Title

Dynamical Methods in Spectral Theory of periodic Schrödinger Operators with Random Noise

Author

Wood, William

Publication Date

2024

UNIVERSITY OF CALIFORNIA,

IRVINE

Dynamical Methods in Spectral Theory of periodic Schrödinger Operators with Random Noise

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

William Wood

Dissertation Committee:
Professor Anton Gorodetski, Chair
Professor Figotin
Professor Klein
Professor Krupchyk

2024

Dedication

To my dear and close friends
who taught me what it is like to have support
and kept me going when I was tired.

To the doctors, nurses, and therapists who saved my life
and helped return it to normal.

To Professor Anton Gorodetski,
whose influence and guidance I will never forget, never be able to repay,
and forever value.

"Oh no no no. I was out within 25 years. It felt like a minute compared to grad school."
- Professor Hubert Farnsworth, Futurama

"Think of the life you have lived until now as over and, as a dead man,
see what's left as a bonus and live it according to Nature.
Love the hand that fate deals you and play it as your own,
for what could be more fitting?"
- Marcus Aurelius, Meditations

"I realize that what I was living for were the questions!...The pursuit of knowledge is hopeless and eternal. Hooray!"
- Professor Hubert Farnsworth, Futurama

Table of Contents

List of Figures

Acknowledgments

It may be most important to properly acknowledge my PhD advisor Anton Gorodetski. I have heard a relationship with one's advisor to be pivotal in academic success. For my time at the University of California at Irvine, I may be most grateful for Professor Anton Gorodetski. I am confident he has been pivotal and mindful in maintaining my well being and goals as a student. I would like to acknowledge financial support through NSF grant DMS-2247966. Additionally, I would like to thank Professor Jake Fillman for his insight and guidance.

I believe it is unlikely that my experiences at the University of California at Irvine have been typical, either as a graduate student or for any individual. Stated by the nurses and doctors at UCSD Medical, one of the most important factors in perseverance and recovery is a supportive environment. During my time at UCI, I have had an incredibly supportive group of colleagues and friends who have helped me be a functional and allowed me to complete my goals.

There have been a variety of doctors and nurses I have had to please to meet and interact with over the past few years. My time here would not be possible without them. There is no way I can properly express my deep appreciation for them.

Finally, I should express my deep gratitude to the Math Department here at UC Irvine. This has been a supportive environment, and I have been happy to make his place my home for the past several years.

Vita

William Wood

B.S. Applied Mathematics, Georgia Institute of Technology, 2011

M.S. Mathematics, University of California, San Diego, 2014

Ph.D. Mathematics, University of California, Irvine, 2024

Field of Study

Dynamical Systems and Spectral Theory

Publications

"On the spectrum of the periodic Anderson-Bernoulli model", J. Math. Phys. 63 (2022), no. 10, Paper No. 102705, 16. MR 4497771

Abstract of the Dissertation

Dynamical Methods in Spectral Theory of periodic Schrödinger Operators with Random Noise

by

William Wood

Doctor of Philosophy in Mathematics

University of California, Irvine, 2024

Professor Anton Gorodetski, Chair

We will study the spectrum of a discrete Schrödinger operator called the periodic Anderson-Bernoulli operator. Because the operator is ergodic, we can use techniques in dynamical systems and apply Johnson's Theorem to better understand this operator. These techniques involve studying the hyperbolic locus of $SL(2, \mathbb{R})$ cocycles and the geometry of the hyperbolic locus in $SL(2, \mathbb{R})^n$. This model has a spectrum that is completely pure-point, and there exist parameters such that the spectrum can be defined as the union of an infinite number of intervals, which is unexpected for multiple reasons. A result of Avila, Damanik, and Gorodetski [2] says that if the Anderson model has a background potential defined by a dynamical system with a continuous phase space, then such a result is impossible. In this model, the background potential is periodic, making that result not applicable. This thesis provides details of this work, as well as insight into where this work may progress.

Introduction

This thesis will cover my work and results while at the University of California, Irvine. The results center around studying the spectrum of a discrete, erogodic Schrödinger operator and hyperbolicity of $SL(2,\mathbb{R})$ cocycles.

Definition 1. *A discrete Schrödinger operator is a mapping $H : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined by*

$$H(\varphi)_n = \varphi_{n-1} + \varphi_{n+1} + V(n)\varphi_n,$$

where $V(n)$ is a bounded real sequence.

The continuous Schrödinger operator $H(\varphi) = \Delta\varphi + P \cdot \varphi$ is more well known, but one can study the discrete version if the laplacian is discretized and the potential is a bounded sequence. In n dimensions, the discrete laplacian is $(\triangle\varphi)_m = \sum_{|m-i|=1} \varphi_i - 2n\varphi_m$. Over 1 dimension, it is $(\triangle\varphi)_m = \varphi_{m+1} + \varphi_{m-1} - 2\varphi_m$. Typically, the 3rd term is expressed as part of the potential and we would get $(\triangle\varphi)_m = \varphi_{m+1} + \varphi_{m-1}$. When studying the behavior of an operator of a Hilbert space, it is useful to study the spectrum of the operator. For instance, the spectrum of a Schrödinger operator gives the potential energies in the system.

Definition 2. *The spectrum of the operator H is defined as the following set*

$$\sigma(H) = \{E : H_E \text{ does not have a bounded inverse}\},$$

where the bound refers to the operator norm being bounded.

The operator studied in this thesis is called the periodic Anderson-Bernoulli operator, and the topology of the spectrum of this operator is the center of focus. This model can most easily be described as the Anderson model with a background periodic potential. The Anderson model is a Schrödinger operator where the potential is a sequence of iid random variables, and in the case of the

Anderson-Bernoulli operator, the distribution is Bernoulli. The research for the main result centers around studying the hyperbolic locus in $SL(2,\mathbb{R})^{16}$, and finding a path defined by the Schrödinger cocycle that passes through an infinite number of connected components of the hyperbolic locus. Additional material on Schrödinger cocycles is in section 2.2.2 and the hyperbolic locus is discussed in section 3. Details of Schrödinger operators, and relevant information, is provided in section 2.

Theorems 1 and 2 pertain to the periodic Anderson-Bernoulli model. By Theorem 1, the spectrum of the periodic Anderson-Bernoulli operator can consist of an infinite number of intervals. For this to be possible, the period has to be greater than two, by Theorem 2. Whether this is possible for period 3 is still currently unknown. We can furthermore show that the spectrum has a dense interior. This is in thanks to Professor Jake Fillman. There are two reasons why Theorem 1 is an interesting result. The first is that the Anderson model has a spectrum that is the union of a finite number of intervals. Explicitly, given the distribution ν that defines the variables of the potential, then the spectrum is entirely pure-point, and is the set $\sigma(H_A) = [-2,2] + \operatorname{supp}\nu$. The same is true for any periodic Schrödinger operator. Explicitly, the potential of the periodic operator, with period n, is the union of n intervals (possibly overlapping). Using Johnson's Theorem, one can show that the spectrum is the set of values such that the trace of a given $SL(2,\mathbb{R})$ matrix is inclusively between -2 and 2. The trace of the matrix is a polynomial of degree equal to the period length. The fact that potential of these two operators, when added together, would have an infinite number of intervals in the spectrum then becomes counterintuitive. The second reason is due to [2], which found that if the Anderson model has an ergodic background potential, defined by a dynamical system with a connected phase space, then the spectrum would also have to have a finite number of intervals. A periodic potential, of period n, is defined by the discrete dynamical system $(\mathbb{Z}_n, T)$, where $T(x) = x + 1$, and does not have a connected phase space.

The second topic of focus is the hyperbolic locus and the notion of uniform hyperbolicity of sets of $SL(2,\mathbb{R})$ matrices. Great sources of information on this topic are [1] and [20].

Definition 3. *Given a set of $SL(2,\mathbb{R})$ matrices, $\mathcal{M}$, the set is uniformly hyperbolic if there exists $\lambda > 1$ such that for any $M_1, M_2, \cdots, M_m \in \mathcal{M}$*

$$||M_1 \cdot M_2 \cdots M_m|| > \lambda^m.$$

This inequality holds for all positive integers m and for any ordering of the product of the matrices. The matrices in the inequality do not need to be distinct from each other.

Definition 4. *For any integer n, the hyperbolic locus is a open subset of $SL(2,\mathbb{R})^n$ (denoted $\mathcal{H}_n \subset SL(2,\mathbb{R})^n$). Given any finite set of $SL(2,\mathbb{R})$ matrices $\mathcal{M} = \{M_1, M_2, \cdots, M_n\}$, the $n-$tuple $(M_1, M_2, \ldots M_n)$ is an element of the hyperbolic locus $\mathcal{H}_n$ if and only if $\mathcal{M}$ is uniformly hyperbolic.*

In $SL(2,\mathbb{R})^2$, the geometry of $\mathcal{H}_2$ is well understood from [1]. The locus $\mathcal{H}_2$ is a countable union of connected components that can be individually defined by their boundaries. For $n \geq 3$, $\mathcal{H}_n$ has a more complicated geometry, and there a few open questions posted by [1], some of which are discussed in section 5. In $SL(2,\mathbb{R})^3$ there exists path $P : [a,b] \to SL(2,\mathbb{R})^3$ which passes through an infinite number of disjoint, connected components of $\mathcal{H}_3$ as defined in 3.5 and Proposition 4.18 of [1]. This path can be lifted to $SL(2,\mathbb{R})^n$ and $\mathcal{H}_n$ for any $n > 3$ and relates to the work involving the spectrum of the Schrödinger operators. Such a path does not exist in $SL(2,\mathbb{R})^2$.

Chapter 1

Outline

1.1 Statement of Results

Theorem 1. *The periodic Anderson-Bernoulli model, with period 4 or greater, can have a spectrum with infinitely many gaps.*

Theorem 2. *The periodic Anderson-Bernoulli model, with period 2, has a spectrum equal to the union of 4 intervals. Explicitly, it is the complement of the union of 4 well-defined intervals, one of which extends to* $-\infty$ *and another extends to* $+\infty$.

1.2 Organization of Thesis

Chapter 2 discusses information and the background of Schrödinger operators, specifically ones with a randomness in the potential. This section outlines the general details of ergodic Schrödinger operators and defines the spectrum of these operators. Schrödinger cocycles and transfer matrices are then introduced as well as Johnson's Theorem, which we can utilize to calculate the spectrum of the operators. Then the concept of Anderson localization is expanded on as well as the RAGE Theorem and details of spectral theory. In the last section of the chapter the periodic Anderson-Bernoulli model is defined, which is the main subject of the results.

Chapter 3 involves the hyperbolic locus details, its various properties, and relevant open questions about the hyperbolic locus. Additionally, the relevant material and calculations for the work involving the periodic Anderson-Bernoulli operator are provided.

The next Chapter, 4, provides the calculations for the the main results about the periodic Anderson-Bernoulli model. This utilizes the material from the previous two sections to calculate the work.

The final Chapter, 5, provides details about a few open questions, and some of the progress that can provide insight into ways to approach answering these questions. A single question is posed about the periodic Anderson-Bernoulli operator, but most of the material centers around material involving questions posed in [1].

Chapter 2

Random Schrödinger Operators

2.1 Basic Details

A discrete Schrödinger operator takes the form is a mapping $H : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ such that

$$H(\varphi)_n = \varphi_{n-1} + \varphi_{n+1} + V(n)\varphi_n.$$

Here $\{V(n)\}$ is a bounded sequence of real numbers. When studying Schrödinger operators, one often wants to study spectrum of the operator $\sigma(H)$.

$$\sigma(H) = \{E : H - E \text{ does not have a bounded inverse}\}$$

The spectrum can be decomposed into 3 different types, an absolutely continuous piece, a singular continuous piece, and a pure point piece. This leads into information about the RAGE Theorem, and for additional details, see [8] and Section 6. When studying the spectrum of a discrete Schrödinger operator, one can study the decomposition into the 3 types, as well as the topological structure of the spectrum. Does the spectrum consist of intervals? Are there finitely many or infinitely many? Are there isolated points? If there are no intervals, then what is the Hausdorff dimension of the spectrum? This work puts more effort in studying the topological structure of a certain Schrödinger operator. We will consider what are called ergodic Schrödinger operators. When studying ergodic Schrödinger operators, there are special tools we can use when studying the spectrum.

Definition 5. *A family of discrete Schrödinger operators* $\{H_V\}$ *is ergodic if there exists an ergodic*[1] *topological dynamical system* (Ω, T, μ) *and* $f : \Omega \to \mathbb{R}$ *such that for all* $\omega \in \Omega$ *there is a unique* $H_V \in \{H_V\}$ *where, for all* $n \geq 0$*, the following equation holds*

$$H(\varphi)_n = \varphi_{n-1} + \varphi_{n+1} + f(T^n(\omega))\varphi_n,$$

ie $V_\omega(n) = f(T^n(\omega))$.

If T is invertible, then n can be any integer. The ergodic family is often denoted $\{H_\omega\}_\Omega$, indexed by the elements of the dynamical system. In the following work, the dynamical system will always be unitary, ie $\mu(\Omega) = 1$.

The simplest example of an ergodic family of Schrödinger operators is one where the potential is a periodic sequence. If the potential has period n, then one can consider the dynamical system $(\mathbb{Z}_n, S, \mu)$, where $S(x) = x + 1 \mod n$ and $\mu(\{x\}) = 1/n$. Considering any sequence $(a_0, a_1, a_2, \cdots a_{n-1})$, one can define $f : \mathbb{Z}_n \to \mathbb{R}$ by $i \mapsto a_i$. With this dynamical system, the ergodic family of Schrödinger operators $\{H_x\}_{x\in\mathbb{Z}_n}$ is the set with that sequence as a potential. The operators in this family all have the same spectrum. If the dynamical system is minimal[2], then all of the operators in the family have the same spectrum. In the case of a periodic Schrödinger operator, with period n, the spectrum consists of a finite union of n intervals.

This brings us to the concept of the almost sure spectrum. When considering an ergodic family $\{H_\omega\}_\Omega$ with the ergodic dynamical system (Ω, T, μ) with unitary μ, there exists $\Omega' \subset \Omega$ such that $\Omega' = \{\omega : \overline{Orb_T(\omega)} = \Omega\}$. Here, Ω' is the subset of elements such that their orbit is dense in Ω. Here $\mu(\Omega') = 1$. All the members of the family $\{H_\omega\}_{\Omega'}$ have the same spectrum, and this is called the almost sure spectrum of $\{H_\omega\}_\Omega$ and is denoted $\sigma_{AS}(\{H_\omega\})$.

There are a variety of examples of ergodic Schrödinger operators that are studied. When studying more complicated, well-known examples of ergodic Schrödinger operators, one should recognize almost Mathieu operator and the Anderson model. The almost-Mathieu operator, H_{AM}, has a potential $P(n) = \lambda \cdot cos(n\pi\alpha)$ such that $\lambda \neq 0$, $\alpha \notin \mathbb{Q}$, and $n \in \mathbb{Z}$. Proven by Avila and Jitomirskaya in [3], the spectrum $\sigma(H_{AM})$ is a Cantor set that can be defined by the Hofstadter's butterfly. The nature of the spectrum was an open problem for several decades, and was dubbed the ten martini

[1] A topological dynamical system (Ω, T, μ) is ergodic if the measure μ is unitary and for all measurable sets A, we have $T^{-1}(A) = A$ if and only if $\mu(A) \in \{0, 1\}$.

[2] A dynamical system (Ω, T) is minimal if it does not contain any closed, nontrivial, proper subsets that are closed under T.

problem.

Additionally there is what is called the Fibonacci Hamiltonian. This Schrödinger operator has a potential $P(n) = \lambda\chi_{[1-\alpha,1]}(n\alpha + \omega \mod 1)$. Here $\lambda > 0$ is the coupling constant, $\omega \in \mathbb{T}$, and $\alpha = \frac{-1+\sqrt{5}}{2}$ is the inverse of the golden ratio. We know that for any coupling constant, the spectrum is a dynamically defined Cantor set[3] For a few additional sources on studied ergodic Schroödinger operators, one can see [10] [19] [21] [16][5][4].

The Anderson model, H_A, is a heavily studied Schrödinger operator named after Philip Anderson. His nobel prize in 1977 came in part from studying what became known as Anderson localization and led to studying random Schrodinger operators. For additional details on the history, one can see [18][12]. The discrete Anderson model has a potential defined by a sequence of iid random variables $\{x_n\}$ defined by a distribution ν. The almost sure spectrum of the Anderson model is the set

$$\sigma(H_A) = supp(\nu) + [-2, 2],$$

where the summation is the Minkowski sum[4]. Because the spectrum is a compact set, it has to be the union of finitely many disjoint intervals. This work will provide an example of a similar operator, one with a potential defined by iid random variables $\{\nu(n)\}$ plus a periodic background potential. The main results are given in Theorem 1, which answers the question asked in [19].

As a 'family' of Schrödinger operators, the Anderson model is the family of all possible operators with a potential being a sequence of values in $supp(\nu)$. The corresponding ergodic dynamical system that defines the Anderson model is $(supp(\nu)^{\mathbb{Z}}, S)$, where S is the right shift, ie for all $\{x_n\} \in supp(\nu)^{\mathbb{Z}}$, we get $S(\{x_n\})_j = x_{j-1}$. The potential can be defined by the mapping $\omega \mapsto \omega_0$, therefore, we get

$$V_\omega(0) = \omega_0,\ V_\omega(1) = \omega_1, \cdots, V_\omega(n) = \omega_n.$$

The right shift has a well-defined inverse operator (the left shift here), so n can be extended to any integer. Any $\omega \in \Omega$ must therefore define a sequence of possible outcomes, and Ω is the set of all possible outcomes. This dynamical system is not minimal, as one can construct closed subsets that are closed under the shift operator, the most obvious examples of which are periodic sequences. Given a Bernoulli distribution, where ν can take on the values of 0 and 1, the almost sure spectrum is $[-2, 3]$. If you consider the specific operator $H(\varphi)_n = \varphi_{n+1} + \varphi_{n-1} + \varphi_n$, the potential would be

[3] A dynamically defined Cantor set is a Cantor set defined by sets $E_1, E_2, \cdots E_n \subset K$ where $K \subset \mathbb{R}$ is an interval and $\varphi : \bigcup E_i \to K$ is an expanding map. The Cantor set can be defined as $\bigcap_{m>0} \varphi^{-m}(K)$.

[4] A Minkowski sum is an operation between sets such that $A + B = \{a + b : a \in A \ \&\ b \in B\}$.

defined by the element $\omega = (\cdots, 1, 1, 1, \cdots)$ and has a spectrum of $[-1, 3]$.

2.2 Schrödinger Cocycle

2.2.1 Introduction

We can begin by providing Katok and Hasselblatt's definition of a cocycle.

Definition 6. *[15] Given a system $(X, \mathcal{B}, \mu)$ with a measure preserving $f : X \to X$, a linear cocycle is a measurable function $A : X \to GL(n, \mathbb{R})$ such that*

$$A(x, i + j) = A(f^j(x), i) \cdot A(x, j)$$

Assuming that f is invertible, then $i, j \in \mathbb{Z}$, $A(x, 0) = Id$, and all possible images of the function are invertible matrices. In the context of this work, we will restrict ourselves to $SL(2, \mathbb{R})$ cocycles and use a slightly different notation provided in the definition below.

Definition 7. *Given a space Ω and map $T : \Omega \to \Omega$ and $A : \Omega \to SL(2, \mathbb{R})$, an $SL(2, \mathbb{R})-$cocycle is the map*

$$B : (\Omega, \mathbb{R}^2) \to (\Omega, \mathbb{R}^2)$$

$$(\omega, \vec{v}) \mapsto (T(\omega), A(\omega)\vec{v}).$$

There are a couple concepts we can define on cocycles. Iterations of a cocycle B^n can be defined by the following formula, for $n > 1$.

$$\begin{aligned} B^n(\omega, \vec{v}) &= B^{n-1}(T(\omega), A(\omega)\vec{v}) \\ &= (T^n(\omega), A(T^{n-1}(\omega)) \cdot A(T^{n-2}(\omega)) \cdots A(T(\omega)) \cdot A(\omega)\vec{v}) \end{aligned}$$

We will use the notation $B^n(\omega, \vec{v}) = (T^n\omega, A_n(\omega)\vec{v})$. If T is invertible, then the equation holds for n being any integer. With this we can introduce the idea of an $SL(2, \mathbb{R})$ cocycle being uniformly hyperbolic.

Definition 8. *A cocycle is uniformly hyperbolic if there exists $\lambda > 1$ and $C > 0$ such that for all $\omega \in \Omega$,*

$$||A_n(\omega)|| > C\lambda^{|n|}.$$

Via [20],

Definition 9. *For any cocycle (T, A), there is a unique continuous section*

$$e_s : X \to \mathbb{P}^1$$

if and only if the cocycle is uniformly hyperbolic. Here $e_s(x)$ is invariant and repelling.

If T is a homomorphism, then there also exists a continuous section $e_u(x) \neq e_s(x)$ which is invariant and contracting. This concept will be expanded on in Chapter 3, and in [20].

2.2.2 Transfer Matrices

As defined in section 2.1, given an ergodic Schrödinger family $\{H_\omega\}_\Omega$, there exists ergodic (Ω, T, μ) and $f : \Omega \to \mathbb{R}$. The function f can be used to define the mapping $A_E : \Omega \to SL(2, \mathbb{R})$ by the following

$$\omega \mapsto \begin{bmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{bmatrix}.$$

Given an ergodic Schrödinger operater, $H_\omega(\varphi)_n = \varphi_{n-1} + \varphi_{n+1} + f(T^n(\omega))\varphi_n$, consider a sequence $\{\varphi_n\}$ such that $H_\omega(\varphi) = E\varphi$. This would give the equation

$$E \cdot \varphi_n = \varphi_{n+1} + \varphi_{n-1} + f(T^n(\omega))\varphi_n.$$

This can be turned into the two equations

$$(E - f(T^n(\omega)) \cdot \varphi_n - 1 \cdot \varphi_{n-1} = \varphi_{n+1} \tag{2.1}$$

$$\varphi_n + 0 \cdot \varphi_{n-1} = \varphi_n \tag{2.2}$$

which gives the matrix shown above, called a transfer matrix

$$\begin{bmatrix} E - f(T^n(\omega)) & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varphi_n \\ \varphi_{n-1} \end{bmatrix} = \begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix}.$$

This brings us to the definition of a Schrödinger cocycle.

Definition 10. *Given a Schrödinger operator H, defined by a dynamical system (Ω, T, μ) where the potential is defined by $f : \Omega \to \mathbb{R}$, the Schrödinger cocycle $B_E : (\Omega, \mathbb{R}^2) \to (\Omega, \mathbb{R}^2)$ is defined as* *the following mapping.*

$$(\omega, \vec{v}) \mapsto \left(T\omega, \begin{bmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{v} \right).$$

Note that the transfer matrices depend on both E and ω, and that a product can be expressed as iterations of the cocycle, ie

$$A_{n,E}(\omega) \cdot \begin{bmatrix} \varphi_0 \\ \varphi_{-1} \end{bmatrix} = \begin{bmatrix} E - f(T^{n-1}(\omega)) & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E - f(T^{n-2}(\omega)) & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varphi_0 \\ \varphi_{-1} \end{bmatrix} = \begin{bmatrix} \varphi_n \\ \varphi_{n-1} \end{bmatrix}.$$

Why is this cocycle so important? The answer is that it can use Johnson's Theorem to help define the spectrum of the relevant model.

Theorem 2.1 (Johnson Theorem (1986))**.** *[13] Consider an ergodic dynamical system* (Ω, T, μ)*, where* Ω *is a compact metric space,* T *is a homeomorphism,* μ *is a* T*-invariant measure with* $\operatorname{supp}(\mu) = \Omega$ *and* $f \in C(\Omega, \mathbb{R})$*. In this case,*

$$\Sigma = \{E \in \mathbb{R} : (T, B_E) \text{ is not uniformly hyperbolic}\}.$$

For additional details of Johnson's Theorem, one can see [13] and [21]. Among the simpler applications of Johnson's Theorem, there is the Anderson model. First it is necessary to provide the following proposition. This follows from definitions 3 and 8 and [1] and [20].

Proposition 2.2. *Given a finite alphabet* $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ *and the dynamical system* $(\Gamma^{\mathbb{Z}}, T)$ *where* $T(\omega)_i = \omega_{i+1}$ *is the shift, any corresponding* $SL(2, \mathbb{R})$ *cocycle* $(\omega, \vec{v}) \mapsto (T(\omega), A(\omega_0)\vec{v})$ *is uniformly hyperbolic if and only if the set of matrices* $\{A(\alpha_i)\}_i$ *is uniformly hyperbolic.*

Given a distribution ν with the support taking on finitely many values $\{\nu_1, \nu_2, \cdots, \nu_n\}$, the corresponding Anderson model would be defined by the ergodic dynamical systems $(\{\nu_i\}^{\mathbb{Z}}, T)$. The corresponding Schrödinger cocycle for the Anderson model is uniformly hyperbolic if and only the following set of matrices is uniformly hyperbolic.

$$\mathcal{M}_E = \left\{ \begin{bmatrix} E - \nu_i & -1 \\ 1 & 0 \end{bmatrix} \right\}_i$$

This follows from Lemma 2.3.

Using material from Chapter 4, we can easily calculate the values for E such that $\mathcal{M}_E$ is uniformly hyperbolic. We first can calculate the eigenvectors of the matrices. Given a transfer matrix,

$A = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}$, it is hyperbolic if and only if $|x| > 2$. If $x > 2$, then the corresponding eigenvectors are

$$u_A = \begin{bmatrix} r_A \\ 1 \end{bmatrix} \quad s_A = \begin{bmatrix} 1 \\ r_A \end{bmatrix}$$

where r_A is the spectral radius, which must be greater than 1. If $x < -2$, then the corresponding eigenvectors are

$$u_A = \begin{bmatrix} -r_A \\ 1 \end{bmatrix} \quad s_A = \begin{bmatrix} -1 \\ r_A \end{bmatrix}.$$

If there exists $M \in \mathcal{M}_E$ such that M is not uniformly hyperbolic (and $|tr(M)| \leq 2$), then the set is not uniformly hyperbolic. If all the matrices are uniformly hyperbolic, then all the eigenvectors are well-defined over $\mathbb{R}^2$. All the unstable eigenvectors are in the set $\mathcal{C} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| > |y| \right\}$, and none of the stable eigenvectors are in $\mathcal{C}$. Considering the projective space $\mathbb{RP}^1$ is defined as

$$\mathbb{RP}^1 = (\mathbb{R}^2 \setminus \{0\})/\sim,$$

where $\sim$ is the relation $\vec{v} \sim \vec{w} \Leftrightarrow \vec{v} = \alpha\vec{w}$ for some $\alpha \neq 0$, the set $\mathcal{C}$ is a single, nontrivial open interval in $\mathbb{RP}^1$. By Proposition 8, if the set $\mathcal{M}_E$ contains hyperbolic matrices, then it forms a principal cone and is uniformly hyperbolic. Thus $\mathcal{M}_E$ is uniformly hyperbolic if and only if it has a principal multicone, and $\mathcal{M}_E$ is uniformly hyperbolic if and only if the cocycle is uniformly hyperbolic. By Johnson's Theorem, we can conclude that $\sigma_{as}(H_A) = \{E : |E - \nu_i| \leq 2 \text{ for some } i\} = \bigcup_i [\nu_i - 2, \nu_i + 2]$.

2.3 Periodic Anderson-Bernoulli Model

The periodic Anderson-Bernoulli model is a discrete ergodic Schrödinger operator and is the subject of the main results in this thesis. Simply put, the periodic Anderson-Bernoulli model is the Anderson model, with a Bernoulli distribution, plus a background periodic potential. We will define a Bernoulli distribution ν such that for a random variable X defined by ν, there exists $p \in (0,1)$ and

$$P(X=1) = p$$
$$P(X=0) = 1-p.$$

The potential of a periodic Anderson-Bernoulli model depends on a finite sequence $(a_0, a_1, \cdots, a_{n-1})$, a sequence of iid random variables $\{x_i\}$ defined by Bernoulli distribution ν, and a constant v. The potential $P(n)$ can then be defined as

$$P(j) = v \cdot x_j + \begin{cases} a_0 & j \equiv 0 \mod n \\ a_1 & j \equiv 1 \mod n \\ \vdots & \vdots \\ a_{n-1} & j \equiv n-1 \mod n \end{cases} . \tag{2.3}$$

It is not immediately obvious that the Schrödinger operator with this potential is ergodic. Given the Bernoulli distribution ν, we can consider the two dynamical systems

$$\begin{aligned} (supp(\mu)^{\mathbb{Z}}, T, \mu_1) &= (\Omega, T, \mu_1) \\ (\mathbb{Z}_n, S, \mu_2)) &= (\Sigma, S, \mu_2) \end{aligned} \tag{2.4}$$

as defined in 2.1. The dynamical system Ω defines the Anderson model (specifically the Anderson-Bernoulli model) and is mixing[5]. The dynamical system Σ defines the potential for periodic Schrödinger operators and is minimal. Here T is the shift operator and $S(x) = x + 1$. due to

[5] A dynamical system (Ω, T, μ) with unitary μ is strongly mixing if given any two measurable sets A and B, we have $\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A) \cdot \mu(B)$. The dynamical sysem is weakly mixing if given sets A and B, we have the $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| = 0$, ie the Cesàro sum of $\{|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)|\}$ is 0. Strongly mixing implies weakly mixing.

to the proposition below, we can consider the ergodic dynamical system

$$(\Omega \times \Sigma, T \times S, \mu_1 \times \mu_2).$$

Proposition 1. *Given a weakly mixing dynamical system (A, f, μ_1) and ergodic (B, g, μ_2) where μ_i are unitary measures, then the dynamical system $(A \times B, f \times g, \mu_1 \times \mu_2)$ is ergodic. Here $\mu_1 \times \mu_2$ is the product measure, and $(f \times g)(a, b) = (f(a), g(b))$.*

Using this product of dynamical systems that is itself ergodic, we can define the potential of the periodic Anderson-Bernoulli operator. Given the functions that define the Anderson model and periodic Schrödinger operator, $f_1 : \Omega \to \mathbb{R}$ by $\omega \mapsto \omega_0$ and $f_2 : \Sigma \to \mathbb{R}$ by $j \mapsto a_{j \mod n}$. We can define the function f by the following.

$$\begin{aligned} f : \Omega \times \Sigma &\to \mathbb{R} \\ (\omega, j) &\mapsto f_1(\omega) + f_2(j). \end{aligned} \tag{2.5}$$

And last this brings us to the Schrödinger cocycle

$$B_E : (\Omega \times \Sigma, \mathbb{R}^2) \to (\Omega \times \Sigma, \mathbb{R}^2)$$

$$\Big((\omega, j), \vec{v}\Big) \mapsto \left((T(\omega), j+1), \begin{bmatrix} E - (v \cdot \omega_0 + a_j) & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{v}\right).$$

In [2], it was found that if you take the Anderson model and add an ergodic background potential with a connected phase space, the potential will consist of a finite number of intervals. Because the dynamical system defining the periodic sequence is not connected, this result does not apply. Theorem 2 shows that if the background potential is of period 2, then the spectrum can consist of at most 4 intervals, which are uniquely defined in [19] and in section 4.1.3. If the period is 4, however, the potential can consist of an infinite number of intervals by Theorem 1. To show why this works, we have Proposition 15 which is a Proposition from [1].

These results rely on Johnson's Theorem, which means we can define the almost sure spectrum as

$$\sigma_{as}(H_{PAB}) = \{E : \text{the cocycle } B_E \text{ is not uniformly hyperbolic}\}$$

as described at the end of section 2.2.2. Calculating the hyperbolicity of this cocycle is a little more complicated, however. The details of the calculations are in Chapter 4, but they rely on a couple of propositions and lemmas.

Lemma 2.3. *Consider an ergodic family of Schrödinger operators with random potentials defined by equation 2.3. The* $SL(2,\mathbb{R})-$*cocycle* $((\omega, j), \vec{v}) \mapsto ((T(\omega), j+1), A_E \cdot \vec{v})$ *is uniformly hyperbolic if and only if the set of distinct matrices*

$$\{A_{n,E}((\omega, 0)) : (\omega, 0) \in \Omega \times \Sigma\}$$

is uniformly hyperbolic for any $n \in \mathbb{Z}^+$. *Note that* n *is the length of the period.*

Remark: There are at most 2 distinct elements in the set $\{A_E((\omega, 0)) : (\omega, 0) \in \Omega \times \Sigma\}$. These elements are expressible as

$$\begin{bmatrix} E - vx_0 - a_0 & -1 \\ 1 & 0 \end{bmatrix}$$

for $x_0 \in \{0, 1\}$. Similarly, $\{A_{2,E}((\omega, 0)) : (\omega, 0) \in \Omega \times \Sigma\}$ has at most 4 distinct elements, expressible as

$$\begin{bmatrix} E - vx_0 - a_0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E - vx_1 - a_1 & -1 \\ 1 & 0 \end{bmatrix}$$

for $x_0, x_1 \in \{0, 1\}$. The set $\{A_{m,E}((\omega, 0) : (\omega, 0) \in \Omega \times \Sigma\}$ has at most 2^m matrices.

Proof.

($\leftarrow$) If the set of matrices is uniformly hyperbolic, then there exists $\lambda > 1$ such that for any ordered set I of the elements in $\{M = A_{m,E}((\omega, 0)) : (\omega, 0) \in \Omega \times \Sigma\}$ where $|I| = k$, $||\prod_I M_i|| > \lambda^k$. Here I can have repeated elements. For any $(\omega, 0) \in \Omega \times \Sigma$ and any k, there exists $|I| = k$, such that $\prod_I M_i = A_{km,E}, (\omega, 0)$. Therefore

$$\big|\big|A_{km,E}(\omega, 0)\big|\big| = \big|\big|\prod_I M_i\big|\big| > \lambda^k = \sqrt[m]{\lambda}^{mk}.$$

It is necessary, however, to bound $\big|\big|A_{j,E}(\omega, 0)\big|\big|$ for any j and any (i, ω). Next, a bound for $\big|\big|A_{j,E}(\omega, 0)\big|\big|$ will be provided for $0 < j < m$. Given that for fixed $E, (\omega, 0)$, and $j \in (0, m)$, we have

$$\inf_{||\vec{v}||=1} \big|\big|A_{j,E}(\omega, 0)(\vec{v})\big|\big| = \big|\big|(A_{j,E}, (\omega, 0))^{-1}\big|\big|^{-1} \in (0, 1)$$

Keeping E fixed, there are at most $\sum_{0<j<m} 2^j$ matrices expressible as $A^j_E(\omega, 0)$. So, there exists C

such that

$$\min_{\substack{0<j<m \\ (\omega,0)\in\Omega\times\Sigma}} \inf_{||\vec{v}||=1} \left|\left|A^j_E, (\omega,0)(\vec{v})\right|\right| / \sqrt[m]{\lambda}^j = C > 0.$$

For $j \in (0,m)$, this gives the inequality

$$\inf_{||\vec{v}||=1} \left|\left|A^j_E(\omega,0)(\vec{v})\right|\right| \geq C \sqrt[m]{\lambda}^j,$$

and this brings the conclusion that for all $\omega \in \Omega$:

$$\left|\left|A^{mk+j}_E(\omega,0)\right|\right| = \sup_{||\vec{v}||=1} \left|\left|A^j_E(\omega, mk \ mod \ m) \cdot A^{mk}_E(\omega,0)\vec{v}\right|\right| \geq$$

$$\inf_{||\vec{v}||=1} \left|\left|A^j_E(\omega,0)(\vec{v})\right|\right| \cdot \left|\left|A^{mk}_E(\omega,0)\right|\right| > C \sqrt[m]{\lambda}^{mk+j}.$$

This proves that $||A_k, E(\omega,0)|| > C\lambda^k$. It is still necessary to prove this statement for any $(\omega, i) \in \Omega \times \Sigma$. For any $i \in (0, m-1)$ recognize that

$$A_{k,E}(\omega,i) = A_{k-m+i,E}(T^{m-i}(\omega),0) \cdot A_{m-i,E}(\omega,i)$$

By the same calculation as above, there exists C such that

$$\inf_{||\vec{v}||=1} \left|\left|A_{m-i,E}(\omega,i)(\vec{v})\right|\right| \geq C \sqrt[m]{\lambda}^{m-i},$$

This gives us that the cocycle is uniformly hyperbolic.

($\rightarrow$) Assume the cocycle (T, A) is uniformly hyperbolic, then there exists a $\lambda > 1$ and a $C > 0$ such that for any $\omega \in \Omega$

$$\left|\left|A^k_{E,\omega}\right|\right| > C\lambda^k$$

For any ordered set I (with $|I| = k$) consisting of the matrices from the set $\{M_i\}$, there exists $(\omega, 0) \in \Omega$ such that $A^{mk}_{E,(\omega,0)} = \prod_I M_i$. Therefore,

$$C\lambda^{mk} < ||\textstyle\prod_I M_i|| \Rightarrow C(\lambda^m)^k < ||\prod_I M_i|| \ .$$

For a set of matrices to be uniformly hyperbolic, we technically need to express such an inequality without a constant C. To address this, we note that for any $\prod_i M_i$, there exists $(\omega, 0) \in \Omega$ such that for any $n \in \mathbb{Z}^+$, $(A^{nmk}_{E,(\omega,0)}) = (\prod_I M_i)^n$, and we have

$$C\lambda^{nmk} < ||A^{nmk}_{E,(\omega,0)}|| = ||(\prod_i M_i)^n|| \leq ||\prod_i M_i||^n.$$

Taking the n^{th} root,

$$C^{1/n}(\lambda^m)^k < ||\prod_i M_i||,$$

and allowing n to be arbitrarily, we get large,

$$(\lambda^m)^k \leq ||\prod_i M_i|| \Rightarrow (\lambda^{m/2})^k < ||\prod_i M_i||.$$

Therefore, the cocycle being uniformly hyperbolic from definition 8 implies the set of matrices is uniformly hyperbolic from definition 3.

□

Chapter 3

Hyperbolic Locus

3.1 Concepts

Before we can discuss the results, we will discuss the hyperbolic locus in $SL(2,\mathbb{R})^n$, which allows us to do the necessary calculations. As previously mentioned in the Introduction, much of the material for this dissertation relies on the notion of uniform hyperbolicity with sets of $SL(2,\mathbb{R})$ matrices. Matrices in $SL(2,\mathbb{R})^2$ can be classified as hyperbolic, parabolic, or elliptic.

- Matrix $A \in SL(2,\mathbb{R})^2$ is hyperbolic if $|tr(A)| > 2$.
- Matrix $A \in SL(2,\mathbb{R})^2$ is elliptic, if $|tra(A)| < 2$.
- Matrix $A \in SL(2,\mathbb{R})^2$ parabolic, if $tr(A) = 2$[1].

Throughout this chapter, a set of matrices $\mathcal{M}$ will be assumed to be $SL(2,\mathbb{R})$ matrices. If you consider a set $\mathcal{M}$ to be uniformly hyperbolic, then the set of matrices are $SL(2,\mathbb{R})$ and products of the matrices will have a lower bound on the norm as in definition 3. Specifically, set of matrices $\mathcal{M}$ is uniformly hyperbolic if there exists $\lambda > 1$ such that for any n and $M_1, M_2, \cdots, M_n \in \mathcal{M}^n$, we have $||M_1 \cdot M_2 \cdots M_n|| > \lambda^n$. If $\{M_1, M_2, \cdots M_n\}$ is uniformly hyperbolic, then all matrices M_i are hyperbolic and all products of the matrices in the set are hyperbolic. The converse of this statement is not necessarily true.

Proposition 2. *Given a finite set of matrices $\mathcal{M}$ with $n = |\mathcal{M}|$, alphabet $A = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ with unitary topology μ, and any bijection $f : A \to \mathcal{M}$, then one can define ergodic dynamical system $(A^{\mathbb{Z}}, T, \mu^{\mathbb{Z}})$ and mapping $g : A^{\mathbb{Z}} \to SL(2,\mathbb{R})$ by $g(\omega) = f(\omega_0)$. Here the dynamical system is*

[1] The identity matrix is sometimes, but not always, considered to be parabolic.

all possible bi-infinite sequences of elements of the alphabet, T is the shift, and the topology is the product topology. The $SL(2,\mathbb{R})$ cocycle $(\omega,\vec{v}) \to (T(\omega), g(\omega)\cdot\vec{v})$ is uniformly hyperbolic if and only the $\mathcal{M}$ is a uniformly set of matrices.

Proof. If the set $\mathcal{M}$ is uniformly hyperbolic, then the cocycle being uniformly hyperbolic follows from the definitions.

If the cocycle is uniformly hyperbolic, then there is $\lambda > 1$ and $C > 0$ such that for all m, $\omega \in Z^{\mathbb{Z}}$, $||A(T^{m-1}(\omega))\cdot A(T^{m-2}(\omega))\cdots A(\omega)|| > C\lambda^m$. We also have that for all $k \in \mathbb{Z}^+$, $i \in \{0,1,\cdots,m-1\}$ and $j \in \{0,1,\cdots,k-1\}$, there exists $\zeta \in A^{\mathbb{Z}}$ such that $\zeta_{i+jm} = \omega_i$. Therefore, we get

$$\left|\left|[A(T^{m-1}(\omega))\cdot A(T^{m-2}(\omega))\cdots A(\omega)]^k\right|\right| = \left|\left|A(T^{km-1}(\zeta))\cdot A(T^{km-2}(\zeta))\cdots A(\zeta)\right|\right| > C\lambda^{km}.$$

By the inequality of products of norms of matrices, we get

$$\left|\left|A(T^{m-1}(\omega))\cdot A(T^{m-2}(\omega))\cdots A(\omega)\right|\right|^k \geq \left|\left|[A(T^{m-1}(\omega))\cdot A(T^{m-2}(\omega))\cdots A(\omega)]^k\right|\right| > C\lambda^{km}$$

$$\left|\left|A(T^{m-1}(\omega))\cdot A(T^{m-2}(\omega))\cdots A(\omega)\right|\right| > C^{1/k}\lambda^m.$$

Because this holds for all positive k, we can let $C = 1$, and the result follows from definition. $\square$

When studying uniformly hyperbolic sets of matrices, we can examine what is the hyperbolic locus, first defined in the Introduction in definition 4. The hyperbolic locus is an open set, and details on the nature of the hyperbolic locus, see [1], [20]. For any n, the set $\mathcal{H}_n$ consists of infinitely many disjoint, connected components. In [1], the geometry of $\mathcal{H}_2$ is defined.

Theorem 3. *Theorem 3.2 of Avila-Bochi-Yoccoz*

Any compact subset of $SL(2,\mathbb{R})^2$ can contain only finitely many components of $\mathcal{H}_2$. None of the connected components of $\mathcal{H}_2$ share boundaries, and if $(A,B) \in \partial\mathcal{H}_2$ then at least one of the conditions holds:

- *There is a sequence of matrices A and B such that the product is parabolic.*
- *$u_A = s_B$ or vice versa.*

If extended to $SL(2,\mathbb{R})^n$ for $n \geq 3$, the boundary of $\mathcal{H}_n$ is more complicated. The details for why it does not hold for higher dimension and what still holds is covered in section 3.5.

Define $\mathcal{E}_n \subset SL(2,\mathbb{R})^n$ to be the set of pairs (A,B) such that there exists some sequence of A and B where the product is elliptic. This set is also open.

Theorem 4. *Theorem 3.3 of Avila-Bochi-Yoccoz*

In $SL(2,\mathbb{R})^2$, *we have* $\partial\mathcal{H}_2 = \partial\mathcal{E}_2$.

In [7], this is shown to not be true in $SL(2,\mathbb{R})^n$ for $n \geq 3$ via an example in $SL(2,\mathbb{R})^3$. In Example 5.1 of this paper, an example is provided of 3 hyperbolic transformations. These three transformations would correspond to the set of $SL(2,\mathbb{R})$ matrices

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & -4 \\ 0 & 1 \end{bmatrix} \right\}.$$

The set of matrices is not uniformly hyperbolic, as there is no lower bound to the norm of the products. Every product of these matrices is hyperbolic, however, so it not an element of $\mathcal{H}_3$ or $\mathcal{E}_3$. If perturbed, the set may correspond to an element of $\mathcal{E}_3$, but it is not on the boundary of $\mathcal{H}_3$.

3.2 Multicone

It is important to introduce the cone condition first, which gives an equivalent definition of uniformly hyperbolic sets of $SL(2,\mathbb{R})$ matrices in terms actions on the projective space $\mathbb{RP}^1$. Formally, the projective space is defined as

$$\mathbb{RP}^1 = (\mathbb{R}^2 \setminus \{0\})/\sim, \text{ where } \vec{v} \sim \vec{w} \Leftrightarrow \vec{v} = \alpha\vec{w} \text{ for some } \alpha \neq 0.$$

When studying $\mathbb{RP}^1$, there is an understood metric applied to the space.

Definition 11 (Cone Condition)**.** *Given a uniformly hyperbolic set $\mathcal{M}$, there exists a proper open subset $\mathcal{C} \subset \mathbb{RP}^1$ such that $\mathcal{C}$ is a finite union of open intervals, and for all $M \in \mathcal{M}$, $M(\overline{\mathcal{C}}) \subset \mathcal{C}^o$.*

This condition states that a set $\mathcal{M}$ is uniformly hyperbolic if and only if there is a nontrivial closed subset of $\mathbb{RP}^1$ (finite number of intervals) over which all $M \in \mathcal{M}$ are contracting over its interior. Because the multicone is a subset of $\mathbb{RP}^1$, all of the relevant computations involve elements of $\mathbb{RP}^1$. Given a hyperbolic matrix M, the stable eigenvector (denoted s_M) and unstable eigenvector (denoted u_M) will be defined as elements of $\mathbb{RP}^1$. Given a finite set $\mathcal{M} \subset SL(2,\mathbb{R})$, define $SG(\mathcal{M}) = \{M : M \text{ is a product of elements of } \mathcal{M}\}$ to be the semigroup[2]. If a set $\mathcal{M}$ is uniformly hyperbolic, then all of the elements of $SG(\mathcal{M})$ are hyperbolic. The converse of this is not true.

Proposition 3. *Given a multicone $\mathcal{C}$ of uniformly hyperbolic $\mathcal{M}$, for all $M \in SG(\mathcal{M})$, then $u_M \in \mathcal{C}$ and $s_M \notin \overline{\mathcal{C}}$.*

Proof. For all $M \in SG(\mathcal{M})$ and disjoint open intervals $\mathcal{A} \ni s_M$ and $\mathcal{B} \ni u_M$, there exists nonnegative integer n such that $M^n(\mathcal{A}) \supset \mathbb{RP}^1 \backslash \mathcal{B}$. Therefore for all $x \in \mathbb{RP}^1 \backslash \{u_M\}$, there exists positive integer n such that $x \in M^n(\mathcal{A})$ and so $\mathcal{A}$ cannot be a subset of the multicone.
For all $M \in SG(\mathcal{M})$ and open sets $\mathcal{C}$ and $\mathcal{B} \ni u_M$, there exists positive integer n such that $M^n(\mathcal{C}) \subset \mathcal{B}$. Therefore $u_M \in \mathcal{C}$. □

By its definition, the multicone of a uniformly hyperbolic set $\mathcal{M}$ does not have to be unique. In fact it is never unique. This brings us to the concept of a skeleton, which is unique.

Definition 12. *Given a uniformly hyperbolic set $\mathcal{M}$, define the skeleton as*

$$\mathcal{S}_{\mathcal{M}} = \overline{\{u_A : A \in SG(\mathcal{M})\}}$$

[2] a *semigroup* is a set that is closed under a binary, associative operation. In this case it is a set matrices that contains any product of its elements

Given a set $\mathcal{M}$, define $\mathcal{M}^{-1} = \{M : M^{-1} \in \mathcal{M}\}$. Given a uniformly hyperbolic set $\mathcal{M}$, denote ${\mathcal{S}_{\mathcal{M}}}^{-1}$ as the skeleton of $\mathcal{M}^{-1}$.

Proposition 4. *Assume $\mathcal{M}$ is uniformly hyperbolic. For all $M \in \mathcal{M}$, $M(\mathcal{S}_{\mathcal{M}}) \subset \mathcal{S}_{\mathcal{M}}$.*

Proof. For all $A \in SG(\mathcal{M})$ and $M \in \mathcal{M}$, we have $u_A, u_{(MA^n)} \in \mathcal{S}_{\mathcal{M}}$ for all positive n. Because $\mathcal{M}$ is uniformly hyperbolic, any multicone $\mathcal{C} \supset \{u_{(MA^n)}\}_{n>0}$ which is contracting under A, therefore for any open set $\mathcal{B} \ni u_A$ there exists $n > 0$ such that $A^n(\mathcal{C}) \subset \mathcal{B}$. Under the metric topology of $\mathbb{RP}^1$, we get the limits

$$A^n \cdot u_{(MA^n)} \xrightarrow[n\to\infty]{} u_A$$

$$u_{(MA^n)} = MA^n \cdot u_{(MA^n)} \xrightarrow[n\to\infty]{} M \cdot u_A.$$

Therefore, $M \cdot u_A \in \overline{\{u_A : A \in SG(\mathcal{M})\}} = \mathcal{S}$ □

Proposition 5. *Assume $\mathcal{M}$ is uniformly hyperbolic, then*

$$\mathcal{S}_{\mathcal{M}} = \overline{\{M \cdot u_N : N \in \mathcal{M},\ M \in SG(\mathcal{M})\}}$$

Proof. Let $\mathcal{C}$ be a cone for $\mathcal{M}$. For all $M \in SG(\mathcal{M})$ and $N \in \mathcal{M}$, $u_N \in \mathcal{C}$, $s_M \notin \mathcal{C}$, and so assuming $M \neq N$ we get

$$M^n \cdot u_N \xrightarrow{n\to\infty} u_M.$$

Therefore $u_M \in \overline{\{M \cdot u_N : N \in \mathcal{M},\ M \in SG(\mathcal{M})\}}$, and so $\mathcal{S}_{\mathcal{M}} \subset \overline{\{M \cdot u_N : N \in \mathcal{M},\ M \in SG(\mathcal{M})\}}$. By Proposition 4, we have $\mathcal{S}_{\mathcal{M}} = \overline{\{M \cdot u_N : N \in \mathcal{M},\ M \in SG(\mathcal{M})\}}$. □

A simple a depiction of a skeleton is for the set $\mathcal{M} = \{M_1, M_2\}$ when it has a principal cone.

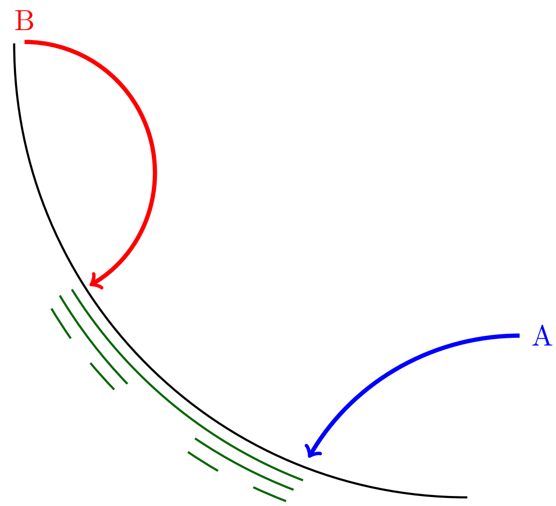

Figure 3.1: Assuming the spectral radius of the matrices A and B are sufficiently large, the skeleton of $\{A, B\}$ is a Cantor set.

In this paper, $\mathbb{RP}^1$ will be depicted as circles, and the matrices will be depicted as arrows pointing from stable eigenvectors to unstable eigenvectors. The top green line in figure 3.1 is simply an interval bounded by $u_B, u_A \in \mathbb{RP}^1$. The green lines below are the interval when acted on by products of A and B. Define the single interval as C_0, and iteratively define $C_i = B(C_{i-1}) \cup A(C_{i-1})$. Assuming $B(C_0) \cap A(C_0) = \emptyset$, we get $\bigcup C_i = \mathcal{S}$ and is a Cantor set. This is the skeleton and is constructed similarly to how middle-α Cantor sets are constructed. if $B(C_0) \cap A(C_0) \neq \emptyset$, then the skeleton is the entire interval.

Proposition 6. *Assuming $\mathcal{M}$ is uniformly hyperbolic and consists of more than one element. The skeleton does not have any isolated points.*

Proof. It is sufficient to show that for all $C \in SG(\mathcal{M})$ we get that u_C is not an isolated points. The result then follows from the definition. For all $C \in SG(\mathcal{M})$ there exists $A \in \mathcal{M}$ such that $A \neq C$. Given $u_C \in \mathcal{S}$, we get $u_{C^nA} \neq u_C$, and for some $n > 0$ we get

$$u_{C^nA} \xrightarrow[n\to\infty]{} u_C.$$

$\square$

This result is similar to Proposition 3.3 from [11] and Proposition 4.5.

Proposition 7. *Given uniformly hyperbolic set $\mathcal{M}$, define the family of sets $\{\mathcal{C}_i\}$ as all possible multicones of $\mathcal{M}$ over $\mathbb{RP}^1$. The skeleton equals the set*

$$\mathcal{S}_{\mathcal{M}} = \overline{\bigcap \mathcal{C}_i}.$$

Proof. $\subseteq$ This holds by definition 12 and Proposition 3.

$\supseteq$ Assume $x \notin \mathcal{S}_{\mathcal{M}}$, therefore there exists open interval $\mathcal{B} \ni x$ such that $\mathcal{B} \cap \overline{\mathcal{S}} = \emptyset$. Because $\mathcal{M}$ is uniformly hyperbolic, so is $\mathcal{M}^{-1}$, so there exists a cone containing the stable eigenvectors and none of the unstable eigenvectors, which we will define $\mathcal{C}^{-1}$. There are finitely many elements of $M_i \in SG(\mathcal{M})$ such that $M_i^{-1}(B) \nsubseteq \mathcal{C}^{-1}$. Therefore $\bigcup_i M_i^{-1}(B) \cup \mathcal{C}^{-1}$ is a cone is $\mathcal{M}^{-1}$. This gives us that the complement of $\overline{[(\bigcup M_i^{-1}(B)) \cup \mathcal{C}^{-1}]}$ is a cone of $\mathcal{M}$, and x is not an element of this set, making $x \notin \overline{\bigcap \mathcal{C}_i}$. □

A skeleton is not a cone, however, as it is a closed set, and does not contain any intervals. This brings us to the next couple concepts about minimal number. For any set $\mathcal{M}$, there is a minimal number n such that any multicone of $\mathcal{M}$ consists of at least n open intervals. If the minimal number is one, then a principal cone can be defined.

Definition 13. *A principal cone of uniformly hyperbolic set $\mathcal{M}$ is a multicone which consists of a single open interval.*

Proposition 8. *Given a uniformly hyperbolic set $\mathcal{M}$, a principal cone can be defined if and only if there exists a single open interval $\mathcal{C} \subset \mathbb{RP}^1$ such that for all $M \in \mathcal{M}$, $u_M \in \mathcal{C}$ and $s_M \notin \mathcal{C}$.*

Proof. To prove one direction, let $\mathcal{C}$ be a single open interval such that for all $M \in \mathcal{M}$, $u_M \in \mathcal{C}$ and $s_M \notin \mathcal{C}$. Therefore $M(\overline{\mathcal{C}}) \subsetneq \mathcal{C}^o$ for all M. By the cone condition definition 11, $\mathcal{C}$ is a multicone. Proving the other direction comes from Proposition 3. □

The next concept is the minimal cone. As previously mentioned, a uniformly hyperbolic set $\mathcal{M}$ does not have a unique multicone. Defining sets that can be uniquely defined allow some of the calculations to be more straightforward.

Definition 14. *Given uniformly hyperbolic set $\mathcal{M}$ with minimal number n and the set $\{\mathcal{C}_i : \mathcal{C}_i$ is a multicone of $\mathcal{M}$ consisting of n intervals$\}$. Define the minimal cone as $\mathcal{C}'_{\mathcal{M}} = \bigcap \mathcal{C}_i$.*

For any uniformly hyperbolic $\mathcal{M}$, the minimal cone $\mathcal{C}'_{\mathcal{M}}$ is a closed set, and bounded by elements of the skeleton $\mathcal{S}_{\mathcal{M}}$. Additionally, the intersection of $\mathcal{C}'_{\mathcal{M}}$ and $\mathcal{C}'_{\mathcal{M}^{-1}}$ (the minimal cone of $\mathcal{M}^{-1}$) is empty.

Proposition 9. *Define $\mathcal{F}$ to be the set of closed sets in $\mathbb{RP}^1$, and $\Psi : \mathcal{H}_n \to \mathcal{F}$ the mapping of uniformly hyperbolic sets to its minimal cone. Given any continuous $P : [0,1] \to \mathcal{H}_n$, there exists homeomorphism $\Sigma_t : \mathbb{RP}^1 \to \mathbb{RP}^1$ continuous under t such that*

$$\Sigma_0 = id$$

$$\Sigma_t \circ \Psi \circ P(0) = \Psi \circ P(t)$$

Proof. This follows from the bounds of the minimal cone being elements of the skeleton, which are defined by continuous mappings $SL(2,\mathbb{R})^n \to \mathbb{RP}^1$. For any given t, there are finite number of boundary points of the minimal cone which are defined by eigenvectors of elements of the semigroup[3]. We can deduce that on any path P, the boundaries of the minimal cone change continuously. These continuous changes on the boundaries can be used to define the homeomorphisms (not uniquely) on $\mathbb{RP}^1$, which have to be continuous under t. □

Corollary 1. *For any 2 elements $X, Y \in \mathcal{H}_n$, if X, Y are elements of the same connected component, then there is a homeomorphism on $\mathbb{RP}^1$ which maps the minimal cone of X to the minimal cone of Y. Furthermore, there is a continuous path of homeomorphisms from the identity to the mapping.*

The intervals of $\mathcal{C}'_{\mathcal{M}}$ and ${\mathcal{C}'_{\mathcal{M}}}^{-1}$ alternate over $\mathbb{RP}^1$. When referring to the minimal cone, there will often be an implied ordering of the intervals made apparent. This means that we will define the minimal cone as a union of disjoint intervals $\mathcal{C}'_{\mathcal{M}} = \bigcup_i \mathcal{C}'_{\mathcal{M},i}$. This ordering would create equivalence classes among the minimal cones with the same minimal number. A simple example is in $SL(2,\mathbb{R})^2$, where the minimal number is two, but there are two different orderings, as shown in figures 3.3 and 3.4. These graphs

To expand on the concept of ordering, we need to impose a relation on $\mathbb{RP}^1$, the space over which the cones are defined. Due to the nature of $\mathbb{RP}^1$, it is not completely straightforward to impose a binary relation. One can consider the mapping $\Phi : \mathbb{RP}^1 \to \mathbb{R} \cup \infty$ by $\begin{bmatrix} x \\ 1 \end{bmatrix} \mapsto x$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \infty$.

[3]There are actually a minimal number of eigenvectors on the boundary of the minimal cone, but the proof of this requires more detail than what has currently been introduced.

A straight inequality relation would not be useful, as $-\infty < 0 < \infty = -\infty$, making $a < b$ for all $a, b \in \mathbb{RP}^1$ by transitivity. We will depict the projective space $\mathbb{RP}^1$ as a circle in graphs, and matrices will be represented by arrows pointing from the stable direction to the unstable direction in $\mathbb{RP}^1$. Given $m \geq 3$ and $p_1, \ldots, p_m \in \mathbb{RP}^1$, we say that the ordered m-tuple $(p_1, p_2, \ldots, p_m)$ is a *chain* if p_j lies in the arc going counterclockwise from p_{j-1} to p_{j+1} for all $1 \leq j \leq m$ (with addition and subtraction of indices done modulo m to deal with the boundary points).

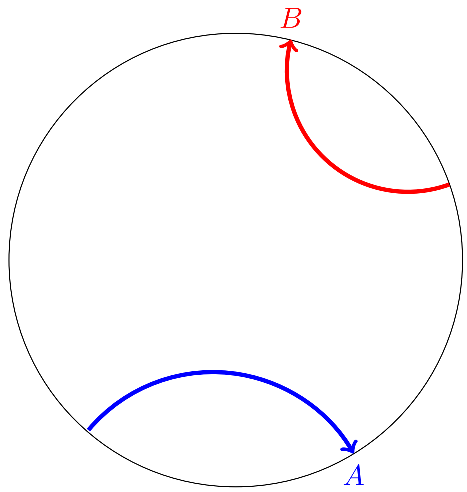

Figure 3.2: Projective space $\mathbb{RP}^1$ with matrices A and B depicted. In this diagram, (s_A, u_A, s_B, u_B) is a chain.

In terms of the calculations, given any chain $(p_1, p_2, \cdots, p_n)$, if $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin \{p_i\}$ then there exists m such that $\Phi p_m \leq \Phi p_{m+1} \leq \cdots \Phi p_n \leq \Phi p_1 \leq \Phi p_2 \leq \cdots \Phi p_{m-1}$, and there are at least three distinct points. If there exists $p_m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\Phi p_{m+1} \leq \cdots \Phi p_n \leq \Phi p_1 \leq \Phi p_2 \leq \cdots \Phi p_{m-1}$. To avoid trivial examples, let at least three of the points in any chain be distinct. We can further define open and closed sets as $]p_i, p_j[= \{p : \text{ the chain } (p_i, p, p_j) \text{ holds } \& \ p \neq p_i, p_j\}$ and $[p_i, p_j] = \{p : \text{ the chain } (p_i, p, p_j) \text{ holds}\}$.

Proposition 10. *For any uniformly hyperbolic $\mathcal{M}$, let n be the minimal number, $\mathcal{C}'_{\mathcal{M}}$ be the minimal cone, and ${\mathcal{C}'_{\mathcal{M}}}^{-1}$ be the minimal cone of $\mathcal{M}^{-1}$. There exists sets of unique, disjoint intervals $\{\mathcal{M}_i\}$ and $\{\mathcal{N}_i\}$ such that $\mathcal{C}'_{\mathcal{M}} = \bigcup_i \mathcal{M}_i$ and ${\mathcal{C}'_{\mathcal{M}}}^{-1} = \bigcup \mathcal{N}_i$ and for any $u_i \in \mathcal{M}_i \cap \mathcal{S}$ and $s_i \in \mathcal{N}_i \cap \mathcal{S}$, we get the chain of distinct points $(u_1, s_1, u_2, s_2, \cdots, u_{n-1}, s_{n-1}, u_n, s_n)$.*

If a set $\mathcal{M} = \{M_1, M_2, \cdots M_n\}$ is uniformly hyperbolic with a principal cone, then there exists $p_1, p_2 \in \mathbb{RP}^1$ such that for all i, $u_{M_i} \in [p_1, p_2]$ and $s_{M_i} \in]p_2, p_1[$.

Definition 15. *The orderings of the minimal cone are the different equivalence classes of elements of $\mathcal{H}_n$ with the same minimal number. Given elements $(X_1, X_2, \cdots X_n), (Y_1, Y_2, \cdots Y_n) \in \mathcal{H}_n$ with minimal number m, define the minimal cone of X as $\bigcup \mathcal{M}_{i,X}$ and inverse cone as $\bigcup \mathcal{N}_{i,X}$, and similarly define the minimal cone of Y as $\bigcup \mathcal{M}_{i,Y}$ and inverse cone as $\bigcup \mathcal{N}_{i,Y}$ as defined in Proposition 10. The equivalence $X \sim Y$ holds if and only if for any finite sequence $(j_1, j_2, \cdots j_k)$ where $j_i \in \{1, 2, \cdots, n\}$ there exist the mappings*

$$\alpha((j_1, j_2, \cdots, j_k)) \in \{1, 2, \cdots, m\}$$

$$\beta((j_1, j_2, \cdots, j_k)) \in \{1, 2, \cdots, m\}$$

such that the following conditions hold.

- *The unstable eigenvector of $X_{j_1} X_{j_2} \cdots X_{j_k}$ is in $\mathcal{M}_{\alpha((j_1, j_2, \cdots, j_k)), X}$.*
- *The unstable eigenvector of $Y_{j_1} Y_{j_2} \cdots Y_{j_k}$ is in $\mathcal{M}_{\alpha((j_1, j_2, \cdots, j_k)), Y}$.*
- *The stable eigenvector of $X_{j_1} X_{j_2} \cdots X_{j_k}$ is in $\mathcal{N}_{\beta((j_1, j_2, \cdots, j_k)), X}$.*
- *The stable eigenvector of $Y_{j_1} Y_{j_2} \cdots Y_{j_k}$ is in $\mathcal{N}_{\beta((j_1, j_2, \cdots, j_k)), Y}$.*

Proposition 11. *Given any connected component of $\mathcal{H}_n$, if X, Y are elements of the connected component, then their minimal cones have the same ordering.*

The proof of this proposition closely resembles the proof of Corollary 1.

3.3 Hyperbolic Locus Introduction

As stated in section 3.1, the hyperbolic locus $\mathcal{H}_n$ is the open set of n-tuples of $SL(2,\mathbb{R})$ matrices such that the corresponding set of matrices is uniformly hyperbolic. The first basic detail to note is that for any n, $\mathcal{H}_n$ is an open set. This follows from 3.2, and is shown below. Given any uniformly hyperbolic set of matrices, it remains uniformly hyperbolic under small perturbations. The set $\mathcal{H}_n$ is a infinite number of connected components, and from Proposition 11, we have that the ordering of the minimal cone is fixed in any connected component.

The geometry of $\mathcal{H}_n$ is largely unknown for $n > 2$, but for $n = 2$, the geometry is detailed in [1]. In their paper, they pose several questions about the geometry of $\mathcal{H}_n$, mostly centering around the boundaries of $\mathcal{H}_n$. Some of the questions are expanded on in Chapter 5. We do know that $\mathcal{H}_n$ is comprised of infinitely many connected components, and the geometry of individual components are fairly well understood. The first most basic detail to address, is that for any n, there are principal components. A component is principal if its elements have a principal cone. In $SL(2,\mathbb{R})^n$ there are 2^n principal components. Over $SL(2,\mathbb{R})^n$, there is the closed set $A = \{(M_1, M_2, \cdots, M_n) : |tr(M_i)| \leq 2 \text{ for some } i\}$, and $A \cap \mathcal{H}_n = \emptyset$. The open set $SL(2,\mathbb{R})^n \backslash A$ is equal to 2^n open disjoint sets, and in each of these disjoint sets, there is a single principal connected component. One can use Propositions 11 and 9 to show this.

The first, and arguably most important detail to mention regarding the geometry of the connected components, is that the connected components of $\mathcal{H}_n$ are semialgebraic sets.

Theorem 3.1 (Theorem 4.1 of [1])**.** *Define $X = (M_1, M_2, \cdots M_n) \in SL(2,\mathbb{R})^n$ and $\mathcal{M} = \{M_1, M_2, \cdots M_n\}$. For all $X \in \partial\mathcal{H}_n$, one of the following conditions holds.*

(a) There exists $\pm id \in SG(\mathcal{M})$.

(b) There exists parabolic $M \in SG(\mathcal{M})$.

(c) What is called a heteroclinic connection occurs, where $M \cdot u_N = s_P$ for some $N, P \in SG(\mathcal{M})$ and $M \in SG(\mathcal{M}) \cup \{Id\}$.

As a note, if condition (b) occurs for some $M \in SG(\mathcal{M})$, then $u_M = s_M$. Condition (b) implies condition (c), where $M = Id$. Condition (c) does not imply condition (b) however, even if $M = Id$[4].

These conditions will be referred to as the boundary conditions a, b, or c in the calculations in the next sections.

[4]Consider $M = Id$, spectral radius of N is 5, and spectral radius of P is 6.

Theorem 3.2 (Corollary 4.5 of [1])**.** *Every connected component of $\mathcal{H}_n$ is semialgebraic.*

This shows us that every connected component of $\mathcal{H}_n$ is an open set for any n. We can then easily deduce that $\mathcal{H}_n$ is an open set.
Thre are is a caveat to Theorem 3.1, which is mentioned below.

Proposition 12. *Condition (a) from Theorem 3.1 can only occur on the boundary of a principal component.*

Proof. For all $\mathcal{M} = (M_1, M_2, \cdots, M_n) \in \mathcal{H}_n$, define $M \in SG(\mathcal{M})$ to be a specific product of the elements of $\{\mathcal{M}_1, \mathcal{M}_2, \cdots \mathcal{M}_n\}$. Given a path $P : [0,1] \to SL(2,\mathbb{R})^n$ in the closure of a connected non-principal component such that $P([0,1))$ is in the component and $P(1)$ is on the boundary such that at $\pm Id = M \in SG(\mathcal{M})$. Given interval $\mathcal{C}$ in the minimal cone $\mathcal{C}'_{P(t)}$ such that $u_M \in \mathcal{C}$, there exists n such that $M^n(\mathcal{C}'_{P(0)}) \subset \mathcal{C}$. Because of continuity, $M^n(\mathcal{C}'_{P(t)}) \subset \mathcal{C}$ for all $t < 1$. Because this is a nonprincipal chain, there exist matrices $N, O \in SG(\mathcal{M})$ such that the chain (u_M, s_N, u_O) holds and $(u_M, M^n \cdot s_N, M^n \cdot u_O, s_N, u_O)$ holds. Because $M^n \to \pm Id$, as $t \nearrow 1$, M preserves orientation and chains, and there exists t such that $M^n \cdot u_O = s_N$ and the path must cross over into the boundary. Therefore the connected component cannot be a nonprincipal component.
Assume $\mathcal{M} = (M_1, M_2, \cdots, M_n) \in \mathcal{H}_n$ is in a principal component, there exists an interval in $\mathbb{RP}^1$ which contains all of the unstable eigenvectors and non of the stable eigenvectors. Take any M_i, and define a path in $SL(2,\mathbb{R})^n$ which only changes the trace (continuously) of M_i and ends at 2 (or -2). So long as the trace is greater than 2 or less than -2, then $\mathcal{M}$ is in the principal component. When the matrix has a trace of ± 2, then the element is no longer in the component and on the boundary. Because the eigenvectors of M_I are fixed on this path, then $M_i = Id$ at the end of the path. $\square$

Proposition 13. *Assume a non-principal connected component $\mathcal{C} \subset \mathcal{H}_n$, a path $P : [0,1] \to SL(2,\mathbb{R})^n$ such that $P(t)|_{0 \le t < 1} \in \mathcal{C}$ and $P(1) \in \partial\mathcal{C}$, $\mathcal{F}$ mapping to the minimal cone $\mathcal{C}'$ as defined in Proposition 9, and $\mathcal{F}'$ mapping to the inverse minimal cone $\mathcal{C}'^{-1}$. The following limit is true*

$$dis(\mathcal{F}(P(t)), \mathcal{F}'(P(t)) \xrightarrow[t \to 1]{} 0.$$

The distance between $\mathcal{C}'$ and $\mathcal{C}'^{-1}$ goes to 0.

Proof. This follows as a Corollary from Theorem 3.1 and Proposition 9. Because condition (a) can occur only on the boundary of principal components by Proposition 12, only conditions (b) and (c)

need to be considered. Either condition (b) and (c) occur if and only if there exists elements $x \in \mathcal{F}$ and $y \in \mathcal{F}'$ such that $x = y$. Because $\mathcal{F}$ and $\mathcal{F}'$ are subject to the continuous homomorphisms defined in Proposition 9, we get $dist(x, y) \to 0$. $\square$

3.4 Hyperbolic Locus $\mathcal{H}_2$

In this section, we will describe the geometry of $\mathcal{H}_2$, pulling from [1]. This is the set of all 2-ples (A, B) such that $A, B \in SL(2,\mathbb{R})$ and $\{A, B\}$ is uniformly hyperbolic.

First, we will consider the case where $\{A, B\}$ is uniformly hyperbolic with a principal cone. Note that if the set has a principal cone, then there is $p_1, p_2 \in \mathbb{RP}^1$ such that we have the chains (p_1, u_A, p_2, s_A) and (p_1, u_B, p_2, s_B). Note that the sets $\{(A, B) \in SL(2,\mathbb{R}) : tr(A) > 2\}$ and $\{(A, B) \in SL(2,\mathbb{R}) : tr(A) < 2\}$ are disjoint, which means there are 4 disjoint sets to consider.

(1) $\{(A, B) \in SL(2,\mathbb{R}) : tr(A), tr(B) > 2\}$

(2) $\{(A, B) \in SL(2,\mathbb{R}) : tr(A), -tr(B) > 2\}$

(3) $\{(A, B) \in SL(2,\mathbb{R}) : -tr(A), tr(B) > 2\}$

(4) $\{(A, B) \in SL(2,\mathbb{R}) : -tr(A), -tr(B) > 2\}$

Proposition 3.3. *In $\mathcal{H}_2$ there are exactly 4 disjoint connected components with a principal cone.*

Proof. We can define the 4 connected components of $\mathcal{H}_2$ as the following.

(1) $\mathcal{H}_{p,1} = \{(A, B) \in SL(2,\mathbb{R}) : tr(A) > 2, tr(B) > 2 : \{A, B\}$ is uniformly hyperbolic$\}$

(2) $\mathcal{H}_{p,2} = \{(A, B) \in SL(2,\mathbb{R}) : tr(A) > 2, tr(B) < 2 : \{A, B\}$ is uniformly hyperbolic$\}$

(3) $\mathcal{H}_{p,3} = \{(A, B) \in SL(2,\mathbb{R}) : tr(A) < 2, tr(B) > 2 : \{A, B\}$ is uniformly hyperbolic$\}$

(4) $\mathcal{H}_{p,4} = \{(A, B) \in SL(2,\mathbb{R}) : tr(A) < 2, tr(B) < 2 : \{A, B\}$ is uniformly hyperbolic$\}$

To show that the 4 sets are connected, given any 2 elements $X, Y \in \mathcal{H}_{p,i}$ one can define a path $P[0, 1] \to SL(2,\mathbb{R})^2$ such that $P(0) = X$, $P(1) = Y$, and for all t, $P(t) = (P_1(t), P_2(t))$ consists of hyperbolic matrices which have a principal cone. The matrices A, B can be uniquely defined by the eigenvectors and trace so long as they are hyperbolic, making the path simpler to define. Because the trace of matrices are the same sign, then $P_1(t)$ and $P_2(t)$ can remain hyperbolic over $0 \leq t \leq 1$. So long as any of the unstable eigenvectors of $P_1(t)$ and $P_2(t)$ do not overlap with the stable eigenvectors of $P_1(t)$ and $P_2(t)$, then $P(t) \in \mathcal{H}_{p,i}$. □

The minimal cone is either the interval $[u_A, u_B]$ or $[u_B, u_A]$, and the boundary is defined in [1]. The connected components are semialgebraic, and on the boundaries $A = Id$, $B = Id$, A is parabolic, B is parabolic, $u_A = s_B$, or $u_B = s_A$.

The more interesting cases are when the there is not a principal cone. We will begin with what [1] define as $\mathcal{H}_{id}$.

$$\mathcal{H}_{id} = \{(A, B) : |tr(A)|, |tr(B)|, |tr(AB)| > 2, \ \& \ tr(A)tr(B)tr(AB) < 0\}$$

This consists of the following 8 connected components.

- $\{(A, B) \in SL(2, \mathbb{R}) : tr(A), tr(B), -tr(AB) > 2 \ \&$ there is the chain $(u_A, s_B, u_B, s_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : tr(A), tr(B), -tr(AB) > 2 \ \&$ there is the chain $(s_A, u_B, s_B, u_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : tr(A), -tr(B), tr(AB) > 2 \ \&$ there is the chain $(u_A, s_B, u_B, s_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : tr(A), -tr(B), tr(AB) > 2 \ \&$ there is the chain $(s_A, u_B, s_B, u_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : -tr(A), tr(B), tr(AB) > 2 \ \&$ there is the chain $(u_A, s_B, u_B, s_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : -tr(A), tr(B), tr(AB) > 2 \ \&$ there is the chain $(s_A, u_B, s_B, u_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : -tr(A), -tr(B), -tr(AB) > 2 \ \&$ there is the chain $(u_A, s_B, u_B, s_A)\}$
- $\{(A, B) \in SL(2, \mathbb{R}) : -tr(A), -tr(B), -tr(AB) > 2 \ \&$ there is the chain $(s_A, u_B, s_B, u_A)\}$

For all $(A, B) \in \mathcal{H}_{id}$, the minimal cone consists of two intervals. The depictions of these sets can be one of two following figures.

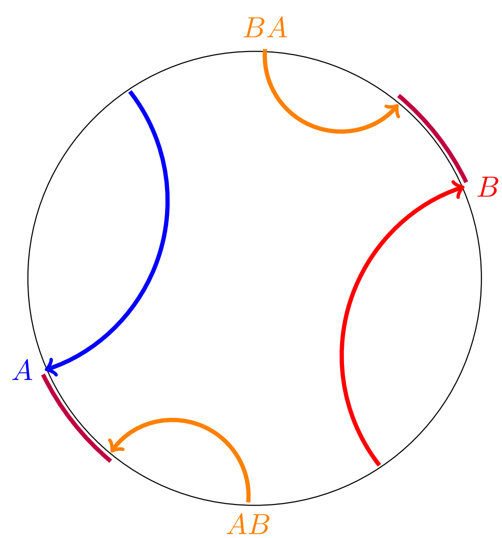

Figure 3.3: H_{id} with chain (u_A, s_B, u_B, s_A). The minimal cone is the set $[u_A, u_{AB}] \cup [u_B, u_{BA}]$, and the inverse minimal cone is $[s_{BA}, s_A] \cup [s_{AB}, s_B]$.

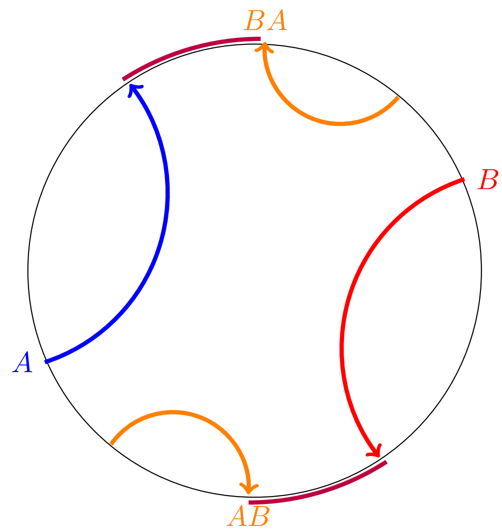

Figure 3.4: H_{id} with chain (u_A, s_A, u_B, s_B). The minimal cone is the set $[u_{AB}, u_B] \cup [u_{BA}, u_A]$.

Note that $B \cdot u_{AB} = BAB \cdot u_{AB} = BA \cdot (Bu_{AB})$. Therefore $B \cdot u_{AB}$ is an eigenvector of BA. Defining $\vec{u_{AB}} \in \mathbb{R}^2$ with unit norm, $|(BA)^n B \cdot \vec{u_{AB}}| = |B(AB)^n \cdot \vec{u_{AB}}| > |B \cdot \vec{u_{AB}}|$, and so we get that $B \cdot u_{AB}$ is the unstable eigenvector of BA. Through similar calculations we get

$$B \cdot s_{AB} = s_{BA},$$

$$B \cdot u_{AB} = u_{BA},$$

$$A \cdot s_{BA} = s_{AB},$$

$$A \cdot s_{BA} = s_{AB}.$$

The following Proposition is a corollary of a Lemma from [1], but is rephrased to fit the wording of this thesis.

Proposition 14. *Assume $(A, B) \in \partial H_{id}$, then*

- *The matrix AB is parabolic and $AB \neq \pm Id$.*
- *The matrix A is parabolic and $A \neq \pm Id$.*
- *The matrix B is parabolic and $B \neq \pm Id$.*

This follows from Proposition 4.15 from [1].

Next we should consider the monoid[5], $< \mathcal{F} >$ generated by the diffeomorphisms $\mathbb{RP}^1 \to \mathbb{RP}^1$

$$\mathcal{F}_+ : (A, B) \mapsto (A, AB)$$

and

$$\mathcal{F}_- : (A, B) \mapsto (BA, B).$$

The elements of this monoid can actually generate all of the nonprinicipal components. The way [1] defined the geometry of $\mathcal{H}_2$ is by the following Theorem.

Theorem 3.4 (3.1 from Avila-Bochi-Yoccoz)**.** *Given any connected component $\mathcal{C} \subset \mathcal{H}_2$, which is not a principal component, then there exists $\mathcal{F} \in < \mathcal{F} >$ such that $\mathcal{F}(\mathcal{C}) \subset \mathcal{H}_{id}$.*

More explicitly, for every nonprincipal connected component, there exists $\mathcal{F} \in < \mathcal{F} >$ such that $\mathcal{F}^{-1}$ is a continuous bijection from $calH_{id}$ to the connected component. Using this, we can explicitly define the boundaries as $\mathcal{F}^{-1}$ acting on the boundaries of $\mathcal{H}_{id}$. This detail will give us that connected components cannot share boundary points, and that any compact subset of $SL(2, \mathbb{R})^2$ can intersect only finitely many connected components. This leads to the question posed in Chapter 5.
Among the simplest examples is the set $\mathcal{F}_-^{-1}\mathcal{H}_{id}$. Technically this is 8 different connected components, but the diagrams are the same modified versions of figure 3.4 and fig 3.3 The minimal cone of this set is 3 intervals rather than 2.

[5] a *monoid* is a semigroup that contains an identity element.

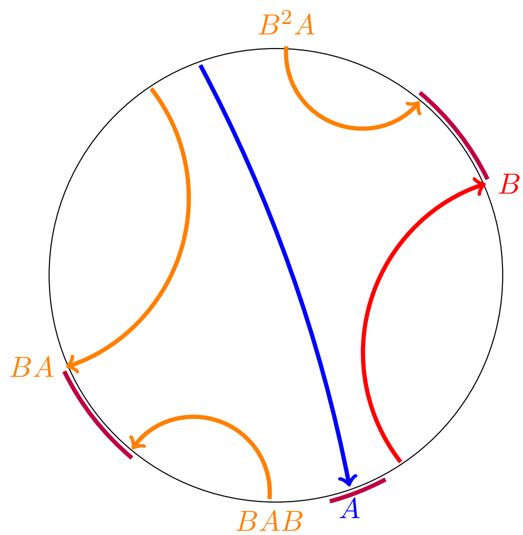

Figure 3.5: Diffeomorphism $\mathcal{F}_{-}^{-1}$ on H_{id} with chain (u_A, s_B, u_B, s_A). The minimal cone is the set $[u_{BA}, u_{BAB}] \cup [u_{AB}, u_{AB^2}] \cup [u_B, u_{B^2A}]$.

Notice that the above figure 3.5 is almost exactly the same as figure 3.3, except A is replaced with BA and a new curve is drawn to depict A. To more accurately depict the cone, the figure 3.6 is provided below.

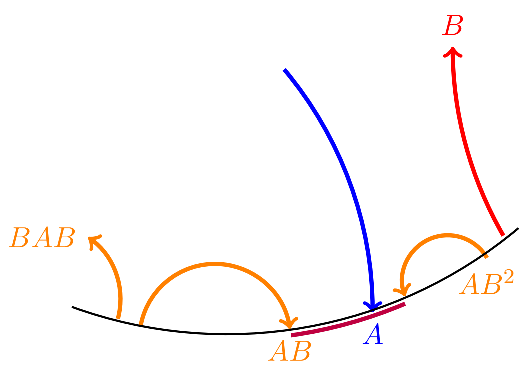

Figure 3.6: Zoomed in detail of minimal cone in $\mathbb{RP}^1$ of $\mathcal{F}_{-}^{-1}$ on H_{id} with chain (u_A, s_B, u_B, s_A).

3.5 Hyperbolic Locus Geometry

The previous section 3.4 outlined the geometry of $\mathcal{H}_2$, in which there are countably many connected components of $\mathcal{H}_2$, and all of the components have unique boundaries. These details are not known to be true or not in $\mathcal{H}_n$ for $n \geq 3$, and answering these questions can shed some light on information related to such topics as Schrödinger cocycles. Several open questions about the geometry will be addressed in the chapter 5.

In $\mathcal{H}_2$, the boundaries of all of the connected components take on the form are explicitly detailed in the previous section. Notice that from 3.1, condition (a) only occurs on the boundary of the principal connected component in $\mathcal{H}_n$ via Proposition 12, and heteronclinic connection (condition (c)) does not occur in $\mathcal{H}_2$ at all.

The next topic to cover is what a heteroclinic connection is and under what conditions can it occur. Simply put, it occurs on the boundary of non-principal components in $\mathcal{H}_n$ for $n \geq 3$. We will consider a connected component $\mathcal{C} \in \mathcal{H}_3$ containing (A, B, C) with the following minimal cone $\mathcal{C}'$ and reverse minimal cone $\mathcal{C}'^{-1}$.

$$\mathcal{C}' = U_1 \cup U_2$$

$$\mathcal{C}'^{-1} = V_1 \cup V_2$$

$$u_A, u_C \in U_1 \qquad u_B \in U_2$$

$$s_A \in V_1 \qquad s_B, s_C \in V_2$$

We can use the example that is depicted in the diagram below an the chain (U_1, V_2, U_2, V_1) holds. The boundaries of the intervals can be calculated. If $U_1 = [p, q]$ for points p and q, then $B \cdot [p, q] \subset U_2$. Due to the location of the eigenvectors of B, we get the chain (u_B, Bp, Bq) This allows us to solve for $U_2 = [u_B, Bq]$. Similarly, if $V_2 = [r, t]$, then $A^{-1}[r, t] \subset V_1$ with the chain $(A^{-1}r, A^{-1}t, s_A)$. We can deduce that $V_1 = [A^{-1}r, s_A]$.

Note that we do not specifically know the ordering of u_C and u_A in the interval U_1, as well as the ordering of s_C and s_B in V_2, so this means there can be different possibilities for all of the bounds. We can get the two chains of elements that are in the interval U_1.

$$(p, Cu_B, CBq, u_C, q)$$

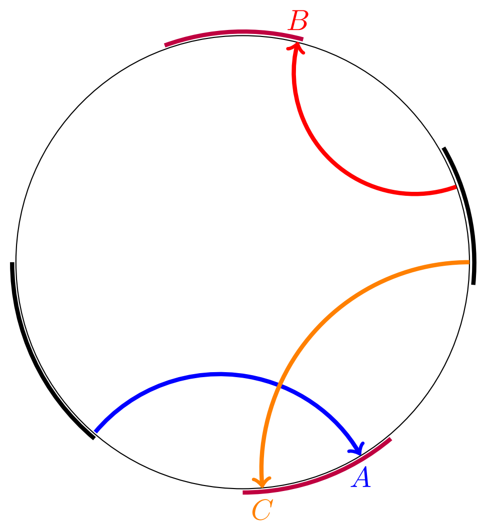

Figure 3.7: Projective space $\mathbb{RP}^1$ of the set $\{A, B, C\}$. The cone of $\{A, B, C\}$ and $\{A^{-1}, B^{-1}, C^{-1}\}$ are depicted.

$$(p, u_A, Au_B, ABq, q)$$

The possible definitions for U_1 are listed below.

$$U_1 = \begin{cases} [C \cdot u_B, u_{AB}] & or \\ [u_A, u_C] \\ [C \cdot u_B, u_C] \\ [u_A, u_{AB}] \end{cases}$$

If the point $q = u_A B$, then the bound of U_2 is $u_B A$. If the point $q = u_C$, then the bound of U_2 is $B \cdot u_C$. Via similar calculations, we can deduced the bounds of V_2, with which the bounds of V_1 can be calculated.

$$V_2 = \begin{cases} [s_C, s_B] & or \\ [s_C, C^{-1} \cdot s_A] \\ [s_{AB}, s_B] \\ [s_{AB}, C^{-1} \cdot s_A] \end{cases}$$

This data can be summarized in the diagram below.

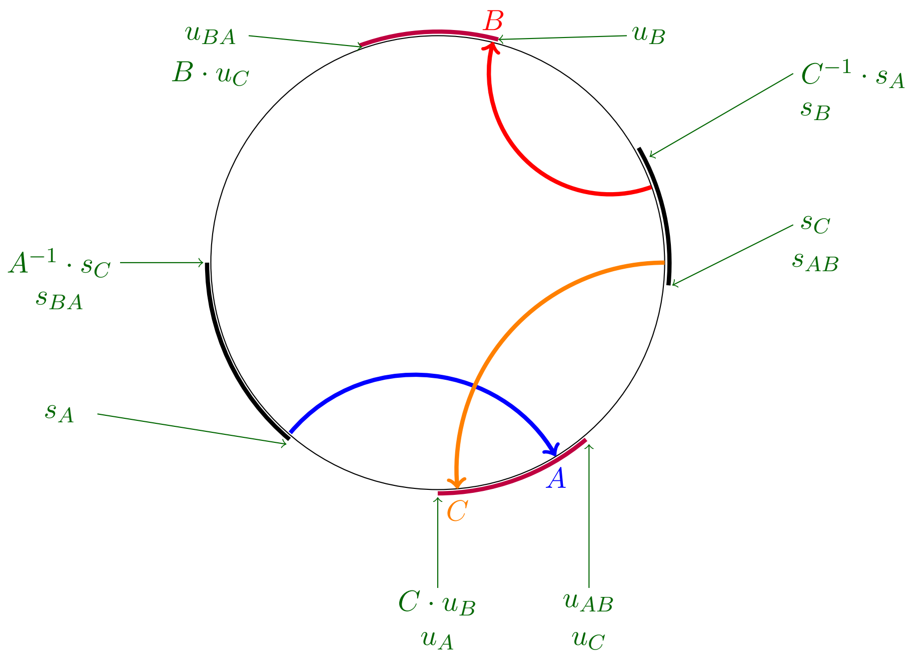

Figure 3.8: Projective space $\mathbb{RP}^1$ of the set $\{A, B, C\}$. The cone of $\{A, B, C\}$ and $\{A^{-1}, B^{-1}, C^{-1}\}$ are depicted.

Using Theorem 3.1, we can recognize the possible boundaries. Because condition (a) cannot be a possible boundary, and via Proposition 9 and Proposition 13, we can see that the boundary conditions are lsited below.

(a) $s_A = C \cdot u_B$

(b) $s_A = u_A$ ie A becomes parabolic

(c) $u_{AB} = s_C$

(d) $u_{AB} = s_{AB}$ ie AB and BA become parabolic

(e) $u_C = s_C$ ie C becomes parabolic

(f) $u_C = s_{AB}$

(g) $s_B = u_B$ ie B becomes parabolic

If we construct a path, fixing A and B, then the path can only meet conditions (a), (c), (e), or (f) are possible. Part of what makes a heteroclinic connection interesting is what the geometry of $\mathcal{H}_n$ in a neighborhood of a heteroclinic connection.

Proposition 15 (Proposition 4.18 from ABY)**.** *There exists path in $SL(2,\mathbb{R})^3$ such that the following points hold.*

- *For $t < 0$, we get $(A, B, C(t)) \in \mathcal{H}$*
- *At $t = 0$, we get $(A, B, C(0)) \in \partial\mathcal{H}$ and $C(0) \cdot u_B = s_A$.*
- *There exists sequence $t_i \searrow 0$ such that $(A, B, C(t_i)) \notin \mathcal{H}$.*
- *There exists sequence $s_i \searrow 0$ such that $(A, B, C(t_i)) \in \mathcal{H}$.*

Note that in the above example, $C \cdot u_B \in \mathcal{S}$ and is not an isolated point in $\mathcal{S}$. The elements of the set $\{u_{CB^n}\}$ are also in the skeleton, and we get the limit

$$u_{CB^n} \xrightarrow[n\to\infty]{} C \cdot u_B.$$

We have from Proposition 6 that $C \cdot u_B$ is not an isolated point, And as C is perturbed, such that scenario (a) occurs from the list above, then on can create a shuffling (of sorts) of the skeleton and the inverse skeleton (specifically intervals V_1 and U_1) such that there can be an infinite different number of orderings and minimal numbers. This can occur at the sequence of values in $\{t_i\}$ mentioned in Proposition 15.

Chapter 4

Results

4.1 Anderson-Bernoulli Model with a Period of 2

The results will be broken into different section involving the different results. The first result will center around the periodic Anderson-Bernoulli model with period 2. This is Theorem 2, stating that there is an explicit bound on the number of intervals in the spectrum of the periodic Anderson-Bernoulli model if the period is 2, which is proved in [19]. The second section addresses the periodic Anderson-Bernoulli model with period 4. This provides the information for Theorem 1, stating that there is not a bound on the number of intervals in the spectrum of the model is unbounded in general [11].

4.1.1 Notation and Basic Details

In this section, the periodic Anderson-Bernoulli model with a background potential of period 2 will be address. The potential will be denoted as

$$P(j) = v \cdot x_j + \begin{cases} a & j \equiv 0 \mod n \\ b & j \equiv 1 \mod n \end{cases}$$

where $\{x_j\}$ is a sequence of $0's$ and $1's$ and $v > 0$. Via Lemma 2.3, we have that the Schrödinger operator is uniformly hyperbolic if and only if the following set is.

$$\left\{ \begin{bmatrix} E-v-a & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-v-b & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} E-v-a & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-b & -1 \\ 1 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} E-a & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-v-b & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} E-a & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-b & -1 \\ 1 & 0 \end{bmatrix} \Bigg\}_E$$

Without loss of generality, we can assume $a < b$ and shift the variable E by a. This gives us

$$\Bigg\{ \begin{bmatrix} E-v & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-v-c & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} E-v & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-c & -1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-v-c & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-c & -1 \\ 1 & 0 \end{bmatrix} \Bigg\}_E .$$

where $c = b - a > 0$. Multiplying these matrices together, we get the following set.

$$\Bigg\{ A_1 = \begin{bmatrix} (E-v)(E-v-c)-1 & v-E \\ E-v-c & -1 \end{bmatrix}, A_2 = \begin{bmatrix} (E-v)(E-c)-1 & v-E \\ E-c & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} E(E-v-c)-1 & -E \\ E-v-c & -1 \end{bmatrix}, A_4 = \begin{bmatrix} E(E-c)-1 & -E \\ E-c & -1 \end{bmatrix} \Bigg\}_E .$$ [1]

Because these are $SL(2,\mathbb{R})$ matrices, their eigenvectors and eigenvalues can be pretty easily evaluated. Here, the eigenvectors are in $\mathbb{RP}^1$, defined in Section 2.2.2. We can define the mapping $\mathbb{RP}^1 \to \mathbb{R} \cup \infty$ where $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{x}{y}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \infty$.

Proposition 16. *Given an $SL(2,\mathbb{R})$ matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, then the spectral radius $\rho_A = \frac{|tr(a)|+\sqrt{tr(A)^2-4}}{2}$. If $tr(A) > 2$ and $A_{22} \neq 0$, then*

$$u_A = \frac{-A_{22} + \rho_A}{A_{21}} \text{ and } s_A = \frac{-A_{22} + \rho_A^{-1}}{A_{21}}.$$

If $tr(A) < -2$ and $A_{22} \neq 0$, then

$$u_A = \frac{-A_{22} - \rho_A}{A_{21}} \text{ and } \frac{-A_{22} - \rho_A^{-1}}{A_{21}}.$$

Using Proposition 16, we can graph out the eigenvectors in $\mathbb{RP}^1$ pretty easily.

[1] For the rest of this section, these matrices will be referred to as $\{A_i\}$ with the appropriate index.

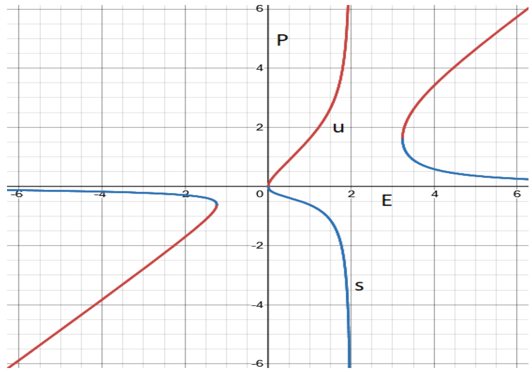

Figure 4.1: Eigenvectors of the first matrix in the set. Here $c = 2$ and $v = 0$. For the first matrix in the set, changing v shifts the graph horizontally.

Note that if $v = 0$, then all four matrices would be equal to each other, and these would be the eigenvalues for all four matrices.

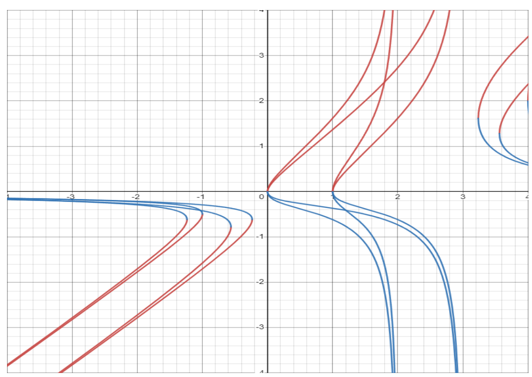

Figure 4.2: Eigenvectors of all four matrices. Here $c = 2$ and $v = 1$.

One can see that as v increase, the curves will change continuously, and this is true for $0 \leq v < c$. It just so happens that for every possible period c and v, if all four matrices are hyperbolic, then their can be a principal cone that can be defined.

4.1.2 Main Propositions

This section will provide the main Lemmas necessary for the proving Theorem 2.

Lemma 4.1. *Given distinct transfer matrices A, B, C, if the matrices AB and AC are hyperbolic, then they cannot share an eigenvector. Similarly, BA and CA cannot share an eigenvector if they are hyperbolic.*

Proof.

- To prove the first statement, given matrices AB and AC, assume they share an eigenvector $\vec{v}$. Then
$$AB\vec{v} = \xi_1 AC\vec{v} = \xi_2\vec{v}$$
for constants ξ_1 and ξ_2. Therefore
$$B\vec{v} = \xi_1 C\vec{v} \Rightarrow (B - \xi_1 C)\vec{v} = 0.$$
Because B and C are transfer matrices we can directly compute the determinant of linear combinations as $det(B-\xi_1 C) = (1-\xi_1)^2$. For this to be zero, $\xi_1 = 1$. This makes $(B-C)\vec{v} = \begin{bmatrix} B_{11} - C_{11} & 0 \\ 0 & 0 \end{bmatrix} \cdot \vec{v} = 0$ in which case $\vec{v} \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For this to be an eigenvalue of AB, we get the equation
$$AB \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \xi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
which gives the equations $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\xi_2 = -1$. This makes AB not hyperbolic.

- To prove the second statement, assume matrices BA and CA which share an eigenvector $\vec{v}$.
$$BA\vec{v} = \xi_1 CA\vec{v} = \xi_2\vec{v}$$
Therefore,
$$B(A\vec{v}) = \xi_1 C(A\vec{v}) \Rightarrow (B - \xi_1 C)(A\vec{v}) = 0.$$
This gives the equation $A\vec{v} \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which implies the equation $\xi_2\vec{v} = B\,(A\vec{v}) \propto B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \propto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Given the statement $A\vec{v} \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then the equations $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\xi_2 = \xi_1 = 1$ are true, making the product parabolic and proves the lemma.

□

Corollary 2. *Assuming all the matrices in $\{A_i\}$ are hyperbolic, if two elements share an eigenvector, then the pair is A_1 and A_4 or the pair A_2 and A_3.*

Lemma 4.2. *The set of matrices*

$$\left\{ AC, AD, BC, BD : \begin{array}{c} A, B, C, D \text{ are transfer matrices and} \\ A_{11} \geq B_{11} \text{ and } C_{11} \geq D_{11} \end{array} \right\}$$

is uniformly hyperbolic if and only if the matrices AC, AD, BC, & BD are individually hyperbolic.

If the set is uniformly hyperbolic, all the matrices are hyperbolic. In the regions where all the matrices are all hyperbolic, they turn out to form a principal cone. The previous lemma and corollary were used to prove this lemma in [19].

Lemma 4.3. *Over E, there are at most 5 intervals over which every matrix of the set $\{A_i\}$ is hyperbolic.*

The paper [19] actually addresses a slightly more complicated model. Rather than looking at the Anderson model with a background potential of period 2, the paper addresses a model with a potential defined by periodic Bernoulli distributions, ie the variables in $\{P(n)\}$ are not iid, but $\{P(2n)\}$ are iid and $\{P(2n+1)\}$ are iid. The material in this thesis, however, addresses the slightly simpler model, such that the variables are iid, but there is a periodic shift.

The set $\{E : A_i \text{ is hyperbolic for all } i\}$ is the intersection of open intervals defined by

$$\bigcap_{i=1}^{4} \{E : |tr(A_i)| > 2\}, \tag{4.1}$$

and this lemma can be proven by explicitly calculating the intervals over E where the statement $|tr(A_i)| > 2$ is true for all i. These intervals depend on the 3 possible orderings of v and c. These orderings are listed below, and within the list are sublists, defining the intervals that make up the set $\{E : A_i \text{ is hyperbolic for all } i\}$. Here, we are assuming $c, v \geq 0$, as mentioned in section 4.1.1.

1. If we assume $v \leq c$, then the list of intervals making up $\{E : A_i \text{ is hyperbolic for all } i\}$ are below.

   a) $E < 0$ and $E(E-c) > 4$

   b) $0 < E < v$ and $(E-v)(E-c) > 4$

   c) $v < E < c$

   d) $c < E < c+v$ and $(E-v)(E-c) > 4$

e) $E > c + v$ and $(E - v)(E - v - c) > 4$

2. If $c < v$, then $\{E : A_i$ is hyperbolic for all $i\}$ is the union of intervals listed below.

a) $E < 0$ and $E(E - c) > 4$

b) $0 < E < c$ and $(E - v)(E - c) > 4$

c) $c < E < v$ and $E(E - c) > 4$ and $(E - v)(E - v - c) > 4$

d) $v < E < c + v$ and $(E - v)(E - c) > 4$

e) $E > c + v$ and $(E - v)(E - v - c) > 4$

This leads us to the conclusion of Theorem 2.

4.1.3 The Explicit Gaps in the Spectrum

We can conclude that for all of the intervals listed as scenarios a to e, the 4 matrices $A_1, A_2, A_3, \& A_4$ are hyperbolic and the set is uniformly hyperbolic. The spectrum therefore consists of at most 4 intervals, which are the complement of the intervals in E over which the set of matrices is uniformly hyperbolic. Below the intervals are explicitly calculated.

Proposition 17. *The complement of the spectrum of H_V from theorem 1 can be explicitly calculated, depending on the ordering of v, and c, where $v, c \geq 0$.*

1. *If we assume $v \leq c$, then*

$$\mathbb{R}\backslash\sigma(H_V) = \{E : tr(A_i) > 2 \;\&\; E < 0\} \cup \{E : tr(A_4) < -2 \;\&\; tr(A_2) > 2\} \cup$$

$$\{E : tr(A_2), tr(A_4) < -2\} \cup \{E : tr(A_4), tr(A_3) > 2\} \cup \{E : tr(A_i) > 2 \;\&\; E > 0\}$$

$$= \Big(-\infty, \frac{1}{2}(c - \sqrt{c^2 + 16}) \Big) \cup \Big(0, \frac{1}{2}(v + c - \sqrt{(v + c)^2 - 4(c \cdot v - 4)}) \Big) \cup (v, c) \cup$$

$$\Big(\frac{1}{2}(v + c + \sqrt{(v + c)^2 - 4(cv - 4)}), c + v \Big) \cup$$

$$\Big(\frac{1}{2}(v + v + c + \sqrt{(v + v + c)^2 - 4(v(v + c) - 4)}), \infty \Big)$$

2. *If we assume $c < v$, then then gaps in the spectrum consist of the intervals*

$$\mathbb{R}\backslash\sigma(H_V) = \Big(-\infty, \frac{1}{2}(c - \sqrt{c^2 + 16}) \Big) \cup \Big(0, \frac{1}{2}(v + c - \sqrt{(v + c)^2 - 4(cv - 4)}) \Big) \cup$$

$$\Big(\frac{1}{2}\big(c+\sqrt{c^2+16}\big),\frac{1}{2}\big(2v+c-\sqrt{(2v+c)^2-4v(v+c)+16}\big)\Big)\cup$$

$$\Big(\frac{1}{2}\big(v+c+\sqrt{(v+c)^2-4(cv-4)}\big),c+v\Big)\cup$$

$$\Big(\frac{1}{2}\big(2v+c+\sqrt{(2v+c)^2-4(v(v+c)-4)}\big),\infty\Big)$$

Note that some of these intervals can overlap, so the actual spectrum can be less than 4 intervals. It is possible to be only 1 interval.

We can conclude that $H_\nu - E$ has a bounded inverse if and only if E is not in one of these, and the spectrum consists of the complement of these 5 intervals. This follows from lemma 2.3 and 2.1 by direct computation.

4.2 Anderson-Bernoulli Model with a Period of 4

In this section, we will address the periodic Anderson-Bernoulli model with a background potential of period 4, specifically this is Theorem 1. As described in Lemma 2.3, there is a corresponding set of matrices that is uniformly hyperbolic if and only if the the corresponding Schrödinger cocycle is uniformly hyperbolic. We can define $\mathbb{R} \to SL(2, \mathbb{R})^n$ as the corresponding set of matrices dependent on the energy E. Based on Lemma 4.1.2, we can see that if the period is 2, then the corresponding path $\mathcal{M}(E)$ only passes through principal components of the hyperbolic locus. As pointed out in section 4.1 the $E \mapsto \mathcal{M}(E) \in SL(2, \mathbb{R})^4$. We can further bound the number of times it will be pass through the boundary.

This property is not true if the period is 4, and one can define the parameters of the potential such that the corresponding path $\mathbb{R} \to SL(2, \mathbb{R})^n$ has the same property as the path in Proposition 15. Because it is period 4, the corresponding set has $2^4 = 16$ elements, ie $E \mapsto \mathcal{M} \in SL(2, \mathbb{R})^{16}$. To prove this is possible for such a specific set of matrices, a more explicit Theorem about the geometry of $\mathcal{H}_{16}$ and paths in $SL(2, \mathbb{R})^{16}$ has to be defined. This will be the next section in this chapter.

4.2.1 Geometric Theorem

Theorem 4.4. *Assume $I = (E_0 - \delta, E_0)$ is an open interval and $A, B, C : I \to \mathrm{SL}(2, \mathbb{R})$ are analytic functions of $E \in I$ with continuous extension to $\overline{I}$ such that A_E, B_E, and C_E are hyperbolic for every $E \in I$ and the following conditions hold:*

(i) *$\{A_E, B_E\}$ is uniformly hyperbolic for every $E \in I$ and the following sequence is a chain in $\mathbb{RP}^1$:*

$$(C_E^2 u_{B_E}, s_{A_E}, u_{A_E}, u_{C_E}, s_{C_E}, s_{B_E}, u_{B_E}).$$

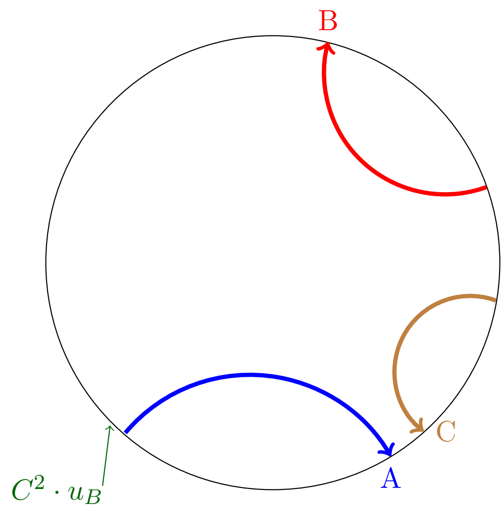

Figure 4.3: $\mathbb{RP}^1$ with $\{A, B, C\}$ and $C^2 \cdot u_B$ depicted for $E \in I$.

(ii) *At* $E = E_0$, *we have* $C_E^2 u_{B_E} = s_{A_E}$.

(iii) *For all* $E \in I$, *one has* $\frac{d}{dE}(C_E^2 u_{B_E}) - \frac{d}{dE}(s_{A_E}) > 0$.

(iv) *There exist* $r, p \in \mathbb{RP}^1$ *such that for* $E = E_0$

$$(u_{B_E}, B_E r, B_E A_E C_E p, r, p, C_E u_{B_E}, u_{C_E}, A_E C_E p, s_{C_E}, s_{B_E})$$

is a chain, $C_E r, C_E p \notin \{u_A, s_A\}$, *and the cross-ratios*

$$\alpha = [u_{A_E}, C_E r, C_E p, s_{A_E}] \quad \beta = [u_{B_E}, r, B_E A_E C_E p, s_{B_E}], \tag{4.2}$$

satisfy the inequality $(\alpha - 1)(\beta - 1) > 1$ *at* $E = E_0$.

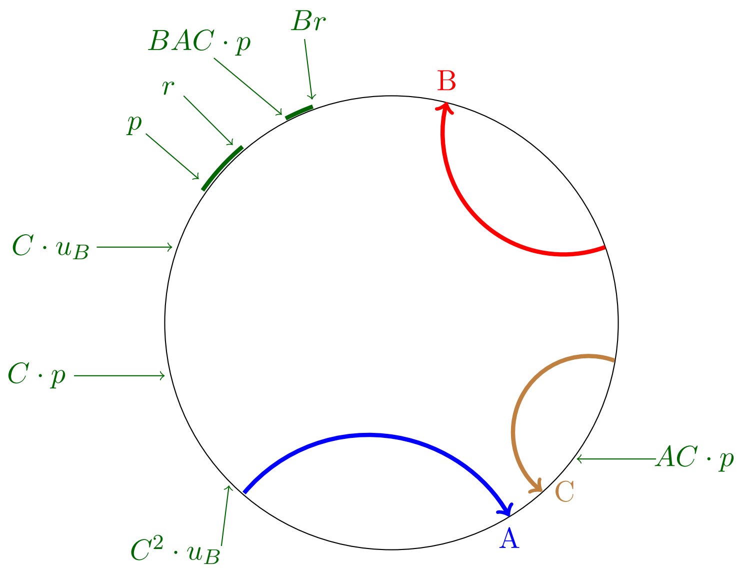

Figure 4.4: $\mathbb{RP}^1$ with points depicted for $E \in (E_0 - \delta, E_0)$.

(v) *At E_0, we have*

$$\frac{\log \operatorname{spr} A_{E_0}}{\log \operatorname{spr} B_{E_0}} \notin \mathbb{Q} \tag{4.3}$$

where spr *denotes the spectral radius.*

Then we have the following:

(a) *There exists a sequence of positive numbers $s_n \to 0$ such that for every n, $\{A_{E_0 - s_n}, B_{E_0 - s_n}, C_{E_0 - s_n}\}$ is not uniformly hyperbolic.*

(b) *There exists a sequence of positive numbers $t_n \to 0$ such that for every n, $\{A_{E_0 - t_n}, B_{E_0 - t_n}, C_{E_0 - t_n}\}$ is uniformly hyperbolic.*

4.2.2 Propositions

This subsection provides a few propositions useful for the proof of Theorem 4.4.

Proposition 4.5. *Suppose A and B are hyperbolic and that (u_A, u_B, s_A) and (u_A, u_B, s_B) are chains. Then, for all $n \in \mathbb{N}$,*

$$(u_A, u_{A^{n+1}B}, u_{A^nB}, \cdots, u_{AB}, u_{BA}, \cdots, u_{B^nA}, u_{B^{n+1}A}, u_B)$$

is a chain. Moreover,

$$\lim_{n\to\infty} u_{B^nA} = u_B$$

Proof. The assumptions imply that the set $\{A, B\}$ is uniformly hyperbolic, since there is a principal multicone, and thus every $C \in SG(\{A, B\})$ is hyperbolic. Moreover, we see that $[u_A, u_B]$ contains u_C and (u_B, u_A) contains s_C for every $C \in SG(\{A, B\})$.
Begin by noticing that

$$Bu_{AB} = BABu_{AB} = BA(Bu_{AB}),$$

which implies that Bu_{AB} is one of the eigendirections of BA. However, it cannot be the stable direction due to uniform hyperbolicity of $\{A, B\}$; indeed $Bu_{AB} = s_{BA}$ would be a heteroclinic connection, so we deduce $Bu_{AB} = u_{BA}$. Thus,

$$(u_A, u_{AB}, u_{BA}, u_B)$$

is a chain. Inductively, we see that

$$(u_A, u_{A^{n+1}B}, u_{A^nB}, u_{AB}, u_{BA}, u_{B^nA}, u_{B^{n+1}A}, u_B)$$

is a chain for every $n \in \mathbb{N}$, which proves the first half of the proposition. The second half follows by noting that $(B^nAu_A, u_{B^nA}, B^nAu_B, u_B)$ is a chain and that $B^nu_A \to u_B$ as $n \to \infty$. $\square$

Next, we need the following proposition, which clarifies how a cross ratio in which two of the directions are the invariant directions of a hyperbolic matrix, computes suitable asymptotic lengths of intervals. To formulate this, let us denote by dist the standard metric on $\mathbb{RP}^1$, that is

$$\operatorname{dist}(v, w) = \min\{|\widetilde{v} - \widetilde{w}| : p_{\mathbb{RP}^1}(\widetilde{v}) = v,\ p_{\mathbb{RP}^1}(\widetilde{w}) = w\}. \tag{4.4}$$

Equivalently, $\operatorname{dist}(v, w) = \angle(v, w)$, the smallest nonnegative angle between $v, w \in \mathbb{RP}^1$. As usual, if B We denote by Leb the standard Lebesgue measure on $\mathbb{RP}^1$, which can be obtained by pushing forward Lebesgue measure on $[0, \pi)$ with the map $p_{\mathbb{RP}^1}$.

Proposition 4.6. *Assume $A \in \mathrm{SL}(2, \mathbb{R})$ is hyperbolic, $u = u_A$, $s = s_A$, x, and y are distinct points* *of* $\mathbb{RP}^1$ *such that* (u, x, y, s) *is a chain. Then*

$$|[u, x, y, s] - 1| = \lim_{n \to \infty} \frac{\mathrm{Leb}(A^{-n} I)}{\mathrm{dist}(s, A^{-n} I)}, \tag{4.5}$$

where $I = [x, y]$.

Proof. By assumption, we have $s, u \notin I$. Since the cross-ratio is invariant under the Möbius action of A, we get $[u, x, y, s] = [u, A^{-n}x, A^{-n}y, s]$ for all n, which in turn yields

$$[u, x, y, s] - 1 = \frac{(\Phi u - \Phi s)(\Phi A^{-n} y - \Phi A^{-n} x)}{(\Phi A^{-n} x - \Phi u)(\Phi A^{-n} y - \Phi s)} \tag{4.6}$$

for every n. Consider first the case $\Phi s \neq \infty$. Since $x \neq u$, we have $A^{-n}x \to s$ as $n \to \infty$, and therefore one has

$$\lim_{n \to \infty} \left| \frac{\Phi u - \Phi s}{\Phi A^{-n} x - \Phi u} \right| = 1,$$

so we focus on the second factors in the numerator and denominator. For large enough n, one has

$$\begin{aligned} |\Phi A^{-n} y - \Phi A^{-n} x| &= |\Phi'(t_1)||A^{-n} I|, \\ |\Phi A^{-n} y - \Phi s| &= |\Phi'(t_2)| \, \mathrm{dist}(s, A^{-n} I) \end{aligned}$$

for some $t_1 \in (A^{-n}x, A^{-n}y)$ and $t_2 \in (A^{-n}y, s)$ by the mean value theorem. Thus, sending $n \to \infty$, we see that $t_j \to s$ and therefore the conclusion follows in this case.

If $\Phi s = \infty$, we can replace A by $A_1 = JAJ^*$ where J is rotation by $\pi/2$ and use

$$[u, x, y, s] = [Ju, Jx, Jy, Js] = [u_{A_1}, Jx, Jy, s_{A_1}]$$

together with the observation

$$\mathrm{Leb}(A_1^{-n}[Jx, Jy]) = \mathrm{Leb}(JA^{-n}[x, y]) = \mathrm{Leb}(A^{-n} I)$$

to conclude. $\square$

Using this, we can deal with the cross ratios from Theorem 4.4.

Proposition 4.7. *Assume* I *is an open interval,* $A : I \to \mathrm{SL}(2, \mathbb{R})$*, is smooth and* $A(t)$ *is hyperbolic* *for every $t \in I$. If $p : I \to \mathbb{RP}^1$ is smooth and $p(t) \neq s_{A(t)}$ for all t, then*

$$\frac{d}{dt} A(t)^n p(t) \xrightarrow[n\to\infty]{} \frac{d}{dt} u_{A(t)}, \tag{4.7}$$

uniformly on compact subsets of I.

Proof. Let $u(t) = u_{A(t)}$ and $s(t) = s_{A(t)}$ denote the unstable and stable directions of $A(t)$, which are associated with eigenvalues $\lambda(t)$ and $1/\lambda(t)$ respectively. Let $\vec{u}$, $\vec{s}$, and $\vec{p}$ be smooth choices of unit vectors in u, s, and p respectively, write

$$\vec{p}(t) = c_+(t)\vec{u}(t) + c_-(t)\vec{s}(t)$$

for C^1 coefficients c_+, c_- and note that the assumptions imply $c_+ \neq 0$. Using

$$A^n \vec{p} = c_+ \lambda^n \vec{u} + c_- \lambda^{-1} \vec{s},$$

and $c_+ \neq 0$, the desired convergence follows. □

Corollary 3. *Assume I is an open interval, $A : I \to \mathrm{SL}(2, \mathbb{R})$, is smooth and $A(t)$ is hyperbolic for every $t \in I$. If $p : I \to \mathbb{RP}^1$ is smooth and $p(t) \neq u_{A(t)}$ for all t, then*

$$\frac{d}{dt} A(t)^{-n} p(t) \xrightarrow[n\to\infty]{} \frac{d}{dt} s_{A(t)}, \tag{4.8}$$

uniformly on compact subsets of I.

Proof. This follows by applying Proposition 4.7 to the inverse. □

Corollary 4. *Assume I is an open interval, $A, B : I \to \mathrm{SL}(2, \mathbb{R})$ are smooth and $A(t)$ is hyperbolic for every $t \in I$. If $p : I \to \mathbb{RP}^1$ is smooth and $p(t) \neq s_{A(t)}$ for all t, then*

$$\frac{d}{dt} B(t) A(t)^n p(t) \xrightarrow[n\to\infty]{} \frac{d}{dt} B(t) u_{A(t)}, \tag{4.9}$$

uniformly on compact subsets of I.

Proof. This follows from Proposition 4.5 and Corollary 3. □

4.2.3 Proof of Theorem 4.4

This subsection will provide the proof for Theorem 4.4.

Proof Theorem 4.4. (a). By assumption (i), $\{A, B\}$ does not enjoy a principal multicone. Consequently, there exists matrix $D \in SG(\{A, B\})$ such that the eigenvectors satisfy the following chain or its reverse.

$$(s_B, u_B, u_D, s_D, s_A, u_A).$$

This comes from the proof of [1, Theorem 3.1]. As defined in the proof, $\mathcal{H}_{id}$ is the union of the free components of the hyperbolic locus in $SL(2, \mathbb{R})^2$ such that for any $(A_0, B_0) \in H_{id}$, $\{A_0, B_0\}$ is uniformly hyperbolic and either the following chain or its reverse holds

$$(u_{A_0}, u_{A_0B_0}, s_{A_0B_0}, s_{B_0}, u_{B_0}, u_{B_0A_0}, s_{B_0A_0}, s_{A_0}).$$

For all connected components $\mathcal{H}$, besides the principal components, there exists some F in the monoid generated by F_+, F_- as defined in [1] such that $F(\mathcal{H})$ is a free component, that is, $F(\{A, B\}) = \{A_0, B_0\} \in \mathcal{H}_{id}$ and $A_0, B_0, A_0B_0 \in SG(\{A, B\})$. For any A, B and $D \in SG(\{A, B\})$ if we define $F_{\pm}^{-1}(\{A, B\}) = \{F_{\pm}^{-1}A, F_{\pm}^{-1}B\}$, if the following chain or its reverse is satisfied

$$(s_B, u_B, u_D, s_D, s_A, u_A),$$

then there exists $D' \in SG(\{F_{\pm}^{-1}A, F_{\pm}^{-1}B\})$ such that the following chain or its reverse holds

$$(s_{F_{\pm}^{-1}B}, u_{F_{\pm}^{-1}B}, u_{D'}, s_{D'}, s_{F_{\pm}^{-1}A}, u_{F_{\pm}^{-1}A}).$$

The matrix D is chosen independently of E.
By Proposition 4.5,

$$\lim_{n\to\infty} s_{DA^n} = s_A$$

and

$$(s_D, s_{DA}, s_{DA^2}, \cdots, s_{DA^n}, s_A).$$

forms a chain for any n. By assumption (i), $(u_{B_E}, C_E^2 u_{B_E}, s_{A_E})$ is a chain for each $E \in (E_0 - \delta, E_0)$.

From assumptions (ii) and (iii), we get

$$\lim_{E \to E_0^-} C_E^2 u_B \nearrow s_A,$$

and so for sufficiently large n there exists s_n such that

$$C^2_{E_0-s_n} \cdot u_{B_{E_0-s_n}} = s_{D_{E_0-s_n} A^n_{E_0-s_n}}.$$

Therefore, for $E = E_0 - s_n$, a heteroclinic connection occurs, so $\{A_E, B_E, C_E\}$ is not uniformly hyperbolic, proving part (a) of Theorem 4.4. □

Proof Theorem 4.4. (b). We will prove this part by showing that for any $\varepsilon > 0$ there exists some $E \in (E_0 - \varepsilon, E_0)$ for which $\{A_E, B_E, C_E\}$ admits an invariant multicone (and hence is uniformly hyperbolic).

Fix $\varepsilon > 0$ small and $n, m \in \mathbb{N}$ large, and define intervals by

$$I_k = [B^k \cdot r, B^k AC \cdot p], \quad J_l = [A^{-l}Cr, A^{-l}Cp], \quad 0 < k < n, \text{ and } 0 < l \leq m.$$

The points $r, p \in \mathbb{RP}^1$ are defined in (iv). It will also be convenient to denote

$$I_n^* = [u_B, B^n AC \cdot p].$$

Note that $I_k = B^k \cdot [r, AC \cdot p]$ and $J_l = A^{-l}[Cr, Cp]$.

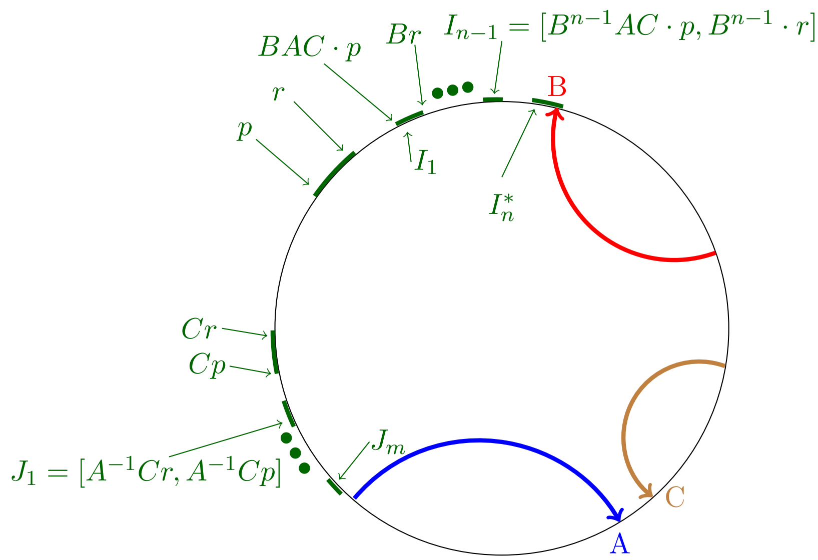

Figure 4.5: $\mathbb{RP}^1$ with intervals in the sets $\{I_k\}$ and $\{J_l\}$ depicted for $E \in (E_0 - \delta, E_0)$.

The constants n and m need to be defined such that for all I_k and J_l will be part of of the multicone for $\{A_E, B_E, C_E\}$ for some $E \in (E_0 - \varepsilon, E_0)$. Define n to be the maximal value such that for all $k \leq n$, $(s_A, C^2B^k \cdot r, u_A)$. Define m such that and $|C^2 I_n^*| < |J_m|$. By assumptions (ii) and (iii), $(u_B, C^2 u_B, s_A)$, so we can deduce $(u_A, C^2 B^n AC \cdot p, s_B)$ and the arrangement of the intervals I_k as depicted in 4.5.

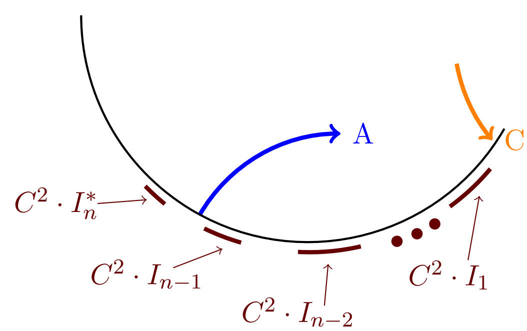

Figure 4.6: Depicting the ordering of the intervals $C^2 \cdot I_k$ and $C^2 \cdot I_n^*$

Also define the sets

$$X = [r, p], \quad X^* = [Cr, Cp],$$

$$Y = (\bigcup_k C \cdot I_k), \quad Y^* = C \cdot I_n^*,$$

$$\& \quad V = [\min\{C^2 B^{n-1} \cdot r, u_A\}, AC \cdot p].$$

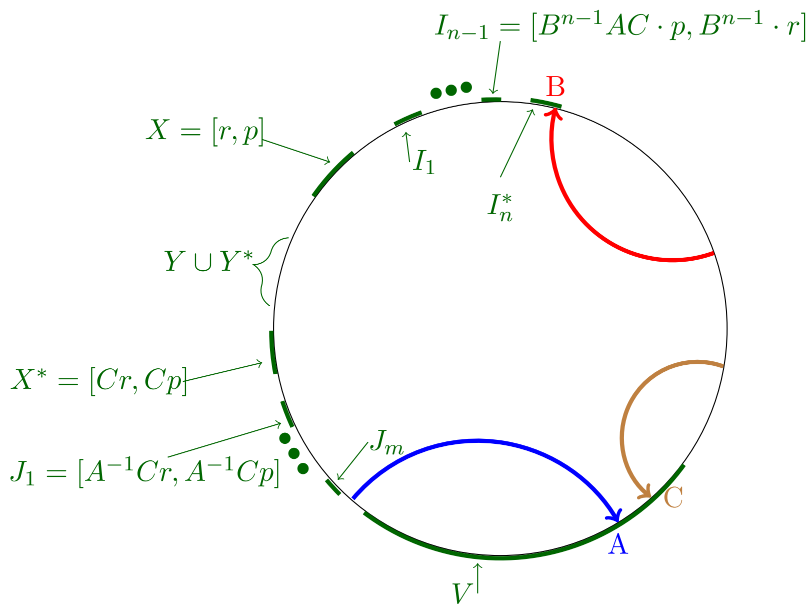

Figure 4.7: $\mathbb{RP}^1$ with intervals X, X^*, Y, Y^*, and V.

We can now define some of the mappings of the intervals defined above.

$$A \cdot \left[V \cup X \cup X^* \cup Y \cup Y^* \cup \left(\bigcup_{k=1}^{n-1} I_k \right) \cup [r,p] \cup I_n^* \right] \subseteq V \quad \& \quad A \cdot J_l = J_{l-1} \text{ for } l > 1$$

$$A(J_1) = X^*$$

The matrix A maps all the set $\bigcup J_l$ into $\bigcup J_l \cup X^*$ and all the other intervals into V.

$$B \cdot \left[V \cup X \cup Y \cup Y^* \cup \left(\bigcup_l J_l \right) \cup X^* \right] \subseteq I_1 \quad \& \quad B \cdot I_k = I_{k+1} \text{ for } 1 \leq k < n-1$$

$$B \cdot I_{n-1} \cup I_n^* \subseteq I_n^*$$

The matrix B maps the set $\bigcup I_k \cup I_n^*$ into itself, and all the other intervals into I_1.

$$C \cdot \left[V \cup \left(\bigcup_l J_l \right) \cup Y \right] \subseteq V \quad \& \quad C \cdot \left(\bigcup_k I_k \right) = Y$$

$$C \cdot X = X^* \quad C \cdot I_n^* = Y^*$$

This shows that the union of the intervals is closed under the actions of A and B. To prove it is closed under C, we need to show that for the given n, m, we have $C^2 \cdot I_n^* = C \cdot Y^* \subseteq J_m$ and

$C \cdot X^* \subseteq V$. If this is true, then the multicone can be defined by the intervals above and the set is uniformly hyperbolic.

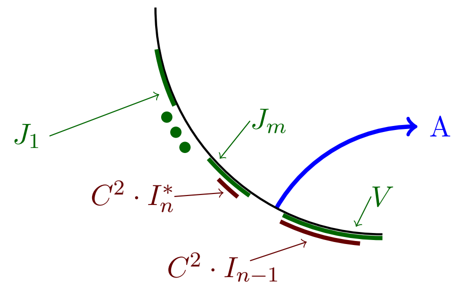

Figure 4.8: Mappings that will allow the intervals to be closed under C

To show $C^2 \cdot I_n^* \subseteq J_m$ and $C \cdot X^* \subseteq V$ for some $E \in (E_0 - \varepsilon, E_0)$, we will prove the inequalities and relation for E.

$$|C^2 \cdot I_n^*| < |J_m| \tag{4.10}$$

$$\operatorname{dist}(C^2 \cdot I_n^*, C^2 \cdot I_{n-1}) > \operatorname{dist}(s_a, J_m) \tag{4.11}$$

$$\text{The chain } (s_A, C^2 B^{n-1} AC \cdot p, u_C) \text{ holds.} \tag{4.12}$$

The first inequality 4.10 is true by the definition of m in terms on n. To prove the second inequality 4.11, we first use Proposition 4.6 and (iv), which gives

$$\operatorname{dist}(C^2 \cdot I_n^*, C^2 \cdot I_{n-1})/|C^2 \cdot I_n^*| = |[B^{n+1}AC \cdot p, B^{n-1} \cdot r]|/|C^2 \cdot I_n^*| \xrightarrow[n\to\infty]{} \beta - 1$$

$$\text{and } |J_m|/|[A^{-m}C \cdot p, s_A]| = |J_m|/\operatorname{dist}(J_m, s_A) \to \alpha - 1.$$

By assumption (iv), for sufficiently large n,

$$\begin{aligned} \operatorname{dist}(C^2 \cdot I_n^*, C^2 \cdot I_{n-1}) &\approx \operatorname{dist}(J_m, s_A) \frac{|C^2 \cdot I_n^*| \cdot (\alpha - 1)(\beta - 1)}{|J_m|} \\ &> \frac{|C^2 \cdot I_n^*| \cdot \operatorname{dist}(J_m, s_A)}{|J_m|}. \end{aligned} \tag{4.13}$$

The approximation is due to $(\alpha - 1)(\beta - 1)$ being a limit as $n, m \to \infty$ and the inequality is due to

$(\alpha-1)(\beta-1)$ being continuous under E and $(\alpha-1)(\beta-1) > 1$ at E_0, so for some $E \in (E_0-\varepsilon, E_0)$, $(\alpha-1)(\beta-1) > 1$ by Assumption (iv).

Last, we need to prove 4.12. Assuming ε is sufficiently small, for $E \in (E_0-\varepsilon, E_0)$, due to assumption (iii) and Corollary 4, as well as assumption (v), the set of distinct values in

$$\left\{\frac{|C_E^2 \cdot (I_n^*)_E|}{|(J_m)_E|}\right\}_{m,n,E}$$

is infinitely large and the values are bounded.

For some $0 < \delta$, there exists m, n, E such that

$$1 > \frac{|C_E^2 \cdot (I_n^*)_E|}{|(J_m)_E|} > \frac{1+\delta}{(\alpha-1)(\beta-1)}\Big|_{E=E_0}. \tag{4.14}$$

Because of continuity, we can define δ such that the inequality holds for all $E \in (E_0 - \varepsilon, E_0)$. This leads us to the conclusion that there exists m, n, E such that

$$\operatorname{dist}(C^2 \cdot I_n^*, C^2 \cdot I_{n-1}) > \operatorname{dist}(J_m, s_A) \tag{4.15}$$

by (4.13) and (4.14) and the definition of m. The inequality $|C^2 \cdot I_n^*| < |J_m|$ and (4.15) Gives us that $C^2 \cdot I_n^* = C \cdot Y^* \subseteq J_m$ and $(s_A, C^2 B^{n-1} AC \cdot p, u_C)$.

So for sufficiently large n, there exists m such that the set

$$V \bigcup (\cup_{l=1}^{m} J_l) \bigcup (\cup_{k=1}^{n-1} I_k) \bigcup I_n^* \bigcup X \bigcup X^* \bigcup Y \bigcup Y^*$$

is closed under the actions A, B, and C. Because the intervals defined are closed intervals, and a multicone consists of open intervals, the multicone can be defined as a union of open intervals by expanding I_n^* and Y^* and shrinking the other intervals by small enough $\delta_i > 0$ such that the mappings are still closed but now act on an open set that maps to its interior.

□

4.2.4 Main Result

The main result is that if the Anderson model is given a background periodic potential, then the spectrum can consist of infinitely many intervals. Specifically this is Theorem 1. Given a Bernoulli distribution (with any nontrivial probability p) that defines the sequence of iid variables $\{x_j\}_{\mathbb{Z}}$ such that $P(x_j = 1) = p$ and $P(x_j = 0) = 1 - p$, then define the potential $\{P(j)\}$ such that

$$P(j) = 9.99 \cdot x_j + \begin{cases} 0 & j \equiv 0 \mod 4 \\ 0.9 & j \equiv 1 \mod 4 \\ -9.7 & j \equiv 2 \mod 4 \\ 2 & j \equiv n-1 \mod 4 \end{cases} . \tag{4.16}$$

Using lemma 2.3, the cocycle is uniformly hyperbolic if and only if the set

$$\left\{ \begin{bmatrix} E-2-\xi_3 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E+9.7-\xi_2 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-0.9-\xi_1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-\xi_0 & -1 \\ 1 & 0 \end{bmatrix} \right\}_{\xi_i \in \{0, 9.99\}}$$

is uniformly hyperbolic. Given the following three matrices,

$$\begin{aligned} \Big\{ A &= \begin{bmatrix} E-2.0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-0.29 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-10.89 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} E-2.0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-0.29 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-0.9 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-9.99 & -1 \\ 1 & 0 \end{bmatrix} \\ C &= \begin{bmatrix} E-2.0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-0.29 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E-0.9 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix} \Big\} \end{aligned}$$

which is a subset of the set fo matrices and E_0 defined by the equation $C_{E_0}^2 \cdot u_{A_{E_0}} = s_{B_{E_0}}$, which can be approximated at $E_0 \approx -0.6005$, theorem 4.4 can be applied. Generally, a set is uniformly hyperbolic implies that any subset is uniformly hyperbolic. The converse is not necessarily true. For this specific set, one can show that there is an open ball around E_0, such that if E is in the ball, then the set of matrices is uniformly hyperbolic if and only if the set of matrices $\{A_E, B_E, C_E\}$ is uniformly hyperbolic. The proof for this detail is in [11], but essentially points out what the cone can be. What this means is that the they hyperbolic locus $\mathcal{H}_{16}$ has infinitely many connected components which can converge to a point. The set of matrices (paramaterized by E) correspond to a path in $SL(2, \mathbb{R})^{16}$ which pass through the components.

4.3 Dense Interior

This last section provides the conceptual explanation for the why the spectrum has a dense interior.[2] This is equivalent to the following Theorem.

Theorem 4.8. *If H is an Anderson model with a background potential of period n. The spectrum can be defined as*

$$\sigma_{as}(H) = \overline{\bigcup_{V \text{ is periodic}} \sigma(H_V)}, \tag{4.17}$$

where the union ranges over all periodic realizations of period divisible by n.

Conceptually, for any periodic realization (of period divisible by n), the period will occur sequentially m times (for any m), with probability 1. Given a period V of length $n \cdot k$ which occurs sequentially m times, there occurs elements in $\varphi \in l^2(\mathbb{Z})$ such that $H(\varphi)$ can be approximated by $H_V(\varphi)$. This points to the spectrum $\sigma(H_V)$ being a subset of $\sigma(H)$. A similar argument can be made going in the reverse direction such that for all $E \in \sigma(H)$, there exists V such that $E \in \sigma(H_V)$. Potential alternate proofs involving this rely on addressing unanswered questions of the geometry of $\mathcal{H}_n$ and relies on question (3) in Section 5.

[2]Special acknowledgement goes to Jake Fillman for providing the proof.

Chapter 5

Conjectures & Open Problems

5.1 Preliminaries

The main conjectures posed and researched have been posed by Avila, Bochi, and Yoccoz in [1] or directly relate to the periodic Anderson-Bernoulli operator. The first three questions are posed in that paper.

(1) Are the boundaries of the connected components of $\mathcal{H}_n$ disjoint?

(2) Is the union of the boundaries of the components equal to the boundary of $\mathcal{H}_n$?

(3) If $\gamma : [a, b] \to SL(2, \mathbb{R})^n$ is an analytic curve, does the set $\gamma^{-1}(\partial\mathcal{H}_n)$ necessarily have countably many components?

The paper [1] shows these to be true in $SL(2, \mathbb{R})^2$, but it is unknown for higher dimension. Details and insight will be provided in the section. I conjecture that this is true for all n. The solution to question 3 depends on question 2. These three questions can provide insight into the topology of the spectrum of a variety of Schrödinger operators. An alternate proof for the spectrum having a dense interior relies on answering questions 2 and 3. Additionally, we pose the following question in regards to the periodic Anderson-Bernoulli model.

(4) Can the periodic Anderson-Bernoulli model have an infinite number of gaps in the spectrum with a background potential of period 3?

Details, progress, and relevant information are provided throughout this section.

5.2 Periodic Anderson-Bernoulli Model

As we have shown in section 4, we have explicitly calculated the almost sure spectrum of the periodic Anderson-Bernoulli model if the period is 2 explicitly. On the other hand, if the period is 4 then there may be an infinite number of intervals in the almost sure spectrum. Results for period 3 are suspiciously absent.

Given the case for period 2 with period (α, β), the set

$$\mathcal{M}_E = \left\{ \begin{bmatrix} E - \lambda\xi_1 0 - \alpha & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} E - \lambda\xi_2 - \beta & -1 \\ 1 & 0 \end{bmatrix} : \alpha, \beta, \lambda \text{ are fixed} \right\}_{\xi_i \in \{0,1\}}$$

is uniformly hyperbolic if and only E is not in the almost sure spectrum. If we examine the corresponding path $P(E) \in SL(2, \mathbb{R})^4$, then $E \in \sigma_{AS}$ if and only if $P(E) \notin \mathcal{H}_4$. The calculations in section 4 show that for any α and β, $P(E)$ intersects only the principal component, and because the path is algebraic, it intersects the component in finitely many intervals. Explicitly, condition (b) from Theorem 3.1 occurs.

Given the case for period 4 with period $(\alpha, \beta, \gamma, \delta)$, there exists values $\alpha_0, \beta_0, \gamma_0$, and δ_0 such that the corresponding path $P(E) \in SL(2, \mathbb{R})^{16}$ can intersect the boundary $\partial\mathcal{H}_{16}$ at a point where a heteroclinic connection occurs. Assuming the answer to question (2) is true, then this condition is necessary for there to be an infinite number of gaps in the spectrum.

Considering the case where the period is 3, this condition still seems to be possible, but whether or not the respective path can pass through an infinite number of components can occur is still not certain.

Consider the following continuous mappings

$$\Psi : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})^3$$

$$\text{where } (A, B, C, \cdots, N) \mapsto (Q, R, S)$$

and

$$\Psi' : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})^2$$

$$\text{where } (A, B, C, \cdots, N) \mapsto (R, S).$$

where $Q, R, S \in SG(\{A, B, C, \cdots, N\})$. Note that for any path $P(E) \in SL(2, \mathbb{R})^n$, if $P(E_0)$ is on the boundary of a connected component, then there exists $P, Q, R \in SG$ such that $\Psi(P(E_0))$ is on the boundary of $\mathcal{H}_3$. If conditions (a) or (b) hold (or condition (c) where some stable eigenvector equals some unstable eigenvector) then there exists $R, S \in SG$ such that $\Psi'(P(E_0))$ is on the boundary of $\mathcal{H}_2$. Because we have a better understanding of the geometry of $\mathcal{H}_2$, this can help us better understand the geometry of $\mathcal{H}_n$ along any path.

It is worth noting that in $SL(2, \mathbb{R})^2$, all of the components of $\mathcal{H}_2$ are closed, disjoint, and disjoint from the closure of the union of all the other components. As pointed out in subsection 3.5, this is not true in $\mathcal{H}_3$. This give us the following Proposition.

Proposition 5.1. *For any path $P(E) \in SL(2, \mathbb{R})^n$, if conditions (a) or (b) from Theorem 3.1 hold or there exists matrices $M, N \in SG$ such that $u_M = s_N$ at $E = E_0$, then there exists small enough δ such that one of the conditions holds true.*

- $P(E)|_{E\in(E_0,E_0+\delta)} \notin \mathcal{H}_n$ *and* $P(E)|_{E\in(E_0-\delta,E_0)} \in \mathcal{H}_n$
- $P(E)|_{E\in(E_0-\delta,E_0)} \notin \mathcal{H}_n$ *and* $P(E)|_{E\in(E_0,E_0+\delta)} \in \mathcal{H}_n$
- $P(E)|_{(E_0-\delta,E_0+\delta)\setminus E_0}$ *is in a single connected component of* $\mathcal{H}_n$.
- $P(E)|_{(E_0-\delta,E_0+\delta)\setminus E_0} \notin \mathcal{H}_n$.

Proof. For any path $P(E) \in SL(2, \mathbb{R})^n$, if $P(E) \in \mathcal{H}_n$, then $\Psi'(P(E)) \in \mathcal{H}_2$ for all possible Ψ'. If any of those conditions hold, then there is Ψ' and E_0 such that $\Psi'(P(E_0)) \in \partial\mathcal{H}_2$. $\square$

It is worth noting that the example in subsection 3.5 of a path of finite length passing through an infinite number of components occurs in $SL(2, \mathbb{R})^3$. This is the lowest dimension where it is possible, and is the lowest dimension such that a heteroclinic connection occurs on a non-principal component. Working under the assumption that the answer to question (2) is true, then for a path of finite length to pass through an infinite number of components, then it must pass through a point where a heteroclinic connection occurs.

5.3 Geometry of Hyperbolic locus

Questions (1), (2), and (3) all center around the geometry of the hyperbolic locus. As made evident previously, the geometry of the hyperbolic locus can shed some light on the spectrum of various Schrödinger operators via Johnson's Theorem. One can construct Schrödinger operators which have a spectrum dependent on $\mathcal{H}_n$ for some specific n (such as the periodic Anderson-Bernoulli model). Further research into the geometry may shed light on a wider array of operators, however. An easy example is the Thue-Morse operator. With details of this operator in [5] and [16], on can see that the spectrum is actually a Cantor set. The Thue-Morse sequence is the set $\{V_n\}_{n\geq 0}$ such that V_n is the sum of the digits of n in binary modulo 2, ie the first couple of terms are $0, 1, 1, 0, 1, 0, 0, 1, ...$ Given a coupling constant λ the Thue-Morse Hamiltonian is the operator H such that

$$H(\varphi)_n = \varphi_{n+1} + \varphi_{n-1} + \lambda \cdot V_n \varphi_n$$

The paper [16] provides a lower bound on the Hausdorff dimension of the spectrum. For a great deal of insight on Hausdorff dimension and fractional dimensions, one can see [17]. The spectrum can be defined, not by a path intersecting $\mathcal{H}_n$, but a sequence of paths in $SL(2,\mathbb{R})^2$ and the intersection of the values in the path that do not intersect $\mathcal{H}_2$.

Remember that any hyperbolic set $\mathcal{M}$ has a minimal cone $\mathcal{C}'_{\mathcal{M}} = \bigcup C_i$ where $\{C_i\}$ is a family of closed intervals. Every minimal cone has a combination that identifies a chain of the the elements of the cone. Conceptually, one can see that for any n, if 2 elements $X, Y \in \mathcal{H}_n$ belong to the same connected component, then there is a path that connects X, Y and is restricted to the connected component. From Proposition 9 and 10 one can see that there is a minimal cone with a single combination for all the elements along the path. If two elements have minimal cones that are not homeomorphisms of each other (or the combinations are different) then they must be in different connected components. Via [1], we will provide some propositions and Theorems to shed light on this.

To begin, it goes by definition that if the set of $\mathcal{M}$ is uniformly hyperbolic, then for any set $\mathcal{N} \subset SG(\mathcal{M})$, we have $\mathcal{N}$ is uniformly hyperbolic.

Proposition 5.2. *Given any $\mathcal{M}$ which is a set of $SL(2,\mathbb{R})$ matrices, there exists nonunique, nontrivial*[1] *sets $\mathcal{N} \subset SG(\mathcal{M})$ such that $\mathcal{N}$ is uniformly hyperbolic if and only if $\mathcal{M}$ is uniformly hyperbolic.*

[1] In this case $\mathcal{M} \subset SG(\mathcal{M})$ would be considered the trivial example.

Proof. Denote $\mathcal{M} = \{M_i\}$. For any positive integer $n > 1$, denote $\mathcal{M}^n = \{M_{i_1} \cdot M_{i_2} \cdots M_{i_n} : M_i \in \mathcal{M}\}$. If a set $\mathcal{N}$ is not uniformly hyperbolic, then for all $\lambda > 1$ there exists positive integer m and $M \in \mathcal{N}^m$ such that $||M|| < \lambda^m$.

Given set $\mathcal{M}$, assume $\mathcal{M}^n$ is not uniformly hyperbolic. Given any $\lambda > 1$, then $\lambda^n > 1$, and there exists $m > 1$ and $M \in (\mathcal{M}^n)^m$ such that $||M|| > (\lambda^n)^m$. We have matrix $M \in \mathcal{M}^{nm}$ and $||M|| > \lambda^{nm}$, so $\mathcal{M}$ is not uniformly hyperbolic.

Assume $\mathcal{M}$ is not uniformly hyperbolic, then for all $\lambda > 1$ there exists positive integer m and $M \in \mathcal{M}^m$ such that $||M|| < \lambda^m$. Therefore $||M^n|| \leq ||M||^n < \lambda^{mn}$. □

This proposition allows us to define mappings $SL(2,\mathbb{R})^n \to SL(2,\mathbb{R})^m$ such that any element $\mathcal{M}$ maps to an element of $SG(\mathcal{M})$ where $\mathcal{H}_n$ maps to $\mathcal{H}_m$ and $\mathcal{H}_n^c$ maps to $\mathcal{H}_m^c$.

Before introduction the next proposition, it would be useful to define a condition.

Definition 16. *We call a uniformly hyperbolic set $\mathcal{M}$ prime if for all $M \in \mathcal{M}$, there exists a unique C_i such that $M(\mathcal{C}') \subset C_i$.*

That is all of the matrices map the entire minimal cone to a single interval. We arrive at the corollary of Propositions 9 and 10.

Corollary 5.3. *Given a connected component of $\mathcal{H}_n$, all of the elements are prime or none of the elements are prime.*

We also have the following useful properties of prime hyperbolic sets.

Proposition 18. *Assume $\mathcal{M}$ is a prime, uniformly hyperbolic set. Then for all $M \in \mathcal{M}$, if the spectral radius of M is increased by any number, the set is still uniformly hyperbolic.*

Proof. If the spectral radius of any $M \in \mathcal{M}$ is increased, then the corresponding interval in the minimal cone may decrease in length, but all of the elements of $\mathcal{M}$ will still map to their corresponding intervals. Therefore a cone can still be defined. □

As a note, Let $P(t)$ is a path in $SL(2,\mathbb{R})^n$ such that $P(0)$ is a prime, uniformly hyperbolic element, and as t increases, all the eigenvectors are fixed but the spectral radius of elements are not decreasing. We get $\mathcal{C}'_{P(t)} \subset \mathcal{C}'_{P(0)}$.

Proposition 19. *Given a connected component $\mathcal{C} \subset \mathcal{H}_n$ with prime elements and any $X \in \mathcal{C}$. All of the prime elements with matching combinatorics for the minimal cone of X are also elements of $\mathcal{C}$.*

Proof. Define two prime, uniformly hyperbolic sets $\mathcal{M}$ and $\mathcal{N}$, such that the combinatorics of the minimal cones are the same. Further define them as the sets

$$\mathcal{M} = \{M_1, M_2, \cdots M_n\}$$

$$\mathcal{N} = \{N_1, N_2, \cdots, N_n\}.$$

For all i, increase the spectral radius of the elements such that N_i and M_i are conjugates of each other. From there, one can define a series of conjugations of the elements of $\mathcal{M}$ (keeping the set uniformly hyperbolic) such that the minimal cones will overlap, and for all i, the eigenvectors of N_i and M_i will be equal. $\square$

The next proposition uses this concept. Given any connected component of $\mathcal{H}_n$, there is a unique minimal cone (up to homeomorphism) with a respective combination. For each $calM$ in the connected component and interval $\mathcal{C}_i$ in the minimal cone $\mathcal{C}'_{\mathcal{M}}$, there exists a subset $\mathcal{N}_i \subset SG(\mathcal{M})$ such that for all $N \in \mathcal{N}_i$, we have $N(\mathcal{C}'_{\mathcal{M}}) \subset \mathcal{C}_i$. There is also a finite subset $\mathcal{N}'_i \subset \mathcal{N}_i$ such that for all $N \in \mathcal{N}_i$, there exists $N' \in \mathcal{N}'_i$ and $M \in SG(\mathcal{M})$ such that $N'M = N$.

Proposition 20. *Any connected component with prime elements do not share boundaries with any other connected components.*

The proof of this proposition is pretty straightforward. From [1], one can see that if two connected components share a boundary, then the minimal cones of the connected components share the same combinatorics. If two elements are prime and share the same combinatorics, then a path can be constructed, restricted to the hyperbolic locus, which will connect the two elements.

Proposition 21. *Given any connected component $\mathcal{U} \subset \mathcal{H}_n$ with element $\mathcal{M} \in \mathcal{U}$ and minimal cone $\mathcal{C}'_M = \bigcup \mathcal{C}_i$, there exists the set $\mathcal{N}' \subset SG(\mathcal{M})$ (where $|\mathcal{N}'| = k$). The continuous mapping $SL(2,\mathbb{R})^n \to SL(2,\mathbb{R})^k$ with $\mathcal{M} \mapsto \mathcal{N}'$ (mapping an element to an element of its semigroup) will map $\mathcal{U}$ to a connected component of $\mathcal{H}_k$. The boundary $\partial\mathcal{U}$ maps to the boundary of the connected component of $\mathcal{H}_k$, and the connected component it maps to will consist of prime elements.*

To prove this, it is enough to show that there is a $\mathcal{N}' \subset SG(\mathcal{M})$ such that $\mathcal{N}$ satisfies Proposition 5.2 and is prime. The simplest example is if $\mathcal{M} \in SL(2,\mathbb{R})^n$ then let $\mathcal{N}'$ be all possible products of k matrices in $\mathcal{M}$ where k is sufficiently large.

An approach to question (1) from [1] is to take this mapping and look to define a series of mappings which could shed a light on whether the boundary of any connected component can be shared with another connected component. If two connected components shared a boundary, then this mapping would map both of those connected components in $\mathcal{H}_n$ to the same connected component in $\mathcal{H}_{n^k}$. This approach can shed some light on answering the questions posed in [1] and the geometry of the hyperbolic locus.

Chapter 6

Addendum: Spectral Theory

In this section, we will discuss a couple concepts involving spectral theory. When studying the spectrum of an operator, we care about two details.

(1) The topology of the spectrum

(2) The decomposition of the spectrum

Point 1 is the subject of interest in this work. Additional information on this material can be found in [9]. When looking at an operator, does the spectrum consist of a finite number of intervals? Does it have a fractional Hausdorff dimension? Does it have a dense interior? As previously mentioned, the Anderson model has a finite number of intervals in the spectrum, as does the periodic Schrödinger operator. This detail is what makes this outcome unique and unexpected. Given the work in [2], if the Anderson model is given a background ergodic potential defined by a dynamical system with a continuous phase space, then the spectrum would still consist of a finite number of intervals. If the spectrum has Cantor set properties, such as the Thue-Morse operator [5] and the Almost Mathieu operator [3], then there are gap labeling Theorems researched extensively by Bellisard [6] [14] [4] can be examined.

Point 2 can be best most simply described by the RAGE Theorem. The spectrum can be decomposed into different types, each of which have their own properties. For instance, the periodic Anderson-Bernoulli model has an almost sure pure-point spectrum, as proven in [12]. This information pulls from [8]. As mentioned previously, we have the spectrum of operator H as the set

$$\sigma(H) = \{E : H - E \text{ does not have a bounded inverse}\},$$

but we also have the resolvent set

$$\rho(H) = \{E : H - E \text{ has a bounded inverse}\}.$$

Note that an operator has a norm

$$||H|| = \lim \sup_{||phi||=1} \frac{||H\varphi||}{||\varphi||}$$

and H is said to be bounded if the norm is bounded. If H is a finite dimensional operator, then the spectrum is simply the set of eigenvalues. If the operator is infinite dimensional, then the spectrum can be more complicated. As an example, let L be the shift operator on $l^2(\mathbb{Z})$, ie $(L\varphi)_n = \varphi_{n-1}$. There are no eigenvectors, but 1 is in the spectrum. We have a couple concepts and tools that will help us understand and study the spectrum of operators.

6.1 Spectral Measures

Let A be a hermitian operator $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ such that $(\varphi, A\varphi) = (A\varphi, \varphi)$ and σ_A be the spectrum of A. Because A is hermitian, then the spectrum is a subset of $\mathbb{R}$. Furthermore, let $\mathcal{B}(\mathbb{C})$ be the borel set of measurable sets of $\mathbb{C}$. Consider that a measure can be seen as a function $\mu : \mathcal{B}(\mathbb{C}) \to \mathbb{C})$ where for any function $f : \mathbb{C} \to \mathbb{C}$ which can be used to define an integral

$$\int_{\mathbb{C}} f d\mu \in \mathbb{C}.$$

A *spectral measure* can be seen as a mapping $\nu : \mathcal{B}(\mathbb{C}) \to l^{2*}(\mathbb{Z})$ such that

$$\int_{\mathbb{C}} d\nu \in l^{2*}(\mathbb{Z}).$$

Theorem 5 (Spectral Theorem)**.** *There exists a spectral measure ν_A supported on σ_A such that*

$$A = \int_{\mathbb{C}} d\nu_A.$$

With this being said, we can apply polynomials g on A to get

$$g(A) = \int_{\mathbb{C}} g \; d\nu_A.$$

There are two additional ways that the integral for the Spectral Theorem can be written. Sometimes this is written such that ν_A maps purely to projections and the integral is written as $A = \int_{\mathbb{C}} t \, d\nu_A(t)$, to better illustrate how the mapping ν is defined. The other way the integral can be written puts focus on the individual $\varphi \in l^2(\mathbb{Z})$. For all $\varphi \in l^2(\mathbb{Z})$ there exists measure ν_φ such that

$$(\varphi, A\varphi) = (\varphi, \int_{\mathbb{C}} d\nu_A \cdot \varphi) = \int_{\mathbb{C}} d\nu_\varphi.$$

6.2 Spectral Decomposition

When studying the spectrum, one can decompose the spectrum into different sets.

- the pure point piece
- the absolutely continuous piece
- the singular continuous piece

For given A and any φ we have $\nu_\varphi = \nu_{\varphi,pp} + \nu_{\varphi,ac} + \nu_{\varphi,sc}$. The pure point piece ($\nu_{\varphi,pp}$) is supported on a countable set. The absolutely continuous piece ($\nu_{\varphi,ac}$) has a weight of zero on sets of measure zero. The singular continuous piece ($\nu_{\varphi,ac}$) is supported on a set of measure zero but has a weight of zero on individual points. This decomposition can be extended to decomposing $l^2(\mathbb{Z})$ into a direct sum of sets with elements that have only a pure point piece or absolutely continuous piece or singular continuous piece. Using the notation from [8], we have

$$l^2(\mathbb{Z})_{pp} = \{\varphi \in l^2(\mathbb{Z}) : \nu_\varphi = \nu_{\varphi,pp}\}$$

$$l^2(\mathbb{Z})_{ac} = \{\varphi \in l^2(\mathbb{Z}) : \nu_\varphi = \nu_{\varphi,ac}\}$$

$$l^2(\mathbb{Z})_{sc} = \{\varphi \in l^2(\mathbb{Z}) : \nu_\varphi = \nu_{\varphi,sc}\}.$$

And we get l^2 as the direct sum of these sets,

$$l^2(\mathbb{Z}) = l^2(\mathbb{Z})_{pp} \oplus l^2(\mathbb{Z})_{ac} \oplus l^2(\mathbb{Z})_{sc}.$$

We can also get a decomposition of the spectrum. If P_{pp} is the projection on $l^2(\mathbb{Z})_{pp}$, then $A \cdot P_{pp}$ has a spectrum denoted σ_{pp}. Similarly, there are projections P_{ac} and P_{sc} and sets σ_{ac} and σ_{sc}.

6.3 RAGE Theorem

We can see [8] for more details on the RAGE Theorem. The Theorem as stated below pulls directly from Dr Damanik's work.

Theorem 6.1. *(a) $\varphi \in l^2(\mathbb{Z})_{pp} \Leftrightarrow$ for every $\varepsilon > 0$, there exists $N > 0$ such that*

$$\sum_{|n| \geq N} |(\delta_n, e^{-itA}\varphi)|^2 < \varepsilon \text{ for all } t \in \mathbb{R}$$

(b) $\varphi \in l^2(\mathbb{Z})_{ab} \cup l^2(\mathbb{Z})_{sc} \Leftrightarrow$ for every $N > 0$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sum_{|n| \leq N} |(\delta_n, e^{-itA}\varphi)|^2 dt = 0$$

(c) $\varphi \in l^2(\mathbb{Z})_{ac} \Rightarrow$ for every $N > 0$

$$\lim_{|t| \to \infty} \sum_{|n| \leq N} |(\delta_n, e^{-itA}\varphi)|^2 = 0$$

Bibliography

Bibliography

[1] Artur Avila, Jairo Bochi, and Jean-Christophe Yoccoz, *Uniformly hyperbolic finite-valued* $\mathrm{SL}(2,\mathbb{R})$*-cocycles*, Comment. Math. Helv. **85** (2010), no. 4, 813–884. MR 2718140

[2] Artur Avila, David Damanik, and Anton Gorodetski, *The spectrum of Schrödinger operators with randomly perturbed ergodic potentials*, Geom. Funct. Anal. **33** (2023), no. 2, 364–375. MR 4578461

[3] Artur Avila and Svetlana Jitomirskaya, *The Ten Martini Problem*, Ann. of Math. (2) **170** (2009), no. 1, 303–342. MR 2521117

[4] Jean Bellissard, *Schrödinger operators with almost periodic potemntial: An overview*, Mathematical Problems in Theoretical Physics (Berlin, Heidelberg) (R. Schrader, R. Seiler, and D. A. Uhlenbrock, eds.), Springer Berlin Heidelberg, 1982, pp. 356–363.

[5] ———, *Spectral properties of schrödinger's operator with a thue-morse potential*, Number Theory and Physics (Berlin, Heidelberg) (Jean-Marc Luck, Pierre Moussa, and Michel Waldschmidt, eds.), Springer Berlin Heidelberg, 1990, pp. 140–150.

[6] ———, *Gap labelling theorems for schrödinger operators*, From Number Theory to Physics (1992).

[7] Argyrios Christodoulou, *Parameter Spaces of Locally Constant Cocycles*, International Mathematics Research Notices **2022** (2021), no. 17, 13590–13628.

[8] David Damanik, *Schrödinger operators with dynamically defined potentials*, Ergodic Theory Dynam. Systems **37** (2017), no. 6, 1681–1764. MR 3681983

[9] David Damanik and Anton Gorodetski, *Must the spectrum of a random schrödinger operator contain an interval?*, Communications in Mathematical Physics **393** (2022), 1583–1613.

[10] David Damanik, Anton Gorodetski, and William Yessen, *The Fibonacci Hamiltonian*, Invent. Math. **206** (2016), no. 3, 629–692. MR 3573970

[11] Jake Fillman and William Wood, *Periodic anderson-bernoulli model with infinitely many spectral gaps*, preprint (2024).

[12] Anton Gorodetski and Victor Kleptsyn, *Non-stationary anderson localization*, preprint (2024).

[13] Russell A. Johnson, *Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients*, J. Differential Equations **61** (1986), no. 1, 54–78. MR 818861

[14] Jerry Kaminker and Ian Putnam, *A proof of the gap labeling conjecture*, Michigan Mathematical Journal **51** (2002).

[15] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1995.

[16] Qing-Hui Liu and Yan-Hui Qu, *On the hausdorff dimension of the spectrum of the thue–morse hamiltonian*, Communications in Mathematical Physics **338** (2014).

[17] Benoit B. Mandelbrot, *The fractal geometry of nature*, Times Books, 1982.

[18] C. Rojas-Molina, *Random schrödinger operators and anderson localization in aperiodic media*, Reviews in Mathematical Physics **33** (2020), no. 01, 2060010.

[19] William Wood, *On the spectrum of the periodic Anderson-Bernoulli model*, J. Math. Phys. **63** (2022), no. 10, Paper No. 102705, 16. MR 4497771

[20] Jean-Christophe Yoccoz, *Some questions and remarks about* $\mathrm{SL}(2,\mathbf{R})$ *cocycles*, Modern dynamical systems and applications, Cambridge Univ. Press, Cambridge, 2004, pp. 447–458. MR 2093316

[21] Zhenghe Zhang, *Uniform hyperbolicity and its relation with spectral analysis of 1D discrete Schrödinger operators*, J. Spectr. Theory **10** (2020), no. 4, 1471–1517. MR 4192758