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Delta eigenoperators and e-positivities in the theory of Macdonald polynomials

A Dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Marino Romero

Committee in charge:

Professor Adriano M. Garsia, Chair
Professor Russell Impagliazzo
Professor Brendon Rhoades
Professor Victor Vianu
Professor Nolan Wallach

2019

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Chair

University of California, San Diego

2019

DEDICATION

To my advisor Adriano Garsia, and to my family. Thank you.

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A note on the kth tensor product of the defining representation (with A. Mendes), *Journal of Combinatorics* 7, 4 (2016).

The Delta Conjecture at $q = 1$, *Trans. Amer. Math. Soc.* 369 (2017), pp. 7509-7530.

Delta operators at $q = 1$ and polyominoes (with A. Hicks). *Séminaire Lotharingien de Combinatoire*, 80B, (2018).

Some new symmetric function tools and their applications, (with A. M. Garsia and J. Haglund) *J. of Combin.* (special issue in honor of Jeff Remmel), to appear.

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ABSTRACT OF THE DISSERTATION

Delta eigenoperators and e-positivities in the theory of Macdonald polynomials

by

Marino Romero

Doctor of Philosophy in Mathematics

University of California, San Diego, 2019

Professor Adriano M. Garsia, Chair

In this dissertation, I give a variety of combinatorial expansions for certain symmetric functions related to the modified Macdonald basis $\{\widetilde{\mathrm{H}}_\mu\}_\mu$. We first prove the Delta Conjecture at $q = 1$, giving an elementary basis expansion for $\Delta_{e_k} e_n \Big|_{q=1}$, where Δ_F is the eigenoperator for the modified Macdonald basis defined by setting $\Delta_F \widetilde{\mathrm{H}}_\mu = F[B_\mu] \widetilde{\mathrm{H}}_\mu$. The main strategy is to expand the symmetric function as a generating series for an infinite, signed and weighted set of objects that give the elementary basis expansion (or e-expansion). To reduce this infinite sum to a finite polynomial with positive integer coefficients, we find a weight-preserving, sign-reversing involution whose fixed points all have positive sign. We then find a bijection to the conjectured combinatorial side of the Delta Conjecture at $q = 1$.

This strategy leads us to give a combinatorial e-expansion of $\Delta_{m_\lambda} e_n\Big|_{q=1}$ in terms of (parallelogram) polyominoes, again by using a weight-preserving, sign-reversing involution. We will show that the coefficient of e_μ in this symmetric function t-counts (by area) the number of polyominoes whose bottom path has horizontal segments rearranging to λ and whose top path has vertical segments rearranging to μ. This result can be used to get combinatorial expansions for $\Delta_F e_\mu\Big|_{q=1}$, where F is any of the classical bases $m_\lambda, f_\lambda, h_\lambda, e_\lambda, s_\lambda$, or p_λ. This result uses the principal evaluation for the forgotten basis giving a new application for the forgotten symmetric functions.

The last result of this kind will give an e-expansion of $\Delta_{m_\lambda}(-1)^{n-1}p_n\Big|_{q=1}$ in terms of polyominoes whose top path ends in two East steps. The only difference is that now every polyomino appears with multiplicity the length of its return, defined as the first row (after the first vertical segment of the top path) at which the top and bottom path are one unit away. We once again get formulas for $\Delta_F(-1)^{n-1}p_n\Big|_{q=1}$, where F is any of the classical bases. In general, $\Delta_{m_\lambda}\omega p_\mu\Big|_{q=1}$ is not e-positive, but we still present a formula in terms of power and elementary symmetric functions.

In the end, we will look at certain e-positivity phenomena and we conjecture that $\Delta_{m_\lambda} e_n$ and $\Delta_{m_\lambda}(-1)^{n-1}p_n$, when expanded in terms of the elementary symmetric function basis, has coefficients which are polynomials in $u = q-1$ and t with positive integer coefficients. We will prove that the plethystic operators B_a satisfy this phenomenon. In turn, we get a homogeneous basis expansion for the Hall-Littlewood polynomials and an elementary basis expansion for $\Delta_{e_k} e_n\Big|_{t=0}$, where the coefficients are polynomials in $u = q-1$ and q with positive integer coefficients.

Chapter 1

Introduction

1.1 A brief history

The symmetric function basis $\{P_\mu\}_\mu$ introduced by Macdonald in [32], now called Macdonald polynomials, are defined by the triangularity $P_\mu = m_\mu + \sum_{\lambda<\mu} u_{\mu,\lambda} m_\lambda$ and the orthogonality $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ for $\lambda \neq \mu$. Macdonald conjectured that the integral form of these polynomials expands in terms of a modified Schur basis as a positive, integral polynomial in q and t. Moreover, these coefficients, $K_{\lambda,\mu}(q,t)$, enumerate the number of standard tableaux of shape λ when $q = t = 1$. Macdonald asked for a combinatorial interpretation of these polynomials by assigning to each standard tableau of shape λ a q,t-weight depending on μ. This remains an open question today.

The $K_{\lambda,\mu}$ were originally defined to be rational functions. Macdonald computed tables that revealed they were in fact polynomials in q,t with integer coefficients. In order to prove the polynomiality and positivity of Macdonald's conjecture using representation theoretical tools, Adriano Garsia and Mark Haiman introduced in [12] the modified version of Macdonald's polynomial $\{\widetilde{\mathrm{H}}_\mu\}_\mu$. When expanded in terms of Schur functions, one has

$$\widetilde{\mathrm{H}}_\mu[X;q,t] = \sum_{\lambda \vdash n} \widetilde{\mathrm{K}}_{\lambda,\mu}(q,t) s_\lambda,$$

where $\widetilde{\mathrm{K}}_{\lambda,\mu}(q,t) = t^{n(\mu)}\,\mathrm{K}_{\lambda,\mu}(q,t^{-1})$ (this definition was first stated in [12]). There are many advantages to using the modified basis over Macdonald's integral basis, but perhaps the most striking part of the modified Macdonald basis is how it gives Macdonald polynomials a representation theoretical setting. Recall that the Frobenius map F, which assigns to a conjugacy class C_λ of S_n corresponding to the partition $\lambda \vdash n$ the power sum symmetric function basis p_λ/z_λ, yields the identity $F(\chi^\lambda) = s_\lambda$ (sending the irreducible character χ^λ to the Schur function s_λ). For any graded module $V = \bigoplus_{r,s\in\mathbb{Z}} V_{r,s}$, one can then define the Frobenius characteristic map by setting

$$\mathbf{F}(V) = \sum_{r,s} t^r q^s F(V_{r,s}).$$

The modules discussed here will be subspaces of $V = \mathbb{C}[X_n, Y_n]$, the space of polynomials in $X_n = x_1, \ldots, x_n$ and $Y_n = y_1, \ldots, y_n$. The homogeneous subspace $V_{r,s}$ of polynomials of degree r in X_n and s in Y_n give a bigrading. Given a partition μ, let $\mathbf{\Delta}_\mu[X_n, Y_n] = \det ||x_i^b y_i^a||_{\substack{i=1,\ldots,n \\ (a,b)\in\mu}}$, and set

$$M_\mu = \mathcal{L}\left[P(\partial_x, \partial_y)\mathbf{\Delta}_\mu : P \in \mathbb{C}[X_n, Y_n]\right]$$

to be the linear span of derivatives of $\mathbf{\Delta}_\mu$. Garsia and Haiman conjectured that

$$\mathbf{F}(M_\mu) = \widetilde{\mathrm{H}}_\mu\,.$$

This would prove that $\widetilde{K}_{\lambda,\mu} \in \mathbb{N}[q,t]$ since the coefficients are enumerating multiplicities of irreducible representations. Furthermore, M_μ is a bigraded version of the left regular representation, meaning this identity would imply that $\widetilde{\mathrm{K}}_{\lambda,\mu}(1,1)$ is the number of standard tableaux of shape λ.

For instance, if $\mu = (2,1)$, then the cells in μ are given by the coordinates $(0,0), (1,0)$, and $(0,1)$. The determinant for $\mathbf{\Delta}_{(2,1)}$ then equals

$$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = x_2y_3 + x_3y_1 + x_1y_2 - (x_2y_1 + x_3y_2 + x_1y_3).$$

Taking a partial derivative with respect to x_1 would give $y_2 - y_3$, with respect to x_2 we would get $y_3 - y_1$ and with respect to x_3, we get

$$\partial_{x_3} \mathbf{\Delta}_{(2,1)} = y_1 - y_2 = -(\partial_{x_1} + \partial_{x_2}) \mathbf{\Delta}_{2,1}.$$

We can similarly take partial derivatives with respect to y_1, y_2 and y_3. This means that $M_{(2,1)}$ is spanned by

$$\mathbf{\Delta}_{(2,1)}, \{y_2 - y_3, y_3 - y_1\}, \{x_2 - x_3, x_3 - x_1\}, 1$$

The first element is an alternant of bidegree $(1,1)$ and therefore corresponds to the sign representation. The two sets are each a basis for a two-dimensional representation of S_3. And the constants give a trivial representation. We then get

$$\mathbf{F}(M_{(2,1)}) = qts_{1,1,1} + (q+t)s_{2,1} + s_3 = \widetilde{\mathrm{H}}_{(2,1)}$$

In order to prove that $\mathbf{F}(M_\mu) = \widetilde{\mathrm{H}}_\mu$, it was left to show that $\dim M_\mu = n!$. This was called the $n!$-Conjecture. Mark Haiman proves this in [29] by algebraic geometrical means, validating the representation theoretical interpretation of the modified Macdonald basis.

A natural space to study in relation to the collection $\{M_\mu\}_{\mu \vdash n}$ is the space of Diagonal Harmonics

$$DH_n = \left\{ P(X_n, Y_n) \in \mathbb{C}[X_n, Y_n] : \text{ for all } r + s > 0, \sum_{i=1}^{n} \partial_{x_i}^r \partial_{y_i}^s P = 0 \right\}.$$

For every partition $\mu \vdash n$, M_μ is a submodule of DH_n. It was conjectured in [11] by Garsia and Haiman that

$$\mathbf{F}(DH_n) = \sum_{\mu \vdash n} T_\mu \frac{MB_\mu \Pi_\mu}{w_\mu} \widetilde{\mathrm{H}}_\mu \,.$$

These quantities will be defined properly in Section 2.3. Upon seeing this formula in relation to

$$e_n = \sum_{\mu \vdash n} \frac{MB_\mu \Pi_\mu}{w_\mu} \widetilde{\mathrm{H}}_\mu,$$

François Bergeron believed that if we set $\nabla \widetilde{\mathrm{H}}_\mu = T_\mu \widetilde{\mathrm{H}}_\mu$, then ∇ must have some amazing properties for it to transform the innocuous elementary symmetric function e_n to the Frobenius characteristic of the space of Diagonal Harmonics. Garsia and Bergeron introduce ∇ in [28], and they indeed show that ∇ has some amazing properties; in this case it can be used to describe the intersections of modules $M_{\alpha^1}, \ldots, M_{\alpha^k}$, where all α^i are obtained from a partition μ by removing a corner cell.

It became essential in [27] to study a different eigenoperator that generalizes ∇. They define the Delta operators Δ_F by setting

$$\Delta_F \widetilde{\mathrm{H}}_\mu = F[B_\mu] \widetilde{\mathrm{H}}_\mu \,.$$

If $\mu \vdash n$, then $\nabla \widetilde{\mathrm{H}}_\mu = \Delta_{e_n} \widetilde{\mathrm{H}}_\mu$. Now we have several families of eigenoperators, and one of the natural questions to then ask is what happens when we apply these operators to classical symmetric functions, like Schur functions or elementary symmetric functions. For example, Mark Haiman shows that in connection to the Hilbert scheme of points on the plane, $\Delta_{s_\lambda} \nabla e_n$ is the Frobenius characteristic of an S_n-module, meaning it is Schur positive. He goes on to conjecture that $\Delta_{s_\lambda} e_n$ is Schur positive as well [30]. In [27] it is conjectured that up to a sign, ∇s_λ is Schur positive.

The main task is to give a combinatorial interpretation for the symmetric functions in question. The first example is the Shuffle Conjecture of [39] and its later compositional form in [36],

stating that when expanded in terms of monomials in $x_1, x_2, \ldots$, we have that

$$\nabla e_n = \Delta_{e_n} e_n = \sum_{L \in LD_n} t^{\text{area}(D)} q^{\text{dinv}(D)} x^L$$

as a weighted sum over labeled Dyck paths. This set and pair of statistics are defined in Section 3.1. The Compositional Shuffle Conjecture was proved by Carlsson and Mellit in [20]. Section 3.1 will also introduce the Delta Conjecture of [37], which states that

$$\Delta_{e_k} e_n = \sum_{L \in LD_n} t^{\text{area}(D)} q^{\text{dinv}(D)} H_{n-k}(L) x^L.$$

Again, the expansion is in terms of monomials. Up to now, this conjecture has been verified for when $q = 1$ [44], $t = 0$ [14], [35], or in the "generalized" case we have the Catalan [45] or the Schröder cases [42]. On the other hand, a Schur positive expression should correspond to some S_n-module. It is conjectured in [46] that a triply-graded space called the "super-diagonal coinvariants" does this for the Delta conjecture. This is the space of polynomials in three sets of variables, two sets commuting and one set anti-commuting with itself, modulo the ideal generated by the invariants under the S_n action.The component of degree k in the anti-commuting variables is then conjectured to give $\Delta'_{e_{n-k-1}} e_n$, for the related operators Δ' discussed in Section 6.2. Proving this conjecture would prove Schur positivity for $\Delta_{e_k} e_n$.

Another example is the Square Paths Conjecture [47], proved in [23], which gives

$$\nabla (-1)^{n-1} p_n = \sum_{P \in \mathcal{LSP}_n} t^{\text{area}(L)} q^{\text{dinv}(L)} x^L,$$

as a sum over labeled square paths. This is described in Subsection 5.2.1. Note that we once again get an expression in terms of monomials, even though it is shown in [51] that $\nabla (-1)^{n-1} p_n$ is the Frobenius image of a certain module (meaning it is Schur positive). There is also the expansion of $\widetilde{\mathrm{H}}_\mu$ in terms of monomials found in [38]. In all, most of the expansions that exist in the literature

are in terms of monomials (or Gessel Fundamental quasisymmetric functions), which is weaker than giving a Schur function expansion.

Garsia and Haiman's modified Macdonald basis and the operator ∇ has now gone beyond the world of symmetric functions and representations of the symmetric group. Mark Haiman's proof of the $n!$-Conjecture was based on Procesi's idea to use the Hilbert scheme of points on the plane. This has now led to a conjectured association between a certain derived category of equivariant sheaves over the Hilbert scheme and complexes of Soergel bimodules [21]. A beautiful picture arises when symmetric functions are translated to the theory of torus knots through this connection, as is described in [31]. Here, Hogancamp shows that

$$\frac{1}{1-q}\sum_{k=0}^{n}\langle \nabla^m e_n, h_k e_{n-k}\rangle a^k$$

is in fact a triply graded invariant of the $(n, nm+1)$-torus knot. He explains as well that the operator ∇ should coincide with the full twist of a braid. It is further shown in [6] how to get this invariant for (m,n)-torus knots using related symmetric function operators appearing in the Rational Shuffle Conjecture ([22], [26]), which was proved in [5].

1.2 The aim

Let us denote by $\widetilde{\Delta}_F$, the operator which sends a symmetric function G to $\Delta_F G\Big|_{q=1}$. In [41], it is shown that $\langle \Delta_{e_{m+n}} e_{m+n}, h_m h_n\rangle = \langle \Delta_{h_m} e_{n+1}, e_{n+1}\rangle$ is a q,t-analogue of the Narayana numbers, written as a sum over polyominoes. Each polyomino has a q,t-weight determined by two statistics: area and bounce (or dinv). The connection between polyominoes and Delta operators was further explored in [40], where it is stated conjecturally that $\widetilde{\Delta}_{h_k}$ can be written as an e-positive sum over polyominoes, enumerated by the area statistic. (This was communicated to the authors by an author of [41].) We will see that this is one of the simple consequences of our work since $h_k = \sum_{\lambda\vdash k} m_\lambda$.

The main goal of this thesis is to give combinatorial expansions for any symmetric function

of the form $\widetilde{\Delta}_F\, e_\mu$ and $\widetilde{\Delta}_F\, \omega p_\mu$, where F is any symmetric function expanded in terms of a classical basis. In the first case, and in the second when $\mu = (n)$, the expansions we get are in terms of the elementary symmetric function basis, giving e-positive expansions whenever F is expanded positively in terms of the monomial basis. In contrast to other results in this area of study, we use the combinatorics of symmetric function theory to guide the combinatorial constructions. The elementary basis expansions give Schur expansions by using Pieri's formula. Since $\Delta_{s_\lambda} e_n$ is conjecturally Schur positive and our construction gives a set that expands $\widetilde{\Delta}_{s_\lambda}\, e_n$ in terms of Schur functions, one can hope to find a second statistic on our objects that can give the full Schur expansions without specializing q.

The last chapter will be dedicated to further e-expansions that arise in this study. In particular, we study a certain e-positivity phenomenon and conjecture that certain symmetric functions are e-positive. We prove some of these cases and are able to give expansions for the Hall-Littlewood polynomials and $\Delta_{e_k} e_n\Big|_{t=0}$ in terms of the homogeneous and elementary symmetric functions, respectively.

1.3 Outline

- An effort is made in Chapter 2 to introduce symmetric function theory with as little prior knowledge as possible. In the second section, we introduce plethystic substitution, which is an incredible device that will allow us to compute the combinatorial formula for the forgotten and monomial symmetric functions (when they are defined as the dual bases for the elementary and homogeneous bases, respectively). Section 2.3 will introduce the modified Macdonald polynomials. We will prove some of their important properties and introduce some new symmetric function tools from [13]. The section that follows will look at some of the basic properties of the Delta operator at $q = 1$, and the last section is a quick note on why our results also work for when $t = 1$ instead.

- Chapter 3 will properly introduce the Delta Conjecture and prove it for when $q = 1$. This

work can be found in [44]. The first three sections will set up the problem, writing the e-expansion of $\widetilde{\Delta}_{e_k} e_n$ as an infinite sum over certain weighted and signed objects $\mathbf{T}_{\mu}^{\lambda}$. To get the final result, in Section 3.4 we introduce the "split" map which ultimately gives us a sign-reversing involution φ. After applying a weight-preserving, sign-reversing involution, only a finite number of positive objects will remain. We then give a weight-preserving bijection to the combinatorial side of the Delta Conjecture at $q = 1$. This bijection can be found in Section 3.5. We include as well a second bijection to the "Leaning Stacks" interpretation of the Delta Conjecture. We end the chapter with some special applications.

- The fourth chapter contains one of the most important observations in this thesis. It relates the q-enumeration of parallelogram polyominoes in a rectangle to the expansion of Delta operators at $q = 1$. This connection lets us ultimately compute $\widetilde{\Delta}_{m_\lambda} e_\mu$. To do this, we define a set $\mathbf{CCT}(\lambda, \mu)$ of Labeled Column-Composition Tableaux, Defintion 4.1.1. This infinite set gives the e-expansion for the symmetric function $\widetilde{\Delta}_{m_\lambda} e_n$. In Section 4.2 we define a weight-preserving, sign-reversing involution ψ on the union of all $\mathbf{CCT}(\lambda, \mu)$ with $\mu \vdash n$. In Section 4.3 we define a bijection Γ from the fixed points of ψ to the set of polyominoes $\mathcal{P}_{\lambda,\mu}$ whose horizontal segments in the bottom path and vertical segments of the top path have lengths that rearrangements to λ and μ respectively. Section 4.3.1 will use the main theorem of the chapter to compute an example in full. The last two sections will explain how to generalize this result: Section 4.4 will tell us how to go from applying the operators on e_n to applying the operators on e_μ, and Section 4.5 will tell us how to change the index F from m_λ to any of the other classical bases.

- The first section of Chapter 5 will give a combinatorial proof of a Newton identity giving the transition from the power symmetric function basis to the homogeneous (or elementary) basis. We use this expansion to construct a set $\mathbf{SCT}(\lambda, \mu)$ similar to the set in the previous chapter, Definition 5.2.1. The symmetric function $\widetilde{\Delta}_{m_\lambda}(-1)^{n-1} p_n$ is given as a sum over these objects. To get a finite, positive sum, we once again use a sign reversing involution, in this

case denoted $\psi_\circ$, Definition 5.2.3. We then biject the collection of fixed points to a set of polyominoes $\mathcal{RP}_{\lambda,\mu}$ with a marked row before their return, Defintion 5.2.7. In the last section of this chapter, we will look at how to generalize to symmetric functions of the form $\widetilde{\Delta}_F \omega p_\mu$. However, this is not in general e-positive. However, when $\mu = (n) \neq 0$, we do get positive e-expansions like in the previous chapter; and for general μ, we get an expansion in terms of power and elementary symmetric functions. We look at the special case when $F = e_n$ and $\mu = (n)$ and compare our result with the Square Paths Theorem.

- In the last chapter we will introduce the e-positivity phenomenon discussed in [2]. We include a new conjecture, namely that $\Delta_{m_\lambda} e_n \Big|_{q=1+u}$ and $\Delta_{m_\lambda}(-1)^{n-1} p_n \Big|_{q=1+u}$ are e-positive in terms of u and t. In the end, we will also get e-expansions for when the operators B_a from [36] are applied to elementary symmetric functions. In turn, this gives an h-basis expansion of the Hall-Littlewood polynomial and an e-basis expansion for $\Delta_{e_k} e_n \Big|_{t=0}$.

Chapter 2

Symmetric Functions

2.1 The basics

In what follows, we will assume an elementary understanding of symmetric function theory in a countable number of indeterminates $x_1, x_2, \ldots$. The usual reference for symmetric functions and Macdonald polynomials is [32], but we should also note [7] is a good source exhibiting some of the strategies used in this thesis. The ring of symmetric functions Λ with coefficients in $\mathbb{Q}$ is composed of the set of all polynomial functions $f(x_1, x_2, \ldots)$ such that for any $n \geq 1$ and any permutation $\sigma \in S_n$, we have

$$f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}, x_{n+1}, \ldots) = f(x_1, x_2, \ldots).$$

By calling such a function "polynomial" we mean that if all but finitely many x_i are set to 0, then the resulting function would be a polynomial.

The six classical symmetric function bases are the "power" $\{p_\lambda\}$, "monomial" $\{m_\lambda\}$, "(complete) homogeneous"$\{h_\lambda\}$, "elementary" $\{e_\lambda\}$, "forgotten" $\{f_\lambda\}$, and "Schur" functions $\{s_\lambda\}$, each indexed by an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$. The number $\ell(\lambda) = \ell$ is called the

length of the partition and $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$ its size. The power, elementary, and homogeneous symmetric functions are multiplicative bases in the sense that for any partition λ,

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}, \qquad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_{\ell(\lambda)}}, \qquad \text{and} \qquad e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{\ell(\lambda)}}.$$

We recall that for any positive integer r, we have

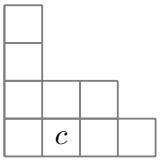
$$p_r = x_1^r + x_2^r + \cdots, \qquad h_r = \sum_{1 \le i_1 \le \cdots \le i_r} x_{i_1} \cdots x_{i_r}, \qquad \text{and} \qquad e_r = \sum_{1 \le i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

The remaining symmetric functions may also be defined combinatorially. We do this in the following section for the monomial and forgotten symmetric functions as a consequence of the Cauchy Identity, Proposition 2.2.1.

When we refer to a partition λ, we will also mean the shape formed by its Young diagram in the French convention. To form the Young diagram, we draw lattice cells in the positive quadrant of the plane so that the lower left corners are given by the coordinates

$$\{(0, i-1), (1, i-1), \ldots, (\lambda_i - 1, i-1) : i = 1, \ldots, \ell(\lambda)\}.$$

In other words, the cells are left and bottom-justified, so that the bottom row has λ_1 cells, the second row up has λ_2 cells, and so on. For example, the partition $(4, 3, 1, 1)$ may be depicted by the following sequence of cells:



For a cell $c \in \lambda$, we denote by $\mathrm{arm}(c)$ the number of cells strictly to the right of c and $\mathrm{leg}(c)$ the number of cells strictly above c. The indices $(i, j) = c$ are also denoted by $(\mathrm{arm}'(c), \mathrm{leg}'(c))$ and are

called the coarm and coleg respectively. For instance, the cell c in the above example is given by the coordinate $c = (1, 0)$, and has arm and leg given by $\text{arm}(c) = 2$ and $\text{leg}(c) = 1$.

We can fill the cells of a partition λ with positive integers. We call such a filling standard if the rows (when read left to right) and columns (when read bottom to top) are increasing. If the rows are weakly increasing, we call the tableau semistandard. For instance, the following tableau is semistandard:

6			
5			
2	4	4	
1	1	3	4

The set of all semistandard Young tableaux of shape λ with positive integer entries will be denoted by $SYT(\lambda)$. Then one can combinatorially define Schur functions by setting

$$s_\lambda = \sum_{T \in SYT(\lambda)} x^T$$

where $x^T = x_1^{a_1} x_2^{a_2} \cdots$ if there are a_1 cells in T with a 1, there are a_2 cells in T with a 2, and so on. In the above example, the semistandard tableau would give the monomial $x^T = x_1^2 x_2 x_3 x_4^3 x_5 x_6$. We can similarly fill any diagram of lattice cells in the plane. For any diagram D, we can define s_D to be the sum over semistandard fillings of D. In particular, when $\mu \subset \lambda$ is a partition contained in the diagram of λ, then the skew Schur function $s_{\lambda/\mu}$ is defined by taking

$$s_{\lambda/\mu} = \sum_{T \in SYT(\lambda/\mu)} x^T,$$

where λ/μ is the shape obtained from λ by removing the cells of μ.

As special cases of Schur functions we have

$$h_n = s_n \qquad \text{and} \qquad e_n = s_{1^n}.$$

Recall the formal series expansion

$$\log \frac{1}{1-x} = \sum_{r \geq 1} \frac{x^r}{r}$$

Then

$$\frac{1}{1-x} = \exp \sum_{r \geq 1} \frac{x^r}{r}.$$

Multiplying exponentials, we have

$$\prod_{i \geq 1} \frac{1}{1-x_i} = \exp \sum_{r \geq 1} \frac{\sum_{i \geq 1} x_i^r}{r} = \exp \sum_{r \geq 1} \frac{p_r}{r}$$

Now using the exponential formula

$$\exp \sum_{r \geq 1} \frac{p_r}{r} = \sum_{k \geq 0} \frac{1}{k!} \sum_{r_1, \ldots, r_k \geq 1} \frac{p_{r_1} \cdots p_{r_k}}{r_1 \cdots r_k}.$$

We now collect terms. If $r_1, \ldots, r_k$ rearranges to the partition λ, then we will write $(r_1, \ldots, r_k) \in R(\lambda)$. Denote by $m_i(\lambda)$ the number of parts in λ of size i. Then the number of rearrangements of $\lambda \vdash n$ is the multinomial coefficient

$$|R(\lambda)| = \binom{\ell(\lambda)}{m_1(\lambda), \ldots, m_n(\lambda)} = \frac{\ell(\lambda)!}{m_1(\lambda)! \cdots m_n(\lambda)!}.$$

Then we have

$$\begin{aligned}\exp\sum_{r\geq 1}\frac{p_r}{r} &= \sum_{n\geq 0}\sum_{\lambda\vdash n} p_\lambda \sum_{(r_1,\ldots,r_k)\in R(\lambda)} \frac{1}{k!\ r_1\cdots r_k}\\ &= \sum_{\lambda} p_\lambda \frac{1}{\ell(\lambda)!\ 1^{m_1(\lambda)}\cdots n^{m_n(\lambda)}} \sum_{(r_1,\ldots,r_k)\in R(\lambda)}\\ &= \sum_{\lambda}\frac{p_\lambda}{z_\lambda},\end{aligned}$$

where we have set

$$z_\lambda = 1^{m_1(\lambda)}\cdots n^{m_n(\lambda)} m_1(\lambda)!\cdots m_n(\lambda)!$$

On the other hand,

$$\prod_{i\geq 1}\frac{1}{1-x_i} = \sum_{n\geq 0} h_n.$$

Comparing the coefficients of homogeneous degree n, we have shown that

$$h_n = \sum_{\lambda\vdash n}\frac{p_\lambda}{z_\lambda}.$$

Now using that

$$1 + x = \exp -\sum_{r\geq 1}\frac{(-x)^r}{r} = \exp\sum_{r\geq 1}\frac{(-1)^{r-1}x^r}{r},$$

we have a similar expression for

$$\prod_{i\geq 1}(1+x_i) = \sum_{n\geq 0} e_n$$

in terms of the power sums. This gives

$$e_n = \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}\frac{p_\lambda}{z_\lambda}.$$

The space of symmetric functions comes equipped with an inner product $\langle\cdot,\cdot\rangle$ (called the

"Hall" or ordinary scalar product) which may be defined by the orthogonality relation

$$\langle p_\lambda, p_\mu \rangle = \chi(\lambda = \mu) z_\lambda,$$

where $\chi(A)$ is the indicator function, giving 1 if A is true and 0 otherwise. The orthogonality relation extends to give

$$\langle h_\lambda, m_\mu \rangle = \chi(\lambda = \mu), \qquad \langle e_\lambda, f_\mu \rangle = \chi(\lambda = \mu) \qquad \text{and} \qquad \langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu),$$

though we will take as definition instead the first two relations, meaning that the monomial symmetric functions are dual to the homogeneous and the forgotten are dual to the elementary, from which their combinatorial formula will follow (using Proposition 2.2.2). The fundamental involution ω is defined by setting

$$\omega(p_k) = (-1)^{k-1} p_k.$$

This is the transformation which sends $\prod_i \frac{1}{1-x_i}$ to $\prod_i (1 + x_i)$ via the exponential formulas above. We get that $\omega(h_n) = e_n$. In terms of the classical bases,

$$\omega(h_\lambda) = e_\lambda, \qquad \omega(m_\lambda) = f_\lambda, \qquad \text{and} \qquad \omega(s_\lambda) = s_{\lambda'}.$$

We write λ' for the conjugate partition obtained from the diagram of λ by reflecting across the line $y = x$.

2.2 Plethystic substitution

An effort has been made to introduce plethystic notation and many of its important uses. Several references can be cited for some of the material in this section (see [48] and [1]), but an effort is made to include and prove all the relevant properties. Our approach is slightly different than previous expositions, and we hope that this introduction not only gives the reader an understanding

but also an appreciation for the power of plethystic notation.

Since the power sums $p_1, p_2, \dots$ are algebraically independent, any identity involving the power sums will remain true under any substitution $p_i \mapsto p_i[A]$. Given a set of variables $t_1, t_2, \dots$, and any expression $E(t_1, t_2, \dots)$, we define

$$p_r[E] = E(t_1^r, t_2^r, \dots) \qquad \text{and} \qquad p_\lambda = p_{\lambda_1}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E].$$

Note that for $X = x_1 + x_2 + \cdots$, we have $p_r[X] = p_r$. It is often useful to treat X as an arbitrary alphabet or expression. One can often prove identities involving arbitrary expressions E with the special case $E = X$, as we will do in what follows.

Note that $p_r[-X] = -p_r[X]$. This means that substituting a variable x by "$-x$", in the usual polynomial sense, does not commute with plethystic substitution. To get a negative sign that corresponds to the usual polynomial substitution we introduce a second negative sign. We let ϵ be a variable within the plethystic brackets and equal to -1 outside the brackets. Then

$$p_r[\epsilon X] = \epsilon^r p_r[X] = (-1)^r p_r[X] \qquad \text{and} \qquad p_r[-\epsilon X] = -\epsilon^r p_r[X] = (-1)^{r-1} p_r[X] = \omega(p_r)[X].$$

This means that for any symmetric function F, we have $F[-\epsilon X] = \omega(F)[X]$. If F is homogeneous of degree d, then $F[-X] = F[-\epsilon(\epsilon X)] = (-1)^d \omega(F)[X]$.

The definition of plethystic substitution immediately gives the following summation and product formulas: For two expressions X and Y, we have

$$p_r[X + Y] = p_r[X] + p_r[Y] \qquad \text{and} \qquad p_r[XY] = p_r[X] p_r[Y].$$

This is just another way of saying what p_r does on a sum or product of expressions: first raise the

variables in the first expression X to the r^{th} power, then raise the variables in the second expression Y to the r^{th} power.

Thinking of X as $x_1 + x_2 + \cdots$ and Y as $y_1 + y_2 + \cdots$ easily gives

$$s_\lambda[X+Y] = \sum_{\mu \subseteq \lambda} s_\mu[X] s_{\lambda/\mu}[Y]$$

where the sum is over partitions μ whose diagram is contained in λ, and λ/μ is the diagram obtained by removing the cells of μ from λ: First, some subshape μ of λ must be filled with the letters $x_1, x_2, \ldots$, and the remaining shape λ/μ must be filled by the letters $y_1, y_2, \ldots$. This summation formula can be used to define a Schur function recursively.

We can also see from this summation formula that

$$s_\lambda[1-u] = \begin{cases} (-u)^r(1-u) & \text{if } \lambda = (n-r, 1^r) \\ 0 & \text{otherwise} \end{cases}. \tag{2.1}$$

First, we must have

$$s_\lambda[1-u] = \sum_{\mu \subseteq \lambda} s_\mu[1] s_{\lambda/\mu}[-u] = \sum_{\mu \subseteq \lambda} s_\mu[1](-u)^{|\lambda/\mu|} s_{(\lambda/\mu)'}[1].$$

The summation term on the right is nonzero only when μ is a single row and λ/μ is a vertical strip. This happens precisely when $\lambda = (n-r, 1^r)$ is a hook and $\mu = (n-r)$ or $(n-r-1)$ (so that $|\lambda/\mu| = r$ or $r+1$). When $\mu = (n-r)$ we get the term $(-u)^r$. When $\mu = (n-r-1)$, we get the term $(-u)^{r+1}$. Combining these two terms gives the above equation.

As special cases for the Schur function summation formula, we have

$$h_n[X+Y] = \sum_{r=0}^{n} h_r[X] h_{n-r}[Y], \qquad \text{and} \qquad e_n[X+Y] = \sum_{r=0}^{n} e_r[X] e_{n-r}[Y].$$

This can be readily proved by observing that $h_n[X+Y]$ is the sum of all monomials of degree n in $x_1, x_2, \ldots, y_1, y_2, \ldots$. Each such monomial is composed of a monomial say of degree r in $x_1, x_2, \ldots$, and degree $n-r$ in $y_1, y_2, \ldots$. This is precisely the summation formula written above. The same can be said about square-free monomials of degree n, giving the equation on the right.

Proposition 2.2.1 (The Cauchy Identity)**.** *For any two dual bases $\{u_\lambda\}_\lambda$ and $\{v_\lambda\}_\lambda$, and for any two expressions X and Y, we have*

$$h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X] v_\lambda[Y].$$

Proof. First, we see that from the product formula for the power sums

$$h_n[XY] = \sum_{\lambda \vdash n} \frac{p_\lambda[XY]}{z_\lambda} = \sum_{\lambda \vdash n} \frac{p_\lambda[X] p_\lambda[Y]}{z_\lambda}.$$

On the other hand,

$$\begin{aligned} \sum_{\lambda \vdash n} u_\lambda[X] v_\lambda[Y] &= \sum_{\lambda \vdash n} \left(\sum_{\mu \vdash n} \langle u_\lambda, p_\mu \rangle \frac{p_\mu[X]}{z_\mu} \right) v_\lambda[Y] = \sum_{\mu \vdash n} \frac{p_\mu[X]}{z_\mu} \left(\sum_{\lambda \vdash n} \langle u_\lambda, p_\mu \rangle v_\lambda[Y] \right) \\ &= \sum_{\mu \vdash n} \frac{p_\mu[X]}{z_\mu} p_\mu[Y]. \end{aligned}$$

□

We can now for example write

$$\begin{aligned} e_n[X] &= h_n \left[-\epsilon \frac{X}{1-q}(1-q) \right] = \sum_{\mu \vdash n} h_\mu \left[\frac{X}{1-q} \right] m_\mu[-\epsilon(1-q)] \\ &= \sum_{\mu \vdash n} h_\mu \left[\frac{X}{1-q} \right] f_\mu[1-q]. \end{aligned}$$

It will be useful to henceforth write $\alpha \cup \beta = \lambda$ if the parts in the partitions α and β together rearrange to λ.

Proposition 2.2.2. *If $\{u_\lambda\}_\lambda$ is a multiplicative basis (i.e. $u_\alpha u_\beta = u_{\alpha\cup\beta}$) and $\{v_\lambda\}_\lambda$ is its dual basis, then*

$$v_\lambda[X+Y] = \sum_{\alpha\cup\beta=\lambda} v_\alpha[X]v_\beta[Y],$$

where the sum is over all (possibly empty) partitions α and β whose union of parts gives λ.

Proof. For any three expressions X, Y and Z, the Cauchy Identity gives

$$h_n[X(Y+Z)] = \sum_{\lambda\vdash n} u_\lambda[X]v_\lambda[Y+Z].$$

On the other hand, first applying the summation formula, then using the Cauchy Identity

$$\begin{aligned} h_n[XY+XZ] &= \sum_{r=0}^{n} h_r[XY]h_{n-r}[XZ] = \sum_{r=0}^{n}\sum_{\alpha\vdash r} u_\alpha[X]v_\beta[Y] \sum_{\beta\vdash n-r} u_\beta[X]v_\beta[Z] \\ &= \sum_{|\alpha|+|\beta|=n} u_{\alpha\cup\beta}[X]v_\alpha[Y]v_\beta[Z]. \end{aligned}$$

Taking the coefficient of $u_\lambda[X]$ on both sides gives the theorem. □

Applying the summation formula to $v_\lambda = v_\lambda[x_1 + x_2 + \cdots]$, we get the following corollary:

Corollary 2.2.3. *Let $\{v_\lambda\}_\lambda$ be as in the previous proposition. Then*

$$v_\lambda = \sum_{\bigcup_{i\geq 1}\alpha^i=\lambda} x_1^{|\alpha^1|}x_2^{|\alpha^2|}\cdots \times v_{\alpha^1}[1]v_{\alpha^2}[1]\cdots$$

This means that in order to get a monomial expansion of v_λ, we need only know the values of v_λ at the alphabet 1. Two special cases follow: the monomial and forgotten symmetric functions. For the monomial, we note that by the Cauchy identity

$$h_n[X(1)] = \sum_{\mu\vdash n} h_\mu[X]m_\mu[1].$$

Comparing the coefficient of h_μ on both sides, we get

$$m_\mu[1] = \chi(\mu = (|\mu|)).$$

Therefore, applying the corollary, we have

$$m_\lambda = \sum_{\bigcup_{i\geq 1}(a_i)=\lambda} x_1^{a_1} x_2^{a_2} \cdots = \sum_{1\leq i_1<\cdots<i_{\ell(\lambda)}} \sum_{\alpha\in R(\lambda)} x_{i_1}^{\alpha_1} \cdots x_{i_{\ell(\lambda)}}^{\alpha_{\ell(\lambda)}},$$

where the last sum is over all compositions $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\lambda)})$ whose parts rearrange to λ. This is the usual definition of the monomial symmetric functions.

Similarly, we have

$$h_n[X(1)] = \sum_{\mu\vdash n} e_\lambda[X] f_\lambda[1].$$

The value for $f_\lambda[1]$ can be computed by using the Jacobi-Trudi determinant. We do this first to get a value, then prove that indeed this value is correct by using a sign-reversing involution.

The Jacobi-Trudi determinant $s_\lambda = \det(e_{\lambda_i+j-i})_{i,j}$ gives

$$s_n = h_n[X] = \det \begin{pmatrix} e_1 & e_2 & \cdots & \cdots & e_n \\ 1 & e_1 & e_2 & \cdots & e_{n-1} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & e_1 \end{pmatrix}$$

In proving a formula for the number of partitions in a given shape, Amdeberhan and Sergel Leven [52] notice that for any matrix whose elements below the sub-diagonal are 0, every term in the determinant is uniquely determined by a selection of elements along the sub-diagonal. For example, suppose we select the $1's$ in rows $2, 3, 5, 6, 7$ in expanding h_8. We have underlined these

entries in the matrix below.

$$\begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \underline{1} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ 0 & \underline{1} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 0 & 0 & 1 & e_1 & e_2 & e_3 & e_4 & e_5 \\ 0 & 0 & 0 & \underline{1} & e_1 & e_2 & e_3 & e_4 \\ 0 & 0 & 0 & 0 & \underline{1} & e_1 & e_2 & e_3 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & e_1 & e_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & e_1 \end{bmatrix}$$

Proceeding from left to right, we first see that columns 1 and 2 have an entry selected. Column 3 has no 1 selected, so it is then forced to take e_3, the top-right corner of the left-most rectangle. Now all first three rows have an element selected. The next column with no 1 selected is 7. The three previous 1's are selected, so we must take e_4, the top-right corner of the middle rectangle. Lastly, we take e_1 at the bottom. Each block forms a cycle in the permutation which selects these elements in the determinantal formula. We can therefore break up the determinant into squares of sizes $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$ along the main diagonal, each contributing a sign of $(-1)^{\lambda_i - 1}$ and a factor of e_{λ_i}. These squares can be rearranged in any order and thus form a rearrangement of the parts of λ.

From this it follows that $f_\lambda[1]$, the coefficient of e_λ in the Jacobi-Trudi determinant, equals

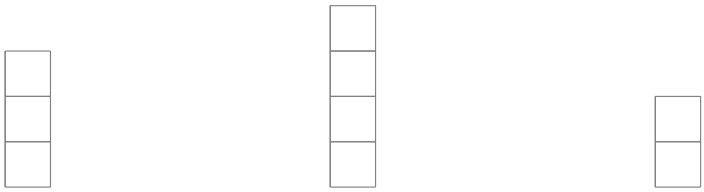
$$f_\lambda[1] = (-1)^{n-\ell(\lambda)} |R(\lambda)|.$$

For completeness we give the following combinatorial proof.

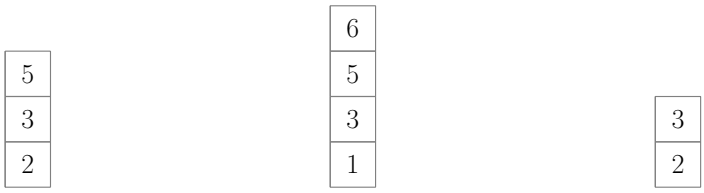
Proposition 2.2.4. *For any nonnegative integer n, we have*

$$h_n = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} |R(\mu)| e_\mu.$$

Proof. We construct a set of of labeled rearrangements $\mathrm{LR}(\mu)$ by first selecting a rearrangement of μ and drawing it as a sequence of columns. For example, if μ is $(4,3,2)$, then we can first select the rearrangement $\rho = (3,4,2)$ and draw it as



We now label the cells with positive integers so that the columns are increasing:



The monomial x^T is computed as with semistandard Young tableaux: $x^T = x_1^{a_1} x_2^{a_2} \cdots$ where a_i is the number of times i occurs as a label in T. The sign is computed by setting

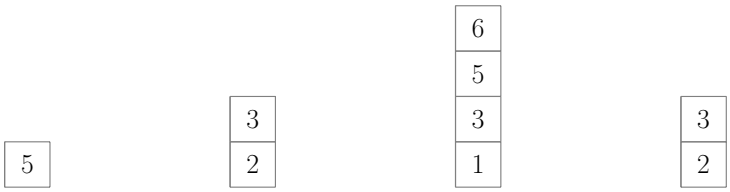
$$\mathrm{sign}(T) = (-1)^{n-\ell(\mu)}.$$

The above example would have a sign of $(-1)^{9-3} = 1$. By the construction, we have

$$\sum_{\mu \vdash n} (-1)^{|\mu|-\ell(\mu)} |R(\mu)| e_\mu = \sum_{\mu \vdash n} \sum_{T \in \mathrm{LR}(\mu)} \mathrm{sign}(T) x^T.$$

A simple sign-reversing involution can now be described. For $T \in \mathrm{LR}(\mu)$, we start at T_1. If T_1 has more than one cell, then remove the top cell of T_1 (along with its label), and place it on the

left of the remaining shape. If T_1 has a single cell with label c_1, and the top cell of T_2 has a label less than c_1, then place the cell with c_1 on top of T_2. For instance, the above example would be sent to



by the first case. The second case would send this sequence of tableaux to the prior example. The number of parts has changed by 1, and so the sign has changed. If neither case holds for T_1, we move by induction to T_2. If every T_i has one cell with label c_i, and $c_i \leq c_{i+1}$, then we leave T fixed. This sign reversing involution leaves as fixed points the set of monomials $x_{c_1} \cdots x_{c_n}$ with $c_1 \leq \cdots \leq c_n$. This is the definition of the homogeneous symmetric function of degree n. We have shown that

$$\sum_{\mu \vdash n} \sum_{T \in \mathrm{LR}(\mu)} \mathrm{sign}(T) x^T = h_n,$$

proving the proposition. □

Now, combining this with Corollory 2.2.3 we get the following combinatorial formula:

$$\begin{aligned} f_\mu[X] = & \sum_{\bigcup_i \lambda^i = \mu} x_1^{|\lambda^1|} x_2^{|\lambda^2|} \cdots \times (-1)^{|\lambda^1| - \ell(\lambda^1)} |R(\lambda^1)| (-1)^{|\lambda^2| - \ell(\lambda^2)} |R(\lambda^2)| \cdots \\ = & (-1)^{|\mu| - \ell(\mu)} \sum_{\bigcup_i \lambda^i = \mu} |R(\lambda^1)| x_1^{|\lambda^1|} |R(\lambda^2)| x_2^{|\lambda^2|} \cdots \end{aligned}$$

where we have used that $|\lambda^1| + |\lambda^2| + \cdots = |\mu|$ and $\ell(\lambda^1) + \ell(\lambda^2) + \cdots = \ell(\mu)$. Now instead of multiplying each x_i by $|R(\lambda^i)|$, we can sum over compositions. We have thus provided combinatorial formulas for the monomial and forgotten symmetric functions:

Theorem 2.2.5. *For a partition $\mu \vdash n$ of length $\ell(\mu) = \ell$, the combinatorial formula for the*

monomial and forgotten symmetric functions is given by

$$m_\mu = \sum_{1\le i_1<\cdots<i_\ell} \sum_{\alpha\in R(\mu)} x_{i_1}^{\alpha_1}\cdots x_{i_\ell}^{\alpha_\ell}$$

$$f_\mu = (-1)^{n-\ell} \sum_{1\le i_1\le\cdots\le i_\ell} \sum_{\alpha\in R(\mu)} x_{i_1}^{\alpha_1}\cdots x_{i_\ell}^{\alpha_\ell}$$

For instance, if $\lambda = (1,2)$, we have

$$f_{(2,1)}[x_1+x_2] = (-1)^{3-2}\left(x_1^{|(2,1)|} + x_1^{|(1,2)|} + x_2^{|(2,1)|} + x_2^{|(1,2)|} + x_1^{|(1)|}x_2^{|(2)|} + x_1^{|(2)|}x_2^{|(1)|}\right)$$

$$= -(2x_1^3 + 2x_2^3 + x_1x_2^2 + x_1^2x_2)$$

We should note here that a combinatorial formula for the forgotten basis is given in [49] in terms of brick tabloids, a useful device which combines the selection $i_1 \le \cdots \le i_\ell$ and $\alpha \in R(\mu)$: Let $\mathrm{CR}(\mu,\lambda)$ be the set of sequences $(\alpha^1,\ldots,\alpha^{\ell(\lambda)})$ of compositions such that α^i is a composition of λ_i and the union of parts $\bigcup_i \alpha^i$ rearranges to μ. Then this set precisely describes the set of brick tabloids of shape λ and content μ. We have from Theorem 2.2.5 that

$$f_\mu = (-1)^{n-\ell} \sum_{\lambda\vdash n} |\,\mathrm{CR}(\mu,\lambda)| \cdot m_\lambda$$

2.3 Modified Macdonald polynomials

In this section, we collect some of the important symmetric function identities and properties surrounding the modified Macdonald polynomials, first introduced in [12]. Some useful references are [1], which gives a "Macdonald polynomial toolkit", and [27], [15] where many important discoveries surrounding these symmetric functions are found.

The modified Macdonald symmetric function basis $\{\widetilde{\mathrm{H}}_\mu[X;q,t]\}_\mu$ can be defined by the

triangularity relations

$$\widetilde{\mathrm{H}}_\mu[X;q,t] = \sum_{\lambda \geq \mu'} a_{\lambda,\mu}(q,t) s_\lambda \left[\frac{X}{1-t}\right], \qquad \widetilde{\mathrm{H}}_\mu[X;q,t] = \sum_{\lambda \geq \mu} b_{\lambda,\mu}(q,t) s_\lambda \left[\frac{X}{1-q}\right],$$

for some $a_{\lambda,\mu}, b_{\lambda,\mu} \in \mathbb{Q}(q,t)$, and the functional relation $\langle \widetilde{\mathrm{H}}_\mu, h_n \rangle = 1$. Note that if we interchanged q and t, then the roles of μ and μ' in the triangularities are flipped. In other words, switching q and t gives the definition of $\widetilde{\mathrm{H}}_{\mu'}$. Therefore

Proposition 2.3.1. *For any partition μ, we have*

$$\widetilde{\mathrm{H}}_\mu[X;t,q] = \widetilde{\mathrm{H}}_{\mu'}[X;q,t].$$

As a consequence of Macdonald-Koorwinder reciprocity (Theorem 2.3.3) we will show that $\langle \widetilde{\mathrm{H}}_\mu, e_k h_{n-k} \rangle = e_k[B_\mu]$ (see Equation (2.4)). For $k = n$, we have

$$\langle \widetilde{\mathrm{H}}_\mu, e_n \rangle = T_\mu = \prod_{(i,j) \in \mu} q^i t^j.$$

Using this, we can now describe what happens when q and t are replaced by their inverses:

Proposition 2.3.2. *For any partition μ,*

$$T_\mu \omega \widetilde{\mathrm{H}}_\mu[X;1/q,1/t] = \widetilde{\mathrm{H}}_\mu[X;q,t].$$

Proof. We must show that the function on the left-hand side satisfies the same defining relations. Note that since

$$\omega s_\lambda \left[\frac{X}{1-1/t}\right] = \omega s_\lambda \left[t\frac{X}{t-1}\right] = (-t)^{|\lambda|} \omega s_{\lambda'} \left[\frac{X}{1-t}\right] = (-t)^{|\lambda|} s_\lambda \left[\frac{X}{1-t}\right],$$

the first triangularity is satisfied by $\omega \widetilde{\mathrm{H}}_\mu[X;1/q,1/t]$. Similarly, the other triangularity holds as

well. Lastly,

$$\langle \omega\, \widetilde{\mathrm{H}}_\mu[X;1/q,1/t], h_n\rangle = \langle \widetilde{\mathrm{H}}_\mu[X;1/q,1/t], e_n\rangle = T_\mu^{-1}.$$

□

A useful, equivalent definition can be made by replacing one of the triangularity relations with the following orthogonality relation:

The $*$-scalar product is defined by setting

$$\langle p_\lambda, p_\mu\rangle_* = z_\mu \chi(\lambda=\mu)(-1)^{|\mu|-\ell(\mu)} \prod_{i=1}^{\ell(\mu)} (1-q^{\mu_i})(1-t^{\mu_i}) = z_\mu \chi(\lambda=\mu) p_\mu[-\epsilon M].$$

with the usual convention of setting $M=(1-t)(1-q)$. The orthogonality relations for the modified Macdonald basis are given by

$$\langle \widetilde{\mathrm{H}}_\mu, \widetilde{\mathrm{H}}_\lambda\rangle_* = \chi(\lambda=\mu) w_\mu \qquad \text{with} \qquad w_\mu(q,t) = \prod_{c\in\mu} \left(q^{\mathrm{arm}(c)} - t^{\mathrm{leg}(c)+1}\right)\left(t^{\mathrm{leg}(c)} - q^{\mathrm{arm}(c)+1}\right).$$

We will also use the usual notation, setting for any partition $\mu \vdash n$

$$B_\mu(q,t) = \sum_{(i,j)\in\mu} q^i t^j \qquad \text{and} \qquad \Pi_\mu(q,t) = \prod_{(i,j)\in\mu/(1)} (1-q^i t^j).$$

Note that we then also have

$$T_\mu = e_n[B_\mu] \qquad \text{and} \qquad \Pi_\mu(q,t) = h_n[2-B_\mu].$$

From here, we can start writing classical symmetric functions in terms of the modified Macdonald basis. First, we have to translate the $*$-scalar product to the Hall scalar product. Note that on the power basis we have

$$\langle p_\lambda, p_\mu\rangle_* = \langle p_\lambda[X], p_\mu[X]\rangle (-1)^{|\mu|-\ell(\mu)} p_\mu[M] = \langle p_\lambda[X], p_\mu[-\epsilon M X]\rangle.$$

So if we define the morphisms ϕ and $*$ by setting $\phi F[X] = F[MX]$ and $\phi^{-1} = *$, we have that for any two symmetric functions F and G,

$$\langle F, G\rangle_* = \langle F, \omega\phi G\rangle \qquad \text{and} \qquad \langle F, G\rangle = \langle F, \omega * G\rangle_*$$

(It is customary to write $*F = F^*$.) This means that under the ordinary scalar product, $\widetilde{\mathrm{H}}_\mu$ is dual to $\omega\phi\,\widetilde{\mathrm{H}}_\mu\,/w_\mu$. It follows from Theorem 2.2.1 that

$$h_n\left[-\epsilon\frac{X}{M}Y\right] = \sum_{\mu\vdash n}\frac{\omega\phi\,\widetilde{\mathrm{H}}_\mu[-\epsilon X/M]\,\widetilde{\mathrm{H}}_\mu[Y]}{w_\mu} = \sum_{\mu\vdash n}\frac{\widetilde{\mathrm{H}}_\mu[X]\,\widetilde{\mathrm{H}}_\mu[Y]}{w_\mu}.$$

Therefore

$$\begin{aligned} e_n\left[\frac{XY}{M}\right] &= \sum_{\mu\vdash n}\frac{\widetilde{\mathrm{H}}_\mu[X]\,\widetilde{\mathrm{H}}_\mu[Y]}{w_\mu} \\ &= \sum_{\mu\vdash n}(-1)^{|\mu|-\ell(\mu)}\frac{p_\mu}{z_\mu}\left[\frac{XY}{M}\right] = \sum_{\mu\vdash n}(-1)^{|\mu|-\ell(\mu)}\frac{p_\mu[X]p_\mu[Y]}{z_\mu p_\mu[M]}. \end{aligned}$$

Setting $Y = M$ we have

$$e_n = \sum_{\mu\vdash n}\frac{\widetilde{\mathrm{H}}_\mu[M]}{w_\mu}\,\widetilde{\mathrm{H}}_\mu\,.$$

One of the most important identities involving Macdonald polynomials is called Macdonald-Koorwinder Reciprocity [32], [15]. It will give us some of the essential expansions of classical bases in terms of the modified Macdonald polynomials and, in particular, the evaluation $\widetilde{\mathrm{H}}_\mu[M]$.

Theorem 2.3.3 (Macdonald-Koorwinder Reciprocity)**.** *For any two partitions α and β*

$$\frac{\widetilde{\mathrm{H}}_\alpha\left[1+u\left(MB_\beta-1\right)\right]}{\prod_{(i,j)\in\alpha}1-uq^it^j} = \frac{\widetilde{\mathrm{H}}_\beta\left[1+u\left(MB_\alpha-1\right)\right]}{\prod_{(i,j)\in\beta}1-uq^it^j}$$

Some special cases follow:

1. Multiplying both sides by $(1-u)$ and sending u to 1, we get

$$\frac{\widetilde{\mathrm{H}}_\alpha[MB_\beta]}{\Pi_\alpha} = \frac{\widetilde{\mathrm{H}}_\beta[MB_\alpha]}{\Pi_\beta}.$$

We will refer to this identity as Macdonald reciprocity. When $\beta = (1)$, we have $\widetilde{\mathrm{H}}_\alpha[M] = MB_\alpha\Pi_\alpha$. The expansion of e_n in terms of the modified Macdonald basis can now be written as

$$e_n = \sum_{\mu\vdash n} \frac{MB_\mu\Pi_\mu}{w_\mu}\,\widetilde{\mathrm{H}}_\mu\,. \tag{2.2}$$

2. On the other hand, replacing u by $1/u$ would give on the left

$$\frac{\widetilde{\mathrm{H}}_\alpha\left[1+\frac{1}{u}\left(MB_\beta-1\right)\right]}{\prod_{(i,j)\in\alpha} 1-q^it^j/u} = u^n\frac{\widetilde{\mathrm{H}}_\alpha\left[1+\frac{1}{u}\left(MB_\beta-1\right)\right]}{\prod_{(i,j)\in\alpha} u-q^it^j} = \frac{\widetilde{\mathrm{H}}_\alpha\left[u+\left(MB_\beta-1\right)\right]}{\prod_{(i,j)\in\alpha} u-q^it^j}.$$

Setting $u=0$ would then give

$$(-1)^{|\alpha|}\frac{\widetilde{\mathrm{H}}_\alpha[MB_\beta-1]}{T_\alpha} = (-1)^{|\beta|}\frac{\widetilde{\mathrm{H}}_\beta[MB_\alpha-1]}{T_\beta}.$$

Since $\widetilde{\mathrm{H}}_0 = 1$ and $MB_0 - 1 = -1$, we can take $\beta = 0$ to get

$$\begin{aligned}\widetilde{\mathrm{H}}_\alpha[1-u] &= \prod_{(i,j)\in\alpha} 1-uq^it^j &&= (1-u)\prod_{(i,j)\in\alpha/(1)}(1-uq^it^j)\\ &= (1-u)\sum_{r\geq 0}(-u)^r e_r[B_\alpha-1] &&= \sum_{r=0}^{n-1}(1-u)(-u)^r e_r[B_\alpha-1].\end{aligned} \tag{2.3}$$

Now, expanding the modified Macdonald polynomial in terms of Schur functions, we get as change of bases constants $\widetilde{\mathrm{K}}_{\lambda,\mu}(q,t)\in\mathbb{Q}(q,t)$, the q,t-Kostka:

$$\widetilde{\mathrm{H}}_\mu[X;q,t] = \sum_{\lambda\vdash n} s_\mu[X]\,\widetilde{\mathrm{K}}_{\lambda,\mu}(q,t).$$

It turns out these are polynomials in q and t (see [10]) with positive integer coefficients (from

[12] and [29]). We can get an explicit formula for $\widetilde{\mathrm{K}}_{\lambda,\mu}$ when λ is a hook by noticing that from Equation (2.1) we have

$$\widetilde{\mathrm{H}}_\mu[1-u;q,t] \quad = \quad \sum_{\lambda\vdash n} s_\mu[1-u]\,\widetilde{\mathrm{K}}_{\lambda,\mu}(q,t) \quad = \quad \sum_{r=0}^{n-1}(-u)^r(1-u)\,\widetilde{\mathrm{K}}_{(n-r,1^r),\mu}(q,t).$$

Comparing this expansion with that of Equation (2.3) gives us

$$\widetilde{\mathrm{K}}_{(n-r,1^r),\mu}(q,t) = e_r[B_\mu - 1],$$

and also

$$\begin{aligned}\langle \widetilde{\mathrm{H}}_\mu, e_r h_{n-r}\rangle &= \langle \widetilde{\mathrm{H}}_\mu, s_{(n-r,1^r)} + s_{(n-r+1,1^{r-1})}\rangle \\ &= e_r[B_\mu - 1] + e_{r-1}[B_\mu - 1] = e_r[B_\mu]. \end{aligned} \tag{2.4}$$

Proposition 2.3.4. *We have*

1. $e_n\left[\frac{X}{M}\right] = \sum_{\mu\vdash n} \frac{\widetilde{\mathrm{H}}_\mu[X]}{w_\mu}$
2. $h_n\left[\frac{X}{M}\right] = \sum_{\mu\vdash n} \frac{T_\mu\,\widetilde{\mathrm{H}}_\mu[X]}{w_\mu}$
3. $h_k\left[\frac{X}{M}\right] e_{n-k}\left[\frac{X}{M}\right] = \sum_{\mu\vdash n} \frac{e_k[B_\mu]\,\widetilde{\mathrm{H}}_\mu[X]}{w_\mu}$
4. $(-1)^{n-1}p_n = (1-t^n)(1-q^n)\sum_{\mu\vdash n} \frac{\widetilde{\mathrm{H}}_\mu[X]\Pi_\mu}{w_\mu}$

Proof. Note that the first two items are specializations of the third: the first is when $k=0$ and the second when $k=n$. To show item 3, we first note that for any symmetric function F, we have

$$F^* = F\left[\frac{X}{M}\right] = \sum_{\mu\vdash n}\frac{\widetilde{\mathrm{H}}_\mu}{w_\mu}\langle \widetilde{\mathrm{H}}_\mu, F^*\rangle_* = \sum_{\mu\vdash n}\frac{\widetilde{\mathrm{H}}_\mu}{w_\mu}\langle \widetilde{\mathrm{H}}_\mu, \omega F\rangle.$$

We then set $F = h_k e_{n-k}$, and use that $\langle \widetilde{\mathrm{H}}_\mu, \omega(h_k e_{n-k})\rangle = \langle \widetilde{\mathrm{H}}_\mu, e_k h_{n-k}\rangle = e_k[B_\mu]$.

For the last equality, we note that since

$$\widetilde{\mathrm{H}}_\mu[1-u] = \prod_{(i,j)\in\mu} 1 - uq^i t^j,$$

we have

$$e_n\left[\frac{X(1-u)}{M}\right] = \sum_{\mu\vdash n} \frac{\widetilde{\mathrm{H}}_\mu[X]\,\widetilde{\mathrm{H}}_\mu[1-u]}{w_\mu} = \sum_{\mu\vdash n} \frac{\widetilde{\mathrm{H}}_\mu[X]}{w_\mu} \prod_{(i,j)\in\mu} 1 - uq^i t^j.$$

From

$$e_n = \sum_{\mu\vdash n} (-1)^{n-\ell(\mu)} \frac{p_\mu}{z_\mu},$$

it follows that

$$e_n\left[\frac{X(1-u)}{M}\right] = \sum_{\mu\vdash n} (-1)^{n-\ell(\mu)} \frac{p_\mu[X]p_\mu[1-u]}{p_\mu[M]z_\mu}.$$

When μ has more than one part, $p_\mu[1-u] = (1-u^{\mu_1})(1-u^{\mu_2})\cdots(1-u^{\mu_{\ell(\mu)}})$ is divisible by $(1-u)^2$. Therefore,

$$e_n\left[\frac{X(1-u)}{M}\right] = (-1)^{n-1}\frac{1-u^n}{(1-t^n)(1-q^n)n}p_n[X] + (1-u)^2 R$$

for some polynomial R in u. Equating the two expressions for $e_n[X(1-u)/M]$ and dividing by $(1-u)$ gives

$$\sum_{\mu\vdash n} \frac{\widetilde{\mathrm{H}}_\mu[X]}{w_\mu} \prod_{(i,j)\in\mu/(1)} 1 - uq^i t^j = (-1)^{n-1}\frac{1+u+\cdots+u^{n-1}}{(1-t^n)(1-q^n)n}p_n[X] + (1-u)R.$$

Setting $u=1$ then gives

$$\sum_{\mu\vdash n} \frac{\widetilde{\mathrm{H}}_\mu[X]}{w_\mu}\Pi_\mu = (-1)^{n-1}\frac{1}{(1-t^n)(1-q^n)}p_n[X],$$

which is the last item. □

The Delta eigenoperators of the modified Macdonald basis were first introduced in [27] and

are defined by setting for any symmetric function F,

$$\Delta_F \widetilde{\mathrm{H}}_\mu = F[B_\mu]\, \widetilde{\mathrm{H}}_\mu \,.$$

Note that for any F, G and constant c, we have

$$c\Delta_F \Delta_G = \Delta_{cFG} \qquad \text{and} \qquad \Delta_F + \Delta_G = \Delta_{F+G}.$$

The following is a list of some simple ways of writing what we have seen so far utilizing the Delta operators.

Proposition 2.3.5. *Let* $\boldsymbol{\alpha}_n = (-1)^{n-1}\frac{M}{(1-q^n)(1-t^n)}$. *Then*

1. $e_n = \Delta_{e_1}\, \boldsymbol{\alpha}_n\, p_n$
2. $h_n e_{n-k} = \phi \Delta_{e_k} e_n^*$
3. $(-qt)^{n-1} h_n = \Delta_{e_n}^{-1} \Delta_{e_{n-1}}\, \boldsymbol{\alpha}_n\, p_n$
4. *If for a given symmetric function* F *there exists a* G *such that* $\langle \widetilde{\mathrm{H}}_\mu, F\rangle = G[B_\mu]$, *then* $\langle \widetilde{\mathrm{H}}_\mu, F\rangle = \langle \Delta_G\, \widetilde{\mathrm{H}}_\mu, h_n\rangle$. *In particular, for any symmetric function* P, *we have*

$$\langle \Delta_{e_k} P, h_n\rangle = \langle P, e_k h_{n-k}\rangle.$$

The following identity is remarkably useful and surprising. It was first found by Jim Haglund and has many consequences. One such consequence is the first identity in Proposition 2.3.7 (previously referred to as Haglund's Identity in [44]). The proofs that follow can be found in a recent submission [13].

Theorem 2.3.6 ([13])**.** *For any two symmetric functions* P *and* Q *of homogeneous degrees* k *and* n *respectively, we have*

$$\langle \Delta_{\omega P}\, \boldsymbol{\alpha}_n\, p_n, Q\rangle = \langle \Delta_{\omega Q}\, \boldsymbol{\alpha}_k\, p_k, P\rangle.$$

Proof. It suffices to prove this over a basis. We will assume $P = \omega\phi\,\widetilde{\mathrm{H}}_\alpha$ and $Q = \omega\phi\,\widetilde{\mathrm{H}}_\beta$ for partitions $\alpha \vdash k$ and $\beta \vdash n$. Since (using item 4 of Proposition 2.3.4 and the definition of Δ_F)

$$\Delta_{\omega P}\,\boldsymbol{\alpha}_n\,p_n = \sum_{\mu\vdash n} \frac{M\Pi_\mu}{w_\mu}(\omega P)[B_\mu]\,\widetilde{\mathrm{H}}_\mu,$$

we have

$$\begin{aligned}\langle \Delta_{\omega P}\,\boldsymbol{\alpha}_n\,p_n, Q\rangle &= \sum_{\mu\vdash n} \frac{M\Pi_\mu}{w_\mu}(\omega P)[B_\mu]\langle \widetilde{\mathrm{H}}_\mu, \omega\phi\,\widetilde{\mathrm{H}}_\beta\rangle \\ &= \sum_{\mu\vdash n} \frac{M\Pi_\mu}{w_\mu}\,\widetilde{\mathrm{H}}_\alpha[MB_\mu]\langle \widetilde{\mathrm{H}}_\mu, \widetilde{\mathrm{H}}_\beta\rangle_* \\ &= M\Pi_\beta\,\widetilde{\mathrm{H}}_\alpha[MB_\beta].\end{aligned}$$

Switching the places of α and β we get $\langle \Delta_{\omega Q}\,\boldsymbol{\alpha}_k\,p_k, P\rangle = M\Pi_\alpha\,\widetilde{\mathrm{H}}_\beta[MB_\alpha]$. Therfore, the theorem holds true if and only if

$$\Pi_\beta\,\widetilde{\mathrm{H}}_\alpha[MB_\beta] = \Pi_\alpha\,\widetilde{\mathrm{H}}_\beta[MB_\alpha],$$

which is another way of writing Macdonald reciprocity. □

Using Proposition 2.3.5, we get as a consequence several inner product identities. The first identity of the following proposition first appears in Haglund's q,t-Schröder paper [33].

Proposition 2.3.7 ([33], [13])**.** *For any $k, n \geq 0$ and Q homogeneous of degree n, we have*

$$\langle \Delta_{e_k} e_n, Q\rangle = \langle \Delta_{\omega Q} e_{k+1}, h_{k+1}\rangle \qquad \textit{and} \qquad \langle \Delta_{h_k} e_n, Q\rangle = (-qt)^k \langle \Delta_{\omega Q} h_{k+1}, e_{k+1}\rangle$$

Proof. Let $P = e_1 h_k$ in the Theorem 2.3.6. Then by using the first item of Proposition 2.3.5, the left-hand side of the first equation becomes $\langle \Delta_{e_k} e_n, Q\rangle = \langle \Delta_{e_k}\Delta_{e_1}\,\boldsymbol{\alpha}_n\,p_n, Q\rangle$. Using the previous theorem, this should equal

$$\langle \Delta_{\omega Q}\,\boldsymbol{\alpha}_{k+1}\,p_{k+1}, e_1 h_k\rangle = \langle \Delta_{e_1}\Delta_{\omega Q}\,\boldsymbol{\alpha}_{k+1}\,p_{k+1}, h_{k+1}\rangle = \langle \Delta_{\omega Q} e_{k+1}, h_{k+1}\rangle.$$

A similar process gives the equation on the right by using the third item of Proposition 2.3.5. □

The first identity of Proposition 2.3.7 is going to be used in the following chapter. It will help us solve the Delta conjecture at $q = 1$, as in [44].

The following inner product identities are quite significant. They suggest some amazing properties of not only symmetric functions, but also the representation theory of the symmetric group surrounding the Delta operators.

Proposition 2.3.8. *For any symmetric functions Q and F, F homogeneous of degree $n - r$,*

$$(\omega F)^{\perp} \Delta_Q \boldsymbol{\alpha}_n \, p_n = \Delta_{\phi \Delta_F Q^*} \boldsymbol{\alpha}_r \, p_r.$$

Proof. Since we have

$$\langle \widetilde{H}_\mu, \widetilde{H}_\lambda[-\epsilon MX] \rangle = w_\mu \chi(\lambda = \mu)$$

and

$$\begin{aligned} \langle \Delta_F \widetilde{H}_\mu, \widetilde{H}_\lambda[-\epsilon MX] \rangle = w_\mu \chi(\lambda = \mu) F[B_\mu] &= \langle \widetilde{H}_\mu, \left(\Delta_F \widetilde{H}_\lambda \right)[-\epsilon MX] \rangle \\ &= \langle \widetilde{H}_\mu, \omega \phi \Delta_F * \omega \widetilde{H}_\lambda[-\epsilon MX] \rangle. \end{aligned}$$

we have that for any three symmetric functions F, G and H,

$$\langle \Delta_F G, H \rangle = \langle G, \omega \phi \Delta_F * \omega H \rangle.$$

Therefore, if Q is of degree k and P is any symmetric function of degree r,

$$\begin{aligned}
\langle \omega(F)^{\perp} \Delta_Q \, \boldsymbol{\alpha}_n \, p_n, P \rangle &= \langle \Delta_Q \, \boldsymbol{\alpha}_n \, p_n, \omega(F) P \rangle \\
\text{(by Theorem (2.3.6))} \quad &= \langle \Delta_{F\omega(P)} \, \boldsymbol{\alpha}_k \, p_k, \omega(Q) \rangle \\
\text{(by the above observation)} \quad &= \langle \Delta_{\omega(P)} \, \boldsymbol{\alpha}_k \, p_k, \omega\phi\Delta_F * \omega\omega(Q) \rangle \\
&= \langle \Delta_{\omega(P)} \, \boldsymbol{\alpha}_k \, p_k, \omega\phi\Delta_F * (Q) \rangle \\
\text{(by Theorem (2.3.6))} \quad &= \langle \Delta_{\omega\omega\phi\Delta_F * (Q)} \, \boldsymbol{\alpha}_r \, p_r, P \rangle \\
&= \langle \Delta_{\phi\Delta_F * (Q)} \, \boldsymbol{\alpha}_r \, p_r, P \rangle.
\end{aligned}$$

Since P is arbitrary, the two symmetric functions must be equal. This proves the theorem. □

For example, suppose we take $Q = e_k$ and $F = e_r$, then

$$h_r^{\perp} \Delta_{e_k} \, \boldsymbol{\alpha}_n \, p_n = \Delta_{\phi\Delta^*_{e_r} e_k} \, \boldsymbol{\alpha}_r \, p_r.$$

Since from Proposition 2.3.4

$$\Delta_{e_r} e_k \left[\frac{X}{M} \right] = h_r \left[\frac{X}{M} \right] e_{k-r} \left[\frac{X}{M} \right],$$

we have

$$h_{n-r}{}^{\perp} \Delta_{e_k} \, \boldsymbol{\alpha}_n \, p_n = \Delta_{\phi^* h_r e_{k-r}} \, \boldsymbol{\alpha}_r \, p_r,$$

or instead

Corollary 2.3.9. *For any r, k and n, we have*

$$h_r^{\perp} \Delta_{e_k} \, \boldsymbol{\alpha}_n \, p_n = \Delta_{h_r e_{k-r}} \, \boldsymbol{\alpha}_{n-r} \, p_{n-r}.$$

In particular, taking $r = 1$ we get

$$e_1^{\perp} \Delta_{e_k} \boldsymbol{\alpha}_n \, p_n = \Delta_{e_{k-1}} e_{n-1}.$$

The amazing representation theoretical implications of this identity are best described by rewriting the last identity of the corollary as

$$e_1^{\perp} \Delta_{e_k} (-1)^{n-1} p_n = [n]_q [n]_t \Delta_{e_{k-1}} e_{n-1}.$$

Suppose for a moment that we have modules $M^{n,k}, N^{n,k}$ so that $\mathbf{F}(M^{n,k}) = \Delta_{e_k}(-1)^{n-1} p_n$ and $\mathbf{F}(N^{n,k}) = \Delta_{e_k} e_n$. Then this equation says that

$$M^{n,k} \Big\downarrow_{S_{n-1}}^{S_n} \simeq \bigoplus_{r,s=1}^{n} N^{n-1,k-1}.$$

In other words, restricting $M^{n,k}$ to the action of S_{n-1} gives n^2 copies of $N^{n-1,k-1}$. If $k = n$, then this says that the module corresponding to $\nabla(-1)^{n-1} p_n$, when restricted to the action of S_{n-1} contains n^2 copies of the space of diagonal harmonics DH_{n-1}.

2.4 Delta operators at $q = 1$

To study the Delta operators at $q = 1$, we are going to need some notation. We will define, for any symmetric functions F and P,

$$\widetilde{\Delta}_F \, P = \Delta_F P \Big|_{q=1}.$$

Proposition 2.4.1. *Let $(q;t)_r = (1-t)(1-qt)\cdots(1-q^{r-1}t)$ be the usual Pochhammer symbol and let $(t;t)_\mu = (t;t)_{\mu_1} \cdots (t;t)_{\mu_{\ell(\mu)}}$. For any partition μ,*

$$\widetilde{\mathrm{H}}_\mu[X;1,t] = (t;t)_{\mu'} h_{\mu'}\left[\frac{X}{1-t}\right].$$

Proof. Following Macdonald's section on Integral Forms on page 364 of [32], we have the equality

$$J_\mu[X;1,t] = \prod_{i=1}^{\ell(\mu')} (t;t)_{\mu'_i} e_{\mu'_i}[X] = (t;t)_{\mu'} e_{\mu'}.$$

The modified Macdonald polynomials of [12] arise from J_μ by setting

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-1/t};q,1/t\right],$$

where $n(\mu) = \sum_{i=1}^{\ell(\mu)}(i-1)\mu_i = \prod_{i=1}^{\ell(\mu')}\binom{\mu'_i}{2}$. Thus, we have

$$\begin{aligned}
\tilde{H}_\mu[X;1,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-1/t};1,1/t\right] &= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} (1/t:1/t)_{\mu'_i} e_{\mu'_i}\left[\frac{X}{1-1/t}\right] \\
&= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} t^{-(1+2+\cdots+\mu'_i)} (-1)^{\mu'_i} (t:t)_{\mu'_i} e_{\mu'_i}\left[-t\frac{X}{1-t}\right] \\
&= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} t^{-(1+2+\cdots+\mu'_i)} (-1)^{\mu'_i} (t:t)_{\mu'_i} (-t)^{\mu'_i} h_{\mu'_i}\left[\frac{X}{1-t}\right] \\
&= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} t^{-\binom{\mu'_i}{2}} (t:t)_{\mu'_i} h_{\mu'_i}\left[\frac{X}{1-t}\right],
\end{aligned}$$

We get the equality

$$\tilde{H}_\mu[X;1,t] = \prod_{i=1}^{\ell(\mu')} (t:t)_{\mu'_i} h_{\mu'_i}\left[\frac{X}{1-t}\right].$$

□

Corollary 2.4.2. *For any symmetric function F, $\widetilde{\Delta}_F$ is the linear operator defined by setting*

$$\widetilde{\Delta}_F\, h_\mu\left[\frac{X}{1-t}\right] = F\left[\sum_{i=1}^{\ell(\mu)} [\mu_i]_t\right] h_\mu\left[\frac{X}{1-t}\right]$$

Proof. Note that over the basis $\{s_\lambda\}$ we have

$$(\Delta_F(s_\lambda)|_{q=1})\Big|_{q=1} = (\Delta_F s_\lambda)\Big|_{q=1} = \widetilde{\Delta}_F\, s_\lambda.$$

By the previous proposition, we evaluate inside first to get

$$\left(\Delta_F(\widetilde{\mathrm{H}}_{\mu'})|_{q=1}\right)\Big|_{q=1} = \widetilde{\Delta}_F(t;t)_\mu h_\mu\left[\frac{X}{1-t}\right].$$

Instead, we can expand in terms of Schur functions first, then evaluate q at 1 after applying the operator:

$$\begin{aligned}\left(\Delta_F(\widetilde{\mathrm{H}}_{\mu'})|_{q=1}\right)\Big|_{q=1} = \widetilde{\Delta}_F\,\widetilde{\mathrm{H}}_{\mu'} &= F[B_{\mu'}]\,\widetilde{\mathrm{H}}_{\mu'}\Big|_{q=1} \\ &= (t;t)_\mu F\left[\sum_{i=1}^{\ell(\mu)}[\mu_i]_t\right] h_\mu\left[\frac{X}{1-t}\right].\end{aligned}$$

The last equality uses the fact that

$$B_{\mu'}(1,t) = \sum_{c\in\mu'} t^{\mathrm{leg}(c)} = \sum_{c\in\mu} t^{\mathrm{arm}(c)} = \sum_{i=1}^{\ell(\mu)}[\mu_i]_t.$$

Putting the two together gives

$$\widetilde{\Delta}_F\, h_\mu\left[\frac{X}{1-t}\right] = F\left[\sum_i[\mu_i]_t\right] h_\mu\left[\frac{X}{1-t}\right].$$

The collection $\{h_\mu[X/(1-t)]\}_\mu$ forms a basis (since the transformation $X \to X/(1-t)$ is invertible and $\{h_\mu\}_\mu$ is a basis). This uniquely determines the operator $\widetilde{\Delta}_F$.

□

In order to compute $\widetilde{\Delta}_F\, P$ for a given F and P, we will start by expanding P in terms of the basis $\{h_\mu[X/(1-t)]\}_\mu$. After applying $\widetilde{\Delta}_F$, we will then expand $h_\mu[X/(1-t)]$ into another

given basis, mostly the elementary basis. Recall that the Cauchy Identity gives

$$h_n\left[\frac{X}{1-t}\right] = \sum_{\mu\vdash n} e_\mu[X] f_\mu\left[\frac{1}{1-t}\right].$$

Then

$$h_\mu\left[\frac{X}{1-t}\right] = \prod_{i=1}^{\ell(\mu)} \sum_{\nu^i\vdash\mu_i} e_{\nu^i}[X] f_{\nu^i}\left[\frac{1}{1-t}\right].$$

We will denote by $\mathrm{PR}(\lambda,\mu)$ the set of sequences of partitions $(\nu^1,\ldots,\nu^{\ell(\mu)})$ satisfying $\nu^i\vdash\mu_i$ for all i and $\cup_i\nu^i=\lambda$. We can also define for compositions α,β the set $\mathrm{PR}(\alpha,\beta)$ just as we did for the partitions. We will adopt the notation of using a vector $\vec{\nu}=(\nu^1,\ldots,\nu^{\ell(\vec{\nu})})$ for a sequence of partitions, and for a family of symmetric functions $\{u_\lambda\}_\lambda$ indexed by partitions, we set $u_{\vec{\nu}} = u_{\nu^1}\cdots u_{\nu^{\ell(\vec{\gamma})}}$. Then

$$\begin{aligned} h_\mu\left[\frac{X}{1-t}\right] &= \sum_{\lambda\vdash n} e_\lambda[X] \sum_{(\nu^1,\ldots,\nu^{\ell(\mu)})\in\mathrm{PR}(\lambda,\mu)} f_{\nu^1}\left[\frac{1}{1-q}\right]\cdots f_{\nu^{\ell(\mu)}}\left[\frac{1}{1-q}\right] \\ &= \sum_{\lambda\vdash n} e_\lambda[X] \sum_{\vec{\nu}\in\mathrm{PR}(\lambda,\mu)} f_{\vec{\nu}}\left[\frac{1}{1-q}\right]. \end{aligned} \tag{2.5}$$

2.5 Delta operators and q,t-symmetry

All of our results for when $q=1$ can also be written for when $t=1$. This is a simple consequence of the fact that most of our symmetric functions are symmetric in q and t. More precisely,

Theorem 2.5.1. *Let $F[X;q,t]$ and $G[X;q,t]$ be symmetric functions that are symmetric in q and t. Then $\Delta_F G$ is also symmetric in q and t.*

Proof. We recall first that

$$\widetilde{\mathrm{H}}_\mu[X;t,q] = \widetilde{\mathrm{H}}_{\mu'}[X;q,t] \qquad \text{and} \qquad B_\mu(t,q) = B_{\mu'}(q,t).$$

Suppose we have the expansion

$$G[X;q,t] = \sum_{\mu\vdash n} \widetilde{\mathrm{H}}_\mu[X;q,t]C_\mu(q,t).$$

The symmetry of G implies that

$$\begin{aligned} G[X;t,q] &= \sum_{\mu\vdash n} \widetilde{\mathrm{H}}_\mu[X;t,q]C_\mu(t,q) \\ &= \sum_{\mu\vdash n} \widetilde{\mathrm{H}}_{\mu'}[X;q,t]C_\mu(t,q) \\ &= G[X;q,t] \end{aligned}$$

meaning $C_\mu(t,q) = C_{\mu'}(q,t)$. Applying a Delta operator, and using the fact that $F[X;q,t] = F[X;t,q]$, we have

$$\begin{aligned} \Delta_F G &= \sum_{\mu\vdash n} F[B_\mu;q,t]\,\widetilde{\mathrm{H}}_\mu[X;q,t]C_\mu(q,t) \\ &= \sum_{\mu\vdash n} F[B_{\mu'}(t,q);t,q]\,\widetilde{\mathrm{H}}_{\mu'}[X;t,q]C_{\mu'}(t,q) \\ &= \sum_{\mu\vdash n} F[B_\mu(t,q);t,q]\,\widetilde{\mathrm{H}}_\mu[X;t,q]C_\mu(t,q) \qquad\qquad = \Delta_F G\Big|_{\substack{q\to t\\ t\to q}}. \end{aligned}$$

□

Whenever F and G are classical symmetric functions, we get that $\Delta_F G$ is symmetric in q and t. The first example we will see is when $F = e_k$ and $G = e_n$. Both of these functions are independent of q and t, meaning $\Delta_{e_k} e_n$ is also symmetric in q and t. We also have q,t symmetry for $\Delta_{m_\lambda} e_\mu$ and $\Delta_{m_\lambda}\omega p_\mu$. Therefore, our formulas for $\widetilde{\Delta}_{m_\lambda}\, e_\mu$ and $\widetilde{\Delta}_{m_\lambda}\,\omega p_\mu$ will also give formulas for when $t = 1$ by simply interchanging t by q.

Acknowledgements. We must thank Adriano Garsia for his many lessons in symmetric functions, plethystic notation, and Macdonald polynomials. Many of the proofs here are due to these

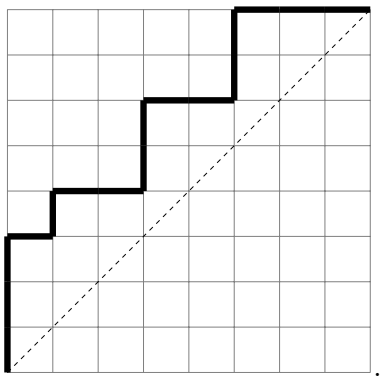
lessons. Section 2.3, is in part comprised of material submitted for publication by the author of this dissertation, Adriano Garsia and Jim Haglund, *Some New Symmetric Function Tools and their Applications* [2]. Adriano Garsia and the current author were supported by NSF grant DMS-1700233, and Jim Haglund was supported by NSF grant DMS-1600670. Some of the introductory material here can also be found in a published work of the current author, *The Delta Conjecture at* $q = 1$ [44]. This work was supported by NSF grant DMS-1362160.

Chapter 3

The Delta Conjecture at $q = 1$

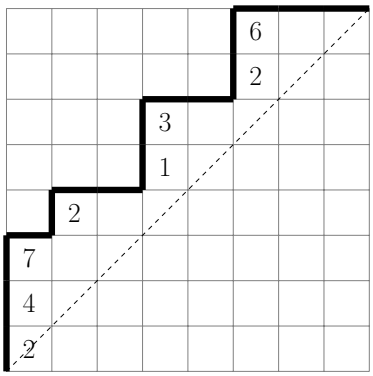
3.1 Introducing the Delta Conjecture

This chapter will essentially describe the contents of [44], where we prove the Delta Conjecture at $q = 1$. We take this opportunity to introduce some of the various combinatorial techniques we will be using in proving our most general results. In order to introduce The Delta Conjecture of Haglund, Remmel and Wilson [37], we have to first introduce Dyck paths and two statistics: area and dinv. To begin, a Dyck path $D \in D_n$ is a lattice path consisting of unit North steps and unit East steps that start at $(0, 0)$, end at (n, n), and never go below the line $y = x$. For example, the sequence of North and East steps $NNNENEENNEENNEEE$ gives the following lattice path:



Reading from bottom to top, let α_i be the number of lattice cells in the i^{th} row of D found between the path and the main diagonal. We call $(\alpha_1, \ldots, \alpha_n)$ the area sequence of D, and let area(D) be the sum $\alpha_1 + \cdots + \alpha_n$. The Dyck path pictured above has area sequence $(0, 1, 2, 2, 1, 2, 1, 2)$.

A labeled Dyck path $L \in LD_n$ is created by starting with the path $D \in D_n$ and labeling the North steps of D by positive integers $l_1, \ldots, l_n > 0$ with the condition that if $\alpha_i > \alpha_{i-1}$ then $l_i > l_{i-1}$. This means the the columns are increasing as we read them from bottom to top. For instance,



is a labeled Dyck path. Denote by x^L the monomial $x_{l_1} \cdots x_{l_n}$. Labeled Dyck paths also have an area, computed by taking the area of the supporting Dyck path. The second statistic we need is the number of diagonal inversions defined as

$$\text{dinv}(L) = \sum_{1 \le i < j \le n} \chi(\alpha_i = \alpha_j)\chi(l_i < l_j) + \chi(\alpha_i = \alpha_j + 1)\chi(l_i > l_j)$$

The critical part of the Delta Conjecture is the Haglund factor

$$H_{n-k}(L) = H_{n-k}(D) = (1+z) \prod_{\alpha_i > \alpha_{i-1}} \left(1 + \frac{z}{t^{\alpha_i}}\right)\Big|_{z^{n-k}}. \tag{3.1}$$

Conjecture 3.1.1 (The Delta Conjecture of Haglund, Remmel, Wilson [37])**.** *For* $0 \le k \le n$

$$\Delta_{e_k} e_n = \sum_{L \in LD_n} q^{\text{dinv}(L)} t^{\text{area}(L)} H_{n-k}(L) x^L.$$

Note that when $q = 1$ we can ignore the statistic dinv . Since the statistic area depends only on the supporting Dyck path, we can group monomials x^L with $L \in LD_n$ by their given Dyck path. If the Dyck path has c_1 North steps in the first column, c_2 North steps in the second column, and so on, we see that the set of labelings of the Dyck path D is the same as a product of semistandard tableaux of shape $1^{c_1}, \cdots, 1^{c_n}$. Therefore the contribution of all labeled Dycks paths of shape D is the e-basis element $e_{c_1} \cdots e_{c_n}$. The partition formed by rearranging the parts $(c_1, \ldots, c_n)$ will be denoted by $\lambda(D)$. The Dyck path above, for example, has vertical segments giving the partition $\lambda(D) = (3, 2, 2, 1)$.

The Delta Conjecture at $q = 1$, may be restated as

Theorem 3.1.2 (The Delta Conjecture at $q = 1$, R. [44])**.**

$$\Delta_{e_k} e_n \Big|_{q=1} = \sum_{D \in D_n} t^{\text{area}(D)} H_{n-k}(D) e_{\lambda(D)}$$

To prove this particular case, we will show that the coefficient of e_λ equals

$$\langle \Delta_{e_k} e_n, f_\lambda \rangle \Big|_{q=1} = \sum_{\substack{D \in D_n \\ \lambda(D)=\lambda}} t^{\text{area}(D)} H_{n-k}(D).$$

3.2 Setting up the proof

To set up the proof, we are going to analyze the symmetric function side first. Even though this proof will be superseded by the main theorem in the following chapter, it is this following transformation that led to the analysis of the symmetric function $\Delta_{m_\lambda} e_n$. We are going to replace k with $k - 1$, and begin by using the first equation of Proposition 2.3.7 to get

$$\langle \Delta_{e_{k-1}} e_n, f_\lambda \rangle = \langle \Delta_{\omega f_\lambda} e_k, h_k \rangle = \langle \Delta_{m_\lambda} e_k, h_k \rangle.$$

Recall that the Cauchy identity gives

$$e_k = \sum_{\mu \vdash k} h_\mu \left[\frac{X}{1-t}\right] f_\mu[1-t].$$

Applying $\widetilde{\Delta}_{m_\lambda}$ we get

$$e_k = \sum_{\mu \vdash k} m_\lambda \left[\sum_i [\mu_i]_t\right] h_\mu \left[\frac{X}{1-t}\right] f_\mu[1-t].$$

Proposition 3.2.1. *For $\mu \vdash n$, we have*

$$\left\langle h_\mu \left[\frac{X}{1-t}\right], h_n \right\rangle = h_\mu \left[\frac{1}{1-t}\right]$$

Proof. First, from the Pieri rules, we see that $\langle h_\mu, h_n \rangle = 1$ for all partitions $\mu \vdash n$. So we can compute the inner product of any symmetric function with h_n by expanding the symmetric function in terms of the homogeneous basis, then setting $h_r \to 1$ for all r. Therefore,

$$\begin{aligned}
\left\langle h_\mu \left[\frac{X}{1-t}\right], h_n \right\rangle &= \left\langle \prod_{i=1}^{\ell(\mu)} h_{\mu_i} \left[\frac{X}{1-t}\right], h_n \right\rangle \\
&= \left\langle \prod_{i=1}^{\ell(\mu)} \sum_{\nu^i \vdash \mu_i} h_{\nu^i}[X] m_{\nu^i} \left[\frac{1}{1-t}\right], h_n \right\rangle \\
&= \prod_{i=1}^{\ell(\mu)} \sum_{\nu^i \vdash \mu_i} m_{\nu^i} \left[\frac{1}{1-t}\right] \\
&= \prod_{i=1}^{\ell(\mu)} h_{\mu_i} \left[\frac{1}{1-t}\right] = h_\mu \left[\frac{1}{1-t}\right]
\end{aligned}$$

□

We get now that

$$\widetilde{\Delta}_{m_\lambda} e_k = \sum_{\mu \vdash k} m_\lambda \left[\sum_i [\mu_i]_t\right] h_\mu \left[\frac{1}{1-t}\right] f_\mu[1-t].$$

3.3 Special evaluations and the set $\mathbf{T}_{\mu}^{\lambda}$

There are three special evaluations above. The first one is $m_\lambda[\sum_i [\mu_i]_t]$. The following way of combinatorially depicting this polynomial will be useful for the following chapters. Recall the combinatorial formula

$$m_\lambda(x_1, \ldots, x_k) = \sum_{(\lambda_{i_1}, \ldots, \lambda_{i_k}) \in R(\lambda, 0^{k-\ell(\lambda)})} x_1^{\lambda_{i_1}} \cdots x_k^{\lambda_{i_k}},$$

where the sum is over all rearrangements $(\lambda_{i_1}, \ldots, \lambda_{i_k})$ of $(\lambda, 0^{k-\ell(\lambda)})$, the sequence λ followed by $k - \ell(\mu)$ 0's. We are labeling the terms $x_1, \ldots, x_k$ with one of the elements in the list $\lambda, 0^{k-\ell(\lambda)}$. Now we substitute $(x_1, x_2, \cdots, x_{\mu_1}) = (t^{\mu_1 - 1}, \ldots, t, 1)$, then $(x_{\mu_1+1}, \ldots, x_{\mu_1+\mu_2}) = (t^{\mu_2 - 1}, \ldots, t, 1)$, and so on. This gives the evaluation $m_\lambda[\sum_i [\mu_i]_t]$.

For example, suppose we are computing $m_{3,2,1,1}[[3]_t + [2]_t + [2]_t]$. Then a monomial is given by a rearrangement of $3, 2, 1, 1, 0, 0, 0$. Selecting the rearrangement $(0, 1, 0, 2, 3, 0, 1)$ corresponds to taking the monomial

$$x_1^0 x_2^1 x_3^0 x_4^2 x_5^3 x_6^0 x_7^1.$$

Writing this rearrangement in the cells of $\mu_1 = 3, \mu_2 = 2$, and $\mu_3 = 2$, we get the filling

0	1	0
t^2	t	1

2	3
t	1

0	1
t	1

(3.2)

We wrote under each cell the term which it represents in $[\mu_1]_t + [\mu_2]_t + [\mu_3]_t$. Reading from left to right, we get the term

$$(t^2)^0 (t)^1 (1)^0 (t)^2 (1)^3 (t)^0 (1)^1.$$

This means $m_\lambda[\sum [\mu_i]_t]$ can be interpreted as the sum over fillings of $\mu_1, \ldots, \mu_{\ell(\mu)}$ by

$\lambda_1, \ldots, \lambda_k$. If μ_i is filled by $(j_1, \ldots, j_{\mu_i})$, the filling of μ_i will contribute by a factor of

$$((t^{\mu_i - 1})^{j_1}(t^{\mu_i - 2})^{j_2} \cdots (t)^{j_{\mu_i - 1}} 1^{j_{\mu_i}})$$

to the term in the sum. This corresponds to saying that if λ_i is placed in a cell with j cells to the right, then it contributes to the term by $(t^j)^{\lambda_i}$. The sum over the contributions of all λ_i gives exactly the power of t contributed by this monomial in the expansion of m_λ.

The second special evaluation is $h_\mu[1/(1-t)]$. First note that for a single homogeneous symmetric function evaluated at $1/(1-t) = 1 + t + t^2 \cdots$, we have

Proposition 3.3.1. *For $r \geq 1$*

$$h_r\left[\frac{1}{1-t}\right] = \sum_{\lambda : \lambda_1 \leq r} t^{|\lambda|} = \frac{1}{(1-t)(1-t^2)\cdots(1-t^r)}$$

is the generating series of all partitions whose largest part is no larger than r.

Proof. The combinatorial formula for the homogeneous basis gives

$$h_r\left[\frac{1}{1-t}\right] = h_r[t^0 + t^1 + t^2 + \cdots] = \sum_{0 \leq i_1 \leq \cdots \leq i_r} t^{i_1} t^{i_2} \cdots t^{i_r}.$$

Now, for a given $i_1 \leq \cdots \leq i_r$, let λ be the partition formed by placing i_r cells in the first column, i_{r-1} cells in the second column, and continue until placing i_1 cells in column r. Then λ's longest part is at most r, and $|\lambda| = i_1 + i_2 + \cdots + i_r$. This bijection is the stated equality of this proposition. □

The final special evaluation involves the forgotten symmetric function. This evaluation is given by a different formula in [11].

Proposition 3.3.2. *For a partition μ,*

$$f_\mu[1-t] = (-1)^{|\mu| - \ell(\mu)} \sum_{\rho \in R(\mu)} (1 - t^{\rho_1}).$$

Proof. The addition formula for the forgotten basis gives

$$\begin{aligned} f_\mu[1-t] &= \sum_{\alpha\cup\beta=\mu} f_\alpha[1]f_\beta[-t] = \sum_{\alpha\cup\beta=\mu} f_\alpha[1](-t)^{|\beta|}m_\beta[1] \\ &= \sum_{\alpha\cup\beta=\mu} (-1)^{|\alpha|-\ell(\alpha)}|R(\alpha)| \ (-t)^{|\beta|}\chi(\beta=(|\beta|)). \end{aligned}$$

If $\beta=(0)$ is the 0 partition, then $\alpha=\mu$ and the term contributes $(-1)^{|\mu|-\ell(\mu)}|R(\mu)|$. If $\beta=(\mu_i)$ is a nonzero part of μ, then the term will contribute

$$(-1)^{|\mu|-\mu_i-(\ell(\mu)-1)}|R(\mu\setminus\mu_i)|(-t)^{\mu_i} = -(-1)^{|\mu|-\ell(\mu)}t^{\mu_i}|R(\mu\setminus\mu_i)|.$$

Note that a rearrangement of μ is uniquely determined by its first part, say ρ_1 from the set of parts $\{\mu\}$ with no multiplicities, and a rearrangement of the remaining parts in $R(\mu\setminus\rho_1)$. The two cases therefore give

$$\begin{aligned} f_\mu[1-t] &= (-1)^{|\mu|-\ell(\mu)}\left(|R(\mu)| - \sum_{\rho_1\in\{\mu\}} t^{\rho_1}|R(\mu\setminus\rho_1)|\right) \\ &= (-1)^{|\mu|-\ell(\mu)}\left(\sum_{\rho\in R(\mu)} 1 - \sum_{\rho\in R(\mu)} t^{\rho_1}\right) \end{aligned}$$

Combining the sums gives the proposed formula. □

We want to combine the three evaluations into a single picture to get an interpretation for the product. To do this, we now create a set $\mathbf{T}_\mu^\lambda$ with a weight w and a sign.

Definition 3.3.3. To create an element $T=(T_1,\ldots,T_{\ell(\mu)})\in\mathbf{T}_\mu^\lambda$ we :

1. Select a rearrangement of μ, say ρ.
2. Draw ρ as a sequence of rows and fill the cells of ρ by the numbers $\lambda_1,\ldots,\lambda_{\ell(\lambda)},0^{k-\ell(\lambda)}$ (as in the example in line (3.2)).
3. Select partitions $\gamma^1,\ldots,\gamma^{\ell(\mu)}$ satisfying $\gamma_1^i\le\rho_i$ with the extra condition $\gamma_1^i<\rho_1$. Create T_i

by drawing γ^i resting above the row ρ_i but right-justified, so that the partition is a French diagram reflected across a vertical axis. We call γ^i the partition of T_i. Such a tableau will be called a "reflected" partition.

4. It will be important to set some notation: Let $\ell(T_i) = \rho_i$ be the length, and let $\mathrm{fc}(T_i)$ be the number of parts in γ^i of size $\ell(T_i)$, that is the number of cells in the first column of T_i from γ^i. (Recall that the partitions are reflected, and thus this is the left-most column, containing the least number of cells). Similarly, let $\mathrm{lc}(T_i)$ be the number of cells in the last column of T_i from γ^i. Lastly, let $|lT_i|$ be the sum of the labels in T_i.

The weight of T is given by setting

$$w(T_i) = |\gamma^i| + \sum_{\text{label } j \text{ in } T_i} j \cdot (\text{ the number of cells to the right of } j \text{ in } T_i\ ),$$

and $w(T) = w(T_1) + \cdots + w(T_{\ell(\mu)})$. The sign of T is given by $\mathrm{sign}(T) = (-1)^{k-\ell(\mu)}$.

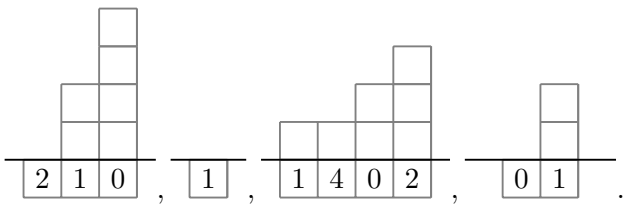
For example, say we are given the partitions $\mu = (4, 3, 2, 1)$, $\lambda = (4, 2, 2, 1, 1, 1, 1)$. We could have

1. selected the rearrangement $\rho = (3, 1, 4, 2)$, and

2. inscribed the cells of ρ by the sequence $((2, 1, 0), (1), (1, 4, 0, 2), (0, 1))$, which we depict by

$$\boxed{2}\boxed{1}\boxed{0}, \boxed{1}, \boxed{1}\boxed{4}\boxed{0}\boxed{2}, \boxed{0}\boxed{1}.$$

3. Lastly, we pick partitions $(\gamma^1, \ldots, \gamma^{\ell(\mu)}) = ((2, 2, 1, 1), (0), (4, 2, 1), (1, 1))$ and place them on top of ρ:



Note here that $\gamma_1^1 = 2 < \rho_1 = 3$. We will always use a horizontal bar between ρ_i and γ^i to distinguish the two. For the above example, our notation would give $|lT_1| = 3$, $\ell(T_3) = 4$, $\mathrm{fc}(T_3) = 1$ and $\mathrm{lc}(T_4) = 2$.

The weight of this tableau is given by summing

$$w(T_1) = 6 + (2 \cdot 2 + 1 \cdot 1 + 0 \cdot 0) \qquad w(T_2) = 0 + (1 \cdot 0)$$
$$w(T_3) = 7 + (1 \cdot 3 + 4 \cdot 2 + 0 \cdot 1 + 2 \cdot 0) \qquad w(T_4) = 2 + (0 \cdot 1 + 1 \cdot 0)$$

giving $w(T) = 11 + 0 + 18 + 2 = 31$. The sign is $(-1)^{10-4} = 1$.

Proposition 3.3.4. *For partitions λ, μ, we have*

$$m_\lambda \left[\sum_i [\mu_i]_t\right] h_\mu \left[\frac{1}{1-t}\right] f_\mu[1-t] = \sum_{T \in \mathbf{T}_\mu^\lambda} \mathrm{sign}(T) t^{w(T)}.$$

Proof. The first observation we must make is that

$$(1 - t^r) h_r \left[\frac{1}{1-t}\right] = h_{r-1} \left[\frac{1}{1-t}\right].$$

Therefore, we have

$$\begin{aligned} h_\mu \left[\frac{1}{1-t}\right] f_\mu[1-t] &= (-1)^{k-\ell(\mu)} \sum_{\rho \in R(\mu)} (1 - t^{\rho_1}) h_{\rho_1} \left[\frac{1}{1-t}\right] \cdots h_{\rho_{\ell(\mu)}} \left[\frac{1}{1-t}\right] \\ &= (-1)^{k-\ell(\mu)} \sum_{\rho \in R(\mu)} h_{\rho_1 - 1} \left[\frac{1}{1-t}\right] \cdots h_{\rho_{\ell(\mu)}} \left[\frac{1}{1-t}\right]. \end{aligned}$$

So we first pick a rearrangement ρ. Then we select partitions $\gamma^1, \ldots, \gamma^{\ell(\mu)}$ satisfying $\gamma_1^1 \leq \rho_1 - 1$ and $\gamma_1^2 \leq \rho_2, \ldots, \gamma_1^{\ell(\mu)} \leq \rho_{\ell(\mu)}$.

Lastly, the term $m_\lambda[\sum_i [\mu_i]_t]$ can be calculated by filling the parts of ρ as described in the beginning of this section. All of these factors are accounted for by the objects in $\mathbf{T}_\mu^\lambda$. The sign for any element of $\mathbf{T}_\mu^\lambda$ is $(-1)^{k-\ell(\mu)}$, matching the sign in the above equation. □

3.4 The split of a tableau and a sign reversing involution

The previous section was devoted to showing that

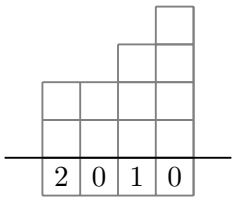
$$\langle \widetilde{\Delta}_{m_\lambda} e_k, s_k \rangle = \sum_{\mu \vdash k} \sum_{T \in \mathbf{T}_\mu^\lambda} \mathrm{sign}(T) t^{w(T)}.$$

To reduce the right hand side from a series with signs to a polynomial with positive integer coefficients, we will apply a very simple involution which we will call the "split" of a tableau.

Definition 3.4.1. Let T be the diagram of a reflected partition γ resting over a row of size m with labels $(c_1, \ldots, c_m)$, in this order from left to right (see line (3.3) below). The split of T at s (for $0 < s < m$), is the pair of diagrams $\mathrm{split}_s(T) = (S_1, S_2)$ formed as follows:

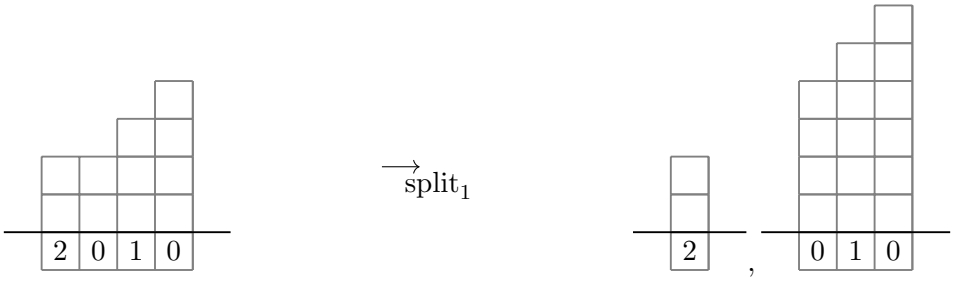
1. S_1 is the portion of T formed by the first s columns.
2. S_2 is the portion of T formed by the remaining $m - s$ columns, but we also add $c_1 + \cdots + c_s$ parts of size $m - s$ to the partition.

Suppose we were given for an example $\gamma = (4, 4, 2, 1)$ and the labels $(c_1, c_2, c_3, c_4) = (2, 0, 1, 0)$:



(3.3)

To apply split_1, we look at the first label: 2. We then make S_1 equal the first column (the partition $(1, 1)$ over the label 2), and we add $c_1 = 2$ rows of length 3 to the partition over the labels $(0, 1, 0)$:



Proposition 3.4.2. *For any s, the map* split_s *is weight preserving.*

Proof. Suppose $\mathrm{split}_s T = (S_1, S_2)$. The partition of S_1 is the same as the partition in the first s columns of T. In creating the partition for S_2, the number of new cells added to the last $\ell(T) - s$ columns of T is $|lS_1| \cdot \ell(S_2)$. The weight contributed by the labels in S_2 go unchanged.

Let c be a label in S_1 with r cells to its right. Then in T, this label contributes to $w(T)$ by $c \cdot (r + \ell(S_2))$. Therefore, when mapped to S_1, the weight labeled c loses to its contribution by $c \cdot \ell(S_2)$. The total loss in contribution by all the labels in S_1 is therefore $|lS_1| \cdot \ell(S_2)$ which is precisely the number of added cells to the partition of S_2. □

Definition 3.4.3. If (S_1, S_2) satisfy $\mathrm{fc}(S_2) - |lS_1| \geq \mathrm{lc}(S_1)$, or in another order

$$\mathrm{lc}(S_1) + |lS_1| \leq \mathrm{fc}(S_2),$$

then we say that we can join S_1 and S_2, and we write $\mathrm{split}^{-1}(S_1, S_2) = T$ if $\mathrm{split}_{\ell(S_1)} T = (S_1, S_2)$.

Lemma 3.4.4. *In the following, T_1, T_2, T_3, S_1, and S_2 are some tableaux like those defined in this section.*

1. *If T_1 cannot join T_2 and T_2 can join T_3, then T_1 cannot join* $\mathrm{split}^{-1}(T_2, T_3)$.

2. *If T_1 cannot join T_2 and T_2 can split as* $\mathrm{split}_s(T_2) = (S_1, S_2)$, *then T_1 cannot join S_1.*

Proof. The main observation to make is that if $\text{split}_s(T_2) = (S_1, S_2)$ then $\text{fc}(T_2) = \text{fc}(S_1)$. Also, T_1 cannot join S_1 if and only if $\text{lc}(T_1) + |lT_1| > \text{fc}(S_1) = \text{fc}(T_2)$, which is the condition that T_1 cannot join T_2. The conclusions of both statements then follows. □

Proposition 3.4.5. *Define a map $\varphi : \bigcup_{\mu\vdash k} \mathbf{T}_\mu^\lambda \to \bigcup_{\mu\vdash k} \mathbf{T}_\mu^\lambda$ by the following procedure: for $T = (T_1, \ldots, T_{\ell(\mu)})$ we begin with T_1. If $\ell(T_1) > 1$ then set $\varphi(T) = (\text{split}_1(T_1), T_2, \ldots, T_{\ell(\mu)})$. If not, then we check if T_1 can join T_2. If T_1 can join T_2, then we set $\varphi(T) = (\text{split}^{-1}(T_1, T_2), T_2, \ldots, T_n)$. If neither is the case, we repeat this process with T_2 and checking if it can split or join with T_3. We continue by induction until either we get an image, or we reach the end, in which case no T_i can split and no T_i can join T_{i+1}. In this case, we set $\varphi(T) = T$. Then φ is a weight-preserving, sign-reversing involution.*

Proof. We have defined split^{-1} so that it will invert splitting a tableau. Therefore, to show that φ is an involution, we need only check that applying φ to $\varphi(T)$ locates the same position i to split or join. If φ splits T_i, then we need to make sure that T_{i-1} cannot join the first part of $\text{split}_1(T_i)$. This is ensured by the previous lemma since φ does not join T_{i-1} with T_i, meaning the two cannot join to begin with. If instead φ joins T_i and T_{i+1}, then we must make sure that T_{i-1} cannot join with $\text{split}^{-1}(T_i, T_{i+1})$. But then again this is ensured by the lemma since we cannot join T_{i-1} and T_i to begin with. Therefore φ is an involution.

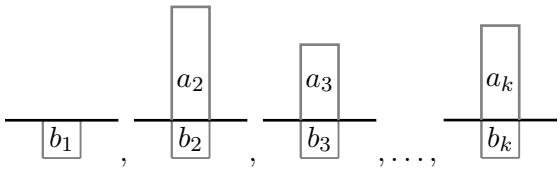
Lastly, split_s and split^{-1} are weight preserving. If T is not fixed then the number of tableau in $\varphi(T)$ is 1 more or less than that of T, meaning the sign changes. □

This proposition allows us to write

$$\langle \widetilde{\Delta}_{m_\lambda} e_k, h_k \rangle = \sum_{\mu\vdash k} \sum_{\substack{T\in\mathbf{T}_\mu^\lambda \\ \varphi(T)=T}} \text{sign}(T) t^{w(T)}$$

Suppose $T_1, \ldots, T_r$ is fixed. If $\ell(T_i) > 1$ for some i, then T is not fixed since T_i can be split. Therefore every fixed object is an element of $\mathbf{T}_{(1,\ldots,1)}^\lambda$ and has associated sign $(-1)^{k-k} = 1$. This

means that we are looking for columns of lengths $(a_1, \ldots, a_k)$ and some underlying labels $(b_1, \ldots, b_k)$ respectively such that no two columns can be combined. In pictures, we have a sequence which looks like



The condition that no two tableaux can be combined is given by $\mathrm{fc}(T_{i+1}) < \mathrm{lc}(T_i) + |lT_i|$. Here, $\mathrm{fc}(T_{i+1}) = a_{i+1}$, $\mathrm{lc}(T_i) = a_i$, and $|lT_i| = b_i$. We have proved the following theorem.

Theorem 3.4.6. *Denote by M_{k-1}^{λ} the set of sequences*

$$S = ((a_1, b_1), \ldots, (a_k, b_k))$$

of nonnegative integer pairs satisfying the following three conditions:

1. $a_1 = 0$;
2. $(b_1, \ldots, b_k)$ *is a rearrangement of* $\lambda, 0^{k-\ell(\lambda)}$;
3. $a_{i+1} < a_i + b_i$.

Define $w(S) = a_1 + \cdots + a_k$. Then

$$\langle \widetilde{\Delta}_{m_\lambda} e_k, h_k \rangle = \sum_{S \in M_{k-1}^{\lambda}} t^{w(S)}.$$

3.5 The bijection

To prove the Delta Conjecture at $q = 1$, we must show that

Theorem 3.5.1. *For any nonnegative integer k and partition λ,*

$$\sum_{S \in M_k^\lambda} t^{w(S)} = \sum_{\substack{D \in D_n \\ \lambda(D)=\lambda}} t^{\mathrm{area}(D)} H_{n-k}(D).$$

Proof. We first write the second sum as

$$\sum_{\substack{D \in D_n \\ \lambda(D)=\lambda}} t^{\mathrm{area}(D)} H_{n-k}(D) = \sum_{\overline{D} \in D_{n-k}^\lambda} t^{\mathrm{area}(\overline{D})},$$

where the objects in D_{n-k}^λ are created in the following way:

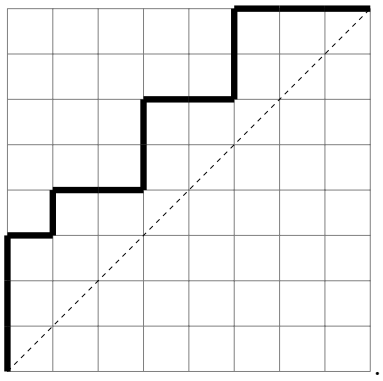
1. Select $D \in D_n$ such that $\lambda(D) = \lambda$. Suppose its area sequence is $\alpha_1, \ldots, \alpha_n$; and define the 0 row to be the origin with area $\alpha_0 = 0$.

2. Select $n-k$ distinct rows $i_1, \ldots, i_{n-k} \in \{0, 2, \ldots, n\}$ such that either $i_j = 0$ or $\alpha_{i_j} > \alpha_{i_j - 1}$.

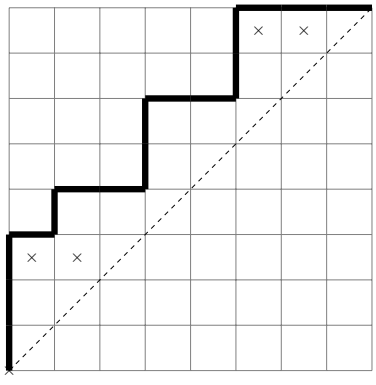
3. For all j, draw an $\times$ in each cell of row i_j that contributes to the area of D. If $i_j = 0$, then mark the origin with an $\times$. After doing this for all j, call this new object $\overline{D}$.

4. Define the area of $\overline{D}$ by setting $\mathrm{area}(\overline{D}) = \alpha_1 + \cdots + \alpha_n - (\alpha_{i_1} + \cdots + \alpha_{i_{n-k}})$. That is, count all the cells with no $\times$ that contribute to the area of D.

Here is an example: Suppose D has area sequence $(0, 1, 2, 2, 1, 2, 1, 2)$ so that the Dyck path is given by

 .

We select rows $0, 3$, and 8 to get the following labeled Dyck path in $D_3^{(3,2,2,1)}$:



Reading the rows bottom to top, the area of this object would be $0+1+0+2+1+2+1+0=7$.

To prove the theorem, we will provide a weight-preserving bijection $\Phi : D_{n-k}^{\lambda} \to M_k^{\lambda}$ so that $\text{area}(\overline{D}) = w(\Phi(\overline{D}))$.

For a given $\overline{D} \in D_{n-k}^{\lambda}$ the image $\Phi(\overline{D})$ is constructed by the following steps:

1. To the left of every North step that is the beginning of a vertical segment in $\overline{D}$, say of length b, we write (a, b), where a is the area number of this row.

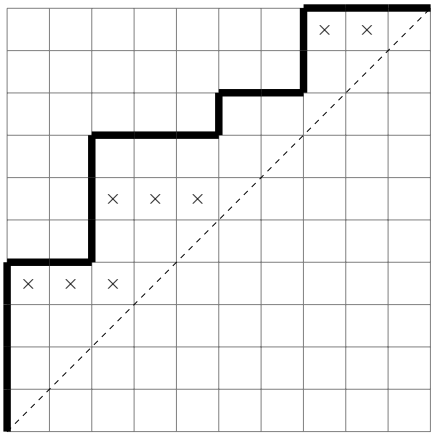
2. To the left of every remaining North step whose row is not marked, write $(c, 0)$, where c is the area number of this row. If the origin is not marked, we also include a $(0, 0)$ at the origin.

3. For each label $(c, 0)$, draw a North-East diagonal line from the beginning of its corresponding North step to the first start of an East step, and write $(c, 0)$ above the East step which precedes the endpoint of the diagonal. This includes the label $(0, 0)$ which, if it exists, will be placed at the end of the path.

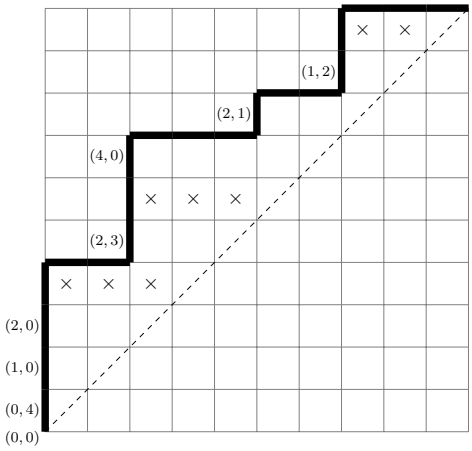
4. Follow the Dyck path from bottom to top writing the pairs in the order they appear.

It is easier to follow the bijection with pictures, so we will select a Dyck path in $D_3^{(4,3,2,1)}$ and provide an element of $M_7^{(4,3,2,1)}$ by following the respective steps from the definition of Φ. Step

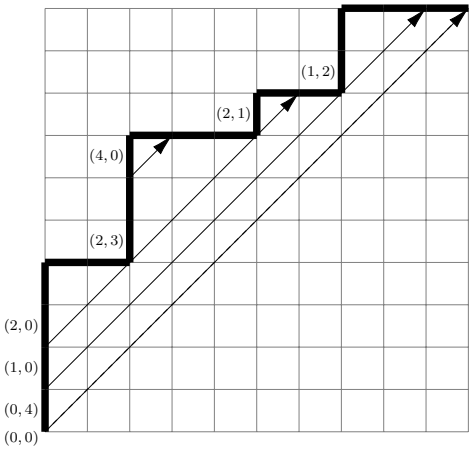
0 is the selection of the marked Dyck path.



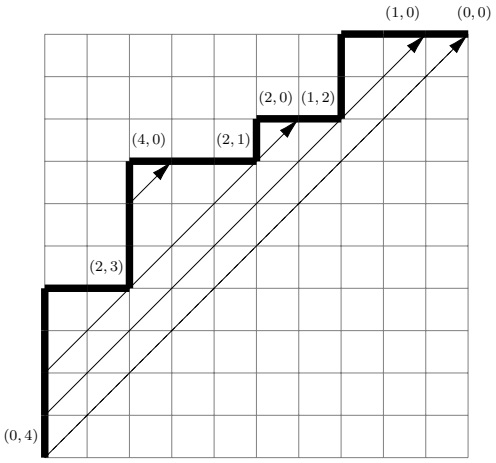
0.



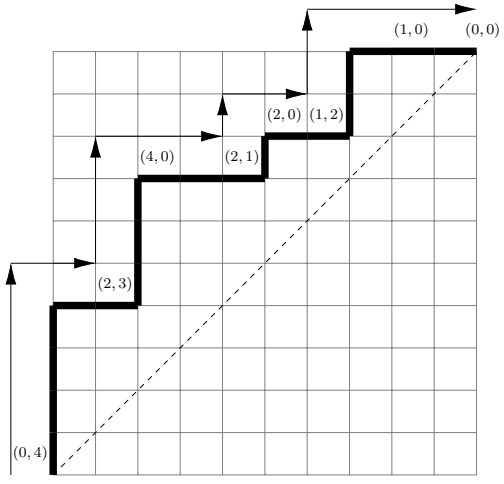
1.



2.



3.

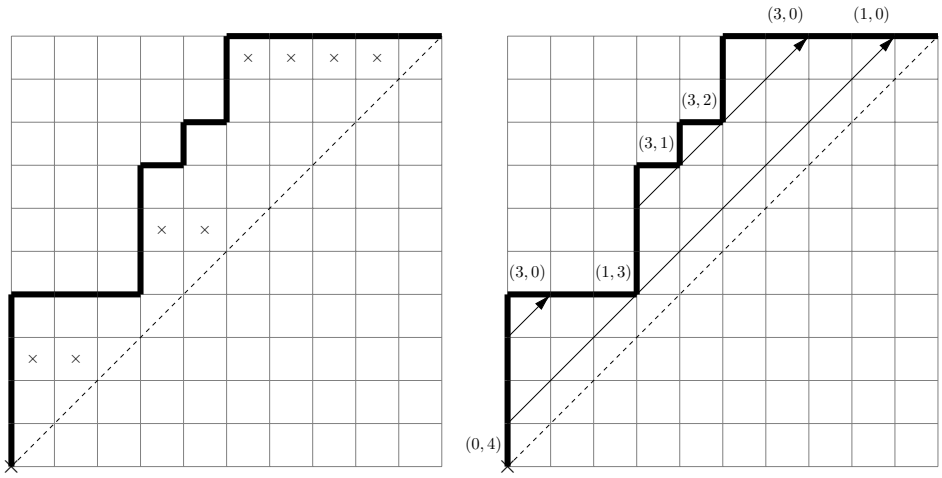


4.

This gives the sequence

$$((0,4),(2,3),(4,0),(2,1),(2,0),(1,2),(1,0),(0,0)).$$

For another example, we have



Since the origin is labeled, we do not include a $(0,0)$ at the end to get

$$((0,4),(3,0),(1,3),(3,1),(3,2),(3,0),(1,0)).$$

We must show that the image of $\overline{D} \in D_{n-k}^{\lambda}$ is an element of M_k^{λ}. It is easy to see that if

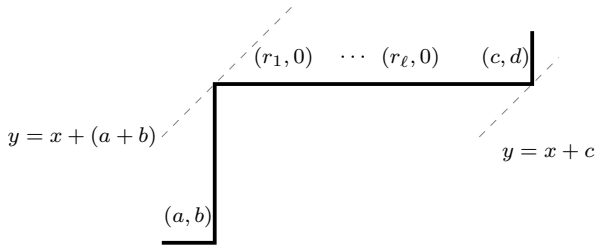
$$\Phi(\overline{D}) = ((a_1,b_1),\ldots,(a_{k+1},b_{k+1})),$$

then by construction $(b_1,\ldots,b_{k+1})$ is a rearrangement of $\lambda, 0^{k+1-\ell(\lambda)}$, since they are the lengths of the vertical segments in $\overline{D}$. We also have $a_1 = 0$. It then suffices to show the inequalities $a_{i+1} < a_i + b_i$ are satisfied. Every two consecutive vertical line segments of lengths b and d (beginning at diagonals

a and c respectively) will create a sequence in the image of the form

$$(a, b), (r_1, 0), \ldots, (r_\ell, 0), (c, d)$$

This is described by the following diagram:



We note that the top corner in the first vertical segment of the picture lies on diagonal $a + b$. Likewise $r_1, \ldots, r_\ell, c$ are the diagonals for the endpoints of the East steps they label. Therefore, since we are moving down diagonals as we proceed left to right, we have the inequalities

$$a + b > r_1 > \cdots > r_\ell > c.$$

This shows that $\Phi(\overline{D})$ provides a sequence which satisfies the defining inequalities of M_k^λ and therefore Φ is properly defined to give an element of M_k^λ. The area of the unmarked rows in $\overline{D}$ are precisely the values of $a_1, \ldots, a_{k+1}$. This means that $\text{area}(\overline{D}) = w\left(\Phi(\overline{D})\right)$.

To prove bijectivity, we need only provide the inverse. We do this first without proof, then show that this operation is indeed valid: Given a sequence $((a_1, b_1), \ldots, (a_{k+1}, b_{k+1})) \in M_k^\lambda$, we do the following:

1. Proceeding from left to right, for each (a, b) with $b \neq 0$ draw a line segment of length b beginning on diagonal a, so that we form a Dyck path.

2. For $(r_i, 0)$ between (a, b) and (c, d), $b, d \neq 0$, draw $(r_i, 0)$ over the east step ending in diagonal

r_i.

3. Move all $(r_i, 0)$ South-West, down their diagonal, until we reach the North end of a North step. Place $(r_i, 0)$, to the left of the North step which lies above. This includes a $(0, 0)$ label, which will move to the origin.

4. Mark with $\times$'s the rows which have no pair to the left.

We are simply doing the exact opposite from the definition of Φ. However, there are two implicit requirements which we have to check. The first step claims that the pairs (a, b) with $b \neq 0$, when read from left to right, describe a Dyck path whose vertical segments rearrange to λ. We again note that the sequence is made up of subsequences of the form

$$(a, b), (r_1, 0), \ldots, (r_\ell, 0), (c, d).$$

The fact that we have the inequality relations $c < r_\ell < \cdots < r_1 < a + b$ means that if we were to draw a vertical line segment of length b beginning on diagonal a, then the top corner of the segment (which lies on diagonal $a + b > c$) appears above diagonal c. So we have to place the segment of length d strictly to the right of the segment of length b. Thus, the first step produces a Dyck path whose vertical segments have lengths rearranging to λ.

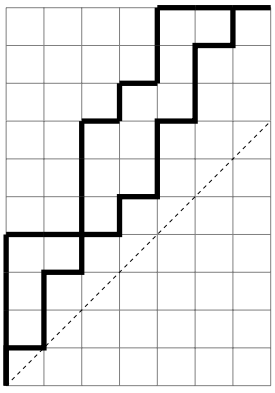
The second place where some detail is needed is in step 2, where we place $(r_i, 0)$ between (a, b) and (c, d). Since $r_1 > \cdots > r_\ell$, no two pairs will be placed above the same east step. Moreover, this portion of the Dyck path is horizontal starting from diagonal $a + b$ and ending on diagonal c, so there is indeed a unique East step which ends on diagonal r_i for each i. This completes the bijection. $\square$

3.5.1 A bijection to Leaning Stacks

The referee of the paper [44] observed the result can be obtained from a different formulation of the Delta Conjecture with certain objects called "Leaning Stacks". The most important part to note here is that this formulation suggests a different kind of structure underlying Delta operators, other than Dyck paths. In fact, one should be quick to notice the similarities between leaning stacks and polyominoes. The referee also noted that we get an important Schur expansion for $\nabla e_n|_{q=1}$ which we present in the last section of this chapter.

Consider the lattice rectangle of height n and length $k+1$, and let $D_{n,k}$ be the set of paths consisting of North and East steps from $(0,0)$ to $(k+1,n)$ which stay weakly above the line $y = x$ and end with an East step. Define the set of leaning stacks $LS_{n,k}$ to be the set of pairs of paths $(P,Q) \in D_{n,k} \times D_{n,k}$ such that P is weakly above Q, and Q has no two consecutive East steps in the first k columns. Let area$((P,Q))$ be the number of lattice cells between P and Q.

For example, we could select the paths (R,S) drawn as



The area of this pair is 14. The "Leaning Stacks" formulation for the Delta Conjecture at $q = 1$ can now be given as follows: Let

$$LS^{\lambda}_{n,k} = \{(P,Q) \in LS_{n,k} : \lambda(P) = \lambda\},$$

so that the vertical segments of P have lengths rearranging to λ. Then

Theorem 3.5.2.

$$\widetilde{\Delta}_{e_k} e_n = \sum_{\lambda \vdash n} e_\lambda \sum_{(P,Q) \in LS^{\lambda}_{n,k}} t^{\text{area}((P,Q))}.$$

There is a second way to describe the pairs (P, Q). First, make a sequence $(b_1, b_2, \ldots, b_{k+1})$ where b_i gives the number of North steps in column i of P (where the North steps of column i are on the line $y = i - 1$). For example, the pair (R, S) given above has 4 North steps in the first column, the second has 0 and the third has 3. Proceeding in this manner we get the sequence

$$(4, 0, 3, 1, 2, 0, 0).$$

Now create the sequence $(a_1, \ldots, a_{k+1})$ where $a_1 = 0$ and a_i gives the number of lattice cells between P and Q in column $i - 1$. The above example has 3 cells in the first column, 1 in the second, 3 in the third, and so on. This will give the sequence

$$(0, 3, 1, 3, 3, 3, 1).$$

Set

$$\Phi^{LS}((P, Q)) = ((a_1, b_1), (a_2, b_2), \ldots, (a_{k+1}, b_{k+1})).$$

Continuing our example, we get $\Phi^{LS}((R, S)) = ((0, 4), (3, 0), (1, 3), (3, 1), (3, 2), (3, 0), (1, 0))$. This is in fact the last example from the previous section.

Suppose $\Phi^{LS}((P, Q)) = ((a_1, b_1), \ldots, (a_{k+1}, b_{k+1}))$. First note that if $\lambda(P) = \lambda$, then $b_1, \ldots, b_{k+1}$ is a rearrangement of $\lambda, 0^{k+1-\ell(\lambda)}$. The fact that Q has no consecutive East steps in the first k columns means that the number of lattice cells in column $i - 1$ plus the number of North steps in column i is larger than the number of lattice cells in column i (for $i \leq k$). In other words:

$$a_i + b_i > a_{i+1}.$$

This gives exactly the definition for elements in M_k^λ. Therefore, Φ defines a bijection between $LS_{n,k}^\lambda$ and M_k^λ. Moreover, since $\text{area}((P,Q)) = w(\Phi^{LS}((P,Q)))$, this bijection is weight preserving. This proves the above expansion.

3.6 Some consequences

Our computation reveals a collection of structures that can give the inner product of $\widetilde{\Delta}_{e_k} e_n$ with any symmetric function F, given that $\omega F = \sum_\lambda c_\lambda m_\lambda$. In fact, given such an F,

$$\left\langle \widetilde{\Delta}_{e_k} e_n, F \right\rangle = \sum_{\lambda \vdash n} c_\lambda \sum_{s \in M_k^\lambda} t^{w(s)}.$$

This immediately gives that $\widetilde{\Delta}_{e_k} e_n$ is t-positive when expanded in terms of the bases $\{s_\lambda\}_\lambda$, $\{m_\lambda\}_\lambda$, $\{e_\lambda\}_\lambda$, and even $\{f_\lambda\}_\lambda$, though these are consequences of the e-positivity of $\Delta_{e_k} e_n$. In fact, every monomial $x_1^{b_1} \cdots x_{k+1}^{b_{k+1}}$ in $F(x_1, \ldots, x_{k+1})$ will contribute to $\left\langle \widetilde{\Delta}_{e_k} e_n, F \right\rangle$ by the weighted sum over sequences $s = ((a_1, b_1), \ldots, (a_{k+1}, b_{k+1}))$, where the a_i are chosen so that $s \in M_k^{\lambda(b_1,\ldots,b_{k+1})}$.

We give here two particular examples which are important in studying the Delta Conjecture.

3.6.1 The Hilbert series for $\widetilde{\Delta}_{e_k} e_n$

In [16], Wilson was able to compute the Hilbert series $\langle \Delta_{e_k} e_n, p_1^n \rangle$ at $q = 1$. His result can be obtained by our methods as follows. We can represent the symmetric function $p_1^n(x_1, \ldots, x_{k+1})$ by writing

$$p_1^n(x_1, \ldots, x_{k+1}) = \sum_{B_1, \ldots, B_{k+1}} x_1^{|B_1|} \cdots x_{k+1}^{|B_{k+1}|}$$

where $B_1, \ldots, B_{k+1}$ are disjoint subsets of $\{1, \ldots, n\}$ whose union is the whole set. In other words, we get an ordered set partition of length $k+1$, allowing empty sets.

Let P_k^n be the set of sequences

$$P = ((a_1, B_1), \ldots, (a_{k+1}, B_{k+1}))$$

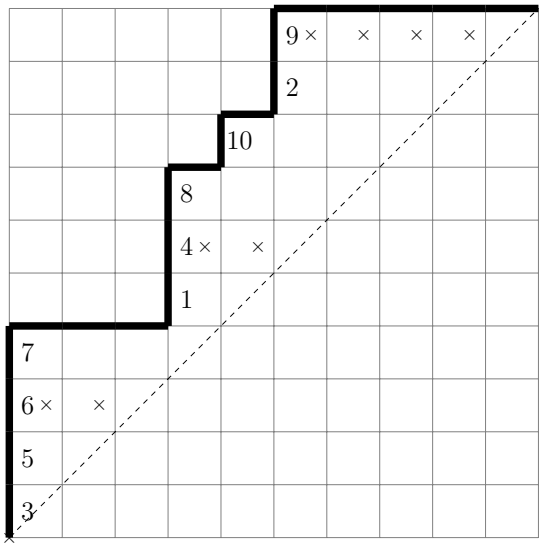
of pairs where $a_1 = 0$; the B_i are disjoint subsets of $\{1, \ldots, n\}$ whose union is all of $\{1, \ldots, n\}$; and $a_{i+1} < a_i + |B_i|$. One such example is the sequence

$$((0, \{3, 5, 6, 7\}), (3, 0), (1, \{1, 4, 8\}), (3, \{10\}), (3, \{2, 9\}), (3, 0), (1, 0)) \in P_6^{10}$$

Letting $w(P) = a_1 + \cdots + a_{k+1}$, we get

$$\left\langle \widetilde{\Delta}_{e_k} e_n, p_1^n \right\rangle = \sum_{P \in P_k^n} t^{w(P)}$$

We can use the same bijection as in the previous section, replacing the b_i with $|B_i|$. Now the sets describe "cars" along north segments. For instance, the previous sequence will be sent to the following marked parking function:



This means that the sum over such sequences is equal to the sum over parking functions in

the $n \times n$ square with $n-k$ marked rows, t-enumerated by the area statistic. That is,

$$\sum_{P \in P_k^n} t^{w(P)} = \sum_{D \in D_n} H_{n-k}(D) \sum_{D(PF)=D} t^{\text{area}(PF)},$$

where the last sum is over all parking functions PF in the $n \times n$ square whose supporting Dyck path is $D(PF) = D$.

3.6.2 A Schur function expansion and diagonal harmonics

The Delta Conjecture also predicts that $\Delta_{e_k} e_n$ is q,t-Schur positive, since it can be written as a sum over LLT-polynomials (see Section 6.2 for more details). We mean by this that the coefficient of s_λ in $\Delta_{e_k} e_n$ is a polynomial in q and t with positive integer coefficients. Using the same methods as before, we get the following expansion at $q = 1$:

For a semistandard tableau T let

$$c_i(T) = (\text{the number of times } i \text{ occurs in } T).$$

Let S_k^λ be the set of objects formed by

1. selecting a semistandard filling T of λ with entries in $1, \ldots, k+1$,
2. setting $a_1 = 0$ and choosing $a_2, \ldots, a_{k+1}$ such that $a_{i+1} < a_i + c_i(T)$, and
3. writing it all together as $S = ((a_1, \ldots, a_{k+1}), T)$.
4. Let $w(S) = a_1 + \cdots + a_{k+1}$.

Then

$$\left\langle \widetilde{\Delta}_{e_k} e_n, \omega s_\lambda \right\rangle = \sum_{S \in S_k^\lambda} t^{w(S)}.$$

For instance, when $\lambda = (3,2)$ and $k = 3$, we can choose the following pair:

$$(0,2,1,0,0), \begin{array}{|c|c|c|}\hline 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 3 \\ \hline \end{array}$$

To look at a specific yet non-trivial case, let us investigate

$$\widetilde{\Delta}_{e_n} e_n = \nabla e_n \Big|_{q=1} = \mathbf{F}DH_n[X;1,t].$$

To get a Schur basis expansion, let us look at one element $S = ((a_1, \ldots, a_{n+1}), T) \in S_n^\lambda$. Note that

$$s = ((a_1, c_1(T)), \ldots, (a_{n+1}, c_{n+1}(T))) \in M_n^\nu$$

where $\nu = \lambda(c_1(T), \ldots, c_{n+1}(T))$. View s by its image $\Phi^{-LS}(s) = (P,Q)$ from the bijection to leaning stacks in the prior section. These paths are from $(0,0)$ to $(n+1,n)$, and Q has no consecutive East steps in the first n columns. This forces Q to be the path with no consecutive North steps: (N,E)^n E. This means that the terms $a_1, \ldots, a_{n+1}$ are completely determined by P, while the vertical segments of P are determined by $c_1(T), \ldots, c_{n+1}(T)$. Therefore, every semistandard tableau T has at most one sequence $(a_1, \ldots, a_{n+1})$ so that $a_{i+1} < a_i + c_i(T)$. This sequence exists, so long as $(c_1(T), \ldots, c_{n+1}(T))$ describes a Dyck path, as given by the map Φ^{LS}. This is to say $c_{n+1} = 0$ and

$$c_1(T) + \cdots + c_i(T) \geq i.$$

Given a partition λ, let $PSYT^\lambda$ be the set of parking semistandard Young tableaux T of λ with entries in $1, \ldots, n$ so that $c_1(T) + \cdots + c_i(T) \geq i$. In other words, when read in increasing order, the i^{th} smallest integer in T is no greater than i. Define the area of T to be

$$\text{area}(T) = n \cdot c_1(T) + (n-1) \cdot c_2(T) + \cdots + c_n(T) - \binom{n+1}{2}.$$

This is simply the area of the Dyck path dictated by the sequence $c_1(T), \ldots, c_n(T)$.

Corollary 3.6.1.

$$DH_n[X; 1, t] = \sum_{\lambda \vdash n} s_\lambda \sum_{T \in PSYT^{\lambda'}} t^{\text{area}(T)}.$$

The usual definition for a parking function was first introduced in [8] and is given as follows: Suppose there are n people numbered $1, \ldots, n$ trying to park their cars on a one-way street with parking positions $1, 2, \ldots, n$ in this order. A word $w_1, \ldots, w_n$ with letters in $\{1, \ldots, n\}$ denotes a preference for each person: person i wants to park in position w_i. On their turn, person i will drive up to position w_i and park if it is empty. Otherwise, they park at the next available spot. If everyone is able to park in this manner, then we say the sequence $w_1, \ldots, w_n$ is a parking function.

It is not hard to see that the necessary and sufficient condition needed for w to be a parking function is that when written in increasing order, the i^{th} element in the list is no greater than i. Now the elements of $PSYT^\lambda$ can be defined as the set of semistandard Young tableaux of shape λ whose entries form a parking function when read in any order. As a simple example, the following is a list of all the elements in $PSYT^{(2,2)}$:

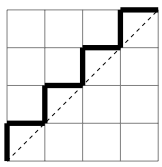
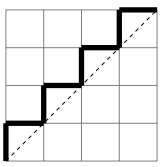
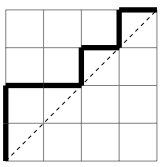
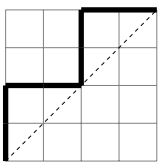
3	4
1	2

2	4
1	3

2	3
1	1

2	2
1	1

Drawing their corresponding Dyck paths, we have

The area number of each Dyck path is $0, 0, 1, 2$ respectively, and we therefore get

$$\left\langle \widetilde{\Delta}_{e_4} e_4, \omega s_{2,2} \right\rangle = 1 + 1 + t + t^2.$$

Acknowledgements. The main component of Chapter 3 can also be found in a published work of the current author, *The Delta Conjecture at $q = 1$* [44]. This work was supported by NSF grant DMS-1362160. Thank you to the referee of this paper, who had many great suggestions and noticed that Theorem 3.5.2 and Corollary 3.6.1 were simple consequences.

Chapter 4

A general approach to Delta operator expansions

After having proved the main result of the previous chapter, it seemed natural to consider the symmetric function $\widetilde{\Delta}_{m_\lambda} e_n$ in its entierty, not only its inner product with h_n. In fact, a full analysis of $\widetilde{\Delta}_{m_\lambda} e_n$ would include the Delta Conjecture at $q = 1$ since $e_k = m_{1^k}$ is a special case of a monomial symmetric function. Many of the results in this chapter are stated in [3].

Recall that for any symmetric function G, we have the expansion

$$\widetilde{\Delta}_G e_n = \sum_{\eta \vdash n} G\left[\sum_i [\eta_i]_t\right] h_\eta \left[\frac{X}{1-t}\right] f_\eta[1-t].$$

To get an expansion in terms of the elementary symmetric function basis, recall that we can use the Cauchy formula to write

$$h_\eta\left[\frac{X}{1-t}\right] \quad = \quad \prod_{i=1}^{\ell(\eta)} h_{\eta_i}\left[\frac{X}{1-t}\right] \quad = \quad \prod_{i=1}^{\ell(\eta)} \sum_{\nu^i \vdash \eta_i} e_{\nu^i}[X] f_{\nu^i}\left[\frac{1}{1-t}\right]$$

Using our notation from the second chapter, Equation (2.5), we can collect terms and write

$$h_\eta\left[\frac{1}{1-t}\right] = \sum_{\mu\vdash n} e_\mu \sum_{\vec{\nu}\in PR(\mu,\eta)} f_{\vec{\nu}}\left[\frac{1}{1-t}\right]$$

This gives the expansion in terms of the elementary basis:

$$\widetilde{\Delta}_G\, e_n = \sum_{\mu\vdash n} e_\mu[X] \sum_{\eta\vdash n} G\left[\sum_i [\eta_i]_t\right] \sum_{\vec{\nu}\in \mathrm{PR}(\mu,\eta)} f_\eta[1-t] f_{\vec{\nu}}\left[\frac{1}{1-t}\right]. \tag{4.1}$$

4.0.1 The principal evaluation of forgotten symmetric functions

Given a partition ν, we are going to compute the symmetric function $f_\nu[1/(1-t)]$. Using Theorem 2.2.5, we can write

$$f_\nu\left[\frac{1}{1-t}\right] = (-1)^{|\nu|-\ell(\nu)} \sum_{0\le i_1\le\cdots\le i_{\ell(\nu)}} \sum_{\alpha\in R(\nu)} (t^{i_1})^{\alpha_1}\cdots(t^{i_{\ell(\nu)}})^{\alpha_{\ell(\nu)}}.$$

Substituting $i_1 = r_1$, $i_2 = i_1 + r_2$, $\ldots$, $i_j = i_{j-1} + r_j$, we can instead sum over all $r_1,\ldots,r_{\ell(\nu)} \ge 0$ to give

$$\begin{aligned}
f_\nu\left[\frac{1}{1-t}\right] &= (-1)^{|\nu|-\ell(\nu)} \sum_{r_1\ge 0,\ldots,r_{\ell(\nu)}\ge 0} \sum_{\alpha\in R(\nu)} (t^{r_1})^{\alpha_1}(t_2^{r_1+r_2})^{\alpha_2}\cdots(t^{r_1+\cdots+r_{\ell(\nu)}})^{\alpha_{\ell(\nu)}} \\
&= (-1)^{|\nu|-\ell(\nu)} \sum_{r_1\ge 0,\ldots,r_{\ell(\nu)}\ge 0} \sum_{\alpha\in R(\nu)} (t^{\alpha_1+\cdots+\alpha_{\ell(\nu)}})^{r_1}(t^{\alpha_2+\cdots+\alpha_{\ell(\nu)}})^{r_2}\cdots(t^{\alpha_{\ell(\nu)}})^{r_{\ell(\nu)}} \\
&= (-1)^{|\nu|-\ell(\nu)} \sum_{\alpha\in R(\nu)} \sum_{r_1\ge 0} (t^{\alpha_1+\cdots+\alpha_{\ell(\nu)}})^{r_1} \sum_{r_2\ge 0} (t^{\alpha_2+\cdots+\alpha_{\ell(\nu)}})^{r_2}\cdots \sum_{r_{\ell(\nu)}\ge 0} (t^{\alpha_{\ell(\nu)}})^{r_{\ell(\nu)}} \\
&= (-1)^{|\nu|-\ell(\nu)} \sum_{\alpha\in R(\nu)} \frac{1}{1-t^{\alpha_1+\cdots+\alpha_{\ell(\nu)}}} \frac{1}{1-t^{\alpha_2+\cdots+\alpha_{\ell(\nu)}}}\cdots\frac{1}{1-t^{\alpha_{\ell(\nu)}}}.
\end{aligned}$$

A rearrangement α has no empty parts, so the numbers $\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{\ell(\nu)}$ are distinct numbers between 1 and n. We now, for convenience, reverse the order of the rearrangements. For a rearrangment α, let $c(\alpha) = \{1,\ldots,n\} \setminus \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{\ell(\alpha)}\}$ be its compliment set.

Then for every α we have

$$\frac{1}{1-t^{\alpha_1}}\frac{1}{1-t^{\alpha_1+\alpha_2}}\cdots\frac{1}{1-t^{\alpha_1+\cdots+\alpha_n}}=\frac{\prod_{i\in c(\alpha)}(1-t^i)}{(1-t)\cdots(1-t^n)}.$$

Multiplying top and bottom by $(1-t)^{|c(\alpha)|}=(1-t)^{|\nu|-\ell(\nu)}$, we can write the principal evaluation as follows:

Theorem 4.0.1. *For a partition $\nu\vdash n$ of length ℓ, we have*

$$f_\nu\left[\frac{1}{1-t}\right]=(t-1)^{n-\ell}\frac{\sum_{\alpha\in R(\nu)}\prod_{i\in c(\alpha)}[i]_t}{(1-t)(1-t^2)\cdots(1-t^n)}$$

We decided to include this formula for two reasons: the first is that this formula will be used later in describing some of the e-positivity phenomena exhibited in Chapter 6. The second is the classical flavor of this result, which seems largely unknown to the general combinatorics community. Typically, the forgotten symmetric function is approached with fear or mystery, but the combinatorics behind this symmetric function are almost as simple as that of the monomial symmetric function. Let us now introduce the more helpful depiction of this principal evaluation in terms of what we will call Column-Composition Tableaux. Recall that

$$f_\nu[X]=(-1)^{|\nu|-\ell(\nu)}\sum_{(\alpha^1,\alpha^2,\dots)\in\mathrm{CR}(\nu)}x_1^{|\alpha^1|}x_2^{|\alpha^2|}\cdots, \tag{4.2}$$

where the sum is over all sequences of compositions $\alpha^1,\alpha^2,\dots$ whose union of parts gives ν. We will again substitute $x_i=t^{i-1}$ for all i to get a formula for $f_\nu[1/(1-t)]$.

For a partition ν construct the set of elements $T\in CC_\nu$, called Column-Composition Tableaux of type ν, by the following steps:

1. Select a composition $\rho\in R(\nu)$. From here on, this will be called the column-composition of T: $\mathrm{ccomp}(T)=\rho$.

2. Begin with a "bottom" row of size $|\nu|$ and draw a horizontal line segment above. Draw vertical line segments between the columns to denote ρ, adding a vertical line after the first ρ_1 columns, then after the next ρ_2 columns, and so on. These vertical lines will be called the "(vertical) bars" of T.

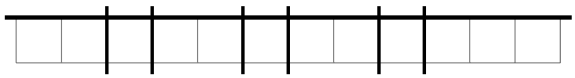
3. Select a reflected partition γ and place it over the bottom row. Such a selection is valid if and only if whenever two columns have no bars between, they have the same number of cells. In other words, the columns of γ, from left to right, only increase at the bars of T.

4. We let $w(T) = |\gamma|$ be the number of squares above the bottom row. We use $\mathrm{fc}(T)$ to denote the number of cells in the first column of T (above the bottom row) as before, and $\mathrm{lc}(T)$ to denote the number of cells in the last column (again above the bottom row).

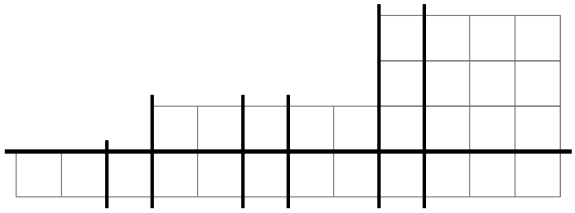
For example, given $\nu = (3, 2, 2, 2, 1, 1, 1)$, we could construct T by

1. first selecting $\rho = (2, 1, 2, 1, 2, 1, 3)$.

2. Next draw the bottom row of T and the vertical bars:



3. Select γ so that the columns increase only at the bars of T:



4. This example has $\mathrm{ccomp}(T) = (2, 1, 2, 1, 2, 1, 3)$, $w(T) = 17$, $\mathrm{fc}(T) = 0$ and $\mathrm{lc}(T) = 3$.

Let $\overline{CC_\nu}$ be the subset of tableaux $T \in CC_\nu$ such that $\mathrm{fc}(T) = 0$. And define

$$F_\nu = (-1)^{|\nu|-\ell(\nu)} f_\nu \left[\frac{1}{1-t}\right] = \sum_{T \in CC_\nu} t^{w(T)}, \text{ and}$$
$$\overline{F_\nu} = (1-t^{|\nu|})(-1)^{|\nu|-\ell(\nu)} f_\nu \left[\frac{1}{1-t}\right] = \sum_{T \in \overline{CC_\nu}} t^{w(T)}.$$

The first equality follows from the construction: Best explained by example, note that the above example has 3 columns with no cells above the bottom row. The composition formed by the bars in this portion of ρ is $(2,1)$. The composition formed by the columns with one cell is $(2,1,2)$, and the composition formed by the columns of size 3 is $(1,3)$. This corresponds to selecting the term $\alpha_1 = (2,1)$, $\alpha_2 = (2,1,2)$, $\alpha_4 = (1,3)$, and $\alpha_j = 0$ for all $j \neq 1,2,4$ in the summation of (4.2).

The second equality can be proved by observing that

$$t^{|\nu|} \sum_{T \in CC_\nu} t^{w(T)} = \sum_{\substack{T \in CC_\nu \\ \mathrm{fc}(T) \geq 1}} t^{w(T)},$$

since adding a row of size $|\nu|$ to the partition of T would increase $\mathrm{fc}(T)$ by 1. Therefore, subtracting this from F_ν would give the generating series for those composition tableau with $\mathrm{fc}(T) = 0$.

4.1 The construction of the set $\mathbf{CCT}(\lambda, \mu)$

We will need the fact from the previous chapter that

$$f_\eta[1-t] = (-1)^{|\eta|-\ell(\eta)} \sum_{\beta \in R(\eta)} (1-t^{\beta_1}).$$

We can now interpret the product of forgotten symmetric functions as a generating series

for sequences of column-composition tableaux:

$$f_\eta[1-t] \sum_{\vec{\nu}\in \mathrm{PR}(\mu,\eta)} f_{\vec{\nu}}\left[\frac{1}{1-t}\right]$$
$$= (-1)^{|\eta|-\ell(\eta)} \sum_{\beta\in R(\eta)} (1-t^{\beta_1}) \sum_{\vec{\nu}\in \mathrm{PR}(\mu,\beta)} f_{\vec{\nu}}\left[\frac{1}{1-t}\right]$$
$$= (-1)^{\ell(\mu)-\ell(\eta)} \sum_{\beta\in R(\eta)} \sum_{\vec{\nu}\in \mathrm{PR}(\mu,\beta)} \overline{F_{\nu^1}} F_{\nu^2} \cdots F_{\nu^{\ell(\gamma)}}$$

Next, setting $G = m_\lambda$ in Equation (4.1), we interpret

$$\left(\widetilde{\Delta}_{m_\lambda}\, e_n\right)\Big|_{e_\mu} = \sum_{\eta\vdash n} m_\lambda\left[\sum [\eta_i]_t\right] \sum_{\vec{\nu}\in \mathrm{PR}(\mu,\eta)} f_\eta[1-t] f_{\vec{\nu}}\left[\frac{1}{1-t}\right] \tag{4.3}$$

by filling the bottom rows of the column composition tableaux.

Definition 4.1.1. We construct the set of $\mathbf{CCT}(\lambda,\mu)$ of labeled Column-Composition Tableaux as follows:

1. Select a partition $\eta \vdash n$ and $\beta \in R(\eta)$.
2. Select $(\nu^1,\ldots,\nu^{\ell(\beta)}) \in \mathrm{PR}(\mu,\beta)$.
3. Select $T_1 \in \overline{CC_{\nu^1}}$; and for $i = 2,\ldots,\ell(\eta)$, select $T_i \in CC_{\nu^i}$.
4. Distribute a rearrangement of the list $\lambda_1,\ldots,\lambda_{\ell(\lambda)}, 0^{n-\ell(\lambda)}$ along the cells in the bottom rows of $T_1,\ldots,T_{\ell(\eta)}$ to obtain a sequence of labeled tableaux $S = (S_1,\ldots,S_{\ell(\eta)})$. We denote by $lS_i = (r_1,\ldots,r_{|\nu^i|})$ the sequence of labels in S_i when read from left to right, and $|lS_i| = r_1+\cdots+r_{|\nu^i|}$ the sum of the labels.

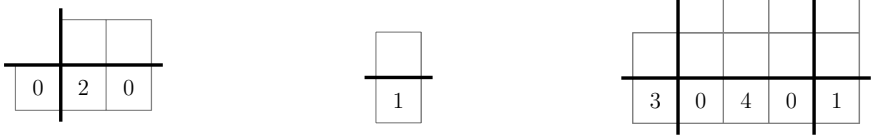
These new labels correspond to the weight contributed by $m_\lambda\left[\sum [\eta_i]_t\right]$, so define

$$w(S_i) = w(T_i) + \sum_{\text{label } j \text{ in } S_i} j \cdot (\text{ the number of cells to the right of } j \text{ in } S_i\ ),$$

and $w(S) = w(S_1) + \cdots + w(S_{\ell(\eta)})$. The sign of S is given by $\text{sign}(S) = (-1)^{\ell(\mu)-\ell(\eta)}$. The most important aspect of the sign is that we can write

$$\text{sign}(S) = (-1)^{\text{number of vertical bars in } S}.$$

For example, suppose $\eta = (5,3,1)$, $\mu = (3,2,1,1,1,1)$, and $\lambda = (4,3,2,1,1)$. Suppose we select $\beta = (3,1,5) \in R(\eta)$, then select $\nu^1 \vdash 3, \nu^2 \vdash 1, \nu^3 \vdash 5$ given by $(2,1), (1)$, and $(3,1,1)$ respectively. The union of parts is $(3,2,1,1,1,1) = \mu$. Finally, we select $T_1 \in \overline{CC_{(2,1)}}$, $T_2 \in CC_{(1)}$, $T_3 \in CC_{(3,1,1)}$ and a rearrangement of $\lambda, 0^{n-\ell(\lambda)} = (4,3,2,1,1,0,0,0,0)$ to place in the bottom rows. For example, let S be the sequence



We can calculate the weight of this list by first counting the number of cells above the bottom rows. In this case we would have $2, 1$, and $5 + 4 = 9$ respectively. The weight corresponding to the new labels would be:

$$0(2) + 2(1) + 0(0) = 2, \qquad 1(0) = 0, \qquad \text{and} \qquad 3(4) + 0(3) + 4(2) + 0(1) + 1(0) = 20.$$

Thus $w(S) = (2+1+9) + (2+0+20) = 34$ and $\text{sign}(S) = (-1)^{(6-3)} = -1$. The number of vertical bars in S is 3, so another way to compute the sign is by taking $(-1)^3 = -1$.

To see why the sign is equal to the number of bars, we note that $\ell(\mu)$ is the number of bars plus the number of tableaux in S. The number of tableau is $\ell(\eta)$. Subtracting $\ell(\eta)$ from $\ell(\mu)$ we get the number of bars.

In conclusion, we have:

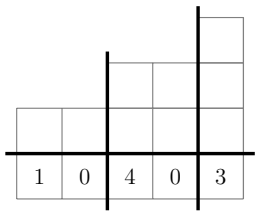
$$\Delta_{m_\lambda} e_n \Big|_{q=1} = \sum_{\mu \vdash n} e_\mu \left(\sum_{S \in \mathbf{CCT}(\lambda,\mu)} \operatorname{sign}(S) t^{w(S)} \right) \tag{4.4}$$

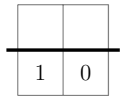
4.2 A weight-preserving, sign-reversing involution ψ

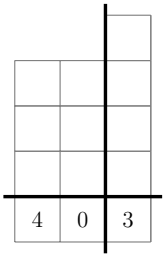
We will find a sign reversing involution $\psi : \bigcup_{\mu \vdash n} \mathbf{CCT}(\lambda, \mu) \to \bigcup_{\mu \vdash n} \mathbf{CCT}(\lambda, \mu)$ which simplifies the right hand side of (4.4), taking us from a series to a polynomial. To define this map, we now introduce the maps $\overline{\text{split}}_s$ and $\overline{\text{split}}^{-1}$ in the same way as in Definition 3.4.1. The only difference, is that we now remove a vertical bar when we split the tableau, and we add a vertical bar when we invert the split.

Definition 4.2.1. A labeled column composition tableau T is said to split, if it has a vertical bar. If the first bar in T (from the left) occurs after c columns, then we write $\text{fb}(T) = c$. If T has a vertical bar after s columns, then we can split T at s and write $\overline{\text{split}}_s(T)$ for the tableaux formed by removing the bar at s and applying split_s. The inverse $\overline{\text{split}}^{-1}$ applies split^{-1} to $\overline{\text{split}}_s(T)$ and then adds a vertical bar after column s.

For example, we have

$\overline{\text{split}}_2$ 

$=$ 

, 

(4.5)

We can now define ψ as follows:

Definition 4.2.2. Given $S = (S_1, \ldots, S_\ell) \in \mathbf{CCT}(\lambda, \mu)$ define $\psi(S)$ by the following process:

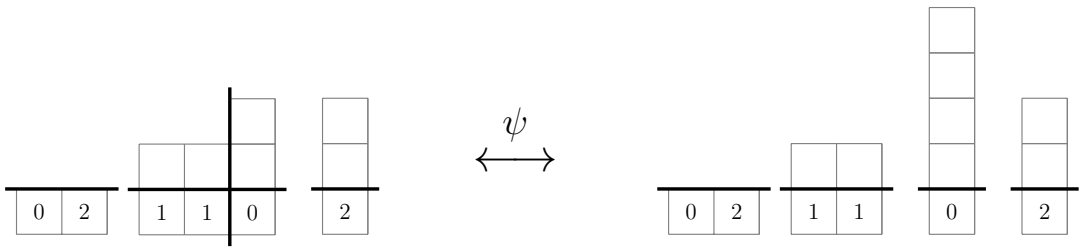
1. If S_1 can split and $\mathrm{fb}(S_1) = c$, then $\psi(S) = (\overline{\mathrm{split}}_c(S_1), S_2, \ldots, S_\ell)$.

2. If S_1 cannot split and S_1 can join S_2 then $\psi(S) = (\overline{\mathrm{split}}^{-1}(S_1, S_2), S_3, \ldots, S_\ell)$.

3. If case 1 or 2 do not hold, then we proceed by induction on $(S_2, \ldots S_\ell)$, so that the first step is to check whether S_2 can split, then if it can join with S_3, and so on.

4. If no S_i can split and no S_i can join S_{i+1} then we set $\psi(S) = S$.

For a simple example, we have



Proposition 4.2.3. *The map ψ is a weight-preserving, sign-reversing involution.*

Proof. The proof follows from an application of Lemma 3.4.4, the same as the proof of Proposition 3.4.5. If S_1 cannot join S_2 and S_2 splits, then S_1 cannot join the first component of the split of S_2. If S_1 cannot join S_2 and S_2 joins S_3 to form T, then S_1 cannot join T. This means that the steps in the definition of ψ locates the same index i in $\psi(S)$ as it does in S, assuring that ψ is an involution. The map is sign reversing because the number of bars changes by 1, unless S is fixed. □

Now we let $\mathbf{M}_{\lambda,\mu}$ be the set of fixed points of $\mathbf{CCT}(\lambda, \mu)$ under the map ψ. An element $S = (S_1, \ldots, S_\ell) \in \mathbf{M}_{\lambda,\mu}$ is fixed by ψ if

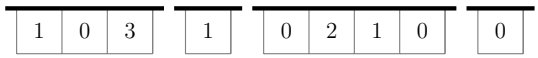
a. For all i S_i cannot split, meaning there are no bars and $\mathrm{lc}\, S_i = \mathrm{fc}\, S_i$.

b. For all $i < \ell$, S_i cannot join S_{i+1}, meaning $\mathrm{fc}(S_{i+1}) < \mathrm{lc}(S_i) + |lS_i|$

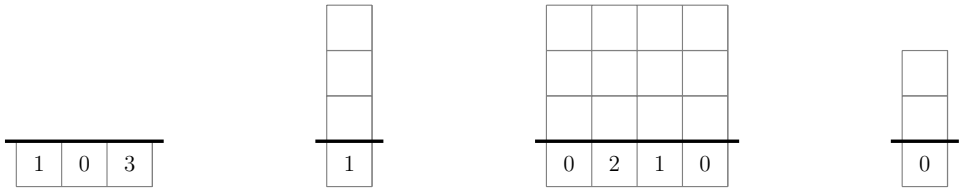
For example, suppose $\mu = (4, 3, 1, 1)$ and $\lambda = (3, 2, 1, 1, 1)$. Then to create an element $S = (S_1, S_2, S_3, S_4) \in \mathbf{M}_{\lambda,\mu}$ first pick a rearrangement of μ: $(3, 1, 4, 1)$. These will be the bottom

rows of the tableaux:

Fill the cells with a rearrangement of $\lambda, 0^{|\mu|-\ell(\lambda)}$, say $(1,0,3,1,0,2,1,0,0)$:



Now, for each $i > 1$ pick a number of rows to place above the bottom row of S_i (that is, pick a value for $\mathrm{fc}(S_i)$). This number must be less than the number of rows in S_{i-1} plus the sum of the labels $|lS_{i-1}|$. For instance, to pick the number of rows above S_2, we have to pick a number less that $|lS_1| = 3+0+1 = 4$. We will pick $\mathrm{fc}(S_2) = 3$. Next we pick $\mathrm{fc}(S_3)$ so that it is less than $\mathrm{fc}(S_2) = 3$ plus $|lS_2| = 1$. We select $3 < 3+1 = 4$. Lastly, we pick $\mathrm{fc}(S_4) = 2 < \mathrm{fc}(S_3)+|lS_3| = 3+(0+1+2+0) = 6$:



The weight of this object is computed by summing the number of cells above the bottom rows $(0+3+12+2\ )$ to the contribution of the labels $(2+0+5+0)$ giving 24. We have proved

Theorem 4.2.4. *For $\mu \vdash n$ and any partition λ, we have*

$$\widetilde{\Delta}_{m_\lambda}\, e_n\Big|_{e_\mu} = \sum_{S \in \mathbf{M}_{\lambda,\mu}} t^{w(S)}.$$

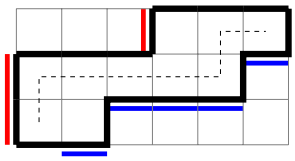
4.3 Parallelogram Polyominoes and the bijection Γ

The set of parallelogram polyominoes, henceforth referred to simply as polyominoes, will give the main combinatorial interpretation for Delta operator expansions.

Definition 4.3.1. A (parallelogram) polyomino $P = (P_1, P_2) \in \mathcal{P}(m, n)$ is a pair of lattice paths consisting of North and East steps such that P_1 and P_2 meet only at the beginning $(0, 0)$ and end $(m + 1, n)$. From here on, when referring to the path P_2, we will always ignore the first East step so that P_2 begins at $(1, 0)$. The area statistic is defined by setting

$$\text{area}(P) = (\text{ the number of lattice cells between the two paths }) - (m + n).$$

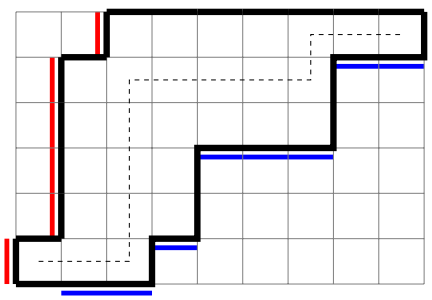
Here is a simple example with area 2:

$(Q_1, Q_2) =$ 

The dashed line represents removing $m + n$ cells from the area cells between the two paths. The vertical (red) segments in Q_1 have lengths $(2, 1)$. The horizontal (blue) segments of Q_2 (ignoring the first one) have lengths $(1, 3, 1)$. We will then say $(Q_1, Q_2) \in \mathcal{P}_{(3,1,1),(2,1)}$.

Definition 4.3.2. For partitions λ, μ the subset $\mathcal{P}_{\lambda,\mu} \subset \mathcal{P}(|\lambda|, |\mu|)$ is the set of polyominoes $P = (P_1, P_2)$ whose horizontal segments of P_2 have lengths that rearrange to λ, and P_1 has vertical segments whose lengths rearrange to μ.

Here is an example of an element in $\mathcal{P}_{(3,2,2,1),(4,1,1)}$ with area 14:



For any partitions λ, μ, set

$$\mathbf{P}_{\lambda,\mu}(t) = \sum_{P \in \mathcal{P}_{\lambda,\mu}} t^{\text{area}(P)}.$$

We now come to main theorem for Delta operator expansions:

Theorem 4.3.3 (Hicks, R. [3])**.** *For any partition λ, we have*

$$\widetilde{\Delta}_{m_\lambda}\, e_n = \sum_{\mu \vdash n} \mathbf{P}_{\lambda,\mu}(t)\ e_\mu.$$

To prove this theorem, we must show that

$$\sum_{P \in \mathcal{P}_{\lambda,\mu}} t^{\text{area}(P)} = \sum_{S \in \mathbf{M}_{\lambda,\mu}} t^{w(S)}.$$

This will be done by providing a weight preserving bijection $\Gamma : \mathcal{P}_{\lambda,\mu} \to \mathbf{M}_{\lambda,\mu}$:

Definition 4.3.4. The middle path of a polyomino P is a walk along the area cells of P, beginning from the bottom left corner in the North direction, ending in the upper-right corner, satisfying the following conditions:

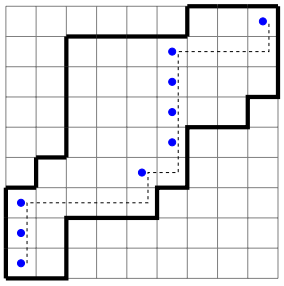
1. If we are traveling North, then we turn East if the current row of cells touches (from below) an East step of P_1.

2. If the current cell is adjacent to a North segment of P_2, then move North.

We call a cell in the middle path a North step if the cell below is also in the middle path, or if it is the first cell of the path.

The dashed line in the previous two examples is this middle path. Here is another example.

The North steps of the middle path are labeled here by blue circles.



Definition 4.3.5. For partitions λ, μ and $P \in \mathcal{P}_{\lambda,\mu}$, we define the element $\Gamma(P) \in \mathbf{M}_{\lambda,\mu}$ as follows:

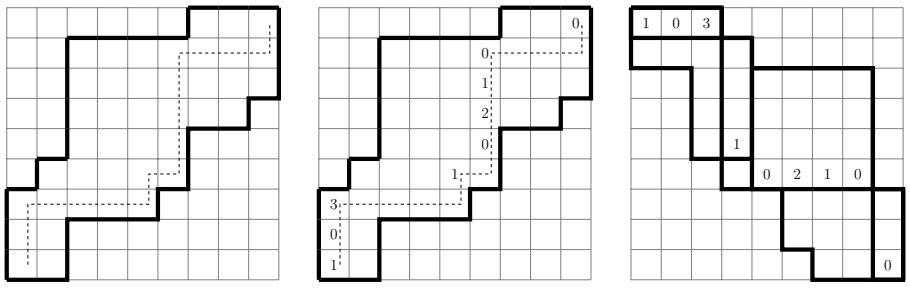
1. Draw the middle path of P.

2. In the i^{th} North step of the middle path of P, write the number of East steps of P_2 along the line $y = i - 1$.

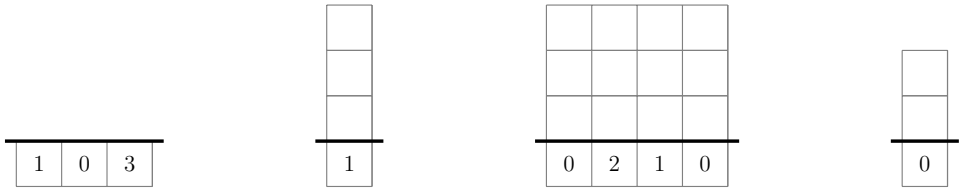
3. Turn the picture 90 degrees clockwise. Now the vertical segments of P_1 are running left to right.

4. To form $\Gamma(P) = (S_1, \ldots, S_{\ell(\mu)})$, we take S_i to be the rectangular tableau delimited from above by the i^{th} vertical segment of P_1 and from below by the labeled cells in the middle path.

Let us take the previous polyomino example and construct its image with its respective steps:



1. 2. 3.



4.

Theorem 4.3.6. *For any λ and μ, $\Gamma : \mathcal{P}_{\lambda,\mu} \to \mathbf{M}_{\lambda,\mu}$ is a weight-preserving bijection.*

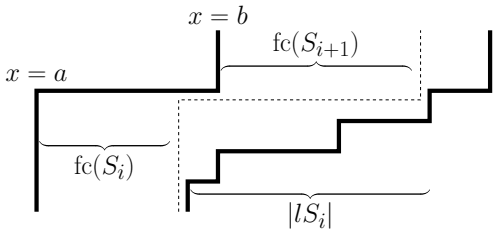
Proof. Let $P = (P_1, P_2) \in \mathbf{M}_{\lambda,\mu}$. First, we will show that $\Gamma(P) = S = (S_1, \ldots, S_\ell)$ is indeed in $\mathbf{M}_{\lambda,\mu}$. The labels in S are given by the horizontal segments of P_2 which, since $P \in \mathcal{P}_{\lambda,\mu}$, rearrange to λ. The lengths of $S_1, \ldots, S_\ell$ are the lengths of the vertical segments of P_1, and therefore rearrange to μ. Furthermore, $\mathrm{fc}(S_i)$ is the number of columns between the i^{th} vertical segment of P_1 and the middle path.

If vertical segment i of P_1 occurs on the line $x = a$, and segment $i + 1$ occurs on $x = b$, we must obviously have $a < b$. The first North step of the middle path occurring on the right of segment $i + 1$ begins after touching a North step of P_2. This occurs at $x = a + \mathrm{fc}(S_i) + |lS_i|$. The distance between the $i + 1^{\mathrm{st}}$ vertical segment of P_1 and the middle path is $\mathrm{fc}(S_{i+1})$. Subtracting this from the the x-coordinate where the middle path hits P_2 we can write b as

$$b = a + \mathrm{fc}(S_i) + |lS_i| - \mathrm{fc}(S_{i+1}).$$

This equality is best described by the following diagram, depicting the i^{th} and $i + 1^{\mathrm{st}}$ vertical

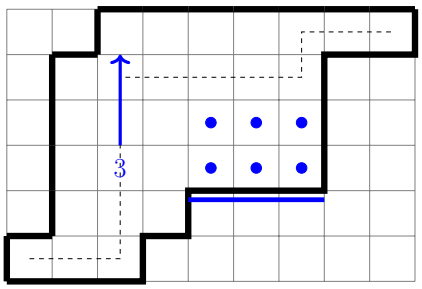
segments of P_1.



Combining this with the inequality we get $a < a + \mathrm{fc}(S_i) + |lS_i| - \mathrm{fc}(S_{i+1})$. In other words,

$$\mathrm{fc}(S_{i+1}) < \mathrm{fc}(S_i) + |lS_i|.$$

Note further that the weight contribution of a label λ_i in the image S is equal to the number of cells between the corresponding horizontal segment in P_2 (of length λ_i) and the middle path. Here, we have depicted a label, in this case 3, and its corresponding horizontal segment in P_2 colored blue. The blue arrow depicts cells in the image that end up on the right-hand side of the label. The blue dots depict the cells between the horizontal segment of P_2 and the middle path (corresponding to the weight contribution of the label).



The weight contributed by the labels is equal to the area of P on the right-hand side of the middle path. Clearly, the block partitions of S give the area to the left of the middle path. So Γ is weight-preserving. Lastly, the middle path begins along the first vertical segment of P_1 and so $\mathrm{fc}(S_1) = 0$. This completes the proof that $\Gamma(P) \in \mathbf{M}_{\lambda,\mu}$.

In some way, we have also shown that this is a bijection. Given $S = (S_1, \ldots, S_\ell) \in \mathbf{M}_{\lambda,\mu}$ we can produce $\Gamma^{-1}(S)$ as follows:

The path P_2 is clearly defined by the sequence $(u_1, \ldots, u_n)$ formed by concatenating the sequences of labels $lS_1, \ldots, lS_\ell$. Then u_i gives the number of East steps in P_2 along the line $y = i - 1$. This uniquely determines P_2. The middle path is drawn by beginning with a vertical segment of length the same as S_1. Then draw East steps until the path is in a cell that touches a North step of P_2. Then go North for the same length as S_2, then travel East until the path touches a North step of P_2, and continue in this way until reaching the last step of P_2. The i^{th} vertical segment of P_1 is then placed $\mathrm{fc}(S_i)$ units left of the middle path. This produces $P = (P_1, P_2)$.

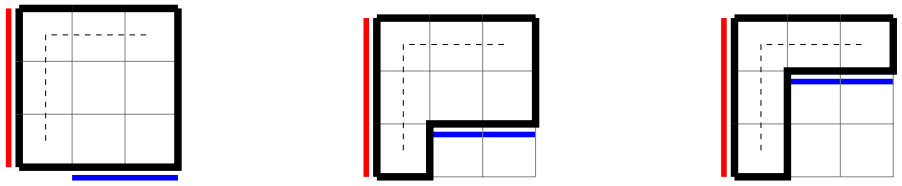
The construction described above forces the paths P_1 and P_2 to only meet at the end, since the middle path bisects P. The only problem that could potentially arise is in the construction of P_1. We must make sure that if the i^{th} vertical segment of P_1 is placed on the line $x = a$ and the $i + 1^{\text{st}}$ segment is placed on the line $x = b$, then $a < b$. We have seen previously that this is equivalent to the condition that $\mathrm{fc}(S_{i+1}) < \mathrm{fc}(S_i) + |lS_i|$. Therefore, since $S \in \mathbf{M}_{\lambda,\mu}$, P is a polyomino. □

This proves Theorem 4.3.3.

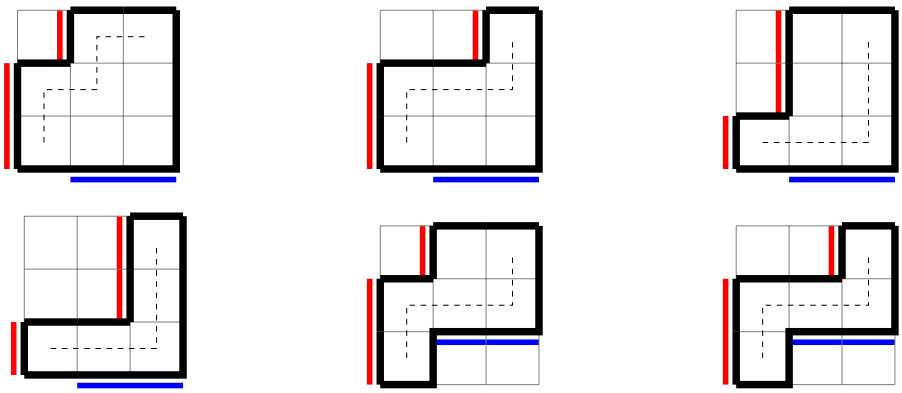
4.3.1 Computing a full example

In this subsection, we are going to take the time to compute in full a symmetric function $\widetilde{\Delta}_{m_\lambda} e_n$ using the combinatorial description we have developed. Let us take $\lambda = (2)$ and $n = 3$. To

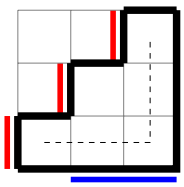
get the coefficient of e_3, we must list all the polyominoes in $\mathcal{P}_{(2),(3)}$:



The area numbers of each polyomino is respectively given by $4, 2$ and 0.To get the coefficient of $e_{2,1}$, we must list all the polyominoes in $\mathcal{P}_{(2),(2,1)}$:



The area numbers here are $3, 2, 2, 0, 1, 0$. Lastly, to get the coefficient of $e_{1,1,1}$, we must list all the polyominoes in $\mathcal{P}_{(2),(1,1,1)}$:

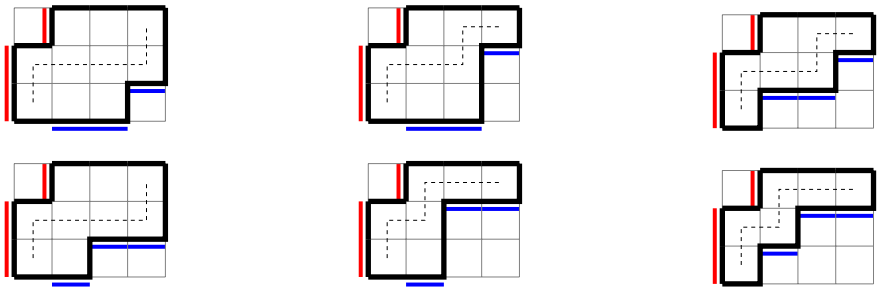


Putting these three cases together, we get the expansion

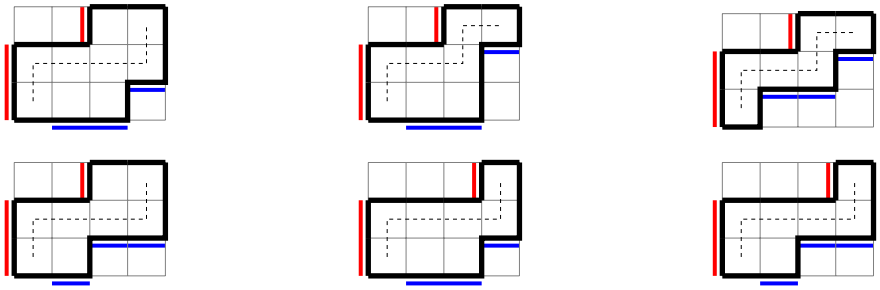
$$\widetilde{\Delta}_{m_2} e_3 \quad = \quad (t^3+t^2+t^2+t+2)e_{2,1} \quad + \quad (t^4+t^2+1)e_3 \quad + \quad te_{1,1,1}.$$

The number of polyominoes grows very fast, so the next most reasonable example would be to list the elements of $\mathcal{P}_{(2,1),(2,1)}$:

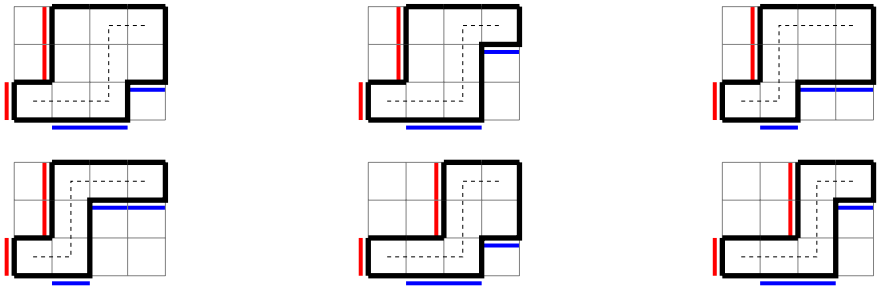
When $P_1 = NNENEEE$, we have



When $P_1 = NNEENEE$ or $NNEEENE$, we have



When $P_1 = NENNEEE$ or $NEENNEE$, we have



All together, we get

$$\widetilde{\Delta}_{m_{2,1}}\, e_3\Big|_{e_{2,1}} = 4 + 4t + 5t^2 + 4t^3 + t^4.$$

4.4 On Delta operators applied to the elementary basis

The purpose of this section is to give a combinatorial interpretation for the symmetric function $\widetilde{\Delta}_{m_\lambda}\, e_\mu$ for any two partitions λ, μ. In some ways, this is our most general result, since it completely describes $\widetilde{\Delta}_F$ on a classical symmetric function basis $\{e_\mu\}_\mu$, whenever F is given by a monomial expansion. There is an important observation we must make first, which describes the multiplicativity of Delta operators at $q = 1$.

Proposition 4.4.1. *Suppose we have a symmetric function H which satisfies some summation formula*

$$H[X+Y] = \sum_{r=1}^{N} F_r[X] G_r[Y].$$

Then for any two symmetric functions f and g, we have

$$\widetilde{\Delta}_H(f \cdot g) = \sum_{r=1}^{N} \widetilde{\Delta}_{F_r}(f) \cdot \widetilde{\Delta}_{G_r}(g).$$

Proof. It suffices to check this over a basis. In particular for $f = h_\lambda[X/(1-t)]$ and $g = h_\mu[X/(1-t)]$, we have

$$\begin{aligned}
\widetilde{\Delta}_H(f \cdot g) \quad &= \quad \widetilde{\Delta}_H\, h_{\lambda \cup \mu}\left[\frac{X}{1-t}\right] \quad = \quad H\left[\sum_{i=1}^{\ell(\lambda)} [\lambda_i]_t + \sum_{j=1}^{\ell(\mu)} [\mu_j]_t\right] h_{\lambda\cup\mu}\left[\frac{X}{1-t}\right] \\
&= \sum_{r=1}^{N} F_r\left[\sum_{i=1}^{\ell(\lambda)} [\lambda_i]_t\right] h_\lambda\left[\frac{X}{1-t}\right] G_r\left[\sum_{j=1}^{\ell(\mu)} [\mu_j]_t\right] h_\mu\left[\frac{X}{1-t}\right] \\
&= \sum_{r=1}^{N} \widetilde{\Delta}_{F_r}\left(h_\lambda\left[\frac{X}{1-t}\right]\right) \widetilde{\Delta}_{G_r}\left(h_\mu\left[\frac{X}{1-t}\right]\right)
\end{aligned}$$

□

For example, recall that the summation formula for the elementary symmetric function is given by

$$e_n[X+Y] = \sum_{i=0}^{n} e_i[X] e_{n-i}[Y].$$

Corollary 4.4.2. *For any two symmetric functions F and G, we have*

$$\widetilde{\Delta}_{e_n}(F \cdot G) = \sum_{i=0}^{n} \widetilde{\Delta}_{e_i}(F)\, \widetilde{\Delta}_{e_{n-i}}(G).$$

If in particular, F is homogeneous of degree $n-k$ and G is homogeneous of degree k (so that $\deg FG = n$*), then*

$$\widetilde{\Delta}_{e_n}(F \cdot G) = \widetilde{\Delta}_{e_{n-k}}(F)\, \widetilde{\Delta}_{e_k}(G).$$

In other words, for any two symmetric functions F and G,

$$\widetilde{\nabla}(F \cdot G) = \widetilde{\nabla}(F)\widetilde{\nabla}(G).$$

The proof of the first equation follows from the previous theorem. The last equality follows from observing that if $i > \deg F$, then $\widetilde{\Delta}_{e_i} F = 0$. Similarly, when $n - i > \deg G$, we have $\widetilde{\Delta}_{e_{n-i}} G = 0$. The only surviving term in the summation formula is when $i = n - k$.

The more useful summation formula for our delta operator expansions is that of the monomial symmetric function. We have

Corollary 4.4.3. *For a partition λ, and any two symmetric functions F and G,*

$$\widetilde{\Delta}_{m_\lambda}(FG) = \sum_{\alpha \cup \beta = \lambda} \widetilde{\Delta}_{m_\alpha}(F)\, \widetilde{\Delta}_{m_\beta}(G).$$

We can now state the main theorem of this section:

Definition 4.4.4. For a composition $\alpha \models n$, $\pi \vdash n$, and any partition λ, let $\mathcal{P}^{\alpha}_{\lambda,\pi}$ be the set of sequences of polyominoes $\vec{P} = (P^1, \ldots, P^{\ell(\alpha)})$ placed corner to corner such that

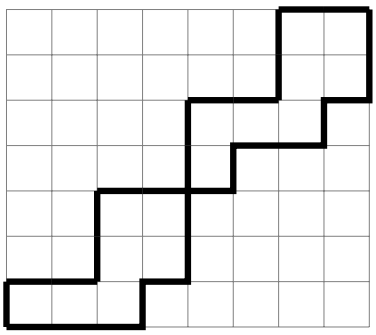
(a) the height of P^i is α_i,

(b) the lengths of the horizontal segments of $P_2^1, \ldots, P_2^{\ell(\alpha)}$ together rearrange to λ, and

(c) the lengths of the vertical segments of $P_1^1, \ldots, P_1^{\ell(\alpha)}$ together rearrange to π.

The area is given by $\text{area}(\vec{P}) = \text{area}(P^1) + \cdots + \text{area}(P^{\ell(\alpha)})$.

For an example, we have the following element of $\mathcal{P}^{(3,4)}_{(2,2,1,1,1),(2,2,2,1)}$ with area 2:



As before, let

$$\mathbf{P}^{\alpha}_{\lambda,\pi}(t) = \sum_{\vec{P} \in \mathcal{P}^{\alpha}_{\lambda,\pi}} t^{\text{area}(\vec{P})}.$$

Theorem 4.4.5. *For any two partitions λ and μ,*

$$\widetilde{\Delta}_{m_\lambda}\, e_\mu = \sum_{\pi \vdash n} e_\pi\, \mathbf{P}^{\mu}_{\lambda,\pi}(t)$$

Proof. Applying the last corollary, we have

$$\widetilde{\Delta}_{m_\lambda}\, e_\mu = \sum_{\bigcup_{j=1}^{\ell(\mu)} \alpha^j = \lambda} \prod_{i=1}^{\ell(\mu)} \widetilde{\Delta}_{m_{\alpha^i}}\, e_{\mu_i}$$

For any such choice of partitions $\alpha^1, \ldots, \alpha^{\ell(\mu)}$, we can use Theorem 4.3.3 to expand the product on the right:

$$\prod_{i=1}^{\ell(\mu)} \widetilde{\Delta}_{m_{\alpha^i}}\, e_{\mu_i} = \prod_{i=1}^{\ell(\mu)} \sum_{\beta^i \vdash \mu_i} e_{\nu^i}\, \mathbf{P}_{\alpha^i,\beta^i}(t).$$

Combining the elementary symmetric functions, we have

$$\prod_{i=1}^{\ell(\mu)} \sum_{\beta^i \vdash \mu_i} e_{\nu^i}\, \mathbf{P}_{\alpha^i,\beta^i}(t) = \sum_{\pi \vdash n} e_\pi \sum_{\vec{\beta} \in \text{PR}(\pi,\mu)} \prod_{i=1}^{\ell(\mu)} \mathbf{P}_{\alpha^i,\beta^i}(t).$$

The result then follows by noting that

$$\mathbf{P}^{\pi}_{\lambda,\mu}(t) = \sum_{\substack{\vec{\alpha}\in \mathrm{PR}(\lambda) \\ \ell(\vec{\alpha})=\ell(\mu)}} \sum_{\vec{\beta}\in \mathrm{PR}(\pi,\mu)} \prod_{j} \mathbf{P}_{\alpha^j,\beta^j}(t),$$

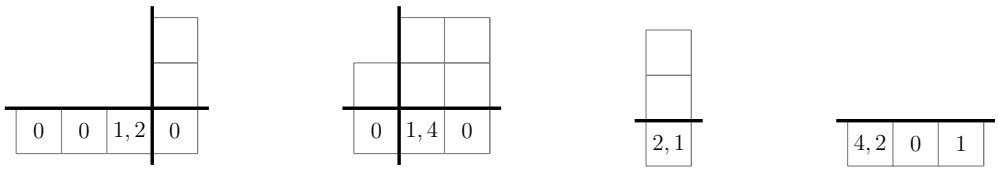
where the first sum ranges over all sequences of (possibly empty) partitions $\vec{\alpha} = (\alpha^1, \dots \alpha^{\ell(\mu)})$ whose union of parts gives λ. □

4.5 Further expansions for other Delta operators

In each of the main results thus far, we have been indexing the Delta operator by a monomial symmetric function m_λ. This led us to filling the bottom rows of the elements in $\mathbf{CCT}(\lambda, \mu)$ by a rearrangement of λ. But we can actually use any other combinatorial symmetric function to fill the cells of the bottom rows. Let us take the forgotten symmetric functions $(-1)^{|\lambda|-\ell(\lambda)} f_\lambda$. Recall that we have

$$(-1)^{|\lambda|-\ell(\lambda)} f_\lambda = \sum_{\vec{\alpha}\in \mathrm{CR}(\lambda)} x_1^{|\alpha^1|} x_2^{|\alpha^2|} \cdots$$

where $\mathrm{CR}(\lambda)$ is the set of sequences $\vec{\alpha} = (\alpha^1, \alpha^2, \dots)$ of (possibly empty) compositions whose union of parts gives λ. Let $\mathbf{CCT}^{f_\lambda}(\mu)$ be the set of column composition tableaux with parts of size μ like in $\mathbf{CCT}(\lambda, \mu)$, but for labels we instead place a composition in each cell so that the union of all compositions rearranges to λ. We define the weight and sign of these labeled composition tableaux in precisely the same manner as in Definition 4.1.1 (every label j contributes to the weight by the term $j \times$ (the number of cells to its right)). Suppose for example that $\mu = (3, 3, 2, 1, 1, 1)$ and $\lambda = (4, 4, 2, 2, 2, 1, 1, 1, 1)$. Then one such object in $\mathbf{CCT}^{f_\lambda}(\mu)$ is $(T_1, T_2, T_3, T_4) =$

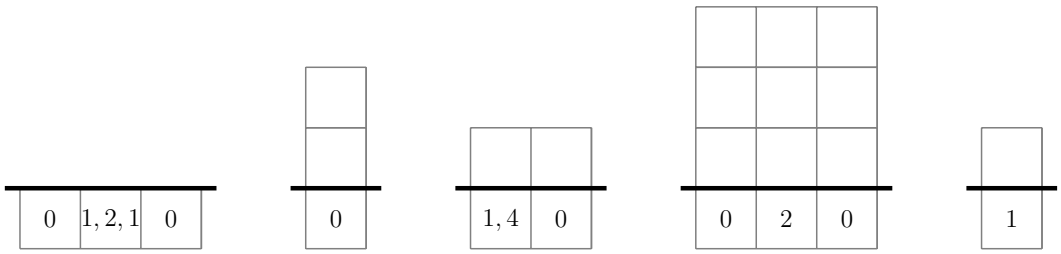


(4.6)

For weights, we have

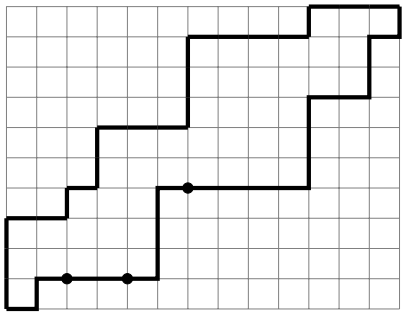
$$w(T_1) = 2 + 3, \qquad w(T_2) = 5 + 5, \qquad w(T_3) = 2 + 0, \qquad w(T_4) = 0 + 12.$$

We listed here the contribution from the partition above and the contribution from the labels in this order, so that in the first tableau, the reflected partition contributes the term 2 and the labels contribute 3. The involution ψ also works precisely in the same way, find the first T_i that can split or combine with T_{i+1}. If no such T_i exists, we leave the object fixed. The fixed points are similarly defined to the elements of $\mathbf{M}_{\lambda,\mu}$. Let $\mathbf{M}_{\mu}^{f_\lambda}$ be the set defined like $\mathbf{M}_{\lambda,\mu}$ only that instead of labeling the cells in the bottom rows with single labels, we label each cell with a composition so that the union of all compositions rearranges to λ. For example, when $\mu = (3, 3, 2, 1, 1)$ and $\lambda = (4, 2, 2, 1, 1, 1, 1)$, we have the following element $S \in \mathbf{M}_{\mu}^{f_\lambda}$:



(4.7)

We can now use the bijection Γ to map these objects to a set of marked polyominoes. Given a polyomino $P = (P_1, P_2)$, we call two consecutive East steps in P_2 a double East step of P. In particular, we will refer to the point at which the two East steps meet as the double East step itself. A polyomino is marked by placing circles on some subset of the double East steps of P. Here is an example of a marked polyomino P' with 3 marks:

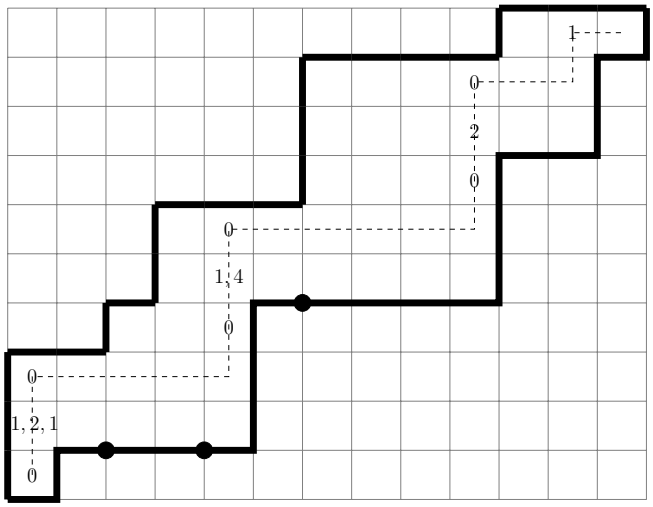


The horizontal segments of P_2' are now split by the marks. Reading from left to right, they have lengths $1, 2, 1, 1, 4, 2, 1$ (recall that we always ignore the first step of the bottom path). The vertical segments of P_1' have lengths $(3, 1, 2, 3, 1)$. We will then write that $P' \in \mathcal{P}^{f_{(4,2,2,1,1,1,1}}_{(3,3,2,1,1)}$. The area statistic is calculated in the same way as ordinary polyominoes. Then

Theorem 4.5.1. *The map Γ extends to a weight-preserving bijection between $\mathbf{M}^{f_\lambda}_\mu$ and $\mathcal{P}^{f_\lambda}_\mu$. As a consequence, we have*

$$(-1)^{|\lambda|-\ell(\lambda)}\Delta_{f_\lambda}e_n = \sum_{\mu\vdash n} e_\mu \sum_{P'\in\mathcal{P}^{f_\lambda}_\mu} t^{\mathrm{area}(P')}.$$

For an example of the bijection, we can look at the two examples above. To see how the labeled column-composition tableaux are mapped to P', we have the following intermediary step of the bijection:



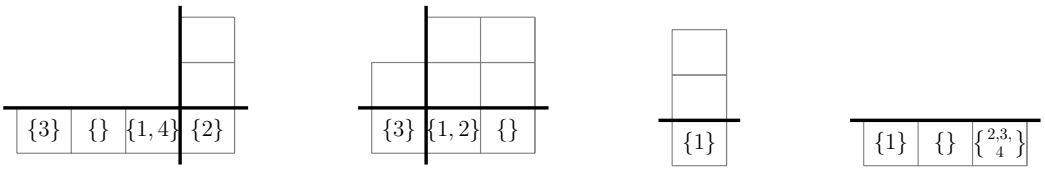
The compositions along the middle path give the marks along the horizontal segments of P_2'.

Now, let us consider the symmetric function $\widetilde{\Delta}_{e_\lambda} e_n$ for a partition λ of length ℓ. Recall that one way of writing the combinatorial formula for the elementary symmetric function is

$$e_\lambda[x_1 + \cdots + x_r] = \sum_{\sqcup_{i=1}^r S_r = \{1^{\lambda_1}, \ldots, \ell^{\lambda_\ell}\}} x_1^{|S_1|} \cdots x_r^{|S_r|},$$

where the sum is over sets S_i whose multiset union is the multiset $\{1^{\lambda_1}, \ldots, \ell^{\lambda_\ell}\}$. Each S_i can only have one occurrence of any particular element.

Let $\mathbf{CCT}^{e_\lambda}(\mu)$ be the set of column-composition tableaux with parts of type μ like in $\mathbf{CCT}(\lambda, \mu)$, but for labels we instead place sets in each cell so that the multiset union of all the sets gives $\{1^{\lambda_1}, \ldots, \ell^{\lambda_\ell}\}$. If a set S is placed in a cell with r cells to its right, then the contribution of S to the total weight is $|S| \cdot r$. This is the only change to calculating the weight contributed by our objects. The sign is calculated in the same way. Suppose for example that $\mu = (3, 3, 2, 1, 1, 1)$ and $\lambda = (4, 3, 3, 2)$. Then one such object in $\mathbf{CCT}^{e_\lambda}(\mu)$ is $(T_1, T_2, T_3, T_4) =$



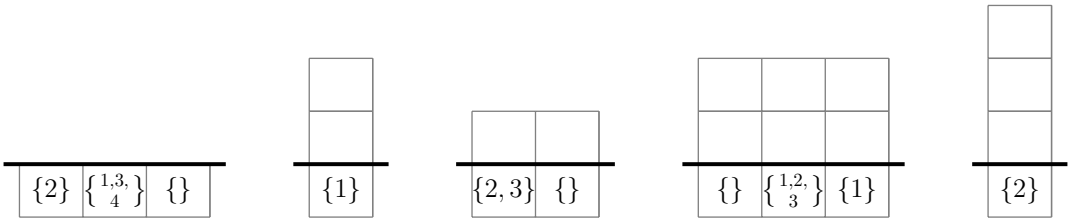
For weights, we have

$$w(T_1) = 2 + (3 + 2), \qquad w(T_2) = 5 + (2 + 2), \qquad w(T_3) = 2 + 0, \qquad w(T_4) = 0 + 2.$$

We listed here the contribution from the partition above and the contribution from the labels in this respective order. For example, the contribution of the first label $\{3\}$ is 3 because the cardinality of this set is 1 and there are 3 cells to its right. The sign is the $(-1)^2 = 1$ since there are 2 vertical

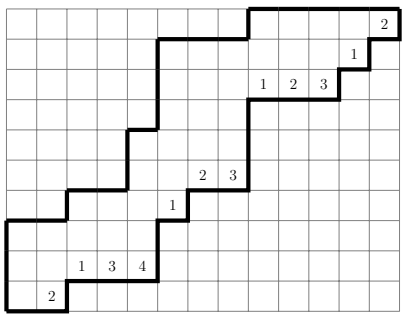
bars in the picture.

The involution ψ works precisely in the same way, but with the replacement of the sets by their cardinalities: find the first T_i that can split or combine with T_{i+1}. If no such T_i exists, we leave the object fixed. The fixed points are similarly defined to the elements of $\mathbf{M}_{\lambda,\mu}$. Let $\mathbf{M}_{\mu}^{e_\lambda}$ be the set defined like $\mathbf{M}_{\lambda,\mu}$ only that instead of labeling the cells in the bottom rows with single labels, we label each cell with a subset of $\{1,\ldots,\ell\}$ so that the multiset union of all sets gives $\{1^{\lambda_1},\ldots,\ell^{\lambda_\ell}\}$. For example, when $\mu=(3,3,2,1,1)$ and $\lambda=(4,4,3,1)$, we have the following element $S\in\mathbf{M}_{\mu}^{e_\lambda}$:



(4.8)

We can now use the bijection Γ to map these objects to a new set of labeled polyominoes. Given a polyomino $P=(P_1,P_2)$, we construct an element of $\mathcal{P}_{\mu}^{e_\lambda}$ by labeling the East steps of P_2 from the set $\{1,\ldots,\ell\}$ with the condition the labels are increasing along the rows from left to right, and that every label i occurs λ_i times. Here is an example of a labeled polyomino in $\mathcal{P}_{\mu}^{e_\lambda}$ for $\lambda=(4,4,3,1)$ and $\mu=(3,3,2,1,1)$:



(4.9)

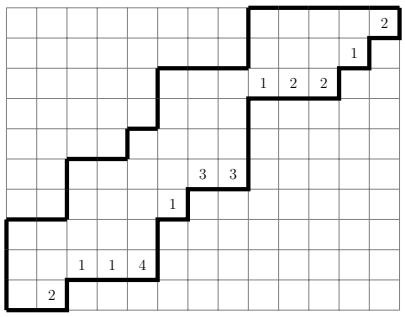
The area statistic is calculated in the same as with ordinary polyominoes.

Theorem 4.5.2. *The map* Γ *extends to a weight-preserving bijection between* $\mathbf{M}_{\mu}^{e_\lambda}$ *and* $\mathcal{P}_{\mu}^{e_\lambda}$. *As a consequence, we have*

$$\Delta_{e_\lambda} e_n = \sum_{\mu \vdash n} e_\mu \sum_{P \in \mathcal{P}_{\mu}^{e_\lambda}} t^{\text{area}(P)}.$$

For an example of the bijection, we can see that the previous polyomino is the image of the tableaux on line (4.9).

Now for the homogeneous symmetric function, we simply allow the rows of P_2 to be weakly increasing. Let $\mathcal{P}_{\mu}^{h_\lambda}$ be defined like the set of labeled polyominoes $\mathcal{P}_{\mu}^{e_\lambda}$ only that we now allow the rows to be weakly increasing. For example, changing the labels of the previous example would give the labeled polyomino which we denote by Q:



Note that the number of 1's is 5, the number of 2's is 4, the number of 3's is 2 and the number of 4's is 1. We then say that $Q \in \mathcal{P}_{(3,2,2,2,1)}^{h_{(5,4,2,1)}}$. Using the same reasoning as before, we have

Theorem 4.5.3. *For any partition* λ, *we have*

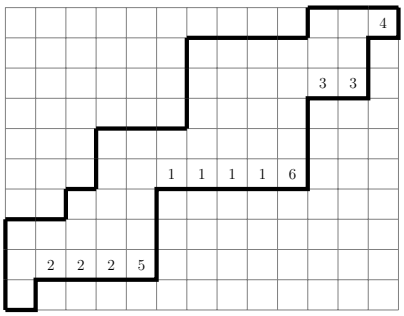
$$\widetilde{\Delta}_{h_\lambda} e_n = \sum_{\mu \vdash n} e_\mu \sum_{Q \in \mathcal{P}_{\mu}^{h_\lambda}} t^{\text{area}(Q)}.$$

Lastly, suppose we would like to index the Delta operator by a power symmetric function.

Recall the combinatorial formula

$$p_\lambda[x_1 + \cdots + x_r] = \sum_{1 \le i_1, \ldots, i_\ell \le r} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{i_\ell}.$$

In other words, for each λ_j, we pick an indeterminate x_{i_j} on which to place the part λ_j as an exponent. We are going to let $\mathcal{P}_\mu^{p_\lambda}$ be the subset of $\mathcal{P}_\mu^{h_\lambda}$ with the extra condition that for each i, the collection of cells containing i are all in the same row. The labeled polyomino



would be an example of an element in $\mathcal{P}_{(3,3,2,2,1,1)}^{p_{(4,3,2,1,1,1)}}$. Then

Theorem 4.5.4. *For any partition λ, we have*

$$\widetilde{\Delta}_{p_\lambda} e_n = \sum_{\mu \vdash n} e_\mu \sum_{Q \in \mathcal{P}_\mu^{p_\lambda}} t^{\mathrm{area}(Q)}.$$

We hope at this point that it is clear how an expansion of a symmetric function into monomials gives a way to label the polyominoes. The last example we will study merits its own subsection.

4.5.1 On Delta operators indexed by Schur functions

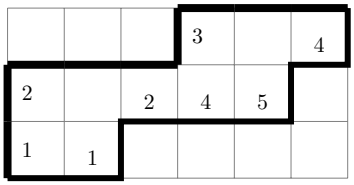
One of the open questions in this area is proving the Schur positivity of the symmetric function $\Delta_{s_\lambda} e_n$, first conjectured in [27]. It would then be nice to find a set of objects $\mathcal{LP}_{\lambda,\mu}$ with

two statistics $\mathrm{stat}_1, \mathrm{stat}_2$ so that we have the Schur expansion

$$\Delta_{s_\lambda} e_n = \sum_{\mu \vdash n} s_\mu \sum_{P \in \mathcal{LP}_{\lambda,\mu}} t^{\mathrm{stat}_1(P)} q^{\mathrm{stat}_2(P)}.$$

If such an expansion were possible, then this chapter has found a set of combinatorial objects and the first statistic. To give a proper Schur expansion, we would then need to find the values for the second statistic. We will now construct $\mathcal{LP}_{\lambda,\mu}$, the set of lattice polyominoes of type λ, μ:

A labeled polyomino P' is constructed from a polyomino $P = (P_1, P_2)$ by labeling the North steps in P_1 and labeling the East steps of P_2 with elements from the alphabet $\{1, 2, \dots\}$ such that the columns of P_1 are increasing from bottom to top, and the rows of P_2 are increasing when read left to right. For a labeled polyomino P', we denote by $\mathrm{cword}\, P'$ the column word, formed by reading the labels of P_1 from bottom to top; and we denote by $\mathrm{rword}\, P'$ the row word formed by reading the labels of P_2 from left to right. For example, we have the following labeled polyomino:



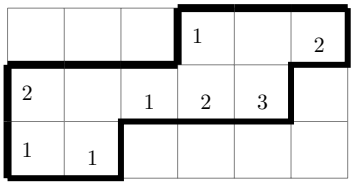
Here, we have $\mathrm{cword}(P') = 1\ 2\ 3$ and $\mathrm{rword}(P') = 1\ 2\ 4\ 5\ 4$. For a word $w = w_1 \cdots w_\ell$ from the alphabet $\{1, 2, \dots\}$, we will say that w is a lattice word if for any $i < j$ and any partial sequence $w_1 \cdots w_k$ the number of occurrences of i is at least the number of occurrences of j. For instance, 1 2 1 is a lattice word, and 1 1 2 3 2 is also a lattice word. A lattice word defines a standard tableau by the following rule: given $w_1 \cdots w_\ell$, construct the tableau $T(w)$ by the rule that if $w_j = i$, then we place j in column i. If the shape of $T(w)$ is the partition λ, then we will say that the type of w

is λ and write type$(w) = \lambda$. Following the previous examples, we have

type (1 2 1) = shape $\begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$ $= (2,1)$, type (1 1 2 3 2) = shape $\begin{array}{|c|c|c|}\hline 2 & 5 & \\ \hline 1 & 3 & 4 \\ \hline \end{array}$ $= (3,2)$.

Definition 4.5.5. A labeled polyomino P' is of type (λ, μ) if type rword $P' = \lambda$ and type cword $P' = \mu$. The collection of lattice polyominoes of type (λ, μ) will be denoted by $\mathcal{LP}_{\lambda,\mu}$.

For example, since the labeled polyomino



has column word 1 2 1 and row word 1 1 2 3 2 (the two lattice words we previously used as examples), we have that this is a lattice polyomino of type $((3,2),(2,1))$. The area statistic is calculated in the same way as the unlabeled polyomino. In the labeled case, it is easy to see that this can be computed by the number of lattice cells in P' with no labels. In this case we have 2 empty cells.

Theorem 4.5.6 (Hicks, R. [3])**.** *For any partition λ and nonnegative integer n,*

$$\widetilde{\Delta}_{s_\lambda} e_n = \sum_{\mu \vdash n} s_\mu \sum_{P' \in \mathcal{LP}_{\lambda,\mu'}} t^{\text{area}(P')}.$$

Proof. Using the two equations

$$e_\eta = \sum_{\mu \vdash n} K_{\mu',\eta} s_\mu \qquad \text{and} \qquad s_\lambda = \sum_{\nu \vdash n} K_{\lambda,\nu} m_\nu$$

we have

$$\begin{aligned}
\widetilde{\Delta}_{s_\lambda}\, e_n &= \sum_{\nu\vdash n} K_{\lambda,\nu}\, \widetilde{\Delta}_{m_\nu}\, e_n \\
&= \sum_{\nu\vdash n} K_{\lambda,\nu} \sum_{\eta\vdash n} e_\eta \sum_{S\in\mathcal{P}_{\nu,\eta}} t^{w(S)} \\
&= \sum_{\nu\vdash n} K_{\lambda,\nu} \sum_{\eta\vdash n}\sum_{\mu\vdash n} K_{\mu',\eta} s_\mu \sum_{S\in\mathcal{P}_{\nu,\eta}} t^{w(S)} \\
&= \sum_{\mu\vdash n} s_\mu \sum_{\nu\vdash n}\sum_{\eta\vdash n}\sum_{S\in\mathcal{P}_{\nu,\eta}} K_{\lambda,\nu}K_{\mu',\eta} t^{w(S)}.
\end{aligned}$$

We can use lattice words to give a nice combinatorial interpretation of

$$\sum_{\nu\vdash n}\sum_{\eta\vdash n}\sum_{S\in\mathcal{P}_{\nu,\eta}} K_{\lambda,\nu}K_{\mu',\eta} t^{w(S)}.$$

Recall that $K_{\lambda,\nu}$ is the number of semistandard tableaux of shape λ and content ν; and for any rearrangement $\alpha \in \mathrm{R}(\nu)$, $K_{\lambda,\nu} = K_{\lambda,\alpha}$. Given $S = (S_1, S_2) \in \mathcal{P}_{\nu,\eta}$, the horizontal segments of S_2, when read from left to right, have lengths given by some $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\nu)}) \in \mathrm{R}(\nu)$. We can represent a semistandard tableau of shape λ and content α by labeling the rows of S_2 in an increasing manner with a lattice word w of type λ. A semistandard tableau R is then formed from $T(w)$ by replacing $1, \ldots, \alpha_1$ by 1, $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$ by 2, and so on until we replace $\alpha_1 + \cdots + \alpha_{\ell(\nu)-1} + 1, \ldots, |\alpha|$ by $\ell(\alpha)$. Since the rows of S_2 are labeled in an increasing fashion, the cells labeled i form a horizontal strip, meaning R is indeed semistandard. Similarly, we can fill the columns of S_1 with a lattice word of type μ'. This proves the theorem.

□

Acknowledgements. We must thank Angela Hicks for the amazing results of this chapter. Chapter 4, in part, is comprised of published work by the author of this dissertation and Angela Hicks as *Delta operators at $q = 1$ and polyominoes*, [3]. The work of the current author was supported by NSF grant DMS-1700233.

Chapter 5

Delta operators on the power basis

In this chapter, we are going to study applications of $\widetilde{\Delta}_{m_\lambda}$ to power basis elements. We will see that for any nonempty partition λ, $\widetilde{\Delta}_{m_\lambda}(-1)^{n-1}p_n$ is e-positive, enumerated by certain marked polyominoes. In general, $\widetilde{\Delta}_{m_\lambda}\,\omega p_\mu$ is not e-positive, but we can still give an expansion using a mixture of the bases $\{\omega p_\mu\}_\mu$ and $\{e_\mu\}_\mu$. For completeness, the first section will give a combinatorial proof of Newton's identity giving the expansion of the power basis in terms of the homogeneous or elementary basis. A short, combinatorial proof for what is more typically called Newton's identity is given in [17].

5.1 A combinatorial proof of Newton's identity

In what follows, we are going to have to interpret the product $n \cdot |R(\eta)|/\ell(\eta)$ for a given partition $\eta \vdash n$. The set of selected rearrangements $SR(\eta)$ is defined by drawing the following pictures:

1. Select a rearrangement $(\rho_1, \ldots, \rho_{\ell(\eta)})$ of η and draw it as a list of rows.

2. Select one of the cells in the part ρ_1 and inscribe the cell with a circle.

For example, if we want to find an element of $SR((4,3,1,1))$, we first have to select a rearrangement of the parts, say $(3,1,4,1)$. This rearrangement leads to 3 options for placing the circle, one choice

of which would be drawn as

Then

$$|SR(\eta)| = \frac{n}{\ell(\eta)}|R(\eta)|.$$

The reason for this is that $n \cdot |R(\eta)|$ is the number of ways of drawing a rearrangement and filling a cell with a circle. There are $\ell(\eta)$ circular rearrangements which can be identified by the selected cell. Therefore, dividing by $\ell(\eta)$ we can select a representative. We pick the representative where the selected cell is in the first part of the rearrangement.

Theorem 5.1.1 (Newton's Identity)**.**

$$p_n = \sum_{\eta \vdash n} (-1)^{\ell(\eta)-1} |SR(\eta)| h_\eta.$$

Proof. Create a set of labeled selected rearrangements $LSR(\eta)$ made by first selecting an element of $SR(\eta)$, say for example

then filling each cell with positive integers so that the labels are weakly increasing from left to right within each part, like in

If $L \in LSR(\eta)$ has a_i cells containing the label i, then we give L a weight of

$$wt(L) = (-1)^{\ell(\eta)-1} x_1^{a_1} x_2^{a_2} \cdots .$$

Then we have

$$\sum_{\eta \vdash n} \sum_{L \in LSR(\eta)} wt(L) = \sum_{\eta \vdash n} (-1)^{\ell(\eta)-1} |SR(\eta)| h_\eta.$$

We can show that after applying a sign-reversing involution, the only remaining tableaux are of the form

whose weight is x_i^n. Summing over all i, we would get $x_1^n + x_2^n + \cdots = p_n$. This would prove the Newton identity.

The sign-reversing involution can be described by the following steps: Let $L = (T_1, \ldots, T_\ell) \in LSR(\eta)$, and let i be the label in T_1 that is contained in the circled cell.

1. If T_1 contains a label less than i, let S_1 be the portion of T_1 with labels less than i, and let S_2 be the remaining part. Then send L to $(S_2, T_2, \ldots, T_\ell, S_1)$. For instance, here $i = 3$:

2. If T_1 does not contain a label less than i, but T_ℓ ends with a label less than i, let S be the concatenation of T_ℓ, T_1 and send L to $(S, T_2, \ldots, T_{\ell-1})$. For an example, reverse the previous example.

3. If neither conditions 1 or 2 are satisfied, and T_1 has labels after the circle, let S_2 be the part of T_1 after the circle, and S_1 the beginning part. Then send L to $(S_1, S_2, T_2, \ldots, T_\ell)$. For example,

4. Otherwise if T_1 has no labels after the circle, but T_2 starts with an element larger than or equal to i, let S be the concatenation of T_1, T_2 and send L to $(S, T_3, \ldots, T_\ell)$. For an example, we can look at the reverse of the previous example.

5. Lastly, if no case above holds, then we apply an involution on the remaining portion $T' = (T_2, \ldots, T_\ell)$:

   (a) If T_2 has length larger than 1, and s_1 is the first column of T_2, s_2 the remaining part, then send T' to $(s_1, s_2, T_3, \ldots, T_\ell)$. For instance,

   (b) If T_2 has length 1 with label c_2, and T_3 begins with a label larger or equal to c_2, then let s be the concatenation of T_2, T_3 and send T' to $(s, T_4, \ldots, T_\ell)$. Again, for an example we can look at the reverse of the picture above.

   (c) If neither of these two subcases hold, proceed by induction on $(T_3, \ldots, T_\ell)$.

6. If no case above holds, leave L fixed.

Suppose $T = (T_1, \ldots, T_\ell)$ is fixed with label i in the circle of T_1. Steps 1 and 2 ensure that T_1 has no labels less than i and T_ℓ ends with something larger than i. Steps 3 and 4 ensure that the circle in T_1 is in the last cell and T_2 begins with a label less than i. Step 5 ensures that $T_2, \ldots, T_\ell$ are single cells with labels $c_2, \ldots, c_\ell$ respectively. Since they cannot combine, we must have $c_2 > c_3 > \cdots > c_\ell$. Note then that $i > c_2 > \cdots > c_\ell > i$, a contradiction if $c_2, \ldots, c_\ell$ exists. Therefore $T = (T_1)$ is a single row with every cell labeled i and a circle in the last square. This proves the identity.

□

5.2 Delta operators on a power symmetric function

To apply a Delta operator to p_n, we must first find an expansion in terms of the eigenbasis. We begin with

$$p_n\left[\frac{X}{1-t}\right] = \frac{p_n[X]}{1-t^n} = -\sum_{\eta\vdash n}(-1)^{\ell(\eta)}|SR(\eta)|h_\eta\left[\frac{X}{1-t}\right]$$

Therefore,

$$(-1)^{n-1}p_n = (1-t^n)\sum_{\eta\vdash n}(-1)^{n-\ell(\eta)}|SR(\eta)|h_\eta\left[\frac{X}{1-t}\right].$$

We can now apply the Delta operator at $q=1$ to get

$$\widetilde{\Delta}_{m_\lambda}(-1)^{n-1}p_n = (1-t^n)\sum_{\eta\vdash n}(-1)^{n-\ell(\eta)}|SR(\eta)|m_\lambda\left[\sum_i[\eta_i]_t\right]h_\eta\left[\frac{X}{1-t}\right].$$

Using again the Cauchy formula to expand $h_\eta[X/(1-t)]$ we have

$$\begin{aligned}\widetilde{\Delta}_{m_\lambda}(-1)^{n-1}p_n &= (1-t^n)\sum_{\eta\vdash n}(-1)^{n-\ell(\eta)}|SR(\eta)|m_\lambda\left[\sum_i[\eta_i]_t\right]\sum_{\mu\vdash n}e_\mu\sum_{\vec{\nu}\in\mathrm{PR}(\mu,\eta)}f_{\vec{\nu}}\left[\frac{1}{1-t}\right]\\ &= \sum_{\mu\vdash n}e_\mu\sum_{\eta\vdash n}\sum_{\vec{\nu}\in\mathrm{PR}(\mu,\eta)}(-1)^{n-\ell(\eta)}|SR(\eta)|m_\lambda\left[\sum_i[\eta_i]_t\right](1-t^n)f_{\vec{\nu}}\left[\frac{1}{1-t}\right]\end{aligned}$$

The only difference in this expression and that of (4.3) is the replacement of

$$f_\eta[1-t] = (-1)^{n-\ell(\eta)}\sum_{\alpha\in R(\eta)}(1-t^{\alpha_1})$$

$$\text{by}\quad (-1)^{n-\ell(\eta)}|SR(\eta)|(1-t^n) = (-1)^{n-\ell(\eta)}\sum_{\alpha\in R(\eta)}\alpha_1\cdot(1-t^n).$$

In the previous chapter, we used the fact that for $\alpha \in R(\eta)$ and $\vec{\nu} \in \mathrm{PR}(\mu, \alpha)$ we have the generating function

$$(1-t^{\alpha_1})f_{\vec{\nu}} = (-1)^{n-\ell(\mu)}\overline{F}_{\nu^1}F_{\nu^2}\cdots F_{\nu^{\ell(\mu)}} = (-1)^{n-\ell(\mu)}\sum_{\substack{T_i \in CC_{\nu^i} \\ \mathrm{fc}\, T_1 = 0}} t^{w(T_1)+\cdots+w(T_{\ell(\mu)})}$$

We now use that $t^n f_{\vec{\nu}} = \prod_i t^{|\nu^i|} f_{\nu^i}$ is the generating function for sequences of column-composition tableaux $(T_1, \ldots, T_{\ell(\mu)})$, $T_i \in CC_{\nu^i}$, with the condition that $\mathrm{fc}(T_i) > 0$ (we can think of the factor $t^{|\nu^i|}$ as adding 1 cell to every column of the tableaux enumerated by F_{ν^i}). This lets us write

$$(1-t^n)f_{\vec{\nu}} = (-1)^{n-\ell(\mu)}\sum_{\substack{T_i \in CC_{\nu^i} \text{ for all } i \\ \mathrm{fc}\, T_j = 0 \text{ for some } j}} t^{w(T_1)+\cdots+w(T_{\ell(\mu)})}.$$

This is the essential difference between the computation here, and that of the previous chapter. We now define the set of selected column-composition tableaux $S = (S_1, \ldots, S_\ell) \in \mathbf{SCT}(\lambda, \mu)$ similar to $\mathbf{CCT}(\lambda, \mu)$ from Definition 4.1.1. The changes can be found in step 3 (where we replace the condition that $\mathrm{fc}\, T_1 = 0$ with $\mathrm{fc}\, T_j = 0$ for some j) and step 4 (where we select one cell in the bottom row of T_1 and inscribe it with a circle).

Definition 5.2.1. A Labeled Selected Column-Composition Tableau $S = (S_1, \ldots, S_\ell) \in \mathbf{SCT}(\lambda, \mu)$ is constructed by the following process:

1. Select a partition $\eta \vdash n$ and $\rho \in R(\eta)$.

2. Select $(\gamma^1, \ldots, \gamma^{\ell(\rho)}) \in \mathrm{PR}(\mu, \rho)$.

3. For $i = 1, \ldots, \ell(\rho)$ select $T_i \in CC_{\gamma^i}$ with the condition that for some j, $\mathrm{fc}\, T_j = 0$.

4. Distribute a rearrangement of the list $\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0^{n-\ell(\lambda)}$ along the cells in the bottom rows of $T_1, \ldots, T_{\ell(\eta)}$, and circle one of the cells in the bottom row of T_1 to obtain a sequence of labeled tableaux $S = (S_1, \ldots, S_{\ell(\eta)})$. From here on, we adopt the following notation: We denote by $lS_i = (r_1, \ldots, r_{|\gamma^i|})$ the sequence of labels in S_i when read from left to right, and

$|lS_i| = r_1 + \cdots + r_{|\gamma^i|}$ the sum of the labels.

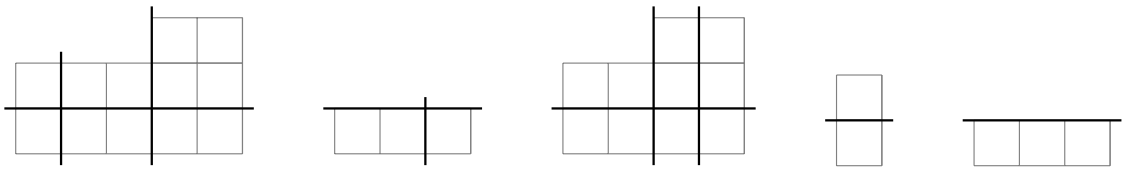
Recall again that we define

$$w(S_i) = w(T_i) + \sum_{\text{label } j \text{ in } S_i} j \cdot (\text{ the number of cells to the right of } j \text{ in } S_i\ ),$$

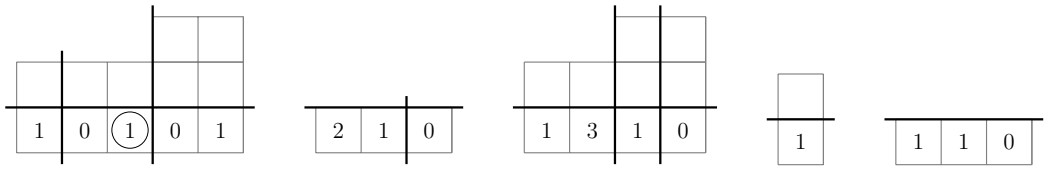
and $w(S) = w(S_1) + \cdots + w(S_{\ell(\eta)})$. The sign of S is again given by

$$\text{sign}(S) = (-1)^{\text{number of vertical bars in } S}.$$

For example, suppose $\mu = (3, 2^4, 1^5)$ and $\lambda = (3, 2, 1^9)$. Then we can first select $\eta = (5, 4, 3^2, 1)$ and the rearrangement $\rho = (5, 3, 4, 1, 3)$. We select $\vec{\nu} = ((2, 2, 1), (2, 1), (2, 1, 1), (1), (3)) \in \text{PR}(\mu, \rho)$, then find column-composition tableaux in $CC_{\nu^1}, \ldots, CC_{\nu^{\ell(\rho)}}$:



Note here that at least one column-composition tableau has no cells in the first column. Lastly, we fill the bottom rows with a rearrangement of λ and select any cell from the first bottom row to inscribe with a circle:

$S' =$ 

This is a labeled selected column-composition tableau.

Proposition 5.2.2. *For any partition λ, we have*

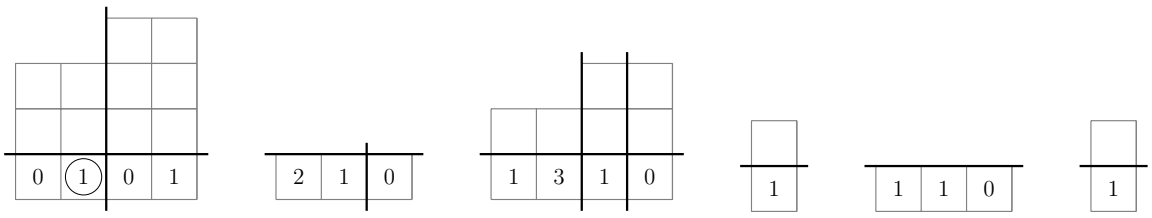
$$\widetilde{\Delta}_{m_\lambda}(-1)^{n-1}p_n = \sum_{\mu\vdash n} e_\mu \sum_{S\in\mathbf{SCT}(\lambda,\mu)} \operatorname{sign}(T)t^{w(T)}.$$

The maps $\overline{\text{split}}_s$ and $\overline{\text{split}}^{-1}$ are defined in the same way as before (in Definition 4.2.1), except now we also have to keep the circle with the label it circumscribes.

Definition 5.2.3. Given $S = (S_1, \ldots, S_\ell) \in \mathbf{SCT}(\lambda, \mu)$, we say that the circle is leading in S if there are no vertical bars on its left (so that it is in the first component of S_1). Define a map $\psi_\circ$ by the following process:

1. If the circle is not leading in S, then there is a right-most vertical bar to the left of the circle, say occurring after column s. Then if $\overline{\text{split}}_s(S_1) = (R_1, R_2)$, set $\psi_\circ(S) = (R_2, S_2, \ldots, S_\ell, R_1)$.
2. If the circle is leading and S_ℓ can join S_1, then we set $\psi_\circ(S) = (\overline{\text{split}}^{-1}(S_\ell, S_1), S_2, \ldots, S_{\ell-1})$.
3. If the circle is leading and S_ℓ cannot join S_1, then we set $\psi_\circ(S) = \psi(S)$.

The above example S' would, for example, be mapped to $\psi_\circ(S') =$



Proposition 5.2.4. *The map $\psi_\circ$ defined above is a weight-preserving, sign-reversing involution whenever λ is nonempty.*

Proof. The proof follows just as in the previous chapter. Clearly, steps 1 and 2 are inverses. And we know Γ in step 3 is a weight-preserving, sign-reversing involution. Thus, we need to show that if S falls into cases 1 or 2, then $\psi_\circ(S)$ also falls into case 1 or 2. And if $S = (S_1, \ldots, S_\ell)$ falls into case

3, then so does $\psi(S)$. It therefore suffices to show that for S in case 3, $\psi(S) = (T_1, \ldots, T_r)$ also falls into case 3.

Suppose the circle is leading in S_1 and S_ℓ cannot join S_1. Then this means that $\mathrm{fc}(S_1) < \mathrm{lc}(S_\ell) + |lS_\ell|$. By definition of ψ, we must always have $\mathrm{lc}(S_\ell) + |lS_\ell| = \mathrm{lc}(T_r) + |lT_r|$. This is because if ψ splits S_ℓ into T_{r-1}, T_r, then the last column of S_ℓ increases by the sum of the labels $|lT_{r-1}|$ in order to create $\mathrm{lc}(T_r)$. In other words, $\mathrm{lc}(T_r) = |lT_{r-1}| + \mathrm{lc}(S_\ell)$. Therefore using that $|lT_{r-1}| + |lT_r| = |lS_\ell|$,

$$\mathrm{lc}(T_r) + |lT_r| = \mathrm{lc}(S_\ell) + |lT_{r-1}| + |lT_r| = \mathrm{lc}(S_\ell) + |lS_\ell|.$$

If instead ψ joins $S_{\ell-1}, S_\ell$ to make T_r, then this argument also works if we interchange T_r, T_{r-1} by $S_\ell, S_{\ell-1}$.

There is a problem with this argument when $S = (S_1)$ consists of a single part, since we would not start with the condition that S_ℓ cannot join S_1. Suppose the circle is leading in S and $\psi(S) = (T_1, T_2)$. We have to make sure that T_2 does not join with T_1, otherwise, we would have something from case 3 landing in case 2. This is only true if λ is nonempty:

If there is a nonempty label i in T_1, then $\mathrm{lc}(T_2) > \mathrm{lc}(S_1)$, since at least i new cells were added to each column to form T_2. In particular $\mathrm{lc}(T_2) > 0$, while (from the definition of $\mathbf{SCT}(\lambda, \mu)$), $\mathrm{fc}(S_1) = \mathrm{fc}(T_1) = 0$. Therefore, T_2 cannot join T_1. On the other hand, if there is a nonzero label in T_2, then to join T_2 and T_1, we would need $\mathrm{fc}(T_1) \geq i > 0$. But since $\mathrm{fc}(T_1) = \mathrm{fc}(S_1) = 0$, we have that this cannot be so.

Therefore, if S falls into case 3 then so will $\psi_\circ(S)$. □

Let $\mathbf{SM}_{\lambda,\mu}$ be the set of fixed points of $\psi_\circ$. Then we have shown that for $\lambda \vdash n$

$$(-1)^{n-1} \widetilde{\Delta}_{m_\lambda} p_n = \sum_{\mu \vdash n} e_\mu \sum_{S \in \mathbf{SM}_{\lambda,\mu}} t^{w(S)}.$$

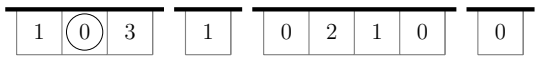
The set $\mathbf{SM}_{\lambda,\mu}$ is very similar to the set $\mathbf{M}_{\lambda,\mu}$. An element $S = (S_1, \ldots, S_\ell) \in \mathbf{SM}_{\lambda,\mu}$ is fixed by ψ° if

a. For all i, S_i cannot split, meaning there are no bars.

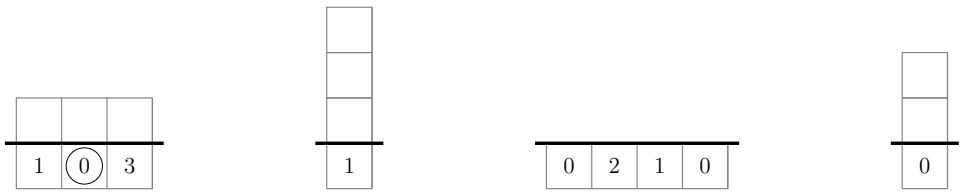
b. For all $i < \ell$, S_i cannot join S_{i+1}, and S_ℓ cannot join S_1.

So each S_i consists of a single part from μ (implying $\mathrm{fc}(S_i) = \mathrm{lc}(S_i)$), $\mathrm{fc}(S_{i+1}) < \mathrm{lc}(S_i) + |S_i|$, and also $\mathrm{fc}(S_1) < \mathrm{lc}(S_\ell) + |lS_\ell|$. For example, suppose $\mu = (4, 3, 1, 1)$ and $\lambda = (3, 2, 1, 1, 1)$. Then to create an element $S = (S_1, S_2, S_3, S_4) \in \mathbf{SM}_{\lambda,\mu}$, first pick a rearrangement of μ, say $(3, 1, 4, 1)$, and select a cell to circle in the first part. These will be the bottom rows of the tableaux.

Fill the cells with a rearrangement of $\lambda, 0^{|\mu|-\ell(\lambda)}$, say $(1, 0, 3, 1, 0, 2, 1, 0, 0)$:



Now, for each i pick a number of rows to place above the bottom row of S_i (which is the number $\mathrm{fc}(S_i)$). This number must be less than the number of rows in S_{i-1} plus the sum of the labels $|lS_i|$. For instance, we will start with $\mathrm{fc}(S_1) = 1$. To pick the number of rows above S_2, we have to pick a number less than $\mathrm{fc}(S_1) + |S_1| = 1 + (3 + 0 + 1) = 5$. We will pick $\mathrm{fc}(S_2) = 3$. Next we pick $\mathrm{fc}(S_3)$ so that it is less than $\mathrm{fc}(S_2) = 3$ plus $|S_2| = 1$. We select $0 < 3 + 1 = 4$. Lastly, we pick $\mathrm{fc}(S_4) = 2 < \mathrm{fc}(S_3) + |S_3| = 0 + (0 + 1 + 2 + 0) = 6$. We must also make sure that $\mathrm{fc}(S_1) = 1 < \mathrm{fc}(S_4) + |lS_4| = 2 + 0$:

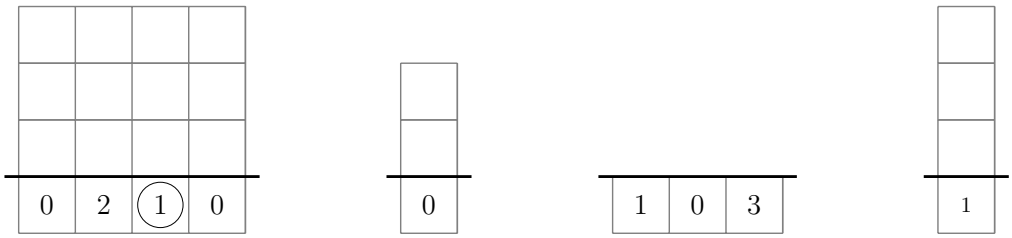


The weight of this object is computed by summing the number of cells above the bottom rows $(3+3+0+2\ )$ to the contribution of the labels $(2+0+5+0)$, giving 24. We have proved

Theorem 5.2.5. *For any nonempty partition* λ

$$\widetilde{\Delta}_{m_\lambda}(-1)^{n-1}p_n = \sum_{\mu\vdash n} e_\mu \sum_{S\in\mathbf{SM}_{\lambda,\mu}} t^{w(S)}.$$

For another example, we have the following fixed point:



Definition 5.2.6. Recall that a polyomino $P=(P_1,P_2)\in\mathcal{P}_{\lambda,\mu}$ consists of two paths that return to each other at $(|\lambda|+1,|\mu|)$. If r is the length of the first vertical segment of P_1, then there must exist some $i\geq r$ for which the distance between P_1 and P_2 along the line $y=i$ is at at most 1 unit. The minimal such i is called the return of P and is denoted by $\mathrm{ret}(P)$. Let $\mathrm{Ret}(P)$ be the set of North steps in the middle path of P that occur before the return of P. The set $\mathcal{RP}_{\lambda,\mu}$ of polyominoes with a marked return is constructed by

1. selecting an element of $P=(P_1,P_2)=\mathcal{P}_{\lambda,\mu}$ where P_1 ends with two East steps, and
2. inscribing a circle in one of the cells of $\mathrm{Ret}(P)$.

The area statistic is calculated the same as in $\mathcal{P}_{\lambda,\mu}$.

Definition 5.2.7. Define the map $\Gamma_\circ:\mathbf{SM}_{\lambda,\mu}\to\mathcal{RP}_{\lambda,\mu}$ by the following process:

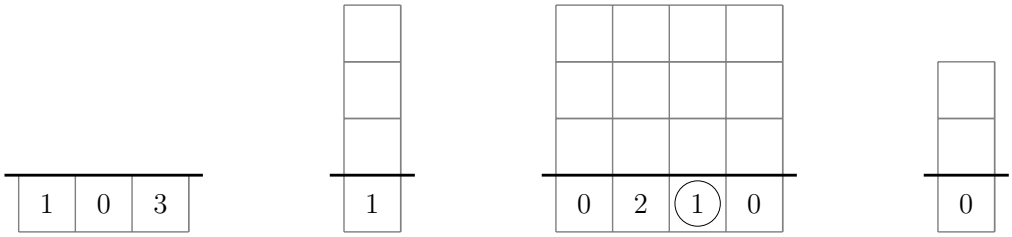
1. If $\mathrm{fc}\,S_1=0$, then set $\Gamma_\circ(S)=\Gamma(S)$. Keep the circle from S_1 inscribed in its corresponding North step along the middle path of $\Gamma(S)$, as is associated by the map Γ from Definition 4.3.5.

2. If $\mathrm{fc}(S_1) > 0$, then there is a largest j such that $\mathrm{fc}(S_j) = 0$. Set

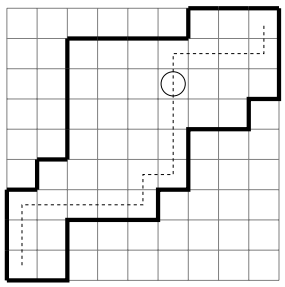
$$\Gamma_\circ(S) = \Gamma(S_j, \ldots, S_\ell, S_1, \ldots, S_{j-1}),$$

again keeping the circle from S_1 in its corresponding North step along the middle path.

For instance, to find the image under $\Gamma_\circ$ for the last example, we first rearrange the parts to give



Applying Γ then gives the following selected polyomino:



Theorem 5.2.8. *The map $\Gamma_\circ$ is a weight-preserving bijection. In particular, if we let*

$$\mathbf{RP}_{\lambda,\mu}(t) = \sum_{P \in \mathcal{RP}_{\lambda,\mu}} t^{\mathrm{area}(P)},$$

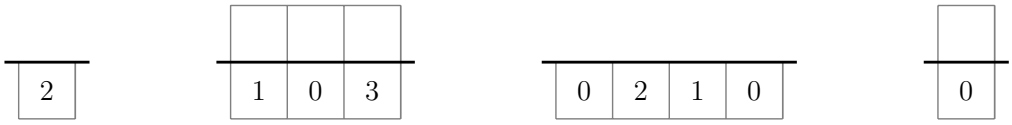
then we have shown that

$$\widetilde{\Delta}_{m_\lambda}\ (-1)^{n-1} p_n = \sum_{\mu \vdash n} e_\mu\ \mathbf{RP}_{\lambda,\mu}(t).$$

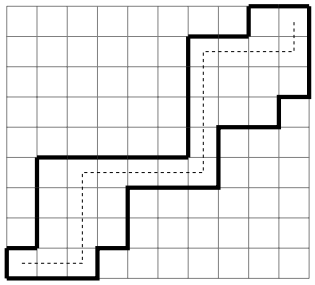
Proof. The bijection Γ gives us an injection into a subset of elements $P \in \mathcal{P}_{\lambda,\mu}$ with a marked

cell in $\mathrm{Ret}(P)$. If $S = (S_1, \ldots, S_\ell)$ is sent to $P = \Gamma_\circ(S) = \Gamma(S_j, \ldots, S_\ell, S_1, \ldots, S_{j-1})$, then we know that $\mathrm{fc}(S_j) = 0$ and $\mathrm{fc}(S_j) < \mathrm{lc}(S_{j-1}) + |lS_{j-1}|$ (since no two consecutive S_i can join). Thus $\mathrm{lc}(S_{j-1}) + |lS_{j-1}| > 0$, meaning one of these quantities is nonzero. If $|lS_{j-1}| > 0$, then the middle path must end in an East step since the distance between the last North step of the middle path and P_2 is nonzero. If $\mathrm{lc}(S_{j-1}) > 0$, then the number of cells between the last vertical segment of P_1 and the middle path is at least 1. This gives that in either case, P_1 must end with two East steps. For an element of $\mathcal{RP}_{\lambda,\mu}$, the inverse is an element of $\mathbf{SM}_{\lambda,\mu}$ for this same reason. This concludes the proof. □

The construction of all elements in $\mathbf{SM}_{\lambda,\mu}$ can be done by first selecting an element of $\mathbf{M}_{\lambda,\mu}$ whose last part has either a nonzero label or at least one cell in each column. For example, suppose we start with the element

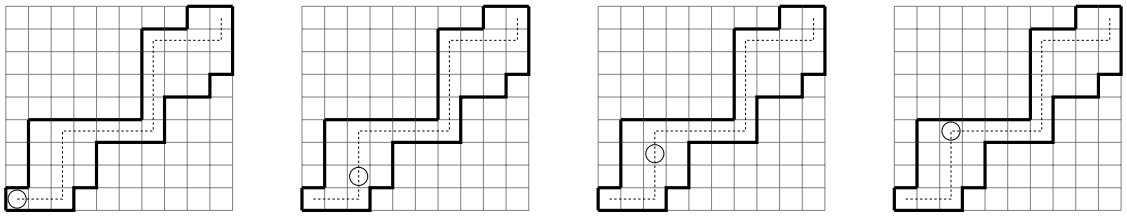


of $\mathbf{M}_{(3,2,2,1,1),(4,3,1,1)}$. Note here that the last part has no nonzero label, but it does have a cell in its column. The corresponding polyomino under the map Γ is



The return is 4. Selecting one of the North steps in the middle path before the return gives the

following four possible elements of $\mathbf{SM}_{(3,2,2,1,1),(4,3,1,1,1)}$:



5.2.1 Some general and special cases

We can define, as we did in Section 4.5, labeled polyominoes $\mathcal{RP}_\mu^{f_\lambda}, \mathcal{RP}_\mu^{e_\lambda}, \mathcal{RP}_\mu^{h_\lambda}$, and $\mathcal{RP}_\mu^{p_\lambda}$ with now a selection before the return.

Theorem 5.2.9. *For any partition λ, we have*

$$(-1)^{|\lambda|-\ell(\lambda)}\,\widetilde{\Delta}_{f_\lambda}(-1)^{n-1}p_n = \sum_{\mu\vdash n} e_\mu \sum_{P\in\mathcal{RP}_\mu^{f_\lambda}} t^{\mathrm{area}(P)}$$

$$\widetilde{\Delta}_{e_\lambda}(-1)^{n-1}p_n = \sum_{\mu\vdash n} e_\mu \sum_{P\in\mathcal{RP}_\mu^{e_\lambda}} t^{\mathrm{area}(P)}$$

$$\widetilde{\Delta}_{h_\lambda}(-1)^{n-1}p_n = \sum_{\mu\vdash n} e_\mu \sum_{P\in\mathcal{RP}_\mu^{h_\lambda}} t^{\mathrm{area}(P)}$$

$$\widetilde{\Delta}_{p_\lambda}(-1)^{n-1}p_n = \sum_{\mu\vdash n} e_\mu \sum_{P\in\mathcal{RP}_\mu^{p_\lambda}} t^{\mathrm{area}(P)}.$$

Similar to before, we can use Corollary 4.4.3 to expand $\widetilde{\Delta}_{m_\lambda}\,\omega p_\mu$, and therefore also get $\widetilde{\Delta}_{f_\lambda}\,p_\mu, \widetilde{\Delta}_{e_\lambda}\,p_\mu, \widetilde{\Delta}_{h_\lambda}\,p_\mu$, and $\widetilde{\Delta}_{p_\lambda}\,p_\mu$ using the above theorem. However, these symmetric functions are not in general e-positive because p_n is not e-positive for $n>1$. Let us look at $\widetilde{\Delta}_{m_\lambda}\,\omega p_\mu$. Theorem 5.2.8, together with Corollary 4.4.3, gives

$$\widetilde{\Delta}_{m_\lambda}\,\omega p_\mu = \sum_{\substack{\vec{\alpha}\in\mathrm{PR}(\lambda)\\ \ell(\vec{\alpha})=\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} \widetilde{\Delta}_{m_{\alpha^i}}(-1)^{\mu_i-1}p_{\mu_i},$$

where we are allowing α^i to be empty. If $\alpha^i=(0)$, then we get a term of $(-1)^{\mu_i-1}p_{\mu_i}$ in the product.

This introduces signs, meaning this symmetric function will not have a positive e-expansion. The best we can do is use a mixture of the bases $\{\omega p_\mu\}_\mu$ and $\{e_\mu\}_\mu$.

We start with a collection of polyominoes, similar to those found in Definition 4.4.4.

Definition 5.2.10. For a composition $\alpha \models n$, $\pi \vdash n$, and any nonempty partition λ, let $\mathcal{RP}^{\alpha}_{\lambda,\pi}$ be the set of sequences of polyominoes with marked returns $\vec{P} = (P^1, \ldots, P^{\ell(\alpha)})$ placed corner to corner such that

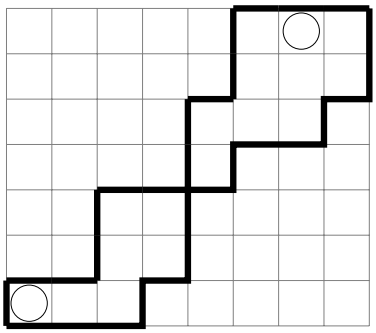
(a) the height of P^i is α_i,

(b) the lengths of the horizontal segments of $P_2^1, \ldots, P_2^{\ell(\alpha)}$ together rearrange to λ, and

(c) the lengths of the vertical segments of $P_1^1, \ldots, P_1^{\ell(\alpha)}$ together rearrange to π.

The area is given by $\operatorname{area}(\vec{P}) = \operatorname{area}(P^1) + \cdots + \operatorname{area}(P^{\ell(\alpha)})$.

For an example, we have the following element of $\mathcal{RP}^{(3,4)}_{(2,2,1,1,1),(2,2,2,1)}$ with area 4:



As before, let

$$\mathbf{P}^{\alpha}_{\lambda,\pi}(t) = \sum_{\vec{P} \in \mathcal{RP}^{\alpha}_{\lambda,\pi}} t^{\operatorname{area}(\vec{P})}.$$

Theorem 5.2.11. *For any partitions λ, μ, we have*

$$\widetilde{\Delta}_{m_\lambda}\, \omega p_\mu = \sum_{S \subseteq \{1,\ldots,\ell(\mu)\}} \omega p_{\mu(S^c)} \sum_{\pi \vdash n} e_\pi\, \mathbf{P}^{\mu(S)}_{\lambda,\pi}(t),$$

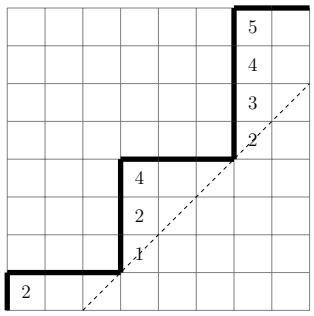
where we have set $\mu(S)$ to be the partition with parts μ_i for $i \in S$, and S^c is the complement of S.

This is proved in the same way as Theorem 4.4.5, by writing the coefficient of e_π as

$$\mathbf{P}_{\lambda,\pi}^{\mu(S)}(t) = \sum_{\substack{\vec{\alpha}\in \mathrm{PR}(\lambda), \alpha^i \neq 0 \\ \ell(\vec{\alpha})=|S|}} \sum_{\vec{\beta}\in \mathrm{PR}(\pi,\mu(S))} \prod_{i=1}^{\ell(\mu)} \mathbf{RP}_{\alpha^i,\beta^i}(t).$$

The only significant result in the literature regarding Delta operators on the power basis are in the form of the Square Paths Conjecture of Loehr and Warrington [47], proved recently by Emily Leven Sergel in her thesis [23] (and also in [24]). The conjecture states that we can write $\nabla(-1)^{n-1}p_n$ as a sum over labeled square paths. We now define these structures and show that at $q = 1$ our results coincide.

A square path $P \in \mathcal{SP}_n^E$ is a lattice path consisting of North and East steps from $(0, 0)$ to (n, n) that ends in an East step. The set of labeled square paths $\mathcal{LSP}_n^E$ is generated by placing positive integers along the north steps of the square path so that the columns are increasing when read from bottom to top. The monomial of a labeled square path $L \in \mathcal{LSP}$ is given by $x^L = x_1^{a_1}x_2^{a_2}\cdots$ if there are a_i occurrences of the label i in L. For example,

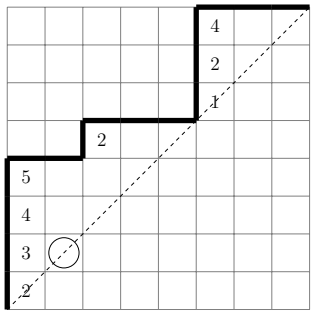


is a square path whose monomial weight is $x_1x_2^3x_3x_4^2x_5$. If the North step on the lowest diagonal starts on diagonal $y = x - d$, then a North step on diagonal $y = x + a$ will contribute $a + d$ units of

area. The above example would then have area 11. The number of diagonal inversions is calculated in a similar way to Dyck paths. Then the Square Paths Theorem can be stated as

$$\nabla(-1)^{n-1}p_n = \sum_{P \in \mathcal{LSP}_n} t^{\text{area}(L)} q^{\text{dinv}(L)} x^L.$$

If we were to circularly rearrange the North and East steps of the square path above, so that the path begins with the right-most, lowest vertical segment, we would get



We have circled a cell on the main diagonal to mark the last column of the original square path. This corresponds to marking a column before the Dyck path returns to the main diagonal. The area is simply the area of the Dyck path. Recall that

$$\widetilde{\nabla}\omega p_n = \widetilde{\Delta}_{m_{1^n}}\, \omega p_n = \sum_{\mu \vdash n} e_\mu \sum_{P \in \mathcal{RP}_{\lambda,1^n}} t^{\text{area}(P)}.$$

The first thing to note is that for $P = (P_1, P_2) \in \mathcal{RP}_{\lambda,1^n}$, P_2 is completely determined: The path must travel from $(1,0)$ to $(n+1,n)$ with only North and East steps; and every East step must be followed by a North step (since $\lambda = 1^n$). This means that P_2 is the path which alternates East, North, East, North, and so on. On the other hand P_1 cannot touch P_2, so it must remain weakly above the line $y = x$ until the last step. Therefore, P_1 is a Dyck path. The return of the polyomino is equal to the return of P_1 as a Dyck path, i.e. the first time P_1 returns to the diagonal $y = x$. This equates the two interpretations when we set $q = 1$.

Chapter 6

An e-positivity phenomenon

In [25], Bergeron observes that certain symmetric functions in the theory of Macdonald polynomials become e-positive after substituting q by $1+u$. For the remainder of this chapter, we set $u = q-1$. The main purpose of this section is to give a new collection of families of symmetric functions that conjecturally have this property. In the end, we will prove several of these cases.

To begin, we say that a symmetric function $F[X;q,t]$ with coefficients in $\mathbb{Q}(q,t)$ exhibits the e-positivity phenomenon if, when expanded in terms of the elementary basis, we have

$$F[X;1+u,t] = \sum_{\mu} c_\mu(u,t) e_\mu,$$

for $c_\mu(u,t) \in \mathbb{N}[u,t]$. One important symmteric function observed to satisfy this property in [25] is ∇e_n. Here, we have the following more general conjecture:

Conjecture 6.0.1. *For any partition λ, the functions $\Delta_{m_\lambda} e_n$ and $\Delta_{m_\lambda}(-1)^{n-1}p_n$ exhibit the e-positivity phenomenon.*

Let us look at the first symmetric function. Recall that from the symmetry of q and t (see

Section 2.5) we have

$$\Delta_{m_\lambda} e_n \Big|_{t=1} = \sum_{\mu \vdash n} e_\mu \sum_{P \in \mathcal{P}_{\lambda,\mu}} (1+u)^{\mathrm{area}(P)}$$

For a polyomino $P = (P_1, P_2)$, let $\mathrm{Area}(P)$ be the set of area cells between P_1 and P_2 not in the middle path. Then since

$$(1+u)^{\mathrm{area}(P)} = \sum_{S \subseteq \mathrm{Area}(P)} u^{|S|},$$

we can rephrase the conjecture as follows:

Conjecture 6.0.2. *Let $\mathcal{P}'_{\lambda,\mu}$ be the set of pairs (P, S) with $P \in \mathcal{P}_{\lambda,\mu}$ and $S \subseteq \mathrm{Area}(P)$. Then there is a statistic,* stat*, on the set $\mathcal{P}'_{\lambda,\mu}$ giving*

$$\Delta_{m_\lambda} e_n = \sum_{\mu \vdash n} e_\mu \sum_{(P,S) \in \mathcal{P}'_{\lambda,\mu}} u^{|S|} t^{\mathrm{stat}(P,S)}.$$

Similarly, define $\mathcal{RP}'_{\lambda,\mu}$ be the set of pairs (P, S) with $P \in \mathcal{RP}_{\lambda,\mu}$ and $S \subseteq \mathrm{Area}(P)$. Then there is a statistic, stat$'$*, on the set $\mathcal{RP}'_{\lambda,\mu}$ giving*

$$\Delta_{m_\lambda} (-1)^{n-1} p_n = \sum_{\mu \vdash n} e_\mu \sum_{(P,S) \in \mathcal{RP}'_{\lambda,\mu}} u^{|S|} t^{\mathrm{stat}'(P,S)}.$$

We should remark that in [43] a statistic is introduced that conjecturally gives $\Delta_{h_k} e_n$ as a sum over labeled polyominoes. This statistic may restrict to give $\Delta_{m_\lambda} e_n$ as a sum over polyominoes in $\bigcup_{\mu \vdash n} \mathcal{P}_{\lambda,\mu}$ and may be the key to finding an expansion like the one conjectured above. These questions are not addressed in this dissertation.

After seeing this e-positivity phenomenon, Adriano Garsia began experimenting and discovered more functions with this phenomenon. The first is the modified Macdonald polynomial when $t = 1$ and will be discussed in the following section. The second section will introduce certain operators $\{B_a\}$ defined in [36] and show that these operators exhibit an e-positivity phenomenon.

These operators are closely related to Hall-Littlewood symmetric functions. This lets us write the Hall-Littlewood polynomial in terms of the h-basis and $\Delta_{e_k} e_n|_{t=0}$ in terms of the elementary basis by playing a combinatorial game. The results from the first few sections can be found in [2].

6.1 The modified Macdonald polynomials at $t = 1$

Recall that we have $\widetilde{\mathrm{H}}_\mu[X;q,t] = \widetilde{\mathrm{H}}_{\mu'}[X;t,q]$. Therefore, from Proposition 2.4.1,

$$\widetilde{\mathrm{H}}_\mu[X;q,1] \;=\; \widetilde{\mathrm{H}}_{\mu'}[X;1,q] \;=\; (q;q)_\mu h_\mu\left[\frac{X}{1-q}\right].$$

The Cauchy formula lets us write

$$\prod_{i=1}^{\ell(\mu)}(q;q)_{\mu_i} h_{\mu_i}\left[\frac{X}{1-q}\right] \qquad = \qquad \sum_{\eta\vdash n} e_\eta \sum_{\vec{\nu}\in \mathrm{PR}(\eta,\mu)} \prod_{i=1}^{\ell(\mu)}(q;q)_{|\nu^i|} f_{\nu^i}\left[\frac{1}{1-q}\right].$$

Theorem 6.1.1 ([2])**.** *The symmetric functions* $\{\widetilde{\mathrm{H}}_\mu[X;q,1]\}_\mu$ *exhibit the e-positivity phenomenon.*

Proof. To show the e-expansion given above is a polynomial in $u = q-1$ with positive integer coefficients, it is sufficient to show that for any partition $\nu \vdash n$, $(q;q)_n f_\nu\left[\frac{1}{1-q}\right]$ is a positive polynomial in u. By Theorem 4.0.1, we can write

$$\begin{aligned}(q;q)_n f_\nu\left[\frac{1}{1-q}\right] \qquad &= \qquad (q;q)_n (q-1)^{n-\ell} \frac{\sum_{\alpha\in R(\nu)} \prod_{i\in c(\alpha)} [i]_q}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \qquad (q-1)^{n-\ell} \sum_{\alpha\in R(\nu)} \prod_{i\in c(\alpha)} [i]_q.\end{aligned}$$

This is clearly a polynomial in u with positive integer coefficients. □

For a special case, when $\mu = (n)$ we have

$$(q;q)_n h_n\left[\frac{X}{1-q}\right] = \sum_{\mu\vdash n} e_\mu \sum_{\alpha\in R(\mu)} \prod_{i\in c(\alpha)} (q^i - 1).$$

For general μ, we then have

$$(q;q)_\mu h_\mu \left[\frac{X}{1-q}\right] = \sum_{\pi\vdash n} e_\pi \sum_{\vec{\alpha}\in \mathrm{CR}(\pi,\mu)} \prod_{i=1}^{\ell(\mu)} \prod_{j\in c(\alpha^i)} (q^j - 1).$$

6.2 The Hall-Littlewood operators

The operators B_a introduced in [36] are defined by setting for any symmetric function F

$$B_a F[X] = F\left[X + \epsilon\frac{1-q}{z}\right] \sum_{k\geq 0} z^k e_k[X]\Big|_{z^a}.$$

For a sequence of integers $p = (p_1, \ldots, p_\ell)$, we set

$$B_p = B_{p_1} \circ \cdots \circ B_{p_\ell} \qquad \text{and} \qquad \mathbf{B}_p = B_p(1).$$

Theorem 6.2.1 ([2])**.** *The family of symmetric functions $\{\mathbf{B}_p\}_p$ exhibits the e-positivity phenomenon.*

To show that $\mathbf{B}_p$ exhibits the e-positivity phenomenon, we first prove the following lemma:

Lemma 6.2.2. *For any partition μ and integer a,*

$$B_a e_\mu = \sum_{\nu\vdash |\mu|+a} e_\nu b_{\mu,\nu}(u)$$

for some $b_{\mu,\nu}(u) \in \mathbb{N}[u]$.

With this given, we can prove the theorem by induction: The base case $B_a(1) = e_a$ is easily verified. Given that

$$\mathbf{B}_p = \sum_{\mu\vdash |p|} e_\mu c_{p,\mu}(u),$$

we can use the lemma to write

$$\begin{aligned} B_a B_p(1) &= \sum_{\mu \vdash |p|} B_a e_\mu c_{p,\mu}(u) \\ &= \sum_{\mu \vdash |p|} \sum_{\nu \vdash |\mu|+a} e_\nu b_{\mu,\nu}(u) c_{p,\mu}(u). \end{aligned}$$

This means that if $\mathbf{B}_p$ is e-positive in terms of u, then so is $B_a \, \mathbf{B}_p$. We now prove the lemma.

Proof. In order to understand the operator B_a, we need to understand how translating an alphabet by another alphabet affects a symmetric function. Let $\mathcal{T}_Y$ be the operator which sends any symmetric function $F[X]$ to $F[X+Y]$. Recall that

$$s_\mu^\perp s_\lambda = s_{\lambda/\mu} = \sum_\nu \langle s_\lambda, s_\mu s_\nu \rangle \; s_\nu.$$

Using the summation formula for Schur functions, we have

$$\mathcal{T}_Y \, s_\lambda = s_\lambda[X+Y] = \sum_{\mu \subseteq \lambda} s_\mu[Y] s_{\lambda/\mu}[X] = \sum_{\mu \subseteq \lambda} s_\mu[Y] s_\mu^\perp s_\lambda[X].$$

We have then showed that

$$\mathcal{T}_Y \, s_\lambda = \sum_\mu s_\mu[Y] s_\mu^\perp s_\lambda.$$

Since $\{s_\lambda\}_\lambda$ is a basis for the space of symmetric functions, the above identity must hold for any symmetric function, meaning we have the operator identity

$$\mathcal{T}_Y = \sum_\mu s_\mu[Y] s_\mu^\perp.$$

When $Y = y$ is a single monomial, then

$$\mathcal{T}_y = \sum_\mu s_\mu[y] s_\mu^\perp = \sum_{k \geq 0} s_{(k)}[y] s_{(k)}^\perp.$$

This means that for any monomial y, we have

$$F[X+y] = \sum_{k\geq 0} y^k h_k^{\perp} F[X] \qquad \text{and} \qquad F[X-y] = \sum_{k\geq 0} (-y)^k e_k^{\perp} F[X]$$

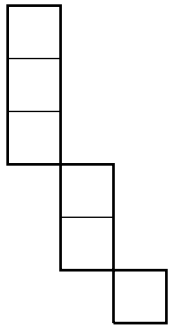
We can then write the shifts in the definition of B_a as

$$F\left[X + \epsilon\frac{1-q}{z}\right] = F\left[X + \frac{\epsilon}{z} - \frac{\epsilon q}{z}\right] = \sum_{r,s} (-1/z)^s (q/z)^r e_r^{\perp} h_s^{\perp} F[X].$$

The operator can then be rewritten as

$$\begin{aligned} B_a &= \sum_{k\geq 0} z^k \underline{e}_k \sum_{r,s} (-1/z)^s (q/z)^r e_r^{\perp} h_s^{\perp}\Big|_{z^a} \\ &= \sum_{r,s} (-1)^s q^r \underline{e}_{a+r+s} e_r^{\perp} h_s^{\perp}, \end{aligned}$$

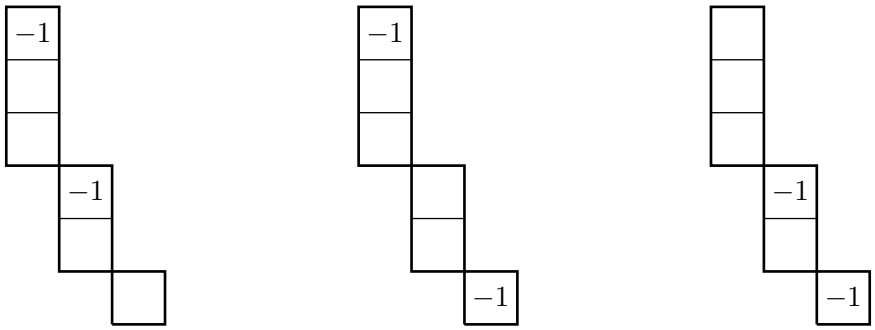
where $\underline{e}_k$ is the operator of multiplication by e_k. To find $B_a e_\mu$, we treat e_μ as a skew Schur function: $e_\mu = s_{(\ell(\mu)^{\mu_{\ell(\mu)}},\ldots,1^{\mu_1})/((\ell(\mu)-1)^{\mu_{\ell(\mu)}},\ldots,1^{\mu_2})}$. The shape $\widehat{\mu} = (\ell(\mu)^{\mu_{\ell(\mu)}},\ldots,1^{\mu_1})/((\ell(\mu)-1)^{\mu_{\ell(\mu)}},\ldots,1^{\mu_2})$ is given by drawing a column for each part of μ and connecting them corner to corner, like below.

$\widehat{(3,2,1)} =$ 

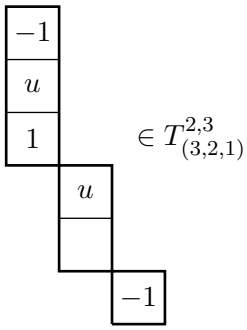
Given r and s, we now construct a set $T_\mu^{r,s}$ of labeled tableaux of shape $\widehat{\mu}$ (drawn in the same way as above). Each element $S \in T_\mu^{r,s}$ will have a weight $wt(S)$ giving

$$B_a e_\mu = \sum_{r+s\leq|\mu|} e_{a+r+s} \sum_{S\in T_\mu^{r,s}} wt(S).$$

To construct $T_\mu^{r,s}$, first select s columns in $\widehat{\mu}$ and inscribe the top cells with a "-1". For instance, if $s = 2$, we have the following three choices for filling $\widehat{(3,2,1)}$:



This describes the effect of applying $(-1)^s h_s^\perp$ to e_μ, in this case giving $(-1)^2 h_2^\perp e_{3,2,1} = e_{2,1,1} + e_{2,2} + e_{3,1}$. Next choose r cells so that they form a vertical strip in the remaining shape and choose for each cell whether to inscribe it with a "1" or "u". One example with $s = 2$ and $r = 3$ is given by

$S =$  $\in T^{2,3}_{(3,2,1)}$

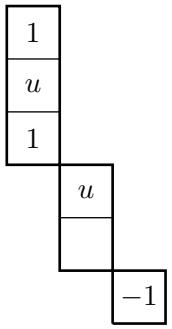
This describes the effect of applying $(1+u)^r e_r^\perp$ to $h_s^\perp e_\mu$. For $i \in \{1, \ldots, \ell(\mu)\}$ let c_i be the number of empty cells in column i of S, and let $\lambda(S)$ be the partition whose parts are given by $c_1, \ldots, c_{\ell(\mu)}$. The above example would then produce the partition (1) since there is one empty cell in column 2. Let $|S|$ be the product of the entries in the cells of S. The weight of this object is defined by the product

$$wt(S) = |S| \cdot e_{\lambda(S)}.$$

The example above would give $wt(S) = (-1) \cdot u \cdot 1 \cdot u \cdot (-1) \cdot e_1 = u^2 e_1$.

We now show $\sum_{S \in T_\mu^{r,s}} wt(S)$ is a positive polynomial in u by using a sign-reversing involution. Given S, scan from left to right for the first top cell in a column that is either inscribed with a 1 or a -1. Switch the 1 into a -1 in the first case, and switch the -1 to a 1 in the second case. If no such entry exists, leave the tableau fixed. This is clearly an involution, and it is sign-reversing since we are negating the value of $|S|$, yet preserving the number of q's. The above example would be paired with

$S =$



Let $U_\mu^{r,s}$ be the subset of $T_\mu^{r,s}$ with the condition that if the top cell of a column is labeled, then it contains a u. We have

$$B_a e_\mu = \sum_{r+s \leq |\mu|} e_{a+r+s} \sum_{S \in U_\mu^{r,s}} wt(S).$$

which is a positive polynomial in u, completing our proof of the lemma. □

We can get an actual formula for the application of B_a on an elementary element e_μ. Note that the set $U_\mu^{r,s}$ is generated by selecting a certain number α_i of labeled cells in the i^{th} column of $\widehat{\mu}$, filling the first cell with a u, and filling the remaining cells with either a 1 or a u. Therefore, each cell that is not first in its column individually contributes a factor of $1 + u = q$. We can write

this as follows:

$$B_a e_\mu = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_{\ell(\mu)}) \\ 0\le\alpha_i\le\mu_i}} e_{a+|\alpha|} \prod_{i=1}^{\ell(\mu)} e_{\mu_i-\alpha_i} \left(uq^{\alpha_i-1}\right)^{\chi(\alpha_i>0)}. \tag{6.1}$$

The Hall-Littlewood polynomial (see [9], [18], [19]) can be attained from the modified Macdonald symmetric function by setting

$$\mathrm{H}_\lambda[X;q] = q^{n(\lambda)}\,\widetilde{\mathrm{H}}_\lambda[X;0,1/q],$$

where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)}(i-1)\cdot\lambda_i$. This version of the Hall-Littlewood polynomial can be written very simply in terms of the B operators:

$$\omega\,\mathbf{B}_\lambda = \mathrm{H}_\lambda$$

Corollary 6.2.3. *The Hall-Littlewood polynomials $\{\mathrm{H}_\lambda\}_\lambda$ are h-positive under the transformation sending q to $1+u$.*

To see how all of this relates to Delta operators, we begin with the symmetric functions $\{E_{n,k}\}$ defined by the expansion

$$e_n\left[X\frac{1-z}{1-q}\right] = \sum_{k=1}^{n} \frac{(q;z)_k}{(q;q)_k} E_{n,k}[X;q].$$

It follows that, setting $z=q$, we have

$$e_n = \sum_{k=1}^{n} E_{n,k}[X;q].$$

It is shown in [36] that

$$E_{n,k} = \sum_{\substack{\alpha\models n \\ \ell(\alpha)=k}} \mathbf{C}_\alpha,$$

where $\mathbf{C}_\alpha = C_{\alpha_1} \circ \cdots \circ C_{\alpha_k}(1)$ is defined setting

$$C_a F[X] \quad = \quad (-1)^{a-1} F\left[X - \frac{1-1/q}{z}\right] \sum_k z^k h_k \Big|_{z^a} \quad = \quad (-q)^{a-1} \omega B_a \Big|^{q \to 1/q} \omega F[X].$$

We have the relation

$$\omega\, \mathbf{B}_\lambda[X;q] = (-q)^{\ell(\lambda)-|\lambda|}\, \mathbf{C}_\lambda[X;1/q].$$

Proposition 5.4 of [36] states that

$$E_{n,k} = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{-n(\mu)-k+M(\mu)} \begin{bmatrix} k \\ m_1(\mu), \ldots, m_n(\mu) \end{bmatrix}_q \mathbf{C}_\mu,$$

with $M(\mu) = \sum_i \binom{m_i(\mu)+1}{2}$ (and again $m_i(\mu)$ is the multiplicity of i in μ). Note first that setting q to be $1/q$ in the coefficient of $\mathbf{C}_\mu$, we get

$$\begin{aligned}
q^{n(\mu)+k-M(\mu)} \begin{bmatrix} k \\ m_1(\mu), \ldots, m_n(\mu) \end{bmatrix}_{1/q} &= q^{n(\mu)+k-M(\mu)} \frac{(1/q;1/q)_k}{(1/q;1/q)_{m_1(\mu)} \cdots (1/q;1/q)_{m_n(\mu)}} \\
&= q^{n(\mu)+k-M(\mu)} \frac{q^{-(1+\cdots+k)}(-1)^k (q;q)_k}{q^{-M(\mu)}(-1)^k (q;q)_{m_1(\mu)} \cdots (q;q)_{m_n(\mu)}} \\
&= q^{n(\mu)-\binom{k}{2}} \frac{(q;q)_k}{(q;q)_{m_1(\mu)} \cdots (q;q)_{m_n(\mu)}} \\
&= q^{n(\mu)-\binom{k}{2}} \begin{bmatrix} k \\ m_1(\mu), \ldots, m_n(\mu) \end{bmatrix}_q
\end{aligned}$$

This gives us

$$\begin{aligned}
\omega E_{n,k}[X;1/q] &= \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{n(\mu)-\binom{k}{2}} \begin{bmatrix} k \\ m_1(\mu), \ldots, m_n(\mu) \end{bmatrix}_q \omega\, \mathbf{C}_\mu[X;1/q] \\
&= \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} q^{n(\mu)-\binom{k}{2}} \begin{bmatrix} k \\ m_1(\mu), \ldots, m_n(\mu) \end{bmatrix}_q (-q)^{n-k}\, \mathbf{B}_\lambda \qquad (6.2)
\end{aligned}$$

The relation between this expansion and the Delta operator at $t = 0$ can be found in [35]. We will need to consider instead the Delta prime operator defined by setting $\Delta'_F\, \widetilde{\mathrm{H}}_\mu = F[B_\mu - 1]\, \widetilde{\mathrm{H}}_\mu$.

To write the Delta prime operator in terms of the ordinary Delta operator, we can use the shift operator $\mathfrak{T}_{-1}$ to write

$$\Delta'_F = \sum_{r\geq 0}(-1)^r \Delta_{e_r^\perp F}.$$

In the case when $F = e_k$ this becomes

$$\Delta'_{e_k} = \sum_{r\geq 0}(-1)^r \Delta_{e_{k-r}}.$$

The only change in the statement of the Delta Conjecture here is by the replacement of the Haglund factor $H_{n-k}(D)$ in Equation (3.1) with

$$H'_{n-k}(D) = \prod_{\alpha_i > \alpha_{i-1}} \left(1 + \frac{z}{t^{\alpha_i}}\right)\Big|_{z^{n-k}}.$$

The combinatorial side of the Delta Conjecture for the Delta prime operator will be denoted by

$$R_{n,k}[X;q] = \sum_{L\in LD_n} q^{\mathrm{dinv}(L)} t^{\mathrm{area}(L)} H'_{n-k}(L) x^L \Big|_{t=0}$$

Though not stated in this way, it was shown in [35] that $\Delta'_{e_{k-1}} e_n \Big|_{t=0}$ is equal to the symmetric function $\widetilde{E}_{n,k} = (-1/q)^{n-k}\omega E_{n,k}[X;1/q]$ (via Equation (6.2)), which can be rewritten as

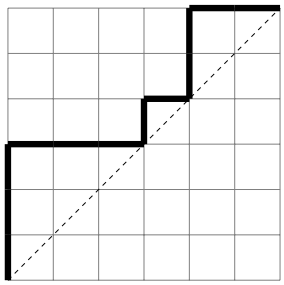
$$\widetilde{E}_{n,k}[X;q] \quad = \quad \sum_{\substack{\alpha\models n \\ \ell(\alpha)=k}} (-1/q)^{n-k}\omega\, \mathbf{C}_\alpha[X;1/q] \quad = \quad \sum_{\substack{\alpha\models n \\ \ell(\alpha)=k}} \mathbf{B}_\alpha\,.$$

Upon the validity of the Delta Conjecture at $t = 0$ (proved in [14], [35]), we have

$$R_{n,k} = \Delta'_{e_{k-1}} e_n \Big|_{t=0} = \widetilde{E}_{n,k}[X;q] = \sum_{\substack{\alpha\models n \\ \ell(\alpha)=k}} \mathbf{B}_\alpha\,.$$

Note that for $t^{\mathrm{area}(L)} H'_{n-k}(L)\Big|_{t=0} \neq 0$ to hold, we require that no area cell of L contributes to the total area. Recall that $H'_{n-k}(L)$ removes the area contribution of $n-k$ rows whose corresponding

North steps are not the beginning of a vertical segment. Any vertical segment of L which begins at a diagonal larger than 0 will therefore force the area to be larger than 1. Therefore, L must have all vertical segments starting on the main diagonal, and the number of vertical segments is precisely k. A Dyck path is called balanced if the first of these conditions hold. For a composition α, we write BD_α for the balanced Dyck path whose steps are $N^{\alpha_1}E^{\alpha_1}\cdots N^{\alpha_{\ell(\alpha)}}E^{\alpha_{\ell(\alpha)}}$. For example, when $\alpha = (3,1,2)$, we have the following balanced path $BD_{(3,1,2)}$:



The sum

$$LLT_D[X;q] = \sum_{L \in LD_n \text{ with path } D} q^{\mathrm{dinv}(L)} x^L$$

is referred to as the LLT polynomial for the Dyck path D (see [4] for their original introduction). This allows us to write the combinatorial side of the Delta Conjecture at $t = 0$ in terms of LLT polynomials of balanced paths:

$$R_{n,k} = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} LLT_{BD_\alpha}.$$

We have from Haglund's book [34] that for any partition λ,

$$LLT_{BD_\lambda} = \mathbf{B}_\lambda\,.$$

Therefore, the LLT polynomials $\{LLT_{BD_\lambda}\}_\lambda$ exhibit the e-positivity phenomenon. It is further conjectured in [2] that LLT_D exhibits the e-positivity phenomenon for any Dyck path D (see [50] for other conjectures).

Putting everything together, we have

$$\Delta'_{e_{k-1}} e_n \Big|_{t=0} = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \mathbf{B}_\alpha = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} LLT_{BD_\alpha} = \widetilde{E}_{n,k}.$$

Corollary 6.2.4. *For $0 \leq k \leq n-1$, the symmetric function $\Delta'_{e_k} e_n \Big|_{t=0}$ exhibits the e-positivity phenomenon.*

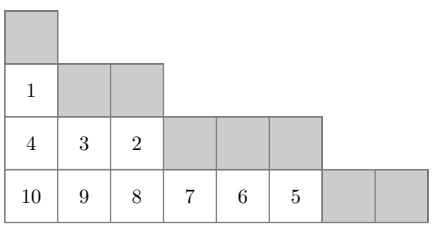
The next section will use Equation (6.1) to give an algorithm for computing the symmetric function $\mathbf{B}_\alpha$ for any composition α. In turn, this gives a precise way of expanding $\Delta_{e_k} e_n \Big|_{t=0}$ in terms of the elementary basis as a polynomial in u and q with positive integer coefficients.

6.3 A combinatorial game

In order to better display the pictures for this game, we are going to work instead with the conjugate partition corresponding to e_μ. Given a composition α, we will consider the partition

$$\widetilde{\alpha} = (n, n-\alpha_1, n-\alpha_1-\alpha_2, \ldots, \alpha_{\ell(\alpha)}).$$

The "game board" GB_α is the partition $(n-\alpha_1, n-\alpha_1-\alpha_2, \ldots, \alpha_{\ell(\alpha)})$ contained in $\widetilde{\alpha}$. For instance, we have displayed here $\widetilde{(2,3,2,1)}$ and its board $GB_{(2,3,2,1)}$ (the region with labels):

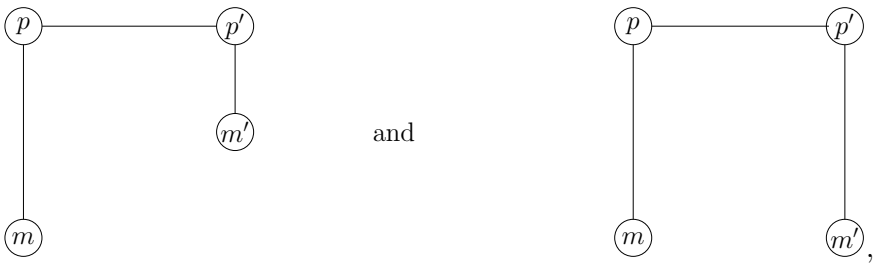


We fix an order on the cells of GB_α by reading the rows from right to left, beginning at the top row. This is depicted by the numbering in the above tableau.

The game begins by placing neutral stones on the boundary $\widetilde{\alpha}/GB_\alpha$ (this is the shaded region above). The game is now played by placing stones on the game board in the fixed order, so that once a stone is placed, the next stone must be placed within the same row and to the left, or in a lower row. Each move, that is each time a single stone is placed, the stone gets a weight of either q or u. A move which places the stone m is said to be valid if it satisfies the following condition:

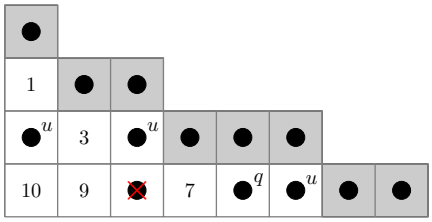
- Let p be the first stone above m. If there exists a stone p' on the right of p then there is a stone m' that is first below p' and placed before m. If m and m' are in the same row, we call m a "connected" move. Otherwise, m is called an "unconnected" move.

An unconnected move will get a weight of u and a connected move will get a weight of q. Diagrammatically, unconnected and connected moves, respectively, can be identified as follows:

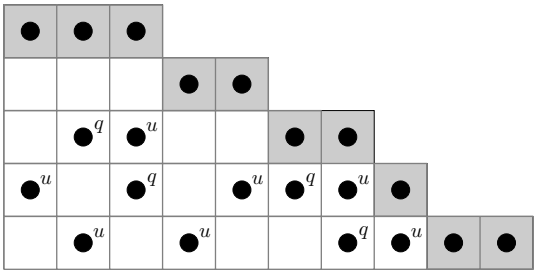


where there are no stones lying on the drawn lines.

Here we have played a game by making 4 valid moves. The next move, marked with a red cross, depicts an illegal move.



The move is illegal because the stone above has a stone directly on its right, yet no move was made on the cell labeled 7. We have labeled the stones with a u if they correspond to unconnected moves and with a q if they are connected. Here is perhaps a more interesting example.



Let $\mathcal{PB}_\alpha$ be the set of valid stone placements on GB_α. For $G \in \mathcal{PB}_\alpha$, let $\mathrm{con}(G)$ be the number of connected moves and $\mathrm{unc}(G)$ the number of unconnected moves. The first example above, after removing the illegal move marked by the symbol $\times$, would have 3 unconnected and 1 connected move. The second example has 7 unconnected and 4 connected moves. Lastly every game $G \in GB_{\mathcal{PB}_\alpha}$ has an associated partition and associated e-basis element. Suppose that the first row (from the bottom) has a stones. Then we can recursively set

$$e_{\lambda(G)} = e_a \ \cdot \ e_{\lambda(\overline{G})},$$

where $\overline{G}$ is obtained from G by erasing all the columns with a stone in the bottom row. The first example would give the e-basis element $e_5 \cdot e_2 \cdot e_1$ and the second would give $e_6 \cdot e_4$. The rules of the game follow the formula from Equation (6.1). To calculate $B_a e_\mu$, we pick some number α_i of

cells to remove from μ_i and add to e_a. In our games, we depict $\mu_1, \mu_2, \ldots$ as rows, drawn from top to bottom. Picking α_i corrseponds to placing α_i connected stones below μ_i, starting from the right. The first stone from the right is called unconnected, contributing the factor u in Equation (6.1). The remaining stones give a factor of q. We then apply this procedure again on the partition formed by counting the stones in each row that have no stone below.

As a result, we have

Theorem 6.3.1. *For any composition α,*

$$\mathbf{B}_\alpha = \sum_{G \in \mathcal{PB}_\alpha} u^{\mathrm{unc}(G)} q^{\mathrm{con}(G)} e_{\lambda(G)}.$$

We then get the following way of writing the Hall-Littlewood polynomials and $\Delta'_{e_k} e_n$ at $t = 0$.

Corollary 6.3.2. *For any partition λ,*

$$H_\lambda = \sum_{G \in \mathcal{PB}_\lambda} u^{\mathrm{unc}(G)} q^{\mathrm{con}(G)} \; h_{\lambda(G)}.$$

Corollary 6.3.3. *For any $0 \leq k \leq n$, we have*

$$\Delta'_{e_{k-1}} e_n \Big|_{t=0} = \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \sum_{G \in \mathcal{PB}_\alpha} u^{\mathrm{unc}(G)} q^{\mathrm{con}(G)} \; e_{\lambda(G)}.$$

In [35], it is shown that

$$e_j^\perp \widetilde{E}_{n,k} = \sum_{r=0}^{j} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+j-r-1 \\ j-r \end{bmatrix}_q \widetilde{E}_{n-j,k-r}$$

For the particular case that $j = 1$, this gives

$$e_1^\perp \widetilde{E}_{n,k} = [k]_q \left(\widetilde{E}_{n-1,k} + \widetilde{E}_{n-1,k-1} \right).$$

An interesting problem would be to find this recursion using only the combinatorial games associated to $\widetilde{E}_{n,k}$.

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