express the composition as $X \xrightarrow{\phi_0} \Omega^r X(r) \xrightarrow{\phi_1} \Omega^r X(r) \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} \Omega^r X(r) \xrightarrow{\phi_{m+1}} \Omega^i X(i)$, where the ϕ_k are compositions of successive h_j.

I claim that for all $1 \leq k \leq m$, ϕ_k lies in $rad(\mathrm{End}_{A\text{-grstab}}(\Omega^r X(r)))$. Note that since f_0 lies in the radical of $\mathrm{End}_{A\text{-grstab}}(X)$ and $\Omega(1)$ is an autoequivalence, we have that $\Omega^r(f)(r)$ lies in the radical of the local ring $\mathrm{End}_{A\text{-grstab}}(\Omega^r X(r))$. Thus, if one of the factors of ϕ_k is $\Omega^r(f_0)(r)$, then we are done. If not, then we must have that ϕ_k factors through $\Omega^j X(j)$ for some $j \neq r$, which again guarantees that ϕ_k lies in the radical.

We have shown that for $n > m(2N+1)$, each component of f^n is a sum of terms of the form $\prod_{i=0}^{m+1} \phi_i$, with $\phi_i \in rad(\mathrm{End}_{A\text{-grstab}}(\Omega^r X(r)))$ for each $1 \leq i \leq m$ and some $-N \leq r \leq N$. But $rad(\mathrm{End}_{A\text{-grstab}}(\Omega^r X(r)))$ is nilpotent for each r, hence f is nilpotent. $\square$

We now define the inclusion functor $F_A : \mathcal{C}(A) \to A$-dgstab. The obvious choice would be for F_A to act as the identity on objects and send the morphism $(f_n)_n : X \to Y$ to the sum of its components $\sum_{n\in\mathbb{Z}} \psi_{n,Y} \circ f_n$, where the $\psi_{n,Y} : \Omega^n Y(n) \to Y$ are isomorphisms chosen so that all the summands share a common domain. However, in order for this process to be functorial, the morphisms $\psi_{n,Y}$ must be satisfy appropriate compatibility conditions.

Lemma 3.1.7. *There exists a family of natural isomorphisms* $\{\psi_n : \Omega^n(n) \to id_{A\text{-dgstab}} \mid n \in \mathbb{Z}\}$ *satisfying:*

i) $\psi_0 = id_{A\text{-dgstab}}$

ii) For all $n, m \in \mathbb{Z}, \psi_m \circ (\psi_n \circ \Omega^m(m)) = \psi_{n+m}$

Proof. Let $\psi_1 : \Omega(1) \to id_{A\text{-dgstab}}$ be the natural isomorphism defined in Section 2.6. Let $\psi_{-1} = (\psi_1 \circ \Omega^{-1}(-1))^{-1} : \Omega^{-1}(-1) \to id_{A\text{-dgstab}}$. Let $\psi_0 = id_{A\text{-dgstab}}$. For $n \geq 2$, recursively define $\psi_n = \psi_1 \circ (\psi_{n-1} \circ \Omega(1))$ and analogously for $n \leq -2$. It is clear that $\{\psi_n\}$ satisfies i) and ii). $\square$

Remark. If we do not assume that $\Omega(1)$ is an autormorphism of $A\text{-dgstab}$, we must again insert appropriately chosen natural isomorphisms $\varepsilon_{n,m} : \Omega^n\Omega^m \to \Omega^{n+m}$ into condition ii).

Definition 3.1.8. Let $\psi_n : \Omega^n(n) \to id_{A\text{-dgstab}}$ be the natural isomorphisms defined in Lemma 3.1.7. Let $F_A : \mathcal{C}(A) \to A$-dgstab be the functor given by:

1) F_A acts as the identity on objects.

2) Given $f = (f_n)_{n\in\mathbb{Z}} \in \mathrm{Hom}_{C(A)}(X, Y)$, let $F_A(f) = \sum_{n\in\mathbb{Z}} \psi_{n,Y} \circ f_n$.

Proposition 3.1.9. *F_A as defined above is a functor.*

Proof. Take $X \in \mathcal{C}(A)$. The identity morphism on X is $(\delta_{0,n}id_X)_n$. Therefore $F_A((\delta_{0,n}id_X)_n) = \sum_{n\in\mathbb{Z}} \psi_{n,X} \circ \delta_{0,n}id_X = \psi_{0,X} = id_X$. Thus F_A preserves identity morphisms.

Given $(f_n)_n : X \to Y$, $(g_m)_m : Y \to Z$ in $\mathcal{C}(A)$, we have

$$\begin{aligned}
F_A(g_m) \circ F_A(f_n) &= (\sum_m \psi_{m,Z} \circ g_m) \circ (\sum_n \psi_{n,Y} \circ f_n) \\
&= \sum_{m,n} \psi_{m,Z} \circ g_m \circ \psi_{n,Y} \circ f_n \\
&= \sum_{m,n} \psi_{m,Z} \circ \psi_{n,\Omega^m Z(m)} \circ \Omega^n g_m(n) \circ f_n \\
&= \sum_{m,n} \psi_{m+n,Z} \circ \Omega^n g_m(n) \circ f_n \\
&= \sum_{j=m+n} \psi_{j,Z} \circ (\sum_n \Omega^n g_{j-n}(n) \circ f_n) \\
&= F_A((g_m) \circ (f_n))
\end{aligned}$$

□

3.1.2 Embedding $\mathcal{C}(A)$ into $A\text{-dgstab}$

We now state the main theorem.

Theorem 3.1.10. *$F_A : \mathcal{C}(A) \to A$-dgstab is fully faithful, and the image of F_A generates A-dgstab as a triangulated category.*

We prove the theorem with a sequence of lemmas below.

Definition 3.1.11. Let $X, Y \in A\text{-grmod}$, viewed as dg-modules with zero differential. If X and Y are nonzero, let $N = N_{X,Y} := max\{n \leq 0 \mid max(\Omega^{-n}Y(-n)) < min(X)\}$. Define the **bridge complex from** X **to** Y to be the complex $R^{\bullet}_{X,Y} = B^{\geq N}_{Y}$ (see Sections 2.2 and 2.6 for notation) if X and Y are both nonzero, and $R^{\bullet}_{X,Y} = 0$ otherwise.

By Proposition 3.1.3, $N_{X,Y}$ is well-defined. We will omit the subscript when it is clear from context.

By unwinding the definitions, we obtain a quasi-isomorphism of complexes $\Omega^{-N}(Y)[-N] \hookrightarrow R^{\bullet}_{X,Y}$. In particular, $H^N(R^{\bullet}_{X,Y}) \cong \Omega^{-N}(Y)$ and $H^k(R^{\bullet}_{X,Y}) = 0$ for $k \neq N$. We also note that $ker(d^n_{R_{X,Y}})(-n) = \Omega^{-n}X(-n)$ for all $n \geq N$.

Morphisms in $A\text{-dgstab}$ can be represented as equivalence classes of roofs $X \xrightarrow{f} M \xleftarrow{s} Y$, where s has perfect cone. The primary challenge in understanding morphisms in $A\text{-dgstab}$ is that perfect dg-modules need not arise from complexes of graded projective modules. However, by restricting our attention to dg-modules with zero differential, we can bypass this difficulty by using the bridge complexes defined above.

Lemma 3.1.12. *Let $X, Y \in A\text{-grmod}$. Then any morphism in $\mathrm{Hom}_{A\text{-dgstab}}(X, Y)$ can be expressed as a roof of the form*

$$X \xrightarrow{f} \widehat{\tau_{\leq 0}R_{X,Y}} \xleftarrow{i} Y$$

where i is induced by the natural map $Y \to R^{\bullet}_{X,Y}$.

Proof. If X or Y is zero, then the result is immediate, so assume neither X nor Y is zero. Any morphism $X \to Y$ in $A\text{-dgstab}$ can be represented as a roof

$$X \xrightarrow{g} M \xleftarrow{s} Y$$

where $M \in A\text{-dgmod}$, $g, s \in \mathrm{Mor}(D^b_{dg}(A))$, and there is an exact triangle $P \xrightarrow{\alpha} Y \xrightarrow{s} M \to P(1)$ in $D^b_{dg}(A)$, with $P \in D^{perf}_{dg}(A)$. By changing P up to quasi-isomorphism, we may assume without loss of generality that P is strictly perfect.

Let p_n denote the natural map of complexes $P_Y^{\geq n} \hookrightarrow P_Y^\bullet \twoheadrightarrow Y$ and let $i_n : Y \hookrightarrow C(p_n)$ denote the natural inclusion of complexes. If $n \geq 1$, note that p_n is the map from the zero complex to Y and i_n is the identity map on Y. Note also that $C(p_{N+1}) = \tau_{\leq 0} R_{X,Y}^\bullet$ and $\widehat{i_{N+1}} = i$.

We first show that every morphism can be expressed as a roof of the form

$$\begin{array}{ccccc} X & & & & Y \\ & \searrow & & \swarrow{\scriptstyle \widehat{i_k}} & \\ & & \widehat{C(p_k)} & & \end{array}$$

for some $k \leq N+1$.

Since $p : P_Y^\bullet \twoheadrightarrow Y$ is a quasi-isomorphism of complexes, we obtain a morphism $\widehat{p}^{-1} \circ \alpha : P \to \widehat{P_Y}$ in $D_{dg}(A)$. Since $P \in D_{dg}^{perf}(A)$, the underlying graded module of P is finitely generated. Thus $supp(P)$ is bounded. Note that $max(P_Y^{-k}) = max(\Omega^k Y)$, thus by Proposition 3.1.3 the sequence $\{max(P_Y^{-k}(k)\}_k$ is strictly decreasing. Then we may choose $k << 0$ such that $k \leq N+1$ and $max(P_Y^{k-1}(-k+1)) < min(P)$. Then the short exact sequence of dg-modules $0 \to \widehat{P_Y^{\geq k}} \hookrightarrow \widehat{P_Y} \twoheadrightarrow \widehat{P_Y^{<k}} \to 0$ yields an exact triangle $\widehat{P_Y^{\geq k}} \to \widehat{P_Y} \to \widehat{P_Y^{<k}} \to \widehat{P_Y^{\geq k}}(1)$ in $D_{dg}(A)$. Since

$$max(\widehat{P^{<k}}) = max(P_Y^{k-1}(-k+1)) < min(P)$$

we have that $\mathrm{Hom}_{Ho_{dg}(A)}(P, \widehat{P_Y^{<k}}) = 0$. Since P is strictly perfect, morphisms in the derived and homotopy categories coincide, and so $\mathrm{Hom}_{D_{dg}(A)}(P, \widehat{P_Y^{<k}}) = 0$.

We obtain a morphism of triangles in $D_{dg}(A)$:

$$\begin{array}{ccccccc} P & \xrightarrow{id} & P & \longrightarrow & 0 & \longrightarrow & P(1) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \widehat{p}^{-1}\circ\alpha} & & \downarrow & & \downarrow{\scriptstyle h(1)} \\ \widehat{P_Y^{\geq k}} & \longrightarrow & \widehat{P_Y} & \longrightarrow & \widehat{P_Y^{<k}} & \longrightarrow & \widehat{P_Y^{\geq k}}(1) \end{array}$$

Postcomposing the left square with $\widehat{P_Y} \xrightarrow{\widehat{p}} X$, we obtain $\alpha = \widehat{p_k} \circ h$. We obtain a morphism of triangles in $D_{dg}^b(A)$:

$$\begin{array}{ccccccc} P & \xrightarrow{\alpha} & Y & \xrightarrow{s} & M & \longrightarrow & P(1) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h(1)} \\ \widehat{P_Y^{\geq k}} & \xrightarrow{\widehat{p_k}} & Y & \xrightarrow{\widehat{i_k}} & \widehat{C(p_k)} & \longrightarrow & \widehat{P_Y^{\leq k}}(1) \end{array}$$

Since $\widehat{P_Y^{\geq k}} \in D_{dg}^{perf}(A)$, the roof $\widehat{i_k}^{-1} \circ (g' \circ g) : X \to Y$ defines a morphism in A-dgstab. It follows immediately from the above diagram that the roofs $s^{-1} \circ g$ and $\widehat{i_k}^{-1} \circ (g' \circ g)$ are equivalent in A-dgstab.

It remains to show that k can be replaced by $N+1$. Since $k \leq N+1$ by definition, we have an exact sequence of dg-modules $0 \to \widehat{C(p_{N+1})} \hookrightarrow \widehat{C(p_k)} \twoheadrightarrow \widehat{(P_Y^{\geq k})^{\leq N}}(1) \to 0$ arising from the underlying exact sequence of complexes. We also have that

$$\begin{aligned} max(\widehat{(P_Y^{\geq k})^{\leq N}}(1)) &= max(P_Y^N(-N)(1)) \\ &= max(\Omega^{-N}Y(-N+1)) \\ &< min(X) \end{aligned}$$

The last inequality is true by definition of N. Thus $\mathrm{Hom}_{Ho_{dg}^b(A)}(X, \widehat{(P_Y^{\geq k})^{\leq N}}(1)) = 0$ and, since $\widehat{(P_Y^{\geq k})^{\leq N}}(1)$ is strictly perfect, $\mathrm{Hom}_{D_{dg}^b(A)}(X, \widehat{(P_Y^{\geq k})^{\leq N}}(1)) = 0$. We obtain a morphism of triangles in $D_{dg}^b(A)$:

$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & X(1) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g \circ g'} & & \downarrow & & \downarrow{\scriptstyle f(1)} \\ \widehat{C(p_{N+1})} & \hookrightarrow & \widehat{C(p_k)} & \longrightarrow & \widehat{(P_Y^{\geq k})^{\leq N}}(1) & \longrightarrow & \widehat{C(p_N)}(1) \end{array}$$

We also have that $\widehat{i_k}$ factors as $Y \xrightarrow{\widehat{i_{N+1}}} \widehat{C(p_{N+1})} \hookrightarrow \widehat{C(p_k)}$. It follows that the roof $\widehat{i_{N+1}}^{-1} \circ f$ defines a morphism in A-dgstab which is equal to $\widehat{i_k}^{-1} \circ (g' \circ g)$. Since $\widehat{C(p_N)} = \widehat{\tau_{\leq 0} R_{X,Y}}$ and $\widehat{i_{N+1}} = i$, we are done. $\square$

Having found a convenient choice of roofs between X and Y, we now investigate maps between X and $\widehat{\tau_{\leq 0} R_{X,Y}}$ in the derived category. This investigation shall yield a method for computing morphisms between zero-differential modules.

Lemma 3.1.13. *Let* $X, Y \in A$-grmod. *Then we have an isomorphism*

$$\begin{array}{rcl} \xi : \mathrm{Hom}_{Ho_{dg}^+(A)}(X, \widehat{R_{X,Y}}) & \xrightarrow{\sim} & \mathrm{Hom}_{A\text{-dgstab}}(X, Y) \\ f & \mapsto & X \xrightarrow{\phi^{-1} \circ f} \widehat{\tau_{\leq 0} R_{X,Y}} \xleftarrow{i} Y \end{array}$$

where ϕ is the natural inclusion of $\widehat{\tau_{\leq 0} R_{X,Y}}$ into $\widehat{R_{X,Y}}$.

Proof. Let $f \in \mathrm{Hom}_{Ho^+_{dg}(A)}(X, \widehat{R_{X,Y}})$. In order for ξ to be well-defined, we must show that $\phi^{-1} \circ f \in Mor(D_{dg}(A))$ can be represented by a roof in $D^b_{dg}(A)$. By Proposition 3.1.3, the sequence $\{min(\widehat{\tau_{>M} R_{X,Y}})\}_M$ strictly increases with M. Since X is finitely generated, there exists $M >> 0$ such that the image of f lies in $\widehat{\tau_{\leq M}(R_{X,Y}})$. It is clear that the inclusion ϕ also factors through $\widehat{\tau_{\leq M}(R_{X,Y}})$, and the inclusion of $\widehat{\tau_{\leq M}(R_{X,Y}})$ into $\widehat{R_{X,Y}}$ is a quasi-isomorphism. Thus $\widehat{R_{X,Y}}$ can be replaced by the bounded dg-module $\widehat{\tau_{\leq M}(R_{X,Y}})$ in the roof $\phi^{-1} \circ f$, and so we may view $\phi^{-1} \circ f$ as a morphism in $Mor(D^b_{dg}(A))$. Thus ξ is well-defined.

We now prove surjectivity of ξ. Since $R^\bullet_{X,Y} \in Ho^+(A\text{-grproj})$, $\mathrm{Hom}_{Ho_{dg}(A)}(X, \widehat{R_{X,Y}}) \cong \mathrm{Hom}_{D_{dg}(A)}(X, \widehat{R_{X,Y}})$. Post-composition with ϕ^{-1} yields an isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{Ho^+_{dg}(A)}(X, \widehat{R_{X,Y}}) & \xrightarrow{\sim} & \mathrm{Hom}_{D^b_{dg}(A)}(X, \widehat{\tau_{\leq 0} R_{X,Y}}) \\ f & \mapsto & \phi^{-1} \circ f \end{array}$$

It follows immediately from Lemma 3.1.12 that the map

$$\begin{array}{ccc} \mathrm{Hom}_{D^b_{dg}(A)}(X, \widehat{\tau_{\leq 0} R_{X,Y}}) & \twoheadrightarrow & \mathrm{Hom}_{A\text{-dgstab}}(X, Y) \\ g & \mapsto & X \xrightarrow{g} \widehat{\tau_{\leq 0} R_{X,Y}} \xleftarrow{i} Y \end{array}$$

is surjective. The composition of these two maps is precisely ξ, which is therefore surjective.

It remains to show injectivity. Suppose that $\xi(f) = 0$. Then there exists a morphism $s : \widehat{\tau_{\leq 0} R_{X,Y}} \to M$ in $D^b_{dg}(A)$ such that $s \circ \phi^{-1} \circ f = 0$ and $C(s)$ is strictly perfect. We obtain a morphism of triangles in $D^b_{dg}(A)$:

$$\begin{array}{ccccccc} X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & X(1) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \phi^{-1} \circ f} & & \downarrow & & \downarrow{\scriptstyle \alpha(1)} \\ C(s)(-1) & \xrightarrow{\beta} & \widehat{\tau_{\leq 0} R_{X,Y}} & \xrightarrow{s} & M & \longrightarrow & C(s) \end{array}$$

Since $C(s)(-1)$ is strictly perfect, we can choose to represent α by a morphism in $Ho^b_{dg}(A)$. From the above diagram and the fact that $R^\bullet_{X,Y} \in Comp^+(A\text{-grproj})$ it follows that $f = \phi \circ \beta \circ \alpha$ in $Ho^b_{dg}(A)$.

Note that the natural inclusion of complexes $\epsilon : \Omega^{-N}Y[-N] \hookrightarrow R^\bullet_{X,Y}$ is a quasi-isomorphism, hence so is $\widehat{\epsilon} : \Omega^{-N}Y(-N) \hookrightarrow \widehat{R_{X,Y}}$. Since $C(s)(-1)$ is strictly perfect, the roof $\widehat{\epsilon}^{-1} \circ \phi \circ \beta : C(s)(-1) \to \Omega^{-N}Y(-N)$ can be represented by a morphism γ in $Ho^b_{dg}(A)$. Repeating the argument of the previous paragraph, we have that $\phi \circ \beta = \widehat{\epsilon} \circ \gamma$ in $Ho^b_{dg}(A)$.

By the above two paragraphs, we have $f = \phi \circ \beta \circ \alpha = \widehat{\epsilon} \circ \gamma \circ \alpha$ in $Ho^b_{dg}(A)$. But this means that f factors through $\Omega^{-N}Y(-N)$ in the homotopy category. By definition of N, $max(\Omega^{-N}Y(-N)) < min(X)$ and so $\mathrm{Hom}_{Ho^b_{dg}}(X, \Omega^{-N}Y(-N)) = 0$. Thus $f = 0$ and ξ is injective. □

In the next three lemmas, we relate morphisms in $\mathcal{C}(A)$ to those in $A\text{-dgstab}$ via the homotopy category.

Lemma 3.1.14. *Let* $X \in A\text{-grmod}$. *Let* $(P^\bullet, d^\bullet_P) \in Comp(A\text{-grmod})$. *Suppose that*

$$\mathrm{Hom}_{A\text{-grmod}}(X, ker(d^n_P)(-n)) = 0$$

for almost all n. *Let* $i_n : ker(d^n_P)(-n) \hookrightarrow \widehat{P}$ *denote the inclusion (of dg-modules). Then the map*

$$\begin{aligned} \Phi : \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{A\text{-grmod}}(X, ker(d^n_P)(-n)) &\to \mathrm{Hom}_{A\text{-dgMod}}(X, \widehat{P}) \\ (f_n)_n &\mapsto \sum_n i_n \circ f_n \end{aligned}$$

is an isomorphism of vector spaces.

Proof. By hypothesis, the sum in the definition of Φ is finite, so Φ is a well-defined k-linear map. It remains to construct Φ^{-1}. Given $f \in \mathrm{Hom}_{A\text{-dgMod}}(X, \widehat{P})$, we have $d_{\widehat{P}} \circ f = f \circ d_X = 0$, since X has zero differential. Thus $im(f) \subset ker(d_{\widehat{P}}) = \bigoplus_n ker(d^n_P)(-n)$. Let π_n denote the projection onto the nth summand, and define $\Phi^{-1}(f) = (\pi_n \circ f)_n$; it is easy to verify that Φ^{-1} is inverse to Φ. □

Lemma 3.1.15. *Let all notation and assumptions be as in Lemma 3.1.14. Assume in addition that* $P^\bullet \in Comp(A\text{-grproj})$ *and that* $P^\bullet$ *is exact at each* n *for which* $\mathrm{Hom}_{A\text{-grmod}}(X, ker(d^n_P)(-n))$

is nonzero. Then Φ *induces an isomorphism:*

$$\begin{aligned} \overline{\Phi} : \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{A\text{-grstab}}(X, ker(d_P^n)(-n)) &\rightarrow \mathrm{Hom}_{Ho_{dg}(A)}(X, \widehat{P}) \\ (f_n)_n &\mapsto \sum_n i_n \circ f_n \end{aligned}$$

Proof. Take $(f_n)_n \in \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{A\text{-grmod}}(X, ker(d_P^n)(-n))$. By Lemma 3.1.14, it suffices to show that $\Phi(f_n)$ is nullhomotopic if and only if f_n factors through a projective module for all n. We also note that $d_{\widehat{P}}$ is A-linear, since $d_A = 0$.

Suppose that $\Phi(f_n)$ is nullhomotopic and fix $k \in \mathbb{Z}$. Let $h : X \to \widehat{P}(-1)$ be a homotopy. Since $d_X = 0$, we have that $\Phi(f_n) = d_{\widehat{P}}(-1) \circ h$ (as morphisms of graded modules). As a graded module, $\widehat{P} = \bigoplus_n P^n(-n)$; let π_n be the projection onto the nth summand. From the proof of Lemma 3.1.14, we have that $f_k = \pi_k \circ \Phi(f_n)$, and so $f_k = \pi_k \circ d_{\widehat{P}}(-1) \circ h$. Thus f_k factors through the graded projective module $\widehat{P}(-1)$.

Now suppose that for each n, f_n factors as $X \xrightarrow{a_n} Q_n \xrightarrow{b_n} ker(d_P^n)(-n)$ for some $Q_n \in A\text{-grproj}$. We shall define a nullhomotopy of $\Phi(f_n)$ by constructing maps $h_n : X \to P^{n-1}(-n)$. If $f_n = 0$, let $h_n = 0$. If f_n is nonzero, then $P^\bullet$ is exact at n, and so $P^{n-1}(-n)$ surjects onto $ker(d_P^n)(-n)$ via the differential. Since Q_n is projective, b_n lifts to $c_n : Q_n \to P^{n-1}(-n)$. Define $h_n = c_n \circ a_n$, as summarized by the diagram below.

$$\begin{array}{ccc} X & \xrightarrow{\quad a_n \quad} & Q_n \\ {\scriptstyle h_n}\downarrow & \swarrow{\scriptstyle c_n} & \downarrow{\scriptstyle b_n} \\ P^{n-1}(-n) & \xrightarrow[d_P^{n-1}(-n)]{} & ker(d_P^n)(-n) \end{array}$$

Viewing $P^{n-1}(-n)$ as a graded submodule of $\widehat{P}(-1)$, define $h := \sum_n h_n : X \to \widehat{P}(-1)$. Since all but finitely many of the h_n are zero, h is a well-defined morphism of graded modules. It is easy to check that $\Phi(f_n) = d_{\widehat{P}}(-1) \circ h$, hence $\Phi(f_n)$ is nullhomotopic. □

Lemma 3.1.16. *Let* $X, Y \in A\text{-grmod}$. *Then there is an isomorphism*

$$\begin{aligned} \chi : \mathrm{Hom}_{\mathcal{C}(A)}(X, Y) &\xrightarrow{\sim} \mathrm{Hom}_{Ho_{dg}(A)}(X, \widehat{R_{X,Y}}) \\ (f_n) &\mapsto \sum_n i_n \circ f_{-n} \end{aligned}$$

where $i_n : \Omega^{-n}Y(-n) = ker(d^n_{R_{X,Y}})(-n) \hookrightarrow \widehat{R_{X,Y}}$ *is the natural inclusion of dg-modules for* $n \geq N$ *and the zero map for* $n < N$*.*

Proof. We may assume that X and Y are nonzero. We first show that the hypotheses of Lemmas 3.1.14 and 3.1.15 are satisfied by X and $R^{\bullet}_{X,Y}$. By Proposition 3.1.3, for all but finitely many n, we have that $\mathrm{Hom}_{A\text{-grmod}}(X, ker(d^n_{R_{X,Y}})(-n)) = 0$. By construction, $R^{\bullet}_{X,Y} \in Comp(A\text{-grproj})$ is exact at all $n \neq N$. By the definition of N, $\mathrm{Hom}_{A\text{-dgmod}}(X, \Omega^{-N}Y(-N)) = 0$. Thus the hypotheses of Lemmas 3.1.14 and 3.1.15 are satisfied.

χ is precisely the composition of the isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}(A)}(X,Y) & \xrightarrow{\sim} & \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{A\text{-grstab}}(X, \Omega^{-n}Y(-n)) \\ (f_n) & \mapsto & (f_{-n}) \end{array}$$

with $\overline{\Phi}$ of Lemma 3.1.15. Thus χ is an isomorphism. □

We are now ready to prove Theorem 3.1.10.

Lemma 3.1.17. *The functor* F_A *is fully faithful.*

Proof. Let $X, Y \in A\text{-grmod}$. From Lemmas 3.1.13 and 3.1.16, we obtain an isomorphism $\xi \circ \chi : \mathrm{Hom}_{\mathcal{C}(A)}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-dgstab}}(X,Y)$. It remains to show that this isomorphism is induced by F_A.

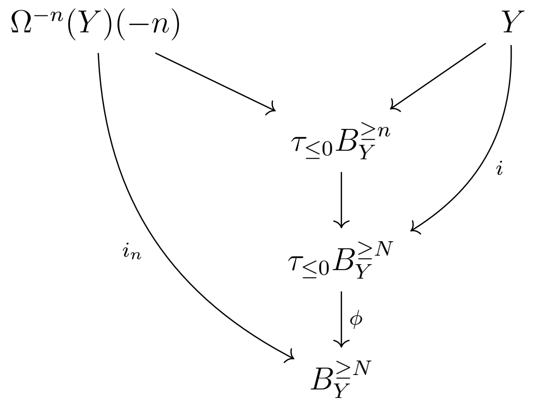
We have that $\xi \circ \chi(f_n) = \sum_n i^{-1} \circ \phi^{-1} \circ i_n \circ f_{-n}$. Since $f_{-n} = 0$ for $n \leq N$, it suffices to show that $i^{-1} \circ \phi^{-1} \circ i_n = \psi_{-n,Y}$ for $n > N$, where the $\psi_{-n,Y}$ are defined in Lemma 3.1.7.

It follows easily from the definitions that $\psi_{-n,Y}$ can be represented by roofs of the form

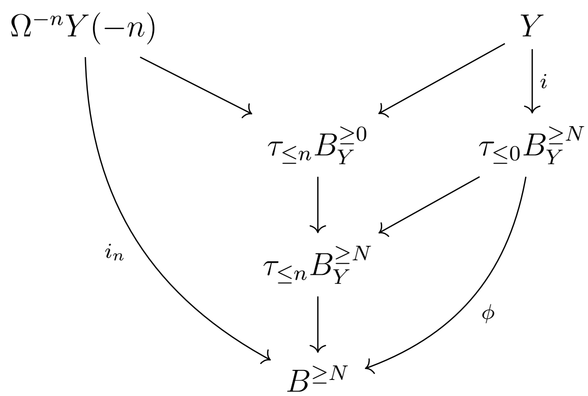
$$\begin{array}{ccccc} \Omega^{-n}Y(-n) & & & & Y \\ & \searrow & & \swarrow & \\ & & \widehat{\tau_{\leq 0}B_Y^{\geq n}} & & \end{array} \qquad \text{for } N < n \leq 0$$

$$\begin{array}{ccccc} \Omega^{-n}Y(-n) & & & & Y \\ & \searrow & & \swarrow & \\ & & \widehat{\tau_{\leq n}B_Y^{\geq 0}} & & \end{array} \qquad \text{for } n \geq 0$$

where all morphisms are inclusions of dg-modules and have either acyclic or perfect cones. We then obtain commutative diagrams of inclusions:



for $N < n \leq 0$



for $n \geq 0$

Every map in the above diagrams is either a quasi-isomorphism or has perfect cone. (This is immediate for all maps except i_n. It then follows that i_n is an isomorphism in $A\text{-dgstab}$, hence has perfect cone.) Thus the above diagrams show that the roof defining $\psi_{n,Y}$ is equivalent to $i^{-1} \circ \phi^{-1} \circ i_n$ for all $n > N$. □

Lemma 3.1.18. *The image of F_A generates $A\text{-dgstab}$ as a triangulated category.*

Proof. Let $\mathcal{T}$ denote the full, replete, triangulated subcategory of $A\text{-dgstab}$ generated by the image of F_A. Let $M \in A\text{-dgstab}$. We have a short exact sequence of dg-modules $0 \to ker(d_M) \hookrightarrow M \twoheadrightarrow M/ker(d_M) \to 0$ which induces an exact triangle in $A\text{-dgstab}$. Both $ker(d_M)$ and $(M/ker(d_M))(-1)$ have zero differential and thus lie in $\mathcal{T}$. It follows immediately that $M \in \mathcal{T}$.

□

We conclude this section with an important consequence of the main theorem.

Corollary 3.1.19. *Let A be a finite-dimensional symmetric algebra with a non-positive grading. Suppose the socle of A is concentrated in degree $-d$ for some $d \geq 0$. Then $A\text{-dgstab}$ is $-(d+1)$-Calabi-Yau.*

Proof. For any self-injective algebra A, $A\text{-stab}$ has Serre functor $\mathbb{S} = \Omega \circ \nu$, where $\nu := A^* \otimes_A -$ is the Nakayama functor. (See [ES06, Proposition 1.2] or [Ben91, Section 4.12].) Similarly, when A is a graded self-injective algebra, $A\text{-grstab}$ has Serre functor $\mathbb{S} = \Omega \circ \nu$; we omit the proof since it is identical to that of the ungraded case. Since A is symmetric with socle concentrated in degree $-d$, we have that $\nu \cong (-d)$, hence $\mathbb{S}$ commutes with $\Omega(1)$. It follows that $\mathbb{S}$ is a Serre functor on $\mathcal{C}(A)$. Furthermore, since $\Omega \cong (-1)$, we have that $\mathbb{S} \cong (-d-1)$ in $\mathcal{C}(A)$. Since $\mathcal{C}(A)$ generates $A\text{-dgstab}$ as a triangulated category, it follows that $(-d-1)$ extends to a Serre functor on all of $A\text{-dgstab}$. □

3.1.3 The Dg-Stable Category as a Triangulated Hull

Theorem 3.1.10 suggests that $A\text{-dgstab}$ the triangulated hull of $\mathcal{C}(A)$. To establish this, we must express our constructions in the language of dg-categories.

Given a category $\mathcal{C}$ with an automorphism Φ, we define a category $\mathcal{C}_{gr}$, enriched in $k\text{-grmod}$, with the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}^n_{\mathcal{C}_{gr}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, \Phi^n(Y))$. The grading shift functor (1) is an automorphism of the categories $A\text{-grmod}$, $A\text{-grstab}$, and $\mathcal{C}(A)$, allowing us to define the corresponding graded categories, which we view as dg-categories. The natural functors between these categories are all compatible with grading shifts and hence lift to functors of dg-categories. Likewise, the functor Ω commutes with grading shifts and thus defines a dg-endofunctor of each of these categories.

We can view the differential graded algebra A as a dg-category with one object, and we denote the associated dg-category of (bounded) dg-modules by $A\text{-dgmod}_{dg}$. Then $Z^0(A\text{-dgmod}_{dg}) \cong A\text{-dgmod}$ and $H^0(A\text{-dgmod}_{dg}) \cong Ho^b_{dg}(A)$. There is a natural dg-functor $A\text{-grmod}_{gr} \to A\text{-dgstab}_{dg}$.

Define the dg-derived category $\mathcal{D}^b_{dg}(A)$ to be the dg-quotient of $A\text{-dgmod}_{dg}$ by the full dg-subcategory of acyclic dg-modules. Let $A\text{-dgstab}_{dg}$ denote the dg-quotient of $A\text{-dgmod}_{dg}$ by the full dg-subcategory of objects which are quasi-isomorphic to a perfect dg-module. We have that $H^0(\mathcal{D}^b_{dg}(A)) \cong D^b_{dg}(A)$, $H^0(A\text{-dgstab}_{dg}) \cong A\text{-dgstab}$, and there are natural dg-functors $A\text{-dgmod}_{dg} \to \mathcal{D}^b_{dg}(A) \to A\text{-dgstab}_{dg}$.

Since the functor Ω is given by tensoring with the bimodule I defined in Section 2.6, it induces a dg-functor $A\text{-dgmod}_{dg} \to A\text{-dgmod}_{dg}$. Since this functor preserves acyclic and perfect dg-modules, by the universal property of the dg-quotient, it descends to a dg-endofunctor of $\mathcal{D}^b_{dg}(A)$ and $A\text{-dgstab}_{dg}$.

Since the natural dg-functor $A\text{-grmod}_{gr} \to A\text{-dgstab}_{dg}$ sends projective modules to zero, we obtain an induced dg-functor $A\text{-grstab}_{gr} \to A\text{-dgstab}_{dg}$ which is the identity on objects. We would like this functor to descend to $\mathcal{C}(A)_{gr}$.

Proposition 3.1.20. *$\mathcal{C}(A)_{gr}$ is the dg-orbit category of $A\text{-grstab}_{gr}$ by the functor $\Omega(1)$.*

Proof. By construction, the projection map $A\text{-grstab} \to \mathcal{C}(A)$ is essentially surjective, and the natural map

$$\bigoplus_{c\in\mathbb{Z}} colim_{r>>0} \operatorname{Hom}_{A\text{-grstab}_{gr}}(\Omega^r X(r), \Omega^{r+c} Y(r+c)) \to \operatorname{Hom}_{\mathcal{C}(A)_{gr}}(X, Y)$$

is an isomorphism of dg k-modules. The result then follows by Keller's characterization of the orbit category ([Kel05], Section 9.3, part d) of the Theorem). □

Write $\mathcal{B} = A\text{-grstab}_{gr}$ and $\mathcal{C} = A\text{-dgstab}_{dg}$. Let $F_{dg} : \mathcal{B} \to \mathcal{C}$ denote the natural dg-functor. Since F_{dg} is the identity on objects, we can identify it with the dg $\mathcal{B}^{op} \otimes \mathcal{C}$-module $(X, Y) \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X \otimes_A A)$. Similarly, we can identify $F_{dg} \circ \Omega(1)$ with the dg bimodule given by $(X, Y) \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X \otimes_A I(1))$. By construction of I, we have a closed morphism $\phi : A \to I(1)$ in $\mathcal{D}^b_{dg}(A^{op} \otimes_k A)$ which has perfect cone in $D^b_{dg}(A^{op} \otimes_k A)$. Thus, ϕ descends to a closed morphism in $\mathcal{C}$ which becomes an isomorphism in $H^0(\mathcal{C}) = A\text{-dgstab}$. Thus ϕ induces a quasi-isomorphism of dg-bimodules from F_{dg} to $F_{dg} \circ \Omega(1)$. Thus, by the universal property of the orbit category, F_{dg} descends to $\mathcal{C}(A)_{dg}$, and it is clear that $H^0(F_{dg})$ is the functor F_A defined in 3.1.8.

Corollary 3.1.21. *$A\text{-dgstab}$ is equivalent to the triangulated hull $Tr(\mathcal{C}(A)_{gr})$.*

Proof. By the universal property of the pretriangulated hull, the dg-functor $F_{dg} : \mathcal{C}(A)_{gr} \to A\text{-dgstab}_{dg} \hookrightarrow PreTr(A\text{-dgstab}_{dg})$ factors through a dg-functor

$$\widehat{F} : PreTr(\mathcal{C}(A)_{gr}) \to PreTr(A\text{-dgstab}_{dg})$$

By construction, $H^0(A\text{-dgstab}_{dg}) \cong A\text{-dgstab}$ and will generate $H^0(PreTr(A\text{-dgstab}_{dg})) = Tr(A\text{-dgstab}_{dg})$ as a triangulated category. Since $A\text{-dgstab}$ is already triangulated, we have that $A\text{-dgstab} \cong Tr(A\text{-dgstab}_{dg})$.

The functor $H^0(\widehat{F}) : Tr(\mathcal{C}(A)_{gr}) \to A\text{-dgstab}$ is exact and restricts to F_A on the image of $\mathcal{C}(A)$ inside $Tr(\mathcal{C}(A)_{gr})$. The image of F_A generates $A\text{-dgstab}$ as a triangulated category, and the image of $H^0(\widehat{F})$ is triangulated, hence $H^0(\widehat{F})$ is essentially surjective. Furthermore, $H^0(\widehat{F})$ is fully faithful on $\mathcal{C}(A)$. Since $\mathcal{C}(A)$ generates $Tr(\mathcal{C}(A))$ as a triangulated category, it follows by a standard argument that $H^0(\widehat{F})$ is fully faithful. Thus $A\text{-dgstab}$ is equivalent to $Tr(\mathcal{C}(A)_{gr})$. □

Having proven Corollary 3.1.21, we make a few brief remarks on when two graded algebras A and B have equivalent dg-stable categories. If $D^b(A\text{-grmod})$ is equivalent to $D^b(B\text{-grmod})$, then the equivalence can be expressed as tensoring by a tilting complex. (See Rickard, [Ric89b]). This functor is still defined on the derived category of dg-modules and remains an equivalence. Furthermore, this equivalence preserves the perfect dg-modules and thus induces an equivalence between the dgstable categories. Thus, graded derived equivalence implies dg-stable equivalence. However, we can say more:

Lemma 3.1.22. *Let A and B be finite-dimensional self-injective algebras, graded in non-positive degree. Suppose there is an equivalence of triangulated categories $G : A\text{-grstab} \to B\text{-grstab}$ which commutes with grading shifts. Then G induces an equivalence between* $A\text{-dgstab}$ *and* $B\text{-dgstab}$.

Proof. Since G is a triangulated equivalence, it commutes with Ω. Thus G commutes with the functor $\Omega(1)$ and induces a functor $\mathcal{C}(A) \to \mathcal{C}(B)$. Given $Y \in B\text{-grstab}$, there exists $X \in A\text{-grstab}$ such that $G(X) \cong Y$ in $B\text{-grstab}$, hence in $\mathcal{C}(B)$. Thus the induced functor on the orbit category is essentially surjective. Given $X, Y \in A\text{-grstab}$, $\mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n(Y)(n)) \to \mathrm{Hom}_{B\text{-grstab}}(G(X), \Omega^n G(Y)(n))$ is bijective for each $n \in \mathbb{Z}$, hence the map $\mathrm{Hom}_{\mathcal{C}(A)}(X, Y) \to$

$\mathrm{Hom}_{\mathcal{C}(B)}(G(X), G(Y))$ is bijective. Thus $G : \mathcal{C}(A) \to \mathcal{C}(B)$ is an equivalence. Similarly, G induces an equivalence of dg-categories $\mathcal{C}(A)_{gr} \to \mathcal{C}(B)_{gr}$.

The composition $\mathcal{C}(A)_{gr} \to \mathcal{C}(B)_{gr} \to PreTr(\mathcal{C}(B)_{gr})$ factors through the pretriangulated hull $PreTr(\mathcal{C}(A)_{gr})$. Applying H^0, we obtain an exact functor $\overline{G} : A\text{-dgstab} \to B\text{-dgstab}$ extending G. Since G is an equivalence, it follows that $\overline{G}$ is an equivalence. □

3.2 Essential Surjectivity

3.2.1 Morphisms Concentrated in One Degree

Let A be a non-positively graded finite-dimensional self-injective algebra over a field k. Let $F_A : \mathcal{C}(A) \to A\text{-dgstab}$ be the functor of Definition 3.1.8. Having shown in the previous section that F_A is fully faithful, we now investigate conditions on A that guarantee essential surjectivity.

Since the image of $\mathcal{C}(A)$ generates $A\text{-dgstab}$ as a triangulated category, F_A is essentially surjective if and only if the essential image $Im(F_A)$ is a triangulated subcategory of $A\text{-dgstab}$, if and only if $\mathcal{C}(A)$ admits a triangulated structure compatible with F_A. In general, this is not the case.

The primary obstacle to essential surjectivity is that there is no natural candidate for the cone of a morphism $(f_n)_n : X \to Y$ for which more than one f_n is nonzero. The cone of such a morphism will correspond to a dg-module that does not arise from a chain complex and need not be isomorphic to a chain complex modulo projectives.

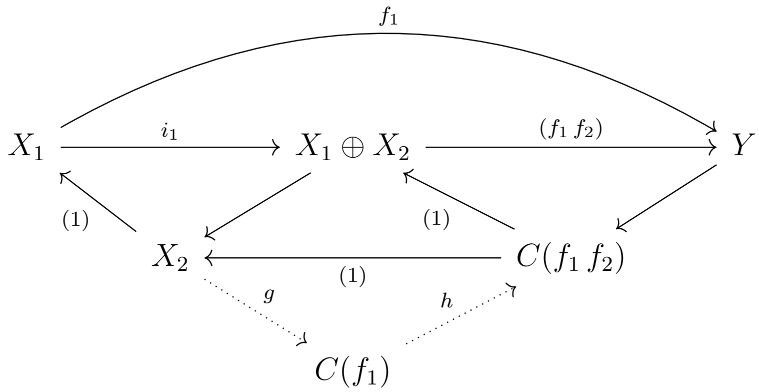
However, by imposing restrictions on the algebra A, we can prevent this scenario from occurring. In this case, we shall see that F_A is essentially surjective, hence an equivalence.

Definition 3.2.1. Let $X, Y \in Im(F_A) \subset A\text{-dgstab}$. We say that a morphism $f : X \to Y$ is **chain-like** if $C(f) \in Im(F_A)$. We say that $X \in Im(F_A)$ is **nice** if $f : X \to Y$ is chain-like for all $Y \in Im(F_A)$ and all $f \in \mathrm{Hom}_{A\text{-dgstab}}(X, Y)$.

Note that $Im(F_A)$ is closed under cones (and thus triangulated) if and only if all of its objects are nice. In fact, it suffices for all the indecomposable objects of $Im(F_A)$ to be nice:

Lemma 3.2.2. *Let $X_1, X_2 \in Im(F_A)$ be nice. Then $X_1 \oplus X_2$ is nice.*

Proof. Take $Y \in Im(F_A)$ and $(f_1\, f_2) : X_1 \oplus X_2 \to Y$. Applying the octahedron axiom to the composition $f_1 = (f_1\, f_2) \circ i_1$, we obtain the following diagram:



The bottom-most triangle is exact, so $C(f_1\, f_2)$ is the cone of $g : X_2 \to C(f_1)$. Since X_1 is nice and $Y \in Im(F_A)$, it follows that $C(f_1) \in Im(F_A)$. Since X_2 is nice, $C(f_1\, f_2) \in Im(F_A)$. Thus $X_1 \oplus X_2$ is nice. □

The following condition is sufficient to guarantee that all indecomposables are nice.

Lemma 3.2.3 (One Morphism Rule)**.** *Suppose, for every pair of indecomposable objects $X, Y \in A\text{-grmod}$, that $\mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n Y(n)) \neq 0$ for at most one $n \in \mathbb{Z}$. Then every indecomposable object of $Im(F_A)$ is nice. In particular, $F_A : \mathcal{C}(A) \to A\text{-dgstab}$ is an equivalence.*

Proof. Let $X \in A\text{-grmod}$ be indecomposable. Let $M = \bigoplus_{i=1}^n Y_i \in Im(F_A)$, with Y_i indecomposable. Changing each Y_i up to isomorphism, we may assume without loss of generality that $\mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n Y_i(n)) = 0$ for $n \neq 0$. Then any morphism $(f_n)_n : X \to M$ in $\mathcal{C}(A)$ is concentrated in degree 0 and thus can be identified with the morphism f_0 in $A\text{-grstab}$. Since F_A is fully faithful, any morphism $f : X \to M$ in $A\text{-dgstab}$ can be represented by a morphism in $A\text{-grmod}$.

Choosing a monomorphism $i : X \hookrightarrow I$, where I is injective, we obtain a short exact sequence of graded A-modules $0 \to X \xrightarrow{(f\, i)} M \oplus I \to C \to 0$ which induces an exact triangle in $D^b_{dg}(A)$, hence in $A\text{-dgstab}$. Since $I \cong 0$ in $A\text{-dgstab}$, this triangle is equivalent to one of the form $X \xrightarrow{f} M \to C \to X(1)$. C is a cone of f and lies in the image of F_A (since it is in $A\text{-grmod}$). Thus X is nice.

The second statement follows immediately from Lemma 3.2.2 and the preceding remarks. □

Remark. The hypotheses of Lemma 3.2.3 are quite restrictive. However, we note that if A is concentrated in degree 0 (that is, ungraded), then the One Morphism Rule is trivially satisfied.

In this case, any indecomposable object $X \in A\text{-grmod}$ is concentrated in a single degree n, and so $\Omega^n X(n)$ is concentrated in degree 0. Thus every object of $\mathcal{C}(A)$ is isomorphic to an object concentrated in degree zero, and $\mathrm{Hom}_{\mathcal{C}(A)}(X,Y) \cong \mathrm{Hom}_{A\text{-stab}}(X,Y)$ for any two such objects X and Y. Thus $\mathcal{C}(A)$ is equivalent to $A\text{-stab}$.

Furthermore, a dg-module over A is the same as a complex of A-modules. Thus $A\text{-dgstab} = D^b_{dg}(A)/D^{perf}_{dg}(A) = D^b(A\text{-mod})/D^{perf}(A\text{-mod})$. Thus, in the case where A an ungraded finite-dimensional, self-injective algebra, Theorem 3.1.10 and Lemma 3.2.3 precisely yield Rickard's Theorem [Ric89a] that $A\text{-stab} \cong D^b(A\text{-mod})/D^{perf}(A\text{-mod})$. Thus it is appropriate to view $\mathcal{C}(A)$ as the differential graded analogue of the additive definition of the stable module category.

3.2.2 Nakayama Algebras

Definition 3.2.4. A **Nakayama algebra** is a finite-dimensional algebra for which all indecomposable projective and injective modules are uniserial.

Since every indecomposable module has an indecomposable projective cover, it follows that every indecomposable module over a Nakayama algebra is uniserial.

Proposition 3.2.5. *Let A be a finite-dimensional, self-injective Nakayama algebra, graded in non-positive degree. Let $X \in A\text{-grmod}$ be indecomposable and not projective. Let $p_X : P_X \twoheadrightarrow X$ be a projective cover of X and let $i_X : X \hookrightarrow I_X$ be an injective hull of X. Let $K = ker(p_X)$ and $C = coker(i_X)$. Then $max(K) \leq min(X)$, and $max(X) \leq min(C)$.*

Proof. For any $k \geq 0$ and $Y \in A\text{-grmod}$, let $L^k(Y) = rad^k(Y)/rad^{k+1}(Y)$ be the k-th radical layer of Y. Let $l(Y)$ denote the length of Y. If Y is indecomposable, then it is uniserial and so $L^k(Y)$ is simple for $0 \leq k < l(Y)$.

Since X is indecomposable, P_X is indecomposable, hence uniserial, and we have that $K = ker(p_X) = rad^{l(X)}(P_X)$ and $X \cong P_X/rad^{l(X)}(P_X)$. Let $M = rad^{l(X)-1}(P_X)/rad^{l(X)+1}(P_X)$.

Then $hd(M) = L^{l(X)-1}(P_X) = soc(X)$ and $soc(M) = L^{l(X)}(P_X) = hd(K)$ are simple. Thus,

$$\begin{aligned} max(K) = max(hd(K)) = max(soc(M)) &= min(soc(M)) = min(M) \\ &\leq max(M) = max(hd(M)) = max(soc(X)) = min(soc(X)) \\ &= min(X) \end{aligned}$$

The proof of the second inequality is precisely dual, using the socle layers of I_X. □

Lemma 3.2.6. *Let A be a non-positively graded, finite-dimensional, self-injective Nakayama algebra. Then the conditions of Lemma 3.2.3 are satisfied. In particular, F_A is an equivalence.*

Proof. Let $X, Y \in A\text{-grmod}$ be indecomposable, and suppose that there is a nonzero morphism $f : X \to \Omega^m(Y)(m)$ in $A\text{-grstab}$ for some $m \in \mathbb{Z}$. Changing Y up to isomorphism in $Im(F_A) \subset A\text{-dgstab}$, we may assume that $m = 0$. Then there is a nonzero morphism from X to Y in $A\text{-grmod}$, and so $max(hd(X)) \geq min(soc(Y))$.

Note that $\Omega(Y) \in A\text{-grmod}$ has a unique (up to isomorphism) non-projective direct summand K, which is the kernel of a projective cover of Y. Then $\Omega(Y) \cong K$ in $A\text{-grstab}$, hence in $Im(F_A)$. Identifying $\Omega(Y)$ with K, Proposition 3.2.5 states that

$$max(\Omega(Y)(1)) < min(soc(Y)) \leq max(hd(X)) = min(hd(X))$$

Thus $\mathrm{Hom}_{A\text{-grstab}}(X, \Omega Y(1)) = 0$ and, by induction, $\mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n Y(n)) = 0$ for all $n > 0$. A dual argument shows that $max(hd(X)) < min(\Omega^{-1}Y(-1))$ and so $\mathrm{Hom}_{A\text{-grstab}}(X, \Omega^n Y(n)) = 0$ for all $n < 0$. □

3.2.3 An Example of the Failure of Essential Surjectivity

Let $A = k[x, y]/(x^2, y^2)$, where $k = \mathbb{C}$. We grade A by putting x in degree 0 and y in degree -1. It is easy to check that A is symmetric, hence self-injective. Up to grading shift, A has a single simple graded module, S, which has dimension one and upon which both x and y act by zero. Therefore, up to grading shift, the only indecomposable projective module is A itself.

The representation theory of A is closely related to that of the Kronecker quiver,

$$v_1 \overset{a}{\underset{b}{\rightrightarrows}} v_2$$

We let B denote the path algebra of this quiver, with a in degree 0 and b in degree -1. B has two simple modules S_1 and S_2, one corresponding to each vertex. There is a one-to-one correspondence between the indecomposable graded A-modules, excluding the projective module, and the graded B-modules, excluding the simple module S_2. (See [Ben91, Chapter 4.3] for the ungraded case. Note that the graded case follows from the same argument.)

The classification of graded indecomposable B-modules is known. (For instance, see [Sei04, Section 4].) Transferring these results to A-modules, we obtain the following classification of the indecomposable graded A-modules. Up to shift, these are:

1) The indecomposable projective module, A.

2) For $n \geq 0$, the module K^n, which is of dimension $2n+1$. As a graded vector space, $K^n = V \oplus W$, where $V = \bigoplus_{i=0}^{n} k(i)$, $W = \bigoplus_{i=1}^{n} k(i)$, and x and y act by mapping V into W via the matrices $\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}$, respectively. Note that in $A\text{-grstab}$ we have that $K^n \cong \Omega^n S$ for all $n \geq 0$. We shall use the notation $\Omega^n S$ going forward.

3) For $n < 0$, the module K^n, which is of dimension $2n+1$. As a graded vector space, $K^n = V \oplus W$, where $V = \bigoplus_{i=1}^{n} k(-i)$, $W = \bigoplus_{i=0}^{n} k(-i)$, and x and y act by mapping V into W via the matrices $\begin{pmatrix} 0 & \dots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \dots & 0 \end{pmatrix}$, respectively. Once again, we note that $K^n \cong \Omega^n S$ in $A\text{-grstab}$ for all $n < 0$. We shall use the notation $\Omega^n S$ going forward.

4) For $n > 0$, the module $M_{0,n}$, which is of dimension $2n$. As a graded vector space, $M_{0,n} = V \oplus W$, where $V = \bigoplus_{i=1}^{n} k(-i)$, $W = \bigoplus_{i=0}^{n-1} k(-i)$, and x and y act by mapping V into W via

the $n \times n$ matrices $\begin{pmatrix} 0 & \dots & & 0 \\ 1 & & 0 & \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, respectively.

5) For $n > 0$, the module $M_{\infty,n}$, which is of dimension $2n$. As a graded vector space, $M_{\infty,n} = V \oplus W$, where $V = W = \bigoplus_{i=0}^{n-1} k(-i)$, and x and y act by mapping V into W via the $n \times n$ matrices $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ & 0 & & 1 \\ 0 & & \dots & 0 \end{pmatrix}$, respectively.

Note that for any of the modules described in 2-5 above, $hd(X) \cong V$ and $soc(X) \cong W$ as graded modules, each with x and y acting by 0.

The following computations are straightforward; we leave them to the reader. Below, $n \geq 0$ and $m \geq 1$.

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(S, \Omega^m S(k)) = \begin{cases} 1 & -m \leq k \leq -1 \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(S, \Omega^{-n} S(k)) = \begin{cases} 1 & 0 \leq k \leq n \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(S, M_{0,m}(k)) = \begin{cases} 1 & 0 \leq k \leq m-1 \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(S, M_{\infty,m}(k)) = \begin{cases} 1 & 0 \leq k \leq m-1 \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(M_{0,m}, S(k)) = \begin{cases} 1 & -m \leq k \leq -1 \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(M_{\infty,m}, S(k)) = \begin{cases} 1 & -m+1 \leq k \leq 0 \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(M_{\infty,m}, M_{\infty,1}(k)) = \begin{cases} 1 & k = 0, -m+1 \\ 0 & o.w. \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(M_{0,m}, M_{0,1}(k)) = \begin{cases} 1 & k = 0, -m \\ 0 & o.w. \end{cases}$$

In $\mathcal{C}(A)$, functors Ω and (-1) are isomorphic, so our list of indecomposable objects shrinks. In A-grstab, note that $\Omega M_{0,n} = M_{0,n}$ and $\Omega M_{\infty,n} = M_{\infty,n}(1)$; thus $M_{0,n} \cong M_{0,n}(1)$ and $M_{\infty,n} \cong M_{\infty,n}(2)$ in $\mathcal{C}(A)$. Thus, a complete list of indecomposable objects in $\mathcal{C}(A)$ up to isomorphism is:

1) $S(n)$, for $n \in \mathbb{Z}$.

2) $M_{0,m}$, for $m > 0$.

3) $M_{\infty,m}(n)$, for $m > 0$ and $n \in \{0, 1\}$.

The sizes of the following Hom sets in A-dgstab are an immediate consequence of the above computations for A-grstab and some simple counting arguments.

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(S, S(n)) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1 & n \geq 0 \\ \lfloor \frac{|n|}{2} \rfloor & n < 0 \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(S, M_{0,m}) = m$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(S, M_{\infty,m}(n)) = \begin{cases} \lfloor \frac{m+1}{2} \rfloor & n \equiv 0 \mod 2 \\ \lfloor \frac{m}{2} \rfloor & n \equiv 1 \mod 2 \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(M_{0,m}, S(n)) = m$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(M_{0,m}, M_{0,1}) = 2$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(M_{\infty,m}(r), S(n)) = \begin{cases} \lfloor \frac{m+1}{2} \rfloor & n - r \equiv 0 \mod 2 \\ \lfloor \frac{m}{2} \rfloor & n - r \equiv 1 \mod 2 \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(M_{\infty,m}, M_{\infty,1}) = \begin{cases} 1 & m \equiv 0 \mod 2 \\ 2 & m \equiv 1 \mod 2 \end{cases}$$

$$dim\,\mathrm{Hom}_{A\text{-dgstab}}(M_{\infty,m}, M_{\infty,1}(1)) = \begin{cases} 1 & m \equiv 0 \mod 2 \\ 0 & m \equiv 1 \mod 2 \end{cases}$$

We are now ready to construct an object K of A-dgstab lying outside of $\mathcal{C}(A)$. From the above computations, we have that $dim\,\mathrm{Hom}_{A\text{-dgstab}}(S, S(2)) = 2$; for a basis we can take the unique (up to a nonzero scalar) morphisms $f_{-1} : S \to \Omega^{-1}(S)(1) \cong S(2)$ and $f_{-2} : S \to \Omega^{-2}S \cong S(2)$. Let $g = f_{-1} + f_{-2}$, and let K be the cone of g in $A\text{-dgstab}$. We shall show that K does not lie in the image of F_A.

Proposition 3.2.7. $dim\,\mathrm{Hom}_{A\text{-dgstab}}(K, S(n)) = 1$ *for all* $n \geq 3$

Proof. Consider the triangle $S \xrightarrow{g} S(2) \to K \to S(1)$ which defines K. Choosing some $n \geq 2$, we apply $\mathrm{Hom}_{A\text{-dgstab}}(-, S(n))$ and observe the resulting long exact sequence. We will show that $g(-k)^* : \mathrm{Hom}_{A\text{-dgstab}}(S(2-k), S(n)) \to \mathrm{Hom}_{A\text{-dgstab}}(S(-k), S(n))$ is injective for all $k \geq 0$. From this, it will follow from the long exact sequence that

$$\begin{aligned} dim\,\mathrm{Hom}_{A\text{-dgstab}}(K(-k-1), S(n)) &= dim\,\mathrm{Hom}_{A\text{-dgstab}}(S(-k), S(n)) \\ &\quad - dim\,\mathrm{Hom}_{A\text{-dgstab}}(S(2-k), S(n)) \\ &= 1 \end{aligned}$$

for all $k \geq 0$, and we will have $dim\,\mathrm{Hom}_{A\text{-dgstab}}(K, S(n)) = 1$ for all $n \geq 3$.

It suffices to show that $g^* : \mathrm{Hom}_{A\text{-dgstab}}(S(2), S(r)) \to \mathrm{Hom}_{A\text{-dgstab}}(S, S(r))$ is injective for all $r \geq 2$, where $r = n + k$. Interpreting f_{-1} and f_{-2} as morphisms in $A\text{-dgstab}$, we have that $g^* = f_{-1}^* + f_{-2}^*$. If we are given a nonzero morphism $h_s : S(2) \to \Omega^s S(r+s)$ in $A\text{-grstab}$, a straightforward computation shows that both $\Omega^{-1}h_s(-1) \circ f_{-1} : S \to \Omega^{s-1}S(r+s-1)$ and $\Omega^{-2}h_s(-2) \circ f_{-2} : S \to \Omega^{s-2}S(r+s-2)$ are nonzero morphisms in $A\text{-grstab}$. It follows immediately that f_{-1}^* and f_{-2}^* are injective.

We now show that g^* is injective. Let $(h_s)_s : S(2) \to S(r)$ in $\mathcal{C}(A)$. Note that h_s can be nonzero only when $-r + 2 \leq s \leq -\lceil \frac{r}{2} \rceil + 1$. Therefore $g^*(h_s)_s = (a_s)_s$, where

$$a_s = \begin{cases} \Omega^{-2}h_{-r+2}(-2) \circ f_{-2} & \text{if } s = -r \\ \Omega^{-1}h_{s+1}(-1) \circ f_{-1} + \Omega^{-2}h_{s+2}(-2) \circ f_{-2} & \text{if } -r < s < -\lceil \frac{r}{2} \rceil \\ \Omega^{-1}h_{-\lceil \frac{r}{2} \rceil + 1}(-1) \circ f_{-1} & \text{if } s = -\lceil \frac{r}{2} \rceil \\ 0 & \text{otherwise} \end{cases}$$

Now suppose that $g^*(h_s) = 0$. If $(h_s)_s \neq 0$, let N be the maximum s such that h_s is nonzero. By injectivity of f_{-1}^*, we have that $N < -\lceil \frac{r}{2} \rceil + 1$, and by injectivity of f_{-2}^*, we have that $N > -r+2$. But then

$$\begin{aligned} 0 = a_{N-1} &= \Omega^{-1} h_N(-1) \circ f_{-1} + \Omega^{-2} h_{N+1}(-2) \circ f_{-2} \\ &= \Omega^{-1} h_N(-1) \circ f_{-1} + 0 \end{aligned}$$

Injectivity of f_{-1}^* implies that $\Omega^{-1} h_N(-1) = 0$, hence $h_N = 0$. As this contradicts the definition of N, we must have that $h_s = 0$ for all s, and so g^* is injective for all $r \geq 2$. Thus $dim\, \mathrm{Hom}_{A\text{-dgstab}}(K, S(n)) = 1$ for all $n \geq 3$. □

Proposition 3.2.7 and the above computations of Hom spaces show that K cannot be isomorphic to any object of $\mathcal{C}(A)$ except possibly $M_{\infty,2}$, $M_{\infty,2}(1)$, $M_{\infty,1} \oplus M_{\infty,1}(1)$, or $M_{0,1}$.

Proposition 3.2.8. $\mathrm{Hom}_{A\text{-dgstab}}(K, M_{\infty,1}(k)) = 0$ *for all* k.

Proof. Again consider the triangle $S \xrightarrow{g} S(2) \to K \to S(1)$ defining K and write $g = f_{-1} + f_{-2}$. Applying $\mathrm{Hom}_{A\text{-dgstab}}(-, M_{\infty,1})$, we again show that $g^*(k) : \mathrm{Hom}_{A\text{-dgstab}}(S(2+k), M_{\infty,1}) \to \mathrm{Hom}_{A\text{-dgstab}}(S(k), M_{\infty,1})$ is an isomorphism for all k. As in Proposition 3.2.7, we shall apply $(-k)$ and work instead with $g^* : \mathrm{Hom}_{A\text{-dgstab}}(S(2), M_{\infty,1}(-k)) \to \mathrm{Hom}_{A\text{-dgstab}}(S, M_{\infty,1}(-k))$. Since $M_{\infty,1} \cong M_{\infty,1}(2)$, it suffices to consider the cases $k = 0$ and $k = 1$.

If $k = 1$, both spaces are zero, and the result is immediate. If $k = 0$, both spaces are one-dimensional, so it is enough to show that g^* is not the zero map. The unique morphism $S(2) \to M_{\infty,1}$ is (up to rescaling) of the form $h_1 : S(2) \to \Omega M_{\infty,1}(1)$. Then $g^*(h_1) = (r_n)_n$, where

$$r_n = \begin{cases} \Omega^{-1} h_1(-1) \circ f_{-1} & \text{if } n = 0 \\ \Omega^{-2} h_1(-2) \circ f_{-2} & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

A simple computation in A-grstab shows that $r_{-1} = 0$ and and r_0 is a nonzero element of $\mathrm{Hom}_{A\text{-grstab}}(S, M_{\infty,1})$, whence g^* is nonzero. Thus g^* is an isomorphism in all cases, and so $\mathrm{Hom}_{A\text{-dgstab}}(K, M_{\infty,1}(k)) = 0$ for all k. □

Proposition 3.2.8 eliminates all remaining possibilities for K except for $M_{0,1}$. This final possibility can be eliminated by proving:

Proposition 3.2.9. $\mathrm{Hom}_{A\text{-dgstab}}(K, M_{0,1}) = 0$

Proof. Once again, we consider the triangle $S \xrightarrow{g} S(2) \to K \to S(1)$ defining K and write $g = f_{-1} + f_{-2}$. We show $g^*(k) : \mathrm{Hom}_{A\text{-dgstab}}(S(2+k), M_{0,1}) \to \mathrm{Hom}_{A\text{-dgstab}}(S(k), M_{0,1})$ is an isomorphism for all k. Applying $(-k)$ and using the identity $M_{0,1} \cong M_{0,1}(1)$, we show that $g^* : \mathrm{Hom}_{A\text{-dgstab}}(S(2), M_{0,1}) \to \mathrm{Hom}_{A\text{-dgstab}}(S, M_{0,1})$ is an isomorphism. Since both spaces are one-dimensional, it suffices to show that the map is nonzero.

The generator of $\mathrm{Hom}_{A\text{-dgstab}}(S(2), M_{0,1})$ is $h_2 : S(2) \to \Omega^2 M_{0,1}(2)$, and so $g^*(h_2) = (r_n)_n$, where

$$r_n = \begin{cases} \Omega^{-1} h_2(-1) \circ f_{-1} & \text{if } n = 1 \\ \Omega^{-2} h_2(-2) \circ f_{-2} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

A straightforward computation shows that $r_1 = 0$ and r_0 generates $\mathrm{Hom}_{A\text{-dgstab}}(S, M_{0,1})$. Thus g^* is an isomorphism and $\mathrm{Hom}_{A\text{-dgstab}}(K, M_{0,1}) = 0$. □

Corollary 3.2.10. *K does not lie in the image of F_A. In particular, F_A is not essentially surjective.*

CHAPTER 4

Perverse Equivalences in a Negative Calabi-Yau Category

4.1 Basic Definitions

4.1.1 Orthogonality and Bases

Let k be a field, and let $(\mathcal{T}, \Sigma)$ be a k-linear, w-Calabi-Yau triangulated category, for some $w < 0$. Fix a positive integer n. Let $\mathcal{S}$ be a collection of objects in $\mathcal{T}$.

For objects $X, Y, Z \in \mathcal{T}$, we say that Y is an **extension of** Z **by** X if there is a distinguished triangle $X \to Y \to Z \to \Sigma X$. Let $\langle \mathcal{S} \rangle$ denote the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and is closed under isomorphisms and extensions.

Following Coelho Simões and Pauksztello [SP18], a tuple $(X_1, \cdots X_n)$ of objects in $\mathcal{T}$ is called $|w|$-**orthogonal** if $dim\, \mathrm{Hom}(X_i, \Sigma^{-m} X_j) = \delta_{i=j}\delta_{m=0}$ for all $1 \leq i, j \leq n$ and $0 \leq m \leq |w| - 1$.

If, in addition, we have that $\mathcal{T} = \langle \{\Sigma^{-m} X_i \mid 1 \leq i \leq n, 0 \leq m < |w|\} \rangle$, we say $(X_1, \cdots, X_n)$ is a $|w|$-**basis** for $\mathcal{T}$.

Definition 4.1.1. Let $\widehat{\mathcal{E}}$ be the set of all $|w|$-orthogonal n-tuples of objects of $\mathcal{T}$ (up to isomorphism). Let $\mathcal{E}$ be the subset of all n-tuples which form a $|w|$-basis.

We shall refer to elements of $\widehat{\mathcal{E}}$ as **orthogonal tuples**. Elements of $\mathcal{E}$ will be referred to as **bases**.

Note that if $(X_i)_i \in \mathcal{E}$ (resp. $\widehat{\mathcal{E}}$) then $(\Sigma^m X_{\sigma(i)})_i \in \mathcal{E}$ (resp. $\widehat{\mathcal{E}}$) for any $m \in \mathbb{Z}$ and any $\sigma \in \mathfrak{S}_n$. We define an equivalence relation $\sim$ on $\mathcal{E}$ (resp. $\widehat{\mathcal{E}}$) by $(X_i)_i \sim (Y_i)_i$ if there exists $m \in \mathbb{Z}, \sigma \in \mathfrak{S}_n$ such that $Y_i = \Sigma^m X_{\sigma(i)}$ for each i. Since we are interested in classifying and counting the members of $\mathcal{E}$, it will frequently be helpful to work modulo these symmetries.

Remark. In practice, the triangulated category $\mathcal{T}$ will usually arise from some category of modules over an algebra A; in this case, the number of simple A-modules is a natural choice for n. For this reason, we suppress the dependence of $\mathcal{E}$ on the choice of n in our notation.

We now introduce some terminology that will be convenient throughout the rest of this paper:

Definition 4.1.2. Let $X \in Ob(\mathcal{T})$. We say X is **elementary** if

$$dim\,\mathrm{Hom}(X, \Sigma^{-m}X) = \delta_{0=m}$$

for all $0 \leq m < |w|$.

Definition 4.1.3. Let X, Y be elementary objects of $\mathcal{T}$. We say X and Y are **independent** if

$$dim\,\mathrm{Hom}(X, \Sigma^{-m}Y) = dim\,\mathrm{Hom}(Y, \Sigma^{-m}X) = \delta_{0=m}\delta_{X \cong Y}$$

for all $0 \leq m < |w|$.

Thus an orthogonal tuple is a tuple of distinct elementary objects which are pairwise independent.

4.1.2 Maximal Extensions

Chuang and Rouquier define an action of perverse equivalences on tilting complexes in the derived category of a finite-dimensional symmetric algebra [CR17, Section 5.2], as well as for bases of (-1)-Calabi-Yau categories [CR17, Section 7]. These actions are defined in terms of maximal extensions.

Definition 4.1.4. [CR17, Definition 3.28] Let $f : X \to Y$ be a morphism in $\mathcal{T}$.

f (or X) is a **maximal extension of** Y **by** $\mathcal{S}$ if $\Sigma^{-1}C(f) \in \mathcal{S}$ and $\mathrm{Hom}(\Sigma^{-1}C(f), S) \xrightarrow{\sim} \mathrm{Hom}(\Sigma^{-1}Y, S)$ is an isomorphism for all $S \in \mathcal{S}$.

f (or Y) is a **maximal $\mathcal{S}$-extension by** X if $C(f) \in \mathcal{S}$ and $\mathrm{Hom}(S, C(f)) \xrightarrow{\sim} \mathrm{Hom}(S, \Sigma X)$ is an isomorphism for all $S \in \mathcal{S}$.

If $X \in {}^{\perp}\mathcal{S} \cap \mathcal{S}^{\perp}$, Chuang and Rouquier [CR17, Lemma 3.29] prove that both maximal extensions of $\mathcal{S}$ by X and maximal X-extensions of $\mathcal{S}$ are unique up to unique isomorphism (if they exist). They also prove the following characterization of maximal extensions:

Proposition 4.1.5. *[CR17, Lemma 3.30] Suppose $\mathcal{S}$ is closed under extensions. Let $f : X \to Y$ be a morphism in $\mathcal{T}$.*

Let $\mathrm{Hom}(Y, \mathcal{S}) = 0$. Then f is a maximal extension of Y by $\mathcal{S}$ if and only if $\Sigma^{-1}C(f) \in \mathcal{S}$ and $\mathrm{Hom}(X, \mathcal{S}) = \mathrm{Hom}(X, \Sigma\mathcal{S}) = 0$.

Let $\mathrm{Hom}(\mathcal{S}, X) = 0$. Then f is a maximal $\mathcal{S}$-extension by X if and only if $C(f) \in \mathcal{S}$ and $\mathrm{Hom}(\mathcal{S}, Y) = \mathrm{Hom}(\mathcal{S}, \Sigma Y) = 0$.

We say that $\mathcal{T}$ **admits $|w|$-orthogonal maximal extensions** if, given a $|w|$-orthogonal n-tuple $(X_1, \cdots, X_n)$, a subset $\mathcal{S} \subset \{X_i\}$, and $X_j \notin \mathcal{S}$, both the maximal extension of X_j by $\mathcal{S}$ and the maximal $\mathcal{S}$-extension by X_j exist. We will generally ignore the dependence of this definition on the integer n, since we will be working with a fixed n throughout the paper.

We are now ready to define the action of perverse equivalences on $\mathcal{E}$. We shall assume that $\mathcal{T}$ admits $|w|$-orthogonal maximal extensions. Let $\mathcal{P}'(n)$ denote the set of proper subsets of $[n] := \{1, \cdots n\}$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{P}'(n)$ in the obvious way, allowing us to define the semi-direct product $\Xi := Free(\mathcal{P}'(n)) \rtimes \mathfrak{S}_n$.

Definition 4.1.6. Define an action of $\Xi := Free(\mathcal{P}'(n)) \rtimes \mathfrak{S}_n$ on $\mathcal{E}, \widehat{\mathcal{E}}$ as follows:
1) $\mathfrak{S}_n$ acts on $(X_i)_i \in \widehat{\mathcal{E}}$ by permutation of indices.
2) Given $S \in \mathcal{P}'(n)$, $(X_i)_i \in \widehat{\mathcal{E}}$, define $S \cdot (X_i)_i = (X_i')_i$ by

$$X_i' = \begin{cases} X_i & i \in S \\ \Sigma^{-1}(X_i)_S & i \notin S \end{cases}$$

where $(X_i)_S$ is the minimal extension of X_i by $\mathcal{S} := \langle X_s \mid s \in S \rangle$.
3) Define $S^{-1} \cdot (X_i)_i = (X_i')_i$ by

$$X_i' = \begin{cases} X_i & i \in S \\ \Sigma(X_i)^S & i \notin S \end{cases}$$

where $(X_i)^S$ is the minimal $\mathcal{S}$-extension by X_i.

It is not obvious that the action of Ξ preserves the property of being an orthogonal tuple or a basis. The former statement is proved below; the latter requires some machinery, which will be developed in Section 4.2.

Proposition 4.1.7. *Suppose $\mathcal{T}$ admits $|w|$-orthogonal maximal extensions. Then the action of Ξ on $\widehat{\mathcal{E}}$ is well-defined.*

Proof. It suffices to show that the action of $Free(\mathcal{P}'(n))$ on $\widehat{\mathcal{E}}$ is well-defined.

Take $(X_i)_i \in \widehat{\mathcal{E}}$, $S \subsetneq [n]$. By assumption, the minimal extension $(X_j)_S$ exists for each $j \notin S$; we now verify that the tuple $S \cdot (X_i)_i$ is $|w|$-orthogonal. Let $\mathcal{S} = \langle \{X_i \mid i \in S\} \rangle$.

Fix $j \notin S$. Let $f_j : (X_j)_S \to X_j$ be the morphism defining the extension. Let $Y \in \mathcal{S}$.

We claim that, for all $0 \leq m \leq |w|$,

$$\mathrm{Hom}(\Sigma^{-1}(X_j)_S, \Sigma^{-m} Y) = 0 \tag{4.1}$$

By Proposition 4.1.5, Equation (4.1) holds for $m = 0, 1$. For $2 \leq m \leq |w|$, apply the functor $\mathrm{Hom}(-, \Sigma^{-m+1} Y)$ to the triangle $\Sigma^{-1} C(f_j) \to (X_j)_S \xrightarrow{f_j} X_j \to C(f_j)$. Since $(X_i)_i$ is $|w|$-orthogonal, $\mathrm{Hom}(X_j, \Sigma^{-m+1} Y) = 0$. Similarly, $\mathrm{Hom}(\Sigma^{-1} C(f_j), \Sigma^{-m+1} Y) = 0$, so $\Sigma^{-1} C(f_j) \in \mathcal{S}$. It follows that $\mathrm{Hom}((X_j)_S, \Sigma^{-m+1} Y) = 0$, hence $\mathrm{Hom}(\Sigma^{-1}(X_j)_S, \Sigma^{-m} Y) = 0$.

Next, we claim that, for all $0 \leq m \leq |w|$,

$$\mathrm{Hom}(Y, \Sigma^{-m-1}(X_j)_S) = 0 \tag{4.2}$$

By Serre duality and Equation (4.1),

$$\begin{aligned} \mathrm{Hom}(Y, \Sigma^{-m-1}(X_j)_S) &\cong \mathrm{Hom}(\Sigma^{-m-1}(X_j)_S, \Sigma^{w} Y)^* \\ &\cong \mathrm{Hom}(\Sigma^{-1}(X_j)_S, \Sigma^{m+w} Y)^* \\ &= 0 \end{aligned}$$

for all $0 \leq m \leq |w|$.

Let $j, l \notin S$. We claim that, for all $0 \leq m \leq |w|-1$,

$$\mathrm{Hom}(\Sigma^{-1}(X_l)_S, \Sigma^{-m-1}(X_j)_S) \cong \mathrm{Hom}(\Sigma^{-1}X_l, \Sigma^{-m-1}X_j) \tag{4.3}$$

Apply $\mathrm{Hom}(\Sigma^{-1}(X_l)_S, -)$ to the triangle

$$\Sigma^{-m-2}C(f_j) \to \Sigma^{-m-1}(X_j)_S \to \Sigma^{-m-1}X_j \to \Sigma^{-m-1}C(f_j)$$

$\Sigma^{-1}C(f_j) \in \mathcal{S}$, so by (4.1), we have, for all $0 \leq m \leq |w|-1$, that

$$\mathrm{Hom}(\Sigma^{-1}(X_l)_S, \Sigma^{-m-2}C(f_j)) = 0 = \mathrm{Hom}(\Sigma^{-1}(X_l)_S, \Sigma^{-m-1}C(f_j))$$

Thus $\mathrm{Hom}(\Sigma^{-1}(X_l)_S, \Sigma^{-m-1}(X_j)_S) \cong \mathrm{Hom}(\Sigma^{-1}(X_l)_S, \Sigma^{-m-1}X_j)$.

Next, we can apply $\mathrm{Hom}(-, \Sigma^{-m-1}X_j)$ to the triangle $\Sigma^{-2}C(f_l) \to \Sigma^{-1}(X_l)_S \to \Sigma^{-1}X_l \to \Sigma^{-1}C(f_l)$. $\Sigma^{-1}C(f_l) \in \mathcal{S}$, so for all $0 \leq m \leq |w|-1$,

$$\mathrm{Hom}(\Sigma^{-2}C(f_l), \Sigma^{-m-1}X_j) = 0 = \mathrm{Hom}(\Sigma^{-1}C(f_l), \Sigma^{-m-1}X_j)$$

Thus $\mathrm{Hom}(\Sigma^{-1}(X_l)_S, \Sigma^{-m-1}X_j) \cong \mathrm{Hom}(\Sigma^{-1}X_l, \Sigma^{-m-1}X_j)$. Combining the two isomorphisms, we obtain the Equation (4.3).

Substituting $j = l$ into Equation (4.3) and using the fact that X_j is elementary, we have that $\Sigma^{-1}(X_j)_S$ is elementary. When $l \neq j$, independence of $\Sigma^{-1}(X_l)_S$ and $\Sigma^{-1}(X_j)_S$ follows from Equation (4.3) and the independence of X_l and X_j. When $j \notin S$ and $l \in S$, independence of $\Sigma^{-1}(X_j)_S$ and X_l follows from Equations (4.1) and (4.2). Thus $\widehat{\mathcal{E}}$ is closed under the action of $S \subsetneq [n]$.

The proof that $\widehat{\mathcal{E}}$ is closed under the action of S^{-1} is dual.

Finally, we must show that the action of S and S^{-1} are mutually inverse. To show that $S^{-1} \cdot (S \cdot (X_i)_i) = (X_i)_i$, it is enough to show that for each $j \notin S$, $(\Sigma^{-1}(X_j)_S)^S \cong \Sigma^{-1}X_j$. It is easy to verify that the map $\Sigma^{-1}(X_j)_S \to \Sigma^{-1}X_j$ satisfies the conditions of Proposition 4.1.5, hence $\Sigma^{-1}X_j$ is isomorphic to $(\Sigma^{-1}(X_j)_S)^S$. The proof that $S \cdot (S^{-1} \cdot (X_i)_i) = (X_i)_i$ is dual. □

4.2 Filtrations

In the previous section, we proved that $\widehat{\mathcal{E}}$ is stable under the action of Ξ. In this section, we show that the subset $\mathcal{E}$ is stable under this action. To accomplish this, we will need a few technical results.

Definition 4.2.1. Let $\mathcal{F}$ be a basis for $\mathcal{T}$. Let $M \in \mathcal{T}$. A **descending $\mathcal{F}$-filtration** is a sequence of morphisms $f_i : M_i \to M_{i-1}$, $1 \leq i \leq m$, with $M_m = M$, $M_0 = 0$ and $\Sigma^{-1}C(f_i) = \Sigma^{-d_i}S_i$ for some $S_i \in \mathcal{F}$ and $1 \leq d_i < |w|$. We say this filtration is **nice** if the sequence $\{d_i\}$ is non-strictly decreasing.

Dually, define an **ascending $\mathcal{F}$-filtration** to be a sequence of morphisms $f_i : M_{i-1} \to M_i$, $1 \leq i \leq m$, with $M_0 = 0$, $M_m = M$, and $C(f_i) = \Sigma^{-d_i}S_i$ for some $S_i \in \mathcal{F}$. We say this filtration is **nice** if the sequence $\{d_i\}$ is non-strictly increasing. For both filtrations, we shall call m the **length** of the filtration.

Given a descending (resp., ascending) filtration of M, we write $M = [\Sigma^{-d_m}S_m, \cdots \Sigma^{-d_1}S_1]_d$ (resp., $M = [\Sigma^{-d_1}S_1, \cdots \Sigma^{-d_m}S_m]_a$). If m is minimal, we shall refer to m as the descending (resp., ascending) **length** of M, which we shall denote by $l_d(M)$ (resp., $l_a(M)$). We shall drop the modifiers and subscripts when there is no risk of confusion and refer simply to "lengths" and "filtrations".

We shall refer to the $\Sigma^{-d_i}S_i$ as the **factors** of M. If a factor appears as the right-most (resp., left-most) term in a nice, minimal descending (resp., ascending) filtration of M, we say that factor **lies in the head** (resp., **socle**), of M.

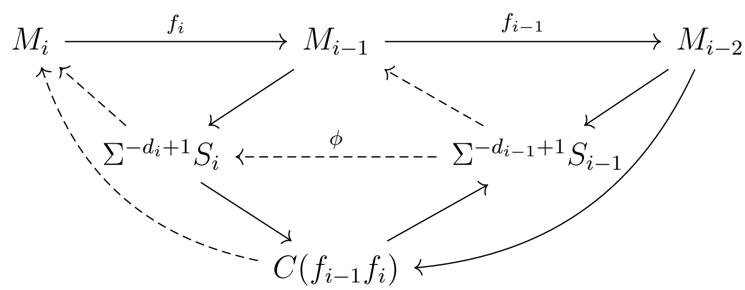
Intuitively, filtrations provide a triangulated analogue of composition series. An object may have many different filtrations relative to a given basis, but filtrations of minimal length are relatively well-behaved. The following lemma is adapted from [CR17, Lemma 7.1].

Lemma 4.2.2. *Let $\mathcal{F}$ be a basis for $\mathcal{T}$. Let $M \in \mathcal{T}$. Then:*
1) M has a descending $\mathcal{F}$-filtration $M = [\Sigma^{-d_m}S_m, \cdots, \Sigma^{-d_1}S_1]$ which is both nice and of minimal length. Given any minimal filtration of M, there is a nice, minimal filtration of M with the same multiset of factors.
2) Any two minimal filtrations of M have the same multiset of factors.
3) Using the notation of part 1), if $\mathrm{Hom}(M, \Sigma^{-d_1}S) \neq 0$ for some $S \in \mathcal{F}$, then $\Sigma^{-d_1}S$ is isomorphic to one of the factors of M, and $\Sigma^{-d_1}S$ lies in the head of M.
4) For any nice, minimal descending filtration, $M = [\Sigma^{-d_m}S_m, \cdots, \Sigma^{-d_1}S_1]$, the composition

$M = M_m \to \cdots \to M_1 \cong \Sigma^{-d_1} S_1$ *is nonzero.*

The dual statements hold for ascending filtrations.

Proof. For 1), since $\mathcal{F}$ is a basis, every object of $\mathcal{T}$ has a finite $\mathcal{F}$-filtration, hence a minimal one. Let $M = [\Sigma^{-d_m} S_m, \cdots, \Sigma^{-d_1} S_1]$ be one such minimal filtration. If this filtration is not nice, there exists i such that $d_i > d_{i-1}$. Consider the following diagram, obtained from the octahedron axiom:



Since $d_i > d_{i-1}$, the morphism $\phi : \Sigma^{-d_{i-1}+1} S_{i-1} \to \Sigma^{-d_i+2} S_i$ is either zero or an isomorphism. If ϕ is an isomorphism, then $C(f_{i-1}f_i) = 0$, hence $f_{i-1}f_i$ is an isomorphism. But then M_i and M_{i-1} can be deleted from the filtration, since the composite map $f_{i-1}f_i f_{i+1} : M_{i+1} \to M_{i-2}$ has cone isomorphic to $C(f_{i+1})$. (If $i = m$, one simply deletes the last two terms, since $M_{m-2} \cong M$.) This contradicts minimality of m, hence we must have $\phi = 0$.

Since $\phi = 0$, $C(f_{i-1}f_i) \cong \Sigma^{-d_i+1} S_i \oplus \Sigma^{-d_{i-1}+1} S_{i-1}$. Let X be the cone of the composition $g : \Sigma^{-d_{i-1}} S_{i-1} \to \Sigma^{-1} C(f_{i-1}f_i) \to M_i$. Let $f_i' : M_i \to X$ be the natural map into the cone. By construction, $\Sigma^{-1} C(f_i') = \Sigma^{-d_{i-1}} S_{i-1}$. Furthermore, applying the octahedron axiom to g yields a map $f_{i-1}' : X \to M_{i-2}$ such that $\Sigma^{-1} C(f_{i-1}') = \Sigma^{-d_i} S_i$. Thus replacing f_i and f_{i-1} with f_i' and f_{i-1}' yields a minimal filtration with $\Sigma^{-d_i} S_i$ and $\Sigma^{-d_{i-1}} S_{i-1}$ swapped. We may repeat this process until there are no more inversions, yielding a nice, minimal filtration. Since the factors have only been permuted, the multiset of factors remains unchanged.

For 3), let $1 \leq r \leq m$ be minimal such that there is a nonzero morphism $M_r \to \Sigma^{-d_1} S$. Consider the triangle $\Sigma^{-d_r} S_r \to M_r \to M_{r-1} \to \Sigma^{-d_r+1} S_r$. Since $\mathrm{Hom}(M_{r-1}, \Sigma^{-d_1} S) = 0$, the nonzero space $\mathrm{Hom}(M_r, \Sigma^{-d_1} S)$ injects into $\mathrm{Hom}(\Sigma^{-d_r} S_r, \Sigma^{-d_1} S)$. Since the filtration is nice, $d_1 \geq d_r$; since the Hom space is nonzero, we deduce that $d_r = d_1$ and $S_r \cong S$. It follows that the composition $\Sigma^{-d_r} S_r \to M_r \to \Sigma^{-d_1} S$ is an isomorphism, hence the above triangle splits and $M_r \cong \Sigma^{-d_1} S \oplus M_{r-1}$.

Define a new filtration of M as follows. For $1 \leq i \leq r-1$, let $M'_i = \Sigma^{-d_1} S \oplus M_{i-1}$ and let $f'_i : M'_i \to M'_{i-1}$ be the direct sum of the identity map and f_{i-1}. Since $M_r \cong \Sigma^{-d_1} S \oplus M_{r-1}$, we can define f'_r in the same way. Let all remaining objects and maps remain the same. It is straightforward to verify that this is a filtration identical to the original, except that the last r factors have been cyclically permuted, so that $\Sigma^{-d_r} S_r \cong \Sigma^{-d_1} S$ is the final term. It is clear that this filtration is nice and minimal, hence $\Sigma^{-d_1} S$ lies in the head of M.

To prove 4), for each $1 \leq i \leq m$ let $g_i : M_i \to M_1 \cong \Sigma^{-d_1} S_1$ be the natural composition. Suppose for a contradiction that $g_m = 0$ and let $i \geq 2$ be minimal such that $g_i = 0$. Decompose g_i as $M_i \xrightarrow{g} M_2 \xrightarrow{f_2} M_1$ and apply the octahedron axiom. We obtain a triangle $\Sigma^{-1} C(g) \to M_i \oplus \Sigma^{d_1 - 1} S_1 \to \Sigma^{-d_2} S_2 \to C(g)$. We have that $\Sigma^{-1} C(g) = [\Sigma^{-d_i} S_i, \cdots \Sigma^{-d_3} S_3]$; by niceness of the filtration it follows that $\mathrm{Hom}(\Sigma^{-1} C(g), \Sigma^{-d_1 - 1} S_1) = 0$. Therefore the morphism $\Sigma^{-1} C(g) \to M_i \oplus \Sigma^{d_1 - 1} S_1$ factors through the inclusion of M_i, hence the cone of this morphism is $M_2 \oplus \Sigma^{d_1 - 1} S_1 \cong \Sigma^{-d_2} S_2$. This contradicts locality of $\mathrm{End}(S_2)$, thus $g_i \neq 0$ for all i.

The proof of 2) is by induction on the length, m, of M. For $m = 0, 1$, the result is clear. Suppose the result holds for all lengths less than m. Given $M = [\Sigma^{-d_m} S_m, \cdots, \Sigma^{-d_1} S_1] = [\Sigma^{-d'_m} S'_m, \cdots, \Sigma^{-d'_1} S'_1]$ two minimal filtrations, by 1) we can rearrange the factors and assume WLOG that both filtrations are nice. Then $\mathrm{Hom}(M, \Sigma^{-d} S) = 0$ for any $S \in \mathcal{F}$, $d_1 < d \leq |w| - 1$. By 4), $\mathrm{Hom}(M, \Sigma^{-d'_1} S'_1) \neq 0$, hence $d'_1 \leq d_1$. A symmetric argument gives the reverse inequality, hence $d_1 = d'_1$. By 3), $\Sigma^{-d_1} S_1 \cong \Sigma^{-d'_r} S'_r$ for some r, and we can rearrange the second filtration so that $\Sigma^{-d_1} S_1$ is the last term. Since both filtrations now end in $\Sigma^{-d_1} S_1$, we obtain two nice, minimal filtrations of $M' = \Sigma^{-1} C(M \to \Sigma^{-d_1} S_1)$ whose factor multisets correspond to the original factor multisets with one copy of $\Sigma^{-d_1} S_1$ removed. Applying the induction hypothesis to M', we are done. □

The following technical lemma describes the interaction between filtrations and maximal extensions.

Lemma 4.2.3. *Let $\mathcal{F}$ be a basis for $\mathcal{T}$ and let $\mathcal{S} \subset \mathcal{F}$. Let $T \in \mathcal{F} - \mathcal{S}$ and let $T_{\mathcal{S}} \to T$ denote the maximal extension of T by $\mathcal{S}$. Suppose this map factors as $T_{\mathcal{S}} \xrightarrow{f} N \xrightarrow{g} T$ for some object $N = [S_k, \cdots, S_2, T]_d$, with $S_i \in \mathcal{S}$. Suppose that* $\mathrm{Hom}(N, \mathcal{S}) = 0$. *Then* $\Sigma^{-1} C(f) \in \langle \mathcal{S} \rangle$.

Dually, let $T \to T^S$ denote the maximal $\mathcal{S}$-extension by T. Suppose this map factors as $T \xrightarrow{g} N \xrightarrow{f} T^S$ for some object $N = [T, S_2, \cdots, S_k]_a$, with $S_i \in \mathcal{S}$. Suppose $\operatorname{Hom}(\mathcal{S}, N) = 0$. Then $C(f) \in \langle \mathcal{S} \rangle$.

Proof. Applying the octahedron axiom to the composition gf, we obtain a triangle $\Sigma^{-2}C(g) \to \Sigma^{-1}C(f) \to \Sigma^{-1}C(gf) \to \Sigma^{-1}C(g)$, where $\Sigma^{-1}C(g), \Sigma^{-1}C(gf) \in \langle \mathcal{S} \rangle$. It follows that $\Sigma^{-1}C(f)$ has a nice, minimal filtration whose factors lie in $\mathcal{S} \cup \Sigma^{-1}\mathcal{S}$. We have that $\operatorname{Hom}(\Sigma^{-1}N, \Sigma^{-1}\mathcal{S}) = 0 = \operatorname{Hom}(T_S, \Sigma^{-1}\mathcal{S})$, hence $\operatorname{Hom}(\Sigma^{-1}C(f), \Sigma^{-1}\mathcal{S}) = 0$. It follows from Lemma 4.2.2 that $\Sigma^{-1}C(f)$ can have no factors lying in $\Sigma^{-1}\mathcal{S}$. Therefore $\Sigma^{-1}C(f) \in \langle \mathcal{S} \rangle$.

The proof of the second statement is dual. □

We are now ready to prove that $\mathcal{E}$ is closed under the action of Ξ. The following result is based on [CR17, Proposition 7.4].

Theorem 4.2.4. *Suppose $\mathcal{T}$ admits $|w|$-orthogonal maximal extensions. Then the action of Ξ on $\mathcal{E}$ is well-defined.*

Proof. We must show that the action of $S \subsetneq [n]$ on an orthogonal tuple preserves the property of being a basis. Let $(X_i)_i \in \mathcal{E}$ and let $(X'_i)_i = S \cdot (X_i)_i \in \widehat{\mathcal{E}}$. Let $\mathcal{S} = \{X_i \mid i \in S\}$, let $\mathcal{F} = \{X_i\}$, and let $\mathcal{F}' = \{X'_i\}$. Let $\mathcal{G} = \bigcup_{i=0}^{|w|-1} \Sigma^{-i}\mathcal{F}$ and $\mathcal{G}' = \bigcup_{i=0}^{|w|-1} \Sigma^{-i}\mathcal{F}'$. Then $\langle \mathcal{G} \rangle = \mathcal{T}$, and we must show that the same holds for $\langle \mathcal{G}' \rangle$.

Take a nonzero object $M \in \mathcal{T}$. We first consider the special case where no $\Sigma^{-i}Y$ lies in the head of ΣM, for any $Y \in \mathcal{S}, 0 \leq i < |w|$. We claim that $M \in \langle \mathcal{G}' \rangle$; the proof will be by induction on the $\mathcal{F}$-length, m, of ΣM. If $m = 1$, then $\Sigma M \cong \Sigma^{-i}T$ for some $T \in \mathcal{F} - \mathcal{S}$. We have a triangle $\Sigma^{-i-1}T_S \to M \to \Sigma^{-i}Y \to \Sigma^{-i}T_S$ for some $Y \in \langle \mathcal{S} \rangle$. For any $0 \leq i < |w|$, $\Sigma^{-i-1}T_S, \Sigma^{-i}Y \in \mathcal{G}'$, hence $M \in \langle \mathcal{G}' \rangle$.

Now suppose $m > 1$ and the result holds for lower lengths. By Lemma 4.2.2, ΣM must have a nice, minimal descending $\mathcal{G}$-filtration ending in some $\Sigma^{-d_1}T_1$, where $T_1 \in \mathcal{F} - \mathcal{S}, 0 \leq d_1 < |w|$. There exists a maximal $0 \leq k \leq m$ such that there exists a minimal filtration of the form

$$\Sigma M = [\Sigma^{-d_m}T_m, \cdots, \Sigma^{-d_{k+1}}T_{k+1}, \Sigma^{-d_k}S_k, \cdots \Sigma^{-d_2}S_2, \Sigma^{-d_1}T_1]$$

where each $S_j \in \mathcal{S}$, and $T_{k+1} \in \mathcal{F} - \mathcal{S}$. (If $k = m$, the filtration starts with $\Sigma^{-d_m} S_m$.) The octahedron axiom gives us a triangle $\Sigma M' \to \Sigma M \to \Sigma M'' \to \Sigma^2 M'$, where $\Sigma M'$ and $\Sigma M''$ have (necessarily minimal) filtrations given by $\Sigma M' = [\Sigma^{-d_m} T_m, \cdots, \Sigma^{-d_{k+1}} T_{k+1}]$ and $\Sigma M'' = [\Sigma^{-d_k} S_k, \cdots \Sigma^{-d_2} S_2, \Sigma^{-d_1} T_1]$.

Note that there is no minimal filtration of $\Sigma M'$ whose last factor is of the form $\Sigma^{-d_{k+1}} S_{k+1}$, with $S_{k+1} \in \mathcal{S}, 0 \leq d_{k+1} < |w|$. If so, we could concatenate this filtration of $\Sigma M'$ with the given filtration $\Sigma M''$ to produce a new minimal filtration for ΣM which would contradict the maximality of k. Since the length of $\Sigma M''$ is at least one, $\Sigma M'$ has length strictly shorter than ΣM. By the induction hypothesis, $M' \in \langle \mathcal{G}' \rangle$.

We now show that $M'' \in \langle \mathcal{G}' \rangle$. By the proof of part 1) of Lemma 4.2.2, by rearranging the S_i we may assume WLOG that the filtration for $\Sigma M''$ expressed above is nice. Let $1 \leq r \leq k$ be maximal such that $d_r = d_1$. We may express $\Sigma M''$ as the triangle $\Sigma N_1 \to \Sigma M'' \to \Sigma N_2 \to \Sigma^2 N_1$, where $\Sigma N_1 = [\Sigma^{-d_k} S_k, \cdots \Sigma^{-d_{r+1}} S_{r+1}]$ and $\Sigma N_2 = \Sigma^{-d_1}([S_r, \cdots S_2, T_1])$ are nice, minimal filtrations. For all $r < i \leq k$, we have that $d_i < d_1 \leq |w| - 1$, thus $N_1 = [\Sigma^{-d_k - 1} S_k, \cdots \Sigma^{-d_{r+1} - 1} S_{r+1}] \in \langle \mathcal{G}' \rangle$.

Next, $\mathrm{Hom}((T_1)_S, \Sigma \mathcal{S}) = 0$, hence the minimal extension $(T_1)_S \to T_1$ factors through the natural map $[S_r, \cdots S_2, T_1] \to T_1$. Note also that $\mathrm{Hom}([S_r, \cdots S_2, T_1], \mathcal{S}) = 0$; otherwise by part 3) of Lemma 4.2.2, there would be some member of $\Sigma^{-d_1} \mathcal{S}$ lying in the head of ΣN_2. But this is impossible, since any factor in the head of ΣN_2 also lies in the head of M' and M, and the head of M contains no such factors by assumption. The hypotheses of Lemma 4.2.3 are satisfied, and so we obtain a triangle $\Sigma^{-1}(T_1)_S \to \Sigma^{-1}[S_r, \cdots S_2, T_1] \to Y \to (T_1)_S$, with $Y \in \langle \mathcal{S} \rangle$. Applying Σ^{-d_1} to this triangle, we obtain $\Sigma^{-d_1 - 1}(T_1)_S \to N_2 \to \Sigma^{-d_1} Y$. Since $0 \leq |d_1| < |w|$, both of the outside terms lie in $\langle \mathcal{G}' \rangle$, hence so does N_2. It follows immediately that M'' and therefore M lie in $\langle \mathcal{G}' \rangle$. This concludes our proof of the special case.

We are now ready to prove the general case; it suffices to show that $\mathcal{G} \subset \langle \mathcal{G}' \rangle$. By definition, $\bigcup_{i=0}^{|w|-1} \Sigma^{-i} \mathcal{S} \subset \mathcal{G}'$. For $T \notin \mathcal{S}, 0 < i \leq |w| - 1$, the triangle $\Sigma^{-i-1} C(f) \to \Sigma^{-i} T_S \to \Sigma^{-i} T \to \Sigma^{-i} C(f)$ shows that $\Sigma^{-i} T \in \langle \mathcal{G}' \rangle$. It remains to show that $T \in \langle \mathcal{G}' \rangle$; we shall reduce this problem to the special case shown above.

Note that $\mathrm{Hom}(\Sigma^i Y, T) = 0$ for all $Y \in \mathcal{S}, 0 \leq i \leq |w| - 1$, hence $\mathrm{Hom}(\Sigma T, \Sigma^{-|w|+1+i} Y) = 0$

by Serre duality. In particular, by Lemma 4.2.2, part 4), ΣT has no nice, minimal descending filtration ending in $\Sigma^i Y$, for any $Y \in \mathcal{S}$, $0 \leq i \leq |w| - 1$. By the special case, $T \in \langle \mathcal{G}' \rangle$, and we are done. $\square$

CHAPTER 5

The Dg-Stable Category of a Brauer Tree Algebra

5.1 Brauer Tree Algebras

In this section we shall prove that the functor F_A of Theorem 3.1.10 is an equivalence whenever the algebra A is any non-positively graded Brauer tree algebra. We shall work over an algebraically closed field k.

A **Brauer tree** consists of the data $\Gamma = (T, e, v, m)$, where T is a tree, e is the number of edges of T, v is a vertex of T, called the **exceptional vertex**, and m is a positive integer, called the **multiplicity** of v. To any Brauer tree Γ, we can associate a basic finite-dimensional symmetric algebra A_Γ. For the details of this process, we refer to [Sch18].

An important special case is $S = (S, n, v, m)$, the star with n edges and exceptional vertex at the center. In this case, the algebra A_S is a Nakayama algebra whose indecomposable projective modules have length $nm + 1$.

The following theorems are due to Bogdanic:

Theorem 5.1.1. *[Bog10, Theorem 4.3 and Lemma 4.9] Let S be the star with n vertices and multiplicity m. Let A_S be graded so that $soc(A_S)$ is in degree nm. Let Γ be any Brauer tree with n vertices and multiplicity m. Then A_Γ admits a non-negative grading such that $soc(A_\Gamma)$ is in degree nm, and there is an equivalence $D^b(A_S\text{-grmod}) \to D^b(A_\Gamma\text{-grmod})$.*

Theorem 5.1.2. *[Bog10, Section 11] Let Γ be a Brauer tree with multiplicity m. Then, up to graded Morita equivalence and rescaling, A_Γ possesses a unique grading. The socle of A_Γ lies in degree dm for some $d \in \mathbb{Z}$, and the grading is determined up to graded Morita equivalence by d. If $d > 0$, the grading can be chosen to be non-negative, and if $d < 0$ the grading can be chosen to* *be non-positive.*

From these facts, we obtain the following result:

Corollary 5.1.3. *Let* Γ *be a Brauer tree. Let* S *be the star with the same multiplicity and number of edges. Let* A_Γ *and* A_S *be graded so that* $soc(A_\Gamma)$ *and* $soc(A_S)$ *lie in degree* d *for some* $d \in \mathbb{Z}$*. Then we have equivalences of triangulated categories* $D^b(A_S\text{-grmod}) \to D^b(A_\Gamma\text{-grmod})$ *and* $A_S\text{-grstab} \to A_\Gamma\text{-grstab}$.

Theorem 5.1.4. *Let* $\Gamma = (T, e, v, m)$ *be a Brauer graph, and let* A_Γ *be non-positively graded with socle in degree* $-d \leq 0$*. Then* F_{A_Γ} *is an equivalence.*

Proof. If Γ is the star, then A_Γ is a Nakayama algebra and the result follows immediately from Lemma 3.2.6. If Γ is not the star, let S denote the star with the same number of edges and multiplicity as Γ. By Theorem 5.1.2 there is a nonpositive grading on A_S such that $soc(A_S)$ is in degree $-d$. Then by Theorem 5.1.1, $D^b(A_\Gamma\text{-grmod})$ and $D^b(A_S\text{-grmod})$ are equivalent as triangulated categories. By a theorem of Rickard [Ric89a], this induces a triangulated equivalence $G : A_\Gamma\text{-grstab} \to A_S\text{-grstab}$ which commutes with grading shifts. By Proposition 3.1.22, G induces an equivalence between $\mathcal{C}(A_\Gamma)$ and $\mathcal{C}(A_S)$. Since A_S is a Nakayama algebra, it satisfies the hypotheses of Lemma 3.2.3, hence A_Γ does as well. Thus F_{A_Γ} is an equivalence. □

Corollary 5.1.5. *Let* Γ *be a Brauer tree, and let* S *be the star with the same multiplicity and number of edges. Let* A_Γ *and* A_S *both be graded with socle in degree* $-d \leq 0$*. Then* $A_\Gamma\text{-dgstab}$ *and* $A_S\text{-dgstab}$ *are equivalent as triangulated categories.*

Proof. This follows from the use of Proposition 3.1.22 in the previous theorem. □

5.2 The Dg-Stable Category of the Star with n Vertices

For $n \geq 2, d \geq 0$, let $A = A_{n,d}$ denote the graded Brauer tree algebra, with socle in degree $-d$, corresponding to the star S with n edges and exceptional vertex of multiplicity one. This specifies A up to graded Morita equivalence; we will choose a specific grading once we have adopted some more notation in the section below. By the results of Section 5.1, $A\text{-dgstab}$ is equivalent to $\mathcal{C}(A)$. We shall identify the two categories throughout this section.

5.2.1 Notation, Indexing, and Grading

We index the edges of S by the set $\mathbb{Z}/n\mathbb{Z} = \{\overline{1}, \cdots, \overline{n}\}$, according to their cyclic order around the center vertex. We define a total order $\leq$ on $\mathbb{Z}/n\mathbb{Z}$ by $\overline{1} < \overline{2} < \cdots < \overline{n}$. This order is of course not compatible with the group operation on $\mathbb{Z}/n\mathbb{Z}$.

If P is a statement with a truth value, we define δ_P to be 1 if P is true and 0 if P is false.

For $x, y \in \mathbb{Z}$, define $\langle x, y\rangle$ to be the closed arc of the unit circle starting at $e^{\frac{2\pi\sqrt{-1}}{n}x}$ and proceeding counterclockwise to $e^{\frac{2\pi\sqrt{-1}}{n}y}$. Thus $\langle x, x\rangle$ denotes a point, rather than the full circle.

With these definitions, the Ext-quiver of A is a directed cycle, C, of length n. C has vertices $e_{\overline{i}}$ and edges $e_{\overline{i}} \xrightarrow{a_{\overline{i}}} e_{\overline{i}+\overline{1}}$ for all $\overline{i} \in \mathbb{Z}/n\mathbb{Z}$. A is isomorphic to, and will be identified with, the quotient of the path algebra of C by the ideal generated by paths of length $n+1$. Changing A up to graded Morita equivalence, we determine the grading on A by defining $deg(a_{\overline{i}}) = -d\delta_{\overline{i}=\overline{n}}$. We denote by $S_{\overline{i}}$ the simple A-module corresponding to $e_{\overline{i}}$, in degree 0. We denote by $P_{\overline{i}}$ the indecomposable projective module with head $S_{\overline{i}}$ and socle $S_{\overline{i}}(d)$.

The indecomposable A-modules are uniserial and determined, up to isomorphism, by their head and socle. For $\overline{i}, \overline{j} \in \mathbb{Z}/n\mathbb{Z}$, let $M_{\overline{j}}^{\overline{i}}$ denote the indecomposable module with head $S_{\overline{i}}$ and socle $S_{\overline{j}}(d\delta_{\overline{j}<\overline{i}})$. More specifically, for $1 \leq i, j \leq n$, we define $M_{\overline{j}}^{\overline{i}}$ to be the module $e_{\overline{i}}A/e_{\overline{i}}J^l$, where J is the Jacobson radical of A and $l = \delta_{i>j}n+1+j-i$ is the length of $M_{\overline{j}}^{\overline{i}}$. The non-projective indecomposable objects of $A\text{-grmod}$, up to grading shifts and isomorphism, are precisely $M_{\overline{j}}^{\overline{i}}$ for $\overline{i}, \overline{j} \in \mathbb{Z}/n\mathbb{Z}$.

Even when working in $A\text{-grstab}$, it will be helpful to define the "length" of $M_{\overline{j}}^{\overline{i}}$, for $1 \leq i, j \leq n$, to be $l(M_{\overline{j}}^{\overline{i}}) = \delta_{i>j}n + 1 + j - i$.

Finally, we note that for $1 \leq r, j \leq n$, the module $M_{\overline{j}}^{\overline{j}+\overline{1}-\overline{r}}(-d\delta_{\overline{j}\neq\overline{r}})$ has length r and socle S_j in degree zero; we shall make extensive use of this module later on.

5.2.2 Structure of A-grstab

One of the desirable features of Brauer tree algebras is that the A-module homomorphisms $X \to Y$ can be determined combinatorially from the composition towers of X and Y, allowing quick and easy computation of morphisms. For a more general and explicit description of this procedure, we refer to Crawley-Boevey [Cra89]. These techniques generalize easily to graded modules.

The following results about A-grstab follow from straightforward computation and are well-known. We state them without proof.

Proposition 5.2.1. *The (distinct) indecomposable objects of* A-grstab *are precisely* $M_{\bar{j}}^{\bar{i}}(k)$*, for any* $\bar{i}, \bar{j} \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}$.

Proposition 5.2.2.

$$dim\, \mathrm{Hom}_{A\text{-grstab}}(M_{\bar{b}}^{\bar{a}}, M_{\bar{j}}^{\bar{i}}(k)) = \begin{cases} 1 & \text{if } \langle a, j\rangle \subset \langle i, b\rangle \text{ and } k = -d\delta_{\bar{a}<\bar{i}} \\ 0 & \text{otherwise} \end{cases}$$

We shall refer to the statement $\langle a, j\rangle \subset \langle i, b\rangle$ as the **arc containment condition**.

For describing composition, it will be helpful to choose a collection of generators for the above Hom spaces. Fortunately, there are natural choices.

Definition 5.2.3. Let $1 \leq a, b, i, j \leq n$, and let l be the length of $M_{\bar{j}}^{\bar{i}}$.

For $\bar{i} \neq \bar{j}$, define the **canonical surjection**

$$p_{\bar{j}}^{\bar{i}} : M_{\bar{j}}^{\bar{i}} = e_{\bar{i}}A/e_{\bar{i}}J^{l} \twoheadrightarrow e_{\bar{i}}A/e_{\bar{i}}J^{l-1} = M_{\bar{j}-\bar{1}}^{\bar{i}}$$

$$e_{\bar{i}} \longmapsto e_{\bar{i}}$$

For $\bar{i} \neq \bar{j} + \bar{1}$, define the **canonical injection**

$$\iota_{\bar{j}}^{\bar{i}} : M_{\bar{j}}^{\bar{i}} = e_{\bar{i}}A/e_{\bar{i}}J^{l} \xrightarrow{\sim} (e_{\bar{i}-\bar{1}}J/e_{\bar{i}-\bar{1}}J^{l+1})(-d\delta_{\bar{i}=\bar{1}}) \hookrightarrow M_{\bar{j}}^{\bar{i}-\bar{1}}(-d\delta_{\bar{i}=\bar{1}})$$

$$e_{\bar{i}} \longmapsto e_{\bar{i}-\bar{1}}a_{\bar{i}-\bar{1}}$$

For $< a, j > \subset < i, b >$, define the **canonical map** $\alpha_{\bar{b},\bar{j}}^{\bar{a},\bar{i}} : M_{\bar{b}}^{\bar{a}} \to M_{\bar{j}}^{\bar{i}}(-d\delta_{\bar{a}<\bar{i}})$ by

$$\alpha_{\bar{b},\bar{j}}^{\bar{a},\bar{i}} = \iota_{\bar{j}}^{\bar{i}+\bar{1}}(-d\delta_{\bar{i}+\bar{1}>\bar{a}}) \cdots \iota_{\bar{j}}^{\bar{a}-\bar{1}}(-d\delta_{\bar{a}-\bar{1}>\bar{a}}) \circ \iota_{\bar{j}}^{\bar{a}} \circ p_{\bar{j}+\bar{1}}^{\bar{a}} \cdots p_{\bar{b}}^{\bar{a}}$$

Note, in particular, that $\alpha^{\bar{a},\bar{a}}_{\bar{b},\bar{b}}$ is the identity map.

Proposition 5.2.4. *The indecomposable maps in* $A\text{-grstab}$ *are precisely the canonical surjections and injections. Composition in* $A\text{-grstab}$ *is given by the formula:*

$$\alpha^{\bar{c},\bar{e}}_{\bar{d},\bar{f}}(-d\delta_{\bar{a}<\bar{c}}) \circ \alpha^{\bar{a},\bar{c}}_{\bar{b},\bar{d}} = \begin{cases} \alpha^{\bar{a},\bar{e}}_{\bar{b},\bar{f}} & \textit{if } \langle a, f\rangle \subset \langle e, b\rangle \\ 0 & \textit{otherwise} \end{cases}$$

Proposition 5.2.5. *In* $A\text{-grstab}$*, the following formulas hold:*

$$\Omega(M^{\bar{i}}_{\bar{j}}) = M^{\bar{j}+\bar{1}}_{\bar{i}}(d\delta_{\bar{j}+\bar{1}\leq\bar{i}}) \tag{5.1}$$

$$\Omega^{-1}(M^{\bar{i}}_{\bar{j}}) = M^{\bar{j}}_{\bar{i}-\bar{1}}(-d\delta_{\bar{i}\leq\bar{j}}) \tag{5.2}$$

$$\Omega^{2k}(M^{\bar{i}}_{\bar{j}}) = M^{\bar{i}+\bar{k}}_{\bar{j}+\bar{k}}(d(k+\delta_{\bar{n}+\bar{1}-\bar{k}\leq\bar{i}})) \qquad \textit{for } 1\leq k\leq n. \tag{5.3}$$

$$\Omega^{2k-1}(M^{\bar{i}}_{\bar{j}}) = M^{\bar{j}+\bar{k}}_{\bar{i}+\bar{k}-\bar{1}}(d(k+\delta_{\bar{n}+\bar{1}-\bar{k}\leq\bar{i}}-\delta_{\bar{i}+\bar{k}\leq\bar{j}+\bar{k}})) \qquad \textit{for } 1\leq k\leq n. \tag{5.4}$$

$$\Omega^{-2k}(M^{\bar{i}}_{\bar{j}}) = M^{\bar{i}-\bar{k}}_{\bar{j}-\bar{k}}(-d(k+\delta_{\bar{i}\leq\bar{k}})) \qquad \textit{for } 1\leq k\leq n. \tag{5.5}$$

$$\Omega^{-2k+1}(M^{\bar{i}}_{\bar{j}}) = M^{\bar{j}-\bar{k}+1}_{\bar{i}-\bar{k}}(-d(k+\delta_{\bar{i}\leq\bar{k}}-\delta_{\bar{j}-\bar{k}+\bar{1}\leq\bar{i}-\bar{k}})) \qquad \textit{for } 1\leq k\leq n. \tag{5.6}$$

Analogous formulas hold for the $\alpha^{\bar{a},\bar{i}}_{\bar{b},\bar{j}}$.

Proposition 5.2.6. $\alpha^{\bar{a},\bar{i}}_{\bar{b},\bar{j}} : M^{\bar{a}}_{\bar{b}} \to M^{\bar{i}}_{\bar{j}}(-d\delta_{\bar{a}<\bar{i}})$ *can be completed into the exact triangle:*

$$\begin{array}{c} M^{\bar{a}}_{\bar{b}} \\ \big\downarrow{\scriptstyle \alpha^{\bar{a},\bar{i}}_{\bar{b},\bar{j}}} \\ M^{\bar{i}}_{\bar{j}}(-d\delta_{\bar{a}<\bar{i}}) \\ \big\downarrow{\scriptstyle h_1} \\ \delta_{\bar{a}\neq\bar{i}}M^{\bar{i}}_{\bar{a}-\bar{1}}(-d\delta_{\bar{a}<\bar{i}}) \oplus \delta_{\bar{b}\neq\bar{j}}M^{\bar{b}}_{\bar{j}}(-d\delta_{\bar{a}\leq\bar{b}}) \\ \big\downarrow{\scriptstyle h_2} \\ M^{\bar{b}}_{\bar{a}-\bar{1}}(-d\delta_{\bar{a}\leq\bar{b}}) \end{array}$$

where $h_1 = \begin{pmatrix} \delta_{\bar{a}\neq\bar{i}}\alpha^{\bar{i},\bar{i}}_{\bar{j},\bar{a}-\bar{1}}(-d\delta_{\bar{a}<\bar{i}}) \\ \delta_{\bar{b}\neq\bar{j}}\alpha^{\bar{i},\bar{b}}_{\bar{j},\bar{j}}(-d\delta_{\bar{a}<\bar{i}}) \end{pmatrix}$*, and*

$h_2 = \begin{pmatrix} \delta_{\bar{a}\neq\bar{i}}\alpha^{\bar{i},\bar{b}}_{\bar{a}-\bar{1},\bar{a}-\bar{1}}(-d\delta_{\bar{a}<\bar{i}}) & \delta_{\bar{b}\neq\bar{j}}\alpha^{\bar{b},\bar{b}}_{\bar{j},\bar{a}-\bar{1}}(-d\delta_{\bar{a}\leq\bar{b}}) \end{pmatrix}$

5.2.3 Structure of A-dgstab

Since $\Omega \cong (-1)$ in $A\text{-dgstab}$, and Ω is periodic in $A\text{-grstab}$, it follows that (1) is periodic in $A\text{-dgstab}$. The period depends both on n and d. This period is the same for all indecomposable modules except when n is odd, in which case the indecomposable modules of length $\frac{n+1}{2}$ have their period halved.

Proposition 5.2.7. *In $A\text{-dgstab}$, $M_{\bar{j}}^{\bar{i}} \cong M_{\bar{j}}^{\bar{i}}((n+1)d+2n)$ for all $\bar{i}, \bar{j} \in \mathbb{Z}/n\mathbb{Z}$. If n is odd, then we also have $M_{\bar{i}+\frac{n-1}{2}}^{\bar{i}} \cong M_{\bar{i}+\frac{n-1}{2}}^{\bar{i}}(\frac{(n+1)d}{2}+n)$.*

Proof. By Proposition 5.2.5 we have that $M_{\bar{j}}^{\bar{i}}(-2n) \cong \Omega^{2n} M_{\bar{j}}^{\bar{i}} = M_{\bar{j}}^{\bar{i}}(d(n+1))$, from which the first formula follows. Similarly, if n is odd, then $M_{\bar{i}+\frac{n-1}{2}}^{\bar{i}}(-n) \cong \Omega^n M_{\bar{i}+\frac{n-1}{2}}^{\bar{i}} = M_{\bar{i}+\frac{n-1}{2}}^{\bar{i}}(\frac{(n+1)d}{2})$, from which the second formula follows. □

Definition 5.2.8. Define the **period** of $r \in \{1, \cdots, n\}$ to be

$$P(r) = \begin{cases} (n+1)d+2n & \text{if } r \neq \frac{n+1}{2} \\ \frac{(n+1)d}{2}+n & \text{if } r = \frac{n+1}{2} \end{cases}$$

We also define the **period** of $M_{\bar{j}}^{\bar{i}}$ to be $P(l(M_{\bar{j}}^{\bar{i}}))$. We define the period $P(X)$ of an arbitrary object X to be the maximum period of its indecomposable components. When we do not wish to emphasize the dependence on the length of the module, we will simply write $P = (n+1)d+2n$.

For any $X \in A\text{-dgstab}$, let $\psi : X \to X((n+1)d+2n)$ denote the map induced by the natural isomorphism $id \to ((n+1)d+2n)$, whose unique nonzero component is the identity map in degree $-2n$. For any $X \in A\text{-dgstab}$ that can be expressed as a direct sum of modules of length $\frac{n+1}{2}$, let $\psi^{1/2} : X \to X(\frac{(n+1)d}{2}+n)$ denote the isomorphism whose unique nonzero component is the identity map in degree $-n$.

Thus Proposition 5.2.7 states that for any $1 \leq i, j \leq n$, $M_{\bar{j}}^{\bar{i}} \cong M_{\bar{j}}^{\bar{i}}(P(r))$ in $A\text{-dgstab}$, where $r = j+1-i$ is the length of $M_{\bar{j}}^{\bar{i}}$.

One consequence of periodicity is that we can express any $M_{\bar{j}}^{\bar{i}}$ as a suitable shift of some $M_{\bar{l}}^{\bar{1}}$. Furthermore, l can always be chosen to lie in the range $1 \leq l \leq \frac{n+1}{2}$, since $l(\Omega M_{\bar{j}}^{\bar{i}}) = n+1-l(M_{\bar{j}}^{\bar{i}})$.

Proposition 5.2.9. *Let $1 \leq i, r \leq n$ and $1 \leq l \leq \frac{n+1}{2}$. The following identities hold in A-dgstab:*

$$M^{\bar{i}}_{\bar{i}+\bar{r}-\bar{1}} \cong M^{\bar{1}}_{\bar{r}}(-(d+2)(i-1)) \tag{5.7}$$

$$M^{\bar{1}}_{\bar{r}} \cong M^{\bar{1}}_{\bar{n}+\bar{1}-\bar{r}}((d+2)(n+1-r)-1) \tag{5.8}$$

$$M^{\bar{i}}_{\bar{i}+\bar{n}-\bar{l}} \cong M^{\bar{1}}_{\bar{l}}(-(d+2)(n+i-l)+1) \tag{5.9}$$

Proof. We first show (5.7). If $i = 1$, we are done. Otherwise, Equation (5.3) of Proposition 5.2.5 yields

$$\begin{aligned} M^{\bar{1}}_{\bar{r}}(-2(i-1)) &\cong \Omega^{2(i-1)} M^{\bar{1}}_{\bar{r}} \\ &= M^{\bar{i}}_{\bar{i}+\bar{r}-\bar{1}}(d(i-1)) \end{aligned}$$

from which (5.7) follows.

Applying (5.7) and (5.2), we obtain

$$\begin{aligned} M^{\bar{1}}_{\bar{n}+\bar{1}-\bar{r}}(1) &\cong \Omega^{-1}(M^{\bar{1}}_{\bar{n}+\bar{1}-\bar{r}}) \\ &= M^{\bar{n}+\bar{1}-\bar{r}}_{\bar{n}}(-d) \\ &\cong M^{\bar{1}}_{\bar{r}}(-(d+2)(n-r)-d) \\ &= M^{\bar{1}}_{\bar{r}}(-(d+2)(n+1-r)+2) \end{aligned}$$

from which (5.8) follows.

(5.9) follows immediately from (5.7) and (5.8). □

We immediately obtain the following corollary:

Proposition 5.2.10. *Every indecomposable object of A-dgstab is isomorphic to one of the following:*

1) $M^{\bar{1}}_{\bar{l}}(k)$ for $1 \leq l < \frac{n+1}{2}$ and $0 \leq k < P$

2) $M^{\bar{1}}_{\frac{n+1}{2}}(k)$ for $0 \leq k < \frac{P}{2}$ (if n is odd)

Remark. The above list of objects are in fact pairwise non-isomorphic. We shall prove this in Theorem 5.2.12.

Proof. By applying the identities in Proposition 5.2.9, we can express any modules $M_{\bar{j}}^{\bar{i}}$ as $M_{\bar{l}}^{\bar{1}}(k)$ for some $k \in \mathbb{Z}$ and $1 \leq l \leq \frac{n+1}{2}$. By Proposition 5.2.7, we can reduce k mod $P(l)$ until k lies in the desired range. $\square$

Since A is a Nakayama algebra, Lemma 3.2.6 guarantees that every morphism between indecomposable objects X and Y can be represented by a map $X \to \Omega^m Y(m)$ in A-grstab for some unique m. To compute $\mathrm{Hom}_{A\text{-dgstab}}(M_{\bar{l}}^{\bar{1}}, M_{\bar{r}}^{\bar{1}}(k))$, we must determine which $M_{\bar{j}}^{\bar{i}}(m)$ admit maps from $M_{\bar{l}}^{\bar{1}}$ in the graded stable category, then express such $M_{\bar{j}}^{\bar{i}}(m)$ as $M_{\bar{r}}^{\bar{1}}(k)$ using the formulas in Proposition 5.2.9.

Proposition 5.2.11. *Let* $1 \leq l, r, j \leq n$. *Then*

$$dim\,\mathrm{Hom}_{A\text{-grstab}}(M_{\bar{l}}^{\bar{1}}, M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(k)) = \begin{cases} 1 & \textit{if } max(1, r+l-n) \leq j \leq min(r, l) \\ & \quad \textit{and } k = -d\delta_{\bar{j} \neq \bar{r}} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. By Proposition 5.2.2, the dimension of the Hom space is nonzero if and only if $\langle 1, j \rangle \subset \langle j+1-r, l \rangle$ and $k = -d\delta_{\bar{1} < \bar{j}+\bar{1}-\bar{r}} = -d\delta_{\bar{j} \neq \bar{r}}$. Thus it is enough to show that the inequalities in the statement are equivalent to the arc containment condition. The arc containment condition holds if and only if $e^{\frac{2\pi\sqrt{-1}}{n}j} \in \langle 1, l \rangle$ and $e^{\frac{2\pi\sqrt{-1}}{n}(j+1-r)} \in \langle l+1, 1 \rangle$. This is equivalent to the chain of inequalities

$$l - n + 1 \leq j + 1 - r \leq 1 \leq j \leq l$$

The first two inequalities are equivalent to $l + r - n \leq j \leq r$. Combining these with the last two inequalities, we see the system is equivalent to $max(1, l+r-n) \leq j \leq min(r, l)$. $\square$

We are now ready to give a complete description of the morphisms $M_{\bar{l}}^{\bar{1}} \to M_{\bar{r}}^{\bar{1}}(k)$ in A-dgstab, for $1 \leq l, r \leq \frac{n+1}{2}$. It is helpful to organize the morphisms into two families. The "short" morphisms are those of the form $f_{l,r,j} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(m)$; note that the codomain is represented by a module of length $r \leq \frac{n+1}{2}$. The "long" morphisms are of the form $g_{l,r,j} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{j}}^{\bar{j}+\bar{r}}(m)$; here the codomain has length $n + 1 - r \geq \frac{n+1}{2}$.

Theorem 5.2.12 (Structure of A-dgstab)**.** *Let* $1 \leq l, r \leq \lfloor \frac{n+1}{2} \rfloor$ *and* $0 \leq k < P(r)$. *Then* $\mathrm{Hom}_{A\text{-dgstab}}(M_{\bar{l}}^{\bar{1}}, M_{\bar{r}}^{\bar{1}}(k))$ *has dimension at most* 1 *and is spanned by:*

$$f_{l,r,j} : M_{\bar{l}}^{\bar{1}} \xrightarrow{\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{1}-\bar{r}}} M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(-d\delta_{\bar{j}\neq\bar{r}}) \cong M_{\bar{r}}^{\bar{1}}(k)$$

$$\text{if } k \equiv (d+2)(r-j) \mod P(r) \text{ for some } 1 \leq j \leq min(r,l)$$

$$g_{l,r,j} : M_{\bar{l}}^{\bar{1}} \xrightarrow{\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{r}}} M_{\bar{j}}^{\bar{j}+\bar{r}}(-d\delta_{\bar{j}\neq\bar{n}+\bar{1}-\bar{r}}) \cong M_{\bar{r}}^{\bar{1}}(k)$$

$$\text{if } k \equiv (d+2)(n+1-j)-1 \mod P(r) \text{ for some } max(1, 1+l-r) \leq j \leq l$$

$$0 \qquad \text{otherwise}$$

$f_{l,r,j}$ *is an isomorphism if and only if* $l = r = j$, *in which case it is the identity map.* $g_{l,r,j}$ *is an isomorphism if and only if* $l = r = j = \frac{n+1}{2}$, *in which case* $g_{l,l,l} = \psi^{1/2}$. *In particular, the indecomposable modules listed in Proposition 5.2.10 are pairwise non-isomorphic.*

For $r = \frac{n+1}{2}$ *and any value of* l, *the morphisms* $f_{l,r,j}$ *and* $g_{l,r,j}$ *are defined for the same values of* j *and represented by the same morphism in* A-grstab. *More specifically, for each such* j,

$$g_{l,r,j} = \psi^{1/2} \circ f_{l,r,j} \tag{5.10}$$

For $l = \frac{n+1}{2}$ *and any value of* r, *the morphisms* $f_{l,r,j+r-\frac{n+1}{2}}$ *and* $g_{l,r,j}$ *are defined for the same values of* j, *and their unique nonzero components differ only by an application of* Ω^n *and a grading shift. More precisely, for each such* j,

$$\psi \circ f_{l,r,j+r-\frac{n+1}{2}} = g_{l,r,j}\left(\frac{(n+1)d}{2} + n\right) \circ \psi^{1/2} \tag{5.11}$$

Apart from the above identities, all the $f_{l,r,j}$ *and* $g_{l,r,j}$ *are distinct, in the sense that their unique nonzero components cannot be transformed into one another by applying powers of* Ω *and grading shifts.*

The indecomposable morphisms in A-dgstab *are, up to shifts,* $f_{l,l-1,l-1}$ *for* $1 < l \leq \lfloor \frac{n+1}{2} \rfloor$, $f_{l,l+1,l}$ *for* $1 \leq l < \lfloor \frac{n+1}{2} \rfloor$, $g_{\frac{n}{2},\frac{n}{2},\frac{n}{2}}$ *for* n *even, and* $g_{\frac{n+1}{2},\frac{n+1}{2}-1,\frac{n+1}{2}}$ *for* n *odd.*

Composition of morphisms in A-dgstab *is determined by composing the unique nonzero morphisms in* A-grstab. *In particular, we have the formulas:*

$$f_{r,c,q}((d+2)(r-j)) \circ f_{l,r,j} = \begin{cases} f_{l,c,q+j-r} & \textit{if } 1 \le q+j-r \le \\ & \quad min(c,l) \\ 0 & \textit{otherwise} \end{cases} \tag{5.12}$$

$$g_{r,c,q}((d+2)(r-j)) \circ f_{l,r,j} = \begin{cases} g_{l,c,q+j-r} & \textit{if } max(1, 1+l-c) \le \\ & \quad q+j-r \le l \\ 0 & \textit{otherwise} \end{cases} \tag{5.13}$$

$$f_{r,c,q}((d+2)(n+1-j)-1) \circ g_{l,r,j} = \begin{cases} g_{l,c,q+j-c} & \textit{if } max(1, 1+l-c) \le \\ & \quad q+j-c \le l \\ 0 & \textit{otherwise} \end{cases} \tag{5.14}$$

$$g_{r,c,q}((d+2)(n+1-j)-1) \circ g_{l,r,j} = \begin{cases} \psi \circ f_{l,c,q+j+c-(n+1)} & \textit{if } l < q+j \le \\ & \quad n+1 \le \\ & \quad q+j+c-1 \\ 0 & \textit{otherwise} \end{cases} \tag{5.15}$$

Proof. By Lemma 3.2.6, $\mathrm{Hom}_{A\text{-grstab}}(M_{\bar{l}}^{\bar{1}}, \Omega^m M_{\bar{r}}^{\bar{1}}(k+m))$ is nonzero for at most one m. Thus by Proposition 5.2.11, $\mathrm{Hom}_{A\text{-dgstab}}(M_{\bar{l}}^{\bar{1}}, M_{\bar{r}}^{\bar{1}}(k))$ is nonzero if and only if $\Omega^m M_{\bar{r}}^{\bar{1}}(k+m)) \cong M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{x}}(-d\delta_{\bar{j}\neq\bar{x}})$ in A-grstab, for some $1 \le x \le n$, $max(1, x+l-n) \le j \le min(x,l)$, and $m \in \mathbb{Z}$. The only possibles values for the length of $\Omega^m M_{\bar{r}}^{\bar{1}}(k+m))$ are r and $n+1-r$, so we need only consider the cases $x = r$ and $x = n+1-r$, and verify that $M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{x}}(-d\delta_{\bar{j}\neq\bar{x}}) \cong M_{\bar{r}}^{\bar{1}}(k)$ for the desired value of k.

If $x = r$, then the condition on j simplifies to $1 \le j \le min(r,l)$. For each such j we obtain the morphism $f_{l,r,j}$ whose nonzero component is $\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{1}-\bar{r}} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(-d\delta_{\bar{j}\neq\bar{r}})$. If $j \neq r$, by substituting $i = n+1-(r-j)$ into Equation (5.7) of Proposition 5.2.9 we obtain

$$\begin{aligned}M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(-d) &\cong M_{\bar{r}}^{\bar{1}}(-(d+2)(n-(r-j))-d)\\ &= M_{\bar{r}}^{\bar{1}}((d+2)(r-j)-(n+1)d-2n)\\ &\cong M_{\bar{r}}^{\bar{1}}((d+2)(r-j))\end{aligned}$$

If $j = r$, then the desired identity is immediate.

If $x = n+1-r$, then the condition on j simplifies to $max(1, 1+l-r) \leq j \leq l$. For each such j we obtain the morphism $g_{l,r,j}$ whose nonzero component is $\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{r}} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{j}}^{\bar{j}+\bar{r}}(-d\delta_{\bar{j}\neq\bar{n}+\bar{1}-\bar{r}})$. If $j \neq n+1-r$, applying Equation (5.9) of Proposition 5.2.9 with the substitutions $i \mapsto j+r$ and $l \mapsto r$, we obtain

$$\begin{aligned}M_{\bar{j}}^{\bar{j}+\bar{r}}(-d) &\cong M_{\bar{r}}^{\bar{1}}(-(d+2)(n+j)+1-d)\\ &\cong M_{\bar{r}}^{\bar{1}}(-(d+2)(n+j)+1-d+2(n+1)d+4n)\\ &= M_{\bar{r}}^{\bar{1}}((d+2)(n+1-j)-1)\end{aligned}$$

If $j = n+1-r$, the restrictions on j, r, and l imply that $j = l = r = \frac{n+1}{2}$; the desired identity then follows from direct substitution and the fact that $P(r) = \frac{(n+1)d}{2} + n$. This establishes the descriptions of the Hom spaces.

We now determine the isomorphisms of A-dgstab. Note that a morphism is an isomorphism in $\mathcal{C}(A)$ if and only if its unique nonzero component is an isomorphism in A-grstab. If $f_{l,r,j}$ is an isomorphism, $\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{1}-\bar{r}} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(-d\delta_{\bar{j}\neq\bar{r}})$ must also be an isomorphism. From Proposition 5.2.1, we have that $j = r = l$. Conversely, direct substitution shows that $f_{l,l,l}$ is the identity map.

Similarly, if $g_{l,r,j}$ is an isomorphism, then its unique nonzero component $\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{r}} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{j}}^{\bar{j}+\bar{r}}(-d\delta_{\bar{j}\neq\bar{n}+\bar{1}-\bar{r}})$ is also an isomorphism. Proposition 5.2.1 then forces $j = r = l = \frac{n+1}{2}$. Conversely, if $j = r = l = \frac{n+1}{2}$, direct substitution shows that the nonzero component of $g_{l,l,l} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{l}}^{\bar{1}}(\frac{(n+1)d}{2} + n)$ is the identity map. Thus $g_{l,l,l} = \psi^{1/2}$.

This completes the description of the isomorphisms of A-dgstab. It follows immediately that $M_{\bar{l}}^{\bar{1}} \cong M_{\bar{r}}^{\bar{1}}(k)$ if and only if $l = r$ and $k \equiv 0 \mod P(l)$. Thus the indecomposable objects of Proposition 5.2.10 are pairwise non-isomorphic.

If $r = \frac{n+1}{2}$, the identity $g_{l,r,j} = \psi^{1/2} \circ f_{l,r,j}$ follows immediately from direct substitution, as does the fact that f and g are defined for the same values of j. (We note that both morphisms have the same domain, codomain, and nonzero component.) If $l = \frac{n+1}{2}$, the corresponding statement follows from similar computations.

To show distinctness, suppose that we can transform the nonzero component $f_{l,r,j}$ into that of $g_{l',r',j'}$ by applying grading shifts and Ω^m for some even integer m. We shall show that $l = l'$, $j = j'$, and $r = r' = \frac{n+1}{2}$. In order for the domains to be equal, we must have that $l = l'$ and m is a multiple of $2n$. By using periodicity of Ω and changing the grading shift, we can also assume without loss of generality that $m = 0$. For the codomains to be equal, we must have in particular that $M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}} \cong M_{\overline{j'}}^{\overline{j'}+\overline{r'}}$ in A-grstab, hence $j = j'$ and $r = r' = \frac{n+1}{2}$.

Similarly, suppose we can transform the nonzero component of $f_{l,r,j}$ into that of $f_{l',r',j'}$ by applying a grading shift and Ω^m for some even m. By observing the domain and codomain we once again see that m can be taken to be zero, and we immediately obtain $l = l'$, $r = r'$, and $j = j'$. The same argument also applies to $g_{l,r,j}$ and $g_{l',r',j'}$.

Next, suppose that we can transform the nonzero component $f_{l,r,j}$ into that of $g_{l',r',j'}$ by applying grading shifts and Ω^m for some odd integer m. We shall show that $l = l' = \frac{n+1}{2}$, $r = r'$, and $j = j' + r' - \frac{n+1}{2}$. In order for the domains to be equal, their lengths, $n + 1 - l$ and l', respectively, must be equal. This implies that n is odd and $l = l' = \frac{n+1}{2}$. Furthermore, m must be an odd multiple of n; without loss of generality, we may assume that $m = n$ by periodicity. Observing the codomains, we must have that $M_{\bar{j}-\bar{r}+\overline{\frac{n+1}{2}}}^{\bar{j}+\overline{\frac{n+1}{2}}} \cong M_{\overline{j'}}^{\overline{j'}+\overline{r'}}$ in A-grstab. Comparing the bottom indices, we have that $\bar{j} = \overline{j'} + \bar{r} - \overline{\frac{n+1}{2}}$. Comparing the top indices and using the previous equation, we have that $\bar{r} = \overline{r'}$, hence $r = r'$. From the restrictions on the ranges of the indices, we deduce that $j = j' + r' - \frac{n+1}{2}$, as desired.

If we can transform the nonzero component of $f_{l,r,j}$ into that of $f_{l',r',j'}$ by applying a grading shift and Ω^m for some odd m, by considering the lengths of the domain and codomain, we must have that $l = l' = r = r' = \frac{n+1}{2}$. We can again assume that $m = n$. Observing the codomains, we must have that $M_{\bar{j}-\bar{r}+\overline{\frac{n+1}{2}}}^{\bar{j}+\overline{\frac{n+1}{2}}} \cong M_{\overline{j'}}^{\overline{j'}+\bar{1}-\overline{r'}}$. It follows that $j = j'$. A parallel argument applies to $g_{l,r,j}$ and $g_{l',r',j'}$. Thus all the morphisms are distinct, except for the listed identities.

A morphism $M_{\bar{l}}^{\bar{1}} \to M_{\bar{r}}^{\bar{1}}(k)$ is indecomposable if and only if its nonzero component is indecomposable in A-grstab. Thus, up to a degree shift, the indecomposable morphisms of A-dgstab are those $f_{l,r,j}$ and $g_{l,r,j}$ whose nonzero component is a canonical injection or surjection. Since the domain must be $M_{\bar{l}}^{\bar{1}}$, the only possible values for these nonzero components are $p_{\bar{l}}^{\bar{1}}$ for $1 < l \leq \frac{n+1}{2}$ and $\iota_{\bar{l}}^{\bar{1}}$ for $1 \leq l \leq \frac{n+1}{2}$. From the definitions, $p_{\bar{l}}^{\bar{1}}$ is the nonzero component of $f_{l,l-1,l-1}$ for all $1 < l \leq \frac{n+1}{2}$ and $\iota_{\bar{l}}^{\bar{1}}$ is the nonzero component of $f_{l,l+1,l}$ for all $1 \leq l < \lfloor \frac{n+1}{2} \rfloor$. When n is even, $\lfloor \frac{n+1}{2} \rfloor = \frac{n}{2}$, and $\iota_{\frac{n}{2}}^{\bar{1}}$ is the nonzero component of $g_{\frac{n}{2},\frac{n}{2},\frac{n}{2}}$. When n is odd, $\lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2}$, and $\iota_{\frac{n+1}{2}}^{\bar{1}}$ is the nonzero component of $g_{\frac{n+1}{2},\frac{n+1}{2}-1,\frac{n+1}{2}}$. These are precisely the indecomposable morphisms listed in the statement.

To verify the composition formulas, we translate them into statements about $A\text{-grstab}$ and use Proposition 5.2.4.

We start with Equation (5.12). A tedious but straightforward computation using Proposition 5.2.5 shows that the only possible nonzero component of

$$f_{r,c,q}((d+2)(r-j)) \circ f_{l,r,j} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{c}}^{\bar{1}}((d+2)(c-(q+j-r)))$$

is $\alpha_{\bar{j},\bar{q}+\bar{j}-\bar{r}}^{\bar{j}+\bar{1}-\bar{r},(\bar{q}+\bar{j}-\bar{r})+\bar{1}-\bar{c}}(-d\delta_{j\neq r}) \circ \alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{1}-\bar{r}}$. Then by Proposition 5.2.4, this composition is nonzero if and only if $\langle 1, q+j-r \rangle \subset \langle (q+j-r)+1-c, l \rangle$, in which case it is equal to $\alpha_{\bar{l},\bar{q}+\bar{j}-\bar{r}}^{\bar{1},(\bar{q}+\bar{j}-\bar{r})+\bar{1}-\bar{c}}$. Since the codomain of this morphism has length c, it follows that the resulting morphism, if nonzero, is equal to $f_{l,c,q+j-r}$. It remains to verify that the arc containment condition is equivalent to the desired inequality. If $q+j-r < 1$, then the restrictions on q, j, r, and l imply that $l-n < q+j-r < 1$, hence both the desired inequality and the arc containment condition are false. If $q+j-r \geq 1$, the restrictions on q, j, l, r, and r imply that $1 \leq q+j-r \leq n$ and $c+l-n \leq 1$. We can then apply Proposition 5.2.11 and conclude the arc containment condition is equivalent to the inequality $1 \leq q+j-r \leq min(c,l)$. Thus Equation (5.12) holds.

Proceeding to Equation (5.13), the only possible nonzero component of

$$g_{r,c,q}((d+2)(r-j)) \circ f_{l,r,j} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{c}}^{\bar{1}}((d+2)(n+1-(q+j-r))-1)$$

is $\alpha_{\bar{j},\bar{q}+\bar{j}-\bar{r}}^{\bar{j}+\bar{1}-\bar{r},(\bar{q}+\bar{j}-\bar{r})+\bar{c}}(-d\delta_{j\neq r}) \circ \alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{1}-\bar{r}}$. This composition is nonzero if and only if $\langle 1, q+j-r \rangle \subset \langle (q+j-r)+c, l \rangle$, in which case it is equal to $\alpha_{\bar{l},\bar{q}+\bar{j}-\bar{r}}^{\bar{1},(\bar{q}+\bar{j}-\bar{r})+\bar{c}}$. The codomain of this component has length $n+1-c$, hence the composition, if nonzero, is equal to $g_{l,c,q+j-r}$. The same argument as above shows that the arc containment condition is equivalent to the inequality $max(1, 1+l-c) \leq q+j-r \leq l$. Thus Equation (5.13) holds.

For Equation (5.14), the only possible nonzero component of

$$f_{r,c,q}((d+2)(n+1-j)-1)\circ g_{l,r,j} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{c}}^{\bar{1}}((d+2)(n+1-(q+j-c))-1)$$

is $\alpha_{\bar{j},\bar{q}+\bar{j}-\bar{c}}^{\bar{j}+\bar{r},\bar{q}+\bar{j}}(-d\delta_{\bar{j}\neq\bar{n}+\bar{1}-\bar{r}})\circ\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{r}}$. This composition is nonzero if and only if $\langle 1, q+j-c\rangle \subset \langle q+j, l\rangle$, in which case it is equal to $\alpha_{\bar{l},\bar{q}+\bar{j}-\bar{c}}^{\bar{1},\bar{q}+\bar{j}}$. The codomain of this component has length $n+1-c$, hence the composition, if nonzero, is equal to $g_{l,c,q+j-c}$. The same argument as above shows that the arc containment condition is equivalent to the inequality $max(1, 1+l-c) \leq q+j-c \leq l$. Thus Equation (5.14) holds.

For Equation (5.15), the only possible nonzero component of

$$g_{r,c,q}((d+2)(n+1-j)-1)\circ g_{l,r,j} : M_{\bar{l}}^{\bar{1}} \to M_{\bar{c}}^{\bar{1}}((d+2)(2(n+1)-q-j)-2)$$

is $\alpha_{\bar{j},\bar{q}+\bar{j}+\bar{c}-\bar{1}}^{\bar{j}+\bar{r},\bar{q}+\bar{j}}(-d\delta_{\bar{j}\neq\bar{n}+\bar{1}-\bar{r}})\circ\alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{r}}$. This composition is nonzero if and only if $\langle 1, q+j+c-1\rangle \subset \langle q+j, l\rangle$, in which case it is equal to $\alpha_{\bar{l},\bar{q}+\bar{j}+\bar{c}-\bar{1}}^{\bar{1},\bar{q}+\bar{j}}$. It is clear that the desired inequality implies the arc containment condition; we now show the converse. Due to the restrictions on q, j, and c, we have that $2 \leq q+j \leq q+j+c-1 \leq l+n$ and $q+j \leq n+1$. Thus if $q+j \leq l$, we have that $1 < q+j \leq l$, and the arc containment condition fails. Thus we must have that $l < q+j \leq n+1$. If $l < q+j+c-1 \leq n+1$, then the arc containment condition fails, hence we must also have $n+1 \leq q+j+c-1$. The desired inequality follows immediately. Thus the arc containment condition and the desired equality are equivalent. If both hold, the codomain of the nonzero component has length c. We also have that $1 \leq q+j+c-(n+1) \leq min(c,l)$. Thus the composition is equal to $\psi \circ f_{l,c,q+j+c-(n+1)}$. To explain the presence of ψ in this formula, note that the grading shift of the codomain of the composition is

$$(d+2)(2(n+1)-q-j)-2 = [(d+2)(c-(q+j+c-(n+1)))] + [(n+1)d+2n]$$

The factor of ψ accounts for the second bracketed term. □

Computing the cones of the morphisms in Theorem 5.2.12 is straightforward, since the computations can be done in $A\text{-grstab}$.

Proposition 5.2.13. *For all values of l, r, and j for which it is defined, $f_{l,r,j}$ can be completed to the exact triangle:*

$$\begin{array}{c} M_{\bar{l}}^{\bar{1}} \\ \Big\downarrow f_{l,r,j} \\ M_{\bar{r}}^{\bar{1}}((d+2)(r-j)) \\ \Big\downarrow h_1 \\ \delta_{\bar{j}\neq\bar{r}} M_{\bar{r}-\bar{j}}^{\bar{1}}((d+2)(r-j)) \oplus \delta_{\bar{j}\neq\bar{l}} M_{\bar{l}-\bar{j}}^{\bar{1}}((d+2)(n+1-j)-1) \\ \Big\downarrow h_2 \\ M_{\bar{l}}^{\bar{1}}(1) \end{array} \tag{5.16}$$

where $h_1 = \begin{pmatrix} \delta_{\bar{j}\neq\bar{r}} f_{r,r-j,r-j}((d+2)(r-j)) \\ \delta_{\bar{j}\neq\bar{l}} g_{r,l-j,r}((d+2)(r-j)) \end{pmatrix}$ *and*
$h_2 = (\psi^{-1}\delta_{\bar{j}\neq\bar{r}} g_{r-j,l,r-j}((d+2)(r-j)), \psi^{-1}\delta_{\bar{j}\neq\bar{l}} f_{l-j,l,l-j}((d+2)(n+1-j)-1))$

For all values of l, r, and j for which it is defined and such that $j+r \geq \frac{n+1}{2}$, $g_{l,r,j}$ can be completed to the exact triangle:

$$\begin{array}{c} M_{\bar{l}}^{\bar{1}} \\ \Big\downarrow g_{l,r,j} \\ M_{\bar{r}}^{\bar{1}}((d+2)(n+1-j)-1) \\ \Big\downarrow h_3 \\ \delta_{\bar{j}+\bar{r}\neq\bar{1}} M_{\bar{n}+\bar{1}-(\bar{j}+\bar{r})}^{\bar{1}}((d+2)(2(n+1)-(j+r))-2) \\ \oplus \\ \delta_{\bar{j}\neq\bar{l}} M_{\bar{l}-\bar{j}}^{\bar{1}}((d+2)(n+1-j)-1) \\ \Big\downarrow h_4 \\ M_{\bar{l}}^{\bar{1}}(1) \end{array} \tag{5.17}$$

where $h_3 = \begin{pmatrix} \delta_{\bar{j}+\bar{r}\neq\bar{1}} g_{r,n+1-(j+r),r}((d+2)(n+1-j)-1) \\ \delta_{\bar{j}\neq\bar{l}} f_{r,l-j,l-j}((d+2)(n+1-j)-1) \end{pmatrix}$ *and*
$h_4 = (\psi^{-2}\delta_{\bar{j}+\bar{r}\neq\bar{1}} g_{n+1-(j+r),l,n+1-(j+r)}((d+2)(2(n+1)-(j+r))-2),$
$\psi^{-1}\delta_{\bar{j}\neq\bar{l}} f_{l-j,l,l-j}((d+2)(n+1-j)-1))$

For all values of l, r, and j for which it is defined and such that $j + r \leq \frac{n+1}{2}$, $g_{l,r,j}$ can be completed to the exact triangle:

$$\begin{array}{c} M_{\bar{l}}^{\bar{1}} \\ \Big\downarrow g_{l,r,j} \\ M_{\bar{r}}^{\bar{1}}((d+2)(n+1-j)-1) \\ \Big\downarrow h_5 \\ \delta_{\bar{j}+\bar{r}\neq\bar{1}} M_{\bar{j}+\bar{r}}^{\bar{1}}((d+2)(n+1)-1) \oplus \delta_{\bar{j}\neq\bar{l}} M_{\bar{l}-\bar{j}}^{\bar{1}}((d+2)(n+1-j)-1) \\ \Big\downarrow h_6 \\ M_{\bar{l}}^{\bar{1}}(1) \end{array} \tag{5.18}$$

where $h_5 = \begin{pmatrix} \delta_{\bar{j}+\bar{r}\neq\bar{1}} f_{r,j+r,r}((d+2)(n+1-j)-1) \\ \delta_{\bar{j}\neq\bar{l}} f_{r,l-j,l-j}((d+2)(n+1-j)-1) \end{pmatrix}$ *and*
$h_6 = (\psi^{-1}\delta_{\bar{j}+\bar{r}\neq\bar{1}} f_{j+r,l,l}((d+2)(n+1)-1), \psi^{-1}\delta_{\bar{j}\neq\bar{l}} f_{l-j,l,l-j}((d+2)(n+1-j)-1))$

Triangles (5.17) and (5.18) are equivalent when $j + r = \frac{n+1}{2}$.

Proof. Note that triangles in A-grstab induce triangles in A-dgstab. By Proposition 5.2.6, the nonzero component of $f_{l,r,j}$ fits into an exact triangle

$$\begin{array}{c} M_{\bar{l}}^{\bar{1}} \\ \Big\downarrow \alpha_{\bar{l},\bar{j}}^{\bar{1},\bar{j}+\bar{1}-\bar{r}} \\ M_{\bar{j}}^{\bar{j}+\bar{1}-\bar{r}}(-d\delta_{\bar{j}\neq\bar{r}}) \\ \Big\downarrow h_1' \\ \delta_{\bar{j}\neq\bar{r}} M_{\bar{n}}^{\bar{j}+\bar{1}-\bar{r}}(-d) \oplus \delta_{\bar{j}\neq\bar{l}} M_{\bar{j}}^{\bar{l}}(-d) \\ \Big\downarrow h_2' \\ M_{\bar{n}}^{\bar{l}}(-d) \end{array}$$

where $h_1' = \begin{pmatrix} \delta_{\bar{j}\neq\bar{r}} \alpha_{\bar{j},\bar{n}}^{\bar{j}+\bar{1}-\bar{r},\bar{j}+\bar{1}-\bar{r}}(-d\delta_{\bar{j}\neq\bar{r}}) \\ \delta_{\bar{j}\neq\bar{l}} \alpha_{\bar{j},\bar{j}}^{\bar{j}+\bar{1}-\bar{r},\bar{l}}(-d\delta_{\bar{j}\neq\bar{r}}) \end{pmatrix}$, and
$h_2' = \begin{pmatrix} \delta_{\bar{j}\neq\bar{r}} \alpha_{\bar{n},\bar{n}}^{\bar{j}+\bar{1}-\bar{r},\bar{l}}(-d) & \delta_{\bar{j}\neq\bar{l}} \alpha_{\bar{j},\bar{n}}^{\bar{l},\bar{l}}(-d) \end{pmatrix}$.

Using the identities in Proposition 5.2.5, it follows immediately that this triangle is isomorphic to the triangle (5.16).

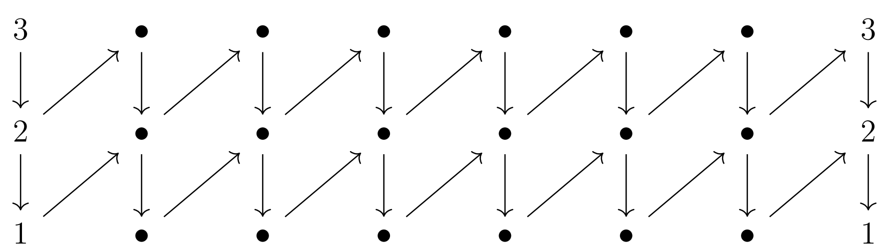
The other two cases are proved analogously. □

We have described the indecomposable objects and morphisms of A-dgstab in Theorem 5.2.12; by two easy counting arguments (n even and n odd), there are $\frac{nP}{2}$ indecomposable objects in A-dgstab. A description of the Auslander-Reiten quiver A-dgstab follows easily.

Definition 5.2.14. For positive integers N, M, let $Q_{N,M}$ denote the quiver with vertex set $V = \mathbb{Z}/N\mathbb{Z} \times \{1, \cdots, M\}$ and arrows of the form $(x, y) \to (x + 1, y + 1)$ for $1 \leq y < M$ and $(x, y) \to (x, y - 1)$ for $1 < y \leq M$.

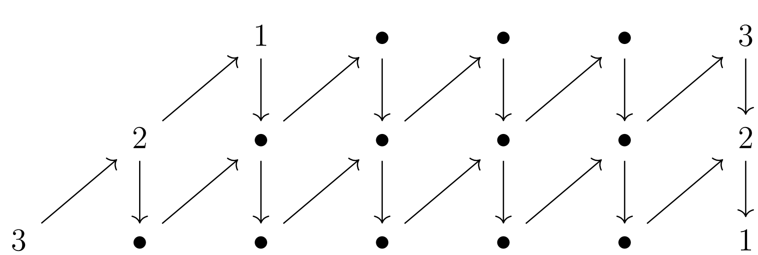
Corollary 5.2.15. *If d is even, the Auslander-Reiten quiver of A-dgstab is the cylinder $Q_{\frac{P}{2},n}$. If d is odd, the Auslander-Reiten quiver is the Möbius strip $Q_{P,n}/\tau$, where τ is the involution sending (x, y) to $(x - y + \frac{P+n+1}{2}, n + 1 - y)$. The indecomposable morphisms representing the arrows can be chosen such that all squares commute (up to powers of ψ), and a composition of arrows out of vertex (x, y) is zero if and only if the composition contains at least y vertical arrows or at least $n + 1 - y$ diagonal arrows.*

Example 5.2.16.



The Auslander-Reiten quiver of A-dgstab for $n = 3, d = 2$.

Corresponding numbered vertices are identified.



The Auslander-Reiten quiver of A-dgstab for $n = 3, d = 1$.

Corresponding numbered vertices are identified.

Proof. An easy counting argument shows that there are exactly $\frac{Pn}{2}$ indecomposable objects in A-dgstab (regardless of parity of n), and that there are $\frac{Pn}{2}$ vertices in the corresponding candidate quivers. We first make explicit the bijection between vertices and indecomposable objects.

For d even, $x \in \mathbb{Z}/\frac{P}{2}\mathbb{Z}$, and $1 \leq y \leq n$, associate the vertex (x, y) with the object $M_{y}^{\bar{1}}((d+2)x)$; the isomorphism class of this object is independent of the choice of representative of x since $d+2$ is even. For $\lfloor \frac{n+1}{2} \rfloor < y \leq n$, note that $M_{y}^{\bar{1}}((d+2)x) \cong M_{n+1-y}^{\bar{1}}((d+2)(x+n+1-y)-1)$, with $1 \leq n+1-y < \lfloor \frac{n+1}{2} \rfloor$. Since $(n+1)(d+2) \equiv 2$ in $\mathbb{Z}/P\mathbb{Z}$ and both d and P are even, we have that $(d+2) = (2)$ is an index 2 subgroup of $\mathbb{Z}/P\mathbb{Z}$. Thus, for fixed $1 < y < \frac{n+1}{2}$, the above mapping sends $\{(x, y) \mid 0 \leq x < \frac{P}{2}\}$ onto $\{M_{y}^{\bar{1}}(2i) \mid 0 \leq i < \frac{P}{2}\}$. For fixed $1 < n+1-y < \frac{n+1}{2}$, the mapping sends $\{(x, y) \mid 0 \leq x < \frac{P}{2}\}$ onto $\{M_{y}^{\bar{1}}(2i+1) \mid 0 \leq i < \frac{P}{2}\}$. If n is odd, then $d+2$ generates $\mathbb{Z}/\frac{P}{2}\mathbb{Z}$ and so $\{(x, \frac{n+1}{2}) \mid 0 \leq x < \frac{P}{2}\}$ maps onto $\{M_{\frac{n+1}{2}}^{\bar{1}}(i) \mid 0 \leq i < \frac{P}{2}\}$. Thus the above mapping establishes a bijection between the indecomposable objects of A-dgstab and the vertices of $Q_{\frac{P}{2},n}$.

When d is odd, associate the vertex (x, y) of $Q_{P,n}$ to the object $M_{y}^{\bar{1}}((d+2)x)$. Note that $d+2$ is odd and $\langle d+2 \rangle \supset \langle 2 \rangle$ in $\mathbb{Z}/P\mathbb{Z}$; therefore $d+2$ generates $\mathbb{Z}/P\mathbb{Z}$. Thus, for each $1 \leq y \leq n$, the above mapping establishes a bijection between $\{(x, y) \mid 0 \leq x < P\}$ and $\{M_{y}^{\bar{1}}(i) \mid 0 \leq i < P\}$. In particular, we have a two-to-one map from the vertices of $Q_{P,n}$ to the indecomposable objects of A-dgstab. Furthermore, a straightforward computation using Equation (5.8) of Lemma 5.2.9 shows that (x, y) and $\tau(x, y)$ have the same image. Thus the map defines a bijection between the vertices of $Q_{P,n}/\tau$ are the indecomposable objects of A-dgstab.

It remains to show that the edges of the candidate quivers correspond to the indecomposable morphisms. For d of arbitrary parity, every length one indecomposable object of A-dgstab has one indecomposable morphism in and out; all other objects have two indecomposable morphisms in and out. Thus the degrees of the vertices in the Auslander-Reiten quiver agree with the degrees of the vertices of the candidate quivers. Note that both candidate quivers are symmetric about the central line $y = \frac{n+1}{2}$. For a vertex (x, y), let $\widetilde{y} = min\{y, n+1-y\}$. For vertices (x, y)

which do not lie on the central line, it is straightforward to verify that shifts of the indecomposable morphism $f_{\widetilde{y},\widetilde{y}-1,\widetilde{y}-1}$ correspond to the arrows exiting (x,y) and pointing away from the central line. Similarly, shifts of the morphism $f_{\widetilde{y},\widetilde{y}+1,\widetilde{y}}$ correspond to the arrows pointing towards the central line, and shifts of $g_{\frac{n}{2},\frac{n}{2},\frac{n}{2}}$ correspond to arrows crossing the central line. For vertices of the form $(x,\frac{n+1}{2})$, when d is even associate $f_{\frac{n+1}{2},\frac{n+1}{2}-1,\frac{n+1}{2}-1}$ to the vertical arrow and $g_{\frac{n+1}{2},\frac{n+1}{2}-1,\frac{n+1}{2}}$ to the diagonal arrow. When d is odd, it is necessary to make this assignment in the quiver $Q_{P,n}$, since τ swaps vertical and diagonal arrows. The resulting assignment is compatible with τ by Equation (5.11). This establishes the isomorphism of directed graphs between the Auslander-Reiten quiver and the candidate quivers.

With the above assignment of morphisms to the arrows of the Auslander-Reiten quiver, it follows directly from the composition formulas in Theorem 5.2.12 that all squares commute, up to powers of ψ.

To determine when a composition is zero, it suffices to consider compositions starting at the vertex $(0,y)$, $1 \leq y \leq n$. Suppose d is even, and choose a composition of morphisms starting at $(0,y)$ and consisting of s vertical arrows and t diagonal arrows. We may choose to represent the vertex $(0,y)$ by the module $M^{\overline{1}}_{\overline{y}}$; in this case a vertical arrow has as its unique nonzero component the canonical morphism $M^{\overline{1}}_{\overline{y}} \to M^{\overline{1}}_{\overline{y-1}}$ in A-grstab, and a diagonal arrow has unique nonzero component $M^{\overline{1}}_{\overline{y}} \to M^{\overline{n}}_{\overline{y}}(-d)$. Thus a morphism consisting of $0 \leq s < y$ vertical arrows and $0 \leq t < n+1-y$ diagonal arrows has the canonical morphism $M^{\overline{1}}_{\overline{y}} \to M^{\overline{1-t}}_{\overline{y-s}}(-d\delta_{t>0})$ as its unique nonzero component. When $s = y$, the composition becomes zero in A-grmod, and when $t = n+1-y$, it becomes zero in A-grstab; in either case, the morphism vanishes in A-dgstab. For $s > y$ or $t > n+1-y$, the morphism factors through a morphism with $s = y$ or $t = n+1-y$, hence is zero.

When d is odd, the automorphism τ sends vertical arrows to diagonal ones and vice-versa, hence the notion of "vertical" and "diagonal" arrows are only locally defined. However, once a local choice of definition is made at the vertex $(0,y)$, the argument in the preceding paragraph applies without change. □

CHAPTER 6

A Combinatorial Model for Perverse Equivalences

6.1 Construction of the Model

In this chapter, we shall study the action of perverse equivalences on the category $A\text{-dgstab}$, where A is the Brauer tree algebra described in Chapter 5. By construction, the socle of A is concentrated in degree $-d$, for a fixed $d \geq 0$. By Corollary 3.1.19, $A\text{-dgstab}$ is a $(-d-1)$-Calabi-Yau category, hence we may study the theory of perverse equivalences as defined in Chapter 4. In this chapter, we shall study the action of the group Ξ on the sets $\mathcal{E}$ (resp., $\widehat{\mathcal{E}}$) of bases (resp., orthogonal tuples) in $A\text{-dgstab}$ (see Definitions 4.1.1 and 4.1.6). To accomplish this, we shall construct a combinatorial model for $A\text{-dgstab}$, in which objects are represented by beads of varying lengths on a circular wire. In this model, elements of $\widehat{\mathcal{E}}$ will correspond to certain maximal non-overlapping arrangements of beads. Our main result is Corollary 6.2.21, in which we show that $\widehat{\mathcal{E}} = \mathcal{E}$.

For the action of Ξ to be well-defined, we must show that $A\text{-dgstab}$ admits $(d+1)$-orthogonal maximal extensions. Rather than do so directly, we will instead define the action of Ξ on the set of bead arrangements, and show that the induced action of $\widehat{\mathcal{E}}$ agrees with our original definition.

All notation will be as in Chapter 5, with one exception. For simplicity, we shall write the indecomposable A-modules as M^i_j, rather than $M^{\bar{i}}_{\bar{j}}$; since we will not use the order on $\mathbb{Z}/n\mathbb{Z}$ in this chapter, the simpler notation will cause no confusion.

6.1.1 Beads on a Wire

We now begin construction of our combinatorial model. We shall associate indecomposable objects of A-dgstab to beads of varying lengths on a circular wire.

We consider the set $\mathbb{Z}/P\mathbb{Z}$, viewed as a collection of evenly-spaced points on a circular wire of length $P = (n+1)(d+2) - 2$. For integers i, j, we shall denote by $[[i, j]]$ the image of the closed interval $[i, j] \cap \mathbb{Z}$ in $\mathbb{Z}/P\mathbb{Z}$.

Definition 6.1.1. Let i, l be integers, with $1 \leq l \leq n$. Define $B_l(i)$ to be the interval $[[i-l(d+2), i]]$. We refer to $B_l(i)$ as a **bead of type l in position i**. We refer to the interval $[[i-l(d+2)+1, i-1]]$ as the **well** of $B_l(i)$. The intervals $[[i-l(d+2), i-l(d+2)+1]]$ and $[[i-1, i]]$ are the **ridges** of $B_l(i)$, and the points $i - l(d+2)$ and i are the **endpoints** of $B_l(i)$.

Remark. We shall often identify the integer i in the above definition with its image in $\mathbb{Z}/P\mathbb{Z}$. This shall cause no confusion, as the definition depends only on the image of i. We shall also view (j) as a shift operator on the set of beads, so that $B_l(i)(j) = B_l(i+j)$.

The total length of a bead of type l is $l(d+2)$. Geometrically, we view the beads as possessing an interior well, a depression of length $l(d+2)-2$ into which other (smaller) beads may be placed. This well is surrounded by two ridges of length one, over which other beads cannot be placed. We give an illustration in Figure 6.1. Since no beads can fit in the well of a bead of type 1, we will depict these beads without ridges or a well; this is a purely aesthetic choice.

Definition 6.1.2. Let $1 \leq r \leq l \leq n$, and let $i, j \in \mathbb{Z}$. We say the beads $B_l(i)$ and $B_r(j)$ **do not overlap** if one of the following holds:
1) $[[j-r(d+2), j]] \subset [[i-l(d+2)+1, i-1]]$; that is, $B_r(j)$ is contained within the well of $B_l(i)$.
2) $[[j-r(d+2), j]] \subset [[i, i-l(d+2)+P]]$; that is, $B_r(j)$ lies outside of $B_l(i)$ (though the beads' endpoints may touch).

Remark. Note that condition 2) is symmetric with respect to $B_l(i)$ and $B_r(j)$, and condition 1) can only occur if $r < l$.

In Figure 6.1, none of the beads overlap with one another. Bead $B_1(5)$ lies inside the well of $B_2(7)$. Bead $B_1(0)$ lies outside of $B_2(7)$.

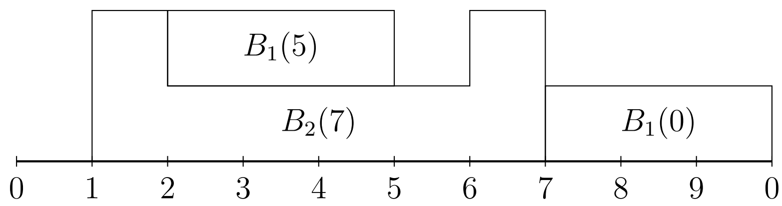


Figure 6.1: Three non-overlapping beads; $n = 3$, $d = 1$

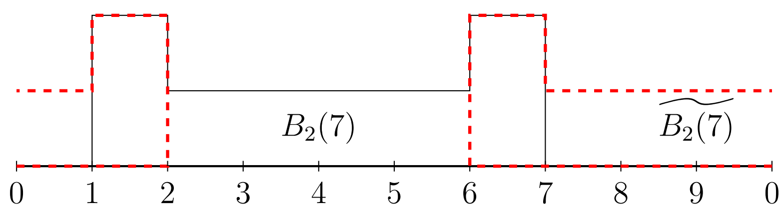


Figure 6.2: A bead and its partner; $n = 3$, $d = 1$

Since the length of the wire is $P = (n+1)(d+2) - 2$, there is not quite enough room on the wire for two beads of types l and $n+1-l$. However, two such beads can be placed on the wire in such a way that they intersect precisely along their ridges. This motivates the following definition and proposition:

Definition 6.1.3. Let $B_l(i)$ be a bead. Define the **partner** of $B_l(i)$ to be the bead $\widetilde{B_l(i)} := B_{n+1-l}(i - l(d+2) + 1)$.

It is easy to verify that $B_l(i)$ and $\widetilde{B_l(i)}$ intersect precisely along the ridges of both beads, and that the function taking a bead to its partner is an involution. See Figure 6.2.

If $B_l(i)$ and $B_r(j)$ are two beads, note that $B_r(j)$ lies in the well of $B_l(i)$ if and only if $B_r(j)$ lies outside $\widetilde{B_l(i)}$. In this case, then $\widetilde{B_r(j)}$ and $B_l(i)$ will necessarily overlap, and $\widetilde{B_l(i)}$ will lie in the well of $\widetilde{B_r(j)}$.

We now relate beads to the indecomposable objects of $A\text{-dgstab}$.

Definition 6.1.4. Given a bead $B_l(i)$, define the **associated object of** $B_l(i)$ to be the object $M_l^1(i) \in A\text{-dgstab}$. Let Φ be denote the function mapping a bead to (the isomorphism class of) its associated object.

Remark. Note that $B_l(i+P) = B_l(i)$ for any i, so that Φ is well-defined when viewed as a function of l and i.

Proposition 6.1.5. Φ *defines a two-to-one map from the set of beads onto the set of (isomorphism classes of) indecomposable objects of* A-dgstab. *Each bead has the same image as its partner.*

Proof. By Proposition 5.2.10, every object of A-dgstab is isomorphic to $M_l^1(i) = \Phi(B_l(i))$ for some $1 \leq l \leq \lfloor \frac{n+1}{2} \rfloor$, $0 \leq i < P$; thus Φ is surjective. A straightforward counting argument shows that there are nP beads, and Corollary 5.2.15 shows that A-dgstab has $\frac{nP}{2}$ indecomposable objects, up to isomorphism.

A straightforward calculation using Proposition 5.2.9, Equation (5.8) shows that $\Phi(\widetilde{M_l^1(i)}) \cong \Omega\Phi(M_l^1(i))(1) \cong \Phi(M_l^1(i))$. Since every indecomposable object has at least two preimages under Φ, it follows by the pigeonhole principal that Φ is two-to-one. $\square$

Remark. Note that if $l < \frac{n+1}{2}$, each $M_l^1(i)$ is the associated object of a unique bead of type l and a unique bead of type $n+1-l$. When $l = \frac{n+1}{2}$ (note this requires n to be odd), both preimages of $M_l^1(i)$ are beads of type $\frac{n+1}{2}$. Taking the partner of a bead corresponds to applying $\Omega(1)$ to its associated object.

Proposition 6.1.6. *Let* $1 \leq r \leq l \leq \lfloor \frac{n+1}{2} \rfloor$. *Then the beads* $B_l(i)$ *and* $B_r(j)$ *do not overlap if and only if* $\Phi(B_l(i))$ *and* $\Phi(B_r(j))$ *are distinct and independent.*

To prove the above Proposition, it will be helpful to reformulate Definition 6.1.2.

Lemma 6.1.7. *Let* $1 \leq r \leq l \leq n$. *Two beads* $B_l(i)$ *and* $B_r(j)$ *do not overlap if and only if* $[[j - r(d+2)+1, j]] \cap \{i - l(d+2)+1, i\} = \emptyset$ *(as subsets of* $\mathbb{Z}/P\mathbb{Z}$*).*

Proof. Note that changing i and j by a multiple of P does not affect the statement of the lemma.

Suppose $B_l(i)$ and $B_r(j)$ do not overlap. Suppose condition 1) of Definition 6.1.2 holds, that is, $B_r(j)$ lies in the well of $B_l(i)$. Then i and j may be chosen so that

$$i - l(d+2)+1 < j - r(d+2)+1 < j < i < i + l(d+2)+1+P$$

Thus neither i nor $i - l(d+2)+1$ lies in $[[j - r(d+2)+1, j]]$, hence the intersection is empty.

If condition 2) holds (i.e., $B_r(j)$ lies outside of $B_l(i)$), then i and j may be chosen so that

$$i < j - r(d+2)+1 < j \leq i - l(d+2)+P < i - l(d+2)+1+P < i+P$$

Once again, the intersection is empty. This proves the forward direction.

For the reverse direction, suppose the intersection is empty. Then, of the six potential cyclic orderings of $\{j - r(d + 2) + 1, j, i - l(d + 2) + 1, i\}$ inside $\mathbb{Z}/P\mathbb{Z}$, the only two consistent possibilities are:

$$i - l(d+2) + 1 < j - r(d+2) + 1 < j < i < i - l(d+2) + 1 + P$$

or

$$j - r(d+2) + 1 < j < i - l(d+2) + 1 < i < j - r(d+2) + 1 + P$$

The first case implies that $B_r(j)$ lies in the well of $B_l(i)$. The second implies that $B_r(j)$ lies outside of $B_l(i)$. In both cases, $B_l(i)$ and $B_r(j)$ do not overlap. □

We now prove Proposition 6.1.6.

Proof. $\Phi(B_l(i))$ and $\Phi(B_r(j))$ are distinct and independent in $A\text{-dgstab}$ if and only if, for all $0 \le m \le d$,

$$\mathrm{Hom}(M_l^1(i), M_r^1(j-m)) = \mathrm{Hom}(M_r^1(j), M_l^1(i-m)) = 0$$

Since $A\text{-dgstab}$ is $-(d+1)$-Calabi-Yau, we can rewrite the above condition as

$$\mathrm{Hom}(M_l^1, M_r^1(j-i-m)) = \mathrm{Hom}(M_l^1, M_r^1(j-i+m-d-1)) = 0$$

for all $0 \le m \le d$. This can be further simplified to

$$\mathrm{Hom}(M_l^1, M_r^1(j-i-m)) = 0$$

for all $0 \le m \le d+1$.

By Theorem 5.2.12 this holds if and only if, for all $0 \le m \le d+1$,

$$\begin{aligned} j-i-m \quad & \notin \{(d+2)(r-k) \mid 1 \le k \le r\} \cup \\ & \quad \{(d+2)(n+1-k) - 1 \mid 1 + l - r \le k \le l\} \\ & \Updownarrow \\ j-i \quad & \notin [[0, (d+2)(r-1) + d + 1]] \cup \\ & \quad [[(d+2)(n+1-l) - 1, (d+2)(n+r-l) + d]] \end{aligned}$$

$$\Updownarrow$$

$$j-i \quad \notin [[0,(d+2)r-1]] \cup [[-(d+2)l+1,(d+2)(r-l)]]$$

$$\Updownarrow$$

$$j-i, j-i+l(d+2)-1 \quad \notin [[0,(d+2)r-1]]$$

$$\Updownarrow$$

$$i, i-l(d+2)+1 \quad \notin [[j-r(d+2)+1, j]]$$

where the above sets are viewed as subsets of $\mathbb{Z}/P\mathbb{Z}$ if $r < \frac{n+1}{2}$ and as subsets of $\mathbb{Z}/(\frac{P}{2})\mathbb{Z}$ if $r = \frac{n+1}{2}$.

If $r < \frac{n+1}{2}$, the last condition is precisely that which appears in Lemma 6.1.7, and we are done.

If $r = \frac{n+1}{2}$, then necessarily $l = \frac{n+1}{2}$. In this case, $B_r(j)$ and $B_l(i)$ always overlap, since the length of both the well and the outside of $B_l(i)$ is $\frac{n+1}{2}(d+2)-2$, which less than the length of $B_r(j)$. Thus it suffices to show that there are no pairs of distinct, independent objects of length $\frac{n+1}{2}$. Since $-r(d+2)+1 = -\frac{P}{2}$, the interval $[[j-r(d+2)+1, j]]$ is equal to $\mathbb{Z}/(\frac{P}{2})\mathbb{Z}$, hence there can be no pairs of distinct, independent objects of length $\frac{n+1}{2}$. □

Proposition 6.1.6 can be partially extended to beads of unrestricted length.

Proposition 6.1.8. *Let* $1 \leq r, l \leq n$. *If* $B_l(i)$ *and* $B_l(j)$ *do not overlap, then* $\Phi(B_l(i))$ *and* $\Phi(B_r(j))$ *are distinct and independent.*

Proof. Without loss of generality, we may assume $r \leq l$. By Proposition 6.1.5, a bead and its partner have the same associated object, so we can replace any bead with its partner without affecting the conclusion of the Proposition. We shall reduce to the case where $r \leq l \leq \lfloor \frac{n+1}{2} \rfloor$ and apply Proposition 6.1.6.

We may assume that $l > \lfloor \frac{n+1}{2} \rfloor$. Since $r \leq l$, note that either $B_r(j)$ is contained in the well of $B_l(i)$ or lies outside. In either case $\widetilde{B_l(i)}$ and $B_r(j)$ do not overlap, and $\widetilde{B_l(i)}$ has type $n+1-l < \lfloor \frac{n+1}{2} \rfloor$. Thus if $r \leq \lfloor \frac{n+1}{2} \rfloor$, we have completed the reduction. Otherwise, $r > \lfloor \frac{n+1}{2} \rfloor$, hence $B_r(j)$ is longer than $\widetilde{B_l(i)}$. Repeating the same argument as above, $\widetilde{B_r(j)}$ and $\widetilde{B_l(i)}$ do not overlap, and both beads have type less than $\lfloor \frac{n+1}{2} \rfloor$, hence their associated objects are distinct and independent. $\square$

Remark. The converse to Proposition 6.1.8 is false. Given a pair of non-overlapping beads, by replacing beads with their partners we can obtain four distinct pairs of beads with the same image under Φ. Of these four pairs, exactly one will overlap.

6.1.2 Bead Arrangements

We now translate the notion of an orthogonal tuple into the language of beads.

Definition 6.1.9. A **colored bead arrangement** is an n-tuple whose entries are mutually non-overlapping beads. An (uncolored) **bead arrangement** is a set of n mutually non-overlapping beads. A **free bead arrangement** is a bead arrangement, taken up to a rotation of the wire. We let CBA (resp. BA, FBA) denote the set of all colored bead arrangements (resp. bead arrangements, free bead arrangements).

Definition 6.1.10. Let B be a bead in a (colored, uncolored, or free) bead arrangment A. Define the **height** $H(B)$ of B in A to be the number of beads B' in A such that $B \subset B'$. If B is in a colored bead arrangement, we define the **color** of B to be the integer $i \in \{1, \cdots, n\}$ such that B is the ith entry of the tuple.

Figure 6.1 shows an uncolored free bead arrangement. The height of $B_1(5)$ is 2, and the height of the other two beads is 1.

The following statement is an immediate corollary of Proposition 6.1.6.

Proposition 6.1.11. Φ *induces surjections*

$$CBA \twoheadrightarrow \widehat{\mathcal{E}}$$

$$FBA \twoheadrightarrow \widehat{\mathcal{E}}/\sim$$

Proof. Choose $(X_i)_i \in \widehat{\mathcal{E}}$. By Proposition 6.1.5, for each i there exists a bead B^i of type l, $1 \leq l \leq \lfloor \frac{n+1}{2} \rfloor$, such that $\Phi(B^i) = X_i$. By Proposition 6.1.6, the B^i are mutually non-overlapping, since the X_i are mutually independent. Thus $(B^i)_i$ is a colored bead arrangement and $\Phi(B^i)_i = (X_i)_i$. If C^i is a bead obtained from B^i by a rotation of the wire, then $\Phi(C^i)$ is a shift of $\Phi(B^i)$. Thus the second function is well-defined, and it is clear from the above argument that it is surjective. □

We would like to further restrict the class of bead arrangements so that the surjective maps defined above become bijections. The above proof suggests that we restrict our attention to arrangements in which only beads of type $1 \leq l \leq \lfloor \frac{n+1}{2} \rfloor$ are permitted. For n even, this is the correct solution, as Φ induces a bijection between the beads of type $1 \leq l < \frac{n+1}{2}$ and isomorphism classes of indecomposable objects of length l. However, if n is odd, and $l = \frac{n+1}{2}$, then Φ is two-to-one on the set of beads of type l. To resolve this issue, we introduce a new object to our combinatorial model.

Definition 6.1.12. A **circlet** is a set $C(i) = \{B_{\frac{n+1}{2}}(i), \widetilde{B_{\frac{n+1}{2}}(i)}\}$ consisting of a bead of type $\frac{n+1}{2}$ and its partner.

Geometrically, we interpret $C(i)$ as both beads, glued along their overlapping boundaries. (See Figure 6.3.) Thus, $C(i)$ divides the ring into two wells of length $\frac{n+1}{2}(d+2) - 2$, separated by the two ridges $[[i-1, i]]$ and $[[i - \frac{n+1}{2}(d+2), i - \frac{n+1}{2}(d+2)+1]]$. Note that we can apply Φ to $C(i)$, since both elements of $C(i)$ have the same image under Φ. Furthermore, Φ establishes a bijection between the set of circlets and the set of isomorphism classes of indecomposable objects of length $\frac{n+1}{2}$.

Definition 6.1.13. A bead $B_r(j)$ and a circlet $C(i)$ **do not overlap** if $B_r(j)$ does not overlap with either bead in $C(i)$.

Note that if $r \geq \frac{n+1}{2}$, $B_r(j)$ will always overlap with at least one of the beads in any circlet $C(i)$, and if $r < \frac{n+1}{2}$, then if $B_r(j)$ does not overlap with one of the beads in $C(i)$, it will not overlap with either.

Definition 6.1.14. A (colored, uncolored, or free) bead arrangement is called **reduced** if all beads in the arrangement have type $1 \leq l \leq \lfloor \frac{n+1}{2} \rfloor$, and any bead $B_{\frac{n+1}{2}}(i)$ is replaced by the corresponding circlet $C(i)$. We denote by $RCBA$ (resp. RBA, $RFBA$) the set of reduced colored bead arrangements (resp. reduced bead arrangements, reduced free bead arrangements).

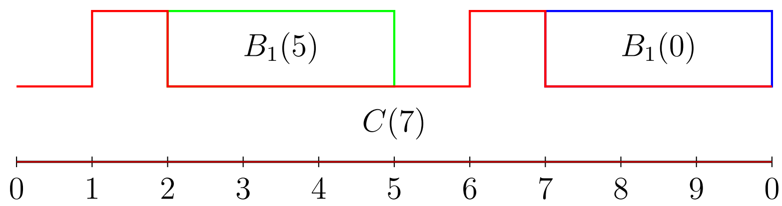


Figure 6.3: The reduced colored bead arrangement $(C(7), B_1(5), B_1(0))$; $n = 3$, $d = 1$

Note that since any two beads of type $\frac{n+1}{2}$ overlap, there can be at most one circlet in any type of reduced bead arrangement. A reduced colored bead arrangement with a circlet is shown in Figure 6.3.

Proposition 6.1.15. Φ *induces bijections*

$$RCBA \leftrightarrow \widehat{\mathcal{E}}$$

$$RFBA \leftrightarrow \widehat{\mathcal{E}}/\sim$$

Proof. Surjectivity of both maps follows immediately from the proof of Proposition 6.1.11. Since Φ induces a bijection between beads of type $1 \leq l < \frac{n+1}{2}$ and isomorphism classes of indecomposable objects of length l, as well as between circlets and isomorphism classes of indecomposable objects of length $\frac{n+1}{2}$, it follows that both maps are injective. □

6.1.3 Counting $\widehat{\mathcal{E}}$

In this section, we determine the cardinality of $\widehat{\mathcal{E}}$. By Proposition 6.1.15, it suffices to count the number of reduced colored bead arrangements. It is easy to reduce the problem to counting the reduced free bead arrangements.

Proposition 6.1.16. $|RCBA| = n! \cdot P \cdot |RFBA|$

Proof. The canonical map $RCBA \to RBA$ sending an n-tuple to a set is clearly surjective and $n!$-to-one. The canonical map $RBA \to RFBA$ sending a bead arrangement to its equivalence class under rotation is clearly surjective and P-to-one. □

Given a bead arrangement, one can draw a plane tree by associating a vertex to each bead and drawing an edge to each bead sitting directly on top of it.

Definition 6.1.17. Given an uncolored bead arrangement A, define the rooted plane tree $(P(A), r_A)$ as follows: The vertices of $P(A)$ are the beads of A, plus a new vertex, r_A, associated to the wire. The vertex r_A is defined to be the root of the tree. Draw edges between r_A and each bead of height 1. Draw an edge between $B_l(i)$ and $B_r(j)$ if and only if the difference in height between the two beads is 1 and one bead contains the other. Associating the vertex $B_l(i)$ with $i \in \mathbb{Z}/P\mathbb{Z}$, the natural cyclic ordering on $\mathbb{Z}/P\mathbb{Z}$ induces a cyclic ordering of all non-root vertices. This induces a cyclic ordering of the edges around each vertex of height $\neq 1$. For a vertex $B_l(i)$ of height one, the edge incident to r_A is ordered as though it had value i.

We refer to $(P(A), r_A)$ as the **tree associated to** A. The isomorphism class of $(P(A), r_A)$ (as a rooted plane tree) is called the **class** of A.

We give two examples of bead arrangements and their associated trees in Figure 6.4. The root of each tree is the bottom-most vertex. Note that the two trees in Figure 6.4 are isomorphic as trees, but not as plane trees, hence the two arrangements do not have the same class. Intuitively, two bead arrangements will have the same class if and only if they differ by a rigid motion, where beads are allowed to move along, but not through, each other. In particular, it is straightforward to check that $P(A)$ is invariant under rotation of the wire, hence the map $A \mapsto (P(A), r_A)$ is defined for free bead arrangements.

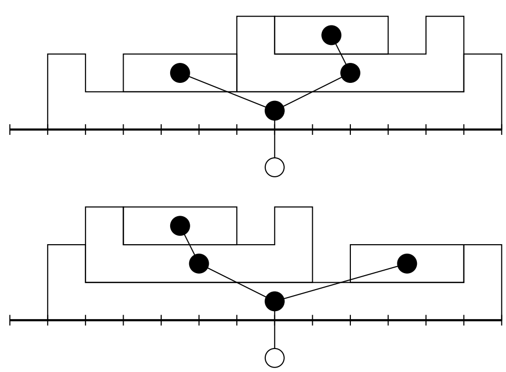


Figure 6.4: Two bead arrangements and their associated plane trees; $n = 4$, $d = 1$

The following properties of the map $A \mapsto (P(A), r_A)$ are straightforward to verify. We refer to Section 2.10 for terminology.

Proposition 6.1.18. *Let A be a (free or uncolored) bead arrangement, and let $B_l(i)$ be a bead in A. Then:*

1) The depth of the vertex $B_l(i)$ in $P(A)$ is equal to the height of $B_l(i)$ in A.

2) The weight of $B_l(i)$ in $(P(A), r_A)$ is l.

3) If $B_l(i)$ has height one, let A' denote the bead arrangement obtained by replacing $B_l(i)$ with its partner. Then there is a canonical isomorphism of plane trees $P(A) \xrightarrow{\sim} P(A')$ induced by identifying the trees' common non-root vertices. Identifying the two trees via this isomorphism, $(P(A'), r_{A'})$ is a rebalancing of $(P(A), r_A)$ in the direction of $B_l(i)$.

Proof. Given two beads B and B' in A, B is an ancestor of B' in $(P(A), r_A)$ if and only if $B' \subsetneq B$. The first statement follows.

For the second statement, we first prove that $W(B_l(i)) \leq l$, by induction on l. If $l = 1$, then the statement is immediate, since $B_1(i)$ contains no bead but itself and is therefore a leaf. Suppose the result holds for beads of type $k < l$. Suppose the children of $B_l(i)$ are $\{B_{x_j}(y_j)\}_{j=1}^{s}$. Since a bead of type k has length $k(d+2)$, and since the beads $B_{x_j}(y_j)$ are mutually non-overlapping beads inside the well of $B_l(i)$, we have that $\sum_j x_j \leq l - 1$. Thus,

$$W(B_l(i)) = 1 + \sum_j W(B_{x_j}(y_j)) \leq 1 + \sum_j x_j \leq l \tag{6.1}$$

and the inductive step is complete.

Note that (6.1) remains true if $B_l(i)$ is replaced by r_A, and l by $n+1$, since the ring has the same length as the well of a (hypothetical) bead of type $n+1$. Furthermore, if equality holds in (6.1), then $W(B_{x_j}) = x_j$ for all j. Thus, equality at a vertex v implies equality at all descendants of v. Equality holds at r_A by construction, hence at all vertices. This proves the second statement.

The isomorphism in the third statement identifies r_A with $\widetilde{B_l(i)}$ and $B_l(i)$ with $r_{A'}$; the remaining vertices are shared by the two trees. The rest of the statement follows directly from definitions. □

Motivated by the previous proposition, if A is a (free or uncolored) bead arrangement, and A' is a bead arrangement obtained from A by replacing a height one bead $B_l(i)$ by its partner, we say that A' is a **rebalancing of** A **in the direction of** $B_l(i)$. Thus, the third statement of the previous proposition can be restated as saying that the rebalancing operation commutes with taking the associated tree of a bead arrangement. By repeatedly rebalancing a tree in the direction of vertices of weight greater than $\lfloor \frac{n+1}{2} \rfloor$, one eventually obtains a balanced tree. Performing the corresponding operation on bead arrangements, we see that reduced bead arrangements are precisely the analogues of balanced trees. More precisely:

Corollary 6.1.19. *Let A be a free bead arrangement. Then A defines a reduced free bead arrangement $\overline{A}$ if and only if $(P(A), r_A)$ is balanced. In this case, $\overline{A}$ contains a circlet $C = \{B, \widetilde{B}\}$ if and only if $P(A)$ has two balancing roots. In this case, let A' be the other free bead arrangement defining $\overline{A}$, with $B \in A$ and $\widetilde{B} \in A'$. After identifying $P(A)$ and $P(A')$ via the canonical isomorphism, B and $\widetilde{B}$ are the two balancing roots of the tree.*

Proof. A defines a reduced free bead arrangement if and only if every bead is of type at most $\frac{n+1}{2}$. By Proposition 6.1.18, this holds if and only if $P(A)$ is balanced. If A defines a reduced free bead arrangement, $\overline{A}$ contains a circlet if and only if A contains a height one bead of type $\frac{n+1}{2}$, if and only if $P(A)$ contains a depth one vertex of weight $\frac{n+1}{2}$. By Proposition 2.10.2, this happens if and only if $P(A)$ has two balancing roots. In this case, A' is a rebalancing of A in the direction of B and A is a rebalancing of A' in the direction of $\widetilde{B}$. Identifying the two trees, B and B' are the vertices incident to the edge of weight $\frac{n+1}{2}$ and thus are the two balancing roots. □

By Corollary 6.1.19, the map $\overline{A} \mapsto P(\overline{A})$ is well-defined for reduced free bead arrangements, if we interpret $P(\overline{A})$ as an isomorphism class of plane trees. We define $P(\overline{A})$ to be the **class** of $\overline{A}$, as for free bead arrangements.

We are now ready to count the reduced free bead arrangements. The key result is the following lemma:

Lemma 6.1.20. *Let (T, r) be a rooted plane tree with n edges. Then the number of free bead arrangements A of class (T, r) is*

$$N_{T,r} = \binom{d + |c(r)| - 1}{d} \prod_{v \in V_T - \{r\}} \binom{d + |c(v)|}{d} \tag{6.2}$$

Proof. We describe a choice procedure for constructing an arbitrary free bead arrangement of class (T, r). Starting with the root and working upwards, we associate beads to the children of each vertex of the tree.

First, we specify the placement of the height one beads. Write $c(r) = \{v_1 < v_2 < \cdots < v_k < v_1\}$ as a cyclically ordered set. (Note $k = |c(r)|$.) We shall place beads of type $W(v_1), W(v_2), \cdots, W(v_k)$ sequentially on the wire, so that their right edges form an increasing sequence in $\mathbb{Z}/P\mathbb{Z}$. Since a free bead arrangement is defined up to a rotation of the wire, we assume without loss of generality that the right edge of the bead corresponding to v_1 is at position 0. Since the ith bead has type $W(v_i)$, and $\sum_{i=1}^{k} W(v_i) = n$, the k beads take up a total of $n(d+2)$ space on the wire, which has total length $n(d+2)+d$. Thus, to uniquely specify the position of the height one beads (up to rotation of the wire), we need to distribute the d empty spaces amongst the k gaps between beads. There are $\binom{d+k-1}{d} = \binom{d+|c(r)|-1}{d}$ such choices.

Next, given any vertex v corresponding to a bead B of type l already placed by our choice procedure, we must place the beads which lie in the well of B and have height $H(B)+1$. It is clear that the number of such placements depends only on the type of B and is independent of its horizontal placement. Thus we may identify the well of B with the interval $[0, l(d+2)-2]$. Since $v \neq r$, $C(v) = \{w_1 < \cdots < w_k\}$ is totally ordered (i.e., the parent of v lies between w_k and w_1 in the cyclic ordering). As before, we place beads of type $W(w_1), \cdots, W(w_k)$ sequentially, from left to right. Once again, there is a total of d empty space in the well of B, and uniquely specifying the position of the beads in the well is equivalent to distributing d empty spaces amongst the $k+1$ gaps found between the k beads and the two walls of B. There are $\binom{d+k}{d} = \binom{d+|c(v)|}{d}$ such choices.

It is clear that this choice procedure uniquely specifies all free bead arrangements of class (T, r). Since the choices made at each vertex are independent of previous choices, this establishes the formula. □

Corollary 6.1.21. *Let (T, r) be a rooted plane tree. The quantity $N_{T,r}$ is independent of r.*

Proof. It suffices to show that $N_{T,r} = N_{T,r'}$ for adjacent vertices r and r'. Let X_r denote the set of free bead arrangements of class (T, r), and similarly for r'. For each arrangement $A \in X_r$, there is a unique bead B of height 1 corresponding to the vertex r'; let A' denote the rebalancing of A

in the direction of B. By Proposition 6.1.18, the map $A \mapsto A'$ defines a function $f : X_r \to X_{r'}$. By rebalancing A' in the direction of $\widetilde{B}$, we recover A; thus $A' \mapsto A$ is a well-defined inverse of f. Thus $N_{T,r} = |X_r| = |X_{r'}| = N_{T,r'}$. □

Remark. In view of Corollary 6.1.21, we shall drop the r from the subscript and simply refer to the quantity N_T. It is not difficult to prove Corollary 6.1.21 directly, without reference to bead arrangements.

We are now ready to state the main result of this section.

Theorem 6.1.22.

$$|\widehat{\mathcal{E}}/\sim| = |RFBA| = \sum_{T \in \mathcal{PT}_n} N_T \tag{6.3}$$

$$|\widehat{\mathcal{E}}| = |RCBA| = (n!) \cdot P \cdot \sum_{T \in \mathcal{PT}_n} N_T \tag{6.4}$$

Proof. The left-hand equalities were proved in Proposition 6.1.15. By Proposition 6.1.16, Equation (6.4) follows immediately from Equation (6.3). Thus it suffices to prove that $|RFBA| = \sum_{T \in \mathcal{PT}_n} N_T$.

If T has a unique balancing root r, then by Corollary 6.1.19 the free bead arrangements of class (T, r) are in bijection with the free reduced bead arrangements of class T, hence by Lemma 6.1.20 there are N_T reduced free bead arrangements of class T.

If T has two balancing roots r and r', then by Corollary 6.1.19 the free bead arrangements of class (T, r) are in bijection with the free bead arrangements of class (T, r') and also with the reduced free bead arrangements of class T. Therefore there are again N_T reduced free bead arrangements of class T. □

6.2 The Action of Perverse Equivalences

6.2.1 Bead Collisions and Mutations

So far, our combinatorial model has allowed us to determine the size of $\widehat{\mathcal{E}}$. Our next task is to describe the action of Ξ on $\widehat{\mathcal{E}}$ (see Definition 4.1.6). We have not yet shown A-dgstab admits

$(d+1)$-orthogonal maximal extensions; instead, we shall define an action on CBA which descends to $\widehat{\mathcal{E}}$, and show that this is the desired action.

The intuition behind this action is as follows: given a colored bead arrangement A, a set $S \subsetneq [n]$ acts on A by sliding all beads of color $i \notin S$ one unit in the counterclockwise direction. The resulting tuple of beads need not be a colored bead arrangement, as some of the beads may now overlap. When this happens, we apply various "mutations" to the moved beads by extending or shrinking them depending on the nature of the collision.

Definition 6.2.1. Let BT be the set of all n-tuples of beads (with overlaps allowed). Define an action of Ξ on BT as follows. Let $\sigma \in \mathfrak{S}_n$, $S \subsetneq [n]$, and $T = (B^1, \cdots, B^n) \in BT$. Then:

$$\sigma \cdot T := (B^{\sigma(1)}, \cdots, B^{\sigma(n)})$$
$$S \cdot T := (B^1(-\delta_{1 \notin S}), \cdots, B^n(-\delta_{n \notin S}))$$
$$S^{-1} \cdot T := (B^1(\delta_{1 \notin S}), \cdots, B^n(\delta_{n \notin S}))$$

We refer to this action as the **naive action** on bead tuples.

Clearly, $CBA \subset BT$ is not stable under the naive action. However, we are able to classify the ways in which collisions can occur between beads.

Definition 6.2.2. Let B^1 and B^2 be beads.

We say that B^1 **has a left collision of Type I** with B^2 if $B^1 \cap B^2$ is precisely the left ridge of B^1 and the right ridge of B^2.

We say that B^1 **has a left collision of Type II** with B^2 if the left ridge of B^1 coincides with the left ridge of B^2, and $B^1(1)$ lies in the well of B^2.

We say that B^1 **has a left collision of Type III** with B^2 if the right ridge of B^1 coincides with the right ridge of B^2, and B^2 lies in the well of $B^1(1)$.

We define the mirror notion of right collisions by reversing all instances of "left" and "right" in the above definitions, and replacing all positive shifts with negative shifts. We shall work almost exclusively with left collisions throughout this paper. When we do not specify left or right, we shall always mean a left collision.

We shall say B^1 is Type I, II, or III **left adjacent** to B^2 if $B(-1)$ has a Type I, II, or III left collision with B_2. We define right adjacency analogously.

Remark. Let $A = (B^1, \cdots B^n)$ be a colored bead arrangement and $S \subsetneq [n]$. It is clear that for any B^i and B^j in A, $B^i(-1)$ and $B^j(-1)$ do not overlap; thus two beads in $S \cdot A$ can overlap only if one is moved by S and the other remains stationary. It is easy to verify that a moved bead $B^i(-1)$ and a stationary bead B^j in $S \cdot A$ overlap if and only if $B^i(-1)$ has a left collision with B^j. Similarly, a moved bead $B^i(1)$ and a stationary bead B^j in $S^{-1} \cdot A$ overlap if and only if $B^i(1)$ has a right collision with B^j.

For Ξ to define an action on CBA, we must develop a means of correcting collisions. For each type of collision, we introduce a corresponding mutation that resolves the collision.

Definition 6.2.3. Let $B^1 = B_{l_1}(i_1)$, $B^2 = B_{l_2}(i_2)$ be beads.

If B^1 has a Type I left collision with B^2, define the **Type I left mutation of** B^1 to be the bead $M_I(B^1) = B_{l_1+l_2}(i_1)$.

If B^1 has a Type II left collision with B^2, define the **Type II left mutation of** B^1 to be the bead $M_{II}(B^1) = B_{l_2-l_1}(i_2 - 1)$.

If B^1 has a Type III left collision with B^2, define the **Type III left mutation of** B^1 to be the bead $M_{III}(B^1) = B_{l_1-l_2}(i_1 - l_2(d+2))$.

Right mutations are defined analogously. As with collisions, we shall simply write "mutations" when referring to left mutations.

If $A = (B^1, \cdots, B^n) \in BT$, and for some i there is a unique j such that B^i has a Type r collision with B^j, define $M_r^i(A)$ to be the tuple obtained from A by replacing B^i with $M_r(B^i)$.

Each of the three types of mutation corresponds to an intuitive physical transformation of the bead.

In a mutation of type I, we extend the length of B^1, keeping the right endpoint fixed, until B^2 lies in its well. B^2 will be Type II left adjacent to $M_I(B^1)$. This process is illustrated in Figure 6.5 below.

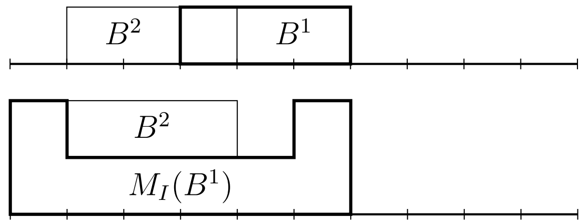


Figure 6.5: Top: A Type I left collision of B^1 with B^2.

Bottom: A Type I mutation of B^1.

$n = 3, d = 1$

In a mutation of type II, we "reflect" B^1 inside the well of B^2. B^2 will be Type III left adjacent to $M_{II}(B^1)$, and the left ridge of $M_{II}(B^1)$ will coincide with the right ridge of B^1. $M_{II}(B^1)$ could be described as the "partner of B^1, relative to B^2". This is illustrated in Figure 6.6.

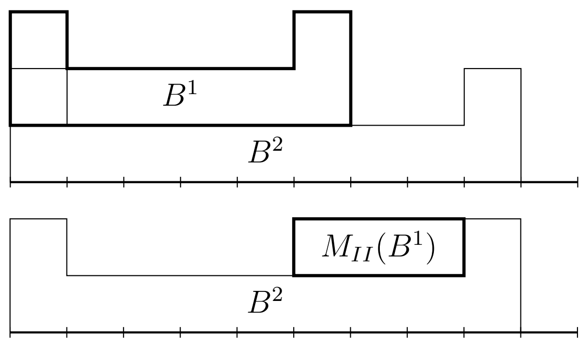


Figure 6.6: Top: A Type II left collision of B^1 with B^2.

Bottom: A Type II mutation of B^1.

$n = 3, d = 1$

In a mutation of type III, we shorten B^1, keeping the left endpoint fixed, until B^2 no longer overlaps with it. B^2 will be Type I left adjacent to $M_{III}(B^1)$. This is illustrated in Figure 6.7.

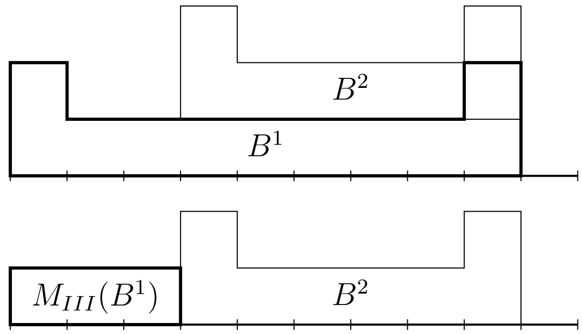


Figure 6.7: Top: A Type III left collision of B^1 with B^2.

Bottom: A Type III mutation of B^1.

$n = 3, d = 1$

We are now ready to define an action of Ξ on CBA.

Definition 6.2.4. Let $\sigma \in \mathfrak{S}_n$, $S \subsetneq \{1, \cdots n\}$, and $A = (B^1, \cdots, B^n) \in CBA$. Define an action of Ξ on CBA according to the following procedure. We denote this action by $\circ$ to distinguish it from the naive action $\cdot$ defined in 6.2.1.

1) Let $S \circ A := S \cdot A$ if $S \cdot A \in CBA$.

2) If $S \cdot A \notin CBA$, apply Type I left mutations to $S \cdot A$ until there are no Type I left collisions between beads in $S \cdot A$. The mutations may be applied in any order. Call the resulting tuple $M_I(S \cdot A)$.

3) Apply Type III left mutations to $M_I(S \cdot A)$ until there are no Type III left collisions between beads in $M_I(S \cdot A)$. The mutations may be applied in any order. Let $M_{III}M_I(S \cdot A)$ denote the resulting tuple.

4) Apply Type II left mutations to $M_{III}M_I(S \cdot A)$ until there are no Type II left collisions between beads in $M_{III}M_I(S \cdot A)$. The mutations may be applied in any order. Call the resulting tuple $M_{II}M_{III}M_I(S \cdot A)$.

5) Apply Type III left mutations to $M_{II}M_{III}M_I(S \cdot A)$ until there are no Type III left collisions between beads in $M_{II}M_{III}M_I(S \cdot A)$. The mutations may be applied in any order. Call the resulting tuple $S \circ A$.

6) Define $S^{-1} \circ A$ in analogy with 1-5) above, but replacing the word "left" with "right".

7) Let $\sigma \circ A := \sigma \cdot A$.

This action is illustrated in Figure 6.8 below. If the red bead in the upper figure is moved to the left, it undergoes mutations of Type I, III, II, and III, which produces the lower figure.

Remark. The procedure for computing the action is chosen to simplify the proof that $S \circ A$ is a well-defined colored bead arrangement. In fact, one can apply mutations to resolve collisions in any order. The final colored bead arrangement remains the same, as does the total number of mutations. However, we do not prove this.

One could define an alternative procedure, in which the Type II mutations occur in Step 3, followed by Type I mutations in Step 4, with the rest of the instructions unchanged. It is easy to check that applying M_{II} to a bead with a Type III collision produces a bead with a Type I collision, and that $M_I M_{II} = M_{II} M_{III}$. Thus the two procedures diverge at Step 3 but reconverge at Step 4. The careful reader may have noticed that the action of S^{-1} does not actually perform the inverse operations of the original procedure in reverse order; instead, it performs the inverse operations of the alternative procedure in reverse order. Since the two procedures are equivalent, we have that S and S^{-1} act inversely.

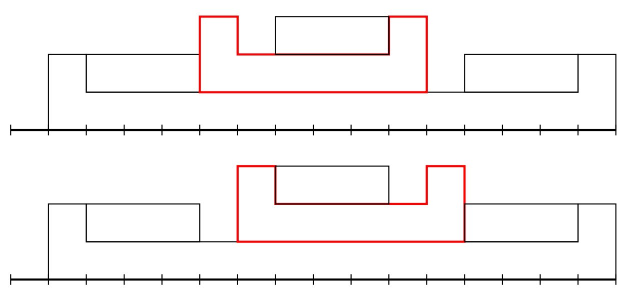


Figure 6.8: Shift the red bead (top figure) one to the left to produce the bottom figure. $n = 5$, $d = 1$

Proposition 6.2.5. *Let all notation be as in Definition 6.2.4. The algorithm defining $S \circ A$ terminates and $S \circ A \in CBA$. The action of Ξ on CBA is well-defined.*

If $S \circ A$ is well-defined, it follows by symmetry that $S^{-1} \circ A$ is well-defined. By above remarks, the actions of S and S^{-1} are mutually inverse, hence the action of $Free(\mathcal{P}'(n))$ on CBA will be well-defined. It is also clear that the actions of $Free(\mathcal{P}'(n))$ and $\mathfrak{S}_n$ induce an action of Ξ on CBA. Thus it is enough to show that $S \circ A$ is well-defined. We prove this with a sequence of lemmas below.

We shall refer to the ith entry of any tuple obtained from $S \cdot A$ via a sequence of mutations as a **moved bead** if $i \notin S$ and a **stationary bead** if $i \in S$. It is clear that the only overlaps between beads in $S \cdot A$ are left collisions of moved beads with stationary beads. Furthermore, a moved bead can have collisions with at most two stationary beads: a Type I or II collision along its left ridge, and a Type III collision along its right ridge.

Lemma 6.2.6. *$M_I(S \cdot A)$ exists; that is, there is a unique tuple in BT which has no Type I left collisions and is obtained from $S \cdot A$ by a finite sequence of Type I left mutations. The only overlaps between beads in $M_I(S \cdot A)$ are Type II or II left collisions of moved beads with stationary beads.*

Proof. Write $A = (B^1, \cdots, B^n)$. Let $B^i(-1)$ be a moved bead in $S \cdot A$ which has a Type I collision with a stationary bead B^j. Apply M_I^i to $S \cdot A$. The right ridge of $M_I(B^i(-1))$ has not moved and thus causes no new overlaps. The left ridge of $M_I(B^i(-1))$ is one unit to the left of the left ridge of B^j, so that B^j lies in the well of $M_I(B^i(-1))$.

If $M_I(B^i(-1))$ overlaps with some bead $B \neq B^i(-1)$ in $S \cdot A$, but not in a Type I, II, or III collision, then one of the ridges of B intersects $M_I(B^i(-1))$ somewhere other than its left ridge. The ridge of B cannot overlap with the left ridge of $B^i(-1)$, since $B^i(-1)$ already has a Type I left collision with B^j; the ridge of B cannot be in any other position, since this would result in a forbidden overlap with either B^j or $B^i(-1)$ in $S \cdot A$. Thus $M_I(B^i(-1))$ overlaps with another bead in $M_I^i(S \cdot A)$ if and only if it has a Type I, II, or III collision.

Since applying M_I^i does not move the right ridge of $B^i(-1)$, Type III collisions are not affected by M_I. There are three possibilities for the left ridge:

The first possibility is that the left ridge of $M_I(B^i(-1))$ does not intersect with any other bead. In this case, $M_I(B^i(-1))$ no longer has a Type I collision with any other bead.

The second possibility is that the left ridge of $M_I(B^i(-1))$ overlaps with the left ridge of a bead B^k. Note that B^k must be a stationary bead, since its left ridge is adjacent to that of the stationary bead B^j. Furthermore, both B^i and B^j must lie in the well of B^k. In this case, $M_I(B^i(-1))$ has a Type II collision with B^k, but no longer has any Type I collisions.

The third possibility is that the left ridge of $M_I(B^i(-1))$ overlaps with the right ridge of a bead B^k. Again B^k must be a stationary bead. Then $M_I(B^i(-1))$ has a Type I collision with B^k. In this case, we apply another Type I mutation to $M_I(B^i(-1))$, repeating the process until we are in either of the first two situations. Each time we apply a Type I mutation, a new stationary bead is added to the well of $B^i(-1)$; since there are only finitely many such beads, this process must terminate after finitely many steps.

We have shown that for each i, there is some $k_i \geq 0$ such that $(M_I^i)^{k_i}(S \cdot A)$ has no Type I collisions with any other entry. Furthermore, the only overlaps between beads in $(M_I^i)^{k_i}(S \cdot A)$ are left collisions of moved beads with stationary beads, and the number of beads with a Type I collision has decreased by one. Note that the above argument applies verbatim if $S \cdot A$ is replaced with $(M_I^i)^{k_i}(S \cdot A)$. Applying the argument at the index of each bead with a Type I collision, we obtain the desired tuple $M_I(S \cdot A)$.

Uniqueness of $M_I(S \cdot A)$ follows from the fact that M_I^i and M_I^j commute for all i and j. □

Lemma 6.2.7. *$M_{III}M_I(S \cdot A)$ exists; that is, there is a unique tuple in BT which has no Type I or III left collisions and is obtained from $M_I(S \cdot A)$ by a finite sequence of Type III left mutations. The only overlaps between beads in $M_{III}M_I(S \cdot A)$ are Type II left collisions of moved beads with stationary beads.*

Proof. We use the same notation as Lemma 6.2.6.

Let $B^{i'} = (M_I)^{k_i}(B^i(-1))$ be a moved bead in $M_I(S \cdot A)$ which has a Type III collision with a stationary bead B^j. Apply M_{III}^i to $M_I(S \cdot A)$. The left ridge of $M_{III}(B^{i'})$ is unchanged and causes no new overlaps; in particular, $M_{III}(B^{i'})$ has no Type I collisions. The right ridge of $M_{III}(B^{i'})$ is one unit to the right of the left ridge of B^j, so the two beads no longer intersect.

Note that it is not possible for the right ridge of $M_{III}(B^{i'})$ to overlap with the left ridge of a bead B in $M_I(S \cdot A)$. If this were the case, then B^k would overlap with either B^j or $B^{i'}$, and neither overlap would be a collision. This contradicts our our construction of $M_I(S \cdot A)$. More generally, it is not possible for $M_{III}(B^{i'})$ to overlap with any bead B in $M_I(S \cdot A)$ except in a left collision. Thus, we need only consider the two possibilities for the right ridge of $M_{III}(B^{i'})$:

The first possibility is that the right ridge of $M_{III}(B^{i'})$ does not overlap with the right ridge of any other bead. In this case, $M_{III}(B^{i'})$ no longer has a Type III collision.

The second possibility is that the right ridge of $M_{III}(B^{i'})$ overlaps with the right ridge of a bead B^k. Since the right ridge of B^k is adjacent to the stationary bead B^j, B^k must be a stationary bead, and $M_{III}(B^{i'})$ has a Type III collision with B^k. Since each Type III mutation removes a bead from the well of $B^{i'}$, there is some l_i such that $(M_{III})^{l_i}(B^{i'})$ has no Type III collisions.

We have shown that for each i, there exists l_i such that $(M^i_{III})^{l_i} M_I(S \cdot A)$ has no Type I or III collisions with any other entry of $M_I(S \cdot A)$. The only overlap between beads in $(M^i_{III})^{l_i} M_I(S \cdot A)$ are Type II collisions of moved beads with stationary beads, and the number of beads with a Type III collision has decreased by one. Once again, we can apply the same argument to $M^{l_i}_{III} M_I(S \cdot A)$ at each remaining bead with a Type III collision. This produces the desired tuple $M_{III} M_I(S \cdot A)$.

Uniqueness of $M_{III} M_I(S \cdot A)$ follows since M^i_{III} and M^j_{III} commute for all i and j. □

Lemma 6.2.8. *$M_{II} M_{III} M_I(S \cdot A)$ exists; that is, there is a unique tuple in BT which has no Type I or II left collisions and is obtained from $M_{III} M_I(S \cdot A)$ by a finite sequence of Type II left mutations. The only overlaps between beads in $M_{II} M_{III} M_I(S \cdot A)$ are Type III left collisions of moved beads with stationary beads.*

Proof. We use the notation of Lemma 6.2.7. Write $B^{i''} = M^{l_i}_{III}(B^{i'})$ for each moved bead in $M_{III} M_I(S \cdot A)$.

If $B^{i''}$ has a Type II collision with the stationary bead B^j, then $M_{II}(B^{i''})$ lies in the well of B^j. The left ridge of $M_{II}(B^{i''})$ coincides with the right ridge of $B^{i''}$, and the right ridge of $M_{II}(B^{i''})$ is adjacent to the right ridge of B^j. Since $B^{i''}$ did not overlap with any bead except B^j, the only possible overlap for $M_{II}(B^{i''})$ is a Type III collision with a bead B^k whose right edge lies at the rightmost point of the well of B^j. Since the right edge of B^k is adjacent to the right ridge of the stationary bead B^j, B^k must be a stationary bead.

Note that $B^{i''}$ is the only bead in $M_{III} M_I(C \cdot A)$ which has a Type II collision with B^j; if another moved bead $B^{r''}$ had a Type II collision with B^j, then $B^{i''}$ and $B^{r''}$ would overlap, a contradiction. In particular, for any $B^{r''}$ which has a Type II collision, $M_{II}(B^{r''})$ and $M_{II}(B^{i''})$ do not overlap.

For each i such that $B^{i''}$ has a Type II collision, apply M_{II}^i to $M_{III}M_I(S \cdot A)$; call the resulting tuple $M_{II}M_{III}M_I(S \cdot A)$. We have shown that the only possible overlaps between beads in $M_{II}M_{III}M_I(S \cdot A)$ are Type III overlaps between moved beads which have undergone a Type II mutation and stationary beads.

Uniqueness of $M_{II}M_{III}M_I(S \cdot A)$ follows since M_{II}^i and M_{II}^j commute for all i and j. □

Lemma 6.2.9. *$S \circ A$ exists; that is, there is a unique tuple in CBA which is obtained from $M_{II}M_{III}M_I(S \cdot A)$ by a finite sequence of Type III left mutations.*

Proof. We use the notation of Lemma 6.2.8. Let $B^{i'''} = M_{II}(B^{i''})$ be a moved bead inside $M_{II}M_{III}M_I(S \cdot A)$ which has a Type III collision with some stationary bead B^k. By the previous lemma, $B^{i'''}$ lies in the well of a stationary bead B^j. The left ridge of $B^{i'''}$ does not overlap with any other bead, and the right ridge of $B^{i'''}$ is adjacent to the right ridge of B^j and coincides with the right ridge of B^k. It is clear that $M_{III}(B^{i'''})$ cannot overlap with another bead in $M_{II}M_{III}M_I(S \cdot A)$, except possibly in a Type III collision with a stationary bead B^l which is right adjacent to B^k. Each application of M_{III} removes a bead from the well of $B^{i'''}$, hence there is some r_i such that $(M_{III})^{r_i}(B_{i'''})$ does not overlap with any bead.

Applying $(M_{III}^i)^{r_i}$ to $M_{II}M_{III}M_I(S \cdot A)$ at each bead with a Type III collision, we obtain the colored bead arrangement $S \circ A$, which is unique since Type III mutations commute. □

6.2.2 Compatibility of Actions

In this section, we show that A-dgstab admits $(d+1)$-orthogonal maximal extensions, hence Ξ acts on $\widehat{\mathcal{E}}$ by perverse tilts. (See Section 4.1 for definitions and terminology.) We shall also see that the two actions are compatible via Φ. We shall need the following three lemmas.

Lemma 6.2.10. *Let B^1, B^2 be non-overlapping beads. Then $B^1(-1)$ has a Type α left collision with B^2 for some $\alpha \in \{$I, II, III$\}$ if and only if dim $\mathrm{Hom}_{A\text{-dgstab}}(\Phi(B^1), \Phi(B^2(1))) = 1$. In this case, any nonzero morphism fits into a triangle*

$$\Phi(B^2) \to \Phi(M_\alpha(B^1(-1)))(1) \to \Phi(B^1) \to \Phi(B^2(1)) \tag{6.5}$$

Dually, $B^1(1)$ *has a Type* α *right collision with the bead* B^2 *if and only if* $dim_{A\text{-dgstab}}\operatorname{Hom}(\Phi(B^2(-1)),\Phi(B^1)) = 1$*. In this case, any nonzero morphism fits into a triangle*

$$\Phi(B^2(-1)) \to \Phi(B^1) \to \Phi(M_\alpha(B^1(1)))(-1) \to \Phi(B^2) \tag{6.6}$$

Proof. Let $B^1 = B_{l_1}(i_1), B^2 = B_{l_2}(i_2)$. If $B^1(-1)$ has a Type I collision with B^2, we must have that $i_2 = i_1 - l_1(d+2)$ and $l_1 + l_2 \leq n$. By Proposition 5.2.9, $M^1_{l_2}(-l_1(d+2)) \cong M^{l_1+1}_{l_1+l_2}$. Thus,

$$\begin{aligned}\operatorname{Hom}_{A\text{-dgstab}}(\Phi(B^1),\Phi(B^2(1))) &\cong \operatorname{Hom}_{A\text{-dgstab}}(M^1_{l_1}, M^1_{l_2}(1-l_1(d+2)))\\ &\cong \operatorname{Hom}_{A\text{-grstab}}(M^1_{l_1}, \Omega^{-1}M^{l_1+1}_{l_1+l_2})\\ &\cong \operatorname{Ext}^1_{A\text{-grmod}}(M^1_{l_1}, M^{l_1+1}_{l_1+l_2})\end{aligned}$$

The last space has dimension one. Any nonzero generator yields a triangle in $D^b(A\text{-grmod})$, which descends to the following triangle in A-dgstab:

$$M^{1+l_1}_{l_1+l_2} \to M^1_{l_1+l_2} \to M^1_{l_1} \to M^{l_1+1}_{l_1+l_2}(1)$$

Applying (i_1) to the triangle, we have $M^1_{l_1+l_2}(i_1) = \Phi(M_I(B^1(-1)))(1)$ and $M^{l_1+1}_{l_1+l_2}(i_1) \cong M^1_{l_2}(i_1 - l_1(d+2)) = \Phi(B^2)$. We have obtained the desired triangle in A-dgstab.

If $B^1(-1)$ has a Type II or III collision with B^2, the proof is analogous.

Conversely, if $B^1(-1)$ has no left collision with B^2, then $B^1(-1)$ and B^2 do not overlap, hence $\operatorname{Hom}_{A\text{-dgstab}}(\Phi(B^1(-1)),\Phi(B^2)) = 0$ by Proposition 6.1.8.

The proof for right mutations is dual. □

Lemma 6.2.11. *Let* $A = (B^1, \cdots, B^n)$ *be a colored bead arrangement. Let* $S \subsetneq [n]$*. Let* $S \circ A = (C^1, \cdots, C^n)$*. Suppose* C^i *is a moved bead and* $C^j = B^j$ *is a stationary bead. Then* $\operatorname{Hom}_{A\text{-dgstab}}(\Phi(C^i)(1), \Phi(B^j)) = 0$.

Dually, if $S^{-1} \circ A = (C^1, \cdots, C^n)$*, with* C^i *a moved bead and* $C^j = B^j$ *a stationary bead, then* $\operatorname{Hom}_{A\text{-dgstab}}(\Phi(B^j), \Phi(C^i)(-1)) = 0$.

Proof. Since C^i and B^j are part of a colored bead arrangement, they do not overlap. Thus either $C^i(1)$ and B^j do not overlap, in which case we are done by Proposition 6.1.8, or $C^i(1)$ has a Type I, II, or III right collision with B^j.

Write $C^i = B_{l_i}(k_i)$, $B^j = B_{l_j}(k_j)$. If there is a Type I right collision, then $k_i = k_j - l_j(d+2)$, hence $\Phi(C^i(1)) = M^1_{l_i}(1 + k_j - l_j(d+2)) \cong M^{1+l_j}_{l_i+l_j}(1+k_j)$ by Proposition 5.2.9. Note that $l_1 + l_2 \leq n$. Then,

$$\begin{aligned}
\mathrm{Hom}_{A\text{-dgstab}}(\Phi(C^i)(1), \Phi(B^j)) &\cong \mathrm{Hom}_{A\text{-dgstab}}(M^{1+l_j}_{l_i+l_j}(1+k_j), M^1_{l_j}(k_j)) \\
&\cong \mathrm{Hom}_{A\text{-dgstab}}(M^{1+l_j}_{l_i+l_j}, M^1_{l_j}(-1)) \\
&= \mathrm{Hom}_{A\text{-grstab}}(M^{1+l_j}_{l_i+l_j}, M^{1+l_j}_1) \\
&= 0
\end{aligned}$$

The other two cases follow by analogous arguments.

The proof for S^{-1} is dual. □

Lemma 6.2.12. *Let $A = (B^1, \cdots B^n) \in CBA$, $S \subsetneq \{1, \cdots, n\}$. Let $\mathcal{S} = \langle \{\Phi(B^j) \mid j \in S\} \rangle$. Write $S \circ A = (C^1, \cdots, C^n)$, and let $i \notin S$. Then $\Phi(C^i)(1)$ is a maximal extension of $\Phi(B^i)$ by $\mathcal{S}$.*

Dually, if $S^{-1} \circ A = (D^1, \cdots, D^n)$, then $\Phi(D^i)(-1)$ is a maximal $\mathcal{S}$-extension by $\Phi(B^i)$.

Proof. To simplify notation, we write $M = \Phi(C^i)(1), N = \Phi(B^i)$.

For any $j \in S$, $\mathrm{Hom}(N, \Phi(B^j)) = 0$, hence $\mathrm{Hom}(N, X) = 0$ for any $X \in \mathcal{S}$. By Proposition 4.1.5, $M = N_{\mathcal{S}}$ if and only if we have a morphism $f : M \to N$ such that $C(f)(-1) \in \mathcal{S}$ and $\mathrm{Hom}(M, X) = \mathrm{Hom}(M(-1), X) = 0$ for all $X \in \mathcal{S}$.

To construct the morphism f, note that C^i is obtained by applying a sequence of mutations $M_{\alpha_1}, \cdots, M_{\alpha_r}$ to $B^i(-1)$. For $1 \leq k \leq r$, write $N_k = \Phi(M_{\alpha_k} \cdots M_{\alpha_1} B^i(-1))(1)$. Define $N_0 = N$ and note that $N_r = M$. For each $1 \leq k \leq r$, by Lemma 6.2.10 we have a morphism $g_k : N_k \to N_{k-1}$ which fits into the triangle

$$\Phi(B^{j_k}) \to N_k \xrightarrow{g_k} N_{k-1} \to \Phi(B^{j_k})(1)$$

with $j_k \in S$. Let $f = g_1 \cdots g_r : M \to N$.

Since $\Phi(B^{j_k}) \in \mathcal{S}$ for each k, it follows from the octahedron axiom and induction on k that $C(f)(-1) \in \mathcal{S}$. Given $j \in S$, we have that $\mathrm{Hom}(M(-1), \Phi(B^j)) = 0$ since the beads C^i and B^j do not overlap. That $\mathrm{Hom}(M, \Phi(B^j)) = 0$ is precisely the statement of Lemma 6.2.11. The corresponding statements with B^j replaced by any $X \in \mathcal{S}$ follow immediately. Thus $M = N_{\mathcal{S}}$.

The proof of the second statement is dual. □

We have established the following theorem:

Theorem 6.2.13. A-dgstab *admits* $(d+1)$*-orthogonal maximal extensions, hence* Ξ *acts on* $\mathcal{E}$ *and* $\widehat{\mathcal{E}}$*, as in Definition 4.1.6. Furthermore,* $\Phi : CBA \to \widehat{\mathcal{E}}$ *is a morphism of* Ξ*-sets.*

Proof. The first statement follows immediately from Lemma 6.2.12. The lemma also shows that $\Phi(S \circ A) = S \cdot \Phi(A)$ for any colored bead arrangement A and $S \subsetneq [n]$. It is clear that any $\sigma \in \mathfrak{S}_n$ commutes with Φ, hence Φ is a morphism of Ξ sets. □

6.2.3 Transitivity

We now prove that the action of Ξ on $\widehat{\mathcal{E}}$ is transitive. It will then follow easily that $\mathcal{E} = \widehat{\mathcal{E}}$, hence every orthogonal tuple is a basis. We shall require two definitions:

Definition 6.2.14. Let A be a colored bead arrangement, and let $S \subsetneq [n]$. If $S \circ A = S \cdot A$ (i.e., no mutations occur), we say that $S \cdot A$ and A **differ by an elementary rigid motion**. We say two colored bead arrangements **differ by a rigid motion** if they are connected by a finite sequence of elementary rigid motions. We say two uncolored or free bead arrangements differ by a rigid motion if they are the images of colored bead arrangements which differ by a rigid motion.

Note that applying a rigid motion does not affect the class of a bead arrangement.

Definition 6.2.15. Let B be a bead in a colored, uncolored, or free bead arrangement. We say B is **right-justified** if $B(1)$ has a Type I or II right collision with another bead. We say a colored, uncolored, or free bead arrangement A is right-justified if there exists a bead B in A such that all beads $B' \neq B$ in A are right-justified.

The parameter d determines the amount of empty space in the well of each bead in the arrangement, as well as on the ring. Thus if $d = 0$, all bead arrangements are right-justified. If $d > 0$, then at most $n - 1$ beads can be simultaneously right-justified, since there will always be a bead on the wire which is not right-justified. Thus in any right-justified bead arrangement, the unique bead which is not right-justified must have height one. Note that $A = (B_1, B_1(-(d+2)), \cdots B_1(-n(d+2)))$ is right-justified, and that $\Phi(A) = (S_1, \cdots, S_n)$.

It is intuitively clear that any bead arrangement can be converted into a right-justified form via a rigid motion: hold one bead on the wire fixed, and slide all other beads to the right as far as they will go. More formally:

Lemma 6.2.16. *Let $A \in CBA$. Then there exists a right-justified bead arrangement A' which differs from A by a rigid motion.*

Proof. Without loss of generality, let $B^1 \in A$ have height 1. I claim that there exist colored bead arrangements $\{A_i\}_{i \geq 0}$ such that $B^1 \in A_i$ for all i, A_{i+1} differs from A_i by a rigid motion, and all beads $B \neq B^1$ in A^i of height at most i are right-justified.

Let $A_0 = A$. Given $A_i = (B^1, \cdots, B^n)$, if every bead $B^j \neq B^1$ of height $i+1$ is right-justified, take $A_{i+1} = A_i$. Otherwise, if some $B^j \neq B^1$ is not right-justified, let $S = \{1, \cdots, n\} - \{r \mid B^r \subset B^j\}$ and repeatedly apply S^{-1} to A_i until B^j becomes right-justified. Each application of S^{-1} is a rigid motion; no Type I or II right collisions occur because B^j is not right-justified, and no Type III collisions occur because all beads in the well of B^j are moved by S^{-1}. Furthermore, all beads of height $\leq i$ (excluding B^1) remain right justified, since we have only moved beads lying inside their wells. Note also that B^1 is never moved.

Repeat the process in the above paragraph until all height $i+1$ (except possibly B^1) beads are right-justified. If $i > 0$, there is finite space in the well of each bead, so the process must terminate after finitely many steps; if $i = 0$, since B^1 is always held fixed and the ring is finite, the process again terminates after finitely many steps. Once all height $i+1$ beads (except possibly B^1) are right-justified, call the resulting bead arrangement A_{i+1}. The existence of the desired family $\{A_i\}$ follows by induction. Then A_n is right-justified and differs from $A = A_0$ by a rigid motion. □

Definition 6.2.17. We say that a right-justified colored bead arrangement $A = (B^1, \cdots, B^n)$ is in **standard form** if:

1) After identifying each bead with its right endpoint, $B^1 > B^2 > \cdots > B^n > B^1$ with respect to the cyclic order on $\mathbb{Z}/P\mathbb{Z}$.

2) For all $j \neq 1$, B^j is right-justified.

Any right-justified colored bead arrangement $A = (B^1, \cdots, B^n)$ can be put in standard form by permuting its entries. Furthermore, if A is in standard form, then A is uniquely determined by two pieces of data: the associated tree $(P(A), r_A)$ and the choice of B^1: Once the position of B^1 is known, one can reconstruct A as an uncolored bead arrangement by simply placing each bead specified by $P(A)$ as far to the right as possible.

Lemma 6.2.18. *Up to permutation of indices, any two right-justified colored bead arrangements of the same class differ by a rigid motion.*

Proof. Let $A, A' \in CBA$ be right-justified colored bead arrangements of the same class. Let $A = (B^1, \cdots, B^n), A' = (C^1, \cdots, C^n)$. Permuting the entries of A, we may assume without loss of generality that A is in standard form. Permuting A', we may assume that the isomorphism mapping $P(A)$ to $P(A')$ sends B^i to C^i for each i. Note that A' need not be in standard form; condition 1) of Definition 6.2.17 will be satisfied, but not necessarily condition 2).

If condition 2) is not satisfied, then some $C^{j_1} \neq C^1$ is the unique bead (necessarily of height 1) which is not right justified. Let $C^{j_1} > C^{j_2} > \cdots C^{j_k} > C^{j_1}$ be the height 1 beads of A', ordered cyclically by their right endpoints; note that $C^1 = C^{j_r}$ for some r. Applying a rigid motion, we may translate C^{j_1} (and all beads lying in its well) until C^{j_1} is right justified, with its right edge adjacent to C^{j_k}. The resulting colored bead arrangement is right justified and C^{j_2} is now the unique bead which is not right-justified. Repeat this procedure until $C^{j_r} = C^1$ is not right-justified. Denote the new arrangement A''; clearly A'' is in standard form.

It is clear that A'' differs from A' by a rigid motion; consequently, A'' has the same class as A' and A. Since C^1 has the same type as B^1, we can write $B^1 = C^1(i)$ for some i. Then $A''(i)$ and A have the same class, are both in standard form, and have the same first bead, hence $A''(i) = A$. Since A and A'' differ by a rigid motion, so do A and A'. □

Lemma 6.2.19. *Two colored bead arrangements have the same class if and only if they differ by a rigid motion and a permutation of indices. Two uncolored or free bead arrangements have the same class if and only if they differ by a rigid motion.*

Proof. Applying an elementary rigid motion does not change the height of any bead, nor the relative ordering of the beads' right edges. Thus two bead arrangements which differ by an elementary rigid motion have the same class, hence also for arbitrary rigid motions.

Conversely, let A, A' be two colored bead arrangements of the same class. By Lemma 6.2.16, after changing A and A' up to a rigid motion, we can assume without loss of generality that both A and A' are right-justified. By Lemma 6.2.18, A and A' differ by a rigid motion and a permutation, and we are done.

The second statement follows immediately from the first. □

Theorem 6.2.20. *The action of Ξ on CBA is transitive.*

Proof. Let $A = (B_1, B_1(-(d+2)), \cdots B_1(-n(d+2)))$. We shall show that the orbit of A is CBA. By Lemma 6.2.19, it suffices to show that the orbit of A contains one representative of every class of colored bead arrangement.

Let (T, r) be a rooted plane tree with $n+1$ vertices. Let $V_{T,i}$ denote the set of vertices of T of depth i. Let $T_{\leq i}$ denote the subtree of T consisting of all vertices of depth $\leq i$. We shall construct a sequence $\{A_i\}_{i\geq 0}$ of bead arrangements with the following properties:

1) $A_{i+1} = \alpha \circ A_i$ for some $\alpha \in \Xi$.

2) For each i, there is an isomorphism $\phi_i : (T_{\leq i}, r) \xrightarrow{\sim} (\mathcal{P}(A_i)_{\leq i}, r_{A_i})$ of rooted plane trees.

3) ϕ_i preserves weight; i.e., $W_T(v) = W_{\mathcal{P}(A_i)}(\phi_i(v))$ for each vertex $v \in T_{\leq i}$.

4) All beads in A_i of height $i+1$ are of type 1.

5) A_i is right-justified and in standard form.

Let $A_0 = A$. Suppose we have constructed $A_i = (B^1, \cdots, B^n)$ for some $i \geq 0$. Let $v \in T$ be a vertex of height i. (If no such vertex exists, let $A_{i+1} = A_i$.) Let $B^j = \phi_i(v)$ (if $i > 0$). Let $c_T(v) = \{v_1 > \cdots > v_r\}$ be the children of v. Let $N_0 = 1$, and let $N_s = 1 + \sum_{c=1}^{s} W_T(v_c)$ for $s \geq 1$. By 3) and 4), $\phi_i(v) \in A_i$ has weight N_r, and there are $N_r - 1$ type 1 beads, $B^{j+1} > B^{j+2} > \cdots > B^{j+N_r-1}$ in the well of B^j. (If $i = 0$, then $\phi_i(v) = r_A$ and the beads B^{j+c} lie on the ring. All superscripts are then taken modulo n.) Since A_i is right-justified in standard form, the beads $B^{j+c}, 1 \leq c < N_r$ are adjacent to one another, so that $B^{j+s}(-1)$ has a Type I left collision with B^{j+s+1}, and B^{j+1} is adjacent to the right ridge of B^j (if $i > 0$).

Let $S = \{j + c \mid 1 \leq c < N_r, c \neq N_s \text{ for any } 0 \leq s < r\}$. When we apply S to A_i, the moved beads are B^j, every bead not in the well of B^j (if $i > 0$), and each bead of the form B^{j+N_s}, for $0 \leq s < r$. Note that none of the beads outside the well of B^j have collisions, since the only stationary beads are in the well of B^j. By the same reasoning, B^j can only have a Type III left collision, and this does not happen since B^{j+1} is also a moved bead. For each $0 \leq s < r$, the bead B^{j+N_s} undergoes $W_T(v_{s+1}) - 1$ Type I left mutations, which place the beads $B^{j+N_s+1}, \cdots B^{j+N_{s+1}-1}$ into its well; no Type II collisions are possible since B^j is moved, and no Type III collisions are possible since the B^{j+N_s} are of type 1. Thus in $\mathcal{P}(S \circ A_i)$, the children of $\phi_i(v)$ have weight $W_T(v_s)$, and are arranged in the same order as the children of v. After performing this process at each height i vertex v of T and converting the tuple into right-justified standard form via a rigid motion and a permutation of indices, denote the resulting colored bead arrangement A_{i+1}.

By construction, A_{i+1} is obtained from A_i by application of an element of Ξ, A_{i+1} is right-justified and in standard form, and all beads of height $i + 2$ in A_{i+1} are of type 1. Note that $\mathcal{P}(A_{i+1})_{\leq i} = \mathcal{P}(A_i)_{\leq i}$. It follows from the preceding paragraph that the map $\phi_i : T_{\leq i} \to \mathcal{P}(A_{i+1})_{\leq i}$ can be extended to an isomorphism $\phi_{i+1} : T_{\leq i+1} \to \mathcal{P}(A_{i+1})_{\leq i+1}$ which preserves the weight of all vertices for which it is defined. Thus A_{i+1} satisfies properties 1-5) above, hence the sequence $\{A_i\}$ exists. Then A_n is of class (T, r) and lies in the orbit of A.

This process is illustrated in Figure 6.9 below. □

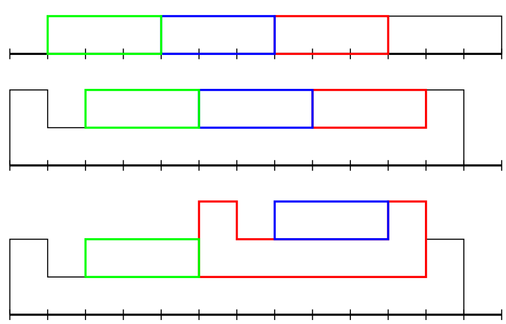


Figure 6.9: Arrangments A_0, A_1, A_2 ; $n = 4$, $d = 1$

Corollary 6.2.21. *The action of Ξ on $\widehat{\mathcal{E}}$ is transitive, and $\widehat{\mathcal{E}} = \mathcal{E}$.*

Proof. Transitivity follows from the fact that Ξ acts transitively on CBA and $\Phi : CBA \to \widehat{\mathcal{E}}$ is a surjective morphism of Ξ-sets. Since $\mathcal{E} \subset \widehat{\mathcal{E}}$ is nonempty and stable under Ξ, it follows from transitivity that the two Ξ-sets are equal. $\square$

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