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The tt-geometry of permutation filtered modules

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The tt-geometry of permutation filtered modules

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Colin Ni

2026

ABSTRACT OF THE DISSERTATION

The tt-geometry of permutation filtered modules

by

Colin Ni
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2026
Professor Paul Balmer, Chair

We study the derived category of permutation filtered modules. We focus on the case $\mathrm{D_b}(\mathsf{pfmod}(E))$ of an elementary abelian group E. Our key insight is to consider for each subgroup $N \leqslant E$ a relative subcategory $\mathsf{pfmod}(E, N) \subset \mathsf{pfmod}(E)$ of permutation filtered modules and a relative category $\mathsf{pgmod}(E, N)$ of permutation graded modules. The case $N = 1$ recovers the original categories. We show that the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$ as a set consists of a copy of the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$ and a colimit $\operatorname{colim}_{L>N} U_L$ of open subsets $U_L \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, L))$ coming from smaller relative subcategories. This allows an inductive analysis of $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$.

The dissertation of Colin Ni is approved.

Raphael Rouquier

Alexander Merkurjev

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Paul Balmer, Committee Chair

University of California, Los Angeles

2026

Dedicated to Casey Zhao.

TABLE OF CONTENTS

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VITA

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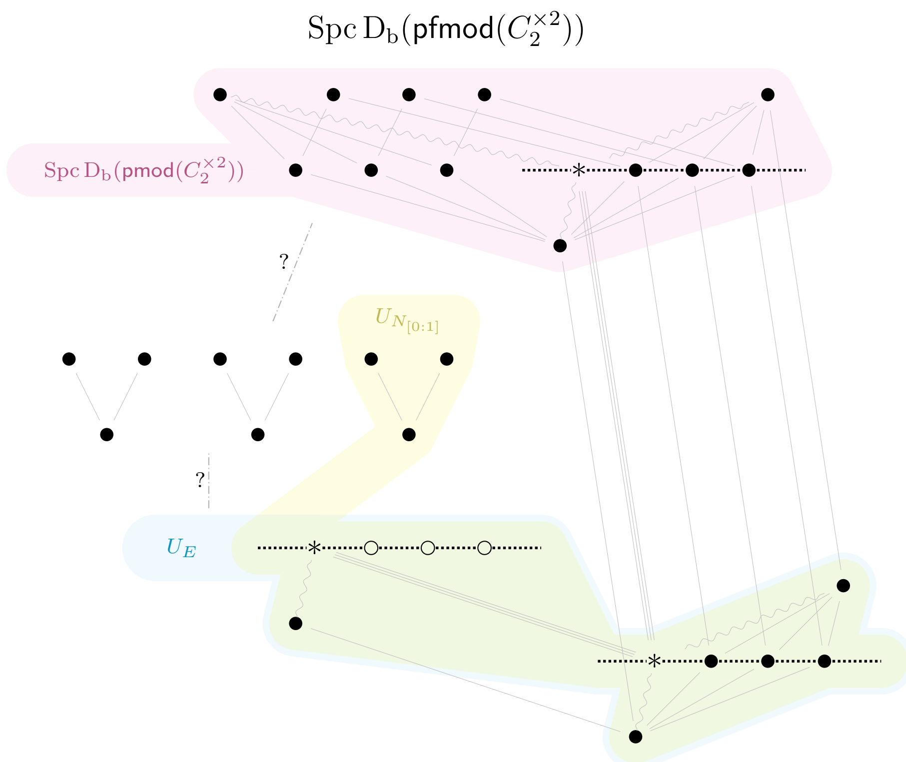
1 Introduction



1.1 Notation. In this introduction, let p be a prime number, let k be a field of characteristic p, let Γ be a profinite group, let G be a finite group, let E be an elementary abelian p-group, let F be a field, and let $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ denote the absolute Galois group of F. By $H \overset{\circ}{\leqslant} \Gamma$ we mean an open subgroup of Γ, that is, a closed subgroup of finite index, and by $N \overset{\circ}{\trianglelefteq} \Gamma$ we mean a normal open subgroup.

Permutation modules

Permutation $k\Gamma$-modules are some of the simplest $k\Gamma$-modules that one can describe: they are the k-linearized Γ-sets or, in other words, a direct sum of $k\Gamma$-modules of the form $k(\Gamma/H)$ for open subgroups $H \overset{\circ}{\leqslant} \Gamma$. The finitely generated permutation $k\Gamma$-modules only form an additive category, whose idempotent completion we denote by $\mathsf{pmod}(\Gamma)$, so the derived category $\mathrm{D_b}(\mathsf{pmod}(\Gamma))$ is nothing but the homotopy category of $\mathsf{pmod}(\Gamma)$.

Nevertheless, permutation modules have been under fire from Balmer–Gallauer for years now. One reason is for their relationship to motives, which we will discuss momentarily. Another reason is that the derived category $\mathrm{D_b}(\mathsf{pmod}(G))$ is shockingly rich and is in fact more complex than the usual derived category $\mathrm{D_b}(\mathsf{mod}(G))$ of the abelian category $\mathsf{mod}(G)$ of finitely generated kG-modules. For example, every object in $\mathsf{mod}(G)$ admits a finite resolution by objects in $\mathsf{pmod}(G)$, and one then deduces (Example 7.14) that $\mathrm{D_b}(\mathsf{mod}(G))$ is the Verdier quotient of $\mathrm{D_b}(\mathsf{pmod}(G))$ at the complexes that are $\mathsf{mod}(G)$-acyclic:

$$\Upsilon\colon\ \mathrm{D_b}(\mathsf{pmod}(G)) \twoheadrightarrow \mathrm{D_b}(\mathsf{mod}(G)).$$

Geometrically $\operatorname{Spc}\mathrm{D_b}(\mathsf{mod}(G))$ embeds as an open set in $\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(G))$, which is called the *cohomological open*.

Balmer–Gallauer determined the spectrum of $\mathrm{D_b}(\mathsf{pmod}(G))$ in its entirety in [BG25a]. To determine it as a set, they introduced the modular fixed points functors

$$\check{\Psi}^H\colon\ \mathrm{D_b}(\mathsf{pmod}(G)) \xrightarrow{\Psi^H} \mathrm{D_b}(\mathsf{pmod}(G/\!\!/H)) \xrightarrow{\Upsilon} \mathrm{D_b}(\mathsf{mod}(G/\!\!/H)) \tag{1.2}$$

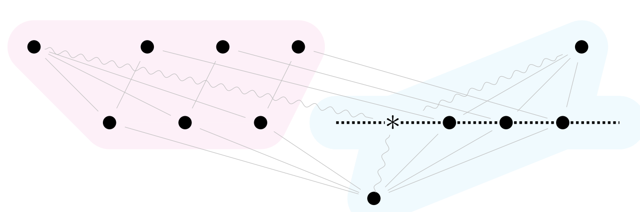
which, as $H \in \mathrm{sub}_p(G)/G$ varies over the p-subgroups of G up to conjugacy, partition the spectrum $\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(G))$ as a set:

$$\bigsqcup_{H\in\mathrm{sub}_p(G)/G} \operatorname{Spc}\mathrm{D_b}(\mathsf{mod}(G/\!\!/H)) \xrightarrow{\cong} \operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(G)). \tag{1.3}$$

To determine specializations, they reduced to the elementary abelian case and used a certain twisted cohomology ring.

For example, in the Klein four case $E = C_2^{\times 2}$ the spectrum is [BG25a, Example 16.16]:

$$\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(E))$$



(1.4)

Here the cyan region is the cohomological open, which is the vanishing locus of the $\mathsf{mod}(G)$-acyclic complexes, and the magenta region is the support of these acyclics. For details about how to read pictures such as these, see Running Example 6.36.

Artin, Tate, and Artin-Tate motives

Understanding Voevodsky's derived category $\mathrm{DM}(F;\mathbb{Z})$ of motives is a grand problem in algebraic geometry. More tractable is the problem of understanding the derived category $\mathrm{DM}^{\mathrm{gm}}(F;k)$ of geometric motives with coefficients in k, and even more tractable are the problems of understanding the thick subcategories of Artin, Tate, and Artin-Tate motives:

$$\begin{aligned}
\mathrm{DAM}^{\mathrm{gm}}(F;k) &= \mathrm{thick}\{\mathrm{M}(L) \mid L/F \text{ finite separable}\} \subset \mathrm{DM}^{\mathrm{gm}}(F;k)\\
\mathrm{DTM}^{\mathrm{gm}}(F;k) &= \mathrm{thick}\{\mathrm{M}(F)(w) \mid w \in \mathbb{Z}\} \subset \mathrm{DM}^{\mathrm{gm}}(F;k)\\
\mathrm{DATM}^{\mathrm{gm}}(F;k) &= \mathrm{thick}\{\mathrm{M}(L)(w) \mid L/F \text{ finite separable}, w \in \mathbb{Z}\} \subset \mathrm{DM}^{\mathrm{gm}}(F;k),
\end{aligned}$$

where given a finite separable extension L of F, we denote by $\mathrm{M}(L) \in \mathrm{DM}^{\mathrm{gm}}(F;k)$ the motive of $\mathrm{Spec}(L)$. Indeed, these derived categories can be expressed algebraically, as we now discuss.

Voevodsky established [Voe00] that the derived category of Artin motives is equivalent to the derived category of finitely generated permutation kG_F-modules:

$$\mathrm{DAM}^{\mathrm{gm}}(F;k) \cong \mathrm{D_b}(\mathsf{pmod}(G_F)).$$

Balmer–Gallauer use this to determine $\mathrm{Spc}\,\mathrm{DAM}^{\mathrm{gm}}(F;k)$ in [BG25b]. Namely, they write the derived category of permutation kG_F-modules as a colimit

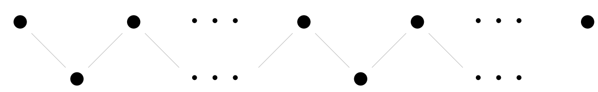
$$\mathrm{D_b}(\mathsf{pmod}(G_F)) \cong \operatorname*{colim}_{N \mathrel{\mathring{\trianglelefteq}} G_F} \mathrm{D_b}(\mathsf{pmod}(G_F/N)) \tag{1.5}$$

of derived categories of permutation modules over finite groups, and then they use continuity of Spc and the computation of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G))$ (see (1.3)) to conclude

$$\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G_F)) \cong \lim_{N \mathrel{\mathring{\trianglelefteq}} G_F} \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G_F/N)). \tag{1.6}$$

For example, in the case where F is a finite field so that $G_F = \hat{\mathbb{Z}}$ is the profinite integers, the spectrum can be illustrated as [BG25b, Theorem 1.4]

$$\operatorname{Spc} \mathrm{DAM}^{\mathrm{gm}}(F;k)$$

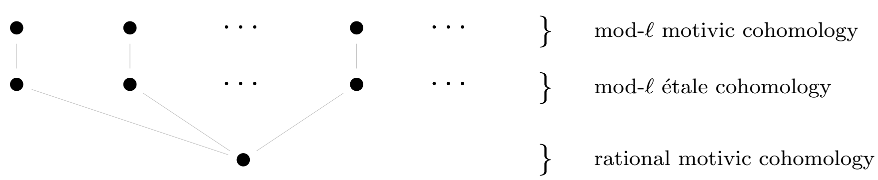


In a similar vein, Positselski considered the exact category $\mathsf{fmod}(G_F, G_F)$ of filtered finitely generated kG_F-modules whose associated graded is a direct sum of twists of k [Pos11, Sections 3 and 9.3] (see also [Gal19, Proposition 7.3]). Positselski established that, in the case $k = \mathbb{F}_p$, the derived category of Tate motives is equivalent

$$\mathrm{DTM}^{\mathrm{gm}}(F;k) \cong \mathrm{D_b}(\mathsf{fmod}(G_F, G_F)) \tag{1.7}$$

to the derived category of this exact category $\mathsf{fmod}(G_F, G_F)$. Gallauer used this in [Gal19, Theorem 8.6] (see also [Gal18]) to determine $\operatorname{Spc} \mathrm{DTM}^{\mathrm{gm}}(F; \mathbb{Z}[1/p])$ when F is algebraically closed and has rational motivic cohomology groups satisfying a certain vanishing condition [Gal19, Hypothesis 6.6]:

$$\operatorname{Spc} \mathrm{DTM}^{\mathrm{gm}}(F; \mathbb{Z}[1/p])$$



Here ℓ runs through the primes different from p.

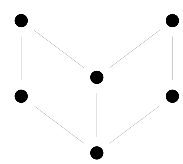
Going one step further, Positselski considered the exact category $\mathsf{pfmod}(C_2)$ of filtered finitely generated kC_2-modules whose associated graded is permutation (vacuously), where a kernel-cokernel pair in $\mathsf{pfmod}(C_2)$ is defined to be short exact if and only if its associated graded in $\mathsf{pmod}(C_2)$ is short exact (split exact) [Pos11, Sections 3 and 9.3]. Balmer–Gallauer established that, in the case $p = 2$ and $F = \mathbb{R}$ (so that $G_F = C_2$), the derived category of

Artin-Tate motives is equivalent

$$\mathrm{DATM}^{\mathrm{gm}}(F;k) \cong \mathrm{D}_{\mathrm{b}}(\mathsf{pfmod}(C_2)) \tag{1.8}$$

to the derived category of this exact category $\mathsf{pfmod}(C_2)$ [BG22b, Proposition 9.2]. They then use this to determine $\mathrm{Spc}\,\mathrm{DATM}^{\mathrm{gm}}(F;k)$, namely by computing $\mathrm{Spc}\,\mathrm{D}_{\mathrm{b}}(\mathsf{pfmod}(C_2))$ [BG22b, Theorem 7.9]:

$\mathrm{Spc}\,\mathrm{D}_{\mathrm{b}}(\mathsf{pfmod}(C_2))$



(1.9)

Permutation filtered modules

In full generality [Pos11, Sections 3 and 9.3], we consider the exact category $\mathsf{pfmod}(G_F)$ of filtered finitely generated kG_F-modules whose associated graded is permutation, where the exact structure is pulled back along the total associated graded functor $\mathrm{ung}\,\mathrm{gr}\colon \mathsf{pfmod}(G_F) \to \mathsf{pmod}(G_F)$, in the sense that a kernel-cokernel pair

$$A_1 \xrightarrow{i} A_2 \xrightarrow{p} A_3$$

in $\mathsf{pfmod}(G_F)$ is short exact if and only if its total associated graded

$$\mathrm{ung}\,\mathrm{gr}(A_1) \xrightarrow{\mathrm{ung}\,\mathrm{gr}(i)} \mathrm{ung}\,\mathrm{gr}(A_2) \xrightarrow{\mathrm{ung}\,\mathrm{gr}(p)} \mathrm{ung}\,\mathrm{gr}(A_3)$$

in $\mathsf{pmod}(G_F)$ is short exact (split exact). The derived category of Artin-Tate motives is expected to be equivalent

$$\mathrm{DATM}^{\mathrm{gm}}(F;k) \cong \mathrm{D}_{\mathrm{b}}(\mathsf{pfmod}(G_F)) \tag{1.10}$$

to the derived category of this exact category $\mathsf{pfmod}(G_F)$. It is true, as we saw, in the case of [BG22b], and more generally it follows from the main conjecture of Positselski in [Pos11] (see [Pos11, Section 9.2]).

In this work, we will not be so concerned with this conjecture, and instead we focus on studying the spectrum of $\mathrm{D_b}(\mathsf{pfmod}(G_F))$. Much like (1.5) and (1.6), we have

$$\mathrm{D_b}(\mathsf{pfmod}(G_F)) \cong \operatorname*{colim}_{N \mathrel{\mathring{\trianglelefteqslant}} G_F} \mathrm{D_b}(\mathsf{pfmod}(G_F/N)), \tag{1.11}$$

hence

$$\operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(G_F)) \cong \lim_{N \mathrel{\mathring{\trianglelefteqslant}} G_F} \operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(G_F/N)). \tag{1.12}$$

In this fashion, we reduce to the case $\operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(G))$ of a finite group G. We may also reduce to the case of a p-group G, for example by observing that $\mathsf{pfmod}(-)$ is a *cohomological Mackey 2-functor* (see for example Proposition 4.1 and [BD24, Theorem 5.10]).

As a reminder, the purpose of computing the spectrum of a tt-category is to classify the objects in the category up to tt-equivalence: two objects have the same support in the spectrum if and only if they can be built from one another using triangles and retracts and by tensoring with arbitrary objects. Thus, in other words we aim to classify the objects of $\mathrm{D_b}(\mathsf{pfmod}(G))$ up to tt-equivalence.

In a nutshell, for an elementary abelian p-group E, our main results in this work imply that two objects $X, Y \in \mathrm{D_b}(\mathsf{pfmod}(E))$ are tt-equivalent only if they satisfy the following three necessary conditions:

(i) They have tt-equivalent total associated graded objects:

$$\operatorname{supp}(\operatorname{ung}\operatorname{gr}(X)) = \operatorname{supp}(\operatorname{ung}\operatorname{gr}(Y)) \quad \text{in} \quad \operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(E)).$$

(ii) They have tt-equivalent underlying objects:

$$\operatorname{supp}(\operatorname{unf}(X)) = \operatorname{supp}(\operatorname{unf}(Y)) \quad \text{in} \quad \operatorname{Spc}\mathrm{D_b}(\mathsf{mod}(E)).$$

(iii) For every subgroup $N \leqslant E$ for which X and Y are complexes of modules over the filtered monoid $\operatorname{radf}_N(kE)$, the objects X and Y have tt-equivalent complexes of associated graded modules over the graded monoid $\operatorname{gr}\operatorname{radf}_N(kE)$: if $X, Y \in \mathrm{D_b}(\mathsf{pfmod}(E, N))$ for some $N \leqslant E$, then

$$\operatorname{supp}(\operatorname{gr}(X)) = \operatorname{supp}(\operatorname{gr}(Y)) \quad \text{in} \quad \operatorname{Spc}\mathrm{D_b}(\mathsf{pgmod}(E, N)),$$

where the relative categories $\mathsf{pfmod}(E, N)$ and $\mathsf{pgmod}(E, N)$ will be discussed in the next subsection and are officially introduced in Sections 14 and 15.

In fact, our main results assert that in some sense these conditions are also sufficient: the spectrum $\operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(E))$ is comprised of points coming from these spectra.

The first two conditions (i) and (ii) are quite natural. Given a filtered object, the two obvious things we can do with it are forget the filtration and take the total associated graded, and we should expect these two operations to preserve tt-equivalence. The third condition (iii) is more striking and contains (i) as the special case $N = 1$. In this introduction we will only give a rough explanation of condition (iii), starting in the setting of our running example.

Introduction to the running example

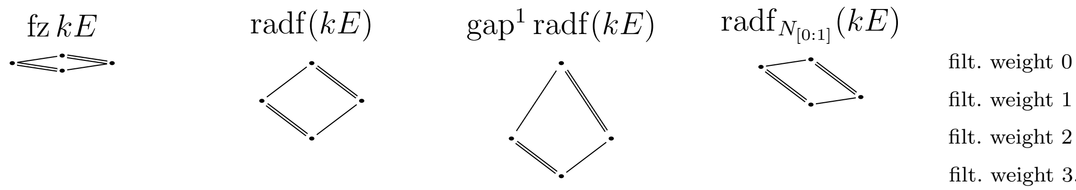
1.13 Running Example. We take our group G to be the Klein four group $C_2^{\times 2}$, and we assume our field k is algebraically closed with characteristic $p = 2$. Note that $E = C_2^{\times 2}$ is an elementary abelian 2-group with rank 2. We fix a minimal set $x, y \in E$ of generators of E, and we denote by $X = x - 1$ and $Y = y - 1$ the corresponding elements of the radical $\operatorname{rad}(kE) \subset kE$ so that

$$kE \simeq \frac{k[X, Y]}{(X^2, Y^2)}.$$

We write $N_{[1:0]} = \langle x \rangle$, $N_{[1:1]} = \langle xy \rangle$, and $N_{[0:1]} = \langle y \rangle$ for the rank 1 subgroups of E generated respectively by x, xy, and y.

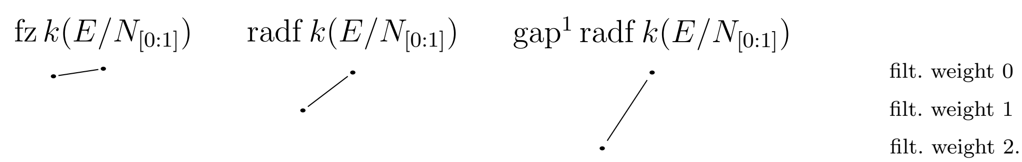
In this subsection, we adopt the notation in Running Example 1.13, and we investigate this example by describing some non-tt-equivalent objects in $\mathrm{D_b}(\mathsf{pfmod}(E))$ using our conditions (i), (ii), and (iii) from the previous subsection. We will see in the next subsection that their supports are indeed different.

The usual way of drawing kE-modules, where the X-action is depicted by a single line and the Y-action is depicted by a double line, extends nicely to a way of drawing objects of $\mathsf{pfmod}(E)$. For example, we can draw the following permutation filtrations of kE:

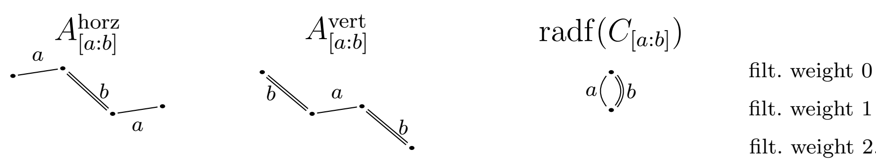


(1.14)

From left to right, we put kE entirely in filtration zero; we filter kE by the powers of the radical $\mathrm{rad}(kE) = \langle g-1 \mid g \in E\rangle = \langle X, Y\rangle$; we insert a gap at filtration weight 1 into the previous filtration; and we filter kE by the powers of the $N_{[0:1]}$-radical $\mathrm{rad}_{N_{[0:1]}}(kE) = \langle n-1 \mid n \in N_{[0:1]}\rangle = \langle Y\rangle$. Similarly, we have the following permutation filtrations of the permutation module $k(E/N_{[0:1]})$:



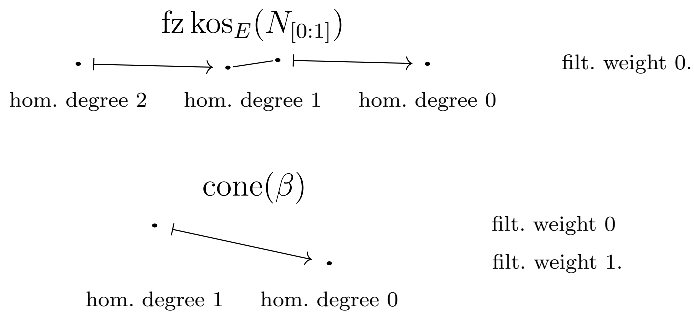
Given a non-$\mathbb{F}_2$-rational point $[a:b] \in \mathbb{P}^1_k$, we also have the following objects in $\mathsf{pfmod}(E)$:



(1.15)

Here $(a,b) \in k^2$ is a representative of $[a:b]$, and the a and b labels on the X- and Y-actions indicate that the actions are scaled by a and b respectively. Up to tt-equivalence, these objects only depend on $[a:b] \in \mathbb{P}^1_k$.

By placing the objects we have seen thus far in homological degree 0, we obtain objects in the derived category $\mathrm{D_b}(\mathsf{pfmod}(E))$, but we also have the following two complexes that are not concentrated in homological degree 0:



(1.16)

We now turn our attention to discussing why these objects in $\mathrm{D_b}(\mathsf{pfmod}(E))$ are all non-tt-equivalent, using conditions (i), (ii), and (iii) from the previous subsection. Consider the table on the following page.

By considering the total associated graded and underlying objects, which are tabulated in the second and third columns and which correspond to conditions (i) and (ii), we easily see that many pairs of these objects are not tt-equivalent. In fact, the only pairs of objects that these invariants cannot distinguish are the pair $\mathrm{radf}(kE)$ and $\mathrm{gap}^1\,\mathrm{radf}(kE)$, the pair $\mathrm{radf}\,k(E/N_{[0:1]})$ and $\mathrm{gap}^1\,\mathrm{radf}\,k(E/N_{[0:1]})$, and the pair $A^{\mathrm{vert}}_{[a:b]}$ and $\mathrm{radf}(C_{[a:b]})$.

Let us now give our rough explanation of condition (iii), which is partially tabulated in the fourth and fifth columns. For details about the relative categories $\mathsf{pfmod}(E, N)$ and $\mathsf{pgmod}(E, N)$, we again refer the reader to Sections 14 and 15. We start with the subgroup $N = E$, in which case $\mathrm{radf}_N(kE) = \mathrm{radf}(kE)$. A module $A \in \mathsf{pfmod}(E)$ being a module over this filtered monoid $\mathrm{radf}(kE)$ means that the elements $X, Y \in kE$ *act with filtration weight* 1 on the module A: if $a \in A^w$, then $Xa, Ya \in A^{w+1}$. We then write $A \in \mathsf{pfmod}(E, E)$. The associated graded module $\mathrm{gr}(A) \in \mathsf{pgmod}(E, E)$ is a graded module over the graded monoid $\mathrm{gr}\,\mathrm{radf}(kE)$, so the elements $X, Y \in \mathrm{gr}\,\mathrm{radf}(kE)$ *act with grading* 1 on $\mathrm{gr}(A)$: if $b \in \mathrm{gr}(A)^w$, then $Xb, Yb \in \mathrm{gr}(A)^{w+1}$. For example, we have $\mathrm{radf}(kE), \mathrm{gap}^1\,\mathrm{radf}(kE) \in \mathsf{pfmod}(E, E)$, and their associated graded modules $\mathrm{gr}\,\mathrm{radf}(kE), \mathrm{gr}\,\mathrm{gap}^1\,\mathrm{radf}(kE) \in \mathsf{pgmod}(E, E)$ are respec-

X $\in \mathrm{D_b}(\mathsf{pfmod}(E))$	$\operatorname{ung}\operatorname{gr}(X)$ $\in \mathrm{D_b}(\mathsf{pmod}(E))$	$\operatorname{unf}(X)$ $\in \mathrm{D_b}(\mathsf{mod}(E))$	$\operatorname{ung}\operatorname{gr}(X)$ $\in \mathrm{D_b}(\mathsf{pmod}(E, N_{[0:1]}))$	$\operatorname{ung}\operatorname{gr}(X)$ $\in \mathrm{D_b}(\mathsf{pmod}(E, E))$
$\operatorname{fz} kE$	kE	kE	not in subcategory	not in subcategory
$\operatorname{radf}(kE)$	$k^{\oplus 4}$	kE	$k(E/N_{[1:0]})^{\oplus 2}$	kE
$\operatorname{gap}^1 \operatorname{radf}(kE)$	$k^{\oplus 4}$	kE	$k^{\oplus 2} \oplus k(E/N_{[1:0]})$	$k \oplus \Omega^1(k)$
$\operatorname{radf}_{N_{[0:1]}}(kE)$	$k(E/N_{[0:1]})^{\oplus 2}$	kE	kE	not in subcategory
$\operatorname{fz} k(E/N_{[0:1]})$	$k(E/N_{[0:1]})$	$k(E/N_{[0:1]})$	$k(E/N_{[0:1]})$	not in subcategory
$\operatorname{radf} k(E/N_{[0:1]})$	$k^{\oplus 2}$	$k(E/N_{[0:1]})$	$k^{\oplus 2}$	$k(E/N_{[0:1]})$
$\operatorname{gap}^1 \operatorname{radf} k(E/N_{[0:1]})$	$k^{\oplus 2}$	$k(E/N_{[0:1]})$	$k^{\oplus 2}$	$k^{\oplus 2}$
$A^{\mathrm{horz}}_{[a:b]}$	$k(E/N_{[0:1]})^{\oplus 2}$	$\operatorname{unf}(A^{\mathrm{horz}}_{[a:b]})$	$\operatorname{unf}(A^{\mathrm{horz}}_{[a:b]})$	not in subcategory
$A^{\mathrm{vert}}_{[a:b]}$	$k^{\oplus 2} \oplus k(E/N_{[0:1]})$	$\operatorname{unf}(A^{\mathrm{vert}}_{[a:b]})$	$\operatorname{unf}(A^{\mathrm{vert}}_{[a:b]})$	not in subcategory
$\operatorname{radf}(C_{[a:b]})$	$k^{\oplus 2}$	$C_{[a:b]}$	$k(E/N_{[1:0]})$	$C_{[a:b]}$
$\operatorname{fz} \operatorname{kos}_E(N_{[0:1]})$	$\operatorname{kos}_E(N_{[0:1]})$	0	$\operatorname{kos}_E(N_{[0:1]})$	not in subcategory
$\operatorname{cone}(\beta)$	$k^{\oplus 2}$	0	$k^{\oplus 2}$	$k^{\oplus 2}$

tively

$$\mathrm{gr}\,\mathrm{radf}(kE) \qquad \mathrm{gr}\,\mathrm{gap}^1\,\mathrm{radf}(kE)$$

grading 0
grading 1
grading 2
grading 3. (1.17)

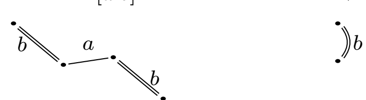
When we forget the gradings, we get the kE-modules

$$\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kE) \simeq kE \qquad \text{and} \qquad \mathrm{ung}\,\mathrm{gr}\,\mathrm{gap}^1\,\mathrm{radf}(kE) \simeq k \oplus \Omega^1(k).$$

For this reason, $\mathrm{radf}(kE)$ and $\mathrm{gap}^1\,\mathrm{radf}(kE)$ are not tt-equivalent.

Similarly, let us now consider the subgroup $N = N_{[0:1]} = \langle y \rangle$. A module $A \in \mathsf{pfmod}(E)$ being a module over the filtered monoid $\mathrm{radf}_{N_{[0:1]}}(kE)$ means that the element $Y \in kE$ acts with filtration weight 1 on A. We then write $A \in \mathsf{pfmod}(E, N_{[0:1]})$. The associated graded module $\mathrm{gr}(A) \in \mathsf{pgmod}(E, N_{[0:1]})$ is a graded module over the graded monoid $\mathrm{gr}\,\mathrm{radf}_{N_{[0:1]}}(kE)$, so the element $Y \in \mathrm{gr}\,\mathrm{radf}_{N_{[0:1]}}(kE)$ acts with grading 1 on $\mathrm{gr}(A)$. For example, we have $A^{\mathrm{vert}}_{[a:b]}, \mathrm{radf}(C_{[a:b]}) \in \mathsf{pfmod}(E, N_{[0:1]})$, and we may compute the associated graded modules $\mathrm{gr}(A^{\mathrm{vert}}_{[a:b]}), \mathrm{gr}\,\mathrm{radf}(C_{[a:b]}) \in \mathsf{pgmod}(E, N_{[0:1]})$ to be respectively

$$\mathrm{gr}\,A^{\mathrm{vert}}_{[a:b]} \qquad \mathrm{gr}\,\mathrm{radf}(C_{[a:b]})$$



grading 0
grading 1
grading 2. (1.18)

When we forget the gradings, we get the kE-modules

$$\mathrm{ung}\,\mathrm{gr}(A^{\mathrm{vert}}_{[a:b]}) \simeq \mathrm{unf}(A^{\mathrm{vert}}_{[a:b]}) \qquad \text{and} \qquad \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(C_{[a:b]}) = k(E/N_{[1:0]}),$$

and this shows that $A^{\mathrm{vert}}_{[a:b]}$ and $\mathrm{radf}(C_{[a:b]})$ are not tt-equivalent.

Intuitively, given $A \in \mathsf{pfmod}(E, N)$, the object $\mathrm{gr}(A) \in \mathsf{pgmod}(E, N)$ and its underlying kE-module $\mathrm{ung}\,\mathrm{gr}(A)$ express how the N-radical elements $\langle n - 1 \mid n \in N \rangle \subset kE$ act across the filtration of A, not just on the total associated graded or the underlying object. This gives rise to a functor $\mathrm{gr}\colon\ \mathsf{pfmod}(E, N) \to \mathsf{pgmod}(E, N)$ defined on the relative subcategory $\mathsf{pfmod}(E, N)$. There is no obvious way to extend this functor to one defined on the entire category $\mathsf{pfmod}(E)$; for example, if we view $A \in \mathsf{pfmod}(E, N)$ as an object in $\mathsf{pfmod}(E)$, then the N-radical elements act trivially on $\mathrm{gr}(A) \in \mathsf{pgmod}(E)$.

Summary of main results

Let us now state our four main theorems. The first two are from Part II, and the final two are from Part III.

Our first main theorem from Part II concerns the categorical properties of our permutation filtered modules $\mathsf{pfmod}(G)$. We would like $\mathsf{pfmod}(G)$ to be Frobenius exact so that, in particular, it has enough projectives. In our setting, this is equivalent to the existence of a projective pre-cover $P \twoheadrightarrow \mathbb{1}$ of just the unit in $\mathsf{pfmod}(G)$ (Proposition 5.29(ii)). One may naively attempt to use the object $\mathrm{fz}\, kG \in \mathsf{pfmod}(G)$ given by putting kG in filtration zero; this is indeed a projective object in $\mathsf{pfmod}(G)$, but $\mathrm{fz}\, kG$ does not admit an admissible epi to the unit because the augmentation $kG \to k$ is not split exact (unless $G = 1$). Luckily, our wish is granted for our elementary abelian p-group E.

1.19 Main Theorem (Theorem 9.13 and Corollary 9.14)**.** *In* $\mathsf{pfmod}(E)$ *there exists a projective pre-cover* $\mathrm{radf}(kE) \twoheadrightarrow \mathbb{1}$ *of the unit. Consequently,* $\mathsf{pfmod}(E)$ *is a Frobenius exact rigid tensor category.*

This projective object $\mathrm{radf}(kE) \in \mathsf{pfmod}(E)$ is the radical filtration (Example 3.44) of the projective module $kE \in \mathsf{mod}(E)$, and the admissible epi $\mathrm{radf}(kE) \twoheadrightarrow \mathbb{1}$ is induced by the augmentation map $kE \to k$. In fact it is even a comonoid in $\mathsf{pfmod}(E)$ (Example 3.45). This theorem seems to be mostly exclusive to the elementary abelian case E. For general groups G, the radical filtration $\mathrm{radf}(kG) \in \mathsf{pfmod}(G)$ is not necessarily projective, and it is unclear whether $\mathsf{pfmod}(G)$ even has enough projectives (Example 9.21).

Our other three main theorems concern the spectrum $\mathrm{Spc}\, \mathrm{D_b}(\mathsf{pfmod}(G))$, so let us first describe our overall strategy for studying this topological space. On the one hand, we make the following assumption, which gives us a complex $P \in \mathrm{D_b}(\mathsf{pfmod}(G))$ concentrated in homological degree 0.

1.20 Notation. For the remainder of this introduction, we assume that there exists a projective comonoid $P \in \mathsf{pfmod}(G)$ whose counit is a pre-cover $\varepsilon\colon P \twoheadrightarrow \mathbb{1}$ of the unit. For example, by the above discussion we may take $G = E$ and $P = \mathrm{radf}(kE)$.

On the other hand, we have the object $\mathrm{cone}(\beta) \in \mathrm{D_b}(\mathsf{pfmod}(G))$ which we depicted in (1.16). To study $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$, we *divide and conquer* using the two objects P and $\mathrm{cone}(\beta)$ (Method 6.11). In particular, we partition $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$ as a set into

$$\mathrm{open}(\mathrm{cone}(\beta)) \sqcup \mathrm{supp}(P \otimes \mathrm{cone}(\beta)) \sqcup (\mathrm{open}(P) \cap \mathrm{supp}(\mathrm{cone}(\beta))). \tag{1.21}$$

Our second main theorem from Part II concerns the subspace

$$\mathrm{open}(\mathrm{cone}(\beta)) \sqcup \mathrm{supp}(P \otimes \mathrm{cone}(\beta)) = \mathrm{open}(\mathrm{cone}(\beta)) \cup \mathrm{supp}(P)$$

of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$. We determine it entirely.

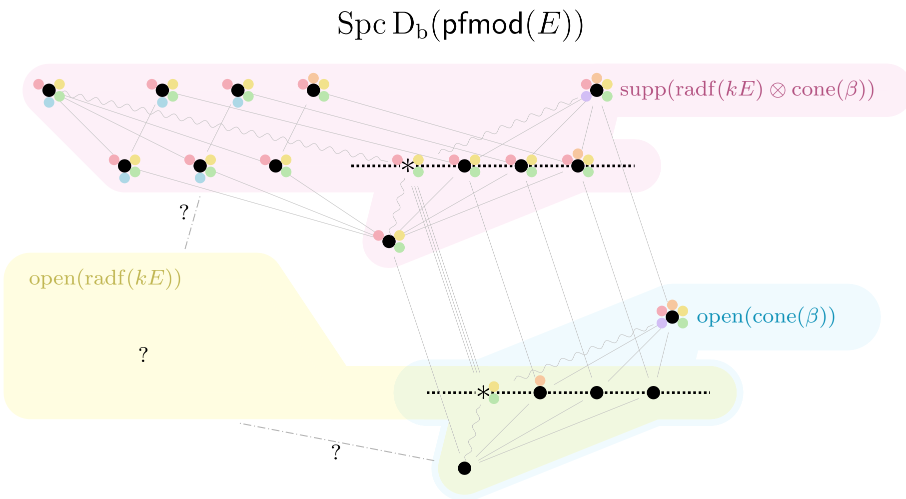
1.22 Main Theorem (Propositions 10.9, 12.11, 13.8, 13.10, and 13.12 and Theorem 11.5)**.** *Assume Notation 1.20. The subspace*

$$\mathrm{open}(\mathrm{cone}(\beta)) \cup \mathrm{supp}(P) \subset \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$$

is a copy of the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G))$ *[BG25a] along with a doubling (downward) of its cohomological open.*

In fact $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G))$ embeds as $\mathrm{supp}(P\otimes\mathrm{cone}(\beta))$, and $\mathrm{open}(\mathrm{cone}(\beta))$ is simply the doubled copy of its cohomological open. They embed respectively via the total associated graded functor $\mathrm{ung}\,\mathrm{gr}\colon \mathsf{pfmod}(G) \to \mathsf{pmod}(G)$ and via the unfiltering functor $\mathrm{unf}\colon \mathsf{pfmod}(G) \to \mathsf{mod}(G)$ (Remark 5.23). This subspace witnesses the conditions (i) and (ii).

For example, in our Running Example 1.13 where we may take $P = \mathrm{radf}(kE)$, we have the following picture:



- supp($\mathrm{gap}^1\,\mathrm{radf}(kE)$)
- supp($\mathrm{fz}\,k(E/N_{[0:1]})$)
- supp($A^{\mathrm{vert}}_{[a:b]}$)
- supp($\mathrm{fz}\,kE$)
- supp($\mathrm{fz}\,\mathrm{kos}_E(N_{[0:1]})$)
- supp($\mathrm{radf}(C_{[a:b]})$).

In this picture, we have highlighted in respectively cyan, magenta, and yellow the three regions $\mathrm{open}(\mathrm{cone}(\beta))$, $\mathrm{supp}(\mathrm{radf}(kE)\otimes\mathrm{cone}(\beta))$, and $\mathrm{open}(\mathrm{radf}(kE))$ (compare with (1.21)). Note that the magenta region, being the complement of

$$\mathrm{open}(\mathrm{radf}(kE)) \cup \mathrm{open}(\mathrm{cone}(\beta)) = \mathrm{open}(\mathrm{radf}(kE) \otimes \mathrm{cone}(\beta)),$$

must be disjoint from the cyan and yellow regions, but the cyan and yellow regions overlap to form the green region. (Contrary to popular belief, mixing blue and yellow does not make green; mixing cyan and yellow does.) We have also depicted with the colored dots around the points the supports of some of the objects from the previous subsection that conditions (i) and (ii) could not distinguish: in this region, $\mathrm{radf}(kE)$ and $\mathrm{gap}^1\,\mathrm{radf}(kE)$ have the same support, as do $A^{\mathrm{vert}}_{[a:b]}$ and $\mathrm{radf}(C_{[a:b]})$.

Our two main theorems in Part III concern the vanishing locus $\mathrm{open}(P)$ in the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$; this vanishing locus is the spectrum $\mathrm{Spc}\,\mathrm{st}(\mathsf{pfmod}(G))$ of the stable category of $\mathsf{pfmod}(G)$. Our key insight is that we should consider, for each normal subgroup $N \trianglelefteq G$, the relative subcategory $\mathsf{pfmod}(G,N) \subset \mathsf{pfmod}(G)$ of permutation filtered modules whose associated graded has trivial N-action (Proposition 14.9). When $N = 1$, we recover the original $\mathsf{pfmod}(G, 1) = \mathsf{pfmod}(G)$.

Having faith that everything will work out, we bravely set out to determine the spectra $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(G, N))$ for all $N \trianglelefteq G$, but we compromise by focusing solely on the elementary abelian case: $G = E$. Our strategy is to divide and conquer the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$ (Method 6.10), but this time we select the single object $\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)$ in $\mathrm{D_b}(\mathsf{pfmod}(E, N))$.

Our first main result in Part III concerns the support of $\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$. As we alluded to in the previous subsection, we have an associated graded functor

$$\mathrm{gr}\colon\ \mathsf{pfmod}(E, N) \longrightarrow \mathsf{pgmod}(E, N)$$

to a relative category $\mathsf{pgmod}(E, N)$ of permutation graded modules where the N-action has grading 1. Passing to their derived categories and then to their spectra, we show that it induces a homeomorphism between $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$ and the support of $\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)$. We then use a graded version of the modular fixed points functors (1.2) to study $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$.

1.23 Main Theorem (Proposition 18.14 and Theorem 19.13)**.** *The support*

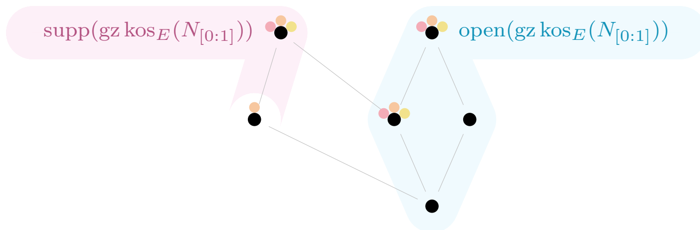
$$\operatorname{supp}(\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)) \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$$

is a copy of the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$*. A graded version of modular fixed points produces points in the spectrum* $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$*, including all of the closed points.*

One of the modular fixed points functors, namely the one for the subgroup $H = 1$, captures our idea from the previous subsection (for example (1.17) and (1.18)) that forgetting the grading on a graded module in $\mathsf{pgmod}(E, N)$ produces a kE-module (see Section 17).

For the rest of this introduction, we assume our conjecture that these graded versions of modular fixed points produce all of the points in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$ (Conjecture 18.15). For example, in our Running Example 1.13, we have the following picture:

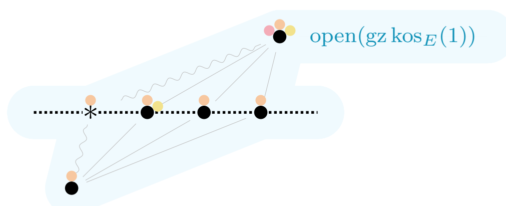
$$\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N_{[0:1]}))$$



• $\operatorname{supp}(\operatorname{gr} \operatorname{radf}(C_{[a:b]}))$ • $\operatorname{supp}(\operatorname{gr}(A^{\mathrm{vert}}_{[a:b]}))$ • $\operatorname{supp}(\operatorname{gr} \operatorname{radf} k(E/N_{[1:0]}))$

We see that the tt-category $\mathrm{D_b}(\mathsf{pgmod}(E, N_{[0:1]}))$ can distinguish $\operatorname{gr} \operatorname{radf}(C_{[a:b]})$ from $\operatorname{gr}(A^{\mathrm{vert}}_{[a:b]})$, even in the stable category whose spectrum is the complement of $\operatorname{supp}(\operatorname{gr} \operatorname{radf} k(E/N_{[1:0]}))$ which is depicted by the yellow markers in the picture. Similarly, for our running example we also have

$$\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, E))$$

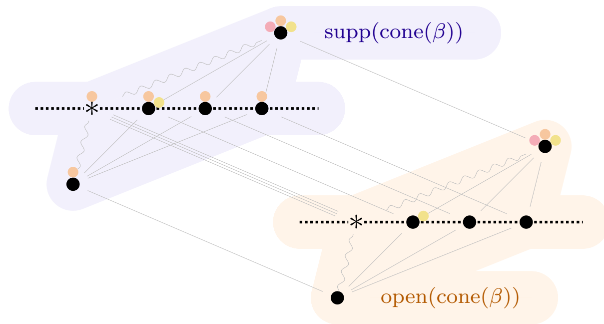


• $\operatorname{supp}(\operatorname{gr} \operatorname{radf}(kE))$ • $\operatorname{supp}(\operatorname{gr} \operatorname{gap}^1 \operatorname{radf}(kE))$ • $\operatorname{supp}(\operatorname{gr} \operatorname{radf} k(E/N_{[1:0]}))$

We see that $\mathrm{D_b}(\mathsf{pgmod}(E, E))$ can distinguish $\operatorname{gr} \operatorname{radf}(kE)$ from $\operatorname{gr} \operatorname{gap}^1 \operatorname{radf}(kE)$.

Our second main result in Part III concerns the vanishing locus of $\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$, namely the complement of the closed subset determined in Main Theorem 1.23. In the case $N = E$, this vanishing locus is simply $\operatorname{open}(\operatorname{cone}(\beta))$ which is as usual a copy of $\operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E))$, so we have already determined $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, E))$ (compare (1.7)) entirely by Main Theorem 1.23. For example, in our running example we have the following picture:

$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,E))$



● $\operatorname{supp}(\operatorname{radf}(kE))$ ● $\operatorname{supp}(\operatorname{gap}^1 \operatorname{radf}(kE))$ ● $\operatorname{supp}(\operatorname{radf} k(E/N_{[1:0]}))$

This kicks off the inductive process described in our final main theorem. For $N < L \leqslant E$, we denote by $\operatorname{incl}_L^N \colon \mathsf{pfmod}(E,L) \rightarrowtail \mathsf{pfmod}(E,N)$ the inclusion.

1.24 Main Theorem (Proposition 20.5 and Corollary 20.16)**.** *Let $N < E$. For each subgroup $L > N$, there exist open sets U_L and V_L, as depicted below, such that* $\operatorname{Spc}(\operatorname{incl}_L^N)$ *is a homeomorphism*

$$\begin{tikzcd}
\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,N)) \arrow[r, "\operatorname{Spc}(\operatorname{incl}_L^N)"] & \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,L)) \\
U_L \arrow[u, phantom, "\cup"] \arrow[r, "\operatorname{Spc}(\overline{\operatorname{incl}_L^N})", "\cong"'] & V_L \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

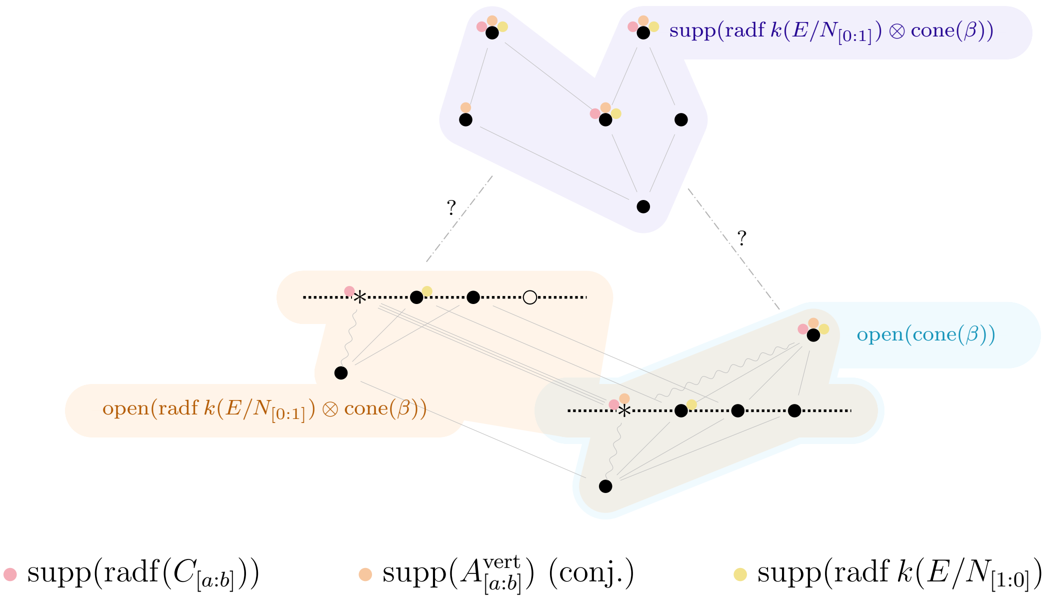
and such that the union of the U_L is the vanishing locus of $\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)$*:*

$$\operatorname{open}(\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)) \cong \operatorname*{colim}_{L>N} U_L \cong \operatorname*{colim}_{L>N} \operatorname{Spc}(\overline{\operatorname{incl}_L^N})^{-1}(V_L).$$

Thus, inductively assuming we have determined $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,L))$ for all $L > N$, this final main theorem asserts that the open sets V_L glue together to give us the vanishing locus of $\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,N))$.

For example, for the subgroup $N = N_{[0:1]}$ in our running example, Main Theorem 1.24 produces the following picture:

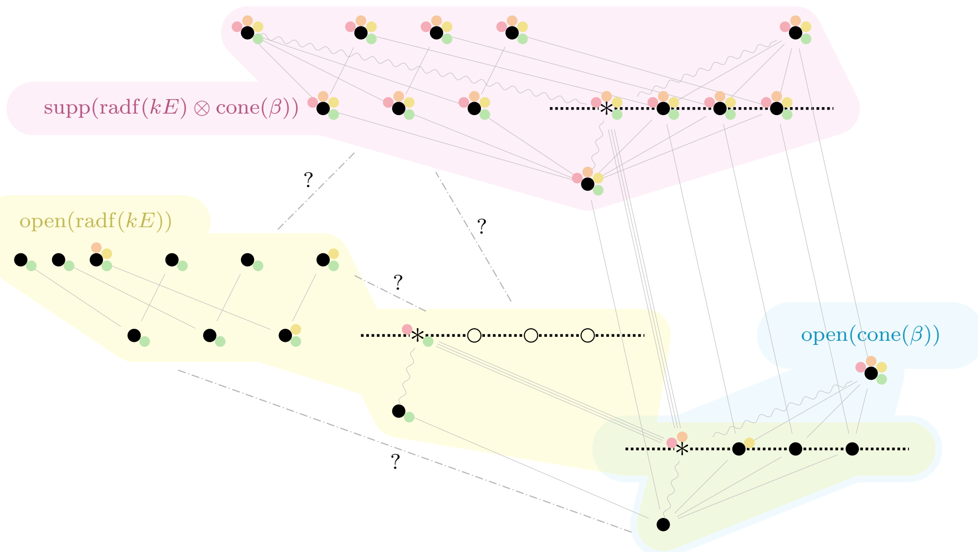
$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N_{[0:1]}))$



Here there is only one subgroup L strictly containing N, namely E, so the orange region comes straight from the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, E))$ and is precisely U_E. The purple region is the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N_{[0:1]}))$. The support of $A^{\mathrm{vert}}_{[a:b]}$ is known in the purple and cyan regions, but it is conjectural in the purely orange region. Nevertheless, this is enough for us to see that the tt-category $\mathrm{D_b}(\mathsf{pfmod}(E, N_{[0:1]}))$ can distinguish $\operatorname{radf}(C_{[a:b]})$ from $A^{\mathrm{vert}}_{[a:b]}$ because their supports differ in the purple region. The spectra $\mathrm{D_b}(\mathsf{pfmod}(E, N))$ for the other rank 1 subgroups $N_{[1:0]}$ and $N_{[1:1]}$ are the same but with the punctures instead at $[1:0]$ and $[1:1]$.

We have determined $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$ for the nontrivial subgroups $N > 1$, so Main Theorem 1.24 enables us to determine the vanishing locus $\operatorname{open}(\operatorname{radf}(kE))$ in the spectrum of $\mathrm{D_b}(\mathsf{pfmod}(E)) = \mathrm{D_b}(\mathsf{pfmod}(E, 1))$:

$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$



● $\operatorname{supp}(\operatorname{radf}(C_{[a:b]}))$ ● $\operatorname{supp}(A^{\mathrm{vert}}_{[a:b]})$ (conj.) ● $\operatorname{supp}(\operatorname{radf} k(E/N_{[1:0]}))$

● $\operatorname{supp}(\operatorname{gap}^1 \operatorname{radf}(kE))$.

This time, we have three open sets U_N corresponding to the three rank 1 subgroups N of E, and they are glued together along U_E to form the yellow region. The picture at the start of this introduction highlights $U_{N_{[0:1]}}$ and U_E more clearly. In the present picture, the open set U_E is the three-punctured projective line in the purely yellow region together with the cyan region.

The nine additional points at the top of the yellow region are the points picked up from the rank 1 subgroups N and come from the stable categories $\operatorname{st}(\mathsf{pgmod}(E, N))$, one triplet from each of these rank 1 subgroups N. The support of $A^{\mathrm{vert}}_{[a:b]}$ is known in the triplet coming from $\operatorname{st}(\mathsf{pgmod}(E, N_{[0:1]}))$, but it is conjectural in the two triplets coming from $\operatorname{st}(\mathsf{pgmod}(E, N_{[1:0]}))$ and $\operatorname{st}(\mathsf{pgmod}(E, N_{[1:1]}))$. Nevertheless, again this is enough for us to see that the tt-category $\mathrm{D_b}(\mathsf{pfmod}(E))$ can distinguish between $\operatorname{radf}(C_{[a:b]})$ and $A^{\mathrm{vert}}_{[a:b]}$.

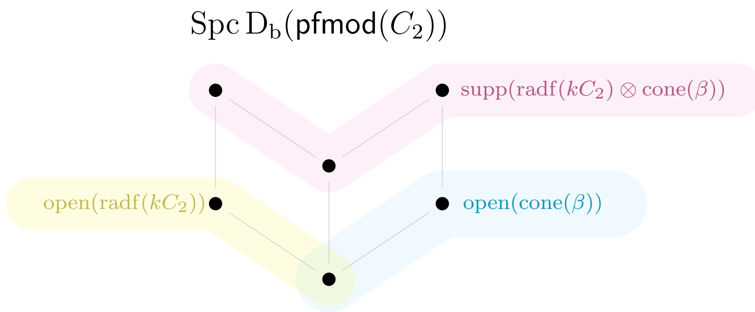
Recapitulation

In this subsection we take a step back in order to highlight why our main results are so surprising.

Several works have computed the spectra of tt-categories of filtered objects of some sort. Most generally, given a big tt-category $\mathcal{T}$, Aoki [Aok23] computed the spectrum of the tt-category $\mathrm{Fun}(\mathbb{Z}, \mathcal{T})^c$: it is a doubling (Terminology 12.8) of the spectrum $\mathrm{Spc}(\mathcal{T}^c)$. With Tate motives in mind (recall (1.7)), Gallauer [Gal18] computed the spectrum of perfect complexes of filtered modules over a ring T: it is a doubling of the Zariski spectrum $\mathrm{Spec}(T)$. Even in this work (Section 12), we compute the spectrum of the derived category $\mathrm{D_b}(\mathsf{fmod}(G))$ of filtered kG-modules where the tensor is over k: it is yet again a doubling, this time of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G))$.

These doublings are perhaps expected. They amount to saying that having tt-equivalent associated graded objects and underlying objects (conditions (i) and (ii)) are not only necessary for tt-equivalence of objects but also sufficient. Since in these cases the associated graded objects and underlying objects live in the same categories, a doubling is the most natural guess for the spectrum, both as a set and as a topological space.

Once we move to permutation filtered modules, one may doubt whether this pattern continues to hold. After all, the associated graded objects live in $\mathrm{D_b}(\mathsf{pmod}(G))$ whereas the underlying objects live in $\mathrm{D_b}(\mathsf{mod}(G))$, which is only homeomorphic to the cohomological open in $\mathrm{D_b}(\mathsf{pmod}(G))$. However, recall (from (1.9)) that Balmer–Gallauer computed the spectrum of $\mathrm{D_b}(\mathsf{pfmod}(C_2))$ with Artin-Tate motives (1.8) in mind and found that it is yet again a doubling:



Even though the associated graded objects live in $\mathrm{D_b}(\mathsf{pmod}(C_2))$ which has a three-point spectrum and the underlying objects live in $\mathrm{D_b}(\mathsf{mod}(C_2))$ which has a two-point spectrum, Balmer–Gallauer found a sixth point which completed the doubling. This sixth point in the purely yellow region was elusive, and pinning it down required intricate ideas, many of which were specific to the case $G = C_2$.

Our main results show that this doubling pattern generalizes to C_p for odd p but breaks down for higher rank elementary abelian groups. Indeed, $\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(E))$ embeds in the spectrum $\operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(E))$ as the closed support $\operatorname{supp}(\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta))$, but it has no hope of being doubled for dimension reasons. On the one hand, the cohomological open in $\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(E))$ has Krull dimension $t-1$, and its complement, the support of the acyclics, consists of pieces that have strictly smaller Krull dimension. On the other hand, already the open set U_E in $\operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(E))$ has two pieces with Krull dimension $t-1$. One of them doubles the cohomological open, but the other is the complement of the rational hyperplanes in $\mathbb{P}^{t-1}_k$ and cannot possibly double the support of the acyclics. We can see this in our running example, where there is a doubled copy of $\mathbb{P}^1_k$ but then an additional three-punctured one.

Outline

We wind down this introduction by giving an outline of this work. It is divided into three parts.

Part I lays the foundation for both Parts II and III. In Section 2 we fix and clarify terminology. In Section 3 we discuss rigid tensor categories of modules over Hopf monoids, and in Section 5 we introduce and study the notion of pulling back exact structures. The motivation is that ultimately we express $\mathsf{pfmod}(G)$ and $\mathsf{pfmod}(G, N)$ (and our other categories) as certain pullbacks

$$\begin{tikzcd} \mathsf{pfmod}(G) \arrow[r, two heads, "\mathrm{gr}"] \arrow[d, tail] & \mathsf{pgmod}(G) \arrow[d, tail] \\ \mathsf{fmod}(G) \arrow[r, two heads, "\mathrm{gr}"] & \mathsf{gmod}(G) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{pfmod}(G, N) \arrow[r, two heads, "\mathrm{Res}^{\sigma}_{G/N}"] \arrow[d, tail] & \mathsf{pfmod}(G/N) \arrow[d, tail] \\ \mathsf{fmod}(G, N) \arrow[r, two heads, "\mathrm{Res}^{\sigma}_{G/N}"] & \mathsf{fmod}(G/N) \end{tikzcd} \tag{1.25}$$

of exact rigid tensor categories, where $\mathsf{fmod}(G)$, $\mathsf{gmod}(G)$, and $\mathsf{gmod}(G, N)$ are categories of modules over Hopf monoids in certain ambient categories.

Sections 4, 6, 7, and 8 in Part I are more optional. In Section 4 we record some computations that we use repeatedly throughout this work, and in Section 6 we gather tt-geometry recollections and put a spin on some of them. In Section 7 we exposit a theory of finite resolutions that is used in Sections 13 and 15, and in Section 8 we present a proof of a general tt-geometry fact that is crucial to Section 20.

Part II determines the subspace $\mathrm{open}(\mathrm{cone}(\beta)) \cup \mathrm{supp}(P)$ (Main Theorem 1.22). In Section 9 we prove that such a P exists in the elementary abelian case (Main Theorem 1.19). In Section 10 we determine the vanishing locus $\mathrm{open}(\mathrm{cone}(\beta))$ by expressing the Verdier quotient $\mathrm{D_b}(\mathsf{pfmod}(G))/\langle \mathrm{cone}(\beta)\rangle$ as a central localization, and in Section 11 we determine $\mathrm{supp}(P \otimes \mathrm{cone}(\beta))$ by proving a detection of nilpotence result for the total associated graded functor

$$\mathrm{ung}\,\mathrm{gr}\colon\ \mathrm{D_b}(\mathsf{pfmod}(G)) \longrightarrow \mathrm{D_b}(\mathsf{pmod}(G)).$$

The remainder of Part II is dedicated to determining the specializations from $\mathrm{open}(\mathrm{cone}(\beta))$ to $\mathrm{supp}(P \otimes \mathrm{cone}(\beta))$. In Section 12 we compute the spectrum of $\mathrm{D_b}(\mathsf{fmod}(G))$ entirely using a filtered version of π-points. In Section 13 we embed this spectrum in $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$ using the theory from Section 7, and once we do this, the specializations follow easily.

Part III studies the vanishing locus $\mathrm{open}(P)$, mainly in the elementary abelian case. In Section 14 we introduce and study the relative subcategories in general, and then in

Section 15 we specialize to the elementary abelian case, where we have full control over the projectives and where we find certain favorable decompositions. In Section 16 we briefly describe the spectrum of $\mathrm{D_b}(\mathsf{mod}(E,N))$ using π-points, and we use this in Section 17 to study the spectrum of $\mathrm{D_b}(\mathsf{gmod}(E,N))$ by studying the surjection

$$\mathrm{Spc}(\mathrm{ung})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(E,N)) \longrightarrow\!\!\!\!\rightarrow \mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}(E,N))$$

induced by forgetting gradings. In Section 18 we study $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}(E,N))$ by introducing our graded version of modular fixed points, and in Section 19 we determine the spectrum of $\mathrm{D_b}(\mathsf{pfmod}(E,N))$ in terms of other spectra and find that it contains a copy of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}(E,N))$ (Main Theorem 1.23). Finally, in Section 20 we establish our reduction theorem to the relative subcategories (Main Theorem 1.24).

Open questions

We conclude this introduction by recording some open questions. Let us start with the most specific ones and zoom out from there.

(1) In Section 17 we get a vivid picture of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}(E,N))$, but we do not get a complete answer. We make conjectures in Conjecture 17.23 about what the complete answer is and a path toward it. Are these conjectures true?

(2) What is $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(E,N))$? It is some open subset of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,N))$ (Proposition 19.6), but it is unclear what its intersection with $\mathrm{open}(\mathrm{radf}\,k(E/N))$ is (this is a subproblem of (3)). In trying to determine it directly, the analogous version of Proposition 15.25 and hence of Section 20 fails.

(3) Leading up to Running Example 20.18, we determine in an ad-hoc manner many of the specializations in $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E))$ that involve the yellow region $\mathrm{open}(\mathrm{radf}(kE))$, and we conjecture the remaining ones in Remark 20.19. Are these conjectures true? More generally,

what are the specializations in $\operatorname{Spc} \mathrm{D}_{\mathrm{b}}(\mathsf{pfmod}(E))$ that involve the spectrum of the stable category?

(4) In this introduction we reduced to the case of a p-group G ((1.11) and (1.12)), and much of this work focuses on the case of an elementary abelian p-group E. How can we reduce from the case G to the case E?

(5) What is the appropriate big tt-category for $\mathrm{D}_{\mathrm{b}}(\mathsf{pfmod}(G))$?

1.26 Convention. In this work, we use $\simeq$ to emphasize non-canonical isomorphisms. We typically use A, B, and C respectively to denote filtered, graded, and plain modules and f, g, and h respectively for morphisms between them.

Part I

Categorical properties

2 Terminology

In this short section, we fix our terminology, especially regarding rigid categories.

2.1 Notation. For this work, let p be a prime number, let k be a field of characteristic p, let G be a p-group, and let E be an elementary abelian p-group.

2.2 Terminology.

$$\begin{tikzcd}
\text{rigid} \arrow[d, dash] & & \text{Frobenius exact} \arrow[d, dash] \\
\text{closed} \arrow[d, dash] & \text{tensor} \arrow[dl, dash] \arrow[dr, dash] & \text{exact} \arrow[d, dash] \\
\text{symmetric monoidal} & & \text{additive}
\end{tikzcd}$$

On the left-hand side, we use the standard notion of a *symmetric monoidal category.* A *closed category* is a symmetric monoidal category with an internal hom $[-,-]\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, and a *rigid category* is a symmetric monoidal category with a dualizing functor $(-)^\vee\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that for all $x \in \mathcal{C}$, the functors

$$\begin{tikzcd}
\mathcal{C} \arrow[d, shift right, "x\otimes-"'] \\
\mathcal{C} \arrow[u, shift right, "x^\vee\otimes-"']
\end{tikzcd} \tag{2.3}$$

are *tensor adjoint*, by which we mean that they are adjoint and that their unit $\eta^{(x)}\colon 1 \Rightarrow x^\vee \otimes x \otimes (-)$ and counit $\varepsilon^{(x)}\colon x \otimes x^\vee \otimes (-) \Rightarrow 1$ satisfy

$$\eta^{(x)}_y = \eta^{(x)}_{\mathbb{1}} \otimes y \quad \text{and} \quad \varepsilon^{(x)}_y = \varepsilon^{(x)}_{\mathbb{1}} \otimes y \tag{2.4}$$

for all objects $y \in \mathcal{C}$.

On the right-hand side, we use the standard notion of an *additive category.* An *exact category* is an additive category with an exact structure, in the sense of [Buh10, Definition 2.1], and a *Frobenius exact category* is an exact category with enough projectives and where the projectives and injectives coincide.

By a *tensor category* we mean a symmetric monoidal additive category $\mathcal{C}$ where the tensor $-\otimes-\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is additive in both variables, and by a *tensor functor* we mean a strong unital symmetric monoidal additive functor. When we combine adjectives, we demand that the structures are cross-compatible. For example, in a *closed tensor category* the internal hom is additive in both variables; in a *rigid tensor category* the dualizing functor is additive; and in an *exact rigid tensor category* the tensor is exact in both variables and the dualizing functor is exact.

When we introduce a functor with an adjective, we mean that it is a functor between categories with those adjectives. For example, by an exact tensor functor we mean an exact tensor functor between exact tensor categories.

Let us clarify our notion of a rigid category.

2.5 Lemma. *Let $\mathcal{C}$ be a rigid tensor category. Then every object in $\mathcal{C}$ is reflexive, so $(-)^\vee$ is a duality of tensor categories.*

Proof. We will show that for any $x \in \mathcal{C}$, the functor $x^\vee \otimes -\colon \mathcal{C} \to \mathcal{C}$ is not only a right adjoint of $x \otimes -\colon \mathcal{C} \to \mathcal{C}$ but also a left adjoint. Then it follows by Yoneda that $x \cong (x^\vee)^\vee$, and one verifies that the isomorphism is given by the canonical map $x \to (x^\vee)^\vee$ and that everything is natural.

For $x \in \mathcal{C}$, let $\eta^{(x)}\colon 1 \Rightarrow x^\vee \otimes x \otimes -$ and $\varepsilon^{(x)}\colon x \otimes x^\vee \otimes - \Rightarrow 1$ denote the unit and counit of the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "x\otimes -"'] \\ \mathcal{C}. \arrow[u, shift right, "x^\vee\otimes -"'] \end{tikzcd}$$

Then for any $y \in \mathcal{C}$, the following compositions are the identities:

$$x \otimes y \xrightarrow{x\otimes\eta^{(x)}_y} x \otimes x^\vee \otimes x \otimes y \xrightarrow{\varepsilon^{(x)}_{x\otimes y}} x \otimes y$$

and

$$x^\vee \otimes y \xrightarrow{\eta^{(x)}_{x^\vee\otimes y}} x^\vee \otimes x \otimes x^\vee \otimes y \xrightarrow{x^\vee\otimes\varepsilon^{(x)}_y} x^\vee \otimes y.$$

We claim that $\alpha^{(x)}\colon 1 \Rightarrow x\otimes x^\vee \otimes (-)$ and $\beta^{(x)}\colon x^\vee\otimes x\otimes - \Rightarrow 1$ defined by

$$\begin{aligned}
\alpha^{(x)}_y\colon y &\xrightarrow{\eta^{(x)}_y} x^\vee\otimes x\otimes y \xrightarrow{(1\,2)} x\otimes x^\vee\otimes y\\
\beta^{(x)}_y\colon x^\vee\otimes x\otimes y &\xrightarrow{(1\,2)} x\otimes x^\vee\otimes y \xrightarrow{\varepsilon^{(x)}_y} y
\end{aligned}$$

satisfy the unit-counit relations for $x^\vee\otimes -$ and $x\otimes -$. This follows from the following commutative diagrams:

$$\begin{tikzcd}
x^\vee\otimes y \arrow[rr, "x^\vee\otimes\alpha^{(x)}_y=\eta^{(x)}_{x^\vee\otimes y}"] \arrow[dr, "x^\vee\otimes\eta^{(x)}_y"'] & & x^\vee\otimes x\otimes x^\vee\otimes y \arrow[rr, "\beta^{(x)}_{x^\vee\otimes y}=x^\vee\otimes\varepsilon^{(x)}_y"] \arrow[dr, "(1\,2)"'] & & x^\vee\otimes y \\
& x^\vee\otimes x^\vee\otimes x\otimes y \arrow[ur, "(2\,3)"'] \arrow[rr, "(1\,2\,3)"'] & & x\otimes x^\vee\otimes x^\vee\otimes y \arrow[ur, "\varepsilon^{(x)}_{x^\vee\otimes y}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
x\otimes y \arrow[rr, "\alpha^{(x)}_{x\otimes y}=x\otimes\eta^{(x)}_y"] \arrow[dr, "\eta^{(x)}_{x\otimes y}"'] & & x\otimes x^\vee\otimes x\otimes y \arrow[rr, "x\otimes\beta^{(x)}_y=\varepsilon^{(x)}_{x\otimes y}"] \arrow[dr, "(2\,3)"'] & & x\otimes y \\
& x^\vee\otimes x\otimes x\otimes y \arrow[ur, "(1\,2)"'] \arrow[rr, "(1\,2\,3)"'] & & x\otimes x\otimes x^\vee\otimes y. \arrow[ur, "x\otimes\varepsilon^{(x)}_y"'] &
\end{tikzcd}$$

The four outer triangles commute by our hypothesis that $x\otimes -$ and $x^\vee\otimes -$ are tensor adjoint (recall (2.4)). ☐

2.6 Remark. A rigid category $\mathcal{C}$ is closed because $[x,y]=x^\vee\otimes y$ defines an internal hom. Given an object $x\in\mathcal{C}$, we denote by $\mathrm{ev}_x=\varepsilon^{(x)}_{\mathbb{1}}\colon x^\vee\otimes x\to\mathbb{1}$ and $\mathrm{coev}_x=\eta^{(x)}_{\mathbb{1}}\colon \mathbb{1}\to x\otimes x^\vee$ the *evaluation* and *coevaluation* morphisms of x, which are given by

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(x^\vee,x^\vee)\cong\mathrm{Hom}_{\mathcal{C}}(x^\vee\otimes x,\mathbb{1}) & & \mathrm{Hom}_{\mathcal{C}}(x,x)\cong\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},x\otimes x^\vee).\\
1\longmapsto \mathrm{ev}_x & \text{and} & 1\longmapsto \mathrm{coev}_x
\end{array}$$

Note that $x=0$ if and only if $\mathrm{ev}_x=0$ if and only if $\mathrm{coev}_x=0$.

2.7 Remark. In a closed category $\mathcal{C}$, we may define a dualizing functor $(-)^\vee\colon \mathcal{C}^{\mathrm{op}}\to\mathcal{C}$ by setting $x^\vee=[x,\mathbb{1}]$. Let us unravel what it means for $\mathcal{C}$ to be a rigid category under this dualizing functor. Let $x\in\mathcal{C}$ be an object, and consider the natural transformation $x^\vee\otimes -\Rightarrow [x,-]$ given by $\varepsilon^{(x)}_{\mathbb{1}}\otimes y$ under the adjunction

$$\mathrm{Hom}(x^\vee\otimes y,[x,y])\cong\mathrm{Hom}(x^\vee\otimes x\otimes y,y).$$

(i) The functors $x \otimes -$ and $x^\vee \otimes -$ are adjoint if and only if the natural transformation $x^\vee \otimes - \Rightarrow [x, -]$ is an isomorphism. Indeed, this is by uniqueness of adjoints. In this case the unit and counit of the adjunction are respectively $\eta^{(x)}_{\mathbb{1}} \otimes (-)$ and $\varepsilon^{(x)}_{\mathbb{1}} \otimes (-)$.

(ii) If $x \otimes -$ and $x^\vee \otimes -$ are adjoint, then they are tensor adjoint (recall (2.4)) if and only if $\eta^{(x)}_{(-)}$ and $\varepsilon^{(x)}_{(-)}$ are the unit and counit of the adjunction. Indeed, this is by uniqueness of units and counits.

2.8 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a conservative closed tensor functor. If $\mathcal{D}$ is rigid, then $\mathcal{C}$ is rigid.*

Proof. Suppose $\mathcal{D}$ is rigid, and let $x \in \mathcal{C}$ be an object.

To see that $x\otimes-$ and $x^\vee\otimes-$ are adjoint, it suffices to show that the natural transformation $x^\vee \otimes - \Rightarrow [x, -]$ is an isomorphism (Remark 2.7(i)). Our functor F takes this natural transformation to $F(x)^\vee \otimes - \Rightarrow [F(x), -]$, where $F(x^\vee) \cong F(x)^\vee$ because F is closed. Since $\mathcal{D}$ is rigid, this natural transformation is an isomorphism (Remark 2.7(i)), so since F is conservative, $x^\vee \otimes - \Rightarrow [x, -]$ is an isomorphism.

It remains to show that $x\otimes-$ and $x^\vee\otimes-$ are in fact tensor adjoint (recall (2.4)). Consider the natural transformations $\eta^{(x)}_{(-)}$ and $\varepsilon^{(x)}_{(-)}$. Our functor F takes these to $F(\eta^{(x)}_{(-)}) \cong \eta^{(F(x))}_{F(-)}$ and $\varepsilon^{(F(x))}_{F(-)}$, which are the unit and counit of the adjunction $F(x) \otimes -$ and $F(x)^\vee \otimes -$ (Remark 2.7(ii)). By conservativity of F, we deduce that $\eta^{(x)}_{(-)}$ and $\varepsilon^{(x)}_{(-)}$ are the unit and counit of the adjunction $x \otimes -$ and $x^\vee \otimes -$. Thus $x \otimes -$ and $x^\vee \otimes -$ are tensor adjoint (Remark 2.7(ii)). $\square$

2.9 Remark. In Lemma 2.8, the converse holds when F is essentially surjective.

We conclude this section by recording the following lemma, which we will use repeatedly throughout this work. Recall that a fully exact subcategory $\mathcal{A} \subset \mathcal{C}$ of an exact category $\mathcal{C}$ is an extension-closed full additive subcategory of $\mathcal{C}$ equipped with the exact structure where a kernel-cokernel pair in $\mathcal{A}$ is short exact if and only if it is short exact in $\mathcal{C}$ [Buh10, Definition 10.21].

2.10 Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be exact rigid tensor categories, and let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact tensor functor. Let $\mathcal{B}$ be a fully exact rigid tensor subcategory of $\mathcal{D}$, and let $J\colon \mathcal{B} \rightarrowtail \mathcal{D}$ denote the inclusion. The pullback, by which we mean the full subcategory $\mathcal{A}$ of objects $C \in \mathcal{C}$ such that $F(C)$ is isomorphic to an object in $\mathcal{B}$, is a fully exact rigid tensor subcategory of $\mathcal{C}$. The diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F_0"] \arrow[d, tail, "I"] & \mathcal{B} \arrow[d, tail, "J"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

is a pullback diagram in the category of exact rigid tensor categories and exact tensor functors. The analogous statements also hold when $\mathcal{C}$ and $\mathcal{D}$ are only exact tensor, exact, tensor, or additive.

Proof. This is straightforward. $\square$

3 Rigid tensor categories

In this section we study rigid tensor categories. In particular, we construct in Definition 3.40 the following diagram of pullbacks of rigid tensor categories upon which Part I and Part II are largely based:

$$\begin{tikzcd} \mathsf{pfmod}(G) \arrow[r, two heads, "\mathrm{gr}"] \arrow[d, tail] & \mathsf{pgmod}(G) \arrow[r, two heads, "\mathrm{ung}"] \arrow[d, tail] & \mathsf{pmod}(G) \arrow[d, tail] \\ \mathsf{fmod}(G) \arrow[r, two heads, "\mathrm{gr}"] & \mathsf{gmod}(G) \arrow[r, two heads, "\mathrm{ung}"] & \mathsf{mod}(G). \end{tikzcd}$$

To read this diagram, we recall that G is a p-group (Notation 2.1).

We start with our ambient categories.

3.1 Notation. We denote by vect the category of finite-dimensional k-vector spaces.

3.2 Notation. We denote by gvect the category of graded finite-dimensional k-vector spaces. An object $B \in \mathsf{gvect}$ consists of a finite-dimensional k-vector space B^w for each $w \in \mathbb{Z}$, where

$B^w = 0$ for all but finitely many w. A morphism $g \colon B_1 \to B_2$ is a k-linear map $g^w \colon B_1^w \to B_2^w$ for all $w \in \mathbb{Z}$.

3.3 Notation. We denote by fvect the category of exhaustive separated filtrations of finite-dimensional k-vector spaces. An object $A \in \mathsf{fvect}$ consists of a finite-dimensional k-vector space $A^{-\infty}$ along with subspaces $A^w \subset A^{-\infty}$ for each $w \in \mathbb{Z}$ such that $A^{w+1} \subset A^w$; because $A^{-\infty}$ is finite-dimensional, exhaustive means $A^w = A^{-\infty}$ for sufficiently small w and separated means $A^w = 0$ for sufficiently large w. A morphism $f \colon A_1 \to A_2$ is a morphism $f^{-\infty} \colon A_1^{-\infty} \to A_2^{-\infty}$ in vect such that $f^{-\infty}(A_1^w) \subset A_2^w$ for all $w \in \mathbb{Z}$.

Let us discuss the basic functors between our ambient categories:

$$\mathsf{vect} \underset{\mathrm{unf}}{\overset{\mathrm{fz}}{\rightleftarrows}} \mathsf{fvect} \underset{\mathrm{trf}}{\overset{\mathrm{gr}}{\rightleftarrows}} \mathsf{gvect} \underset{\mathrm{gz}}{\overset{\mathrm{ung}}{\rightleftarrows}} \mathsf{vect}. \tag{3.4}$$

3.5 Notation. We denote by $\mathrm{gr} \colon \mathsf{fvect} \to \mathsf{gvect}$ the *associated graded* functor

$$\mathrm{gr}(A)^w = A^w / A^{w+1}$$

and by $\mathrm{ung} \colon \mathsf{gvect} \to \mathsf{vect}$ and $\mathrm{unf} \colon \mathsf{fvect} \to \mathsf{vect}$ the forgetful functors

$$\mathrm{ung}(B) = \bigoplus_{w \in \mathbb{Z}} B^w \qquad \text{and} \qquad \mathrm{unf}(A) = A^{-\infty} = \bigcup_{w \in \mathbb{Z}} A^w,$$

which we call the *ungrading* and *unfiltering* functors. We denote by $\mathrm{gz} \colon \mathsf{vect} \to \mathsf{gvect}$ the section of $\mathrm{ung} \colon \mathsf{gvect} \to \mathsf{vect}$ that puts an object in grading zero:

$$\mathrm{gz}(C)^w = \begin{cases} C & w = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we denote by $\mathrm{fz} \colon \mathsf{vect} \to \mathsf{fvect}$ the section of $\mathrm{unf} \colon \mathsf{fvect} \to \mathsf{vect}$ and $\mathrm{ung}\,\mathrm{gr} \colon \mathsf{fvect} \to \mathsf{vect}$ that puts an object in filtration zero:

$$\mathrm{fz}(C)^w = \begin{cases} C & w \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we denote by $\operatorname{trf}\colon \mathsf{gvect} \to \mathsf{fvect}$ the section of gr that trivially filters a graded object:

$$\operatorname{trf}(B)^w = \bigoplus_{v \geq w} B^v.$$

3.6 Remark. In the diagram (3.4), everything that can reasonably be a section is a section: gz is a section of ung, trf is a section of gr, fz is a section of unf and ung gr, and trf gz is a section of unf. No other paths commute.

3.7 Recollection. Our ambient categories vect, gvect, fvect are all tensor categories. The tensor in vect is of course over k. The tensor in gvect is induced by Day convolution:

$$(B_1 \otimes B_2)^w = \bigoplus_{i \in \mathbb{Z}} B_1^i \otimes B_2^{w-i}.$$

The tensor in fvect is also induced by Day convolution:

$$(A_1 \otimes A_2)^w = \sum_{i \in \mathbb{Z}} A_1^i \otimes A_2^{w-i} \subset \operatorname{unf}(A_1) \otimes \operatorname{unf}(A_2).$$

The functors fz, gr, ung, unf, trf, and gz in (3.4) are all tensor functors, so in particular the unit in gvect is $\mathbb{1} = \operatorname{gz} \mathbb{1}$ and the unit in fvect is $\mathbb{1} = \operatorname{fz} \mathbb{1}$.

We now discuss twists and the morphisms between them.

3.8 Notation. We can twist objects in gvect and fvect. For $w \in \mathbb{Z}$, the *twist* $B(w) \in \mathsf{gvect}$ of an object $B \in \mathsf{gvect}$ is defined by $B(w)^v = B^{v-w}$. Similarly, for $w \in \mathbb{Z}$, the *twist* $A(w) \in \mathsf{fvect}$ of an object $A \in \mathsf{fvect}$ is defined by $A(w)^v = A^{v-w}$.

3.9 Notation. For any filtered object $A \in \mathsf{fvect}$ and integers $v < w$, we denote by $\beta\colon A(v) \to A(w)$ the morphism in fvect whose underlying morphism $\operatorname{unf}(\beta)\colon \operatorname{unf}(A) \to \operatorname{unf}(A)$ is the identity. By default β means $\beta\colon \mathbb{1} \to \mathbb{1}(1)$.

3.10 Remark. In fvect, there are no interesting canonical maps to smaller twists; for example, if $v > w$, then the only morphism $\mathbb{1}(v) \to \mathbb{1}(w)$ is zero. In gvect, there are no interesting canonical morphisms between different twists; for example, if $v \neq w$, then the only morphism $\mathbb{1}(v) \to \mathbb{1}(w)$ in gvect is zero. This in fact shows that none of the five pairs of functors depicted in (3.4) that can be adjoint are adjoint.

3.11 Remark. In fvect, a morphism f is a mono (resp. epi) if and only if $\operatorname{unf}(f)$ is a mono (resp. epi). For example, β is a mono and epi, but it is certainly not an isomorphism. In particular, fvect is not abelian.

3.12 Lemma. *Let $f\colon A_1 \to A_2$ be a morphism in fvect. The following are equivalent:*

(i) *We have $\operatorname{gr}(f) = 0$ in gvect.*

(ii) *The morphism f factors through $\beta\colon A_2(-1) \to A_2$:*

$$\begin{tikzcd} & A_2(-1) \arrow[d, "\beta"] \\ A_1 \arrow[ur, dotted] \arrow[r, "f"'] & A_2. \end{tikzcd}$$

(iii) *The morphism f factors through $\beta\colon A_1 \to A_1(1)$:*

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "\beta"'] & A_1 \\ A_1(1). \arrow[ur, dotted] & \end{tikzcd}$$

Thus we may simply say that f factors through β.

Proof. Condition (i) is equivalent to having $\operatorname{unf}(f)(A_1^w) \subset A_2^{w+1}$ for all $w \in \mathbb{Z}$, so we have $\operatorname{unf}(f)(A_1^w) \subset A_2(-1)^w$ and $\operatorname{unf}(f)(A_1(1)^w) \subset A_2^w$ for all $w \in \mathbb{Z}$, which give (ii) and (iii). Conversely, $\operatorname{gr}(\beta) = 0$. $\square$

It will be useful to know which forgetful functors are conservative and/or faithful.

3.13 Lemma. *Associated graded* $\operatorname{gr}\colon \mathsf{fvect} \to \mathsf{gvect}$ *is conservative but not faithful. Ungrading* $\operatorname{ung}\colon \mathsf{gvect} \to \mathsf{vect}$ *is conservative and faithful. Unfiltering* $\operatorname{unf}\colon \mathsf{fvect} \to \mathsf{vect}$ *is not conservative but is faithful.*

Proof. The functor gr is conservative by the five lemma and induction on filtration length, but it is not faithful because $\operatorname{gr}(\beta) = 0$ and $\beta \neq 0$. The functor ung is clearly conservative and faithful. The functor unf is not conservative because $\operatorname{unf}(\beta)$ is an isomorphism but β

is not (Remark 3.11), but it is faithful because morphisms in fvect are determined on unf (Notation 3.3). $\square$

3.14 Remark. Although unf is not conservative in general, it is conservative on endomorphisms: if $f\colon A \to A$ is an endomorphism in fvect and $\mathrm{unf}(f)$ is an isomorphism, then f is an isomorphism. Indeed, from $f^w(A^w) = \mathrm{unf}(f)(A^w) \subset A^w$ and $\mathrm{unf}(f)$ being injective, we deduce that f^w is an isomorphism for all $w \in \mathbb{Z}$, whence f is an isomorphism.

Our ambient categories are closed, and we use this to deduce that they are rigid.

3.15 Lemma. *Let $A_1, A_2 \in \mathsf{fvect}$. For sufficiently large $w \in \mathbb{Z}$, the functor* unf *induces an isomorphism*

$$\mathrm{Hom}_{\mathsf{fvect}}(A_1, A_2(w)) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{vect}}(\mathrm{unf}(A_1), \mathrm{unf}(A_2)).$$

Proof. This is due to finiteness of our filtrations. Spelling it out, the depicted map is an injection because unf is faithful (Lemma 3.13), so let us show it is surjective. By finiteness of our filtrations, there exists a large $w_1 \in \mathbb{Z}$ such that $A_1^{w_1} = 0$, as well as a small $w_2 \in \mathbb{Z}$ such that $A_2^{w_2} = \mathrm{unf}(A_2)$. Take $w = w_1 - w_2$. Now, if $A_1^v \neq 0$ for some $v \in \mathbb{Z}$, then $v \leq w_1$ by definition of w_1, so

$$A_2(w)^v = A_2^{v-w} \supset A_2^{w_1-w} = A_2^{w_2} = \mathrm{unf}(A_2)$$

by the definitions of w_1 and w_2. Thus trivially any morphism $h\colon \mathrm{unf}(A_1) \to \mathrm{unf}(A_2)$ in vect induces a morphism $A_1 \to A_2(w)$ because $h(A_1^v) \subset \mathrm{unf}(A_2) = A_2(w)^v$ whenever $A_1^v \neq 0$. $\square$

3.16 Recollection. Our ambient categories vect, gvect, and fvect and the functors fz, gr, ung, unf, trf, and gz in (3.4) are all closed. The internal homs in our ambient categories are

$$\begin{aligned} [A_1, A_2]^w &= \mathrm{Hom}_{\mathsf{fvect}}(A_1, A_2(-w)) \\ [B_1, B_2]^w &= \mathrm{Hom}_{\mathsf{gvect}}(B_1, B_2(-w)) \\ [C_1, C_2] &= \mathrm{Hom}_{\mathsf{vect}}(C_1, C_2), \end{aligned} \tag{3.17}$$

where we are using that vect, gvect, and fvect are enriched over vect and where β being a mono in fvect (Remark 3.11) ensures that

$$\begin{aligned} [A_1, A_2]^{w+1} &\rightarrowtail [A_1, A_2]^w \\ f &\longmapsto \beta f \end{aligned}$$

is an injection. The tensor functors in (3.4) are closed, where for unf we are using that $\mathrm{unf}([A_1, A_2]) = [\mathrm{unf}(A_1), \mathrm{unf}(A_2)]$ by Lemma 3.15.

3.18 Recollection. Our ambient tensor categories vect, gvect, and fvect are all rigid. Indeed, vect is the classic example of a rigid category. Since gr and ung are conservative (Lemma 3.13) closed (Recollection 3.16) tensor functors, we deduce (Lemma 2.8) that gvect and therefore fvect are rigid as well.

We now discuss monoids and modules.

3.19 Notation. Let $\mathcal{C}$ be a tensor category, and let $R \in \mathcal{C}$ be a monoid. We denote by $\mathrm{mod}_{\mathcal{C}}(R)$ the additive category of left R-modules in $\mathcal{C}$ [Bra14, Section 4.1]. A tensor functor $F\colon \mathcal{C} \to \mathcal{D}$ induces an additive functor on module categories, which we still denote by $F\colon \mathrm{mod}_{\mathcal{C}}(R) \to \mathrm{mod}_{\mathcal{D}}(F(R))$, and a morphism $f\colon R_1 \to R_2$ of monoids in $\mathcal{C}$ gives rise to an additive restriction of scalars functor which we denote by $f^*\colon \mathrm{mod}_{\mathcal{C}}(R_2) \to \mathrm{mod}_{\mathcal{C}}(R_1)$. We may abbreviate $\mathrm{mod}_{\mathsf{vect}} = \mathrm{mod}$.

3.20 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a tensor functor, and let $f\colon R_1 \to R_2$ be a morphism of monoids in $\mathcal{C}$. Then $F(f)$ is a morphism of monoids in $\mathcal{D}$, and we have the following commutative diagram of additive functors:*

$$\begin{array}{ccc} \mathrm{mod}_{\mathcal{C}}(R_2) & \xrightarrow{F} & \mathrm{mod}_{\mathcal{D}}(F(R_2)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle F(f)^*} \\ \mathrm{mod}_{\mathcal{C}}(R_1) & \xrightarrow{F} & \mathrm{mod}_{\mathcal{D}}(F(R_1)). \end{array}$$

Proof. This is straightforward. ☐

3.21 Example. Let us discuss which functors from (3.4) we get on module categories. It depends on whether we start with a monoid in fvect, gvect, or vect.

(i) Given a monoid $R \in \mathsf{fvect}$, we only have the tensor functors

$$\mathrm{mod}_{\mathsf{vect}}(\mathrm{unf}(R)) \xleftarrow[\mathrm{unf}]{} \mathrm{mod}_{\mathsf{fvect}}(R) \xrightarrow{\mathrm{gr}} \mathrm{mod}_{\mathsf{gvect}}(\mathrm{gr}(R)) \xrightarrow{\mathrm{ung}} \mathrm{mod}_{\mathsf{vect}}(\mathrm{ung}\,\mathrm{gr}(R))$$

that start from fvect, with no sections and no guarantees about essential surjectivity or full faithfulness.

(ii) Given a monoid $S \in \mathsf{gvect}$, we have the tensor functors

$$\mathrm{mod}_{\mathsf{vect}}(\mathrm{ung}(S)) \xleftarrow[\mathrm{unf}]{} \mathrm{mod}_{\mathsf{fvect}}(\mathrm{trf}(S)) \underset{\mathrm{trf}}{\overset{\mathrm{gr}}{\rightleftarrows}} \mathrm{mod}_{\mathsf{gvect}}(S) \xrightarrow{\mathrm{ung}} \mathrm{mod}_{\mathsf{vect}}(\mathrm{ung}(S))$$

that start from gvect, along with the one retract gr and no other guarantees about essential surjectivity or full faithfulness. We can think of this case as being a specific case of (i) where the monoid $R \in \mathsf{fvect}$ is the trivial filtration $\mathrm{trf}(S)$ of a monoid $S \in \mathsf{gvect}$.

(iii) Given a monoid $T \in \mathsf{vect}$, we have everything:

$$\mathrm{mod}_{\mathsf{vect}}(T) \underset{\mathrm{unf}}{\overset{\mathrm{fz}}{\rightleftarrows}} \mathrm{mod}_{\mathsf{fvect}}(\mathrm{fz}(T)) \underset{\mathrm{trf}}{\overset{\mathrm{gr}}{\rightleftarrows}} \mathrm{mod}_{\mathsf{gvect}}(\mathrm{gz}(T)) \underset{\mathrm{gz}}{\overset{\mathrm{ung}}{\rightleftarrows}} \mathrm{mod}_{\mathsf{vect}}(T).$$

3.22 Lemma. *Let $\mathcal{C}$ be a tensor category where the tensor $-\otimes-$ preserves epis in both variables, and let $\varphi\colon R_1 \to R_2$ be a morphism of monoids in $\mathcal{C}$ such that φ is an epi in $\mathcal{C}$. Then the restriction of scalars $\varphi^*\colon \mathrm{mod}_{\mathcal{C}}(R_2) \to \mathrm{mod}_{\mathcal{C}}(R_1)$ is fully faithful.*

Proof. Let $f\colon A_1 \to A_2$ be a morphism in $\mathcal{C}$, and consider the following diagram where the morphisms $\varphi \otimes 1$ are epis by our hypothesis and where the left-hand square automatically commutes:

$$\begin{tikzcd} R_1 \otimes A_1 \arrow[r, two heads, "\varphi\otimes 1"] \arrow[d, "1\otimes f"] & R_2 \otimes A_1 \arrow[r, "\mu"] \arrow[d, "1\otimes f"] & A_1 \arrow[d, "f"] \\ R_1 \otimes A_2 \arrow[r, two heads, "\varphi\otimes 1"'] & R_2 \otimes A_2 \arrow[r, "\mu"'] & A_2. \end{tikzcd}$$

Unraveling definitions, $f\colon A_1 \to A_2$ is a morphism in $\mathrm{mod}_{\mathcal{C}}(R_2)$ if and only if the right-hand square commutes, and $\varphi^*(f)\colon \varphi^*(A_1) \to \varphi^*(A_2)$ is a morphism in $\mathrm{mod}_{\mathcal{C}}(R_1)$ if and only if the outer rectangle commutes. Thus if f is a morphism in $\mathrm{mod}_{\mathcal{C}}(R_2)$, then $\varphi^*(f)$ is a morphism in $\mathrm{mod}_{\mathcal{C}}(R_1)$, and the converse is also true because $\varphi \otimes 1\colon R_1 \otimes A_1 \twoheadrightarrow R_2 \otimes A_1$ is an epi. $\square$

3.23 Terminology. We say that $B \in \mathsf{gvect}$ is *effective* if $B^w = 0$ for all $w < 0$, and we say that $A \in \mathsf{fvect}$ is *effective* if $\mathrm{gr}(A) \in \mathsf{gvect}$ is effective. The latter means that $A^w = \mathrm{unf}(A)$ for all $w \leq 0$.

3.24 Example. We can truncate either end of a module over an effective monoid in fvect or gvect. Indeed, let $R \in \mathsf{fvect}$ be an effective monoid, and let $A \in \mathrm{mod}_{\mathsf{fvect}}(R)$ be an R-module. For any $w \in \mathbb{Z}$, we have two types of truncations: $A^{\leq w} \in \mathrm{mod}_{\mathsf{fvect}}(R)$ and $A^{\geq w} \in \mathrm{mod}_{\mathsf{fvect}}(R)$ defined respectively by

$$(A^{\leq w})^v = \begin{cases} A^v/A^{w+1} & v \leq w \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad (A^{\geq w})^v = \begin{cases} A^v & v \geq w \\ A^w & \text{otherwise.} \end{cases}$$

Both truncations rely on R being effective in order to be well-defined. Similarly, let $S \in \mathsf{gvect}$ be an effective monoid, and let $B \in \mathrm{mod}_{\mathsf{gvect}}(S)$ be an S-module. For any $w \in \mathbb{Z}$, we have two types of truncations: $B^{\leq w} \in \mathrm{mod}_{\mathsf{gvect}}(S)$ and $B^{\geq w} \in \mathrm{mod}_{\mathsf{gvect}}(S)$, which we can cheaply define by

$$B^{\leq w} = \mathrm{gr}(\mathrm{trf}(B)^{\leq w}) \qquad \text{and} \qquad B^{\geq w} = \mathrm{gr}(\mathrm{trf}(B)^{\geq w}).$$

These truncations are compatible in the sense that

$$\mathrm{gr}(A^{\leq w}) = \mathrm{gr}(A)^{\leq w} \qquad \text{and} \qquad \mathrm{gr}(A^{\geq w}) = \mathrm{gr}(A)^{\geq w}. \tag{3.25}$$

3.26 Example. We can insert a gap into a module over an effective monoid in fvect. Indeed, let $R \in \mathsf{fvect}$ be an effective monoid, let $A \in \mathrm{mod}_{\mathsf{fvect}}(R)$ be an R-module, and let $w \in \mathbb{Z}$ be a filtration weight where we will insert the gap. We define $\mathrm{gap}^w(A) \in \mathrm{mod}_{\mathsf{fvect}}(R)$ via

$$\mathrm{gap}^w(A)^v = \begin{cases} A^v & v \leq w \\ A^{v-1} & v > w. \end{cases}$$

This definition relies on R being effective in order to be well defined, and we have

$$\mathrm{gr}(\mathrm{gap}^w(A))^v = \begin{cases} \mathrm{gr}^v(A) & v < w \\ 0 & v = w \\ \mathrm{gr}^{v-1}(A) & v > w. \end{cases} \tag{3.27}$$

3.28 Lemma. *Let S be a monoid in* gvect, *and consider the diagrams*

$$\begin{tikzcd} B_1 \arrow[rr, "g"] \arrow[dr, dotted, "g_1"'] & & B_3 \\ & B_2 \arrow[ur, "g_2"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{ung}(B_1) \arrow[rr, "\mathrm{ung}(g)"] \arrow[dr, dotted, "h"'] & & \mathrm{ung}(B_3) \\ & \mathrm{ung}(B_2) \arrow[ur, "\mathrm{ung}(g_2)"'] & \end{tikzcd}$$

in $\mathrm{mod}_{\mathsf{gvect}}(S)$ *and* $\mathrm{mod}_{\mathsf{vect}}(\mathrm{ung}(S))$ *respectively. If there exists h making the diagram on the right-hand side commute, then it induces g_1 making the diagram on the left-hand side commute. Similarly, for the diagrams*

$$\begin{tikzcd} B_1 \arrow[rr, "g"] \arrow[dr, "g_1"'] & & B_3 \\ & B_2 \arrow[ur, dotted, "g_2"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{ung}(B_1) \arrow[rr, "\mathrm{ung}(g)"] \arrow[dr, "\mathrm{ung}(g_1)"'] & & \mathrm{ung}(B_3) \\ & \mathrm{ung}(B_2) \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

if there exists h making the diagram on the right-hand side commute, then it induces g_2 making the diagram on the left-hand side commute.

Proof. For the first case, we define $g_1 \colon B_1 \to B_2$ at the level of k-vector spaces via

$$g_1^w \colon B_1^w \rightarrowtail \mathrm{ung}(B_1) \xrightarrow{h} \mathrm{ung}(B_2) \xrightarrow{\mathrm{proj}_w} B_2^w,$$

where proj_w is the projection onto the wth graded piece. To verify that this is a morphism in $\mathrm{mod}_{\mathsf{gvect}}(S)$, it suffices to observe that for homogeneous elements $s \in S^v$ and $b \in B_1^w$, we have

$$sg_1(b) = sg_1^w(b) = s\,\mathrm{proj}_w(h(b)) = \mathrm{proj}_{w+v}(sh(b)) = \mathrm{proj}_{w+v}(h(sb)) = g_1(sb).$$

For the other case, we similarly define $g_2 \colon B_2 \to B_3$ via

$$g_2^w \colon B_2^w \rightarrowtail \mathrm{ung}(B_2) \xrightarrow{h} \mathrm{ung}(B_3) \xrightarrow{\mathrm{proj}_w} B_3^w,$$

and now

$$sg_2(b) = sg_2^w(b) = s\,\mathrm{proj}_w(sh(b)) = \mathrm{proj}_{w+v}(sh(b)) = \mathrm{proj}_{w+v}(h(sb)) = g_2(sb),$$

as before. $\square$

3.29 Notation. Let $S \in \mathsf{gvect}$ be a monoid, and let $B \in \mathrm{mod}_{\mathsf{gvect}}(S)$. Then $S^0 \in \mathsf{vect}$ is a monoid, and for any $w \in \mathbb{Z}$, the k-vector space $B^w \in \mathsf{vect}$ inherits an S^0-action: $B^w \in \mathrm{mod}_{\mathsf{vect}}(S^0)$.

Hopf monoids give us rigidity for free.

3.30 Recollection. A Hopf monoid $R \in \mathcal{C}$ in a tensor category $\mathcal{C}$ is a bimonoid with a morphism $S : R \to R$ called the antipode such that

$$\begin{tikzcd} R \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{1} \arrow[r, "\eta"] & R \\ R \otimes R \arrow[rr, "1\otimes S"'] & & R \otimes R \arrow[u, "\nabla"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{1} \arrow[r, "\eta"] & R \\ R \otimes R \arrow[rr, "S\otimes 1"'] & & R \otimes R, \arrow[u, "\nabla"'] \end{tikzcd}$$

where Δ, ∇, ε, and η respectively denote the comutiplication, multiplication, counit, and unit of R.

The category of modules $\mathrm{mod}_{\mathcal{C}}(R)$ over R inherits a tensor structure through the comultiplication on R, where given $A_1, A_2 \in \mathrm{mod}_{\mathcal{C}}(R)$, the R-action on $A_1 \otimes A_2$ is given by

$$R \otimes (A_1 \otimes A_2) \xrightarrow{\Delta\otimes 1} R \otimes R \otimes A_1 \otimes A_2 \cong (R \otimes A_1) \otimes (R \otimes A_2) \xrightarrow{\mu\otimes\mu} A_1 \otimes A_2.$$

It is classical [JS06, Proposition 6] that $\mathrm{mod}_{\mathsf{vect}}(T)$ is rigid when T is a Hopf monoid in vect.

3.31 Lemma. *Let R be a Hopf monoid in* fvect. *Then*

$$\mathrm{mod}_{\mathsf{fvect}}(R) \xrightarrow{\mathrm{gr}} \mathrm{mod}_{\mathsf{gvect}}(\mathrm{gr}(R)) \xrightarrow{\mathrm{ung}} \mathrm{mod}_{\mathsf{vect}}(\mathrm{ung}\,\mathrm{gr}(R))$$

is a sequence of rigid tensor categories and tensor functors.

Proof. Our ambient categories have equalizers, so these are closed tensor categories and closed tensor functors [Bra14, Remark 4.1.18]. Moreover since R is a Hopf monoid in fvect, we have that $\mathrm{gr}(R)$ and $\mathrm{ung}\,\mathrm{gr}(R)$ are Hopf monoids respectively in gvect and vect. Thus $\mathrm{mod}_{\mathsf{vect}}(\mathrm{ung}\,\mathrm{gr}(R))$ is rigid (Recollection 3.30), so $\mathrm{mod}_{\mathsf{gvect}}(\mathrm{gr}(R))$ and $\mathrm{mod}_{\mathsf{fvect}}(R)$ are rigid as well (Lemma 2.8) because gr and ung are conservative. $\square$

3.32 Lemma. *Let $\mathcal{C}$ be a rigid tensor category, and let R and S be Hopf monoids in $\mathcal{C}$. On the one hand, the tensor product $R \otimes S$ of Hopf monoids is a Hopf monoid in $\mathcal{C}$, so we may consider the rigid tensor category $\mathrm{mod}_{\mathcal{C}}(R \otimes S)$ of modules in $\mathcal{C}$ over $R \otimes S$. On the other hand, the inflation of S along the counit $\varepsilon\colon R \to \mathbb{1}$ is a Hopf monoid in the rigid tensor category $\mathrm{mod}_{\mathcal{C}}(R)$, so we may consider the rigid tensor category $\mathrm{mod}_{\mathrm{mod}_{\mathcal{C}}(R)}(S)$ of modules in $\mathrm{mod}_{\mathcal{C}}(R)$ over that S. Then*

$$\mathrm{mod}_{\mathcal{C}}(R \otimes S) \cong \mathrm{mod}_{\mathrm{mod}_{\mathcal{C}}(R)}(S).$$

Proof. Indeed, on the left-hand side $(R \otimes S)$-action is $(r \otimes s)c = rsc$, and it is associative because the S-action is R-linear:

$$((r \otimes s)(r' \otimes s'))c = (rr' \otimes ss')c = rr'ss'c = rsr's'c = (r \otimes s)((r' \otimes s')c).$$

On the right-hand side we have the R-action $rc = (r \otimes 1)c$ and the S-action $sc = (1 \otimes s)c$, and the S-action is R-linear because S is inflated along $\varepsilon\colon R \to \mathbb{1}$:

$$s \otimes rc = s \otimes \sum \varepsilon(r_{(1)})r_{(2)}c = \sum \varepsilon(r_{(1)})s \otimes r_{(2)}c = \sum r_{(1)}s \otimes r_{(2)}c = r(s \otimes c)$$

using Sweedler notation. $\square$

We now introduce the basic Hopf monoids we will be using in Part I and Part II. In Part III, we will introduce more interesting ones.

3.33 Recollection. The group algebra kG is a Hopf monoid in vect. The comultiplication and antipode in kG are given by $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$ for $g \in G$.

3.34 Notation. We denote by

$$\mathrm{rad}(kG) = \ker(\varepsilon\colon kG \to k)$$

the augmentation ideal of kG. To justify the notation, we recall that G being a p-group is equivalent to kG being local: $kG/\operatorname{rad}(kG) \cong k$. Thus $\mathrm{rad}(kG)$ is the Jacobson radical of kG and hence is a nilpotent ideal. Given $g \in G$, we denote by $X_g = g - 1 \in \mathrm{rad}(kG)$

the associated element of the radical. The comultiplication Δ and antipode S can also be described in terms of the elements $X_g \in \mathrm{rad}(kG)$:

$$\Delta(X_g) = X_g \otimes 1 + 1 \otimes X_g + X_g \otimes X_g \quad \text{and} \quad S(X_g) = X_{g^{-1}}. \tag{3.35}$$

Consequently, if $C_1, C_2 \in \mathsf{mod}(G)$ and $g \in G$, then

$$X_g(c_1 \otimes c_2) = X_g c_1 \otimes c_2 + c_1 \otimes X_g c_2 + X_g c_1 \otimes X_g c_2. \tag{3.36}$$

for any $c_1 \in C_1$ and $c_2 \in C_2$.

3.37 Example. Since kG is a Hopf monoid in vect, we see that $\mathrm{gz}\, kG$ is a Hopf monoid in gvect and $\mathrm{fz}\, kG$ is a Hopf monoid in fvect.

3.38 Definition. We define

$$\mathsf{mod}(G) = \mathrm{mod}_{\mathsf{vect}}(kG)$$
$$\mathsf{gmod}(G) = \mathrm{mod}_{\mathsf{gvect}}(\mathrm{gz}\, kG)$$
$$\mathsf{fmod}(G) = \mathrm{mod}_{\mathsf{fvect}}(\mathrm{fz}\, kG)$$

to be the categories of modules over the Hopf monoids kG, $\mathrm{gz}\, kG$, and $\mathrm{fz}\, kG$, respectively. We have the sequence

$$\mathsf{fmod}(G) \xrightarrow{\mathrm{gr}} \mathsf{gmod}(G) \xrightarrow{\mathrm{ung}} \mathsf{mod}(G)$$

of rigid tensor categories and tensor functors (Lemma 3.31 and Example 3.21(*iii*)).

We now introduce our permutation variants. In short, $\mathsf{pgmod}(G)$ is the category of graded permutation kG-modules, and $\mathsf{pfmod}(G)$ is the category of filtered kG-modules whose total associated graded is permutation. Note that because G is a p-group (Notation 2.1), we do not need to worry about idempotent completions.

3.39 Recollection. The rigid tensor category $\mathsf{pmod}(G)$ of finitely generated permutation kG-modules is the full rigid tensor subcategory of $\mathsf{mod}(G)$ consisting of the k-linearized finite G-sets. As an additive category it is generated by $k(G/H)$ for $H \leqslant G$.

3.40 Definition. We define the rigid tensor categories $\mathsf{pgmod}(G)$ and $\mathsf{pfmod}(G)$ of *permutation graded modules* and *permutation filtered modules* respectively to be the full rigid tensor subcategories of $\mathsf{gmod}(G)$ and $\mathsf{fmod}(G)$ given by the pullbacks (Lemma 2.10)

$$\begin{tikzcd} \mathsf{pfmod}(G) \arrow[r, two heads, "\mathrm{gr}"] \arrow[d, hook] & \mathsf{pgmod}(G) \arrow[r, two heads, "\mathrm{ung}"] \arrow[d, hook] & \mathsf{pmod}(G) \arrow[d, hook] \\ \mathsf{fmod}(G) \arrow[r, two heads, "\mathrm{gr}"] & \mathsf{gmod}(G) \arrow[r, two heads, "\mathrm{ung}"] & \mathsf{mod}(G). \end{tikzcd} \tag{3.41}$$

Each of these categories is Krull–Remak–Schmidt. In other words, any object decomposes into a direct sum of indecomposable objects uniquely up to permutation of summands.

3.42 Terminology. A *pre-abelian category* is an additive category that has all kernels and cokernels, and a *pre-abelian functor* is an additive functor between pre-abelian categories that preserves kernels and cokernels.

3.43 Lemma. *The additive categories in (3.41) are all Krull–Remak–Schmidt.*

Proof. We appeal to [Sha23, Theorem 6.1]. They are all k-linear and have finite-dimensional homs. Since the module categories $\mathsf{fmod}(G)$, $\mathsf{gmod}(G)$, and $\mathsf{mod}(G)$ are all pre-abelian (Terminology 3.42), they are idempotent complete [Sha23, Proposition 3.10]. The permutation variants $\mathsf{pfmod}(G)$, $\mathsf{pgmod}(G)$, and $\mathsf{pmod}(G)$ are thus idempotent complete since G is a p-group (Notation 2.1). $\square$

Let us now discuss some important examples of objects in these rigid tensor categories.

3.44 Example. The *radical filtration* $\mathrm{radf}(C)$ of a module $C \in \mathsf{mod}(G)$ is in the permutation subcategory $\mathsf{pfmod}(G)$. It is defined to be the filtration of C by powers of the radical $\mathrm{rad}(kG)$; concretely $\mathrm{radf}(C)^w = \mathrm{rad}(kG)^w C$, with the convention that $\mathrm{rad}(kG)^w = kG$ for $w < 0$. This produces a well-defined object $\mathrm{radf}(C)$ in $\mathsf{fmod}(G)$ because $\mathrm{rad}(kG)$ is nilpotent (Notation 3.34). Moreover, since $kG/\mathrm{rad}(kG) \cong k$, we have that $\mathrm{gr}\,\mathrm{radf}(C)^w$ is a sum of copies of k, hence permutation, so $\mathrm{radf}(C) \in \mathsf{pfmod}(G)$.

3.45 Example. The radical filtration of kG is a comonoid in $\mathsf{pfmod}(G)$. Indeed, recall that the comonoid kG in vect is automatically a comonoid in $\mathsf{mod}(G)$ because its comultiplication and counit are both kG-linear:

$$\Delta(gh) = gh \otimes gh = g(h \otimes h) = g\Delta(h) \quad \text{and} \quad \varepsilon(gh) = 1 = g1 = g\varepsilon(h)$$

for any $g, h \in G$. Since $\Delta\colon kG \to kG \otimes kG$ induces a morphism $\Delta\colon \operatorname{radf}(kG) \to \operatorname{radf}(kG) \otimes \operatorname{radf}(kG)$ by the computation (3.35), it follows that $\operatorname{radf}(kG)$ is a comonoid in $\mathsf{pfmod}(G)$.

3.46 Recollection. In this recollection, we allow G be an arbitrary finite group, as opposed to a p-group (Notation 2.1). Every Hopf ideal of kG is of the form

$$\ker(\varepsilon\colon kG \to k(G/N)) = I(kN)kG$$

for some normal subgroup $N \trianglelefteqslant G$; this is the ideal in kG generated by the augmentation ideal $I(kN)$ of kN [Nat15, Example 4.6]. The ideal $I(kN)$ is nilpotent if and only if N is a p-group [Pas11, Lemma 3.1.6]. Thus since $(I(kN)kG)^w = I(kN)^w kG$ by normality of N, the ideal $I(kN)kG$ is nilpotent when N is a p-group.

3.47 Example. For a normal subgroup $N \trianglelefteqslant G$ and a module $C \in \mathsf{mod}(G)$, we have the N-radical filtration $\operatorname{radf}_N(C)$ of C in $\mathsf{fmod}(G)$. To define $\operatorname{radf}_N(C)$, we denote by

$$\operatorname{rad}_N(kG) = \ker(\varepsilon\colon kG \to k(G/N)) = \operatorname{rad}(kN)kG$$

the *N-radical of kG*, which is the ideal in kG generated by $\operatorname{rad}(kN)$, and we define $\operatorname{radf}_N(C)$ to be the filtration of C by the powers of $\operatorname{rad}_N(kG)$. By Recollection 3.46 and our assumption in Notation 2.1 that G is a p-group, we have

$$\operatorname{rad}_N(kG)^w = \operatorname{rad}(kN)^w kG \tag{3.48}$$

since $\operatorname{rad}(kN)$ is the augmentation ideal of kN, so $\operatorname{rad}_N(kG)$ is a nilpotent ideal. Thus $\operatorname{radf}_N(C)$ is a well-defined object in $\mathsf{fmod}(G)$.

3.49 Remark. In Example 3.47, the object $\operatorname{radf}_N(C)$ is not necessarily in $\mathsf{pfmod}(G)$ (compare with Example 3.44). For example, take $E = C_3^2$, let $x, y \in E$ be a minimal set of generators so that $kE = k[X,Y]/(X^3,Y^3)$, take $N = \langle y\rangle \trianglelefteq E$, and let $C = \langle X\rangle \subset kE$ be the submodule of kE generated by X. Then

$$\operatorname{gr}\operatorname{radf}_N(C) = \frac{k[X]}{(X^2)}(1) \oplus \frac{k[X]}{(X^2)}(2) \oplus \frac{k[X]}{(X^2)}(3)$$

is not in $\mathsf{pgmod}(E)$ because the only indecomposable permutation modules in $\mathsf{pmod}(E)$ have k-dimension 1, 3, or 9.

3.50 Example. In contrast with Remark 3.49, the object $\operatorname{radf}_N(kG)$ is in $\mathsf{pfmod}(G)$. Indeed, the multiplication map $\operatorname{rad}(kN)^w \otimes_{kN} kG \to \operatorname{rad}(kN)^w kG$ is an isomorphism because applying $-\otimes_{kN} kG$ to the inclusion $\operatorname{rad}(kN)^w \rightarrowtail kN$ produces an injection $\operatorname{rad}(kN)^w \otimes_{kN} kG \rightarrowtail kG$ (recall that kG is free over kN) with image $\operatorname{rad}(kN)^w kG$. Thus, applying $-\otimes_{kN} kG$ to the short exact sequence

$$0 \longrightarrow \operatorname{rad}(kN)^{w+1} \longrightarrow \operatorname{rad}(kN)^w \longrightarrow \operatorname{gr}\operatorname{radf}(kN)^w \longrightarrow 0$$

in $\mathsf{mod}(N)$ produces the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad}(kN)^{w+1} \otimes_{kN} kG \arrow[r] \arrow[d, "\cong"] & \operatorname{rad}(kN)^w \otimes_{kN} kG \arrow[r] \arrow[d, "\cong"] & \operatorname{gr}\operatorname{radf}(kN)^w \otimes_{kN} kG \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{rad}(kN)^{w+1} kG \arrow[r] & \operatorname{rad}(kN)^w kG \arrow[r] & \operatorname{gr}\operatorname{radf}(kN)^w \otimes_{kN} kG \arrow[r] & 0
\end{tikzcd}$$

in $\mathsf{mod}(G)$. We deduce that

$$\begin{aligned}
\operatorname{gr}\operatorname{radf}_N(kG)^w &= \operatorname{rad}_N(kG)^w / \operatorname{rad}_N(kG)^{w+1} \\
&\cong \operatorname{rad}(kN)^w kG / \operatorname{rad}(kN)^{w+1} kG && \text{(by (3.48))} \\
&\cong \operatorname{gr}\operatorname{radf}(kN)^w \otimes_{kN} kG && \text{(previous diagram)} \\
&\cong k(G/N)^{\oplus \dim_k \operatorname{gr}\operatorname{radf}(kN)^w}
\end{aligned}$$

because $\operatorname{gr}\operatorname{radf}(kN)^w$ is a direct sum of copies of k.

4 Elementary abelian computations

In this short section we perform some computations, especially in the elementary abelian case. We recall that G denotes a p-group and that E denotes an elementary abelian p-group

(Notation 2.1).

4.1 Proposition. *Let $H \leqslant G$. Frobenius reciprocity extends to* pfmod*:*

$$\begin{tikzcd} \mathsf{pfmod}(H) \arrow[d, shift right, "\mathrm{Ind}_H^G(-)"'] \\ \mathsf{pfmod}(G). \arrow[u, shift right, "\mathrm{Res}_H^G(-)"'] \end{tikzcd}$$

Induction $\mathrm{Ind}_H^G(-)$ *and restriction* $\mathrm{Res}_H^G(-)$ *are defined weight-wise: for* $A_1 \in \mathsf{pfmod}(H)$ *and* $A_2 \in \mathsf{pfmod}(G)$*, we have*

$$\mathrm{Ind}_H^G(A_1)^w = \mathrm{Ind}_H^G(A_1^w) \quad \text{and} \quad \mathrm{Res}_H^G(A_2)^w = \mathrm{Res}_H^G(A_2^w). \tag{4.2}$$

In particular

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G(A_2) \cong \mathrm{fz}\, k(G/H) \otimes A_2. \tag{4.3}$$

Proof. In this proof, we denote by $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{mod}(G))$ the category of functors from $\mathbb{Z}^{\mathrm{op}}$ to $\mathsf{mod}(G)$. Observe that $\mathsf{fmod}(G)$ is equivalent to the full subcategory of $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{mod}(G))$ consisting of the functors that take morphisms to monos. Now, we upgrade the usual Frobenius reciprocity, depicted on the left, as follows:

$$\begin{tikzcd} \mathsf{mod}(H) \arrow[d, shift right, "\mathrm{Ind}_H^G"'] \\ \mathsf{mod}(G) \arrow[u, shift right, "\mathrm{Res}_H^G"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{mod}(H)) \arrow[d, shift right, "\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Ind}_H^G)"'] \\ \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{mod}(G)) \arrow[u, shift right, "\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Res}_H^G)"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{fmod}(H) \arrow[d, shift right, "\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Ind}_H^G)"'] \\ \mathsf{fmod}(G). \arrow[u, shift right, "\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Res}_H^G)"'] \end{tikzcd}$$

Applying $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, -)$ preserves the adjunction because $\mathbb{Z}^{\mathrm{op}}$ is small, so this gives the adjunction in the middle and explains (4.2). Now recall that Res_H^G and Ind_H^G are both exact. Thus we may restrict to fmod using that Res_H^G and Ind_H^G preserve monos, and this gives the adjunction on the right. Moreover, we may restrict to pfmod using that Res_H^G and Ind_H^G preserve cokernels and take permutation modules to permutation modules. To see (4.3), observe that

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G(A_2)^w = \mathrm{Ind}_H^G \mathrm{Res}_H^G(A_2^w) \cong k(G/H) \otimes A_2^w = (\mathrm{fz}\, k(G/H) \otimes A_2)^w$$

for any $w \in \mathbb{Z}$ by (4.2) and [Bal13, Proposition 3.16(c)]. $\square$

4.4 Lemma. *Let $N \trianglelefteqslant G$. If $H \trianglelefteqslant G$ is normal, then the tensor product* $\mathrm{fz}\, k(G/H) \otimes \mathrm{radf}_N(k(G/H))$ *is a direct sum of twists of* $\mathrm{fz}\, k(G/H)$.

Proof. Write (Proposition 4.1)

$$\mathrm{fz}\, k(G/H) \otimes \mathrm{radf}_N(k(G/H)) = \mathrm{Ind}_H^G \mathrm{Res}_H^G(\mathrm{radf}_N(k(G/H))),$$

and observe that $\mathrm{Res}_H^G(\mathrm{radf}_N(k(G/H)))$ is a direct sum of twists of the unit because H is normal. The result then follows from the formula (4.2) for $\mathrm{Ind}_H^G(-)$. □

Let us begin discussing the elementary abelian case, which we will eventually focus on. In this case, there is a lever which lets us make concrete computations: we can pick minimal sets of generators of our group E.

4.5 Notation. If we denote an element of E with a lowercase letter, then we denote by the uppercase letter the corresponding element of the radical (recall Notation 3.34). For example if $z \in E$, then $Z = X_z = z - 1 \in \mathrm{rad}(kE)$.

4.6 Example. Let $N, H \leqslant E$. Recalling that E is the same data as a finite-dimensional $\mathbb{F}_p$-vector space, we may pick a minimal set of generators w_i, x_i, y_i, z_i of E such that the elements z_i form a minimal set of generators of $N \cap H$, and similarly y_i, z_i for N and x_i, z_i for H. Then

$$\begin{aligned} kE &\simeq \frac{k[W_i, X_i, Y_i, Z_i]}{(W_i^p, X_i^p, Y_i^p, Z_i^p)} & & \\ k(E/N) &\simeq \frac{k[W_i, X_i]}{(W_i^p, X_i^p)} & k(E/H) &\simeq \frac{k[W_i, Y_i]}{(W_i^p, Y_i^p)} \\ k(E/(N \cap H)) &\simeq \frac{k[W_i, X_i, Y_i]}{(W_i^p, X_i^p, Y_i^p)} & k(E/NH) &\simeq \frac{k[W_i]}{(W_i^p)} \end{aligned}$$

not only as monoids in vect but also as modules in $\mathsf{mod}(E)$ where the actions of W_i, X_i, Y_i, and Z_i are the obvious ones.

4.7 Example. In Example 4.6, the radical and N-radical ideals of kE are respectively

$$\mathrm{rad}(kE) = (W_i, X_i, Y_i, Z_i) \quad \text{and} \quad \mathrm{rad}_N(kE) = (Y_i, Z_i).$$

Thus for example

$$\operatorname{radf} k(E/H) \simeq \operatorname{radf}\left(\frac{k[W_i, Y_i]}{(W_i^p, Y_i^p)}\right) \in \mathsf{pfmod}(E)$$

is the filtration of $k[W_i, Y_i]/(W_i^p, Y_i^p)$ by the powers of the ideal (W_i, Y_i), whereas

$$\operatorname{radf}_N(k(E/H)) \simeq \operatorname{radf}_N\left(\frac{k[W_i, Y_i]}{(W_i^p, Y_i^p)}\right) \cong \operatorname{fz}\left(\frac{k[W_i]}{(W_i^p)}\right) \otimes \operatorname{radf}\left(\frac{k[Y_i]}{(Y_i^p)}\right) \in \mathsf{pfmod}(E)$$

is the filtration of $k[W_i, Y_i]/(W_i^p, Y_i^p)$ by the powers of the ideal (Y_i), both with the obvious kE-actions.

4.8 Lemma. *Let $N, H \leqslant E$. The tensor product* $\operatorname{fz} k(E/NH) \otimes \operatorname{radf} k(E/H)$ *is a direct sum of twists of* $\operatorname{radf}_N(k(E/H))$.

Proof. We use the identifications in Example 4.6. Let us denote by W^a a monomial in the variables W_i; here a is a tuple of nonnegative integers a_i, one for each W_i, and we write $|a| = \sum_i a_i$. Note that the x_i, y_i, z_i form a minimal set of generators for NH, so restricting to NH amounts to forgetting the W_i-actions; for example

$$\operatorname{Res}^E_{NH} \frac{k[W_i, Y_i]}{(W_i^p, Y_i^p)} = \bigoplus_{W^a} \frac{k[Y_i]}{(Y_i^p)}, \tag{4.9}$$

where the sum is over the monomials W^a. Thus

$$\begin{aligned}
\operatorname{fz} k(E/NH) \otimes \operatorname{radf} k(E/H) &\simeq \operatorname{Ind}^E_{NH} \operatorname{Res}^E_{NH} \operatorname{radf}\left(\frac{k[W_i, Y_i]}{(W_i^p, Y_i^p)}\right) && \text{(by (4.3))}\\
&\cong \operatorname{Ind}^E_{NH} \bigoplus_{W^a} \operatorname{radf}\left(\frac{k[Y_i]}{(Y_i^p)}\right)(|a|) && \text{(filtered version of (4.9))}\\
&\cong \bigoplus_{W^a} \operatorname{fz}\left(\frac{k[W_i]}{(W_i^p)}\right) \otimes \operatorname{radf}\left(\frac{k[Y_i]}{(Y_i^p)}\right)(|a|) \\
&\simeq \bigoplus_{W^a} \operatorname{radf}_N(k(E/H))(|a|), && \text{(Example 4.7)}
\end{aligned}$$

as claimed. □

4.10 Lemma. *Let $N, H \leqslant E$. The tensor product* $\operatorname{radf} k(E/N) \otimes \operatorname{radf} k(E/H)$ *in* $\mathsf{pfmod}(E)$ *is a direct sum of twists of* $\operatorname{radf} k(E/(N \cap H))$.

Proof. We use the identifications in Example 4.6, and we denote by X^b a monomial in the variables X_i and similarly for Y^c, as in the proof of Lemma 4.8. Since $k(E/N)$, $k(E/H)$, and $k(E/(N\cap H))$ all involve the variables W_i, for the sake of clarity we denote by W^a a monomial in $k(E/N)$ or $k(E/(N\cap H))$ and by W^d a monomial in $k(E/H)$. We define a morphism

$$f\colon \bigoplus_{W^d} \operatorname{radf} k(E/N\cap H)(|d|) \longrightarrow \operatorname{radf} k(E/N)\otimes \operatorname{radf} k(E/H) \tag{4.11}$$

in $\mathsf{pfmod}(E)$ by defining on underlying modules

$$\begin{aligned} \operatorname{unf}(f)\colon \bigoplus_{W^d} \frac{k[W_i,X_i,Y_i]}{(W_i^p,X_i^p,Y_i^p)} &\longrightarrow \frac{k[W_i,X_i]}{(W_i^p,X_i^p)}\otimes\frac{k[W_i,Y_i]}{(W_i^p,Y_i^p)} \\ 1_{W^d} &\longmapsto 1\otimes W^d \end{aligned} \tag{4.12}$$

in $\mathsf{mod}(E)$, where the direct sum in the source is taken over all monomials W^d in $k(E/H)$ and where 1_{W^d} denotes the element 1 in the copy of $k(E/(N\cap H))$ corresponding to W^d. Note that f is indeed a morphism in $\mathsf{pfmod}(E)$ by (3.36).

To show that f is an isomorphism, it suffices (Lemma 3.13) to show that $\operatorname{gr}(f)$ is an isomorphism. Consider the wth graded component of $\operatorname{gr}(f)$:

$$\operatorname{gr}(f)^w\colon \bigoplus_{W^d} \operatorname{gr}\operatorname{radf} k(E/N\cap H)(|d|)^w \longrightarrow \operatorname{gr}(\operatorname{radf} k(E/N)\otimes\operatorname{radf} k(E/H))^w.$$

The monomials $W^aX^bY^c1_{W^d}$ where $|a|+|d|=w$ form a basis of the left-hand side, and the monomials $W^aX^b\otimes Y^cW^d$ where $|a|+|d|=w$ form a basis of the right-hand side.

Observe that the element $\operatorname{unf}(f)(W^a1_{W^d}) = W^a(1\otimes W^d)$, which we can expand using (3.36), has a nonzero coefficient for $W^{a'}\otimes W^d$ if and only if $a=a'$. It follows that

$$\operatorname{unf}(f)(W^aX^bY^c1_{W^d}) = W^aX^bY^c(1\otimes W^d) = W^a(X^b\otimes Y^cW^d)$$

has a nonzero coefficient for $W^{a'}X^{b'}\otimes Y^{c'}W^d$ if and only if $(a,b,c)=(a',b',c')$. Since moreover the coefficient of $f(W^aX^bY^c1_{W^d})$ for $W^{a'}X^{b'}\otimes Y^{c'}W^{d'}$ can be nonzero only for $d'\geq d$, we deduce that $\operatorname{gr}(f)$ is an isomorphism. $\square$

4.13 Remark. The isomorphism (4.12) is not the isomorphism

$$\begin{aligned} \bigoplus_{g\in N\backslash G/H} k(G/(N\cap {}^g H)) &\xrightarrow{\cong} k(G/N)\otimes k(G/H) \\ 1_g &\longmapsto 1\otimes g \end{aligned} \tag{4.14}$$

given by the Mackey formula. Indeed, in terms of our identifications (4.14) reads

$$\begin{aligned} \bigoplus_{w^d} \frac{k[W_i,X_i,Y_i]}{(W_i^p,X_i^p,Y_i^p)} &\xrightarrow{\cong} \frac{k[W_i,X_i]}{(W_i^p,X_i^p)}\otimes\frac{k[W_i,Y_i]}{(W_i^p,Y_i^p)}, \\ 1_{w^d} &\longmapsto 1\otimes w^d \end{aligned}$$

where w^d denotes a monomial in the elements $w_i \in E$. This does not induce a morphism in $\mathsf{pfmod}(E)$ of the form in (4.11) because $1\otimes w^d$ is nonzero in $\mathrm{gr}(-)^0$; it only induces a morphism

$$\bigoplus_{W^d} \mathrm{radf}\, k(E/N\cap H) \longrightarrow \mathrm{radf}\, k(E/N)\otimes \mathrm{radf}\, k(E/H)$$

in $\mathsf{pfmod}(E)$, which cannot be an isomorphism in general because the k-dimensions of $\mathrm{gr}(-)^0$ do not agree.

5 Frobenius exact structures

In this section, we study pulled back exact structures and Frobenius exact rigid tensor categories. In particular, in Definition 5.21 we upgrade our diagram

$$\begin{tikzcd} \mathsf{pfmod}(G) \arrow[r, two heads, "\mathrm{gr}"] \arrow[d, tail] & \mathsf{pgmod}(G) \arrow[r, two heads, "\mathrm{ung}"] \arrow[d, tail] & \mathsf{pmod}(G) \arrow[d, tail] \\ \mathsf{fmod}(G) \arrow[r, two heads, "\mathrm{gr}"] & \mathsf{gmod}(G) \arrow[r, two heads, "\mathrm{ung}"] & \mathsf{mod}(G) \end{tikzcd} \tag{5.1}$$

to a diagram of pullbacks of exact rigid tensor categories and exact tensor functors where the exact structures are pulled back along gr and ung. In reading this diagram, we recall that G is a p-group (Notation 2.1).

We introduce the notion of pulling back an exact structure.

5.2 Notation. Let $\mathcal{A}$ be an additive category. By a *short complex* (i,p) *in* $\mathcal{A}$ we mean a pair

$$A_1 \xrightarrow{i} A_2 \xrightarrow{p} A_3$$

of composable morphisms in $\mathcal{A}$ such that $pi = 0$.

5.3 Definition. Let $\mathcal{A}$ be an additive category, let $\mathcal{B}$ be an exact category, and let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor. A short complex (i,p) in $\mathcal{A}$ is a *pulled back short exact sequence* if $(F(i), F(p))$ is short exact in $\mathcal{B}$. If the pulled back short exact sequences form an exact structure on $\mathcal{A}$, then we say that we can *pull back the exact structure*, and we call the resulting exact structure *the pulled back exact structure.*

5.4 Remark. In Definition 5.3, we can pull back the exact structure only if $F \colon \mathcal{A} \to \mathcal{B}$ is conservative. Indeed, if $f \colon A_1 \to A_2$ is a morphism in $\mathcal{A}$ such that $F(f)$ is an isomorphism, then $F(A_1) \rightarrowtail F(A_2) \twoheadrightarrow 0$ is short exact in $\mathcal{B}$, so $(f, 0)$ is short exact in $\mathcal{A}$, which implies f is an isomorphism.

5.5 Lemma. *In the setting of Definition 5.3, if we can pull back the exact structure along F, then the pulled back exact structure on $\mathcal{A}$ is the maximal one among those making $F \colon \mathcal{A} \to \mathcal{B}$ exact.*

Proof. This follows easily from the definition. □

5.6 Lemma. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, rightarrowtail, "F"] \arrow[dr, rightarrowtail, "G"'] & & \mathcal{C} \\ & \mathcal{B} \arrow[ur, rightarrowtail, "H"'] & \end{tikzcd}$$

be a commutative diagram of exact categories. If $\mathcal{A}$ and $\mathcal{B}$ have the pulled back exact structures (Definition 5.3) along G and H respectively, then $\mathcal{A}$ has the pulled back exact structure along F. If $\mathcal{A}$ and $\mathcal{B}$ have the pulled back exact structures along F and H respectively, then $\mathcal{A}$ has the pulled back exact structure along G.

Proof. This follows easily from the definition. □

5.7 Recollection. In an exact category, a morphism $f\colon A_1 \to A_2$ is *strict* if it admits a factorization

$$\begin{tikzcd} A_1 \arrow[rr, "f"] \arrow[dr, two heads] & & A_2. \\ & I \arrow[ur, tail] & \end{tikzcd}$$

If the factorization exists, it is unique up to unique isomorphism [Buh10, Lemma 8.4]. A sequence (f_1, f_2) of strict morphisms is *acyclic* if in

$$\begin{tikzcd} A_1 \arrow[rr, "f_1"] \arrow[dr, two heads] & & A_2 \arrow[rr, "f_2"] \arrow[dr, two heads] & & A_3 \\ & I_1 \arrow[ur, tail] & & I_2 \arrow[ur, tail] & \end{tikzcd}$$

the sequence $I_1 \rightarrowtail A_2 \twoheadrightarrow I_2$ is short exact, and a complex is acyclic if every consecutive pair of morphisms is acyclic.

5.8 Recollection. Every morphism $f\colon A_1 \to A_2$ in a pre-abelian (Terminology 3.42) exact category has an *analysis*

$$\begin{tikzcd} & A_1 \arrow[rrr, "f"] \arrow[dr, two heads] & & & A_2 \arrow[dr, two heads] & \\ \ker(f) \arrow[ur, tail] & & \operatorname{coim}(f) \arrow[r] & \operatorname{im}(f) \arrow[ur, tail] & & \operatorname{coker}(f). \end{tikzcd}$$

By the uniqueness in Recollection 5.7, it follows that f is strict if and only if the map $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.

5.9 Lemma. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a pre-abelian (Terminology 3.42) exact functor where $\mathcal{A}$ has the pulled back exact structure (Definition 5.3).*

(i) A morphism f in $\mathcal{A}$ is an admissible mono (resp. admissible epi) in $\mathcal{A}$ if and only if $F(f)$ is an admissible mono (resp. admissible epi) in $\mathcal{B}$.

(ii) A morphism f in $\mathcal{A}$ is strict in $\mathcal{A}$ if and only if $F(f)$ is strict in $\mathcal{B}$.

(iii) A sequence (f_1, f_2) of strict morphisms in $\mathcal{A}$ is acyclic in $\mathcal{A}$ if and only if $(F(f_1), F(f_2))$ is acyclic in $\mathcal{B}$.

Proof. For (i), let us prove the admissible mono case, as the other one is dual. The forward direction is immediate by exactness of F, so let f be a morphism in $\mathcal{A}$ such that $F(f)$ is an admissible mono in $\mathcal{B}$. On the one hand, there exists a morphism p' in $\mathcal{B}$ such that $(F(f), p')$ is short exact in $\mathcal{B}$. On the other hand, there exists a cokernel p of f in $\mathcal{A}$ because $\mathcal{A}$ is pre-abelian, and $F(p) \cong p'$ because F is pre-abelian. Thus, since (f, p) is a short complex in $\mathcal{A}$ and $(F(f), F(p)) = (F(f), p')$ is short exact in $\mathcal{B}$, we see that (f, p) is short exact in $\mathcal{A}$ by virtue of $\mathcal{A}$ having the pulled back exact structure. Therefore f is an admissible mono.

For (ii), since F is pre-abelian, it takes an analysis of f to an analysis of $F(f)$, so the statement follows from Recollection 5.8 and conservativity of F (Remark 5.4).

For (iii), the statement is well-defined by (ii) and immediate from $\mathcal{A}$ having the pulled back exact structure and conservativity of F (Remark 5.4). $\square$

Let us point out a couple of obstructions to being able to pull back an exact structure.

5.10 Remark. In Definition 5.3, in order for the pulled back short exact sequences to form an exact structure on $\mathcal{A}$, the pulled back short exact sequences (i, p) need to be kernel-cokernel pairs.

5.11 Remark. The existence of certain pullbacks and pushouts is also an obstruction for being able to pull back an exact structure. For example, let $\mathcal{A}$ be the additive category of abelian groups whose torsion is only 2-torsion, let $\mathcal{B}$ be the exact category of abelian groups with the abelian exact structure, and let $F\colon \mathcal{A} \to \mathcal{B}$ be the inclusion. Then

$$\mathbb{Z} \overset{\times 2}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$$

is a pulled back short exact sequence in $\mathcal{A}$, but the pushout

$$\begin{tikzcd} \mathbb{Z} \arrow[r, tail, "\times 2"] \arrow[d, two heads] & \mathbb{Z} \arrow[d, two heads] \\ \mathbb{Z}/2 \arrow[r, tail] & \mathbb{Z}/4 \end{tikzcd}$$

does not exist in $\mathcal{A}$.

Let us discuss the exact structures on $\mathrm{mod}_{\mathsf{fvect}}(R)$, $\mathrm{mod}_{\mathsf{gvect}}(S)$, and $\mathrm{mod}_{\mathsf{vect}}(T)$, where R, S, and T are monoids in respectively fvect, gvect, and vect.

5.12 Lemma. *Let $\mathcal{C}$ be a pre-abelian (Terminology 3.42) exact tensor category, and let R be a monoid in $\mathcal{C}$. We can pull back the exact structure (Definition 5.3) along the forgetful functor $U\colon \operatorname{mod}_{\mathcal{C}}(R) \to \mathcal{C}$.*

Proof. See [Kel18, Proposition 2.1.69] for a more general statement in terms of monads. Alternatively, this is a straightforward verification. For example, suppose (i, p) is a pulled back short exact sequence in $\operatorname{mod}_{\mathcal{C}}(R)$ where $i\colon A_1 \to A_2$, and let $f\colon A_1 \to A_3$ be a morphism in $\operatorname{mod}_{\mathcal{C}}(R)$. Consider the following commutative diagram in $\mathcal{C}$ where the front face is a pushout:

$$\begin{tikzcd}
R \otimes U(A_1) \arrow[rr, tail, "1\otimes U(i)"] \arrow[dr, "\mu"] \arrow[dd, "1\otimes U(f)"'] & & R \otimes U(A_2) \arrow[dr, "\mu"] \arrow[dd, "1\otimes f'" near start] & \\
& U(A_1) \arrow[rr, tail, "U(i)" near start] \arrow[dd, "U(f)"' near start] & & U(A_2) \arrow[dd, "f'"] \\
R \otimes U(A_3) \arrow[rr, tail, "1\otimes i'"' near end] \arrow[dr, "\mu"'] & & R \otimes A_4 \arrow[dr, dotted, "\mu"] & \\
& U(A_3) \arrow[rr, tail, "i'"'] & & A_4.
\end{tikzcd}$$

Note that A_4 is the cokernel of $d = \begin{pmatrix} U(i) \\ -U(f) \end{pmatrix}$. Since $R \otimes -$ is pre-abelian, $R \otimes A_4$ is the cokernel of $R \otimes d$, so the back face is also a pushout, which produces the dotted morphism μ. One verifies that μ endows the pushout A_4 in $\mathcal{C}$ with an R-module structure and that it makes the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "i"] \arrow[d, "f"'] & A_2 \arrow[d, "f'"] \\
A_3 \arrow[r, "i'"'] & A_4
\end{tikzcd}$$

in $\operatorname{mod}_{\mathcal{C}}(R)$ commute. This diagram in $\operatorname{mod}_{\mathcal{C}}(R)$ is a pushout: given a test object A and morphisms $A_2 \to A$ and $A_3 \to A$ in $\operatorname{mod}_{\mathcal{C}}(R)$, we use the pushout A_4 in $\mathcal{C}$ to get the factoring morphism $U(A_4) \to U(A)$, and we check that is a morphism in $\operatorname{mod}_{\mathcal{C}}(R)$ by again using that $R \otimes A_4$ is a pushout in $\mathcal{C}$. $\square$

5.13 Example. We equip vect with its abelian exact structure, and we equip gvect and fvect with the pulled back exact structures along $\operatorname{ung}\colon \mathsf{gvect} \to \mathsf{vect}$ and then along $\operatorname{gr}\colon \mathsf{fvect} \to$

gvect:

$$\begin{array}{ccc} \operatorname{mod}_{\mathsf{fvect}}(R) \xrightarrow{\mathrm{gr}} \operatorname{mod}_{\mathsf{gvect}}(\operatorname{gr} R) & \operatorname{mod}_{\mathsf{gvect}}(S) \xrightarrow{\mathrm{ung}} \operatorname{mod}_{\mathsf{vect}}(\operatorname{ung} S) & \operatorname{mod}_{\mathsf{vect}}(T) \\ \downarrow \qquad\qquad \downarrow & \downarrow \qquad\qquad \downarrow & \downarrow \\ \mathsf{fvect} \xrightarrow[\mathrm{gr}]{} \mathsf{gvect} & \mathsf{gvect} \xrightarrow[\mathrm{ung}]{} \mathsf{vect} & \mathsf{vect}. \end{array}$$

It is standard that these are well defined; see for example [NO82, Theorem III.3] and [SS16, Theorem 3.9]. We equip $\operatorname{mod}_{\mathsf{fvect}}(R)$, $\operatorname{mod}_{\mathsf{gvect}}(S)$, and $\operatorname{mod}_{\mathsf{vect}}(T)$ with the exact structures pulled back along the forgetful functors to our ambient categories (Lemma 5.12). It follows (Lemma 5.6) that the exact structures on $\operatorname{mod}_{\mathsf{gvect}}(S)$ and $\operatorname{mod}_{\mathsf{fvect}}(R)$ are also pulled back along $\mathrm{ung}\colon \operatorname{mod}_{\mathsf{gvect}}(S) \to \operatorname{mod}_{\mathsf{vect}}(\operatorname{ung} S)$ and $\mathrm{gr}\colon \operatorname{mod}_{\mathsf{fvect}}(R) \to \operatorname{mod}_{\mathsf{gvect}}(\operatorname{gr} R)$.

5.14 Example. Our categories $\operatorname{mod}_{\mathsf{fvect}}(R)$, $\operatorname{mod}_{\mathsf{gvect}}(S)$, and $\operatorname{mod}_{\mathsf{vect}}(T)$ are all pre-abelian, so we can check admissibility of monos and epis separately, as described in Lemma 5.9(i). Indeed, $\operatorname{mod}_{\mathsf{gvect}}(S)$ and $\operatorname{mod}_{\mathsf{vect}}(T)$ are abelian, and $\operatorname{mod}_{\mathsf{fvect}}(R)$ is quasi-abelian.

5.15 Remark. Lemma 3.28 gives a quick proof of the fact that

$$\mathrm{ung}\colon \operatorname{mod}_{\mathsf{gvect}}(S) \to \operatorname{mod}_{\mathsf{vect}}(\operatorname{ung}(S))$$

reflects projectives and injectives. Indeed, since the exact structure on $\operatorname{mod}_{\mathsf{gvect}}(S)$ is pulled back along ung, we may easily verify the lifting property against admissible epis and extension property along admissible monos using the lemma.

5.16 Example. By truncating (Example 3.24), we obtain short exact sequences that will let us induct on filtration or grading length. Let R be an effective monoid in fvect. For $A \in \operatorname{mod}_{\mathsf{fvect}}(R)$ and $w \in \mathbb{Z}$, we have a short exact sequence $A^{\geq w} \rightarrowtail A \twoheadrightarrow A^{\leq w-1}$, functorial in A. By choosing w to be the maximal integer such that $A^w \neq 0$, we obtain a short exact sequence

$$\operatorname{fz}(A^w)(w) \rightarrowtail A \twoheadrightarrow A^{\leq w-1}. \tag{5.17}$$

Similarly, let S be an effective monoid in gvect. For $B \in \operatorname{mod}_{\mathsf{gvect}}(S)$ and $w \in \mathbb{Z}$, we have a short exact sequence $B^{\geq w} \rightarrowtail B \twoheadrightarrow B^{\leq w-1}$ in $\operatorname{mod}_{\mathsf{gvect}}(S)$. By choosing w to be the maximal integer such that $B^w \neq 0$, we obtain a short exact sequence

$$\operatorname{gz}(B^w)(w) \rightarrowtail B \twoheadrightarrow B^{\leq w-1}. \tag{5.18}$$

We can pull back a pulled back exact structure in a pullback.

5.19 Proposition. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F_0"] \arrow[d, hook, "I"'] & \mathcal{B} \arrow[d, hook, "J"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd} \tag{5.20}$$

be a pullback of additive categories (Lemma 2.10) where J and I are fully faithful. Suppose that F and J are exact functors and that the exact structure on $\mathcal{C}$ is pulled back from $\mathcal{D}$ along F.

(i) *We can pull back the exact structure (Definition 5.3) from $\mathcal{B}$ to $\mathcal{A}$ along F_0.*

(ii) *The exact structure on $\mathcal{A}$ that is pulled back along F_0 is the unique one that makes (5.20) a pullback diagram of exact categories (Lemma 2.10).*

(iii) *Let us equip $\mathcal{A}$ with the pulled back exact structure along F_0 and assume F is pre-abelian (Terminology 3.42). Then a morphism f in $\mathcal{A}$ is an admissible mono (resp. admissible epi) if and only if $F_0(f)$ is an admissible mono (resp. admissible epi).*

Proof. For (i), we will show that the pulled back short exact sequences are stable under pushouts; the other verifications are dual or much easier. Let (i, p) be a pulled back short exact sequence in $\mathcal{A}$ from $\mathcal{B}$ where $i \colon A_1 \to A_2$, and let $f \colon A_1 \to A$ be a morphism. We need to show that the pushout i' of i against f exists in $\mathcal{A}$, that i' is part of a pulled back short exact sequence

$$A_1 \xrightarrow{i'} C_2 \xrightarrow{p'} C_3$$

in $\mathcal{A}$, and that (i, p) and (i', p') are both kernel-cokernel pairs.

For the first claim, consider the following diagram:

$$
\begin{tikzcd}
A_1 \arrow[r, "i"] \arrow[d, "f"'] & A_2 \arrow[r, "p"] \arrow[d] & A_3 \arrow[d] & {} \arrow[r, mapsto, "F_0"] & {} & F_0(A_1) \arrow[r, tail, "F_0(i)"] \arrow[d, "F_0(f)"'] & F_0(A_2) \arrow[r, two heads, "F_0(p)"] \arrow[d] & F_0(A_3) \arrow[d, "\cong"] \\
A \arrow[r, "i'"'] & C_2 \arrow[r, "p'"'] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dd, mapsto, "I"'] & C_3 & & & F(A) \arrow[r, tail] & B_2 \arrow[r, two heads] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dd, mapsto, "J"] & B_3 \\
& & & & & & & \\
I(A_1) \arrow[r, tail, "I(i)"] \arrow[d, "I(f)"'] & I(A_2) \arrow[r, two heads, "I(p)"] \arrow[d] & I(A_3) \arrow[d, "\cong"] & {} \arrow[r, mapsto, "F"'] & {} & JF_0(A_1) \arrow[r, tail, "JF_0(i)"] \arrow[d, "JF_0(f)"'] & JF_0(A_2) \arrow[r, two heads, "JF_0(p)"] \arrow[d] & JF_0(A_3) \arrow[d, "\cong"] \\
I(A) \arrow[r, tail, "i'"'] & C_2 \arrow[r, two heads, "p'"'] \arrow[ul, phantom, "\ulcorner", very near start] & C_3 & & & JF_0(A_1) \arrow[r, tail] & J(B_2) \arrow[r, two heads] \arrow[ul, phantom, "\ulcorner", very near start] & J(B_3).
\end{tikzcd}
$$

On the one hand, $(F_0(i), F_0(p))$ is short exact in $\mathcal{B}$ since (i,p) is a pulled back short exact sequence, so we may push out against $F_0(f)$ to construct the top-right corner (see [Buh10, Proposition 2.12] if necessary). Applying J to this diagram produces the bottom-right corner, using that exact functors preserve pushouts of admissible monos. On the other hand, consider $(I(i), I(p))$, which is short exact since $\mathcal{C}$ has the pulled back exact structure and since (5.20) commutes. We may push out against $I(f)$, which produces the bottom-left corner. Since $F(C_2)$ is a pushout of $FI(i)$ and $FI(f)$, again using that exact functors preserve pushouts of admissible monos, we have $F(C_2) \cong J(B_2)$ by uniqueness of the pushout. Thus $C_2, C_3 \in \mathcal{A}$ because (5.20) is a pullback, and moreover C_2 is the pushout of i and f in $\mathcal{A}$ because I is fully faithful. We have constructed the diagram, hence shown that the pushout i' of i against f exists in $\mathcal{A}$.

Let us justify the other two claims. The short complex $A \to C_2 \to C_3$ is a pulled back short exact sequence from $\mathcal{B}$ because applying $FI = JF_0$ to it produces a short exact sequence in $\mathcal{D}$ and because J is fully faithful. Moreover, (i,p) and (i',p') are both kernel-cokernel pairs because $(I(i), I(p))$ and $(I(i'), I(p'))$ are short exact and hence kernel-cokernel pairs in $\mathcal{C}$ and because fully faithful functors reflect limits and colimits.

For (ii), suppose we equip $\mathcal{A}$ with the pulled back exact structure along F_0, and suppose we are given a test exact category $\mathcal{T}$ and exact functors G and H as follows:

$$\begin{tikzcd}
\mathcal{T} \arrow[drr, bend left, "G"] \arrow[dr, dotted, "T"] \arrow[ddr, bend right, "H"'] & & \\
& \mathcal{A} \arrow[r, "F_0"] \arrow[d, tail, "I"'] & \mathcal{B} \arrow[d, tail, "J"] \\
& \mathcal{C} \arrow[r, "F"'] & \mathcal{D}.
\end{tikzcd}$$

There exists an additive functor T making the diagram commute because (5.20) is a pullback of additive categories. It is straightforward to check that T is exact. Indeed, if (i, p) is short exact in $\mathcal{T}$, then $(G(i), G(p)) = (F_0T(i), F_0T(p))$ is exact in $\mathcal{B}$, so since $(T(i), T(p))$ is a short complex and $\mathcal{A}$ has the pulled back exact structure, $(T(i), T(p))$ is short exact. Conversely, if (5.20) is a pullback of exact categories for some exact structure on $\mathcal{A}$, then testing with

$$\begin{tikzcd}
\mathcal{T} \arrow[drr, bend left, "F_0"] \arrow[dr, dotted, "1"] \arrow[ddr, tail, bend right, "I"'] & & \\
& \mathcal{A} \arrow[r, "F_0"] \arrow[d, tail, "I"'] & \mathcal{B} \arrow[d, tail, "J"] \\
& \mathcal{C} \arrow[r, "F"'] & \mathcal{D},
\end{tikzcd}$$

where $\mathcal{T}$ is the additive category $\mathcal{A}$ equipped with the pulled back exact structure along F_0, shows that the exact structure on $\mathcal{A}$ is at least as big as the one on $\mathcal{T}$. By maximality (Lemma 5.5), we conclude that $\mathcal{A}$ has the pulled back exact structure.

For (iii), we will only show the admissible mono case, as the other one is dual. The forward direction is immediate, so let f be a morphism in $\mathcal{A}$ such that $F_0(f)$ is an admissible mono in $\mathcal{B}$. Consider the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "f"] & A_2 \arrow[r, "p"] \arrow[d, mapsto, "I"'] & C \arrow[r, mapsto, "F_0"] & F_0(A_1) \arrow[r, tail, "F_0(f)"] & F_0(A_2) \arrow[r, two heads, "p'"] \arrow[d, mapsto, "J"] & B \\
I(A_1) \arrow[r, tail, "I(f)"] & I(A_2) \arrow[r, two heads, "p"] & C \arrow[r, mapsto, "F"'] & JF_0(A_1) \arrow[r, tail, "JF_0(f)"] & JF_0(A_2) \arrow[r, two heads, "J(p')"] & J(B).
\end{tikzcd}$$

Let p' be a morphism in $\mathcal{B}$ such that $(F_0(f), p')$ is short exact in $\mathcal{B}$. Since $FI(f) = JF_0(f)$ is an admissible mono and using that F is pre-abelian, Lemma 5.9(i) tells us $I(f)$ is an admissible mono, so let p be a morphism in $\mathcal{C}$ such that $(I(f), p)$ is short exact in $\mathcal{C}$. By uniqueness of the cokernel of $FI(f) = JF_0(f)$ and again using that F is pre-abelian, we deduce that $F(p) = J(p')$, so p is in $\mathcal{A}$ because (5.20) is a pullback. Since $JF_0(p) = F(p) = J(p')$ and since J is fully faithful, we see that $F_0(p) = p'$. We now have a short complex (f, p) in $\mathcal{A}$ such that $(F_0(f), F_0(p)) = (F_0(f), p')$ is short exact in $\mathcal{B}$, so because $\mathcal{A}$ has the pulled back exact structure along F_0, we conclude that (f, p) is short exact in $\mathcal{A}$, hence that f is an admissible mono in $\mathcal{A}$. $\square$

We now officially define our exact structures on our main rigid tensor categories.

5.21 Definition. We equip $\mathsf{mod}(G)$, $\mathsf{gmod}(G)$, and $\mathsf{fmod}(G)$ with their usual exact structures (Example 5.13), and we equip $\mathsf{pmod}(G)$ with its split exact structure. To equip $\mathsf{pgmod}(G)$ and $\mathsf{pfmod}(G)$ with exact structures, we use the diagram

$$\begin{tikzcd} \mathsf{pfmod}(G) \arrow[r, two heads, "\mathrm{gr}"] \arrow[d, tail] & \mathsf{pgmod}(G) \arrow[r, two heads, "\mathrm{ung}"] \arrow[d, tail] & \mathsf{pmod}(G) \arrow[d, tail] \\ \mathsf{fmod}(G) \arrow[r, two heads, "\mathrm{gr}"] & \mathsf{gmod}(G) \arrow[r, two heads, "\mathrm{ung}"] & \mathsf{mod}(G) \end{tikzcd} \tag{5.22}$$

of pullbacks of rigid tensor categories (Lemma 2.10) and Proposition 5.19(i) to pull back the exact structure (Definition 5.3) along $\mathrm{ung}\colon \mathsf{pgmod}(G) \to \mathsf{pmod}(G)$ and then along $\mathrm{gr}\colon \mathsf{pfmod}(G) \to \mathsf{pgmod}(G)$. In other words, the squares in (5.22) are pullbacks of exact rigid tensor categories (Proposition 5.19(ii)).

5.23 Remark. We have the following exact functors

$$\begin{tikzcd} \mathsf{pmod}(G) \arrow[r, tail, "\mathrm{fz}"] \arrow[d, tail] & \mathsf{pfmod}(G) \arrow[r, two heads, shift left, "\mathrm{gr}"] \arrow[d, tail] \arrow[dl, two heads, "\mathrm{unf}"'] & \mathsf{pgmod}(G) \arrow[l, tail, shift left, "\mathrm{trf}"] \arrow[r, two heads, shift left, "\mathrm{ung}"] \arrow[d, tail] & \mathsf{pmod}(G) \arrow[l, tail, shift left, "\mathrm{gz}"] \arrow[d, tail] \\ \mathsf{mod}(G) \arrow[r, tail, shift left, "\mathrm{fz}"] & \mathsf{fmod}(G) \arrow[l, two heads, shift left, "\mathrm{unf}"] \arrow[r, two heads, shift left, "\mathrm{gr}"] & \mathsf{gmod}(G) \arrow[l, tail, shift left, "\mathrm{trf}"] \arrow[r, two heads, shift left, "\mathrm{ung}"] & \mathsf{mod}(G). \arrow[l, tail, shift left, "\mathrm{gz}"] \end{tikzcd}$$

Note that both unf functors are exact by the five-lemma and induction on filtration length and that $\mathrm{unf}\colon \mathsf{pfmod}(G) \to \mathsf{mod}(G)$ escapes the permutation variants.

5.24 Example. We have the following generators as extension-closed subcategories:

$\mathsf{pfmod}(G)$: twists of $\mathrm{fz}\, k(G/H)$ for $H \leqslant G$	$\mathsf{fmod}(G)$: twists of the unit $\mathbb{1}$
$\mathsf{pgmod}(G)$: twists of $\mathrm{gz}\, k(G/H)$ for $H \leqslant G$	$\mathsf{gmod}(G)$: twists of the unit $\mathbb{1}$
$\mathsf{pmod}(G)$: $k(G/H)$ for $H \leqslant G$	$\mathsf{mod}(G)$: the unit $\mathbb{1}$

This follows from our short exact sequences (5.17) and (5.18) which are still short exact in respectively $\mathsf{pfmod}(G)$ and $\mathsf{pgmod}(G)$.

Let us transition to discussing Frobenius exact structures.

5.25 Lemma. *In an exact rigid tensor category $\mathcal{E}$, the dual of a projective object is injective, and the dual of an injective object is projective.*

Proof. If P is projective, then by rigidity we have $\mathrm{Hom}_{\mathcal{E}}(P, (-)^\vee) \cong \mathrm{Hom}_{\mathcal{E}}(-, P^\vee)$. This is exact because P is projective and $(-)^\vee$ is exact (Terminology 2.2), and this shows $P^\vee$ is injective. Similarly, if I is injective, then we have $\mathrm{Hom}_{\mathcal{E}}(I^\vee, -) \cong \mathrm{Hom}_{\mathcal{E}}((-)^\vee, I)$ by rigidity and using that $(-)^\vee$ is a duality (Lemma 2.5). Again this is exact because I is injective and $(-)^\vee$ is exact (Terminology 2.2), and this shows $I^\vee$ is projective. $\square$

5.26 Lemma. *For an exact category $\mathcal{E}$ and a set of objects $J \subset \mathcal{E}$, the extension-closed subcategory of $\mathcal{E}$ generated by J is comprised of the objects $A \in \mathcal{E}$ that admit a sequence of admissible monos*

$$0 = A_{n+1} \rightarrowtail A_n \rightarrowtail \cdots \rightarrowtail A_1 \rightarrowtail A_0 = A \tag{5.27}$$

such that $\operatorname{coker}(A_{i+1} \rightarrowtail A_i) \in J$ *for all* $i = 0, \dots, n$.

Proof. Let us call a sequence of admissible monos as in (5.27) a J-filtration of length n. Observe that if $A \twoheadrightarrow B$ is an admissible epi with kernel in J and B admits a J-filtration of length n, then A admits a J-filtration of length $n+1$ by pulling back the filtration.

We can describe the objects in the extension-closed subcategory generated by J as $\bigcup_{n\geq 0} J_n$ where $J_0 = J$ and J_{n+1} is the set of objects $B \in \mathcal{E}$ that fit in a short exact sequence of the form $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ where $A, C \in J_n$. We will show by induction that every object in J_n admits a finite J-filtration. Certainly this is true for $n = 0$, so suppose $n \geq 1$, and let $B \in J_{n+1}$ fit in a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ where $A, C \in J_n$.

Consider the following diagram:

$$\begin{tikzcd}
A_1 \arrow[d, tail] & & \\
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] \arrow[ur, phantom, "\ulcorner"] & B' \arrow[r, two heads] & C \\
A' \arrow[r, tail] & B_1' \arrow[r, two heads] \arrow[u, tail, "i"'] \arrow[ur, phantom, "\urcorner"] & C_1. \arrow[u, tail, "j"']
\end{tikzcd}$$

Using our inductive hypothesis, let $A_1 \rightarrowtail A$ be the first step in a finite J-filtration of A, and let A' denote the cokernel in J. Pushing out produces the short exact sequence $A' \rightarrowtail B' \twoheadrightarrow C$ where $A' \in J$ and $C \in J_n$. Again by our inductive hypothesis, let $j\colon C_1 \rightarrowtail C$ be the first step in a finite J-filtration of C. Pulling back produces the short exact sequence $A' \rightarrowtail B_1' \twoheadrightarrow C_1$ where $C_1 \in J_{n-1}$ and $A' \in J$. Thus $B_1' \in J_n$. Using our inductive hypothesis one last time to get a finite J-filtration of B_1', we may compose it with i to get a finite J-filtration of B'. Finally, by our observation at the beginning of this proof, we are done. $\square$

5.28 Terminology. For an exact category $\mathcal{E}$ and an object $A \in \mathcal{E}$, a *projective pre-cover of A* is an admissible epi $P \twoheadrightarrow A$ from a projective object $P \in \mathcal{E}$, and an *injective pre-envelope of A* is an admissible mono $A \rightarrowtail I$ into an injective object $I \in \mathcal{E}$.

5.29 Proposition. *Let $\mathcal{E}$ be an exact rigid tensor category whose unit admits a projective pre-cover $\pi\colon P \twoheadrightarrow \mathbb{1}$.*

(i) The projectives form a thick additive tensor ideal $\langle P \rangle$ generated by P.

(ii) The exact category $\mathcal{E}$ is Frobenius exact.

(iii) Let $J \subset \mathcal{E}$ be a set of generators of $\mathcal{E}$ as an extension-closed subcategory. Then $\langle P \rangle$ is the thick additive subcategory generated by $J \otimes P$.

Proof. For (i), the projectives certainly form a thick additive subcategory, and they form a tensor ideal by rigidity: if $Q \in \mathcal{E}$ is projective and $A \in \mathcal{E}$, then

$$\mathrm{Hom}_{\mathcal{E}}(A \otimes Q, -) \cong \mathrm{Hom}_{\mathcal{E}}(Q, A^\vee \otimes -)$$

is exact because Q is projective and $-\otimes-$ is exact in both variables, so $A\otimes Q$ is projective. Since P is projective, it suffices to show that any projective is in $\langle P\rangle$. Observe that in fact any object $A \in \mathcal{E}$ receives an admissible epi from a projective, namely

$$1\otimes\pi\colon A\otimes P \twoheadrightarrow A. \tag{5.30}$$

In particular, if Q is projective, then $1\otimes\pi\colon Q\otimes P \twoheadrightarrow Q$ admits a section, so $Q \mid Q\otimes P \in \langle P\rangle$.

For (ii), the projective pre-cover (5.30) shows that $\mathcal{E}$ has enough projectives, and the projectives and injectives coincide by Lemma 5.25 and Lemma 2.5.

For (iii), let us denote by $\mathcal{S}$ the thick additive subcategory generated by $J\otimes P$. Then $\langle P\rangle \supset \mathcal{S}$ because $\langle P\rangle$ is thick and additive by (i) and because $\langle P\rangle \supset J\otimes P$ by virtue of every object in $J\otimes P$ being projective. For the converse, it suffices to show that for any $A\in\mathcal{E}$, we have $A\otimes P\in\mathcal{S}$. Indeed, then if $Q\in\langle P\rangle$ is projective, we would have $Q \mid Q\otimes P\in\mathcal{S}$. Let $A\in\mathcal{E}$. Since J generates $\mathcal{E}$ as an extension-closed subcategory, by Lemma 5.26 there exists a sequence of admissible monos

$$0 = A_{n+1} \rightarrowtail A_n \rightarrowtail \cdots \rightarrowtail A_1 \rightarrowtail A_0 = A$$

such that $\operatorname{coker}(A_{i+1}\rightarrowtail A_i)\in J$ for all $i=0,\dots,n$. Tensoring this sequence with P produces a sequence

$$0 = A_{n+1}\otimes P \rightarrowtail A_n\otimes P \rightarrowtail \cdots \rightarrowtail A_1\otimes P \rightarrowtail A_0\otimes P = A\otimes P$$

of split monos where $\operatorname{coker}(A_{i+1}\otimes P\rightarrowtail A_i\otimes P)\in J\otimes P$. Thus $A\otimes P\in\mathcal{S}$. $\square$

We now show that $\mathsf{pgmod}(G)$, $\mathsf{pmod}(G)$, $\mathsf{fmod}(G)$, $\mathsf{gmod}(G)$, and $\mathsf{mod}(G)$ are Frobenius exact. We will show Section 9 that $\mathsf{pfmod}(E)$ is Frobenius exact in the elementary abelian case.

5.31 Example. Let R, S, and T be monoids in respectively fvect, gvect, and vect. In the categories $\mathrm{mod}_{\mathsf{fvect}}(R)$, $\mathrm{mod}_{\mathsf{gvect}}(S)$, and $\mathrm{mod}_{\mathsf{vect}}(T)$, the objects R, S, and T are projective because as k-vector spaces

$$\mathrm{Hom}_{\mathrm{mod}_{\mathsf{fvect}}(R)}(R, A) \cong A^0$$
$$\mathrm{Hom}_{\mathrm{mod}_{\mathsf{gvect}}(S)}(S, B) \cong B^0$$
$$\mathrm{Hom}_{\mathrm{mod}_{\mathsf{vect}}(T)}(T, C) \cong C.$$

Thus when R, S, and T are Hopf monoids, the counits $\varepsilon\colon R \twoheadrightarrow \mathbb{1}$, $\varepsilon\colon S \twoheadrightarrow \mathbb{1}$, and $\varepsilon\colon T \twoheadrightarrow \mathbb{1}$ are projective pre-covers of the units.

5.32 Example. The exact categories $\mathsf{pgmod}(G)$, $\mathsf{pmod}(G)$, $\mathsf{fmod}(G)$, $\mathsf{gmod}(G)$, and $\mathsf{mod}(G)$ are all Frobenius exact because we have the following projective pre-covers of units (Proposition 5.29(ii)):

$$\begin{array}{llll} & & \varepsilon\colon\ \mathrm{fz}\, kG \twoheadrightarrow \mathbb{1} & \text{in } \mathsf{fmod}(G) \\ 1\colon \mathbb{1} \twoheadrightarrow \mathbb{1} & \text{in } \mathsf{pgmod}(G) & \varepsilon\colon\ \mathrm{gz}\, kG \twoheadrightarrow \mathbb{1} & \text{in } \mathsf{gmod}(G) \\ 1\colon \mathbb{1} \twoheadrightarrow \mathbb{1} & \text{in } \mathsf{pmod}(G) & \varepsilon\colon kG \twoheadrightarrow \mathbb{1} & \text{in } \mathsf{mod}(G). \end{array}$$

Indeed $\mathsf{pgmod}(G)$ and $\mathsf{pmod}(G)$ have the split exact structure, so every object is projective. The other three are from Example 5.31.

5.33 Remark. The projective pre-covers $\mathrm{fz}\, kG \twoheadrightarrow \mathbb{1}$ in $\mathsf{fmod}(G)$ and $\mathrm{gz}\, kG \twoheadrightarrow \mathbb{1}$ in $\mathsf{gmod}(G)$ are not projective pre-covers in $\mathsf{pfmod}(G)$ and $\mathsf{pgmod}(G)$ when $G \neq 1$ because in these exact subcategories they are not admissible epis. Indeed, the augmentation map $kG \twoheadrightarrow k$ is not split exact in $\mathsf{mod}(G)$ because G is a p-group.

5.34 Example. The projectives in $\mathsf{fmod}(G)$ are the direct sums of twists of $\mathrm{fz}\, kG$, and the projectives in $\mathsf{gmod}(G)$ are the direct sums of twists of $\mathrm{gz}\, kG$. This follows from Proposition 5.29(iii) and Example 5.24. In particular, if $A \in \mathsf{fmod}(G)$ is projective, then $\mathrm{unf}(A) \in \mathsf{mod}(G)$ is projective, and if $B \in \mathsf{gmod}(G)$ is projective, then $\mathrm{ung}(B) \in \mathsf{mod}(G)$ is projective.

6 tt-geometry recollections

In this section we recall the tools from tt-geometry that we will be using for the rest of this work. In particular, we discuss π-points, and we upgrade some of our results from the previous sections into the world of tt-categories.

6.1 Recollection. A *tensor triangulated category* is a triangulated category with a tensor structure where the tensor $-\otimes-$ is triangulated in each variable (and the additive structures agree). By a *tt-category* $\mathcal{K}$ we mean an essentially small tensor triangulated category. A *tt-ideal* of $\mathcal{K}$ is a thick triangulated tensor ideal of $\mathcal{K}$, and for objects $x_1, \ldots, x_n \in \mathcal{K}$, we write $\langle x_1, \ldots, x_n \rangle$ for the tt-ideal they generate. We say a tt-ideal is *finitely generated* if there exist such objects. A *prime tt-ideal* of $\mathcal{K}$ is a proper tt-ideal $\mathcal{P} \subset \mathcal{K}$ such that if $x \otimes y \in \mathcal{P}$ for some objects $x, y \in \mathcal{K}$, then $x \in \mathcal{P}$ or $y \in \mathcal{P}$. By a *tt-functor* we mean a tensor triangulated functor.

6.2 Recollection. The tt-categories in this work all arise from bounded derived categories of exact tensor categories or stable categories of Frobenius exact tensor categories in the following fashion:

$$\begin{tikzcd}[row sep=small]
\text{exact rigid tensor} \arrow[r, mapsto, "\mathrm{D_b}"] \arrow[d, dash] & \text{rigid tt-} \arrow[d, dash] & \text{Frobenius exact rigid tensor} \arrow[l, mapsto, "\mathrm{st}"'] \arrow[d, dash] \\
\text{exact tensor} \arrow[r, mapsto, "\mathrm{D_b}"] \arrow[d, dash] & \text{tt-} \arrow[d, dash] & \text{Frobenius exact tensor} \arrow[l, mapsto, "\mathrm{st}"'] \arrow[d, dash] \\
\text{exact} \arrow[r, mapsto, "\mathrm{D_b}"] & \text{triangulated} & \text{Frobenius exact.} \arrow[l, mapsto, "\mathrm{st}"']
\end{tikzcd}$$

Given an exact category $\mathcal{E}$, the *bounded derived category* $\mathrm{D_b}(\mathcal{E})$ *of* $\mathcal{E}$ is the triangulated category given by the Verdier quotient of the bounded homotopy category $\mathrm{K_b}(\mathcal{E})$ of $\mathcal{E}$ by the triangulated subcategory of acyclic complexes [Buh10, Section 10.4]. Given a Frobenius exact tensor category $\mathcal{E}$, the *stable category* $\mathrm{st}(\mathcal{E})$ is the triangulated category is given by the additive quotient of $\mathcal{E}$ by the full subcategory of projective objects [Kra21, Section 3.3]. An exact tensor functor $F\colon \mathcal{E} \to \mathcal{F}$ induces a tt-functor $F\colon\ \mathrm{D_b}(\mathcal{E}) \to \mathrm{D_b}(\mathcal{F})$; if $\mathcal{E}$ and $\mathcal{F}$ are Frobenius exact and F preserves projectives, then F induces a tt-functor $F\colon\ \mathrm{st}(\mathcal{E}) \to \mathrm{st}(\mathcal{F})$ on stable categories.

6.3 Recollection. The *spectrum* $\mathrm{Spc}(\mathcal{K})$ of a tt-category $\mathcal{K}$ is the topological space of prime tt-ideals in $\mathcal{K}$, as follows. For an object $x \in \mathcal{K}$, the *support* $\mathrm{supp}(x)$ *of* x is

$$\mathrm{supp}(x) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\};$$

we say that $x \in \mathcal{K}$ has *full support* if $\mathrm{supp}(x) = \mathrm{Spc}(\mathcal{K})$, for example the unit $x = \mathbb{1}$. The topology on $\mathrm{Spc}(\mathcal{K})$ is generated by the basis of closed subsets $\mathrm{supp}(x)$ for $x \in \mathcal{K}$. We say that $\mathcal{K}$ is *local* if its spectrum $\mathrm{Spc}(\mathcal{K})$ has a unique closed point. The spectrum Spc is functorial. In particular, a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a continuous map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ on spectra given by pulling back primes: $\mathrm{Spc}(F)(\mathcal{Q}) = F^{-1}(\mathcal{Q})$. Moreover, an equivalence of tt-categories induces a homeomorphism on spectra because naturally isomorphic tt-functors induce the same map on spectra.

6.4 Recollection. Given a tt-ideal $\mathcal{J} \subset \mathcal{K}$ in a tt-category $\mathcal{K}$, we denote by $U(\mathcal{J}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{J} \subset \mathcal{P}\}$ the subspace of primes containing $\mathcal{J}$. If $\mathcal{J} = \langle x_1, \ldots, x_n\rangle$ is *finitely generated* as a tt-ideal, then by thickness $\mathcal{J} = \langle x\rangle$ where $x = x_1 \oplus \cdots \oplus x_n$. Thus $U(\mathcal{J})$ is an open set in $\mathrm{Spc}(\mathcal{K})$, and we write

$$\mathrm{open}(x) = \mathrm{open}(\mathcal{J}) = \mathrm{Spc}(\mathcal{K}) \setminus \mathrm{supp}(x) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \in \mathcal{P}\}.$$

We call this the *vanishing locus of* x.

6.5 Recollection. The induced map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ on spectra by a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ satisfies the following formula for any $x \in \mathcal{K}$:

$$\mathrm{supp}(F(x)) = \mathrm{Spc}(F)^{-1}(\mathrm{supp}(x)).$$

Thus we have the following:

(i) If $F(x) \in \mathcal{L}$ has full support, then $\mathrm{Spc}(F)$ factors through $\mathrm{supp}(x)$:

$$\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \longrightarrow \mathrm{supp}(x) \subset \mathrm{Spc}(\mathcal{K}).$$

(*ii*) If $F(x) = 0 \in \mathcal{L}$ for some $x \in \mathcal{K}$, then $\mathrm{Spc}(F)$ factors through $\mathrm{open}(x)$:

$$\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \longrightarrow \mathrm{open}(x) \subset \mathrm{Spc}(\mathcal{K}).$$

We will frequently need to know when a tt-functor induces an injective or surjective map on spectra of some sort.

6.6 Recollection. In the following situations, a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ induces an injection on spectra of some sort:

(*i*) If F is essentially surjective, then $\mathrm{Spc}(F)$ is an injection [Bal05, Corollary 3.8].

(*ii*) If F admits a section (thus is essentially surjective), then $\mathrm{Spc}(F)$ is a split injection by functoriality, hence is a homeomorphism onto its image.

(*iii*) If F is the Verdier quotient $F\colon \mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{J} = \mathcal{L}$ at a finitely generated tt-ideal $\mathcal{J} \subset \mathcal{K}$, then $\mathrm{Spc}(F)$ induces a homeomorphism

$$\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \xrightarrow{\cong} \mathrm{open}(\mathcal{J}) \subset \mathrm{Spc}(\mathcal{K})$$

of $\mathrm{Spc}(\mathcal{L})$ with the vanishing locus of $\mathcal{J}$ [Bal05, Proposition 3.11].

6.7 Recollection. Let $F\colon \mathcal{K} \to \mathcal{L}$ be a tt-functor, and let $x \in \mathcal{K}$ be an object in $\mathcal{K}$. We say that a morphsim f in $\mathcal{K}$ *vanishes on* x if $f \otimes x = 0$. We say that F *detects nilpotence on* x if for any morphism f in $\mathcal{K}$ such that $F(f) = 0$, there exists $m \geq 0$ such that $f^{\otimes m}$ vanishes on x. If we only need $m = 1$, then we say that F *is faithful on* x, and if $x = \mathbb{1}$, then we say that F *detects nilpotence.*

6.8 Recollection. Let $\mathcal{K}$ be a rigid tt-category. In the following situations, a tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a surjection on spectra of some sort:

(*i*) Let $x \in \mathcal{K}$. If F detects nilpotence on x, then $\mathrm{Spc}(F)$ surjects $\mathrm{supp}(x)$ ([Bal18, Theorem 1.3] and [BG22b, Corollary 7.8]), in the sense that the image of $\mathrm{Spc}(F)$ contains $\mathrm{supp}(x)$. For example, if F is faithful, then $\mathrm{Spc}(F)$ is surjective.

(*ii*) The tt-functor F is conservative if and only if $\mathrm{Spc}(F)$ surjects the closed points of $\mathrm{Spc}(\mathcal{K})$ [Bal18, Theorem 1.2].

We record some reductions for later use.

6.9 Lemma. *Let $\mathcal{K}$ be a rigid tt-category, let $F\colon \mathcal{K} \to \mathcal{L}$ be a tt-functor, and let $x \in \mathcal{K}$ be an object in $\mathcal{K}$.*

(*i*) *The tt-functor F detects nilpotence on x if and only if for any morphism $f\colon \mathbb{1} \to y$ in $\mathcal{K}$ such that $F(f) = 0$, there exists $m \geq 0$ such that $f^{\otimes m}$ vanishes on x.*

(*ii*) *Let $c \in \mathcal{K}$ be another object. Then F detects nilpotence on $x \otimes c$ if and only if for any morphism $f\colon \mathbb{1} \to y$ in $\mathcal{K}$ such that $F(f) = 0$, there exists $m \geq 0$ such that*

$$x^\vee \otimes x \xrightarrow{\mathrm{ev}_x} \mathbb{1} \xrightarrow{f^{\otimes m}} y^{\otimes m}$$

(Remark 2.6) vanishes on c

Proof. For (*i*), the forward implication is trivial. For the reverse implication, let $f\colon y \to z$ be a morphism in $\mathcal{K}$, and consider

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}}(y, z) \;&\cong\; \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, z \otimes y^\vee) \\ f \;&\longmapsto\; (f \otimes y^\vee)\mathrm{coev}_y \end{aligned}$$

where $(f \otimes y^\vee)\mathrm{coev}_y$ is the morphism

$$\mathbb{1} \xrightarrow{\mathrm{coev}_y} y \otimes y^\vee \xrightarrow{f \otimes y^\vee} z \otimes y^\vee.$$

By our hypothesis, we have

$$(f^{\otimes m} \otimes (y^{\otimes m})^\vee \otimes x)(\mathrm{coev}_{y^{\otimes m}} \otimes x) \cong ((f \otimes y^\vee)\mathrm{coev}_y)^{\otimes m} \otimes x = 0$$

for some $m \geq 0$. Thus

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}}(y^{\otimes m} \otimes x, z^{\otimes m} \otimes x) \quad &\cong \quad \mathrm{Hom}_{\mathcal{K}}(x, (y^{\otimes m})^\vee \otimes z^{\otimes m} \otimes x), \\ f^{\otimes m} \otimes x \;&\longmapsto\; (f^{\otimes m} \otimes (y^{\otimes m})^\vee \otimes x)(\mathrm{coev}_{y^{\otimes m}} \otimes x) = 0 \end{aligned}$$

so $f^{\otimes m} \otimes x = 0$.

For (ii), let $f \colon \mathbb{1} \to y$ be a morphism in $\mathcal{K}$ such that $F(f) = 0$, and let $m \geq 0$. By (i), it suffices to show that $f^{\otimes m} \otimes x \otimes c = 0$ if and only if $(f^{\otimes m} \otimes c)(\mathrm{ev}_x \otimes c) = 0$. Consider the following commutative diagram in $\mathcal{K}$:

$$\begin{tikzcd}
x^\vee \otimes x \otimes c \arrow[r, "\mathrm{ev}_x \otimes c"] \arrow[d, "f^{\otimes m} \otimes x^\vee \otimes x \otimes c"'] & c \arrow[d, "f^{\otimes m} \otimes c"] \\
y^{\otimes m} \otimes x^\vee \otimes x \otimes c \arrow[r, "y^{\otimes m} \otimes \mathrm{ev}_x \otimes c"'] & y^{\otimes m} \otimes c.
\end{tikzcd}$$

Since

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{K}}(x \otimes c, y^{\otimes m} \otimes x \otimes c) & \cong & \mathrm{Hom}_{\mathcal{K}}(x^\vee \otimes x \otimes c, y^{\otimes m} \otimes c), \\
f^{\otimes m} \otimes x \otimes c & \longmapsto & (y^{\otimes m} \otimes \mathrm{ev}_x \otimes c)(f^{\otimes m} \otimes x^\vee \otimes x \otimes c)
\end{array}$$

we are done. $\square$

We will frequently combine these tools as follows.

6.10 Method. Let us describe a *divide and conquer* strategy that we will repeatedly employ in this work. Given an object $x \in \mathcal{K}$, its support and vanishing locus partition the spectrum: $\mathrm{Spc}(\mathcal{K}) = \mathrm{supp}(x) \sqcup \mathrm{open}(x)$. Thus we can determine the subspaces $\mathrm{supp}(x)$ and $\mathrm{open}(x)$ separately and then determine the specializations from $\mathrm{open}(x)$ to $\mathrm{supp}(x)$; there are no specializations the other way because $\mathrm{supp}(x)$ is closed. To determine $\mathrm{supp}(x)$ and $\mathrm{open}(x)$ we typically do the following:

(i) To determine $\mathrm{supp}(x)$, we find a tt-functor $G \colon \mathcal{K} \to \mathcal{L}$ such that $G(x)$ has full support, G admits a section, and G detects nilpotence on x. Then G induces a homeomorphism (Recollection 6.5(i), Recollection 6.6(ii), and Recollection 6.8(i))

$$\mathrm{Spc}(G) \colon \mathrm{Spc}(\mathcal{L}) \xrightarrow{\cong} \mathrm{supp}(x) \subset \mathrm{Spc}(\mathcal{K}).$$

(ii) To determine $\mathrm{open}(x)$, we find a tt-functor $U \colon \mathcal{K} \to \mathcal{M}$ such that $U(x) = 0$ and such that the induced tt-functor $\overline{U} \colon \mathcal{K}/\langle x \rangle \to \mathcal{M}$ is an equivalence. Then $\overline{U}$ induces a homeomorphism (Recollection 6.5(ii) and Recollection 6.3)

$$\mathrm{Spc}(\overline{U}) \colon \mathrm{Spc}(\mathcal{M}) \xrightarrow{\cong} \mathrm{open}(x) \subset \mathrm{Spc}(\mathcal{K}).$$

6.11 Method. We may even divide and conquer using two objects. Given objects $x, y \in \mathcal{K}$, we have

$$\mathrm{Spc}(\mathcal{K}) = (\mathrm{open}(x) \cup \mathrm{open}(y)) \sqcup \mathrm{supp}(x \otimes y).$$

In this situation, we only need to determine the specializations from $\mathrm{open}(x, y)$ to $\mathrm{open}(x) \setminus \mathrm{open}(x, y)$ and $\mathrm{open}(y) \setminus \mathrm{open}(x, y)$ and from $\mathrm{open}(x)$ and $\mathrm{open}(y)$ to $\mathrm{supp}(x \otimes y)$. Note that $\mathrm{open}(x) \setminus \mathrm{open}(x, y)$ is the support of y in the Verdier quotient $\mathcal{K}/\langle x \rangle$ (Recollection 6.6(*iii*)) and similarly for $\mathrm{open}(y) \setminus \mathrm{open}(x, y)$.

We preempt our discussion about π-points by discussing the effect of rigidity on the lattice of tt-ideals.

6.12 Recollection. Let $\mathcal{K}$ be a tt-category. Recall that the *radical* $\sqrt{\mathcal{J}}$ of a tt-ideal $\mathcal{J} \subset \mathcal{K}$ is the intersection of the primes $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ that contain $\mathcal{J}$:

$$\sqrt{\mathcal{J}} = \bigcap_{\mathcal{J} \subset \mathcal{P} \in \mathrm{Spc}(\mathcal{K})} \mathcal{P}.$$

(*i*) We have $\sqrt{\langle a \rangle} \cap \sqrt{\langle b \rangle} = \sqrt{\langle a \otimes b \rangle}$ for any objects $a, b \in \mathcal{K}$. This follows from [Bal05, Theorem 4.10].

(*ii*) If $\mathcal{K}$ is rigid, then every tt-ideal is its own radical [Bal05, Remark 4.3 and Proposition 4.4].

6.13 Notation. Let $\mathcal{K}$ be a rigid tt-category. We denote by $\mathrm{ideals}(\mathcal{K})$ the lattice of tt-ideals in $\mathcal{K}$. Given $\mathcal{J}_1, \mathcal{J}_2 \in \mathrm{ideals}(\mathcal{K})$, their join is the tt-ideal generated by $\mathcal{J}_1$ and $\mathcal{J}_2$ and their meet is their intersection. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are finitely generated, then there exist objects $x_1, x_2 \in \mathcal{K}$ such that $\mathcal{J}_1 = \langle x_1 \rangle$ and $\mathcal{J}_2 = \langle x_2 \rangle$ (Recollection 6.4), in which case the join of $\mathcal{J}_1$ and $\mathcal{J}_2$ is $\langle x_1 \oplus x_2 \rangle$ and the meet is $\langle x_1 \otimes x_2 \rangle$ (Recollection 6.12(*i*) and Recollection 6.12(*ii*)).

6.14 Lemma. *Let $\mathcal{K}$ be a rigid tt-category. If $\mathcal{K}$ is generated as a triangulated subcategory by its unit, then the prime tt-ideals of $\mathcal{K}$ are precisely the prime elements in the lattice of thick subcategories of $\mathcal{K}$.*

Proof. Note that in this situation, the tt-ideals of $\mathcal{K}$ are precisely the thick subcategories of $\mathcal{K}$. Thus we are reduced to showing that the prime tt-ideals of $\mathcal{K}$ are precisely the prime elements in $\mathrm{ideals}(\mathcal{K})$. Let $\mathcal{P}$ be a prime tt-ideal, and suppose $\mathcal{I}, \mathcal{J} \in \mathrm{ideals}(\mathcal{K})$ are tt-ideals of $\mathcal{K}$ such that $\mathcal{I} \not\subset \mathcal{P}$ and $\mathcal{J} \not\subset \mathcal{P}$. Then for $a \in \mathcal{I} \setminus \mathcal{P}$ and $b \in \mathcal{J} \setminus \mathcal{P}$, we have $a \otimes b \in (\mathcal{I} \cap \mathcal{J}) \setminus \mathcal{P}$ because $\mathcal{I}$ and $\mathcal{J}$ are tt-ideals and $\mathcal{P}$ is prime. Thus $\mathcal{I} \cap \mathcal{J} \not\subset \mathcal{P}$. Since $\mathcal{I}$ and $\mathcal{J}$ were arbitrary, we conclude that $\mathcal{P}$ is a prime element in $\mathrm{ideals}(\mathcal{K})$. Conversely, let $\mathcal{P} \in \mathrm{ideals}(\mathcal{K})$ be a prime element, and suppose $a, b \in \mathcal{K}$ are such that $a \otimes b \in \mathcal{P}$. Then by Notation 6.13 we have $\langle a \rangle \cap \langle b \rangle = \langle a \otimes b \rangle \subset \mathcal{P}$, so $\langle a \rangle \subset \mathcal{P}$ or $\langle b \rangle \subset \mathcal{P}$ by our hypothesis on $\mathcal{P}$. Hence $a \in \mathcal{P}$ or $b \in \mathcal{P}$. □

We now discuss the example $\mathrm{D_b}(\mathrm{mod}(k\mathcal{G}))$ where $\mathcal{G}$ is a finite group scheme.

6.15 Notation. Given a topological space X, we denote by $\dot{X}$ the one-point compactification of X and by $0 \in \dot{X}$ the unique closed point.

6.16 Recollection. Let $\mathcal{G}$ be a finite group scheme over k. We denote by $k\mathcal{G}$ the group algebra of $\mathcal{G}$; this is a finite-dimensional co-commutative Hopf algebra over k. We denote by

$$\mathrm{H}^\bullet(\mathcal{G}, k) = \begin{cases} \mathrm{Ext}^\bullet_{k\mathcal{G}}(k,k) & p = 2 \\ \text{even part of } \mathrm{Ext}^\bullet_{k\mathcal{G}}(k,k) & p > 2 \end{cases}$$

(the commutative part of) the cohomology of $\mathcal{G}$; this is a commutative finitely-generated algebra over k.

The spectrum of $\mathrm{D_b}(\mathrm{mod}(k\mathcal{G}))$ is homeomorphic via the comparison map to the homogeneous spectrum $\mathrm{Spec^h}\,\mathrm{H}^\bullet(\mathcal{G}, k)$ of the cohomology of $\mathcal{G}$ [Bal10a, Proposition 8.5]:

$$\begin{tikzcd}
 & \{M \mid \mathrm{Ann}_{H^\bullet(\mathcal{G},k)}(H^\bullet(\mathcal{G},M)) \not\subset \mathfrak{p}\} & \mathfrak{p} \arrow[l, mapsto] \\
 & \operatorname{Spc}\operatorname{st}(\mathsf{mod}(k\mathcal{G})) \arrow[dl, tail] & \operatorname{Proj}\mathrm{H}^\bullet(\mathcal{G},k). \arrow[l, "\cong"'] \arrow[dl, tail] \\
\operatorname{Spc}\mathrm{D_b}(\mathrm{mod}(k\mathcal{G})) \arrow[r, "\cong", "\mathrm{comp}"'] & \mathrm{Spec^h}\,\mathrm{H}^\bullet(\mathcal{G},k) &
\end{tikzcd}$$

Our motivation for using π-points is to give a more explicit description of the spectrum $\operatorname{Spc}\mathrm{D_b}(\mathsf{mod}(k\mathcal{G}))$. We begin with the most general case.

6.17 Recollection. A π-point (α, K) of $\mathcal{G}$ is the data of a field extension K/k and a flat map of K-algebras $\alpha\colon KC_p \to K\mathcal{G}_K$ that factors through the group algebra $KC_K \subset K\mathcal{G}_K$ of some abelian subgroup scheme $C_K \subset \mathcal{G}_K$. We emphasize that the factorizing condition is the only part of the definition that uses the coalgebra structure on $k\mathcal{G}$.

6.18 Notation. For a field extension K/k, we write vect_K, gvect_K, and fvect_K for the ambient categories over K and $\mathsf{mod}_K(G) = \mathrm{mod}_{\mathsf{vect}_K}(KG)$ and so on. We also use a subscript K to denote a base change to K; for example for $C \in \mathsf{mod}(G)$, we write $C_K = C \otimes_k K \in \mathsf{mod}_K(G)$. We implicitly base change constant group schemes.

6.19 Recollection. On the one hand, a π-point (α, K) induces a restriction map on cohomology which we denote by

$$\mathrm{H}^\bullet(\alpha, K)\colon\ \mathrm{H}^\bullet(\mathcal{G}_K, K) \longrightarrow \mathrm{H}^\bullet(C_p, K),$$

and its kernel in $\mathrm{H}^\bullet(\mathcal{G}, k)$ is a homogeneous prime ideal:

$$\mathrm{H}^\bullet(\mathcal{G}, k) \cap \ker \mathrm{H}^\bullet(\alpha, K) \in \operatorname{Proj} \mathrm{H}^\bullet(\mathcal{G}, k).$$

On the other hand, a π-point (α, K) induces a triangulated functor which we denote by

$$(\alpha, K)^*\colon\ \mathrm{st}(\mathrm{mod}(k\mathcal{G})) \xrightarrow{-\otimes_k K} \mathrm{st}(\mathrm{mod}_{\mathsf{vect}_K}(K\mathcal{G}_K)) \xrightarrow{\alpha^*} \mathrm{st}(\mathsf{mod}_K(C_p)), \tag{6.20}$$

and its kernel in $\mathrm{st}(\mathrm{mod}(k\mathcal{G}))$ is a prime tt-ideal:

$$\ker(\alpha, K)^* \in \operatorname{Spc} \mathrm{st}(\mathrm{mod}(k\mathcal{G})).$$

These associations are surjective, and we define the space $\Pi(\mathcal{G})$ of π-points of $\mathcal{G}$ to be the set of π-points of $\mathcal{G}$ modulo the equivalence relation and equipped with the topology that makes the back face of the following diagram commute:

$$\begin{tikzcd}
 & & \Pi(\mathcal{G}) \arrow[dll, tail] \arrow[d, "\ker(-)^*"', "\cong"] \arrow[drr, "\mathrm{H}^\bullet(\mathcal{G},k)\cap\ker\mathrm{H}^\bullet(-)", "\cong"'] & & \\
\dot{\Pi}(\mathcal{G}) \arrow[d, "\cong"'] \arrow[drr, "\cong"] & & \operatorname{Spc}\mathrm{st}(\mathrm{mod}(k\mathcal{G})) \arrow[dll, tail, crossing over] & & \operatorname{Proj}\mathrm{H}^\bullet(\mathcal{G},k) \arrow[ll, "\cong"] \arrow[dll, tail] \\
\operatorname{Spc}\mathrm{D_b}(\mathrm{mod}(k\mathcal{G})) \arrow[rr, "\cong", "\mathrm{comp}"'] & & \operatorname{Spec^h}\mathrm{H}^\bullet(\mathcal{G},k). & &
\end{tikzcd}$$

We artificially map $0 \in \dot{\Pi}(\mathcal{G})$ (Notation 6.15) to $0 \in \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(k\mathcal{G}))$ and to the irrelevant ideal $\mathrm{H}^+(\mathcal{G}, k) \in \operatorname{Spec}^{\mathrm{h}} \mathrm{H}^\bullet(\mathcal{G}, k)$. For details, see [Bal05, Corollary 5.10] and [FP07].

We now specialize all the way to the elementary abelian case.

6.21 Notation. Once and for all, we let t denote the rank of E, and we pick a minimal set of generators $z_1, \dots, z_t$ of E.

6.22 Notation. We denote

$$q = \begin{cases} 1 & p = 2 \\ p & p > 2. \end{cases}$$

6.23 Recollection. Every π-point for E (thought of as a constant group scheme) is equivalent to one of the form (α_c, K) [FP05, Proposition 2.2 and Lemma 4.1] where $c \in K^t \setminus \{0\}$ and

$$\begin{array}{rcccccc} \alpha_c \colon KC_p & \simeq & \dfrac{K[T]}{(T^p)} & \longrightarrow & \dfrac{K[Z_1, \dots, Z_t]}{(Z_1^p, \dots, Z_t^p)} & \simeq & KE. \\ & & T & \longmapsto & c_1 Z_1 + \cdots + c_t Z_t & & \end{array}$$

We can describe the kernel $\ker(\alpha_c, K)^* \in \operatorname{Spc} \operatorname{st}(\mathsf{mod}(E))$ concretely as the full subcategory of objects $C \in \mathsf{mod}(E)$ such that the K-linear map given by multiplication by $c_1 Z_1 + \cdots + c_t Z_t$ on C_K has rank precisely

$$\operatorname{rank}_K (c_1 Z_1 + \cdots + c_t Z_t \colon C_K \to C_K) = \frac{p-1}{p} \dim_K(C_K). \tag{6.24}$$

It is standard [Bal21, Lemma 3.11] that

$$\mathrm{H}^\bullet(E, k) \simeq k[U_1, \dots, U_t] \quad \text{where} \quad |U_i| = \begin{cases} 1 & p = 2 \\ 2 & p > 2 \end{cases} \tag{6.25}$$

for some $U_1, \dots, U_t$ such that the induced map on cohomology by the π-point (α_c, K) is

$$\begin{array}{rcccccc} \mathrm{H}^\bullet(\alpha_c, K) \colon \mathrm{H}^\bullet(E_K, K) & \simeq & K[U_1, \dots, U_t] & \longrightarrow & K[T] & \simeq & \mathrm{H}^\bullet(C_p, K). \\ & & U_i & \longmapsto & c_i^q T & & \end{array}$$

Thus $\ker(\mathrm{H}^\bullet(\alpha_c, K)) = (U_1 - c_1^q, \dots, U_t - c_t^q)$.

6.26 Recollection. For any point $x \in \mathbb{P}_k^{t-1}$, there exists a field extension K/k and a K-rational closed point $[c] \in \mathbb{P}_K^{t-1}(K)$ such that

$$\begin{array}{ccc} [c] & \longmapsto & x \\ \mathbb{P}_K^{t-1} & \longrightarrow & \mathbb{P}_k^{t-1} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(k). \end{array} \tag{6.27}$$

Indeed, we may take for K the residue field $\kappa(x)/k$ of x. We denote $[c]_{K/k} = x$.

6.28 Notation. For k algebraically closed, we denote

$$\begin{array}{rccccccc} \mathcal{P}\colon & \mathbb{P}_k^{t-1} & \xrightarrow[\cong]{\text{Frob}} & \mathbb{P}_k^{t-1} & \xrightarrow{\cong} & \mathrm{Proj}\,\mathrm{H}^\bullet(E,k) & \xrightarrow{\cong} & \mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(E)) \\ & [c]_{K/k} & \longmapsto & [c^q]_{K/k} & \longmapsto & (U_1 - c_1^q, \ldots, U_t - c_t^q) & \longmapsto & \ker(\alpha_c, K)^* \end{array}$$

using Recollection 6.26. An object $C \in \mathsf{mod}(E)$ is in $\mathcal{P}([c]_{K/k})$ if and only if its base change C_K satisfies (6.24), and this in particular shows that $\mathcal{P}$ is well defined.

As an example of this notation, we recall how the case of a finite group reduces to the case of an elementary abelian group.

6.29 Recollection. In this recollection, we assume k is algebraically closed, and we allow G to be any finite group. Quillen stratification asserts that, roughly speaking, $\mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(G))$ is a gluing of the spectra $\mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(E'))$ where $E' \leqslant G$ runs through the elementary abelian p-subgroups of G; more precisely

$$\mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(G)) = \operatorname*{colim}_{E' \in \mathrm{Or}(G,\mathrm{elab})} \mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(E'))$$

as described in [Bal16, Theorem 0.1]. In particular, in $\mathrm{st}(\mathsf{mod}(G))$ every prime tt-ideal is of the form $\mathrm{Spc}(\mathrm{Res}_{E'})(\mathcal{P}([c]_{K/k}))$, where $[c]_{K/k} \in \mathbb{P}_k^{r-1}$ and $E' \leqslant G$ is an elementary abelian p-subgroup of rank r.

6.30 Remark. In general, π-points depend on the comultiplication on $k\mathcal{G}$, as we emphasized in Recollection 6.17. However, in the case of a finite group G (Recollection 6.29), we observe that the π-points do not depend on the comultiplication on kG at all. We can explain this phenomenon using tt-geometry in the case where G is a p-group. Indeed, in this case the unit generates $\mathrm{st}(\mathrm{mod}(kG))$, so the prime tt-ideals can be characterized (Lemma 6.14) without reference to the tensor structure.

We describe an alternative to Carlson modules for linear subspaces.

6.31 Lemma. *Assume k is algebraically closed. Then closed points and rational points are the same thing, and the closed subsets of $\mathbb{P}^{t-1}_k$ are determined by the rational points that they contain.*

Proof. Here rational points are the same thing as closed points since $\mathbb{P}^{t-1}_k$ is a scheme over k locally of finite type and k is algebraically closed. Thus the second statement is claiming that the closed subsets are determined by the closed points that they contain. This is because for any field k, the underlying space of the scheme $\mathbb{P}^{t-1}_k$ is Jacobson. □

6.32 Example. Assume k is algebraically closed. Let $[c] \in \mathbb{P}^{t-1}_k$ be a closed point, and let us denote by $[c]^\perp \subset \mathbb{P}^{t-1}_k$ the closed subset given by the orthogonal complement of $[c]$ (recall Lemma 6.31). There exists a p-dimensional module $C_{[c]^\perp} \in \mathsf{mod}(E)$ such that $\mathrm{supp}(C_{[c]^\perp}) = \mathcal{P}([c]^\perp)$. Indeed, pick a representative $c \in K^t \setminus \{0\}$ of $[c]$. We may let $C_{[c]^\perp}$ be the p-dimensional k-vector space with basis $v_1, \ldots, v_p$ and with kE-action defined by

$$Z_j v_i = \begin{cases} c_j v_{i+1} & 1 \leq i < p \\ 0 & i = p \end{cases}$$

for $j = 1, \ldots, t$. Then for $d \in K^t \setminus \{0\}$, we have

$$(d_1 Z_1 + \cdots + d_t Z_t) v_i = (c_1 d_1 + \cdots + c_t d_t) v_{i+1} \tag{6.33}$$

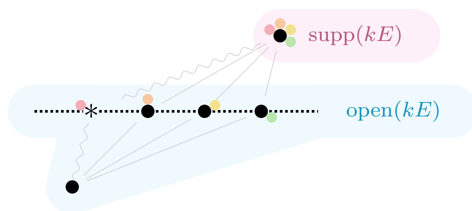
for $1 \leq i < p$, so $\mathcal{P}([d]) \in \mathrm{supp}(C_{[c]^\perp})$ if and only if $[d]$ is orthogonal to $[c]$.

6.34 Example. Continuing Example 6.32, for any linear subspace $P \subset \mathbb{P}^{t-1}_k$, we can tensor together objects of the form $C_{[c]^\perp} \in \mathsf{mod}(E)$ to get an object $C_P \in \mathsf{mod}(E)$ whose support is P. Indeed, any hyperplane in $\mathbb{P}^{t-1}_k$ is the orthogonal complement of a point, and P is an intersection of hyperplanes. If $P = \{[c]\}$ for some $[c] \in \mathbb{P}_k^{t-1}$, then we simply write $C_{[c]} = C_{\{[c]\}}$.

6.35 Remark. In fact, one can show that if P is spanned by $\mathbb{F}_p$-rational points, then $C_P \in \mathsf{pmod}(E)$ is an indecomposable permutation module.

6.36 Running Example. We return to our running example, which we set up in Running Example 1.13. We take our generators $z_1, z_2 \in C_2^{\times 2}$ (Notation 6.21), to be $x, y \in C_2^{\times 2}$ respectively. Then we have the following picture of $\operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E)) = \dot{\mathbb{P}}_k^1$.

$\operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E))$



● $\operatorname{supp}(C_{[a:b]})$ ● $\operatorname{supp}(k(E/N_{[1:0]}))$ ● $\operatorname{supp}(k(E/N_{[1:1]}))$

● $\operatorname{supp}(k(E/N_{[0:1]}))$.

Let us explain this picture in detail, as we will be re-using the notations for future pictures.

The long dotted line represents the closed points of $\mathbb{P}_k^1$, and the three $\bullet$ points on it are in particular the three $\mathbb{F}_2$-rational points $[1 : 0]$, $[1 : 1]$, and $[0 : 1]$ in that order. The $*$ point represents an arbitrary non-$\mathbb{F}_2$-rational point $[a : b] \in \mathbb{P}_k^1$. The point at the bottom is the generic point of $\mathbb{P}_k^1$, and the point at the top is the unique closed point $0 \in \dot{\mathbb{P}}_k^1$ in the one-point compactification (Notation 6.15) of $\mathbb{P}_k^1$.

The grey lines indicate specializations. They always connect a lower point to a higher point, and this conveys that the lower one specializes to the higher one. The undulated (squiggly) line to and from the $*$ point means that all of the non-$\mathbb{F}_2$-rational points have that behavior. In this case, all non-$\mathbb{F}_2$-rational points on $\mathbb{P}_k^1$ all generalize to the generic point of $\mathbb{P}^1_k$ and all specialize to the unique closed point $0 \in \dot{\mathbb{P}}^1_k$. Of course, having no grey line between two distinct points indicates that neither specializes to the other.

Finally, the highlighted regions illustrate our divide and conquer strategy, and the colored markers on the points display the supports of some important objects. In this case, we are displaying the supports of the 2-dimensional indecomposable permutation modules and the support of the module $C_{[a:b]}$ from Example 6.34 which was constructed to have support $\{[a:b]\}$ in $\operatorname{Spc}\operatorname{st}(\mathsf{mod}(E))$.

We wrap up this section by upgrading some of our previous results to derived categories.

6.37 Lemma. *Let $F\colon \mathcal{E} \to \mathcal{F}$ be a pre-abelian (Terminology 3.42) exact functor where the exact structure on $\mathcal{E}$ is pulled back along F. Then $F\colon \mathrm{D_b}(\mathcal{E}) \to \mathrm{D_b}(\mathcal{F})$ is conservative.*

Proof. Recall that $X \in \mathrm{Ch_b}(\mathcal{E})$ is zero in $\mathrm{D_b}(\mathcal{E})$ if and only if it is acyclic in $\mathcal{E}$, which means that every differential is strict and every consecutive pair of differentials is acyclic (Recollection 5.7). Thus the lemma is immediate from Lemma 5.9(iii). $\square$

6.38 Lemma. *Let $\mathcal{E}$ be a rigid tensor exact category whose unit admits a projective pre-cover $P \twoheadrightarrow \mathbb{1}$. Then $\mathcal{E}$ is Frobenius exact, and the composite $\mathcal{E} \rightarrowtail \mathrm{D_b}(\mathcal{E}) \twoheadrightarrow \mathrm{D_b}(\mathcal{E})/\langle P\rangle$ induces an equivalence of tt-categories* $\operatorname{st}(\mathcal{E}) \cong \mathrm{D_b}(\mathcal{E})/\langle P\rangle$.

Proof. This is Proposition 5.29(i) and Proposition 5.29(ii) together with [Kra21, Proposition 4.4.18]. $\square$

6.39 Example. Let R and S be effective monoids in fvect and gvect respectively. For a complex $X \in \mathrm{Ch_b}(\mathrm{mod}_{\mathsf{fvect}}(R))$, applying our short exact sequence (5.17) degree-wise to X produces a distinguished triangle in $\mathrm{D_b}(\mathrm{mod}_{\mathsf{fvect}}(R))$. In particular, if w is the maximal integer such that $X^w \neq 0 \in \mathrm{Ch_b}(\mathrm{mod}_{\mathsf{fvect}}(R))$, then

$$\mathrm{fz}(X^w)(w) \longrightarrow X \longrightarrow X^{\leq w-1} \longrightarrow \mathrm{fz}(X^w)(w)[1] \tag{6.40}$$

is a distinguished triangle in $\mathrm{D_b}(\mathrm{mod}_{\mathsf{fvect}}(R))$. Similarly, for $Y \in \mathrm{Ch_b}(\mathrm{mod}_{\mathsf{gvect}}(S))$, if w is the maximal integer such that $Y^w \neq 0 \in \mathrm{Ch_b}(\mathrm{mod}_{\mathsf{gvect}}(S))$, then applying (5.18) degree-wise produces a distinguished triangle

$$\mathrm{gz}(Y^w)(w) \longrightarrow Y \longrightarrow Y^{\leq w-1} \longrightarrow \mathrm{gz}(Y^w)(w)[1] \tag{6.41}$$

in $\mathrm{D_b}(\mathrm{mod}_{\mathsf{gvect}}(S))$.

6.42 Recollection. Let $\mathcal{E}$ be an exact category, and let J be a set of generators of $\mathcal{E}$ as an extension-closed subcategory. Then the set J generates $\mathrm{D_b}(\mathcal{E})$ as a triangulated subcategory. Indeed, a short exact sequence $A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$ in $\mathcal{E}$ induces a distinguished triangle $A_1 \to A_2 \to A_3 \to A_1[1]$ in $\mathrm{D_b}(\mathcal{E})$, so by Lemma 5.26 and induction on the number of admissible monos in the sequence (5.27) we see that any object in homological degree zero is in the triangulated subcategory generated by J. The claim then follows by induction on complex length, by taking cones and suspensions.

6.43 Example. We have the following generators as triangulated subcategories:

$$\begin{aligned}
\mathrm{D_b}(\mathsf{pfmod}(G)) &: \text{twists of } \mathrm{fz}\, k(G/H) \text{ for } H \leqslant G \\
\mathrm{D_b}(\mathsf{pgmod}(G)) &: \text{twists of } \mathrm{gz}\, k(G/H) \text{ for } H \leqslant G \\
\mathrm{D_b}(\mathsf{pmod}(G)) &: k(G/H) \text{ for } H \leqslant G \\
\mathrm{D_b}(\mathsf{fmod}(G)) &: \text{twists of the unit } \mathbb{1} \\
\mathrm{D_b}(\mathsf{gmod}(G)) &: \text{twists of the unit } \mathbb{1} \\
\mathrm{D_b}(\mathsf{mod}(G)) &: \text{the unit } \mathbb{1}.
\end{aligned}$$

Indeed, this follows from Recollection 6.42 and Example 5.24.

7 Finite resolutions

7.1 Notation. For this section, let $\Upsilon\colon \mathcal{E} \to \mathcal{F}$ be a fully faithful exact functor.

7.2 Terminology. The kernel $\ker(\Upsilon)$ of the triangulated functor $\Upsilon\colon \mathrm{D_b}(\mathcal{E}) \to \mathrm{D_b}(\mathcal{F})$ is a thick subcategory. We call the objects $X \in \mathrm{Ch_b}(\mathcal{E})$ in this kernel the $\mathcal{F}$-*acyclic objects*, and we denote by

$$\overline{\Upsilon}\colon \frac{\mathrm{D_b}(\mathcal{E})}{\ker(\Upsilon)} \longrightarrow \mathrm{D_b}(\mathcal{F})$$

the induced functor from the Verdier quotient at the $\mathcal{F}$-acyclic objects.

In this section we mildly generalize an idea from [BG23] in order to give a sufficient condition for $\overline{\Upsilon}$ to be an equivalence. The main result is Proposition 7.13. In [BG23], the authors were considering the case $\Upsilon\colon \mathsf{pmod}(G) \to \mathsf{mod}(G)$.

We begin by giving some motivation.

7.3 Recollection. A morphism f in $\mathrm{Ch}(\mathcal{F})$ is a *quasi-isomorphism* if $\mathrm{cone}(f) \in \mathrm{Ch}(\mathcal{F})$ is acyclic (Recollection 5.7) with respect to the exact structure on $\mathcal{F}$.

7.4 Recollection. Suppose $\mathcal{F}$ has enough projectives. Then any complex $Y \in \mathrm{Ch_b}(\mathcal{F})$ admits a bounded-below projective resolution, which is a bounded-below complex $P \in \mathrm{Ch_+}(\mathcal{F})$ of projectives along with a quasi-isomorphism $\pi\colon P \to Y$ in $\mathrm{Ch_+}(\mathcal{F})$. Thus we have an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D_b}(\mathcal{F})}(Y, Y') &\xrightarrow{\;\cong\;} \mathrm{Hom}_{\mathrm{K_+}(\mathcal{F})}(P, Y'). \\ f &\longmapsto f\pi \end{aligned} \tag{7.5}$$

7.6 Remark. The dream case to keep in mind is where every object in $\mathcal{F}$ admits a finite resolution by projective objects that are in the essential image of Υ. In this case $\Upsilon\colon \mathrm{D_b}(\mathcal{E}) \to \mathrm{D_b}(\mathcal{F})$ is certainly essentially surjective, and it is fully faithful by Recollection 7.4, noting that an object in $\mathcal{E}$ is projective if it is projective in $\mathcal{F}$. Thus Υ is an equivalence, and automatically $\overline{\Upsilon}$ is as well. However, if $\mathcal{F}$ is Frobenius exact, then an object $A \in \mathcal{F}$ admits a finite resolution by projective objects if and only if A is itself projective. Indeed, given such a resolution

$$P_n \overset{d_n}{\rightarrowtail} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \twoheadrightarrow A,$$

the admissible mono d_n admits a retract because P_n is injective, and we argue by induction.

Thus for the original example $\Upsilon\colon \mathsf{pmod}(G) \to \mathsf{mod}(G)$ where $\mathsf{mod}(G)$ is Frobenius exact, the authors of [BG23] needed to relax the condition of having a finite projective resolution.

We now proceed to give the sufficient condition.

7.7 Definition. Let $Y \in \mathrm{Ch_b}(\mathcal{F})$ be a complex in $\mathcal{F}$. An *$\mathcal{E}$-resolution of Y* is a complex $X \in \mathrm{Ch_b}(\mathcal{E})$ together with a quasi-isomorphism $\Upsilon(X) \to Y$ in $\mathrm{Ch_b}(\mathcal{F})$.

7.8 Remark. Let $A \in \mathcal{F}$ be an object. An $\mathcal{E}$-resolution $\Upsilon(X) \to A$ of the object A, viewed as a complex in $\mathrm{Ch_b}(\mathcal{F})$ concentrated in homological degree zero, is a weaker notion than a resolution of A by objects in $\mathcal{E}$ because the complex X need not be concentrated in nonnegative homological degrees.

7.9 Definition. For $m \in \mathbb{Z}$, we say that a complex in $\mathrm{Ch_b}(\mathcal{F})$ is *m-projective* if it is projective in homological degrees $i \leq m$ for all i. We say that an $\mathcal{E}$-resolution $\Upsilon(X) \to Y$ is *m-projective* if $\Upsilon(X) \in \mathrm{Ch_b}(\mathcal{F})$ is m-projective.

7.10 Notation. For this section, we denote by $\mathcal{S} \subset \mathrm{D_b}(\mathcal{F})$ the full subcategory of objects $Y \in \mathrm{Ch_b}(\mathcal{F})$ that admit m-projective $\mathcal{E}$-resolutions for all $m \geq 0$.

7.11 Lemma. *The subcategory $\mathcal{S}$ of $\mathrm{D_b}(\mathcal{F})$ is a triangulated subcategory contained in the essential image of $\overline{\Upsilon}$.*

Proof. The subcategory $\mathcal{S}$ is triangulated by [BG23, Lemma 2.6, Proposition 2.7, and Proposition 2.8], which go through almost word-for-word. That $\mathcal{S}$ is contained in the essential image of $\overline{\Upsilon}$ is clear from the definition of $\mathcal{S}$. □

7.12 Lemma. *Consider the following pullback (Lemma 2.10) of triangulated categories:*

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\overline{\Upsilon}"] \arrow[d, hook] & \mathcal{S} \arrow[d, hook] \\ \dfrac{\mathrm{D_b}(\mathcal{E})}{\ker(\Upsilon)} \arrow[r, "\overline{\Upsilon}"] & \mathrm{D_b}(\mathcal{F}). \end{tikzcd}$$

The restriction $\overline{\Upsilon}\colon \mathcal{R} \to \mathcal{S}$ is an equivalence of triangulated categories.

Proof. Lemma 7.11 ensures that it is essentially surjective. It is fully faithful by the second half of the proof of [BG23, Theorem 4.3], which again goes through almost word-for-word. ☐

7.13 Proposition. *Suppose there exists a set J of generators of $\mathrm{D_b}(\mathcal{F})$ as a triangulated category such that for every $Y \in J$, the complex Y admits an m-projective $\mathcal{E}$-resolution for all $m \geq 0$. Then*

$$\overline{\Upsilon}\colon \frac{\mathrm{D_b}(\mathcal{E})}{\ker(\Upsilon)} \longrightarrow \mathrm{D_b}(\mathcal{F})$$

is an equivalence of triangulated categories.

Proof. Immediate from Lemma 7.11 and Lemma 7.12. ☐

Let us revisit the original example.

7.14 Example. The inclusion $\Upsilon\colon \mathsf{pmod}(G) \to \mathsf{mod}(G)$ induces an equivalence

$$\overline{\Upsilon}\colon \frac{\mathrm{D_b}(\mathsf{pmod}(G))}{\ker(\Upsilon)} \xrightarrow{\cong} \mathrm{D_b}(\mathsf{mod}(G)) \tag{7.15}$$

of triangulated categories. Indeed, the unit in $\mathsf{mod}(G)$ admits an m-projective $\mathsf{pmod}(G)$-resolution for all $m \geq 0$ [BG23, Theorem 3.1 and Prop 3.10]. Since the unit generates $\mathrm{D_b}(\mathsf{mod}(G))$ as a triangulated subcategory (Example 6.43), we conclude (Proposition 7.13) that $\overline{\Upsilon}$ is an equivalence.

7.16 Remark. Example 7.14 was a result in [BG23] that was independently discovered by Rouquier in [Rou06]. In fact [BG23] proves the following remarkable theorem that gives the paper its name: every finitely generated kG-module admits a finite resolution by permutation modules. This is somewhat stronger than Example 7.14 by Remark 7.8.

We record a tangential fact for later use.

7.17 Lemma. *Suppose $\mathcal{E}$ has enough projectives and that $\Upsilon\colon \mathcal{E} \to \mathcal{F}$ preserves projectives. Then $\Upsilon\colon \mathrm{D_b}(\mathcal{E}) \to \mathrm{D_b}(\mathcal{F})$ is fully faithful.*

Proof. Let $X_1, X_2 \in \mathrm{Ch}_\mathrm{b}(\mathcal{E})$, and let $P \to X_1$ in $\mathrm{K}_+(\mathcal{E})$ be a projective resolution (Recollection 7.4) of X_1, using that $\mathcal{E}$ has enough projectives. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}_\mathrm{b}(\mathcal{E})}(X_1, X_2) &\cong \mathrm{Hom}_{\mathrm{K}_+(\mathcal{E})}(P, X_2) && \text{(by (7.5))}\\
&\cong \mathrm{Hom}_{\mathrm{K}_+(\mathcal{F})}(\Upsilon(P), \Upsilon(X_2)) && \text{(follows from } \Upsilon \text{ being fully faithful)}\\
&\cong \mathrm{Hom}_{\mathrm{D}_\mathrm{b}(\mathcal{F})}(\Upsilon(X_1), \Upsilon(X_2)),\\
& && \text{(by (7.5), using that } \Upsilon \text{ preserves projectives)}
\end{aligned}$$

as we claimed. □

8 Localizing a fully faithful tt-functor

8.1 Notation. In this section, let $\mathcal{K}$ be a rigid tt-category, let $F\colon \mathcal{K} \to \mathcal{L}$ be a fully faithful tt-functor, and let $x \in \mathcal{K}$ be a fixed object in $\mathcal{K}$.

In this section, we show that the Verdier quotient of F at the object x is fully faithful (Proposition 8.9).

Our main ingredient is the following key observation of [Bal18].

8.2 Notation. Choose a homotopy fiber $\xi_x\colon w_x \to \mathbb{1}$ of the coevaluation morphism $\mathrm{coev}_x\colon \mathbb{1} \to x^\vee \otimes x$ (Remark 2.6) so that we have the following distinguished triangle in $\mathcal{K}$:

$$w_x \xrightarrow{\xi_x} \mathbb{1} \xrightarrow{\mathrm{coev}_x} x^\vee \otimes x \longrightarrow \Sigma w_x. \tag{8.3}$$

8.4 Recollection. The tt-ideal $\langle x \rangle \subset \mathcal{K}$ generated by x is the full subcategory of $\mathcal{K}$ on which ξ_x is $\otimes$-nilpotent [Bal18, Corollary 2.11]:

$$\langle x \rangle = \{z \in \mathcal{K} \mid \xi_x^{\otimes n} \otimes z = 0 \text{ for some } n \geq 1\}.$$

The morphism ξ_x in $\mathcal{K}$ is invertible in $\mathcal{K}/\langle x \rangle$ because $\mathrm{cone}(\xi_x) = x^\vee \otimes x \in \langle x \rangle$. In fact, ξ_x is universal with respect to such morphisms, in the following sense.

8.5 Lemma (Balmer). *Let $s\colon c \to a$ be a morphism in $\mathcal{K}$ such that* $\mathrm{cone}(s) \in \langle x \rangle$. *Then there exists $n \geq 1$ and a morphism $t\colon w_x^{\otimes n} \otimes a \to c$ in $\mathcal{K}$ such that the following diagram commutes:*

$$\begin{tikzcd} & w_x^{\otimes n} \otimes a \arrow[dl, "t"'] \arrow[d, "\xi_x^{\otimes n} \otimes 1"] \\ c \arrow[r, "s"'] & a. \end{tikzcd}$$

Proof. Since $\mathrm{cone}(s) \in \langle x \rangle$, there exists $n \geq 1$ such that (Recollection 8.4)

$$\xi_x^{\otimes n} \otimes \mathrm{cone}(s) = 0. \tag{8.6}$$

Tensoring the distinguished triangle

$$c \xrightarrow{\ s\ } a \xrightarrow{\ q\ } \mathrm{cone}(s) \longrightarrow \Sigma c$$

with the morphism $\xi_x^{\otimes n}\colon w_x^{\otimes n} \to \mathbb{1}$ produces the morphism

$$\begin{tikzcd} w_x^{\otimes n} \otimes c \arrow[r, "w_x^{\otimes n} \otimes s"] \arrow[d, "\xi_x^{\otimes n} \otimes 1"'] & w_x^{\otimes n} \otimes a \arrow[r, "w_x^{\otimes n} \otimes q"] \arrow[d, "\xi_x^{\otimes n} \otimes 1"] \arrow[dl, dotted, "t"'] \arrow[dl, phantom, "(*)"] & w_x^{\otimes n} \otimes \mathrm{cone}(s) \arrow[r] \arrow[d, "\xi_x^{\otimes n} \otimes 1 = 0"] & \Sigma(w_x^{\otimes n} \otimes c) \arrow[d, "\Sigma(\xi_x^{\otimes n} \otimes 1)"] \\ c \arrow[r, "s"'] & a \arrow[r, "q"] & \mathrm{cone}(s) \arrow[r] & \Sigma c \end{tikzcd}$$

of distinguished triangles. The third vertical morphism vanishes by (8.6), so the dotted morphism t exists and makes the triangle $(*)$ commute because s is a weak kernel of q. $\square$

We then have the following description of morphisms in $\mathcal{K}/\langle x \rangle$.

8.7 Lemma (Balmer). *For any objects $a, b \in \mathcal{K}$, we have*

$$\mathrm{Hom}_{\mathcal{K}/\langle x \rangle}(a, b) \cong \operatorname*{colim}_n \mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes n} \otimes a, b),$$

where the transition maps in the colimit are given by tensoring with $\xi_x\colon w_x \to \mathbb{1}$:

$$\begin{aligned} \xi_x \otimes -\colon\ \mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes n} \otimes a, b) &\longrightarrow \mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes (n+1)} \otimes a, b). \\ f &\longmapsto \xi_x \otimes f \end{aligned}$$

Proof. Consider the map φ defined by

$$\begin{aligned} \varphi\colon \operatorname*{colim}_n \operatorname{Hom}_{\mathcal{K}}(w_x^{\otimes n}\otimes a, b) &\longrightarrow \operatorname{Hom}_{\mathcal{K}/\langle x\rangle}(a,b). \\ (w_x^{\otimes n}\otimes a \xrightarrow{f} b) &\longmapsto (a \xleftarrow{\xi_x^{\otimes n}\otimes 1} w_x^{\otimes n}\otimes a \xrightarrow{f} b) \end{aligned} \tag{8.8}$$

This is well-defined: the target morphism is well-defined because ξ_x is invertible in $\mathcal{K}/\langle x\rangle$, and this indeed induces a map on the colimit since the commutative diagram

$$\begin{tikzcd} & w_x^{\otimes(n+1)}\otimes a \arrow[dl, "\xi_x^{\otimes(n+1)}\otimes 1"'] \arrow[dd, "\xi_x\otimes 1"] \arrow[dr, "\xi_x\otimes f"] & \\ a & & b \\ & w_x^{\otimes n}\otimes a \arrow[ul, "\xi_x^{\otimes n}\otimes 1"] \arrow[ur, "f"'] & \end{tikzcd}$$

shows that $\varphi(\xi_x\otimes f)=\varphi(f)$.

To see that φ is injective, let $f\colon w_x^{\otimes n}\otimes a\to b$ be a morphism in $\mathcal{K}$ such that

$$\varphi(f)\colon (a \xleftarrow{\xi_x^{\otimes n}\otimes 1} w_x^{\otimes n}\otimes a \xrightarrow{f} b)$$

is zero in $\mathcal{K}/\langle x\rangle$. Then there exists $s\colon c\to w_x^{\otimes n}\otimes a$ such that $\operatorname{cone}(s)\in\langle x\rangle$ and $fs=0$ in $\mathcal{K}$ (see [Nee01, Lemma 2.1.26] if necessary). Using Lemma 8.5, we may amplify this to

$$\begin{tikzcd} & w_x^{\otimes m}\otimes a \arrow[dl, "t"'] \arrow[d, "\xi_x^{\otimes m}\otimes 1"] \arrow[dr, "\xi_x^{\otimes m}\otimes f"] & \\ c \arrow[r, "s"'] & w_x^{\otimes n}\otimes a \arrow[r, "f"'] & b \end{tikzcd}$$

in $\mathcal{K}$ for some $m\geq 1$, so $\xi_x^{\otimes m}\otimes f=0$ by commutativity of the diagram and because $fs=0$. We conclude that $f=0$ in the colimit.

Similarly, to see that φ is surjective, observe that any morphism $fs^{-1}\colon a\leftarrow c\to b$ in $\mathcal{K}/\langle x\rangle$ where $\operatorname{cone}(s)\in\langle x\rangle$ can be amplified using Lemma 8.5 to

$$\begin{tikzcd} & w_x^{\otimes n}\otimes a \arrow[dl, "\xi_x^{\otimes n}\otimes 1"'] \arrow[d, "t"] \arrow[dr, "ft"] & \\ a & c \arrow[l, "s"] \arrow[r, "f"'] & b \end{tikzcd}$$

for some $n\geq 1$, so $\varphi(ft)=fs^{-1}$. $\square$

8.9 Proposition (Balmer)**.** *The induced functor on Verdier quotients*

$$\overline{F}\colon \frac{\mathcal{K}}{\langle x\rangle} \longrightarrow \frac{\mathcal{L}}{\langle F(x)\rangle}$$

is fully faithful.

Proof. Applying our tt-functor F to our distinguished triangle (8.3) produces a distinguished triangle in $\mathcal{L}$, namely the bottom row of the following diagram:

$$\begin{tikzcd}
w_{F(x)} \arrow[r, "\xi_{F(x)}"] \arrow[d, dotted, "u"', "\cong"] & \mathbb{1} \arrow[r, "\mathrm{coev}_{F(x)}"] \arrow[d, equal] & F(x)^\vee \otimes F(x) \arrow[r] \arrow[d, "\cong"] & \Sigma w_{F(x)} \arrow[d, dotted, "\cong"', "\Sigma u"] \\
F(w_x) \arrow[r, "F(\xi_x)"] & \mathbb{1} \arrow[r, "F(\mathrm{coev}_x)"] & F(x^\vee \otimes x) \arrow[r] & \Sigma F(w_x).
\end{tikzcd}$$

The top row is constructed as in Notation 8.2 by taking a homotopy fiber of $\mathrm{coev}_{F(x)}$. By basic triangulated category theory, the morphism u exists and is an isomorphism.

Let $a, b \in \mathcal{K}$, and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes n} \otimes a, b) \arrow[r, "\xi_x \otimes -"] \arrow[d, "\cong"', "F"] & \mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes (n+1)} \otimes a, b) \arrow[d, "\cong"', "F"] \\
\mathrm{Hom}_{\mathcal{L}}(F(w_x)^{\otimes n} \otimes F(a), F(b)) \arrow[r, "F(\xi_x)\otimes -"] \arrow[d, "\cong"', "-\circ(u^{\otimes n}\otimes 1)"] & \mathrm{Hom}_{\mathcal{L}}(F(w_x)^{\otimes (n+1)} \otimes F(a), F(b)) \arrow[d, "\cong"', "-\circ(u^{\otimes (n+1)}\otimes 1)"] \\
\mathrm{Hom}_{\mathcal{L}}(w_{F(x)}^{\otimes n} \otimes F(a), F(b)) \arrow[r, "\xi_{F(x)}\otimes -"] & \mathrm{Hom}_{\mathcal{L}}(w_{F(x)}^{\otimes (n+1)} \otimes F(a), F(b)).
\end{tikzcd}$$

The vertical maps are isomorphisms because F is fully faithful and u is an isomorphism. By Lemma 8.7, the colimit of the first row is isomorphic to $\mathrm{Hom}_{\mathcal{K}/\langle x\rangle}(a, b)$ via (8.8) and the colimit of the third row is isomorphic to $\mathrm{Hom}_{\mathcal{L}/\langle F(x)\rangle}(F(a), F(b))$ via the analogous (8.8) using the homotopy fiber $\xi_{F(x)}$. Thus the induced map on colimits is an isomorphism, and it remains to verify that it is the same map as the one induced by $\overline{F}$, which amounts to verifying that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes n} \otimes a, b) \arrow[r, "\varphi"] \arrow[d, "\cong"', "F"] & \mathrm{Hom}_{\mathcal{K}/\langle x\rangle}(a, b) \arrow[dd, "\overline{F}"] \\
\mathrm{Hom}_{\mathcal{L}}(F(w_x)^{\otimes n} \otimes F(a), F(b)) \arrow[d, "\cong"', "-\circ(u^{\otimes n}\otimes 1)"] & \\
\mathrm{Hom}_{\mathcal{L}}(w_{F(x)}^{\otimes n} \otimes F(a), F(b)) \arrow[r, "\varphi"] & \mathrm{Hom}_{\mathcal{L}/\langle F(x)\rangle}(F(a), F(b)).
\end{tikzcd}$$

This is straightforward: for $f \in \mathrm{Hom}_{\mathcal{K}}(w_x^{\otimes n} \otimes a, b)$, we have

$$\begin{tikzcd}
 & F(w_x^{\otimes n} \otimes a) \arrow[dl, "F(\xi_x^{\otimes n} \otimes 1)"'] \arrow[dr, "F(f)"] & \\
F(a) & F(w_x)^{\otimes n} \otimes F(a) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & F(b) \\
 & w_{F(x)}^{\otimes n} \otimes F(a) \arrow[ul, "\xi_{F(x)}^{\otimes n} \otimes 1"] \arrow[ur, "F(f) \circ (u^{\otimes n} \otimes 1)"'] &
\end{tikzcd}$$

where the morphisms in the middle are essentially forced. $\square$

8.10 Corollary (Balmer)**.** *Let $\mathcal{I}$ be a finitely generated tt-ideal in $\mathcal{K}$. Then the induced functor on Verdier quotients*

$$\overline{F} \colon \frac{\mathcal{K}}{\mathcal{I}} \longrightarrow \frac{\mathcal{L}}{\langle F(\mathcal{I}) \rangle}$$

is fully faithful.

Proof. See Recollection 6.4 if necessary. $\square$

Part II

The support of the projectives

9 Projective radical filtrations

In this section, we prove the crucial fact (Theorem 9.13) that the radical filtration $\mathrm{radf}(kE)$ (Example 3.44) is projective in $\mathsf{pfmod}(E)$. This implies (Corollary 9.14) that $\mathsf{pfmod}(E)$ is Frobenius exact. Here E denotes our elementary abelian p-group E (Notation 2.1), and at the end of the section we discuss why these results seem to be exclusive to E (Example 9.21).

We start by establishing a sufficient condition in terms of a normal subgroup $N \trianglelefteqslant G$ in order for $\mathrm{radf}_N(kG) \in \mathsf{pfmod}(G)$ to be projective. We recall that G denotes a p-group (Notation 2.1) and that for any normal subgroup $N \trianglelefteqslant G$, the N-radical filtration $\mathrm{radf}_N(kG)$ is a well defined object in $\mathsf{pfmod}(G)$ (Example 3.50).

9.1 Recollection. Let $T \in \mathsf{vect}$ be a monoid, let $I \subset T$ be a two-sided ideal, and let $C \in \mathrm{mod}_{\mathsf{vect}}(T)$. The annihilator $\mathrm{Ann}_I(C) = \{c \in C \mid Ic = 0\}$ of I in C is a submodule of C.

9.2 Notation. For $C \in \mathsf{mod}(G)$ and a normal subgroup $N \trianglelefteqslant G$, we denote by $\mathrm{soc}_N(C) = \mathrm{Ann}_{\mathrm{rad}_N(kG)}(C)$ the annihilator of the ideal $\mathrm{rad}_N(kG)$ in C.

9.3 Definition. Let $N \trianglelefteqslant G$ be a normal subgroup. In this definition, we describe a technical condition for N that we leave unnamed. The condition is that for any subgroup $H \leqslant G$, integer $w \geq 1$, and morphism $h\colon \mathrm{rad}_N(kG)^w \to k(G/H)$ in $\mathsf{mod}(G)$, there exists a morphism $\widetilde{h}\colon kG \to k(G/H)$ in $\mathsf{mod}(G)$ making the following diagram commute:

$$\begin{tikzcd}
kG \arrow[dr, dotted, "\widetilde{h}"] & & \\
& k(G/H) \arrow[dr, two heads] & \\
\mathrm{rad}_N(kG)^w \arrow[uu, tail] \arrow[r, "h"] & k(G/H) \arrow[r, two heads] & k(G/H)/\operatorname{soc}_N(k(G/H)).
\end{tikzcd} \tag{9.4}$$

Let us isolate how we will use this technical condition. In the following lemma, we think of the $\mathrm{obs}_i(h)$ as being the obstructions in a particular filtration weight to h being a morphism of filtered modules.

9.5 Lemma. *Let $A \in \mathsf{pfmod}(G)$, let $w \geq 1$, and let $h\colon kG \to \mathrm{unf}(A)$ be a morphism in $\mathsf{mod}(G)$ such that h induces a morphism $\mathrm{radf}_N(kG)^{\leq w-1} \to A^{\leq w-1}$ in $\mathsf{pfmod}(G)$. Write*

$\mathrm{gr}(A)^{w-1}$ *as a finite direct sum of permutation modules* $k(G/H_i)$, *using that* $A \in \mathsf{pfmod}(G)$, *and consider the morphisms*

$$\mathrm{obs}_i(h)\colon\ \mathrm{rad}_N(kG)^{w-1} \subset kG \xrightarrow{h} A^{w-1} \twoheadrightarrow \mathrm{gr}(A)^{w-1} \xrightarrow{\mathrm{proj}_i} k(G/H_i)$$

in $\mathsf{mod}(G)$ *where* proj_i *denotes the projection onto the* i*th factor. Then* h *induces a morphism* $\mathrm{radf}_N(kG)^{\leq w} \to A^{\leq w}$ *in* $\mathsf{pfmod}(G)$ *if and only if*

$$q_i\mathrm{obs}_i(h)\colon\ \mathrm{rad}_N(kG)^{w-1} \xrightarrow{\mathrm{obs}_i(h)} k(G/H_i) \xrightarrow{q_i} \frac{k(G/H_i)}{\mathrm{soc}_N(k(G/H_i))}$$

vanishes for all i.

Proof. Recall that h induces a morphism $\mathrm{radf}_N(kG)^{\leq w} \to A^{\leq w}$ in $\mathsf{pfmod}(G)$ if and only if $h(\mathrm{rad}_N(kG)^v) \subset A^v$ for all $0 \leq v \leq w$. This is already satisfied for $0 \leq v \leq w-1$ by our assumption, and the case $v = w$ being equivalent to $q_i\mathrm{obs}_i(h)$ vanishing for all i is clear from Notation 9.2. □

9.6 Proposition. *Let* $N \trianglelefteq G$ *be a normal subgroup satisfying Definition 9.3, let* $A \in \mathsf{pfmod}(G)$, *and let* $g\colon \mathrm{gz}\,k(G/N) \to \mathrm{gr}(A)$ *be a morphism in* $\mathsf{pgmod}(G)$. *Then there exists a morphism* $f\colon \mathrm{radf}_N(kG) \to A$ *in* $\mathsf{pfmod}(G)$ *such that* $gr(f)^0 = g^0$ *in* $\mathsf{pmod}(G)$ *or, in other words, such that the following diagram in* $\mathsf{pgmod}(G)$ *commutes:*

$$\begin{tikzcd}
\mathrm{gz}\,k(G/N) \arrow[d, hook] \arrow[dr, "g"] & \\
\mathrm{gr}\,\mathrm{radf}_N(kG) \arrow[r, "\mathrm{gr}(f)"'] & \mathrm{gr}(A).
\end{tikzcd}$$

Proof. We will show that for all $w \geq 1$, there exists a morphism $h_w\colon kG \to A^0$ in $\mathsf{mod}(G)$ such that h_w induces a morphism $\mathrm{radf}_N(kG)^{\leq w} \to A^{\leq w}$ in $\mathsf{pfmod}(G)$ and such that the following diagram commutes in $\mathsf{mod}(G)$:

$$\begin{tikzcd}
\mathrm{rad}_N(kG) \arrow[r, "h_w"] \arrow[d, hook] & A^1 \arrow[d, hook] \\
kG \arrow[r, "h_w"] \arrow[d, two heads, "\varepsilon"] & A^0 \arrow[d, two heads] \\
k(G/N) \arrow[r, "g^0"'] & \mathrm{gr}(A)^0.
\end{tikzcd} \tag{9.7}$$

This suffices because for sufficiently large w, the morphism h_w in $\mathsf{mod}(G)$ induces a morphism $f\colon \operatorname{radf}_N(kG) \to A$ in $\mathsf{pfmod}(G)$, and (9.7) ensures that $\operatorname{gr}(f)^0 = g$. We proceed by induction on w.

For $w = 1$, we pick any lift $\widetilde{g^0(1)} \in A^0$ of $g^0(1) \in \operatorname{gr}(A)^0$ and set $h_1(1) = \widetilde{g^0(1)}$. Then (9.7) of course commutes, so $h_1(\operatorname{rad}_N(kG)) \subset A^1$ automatically because the columns are short exact. Thus h_1 induces a morphism $\operatorname{radf}_N(kG)^{\leq 1} \to A^{\leq 1}$.

Let $w \geq 2$, and suppose inductively that we have $h_{w-1}\colon kG \to A$. As in Lemma 9.5, write $\operatorname{gr}(A)^{w-1}$ as a finite direct sum of permutation modules $k(G/H_i)$, and consider the obstructions

$$\operatorname{obs}_i(h_{w-1})\colon\ \operatorname{rad}_N(kG)^{w-1} \subset kG \xrightarrow{h_{w-1}} A^{w-1} \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(A)^{w-1} \xrightarrow{\operatorname{proj}_i}\!\!\!\!\!\rightarrow k(G/H_i).$$

We now clear these obstructions. For each i, we do the following. Using that N satisfies Definition 9.3, let $c_i\colon kG \to k(G/H_i)$ be a morphism in $\mathsf{mod}(G)$ making the following diagram commute:

$$\begin{tikzcd}
 & & A^{w-1} \arrow[d, two heads] & & \\
kG \arrow[urr, dotted, "\widetilde{c_i}"] \arrow[drr, "c_i"'] & & \operatorname{gr}(A)^{w-1} \arrow[d, two heads, "\operatorname{proj}_i"] & & \\
 & & k(G/H_i) \arrow[drr, two heads, "q_i"] & & \\
\operatorname{rad}_N(kG)^{w-1} \arrow[uu, tail] \arrow[rr, "\operatorname{obs}_i(h_{w-1})"'] & & k(G/H_i) \arrow[rr, two heads, "q_i"'] & & k(G/H_i)/\operatorname{soc}_N(k(G/H_i)).
\end{tikzcd} \tag{9.8}$$

We take a lift $\widetilde{c_i}$ of c_i as depicted, using that kG is projective. We now define

$$h_w = h_{w-1} - \sum_i \widetilde{c_i}\colon kG \longrightarrow A^0.$$

Let us verify that this works. It is important that the target of $\widetilde{c_i}$ is $A^{w-1} \subset A^1$. Indeed, then (9.7) commutes for our h_w due to our inductive hypothesis that it commutes for h_{w-1}, and moreover h_w still induces a morphism $\operatorname{radf}_N(kG)^{\leq w-1} \to A^{\leq w-1}$. To see that h_w induces a morphism $\mathrm{radf}_N(kG)^{\leq w} \to A^{\leq w}$, we compute

$$\begin{aligned}
q_i \mathrm{obs}_i(h_w) &= q_i \mathrm{obs}_i\left(h_{w-1} - \sum_j \widetilde{c}_j\right) \\
&= q_i \mathrm{obs}_i(h_{w-1}) - \sum_j q_i \mathrm{obs}_i(\widetilde{c}_j) && (\mathrm{obs}_i(-) \text{ is } k\text{-linear}) \\
&= q_i \mathrm{obs}_i(h_{w-1}) - q_i \mathrm{obs}_i(\widetilde{c}_i) && (\widetilde{c}_j \text{ factors through } \pi_j) \\
&= 0. && (\text{by } (9.8))
\end{aligned}$$

We now appeal to Lemma 9.5. □

9.9 Proposition. *If $N \trianglelefteq G$ is a normal subgroup satisfying Definition 9.3, then the N-radical filtration $\mathrm{radf}_N(kG)$ is projective in $\mathsf{pfmod}(G)$.*

Proof. Let $p\colon A \twoheadrightarrow \mathrm{radf}_N(kG)$ be an admissible epi in $\mathsf{pfmod}(G)$; it suffices [Buh10, Proposition 11.3] to show that p has a section. Since p is an admissible epi, $\mathrm{gr}(p)^0$ is a split epi. Pick a section $s\colon k(G/N) \to \mathrm{gr}(A)^0$ of $\mathrm{gr}(p)^0\colon\ \mathrm{gr}(A)^0 \twoheadrightarrow k(G/N)$ in $\mathsf{pmod}(G)$, and consider the morphism

$$g\colon\ \mathrm{gz}\, k(G/N) \xrightarrow{\mathrm{gz}\, s} \mathrm{gz}\,\mathrm{gr}(A^0) \rightarrowtail \mathrm{gr}(A)$$

in $\mathsf{pgmod}(G)$. By applying Proposition 9.6 to the morphism g, we obtain a morphism $f\colon\ \mathrm{radf}_N(kG) \to A$ in $\mathsf{pfmod}(G)$ such that $\mathrm{gr}(f)^0 = g^0 = s$ in $\mathsf{pmod}(G)$. Thus we have

$$\mathrm{gr}(pf)^0 = \mathrm{gr}(p)^0\,\mathrm{gr}(f)^0 = \mathrm{gr}(p)^0 s = 1\colon\ k(G/N) \to k(G/N).$$

We have shown that $\mathrm{unf}(pf)\colon kG \to kG$ is an isomorphism modulo $\mathrm{rad}_N(kG)$, so $\mathrm{unf}(pf)$ is an isomorphism modulo $\mathrm{rad}(kG)$. This implies $\mathrm{unf}(pf)$ is an isomorphism because $\mathrm{rad}(kG)$ is the Jacobson radical of kG. We conclude (Remark 3.14) that pf is an isomorphism in $\mathsf{pfmod}(G)$. □

We now seek to apply this sufficient condition in the elementary abelian case.

9.10 Proposition. *For E elementary abelian, any subgroup $N \trianglelefteq E$ satisfies Definition 9.3.*

Proof. Let $H \leqslant E$, let $w \geq 1$, and let $h\colon \operatorname{rad}_N(kE)^w \to k(E/H)$ be a morphism in $\mathsf{mod}(E)$. We want to show that there exists a morphism $\widetilde{h}\colon kE \to k(E/H)$ in $\mathsf{mod}(E)$ making the diagram (9.4) commute.

We start by making some identifications. Let T denote the monoid $k(E/H)$ in vect, and let us also write $T \in \mathsf{mod}(E)$ for the inflation along the augmentation $kE \to T$. We adopt Example 4.6 so that in particular

$$kE \simeq \frac{k[W_i, X_i, Y_i, Z_i]}{(W_i^p, X_i^p, Y_i^p, Z_i^p)}, \qquad T \simeq \frac{k[W_i, Y_i]}{(W_i^p, Y_i^p)}, \qquad \text{and} \qquad \operatorname{soc}_N(T) \simeq \left(\prod_i Y_i^{p-1}\right)$$

in $\mathsf{mod}(E)$. The diagram (9.4) becomes

$$\begin{tikzcd}
kE \arrow[dr, dotted, "\widetilde{h}"] & & \\
& T \arrow[dr, two heads] & \\
(Y_i, Z_i)^w \arrow[uu, hook, "i"] \arrow[r, "h"] & T \arrow[r, two heads] & \dfrac{T}{(\prod_i Y_i^{p-1})}
\end{tikzcd} \tag{9.11}$$

in $\mathsf{mod}(E)$. Moreover, under these identifications T is also a subalgebra of kE, so we let $\operatorname{Res}_T\colon \mathsf{mod}(E) \to \operatorname{mod}_{\mathsf{vect}}(T)$ denote the restriction of scalars to T.

We now study the morphism $\operatorname{Res}_T(h)\colon \operatorname{Res}_T((Y_i, Z_i)^w) \to T$ in $\operatorname{mod}_{\mathsf{vect}}(T)$. Let us call a monomial *pure* if it only involves the variables W_i, Y_i and *contaminated* (by the X_i, Z_i) otherwise. Observe that $\operatorname{Res}_T((Y_i, Z_i)^w)$ decomposes as a direct sum $P \oplus C$ in $\operatorname{mod}_{\mathsf{vect}}(T)$ where P is spanned by the pure monomials in $(Y_i, Z_i)^w$ and C is spanned by contaminated monomials in $(Y_i, Z_i)^w$. Indeed, multiplying by $W_i, Y_i \in T$ does not contaminate a pure monomial and cannot purify a contaminated monomial. We may therefore write

$$\operatorname{Res}_T(h)\colon \operatorname{Res}_T(Y_i, Z_i)^w \ \cong \ P \oplus C \xrightarrow{(u\ v)} T$$

for some morphisms u, v in $\operatorname{mod}_{\mathsf{vect}}(T)$. Moreover, since P is canonically a submodule of $T = k[W_i, Y_i]/(W_i^p, Y_i^p)$ in $\operatorname{mod}_{\mathsf{vect}}(T)$ and since T is a self-injective algebra, we may extend

$u\colon P \to T$ to a morphism $\widetilde{u}\colon T \to T$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, tail] & T \\ T \arrow[ur, "\widetilde{u}"'] & \end{tikzcd}$$

commutes in $\mathrm{mod}_{\mathsf{vect}}(T)$.

Finally, we define

$$\widetilde{h}\colon kE \twoheadrightarrow T \xrightarrow{\widetilde{u}} T$$

in $\mathsf{mod}(E)$, where again we write $\widetilde{u}$ for the inflation of $\widetilde{u}$ along the augmentation $kE \twoheadrightarrow T$.

We now argue that this makes our diagram (9.11) commute. Observe that by construction it commutes on the pure monomials in $(Y_i, Z_i)^w$, so it suffices to show that h sends any contaminated monomial m in $(Y_i, Z_i)^w$ to the socle $(\prod_i Y_i^{p-1}) \subset T$. Let us fix Y_i so that we are reduced to showing $Y_i h(m) = 0$. Since m is contaminated, some X_j divides m or some Z_j divides m. If X_j divides m, then we may write $m = X_j m'$ for some monomial $m' \in (Y_i, Z_i)^w$, whence

$$Y_i h(m) = Y_i h(X_j m') = X_j Y_i h(m') = 0$$

since X_j acts trivially on $T \in \mathsf{mod}(E)$. Suppose Z_j divides m so that $m = Z_j m'$ for some monomial $m' \in (Y_i, Z_i)^{w-1}$. Note that the exponent has decreased by one to $w - 1 \geq 0$, so we now need to be mindful that the source of the morphism h is $(Y_i, Z_i)^w \subset (Y_i, Z_i)^{w-1}$. We have

$$Y_i h(m) = Y_i h(Z_j m') = h(Y_i Z_j m') = Z_j h(Y_i m') = 0$$

since Z_j acts trivially on $T \in \mathsf{mod}(E)$ and where importantly $Y_i m' \in (Y_i, Z_i)^w$. □

9.12 Proposition. *For any subgroup $N \leqslant E$, the N-radical filtration* $\mathrm{radf}_N(kE)$ *is projective in* $\mathsf{pfmod}(E)$.

Proof. Immediate from Proposition 9.9 and Proposition 9.10. □

9.13 Theorem. *The radical filtration* $\mathrm{radf}(kE)$ *is projective in* $\mathsf{pfmod}(E)$.

Proof. This is the special case $N = E$ in Proposition 9.12. $\square$

We record some corollaries of Theorem 9.13.

9.14 Corollary. *The exact category* $\mathsf{pfmod}(E)$ *is Frobenius, and the projective-injective objects form a thick additive tensor ideal* $\langle \mathrm{radf}(kE) \rangle$ *generated by* $\mathrm{radf}(kE)$.

Proof. The augmentation $\varepsilon\colon \mathrm{radf}(kE) \twoheadrightarrow \mathbb{1}$ is a projective (Theorem 9.13) pre-cover of the unit, so this follows from Proposition 5.29(ii). $\square$

9.15 Corollary. *The projective objects in* $\mathsf{pfmod}(E)$ *are precisely the sums of twists of objects of the form* $\mathrm{radf}_N(kE)$ *for* $N \leqslant E$.

Proof. We need to show that $\langle \mathrm{radf}(kE) \rangle = S$, where S denotes the set of sums of twists of objects of the form $\mathrm{radf}_N(kE)$ (Corollary 9.14). Note that $\langle \mathrm{radf}(kE) \rangle \supset S$; indeed, $\mathrm{fz}\, k(E/N) \otimes \mathrm{radf}(kE) \in \langle \mathrm{radf}(kE) \rangle$ is a sum of twists of $\mathrm{radf}_N(kE)$ by Lemma 4.8, so $\mathrm{radf}_N(kE) \in \langle \mathrm{radf}(kE) \rangle$.

To see that $\langle \mathrm{radf}(kE) \rangle \subset S$, it suffices to show that S is a thick additive tensor ideal since $\mathrm{radf}(kE) \in S$. Obviously S is additive, and it is thick because the objects $\mathrm{radf}_N(kE)$ are indecomposable (its underlying modules kE are indecomposable) and $\mathsf{pfmod}(E)$ is Krull–Remak–Schmidt (Lemma 3.43). To show that it is a tensor ideal, let $A \in \mathsf{pfmod}(E)$. Since S is additive and closed under twists, it suffices to fix $N \leqslant E$ and show that $\mathrm{radf}_N(kE) \otimes A \in S$. For the same reason, we may assume A is effective. We proceed by induction on the filtration length of A. If A has filtration length 1, then A is a sum of $\mathrm{fz}\, k(E/H)$ for $H \leqslant E$, so one verifies using Lemma 4.8 again that $\mathrm{radf}_N(kE) \otimes \mathrm{fz}\, k(E/H) \in S$. To induct on filtration length, we tensor (5.17) with $\mathrm{radf}_N(kE)$. $\square$

9.16 Notation. We let ℓ_G denote the Loewy length of kG, which is the minimal integer $\ell_G \in \mathbb{Z}$ such that $\mathrm{rad}(kG)^{\ell_G} = 0$. Thus for $\mathrm{radf}(kG) \in \mathsf{pfmod}(G)$ we have $\mathrm{gr}\,\mathrm{radf}(kG)^0 = \mathrm{gr}\,\mathrm{radf}(kG)^{\ell_G - 1} = k$.

9.17 Corollary. *The dual* $\mathrm{radf}(kE)^\vee$ *of the projective object* $\mathrm{radf}(kE) \in \mathsf{pfmod}(E)$ *is* $\mathrm{radf}(kE)(-\ell_E + 1)$.

Proof. The dual $\mathrm{radf}(kE)^\vee$ is projective by Corollary 9.14 (see Lemma 5.25 if necessary), and $\mathrm{gr}(\mathrm{radf}(kE)^\vee) = \mathrm{gr}(\mathrm{radf}(kE))^\vee$ is a sum of twists of the unit concentrated in degrees $-\ell_E + 1, \ldots, 0$ (see (3.17) if necessary). Thus Corollary 9.15 implies it is a twist of $\mathrm{radf}(kE)$, and by definition of ℓ_E it must be the twist $-\ell_E + 1$. $\square$

We conclude this section with some counterexamples that elucidate why we are focusing on the elementary abelian case.

9.18 Example. Let $n \geq 2$, let $G = C_{p^n}$ be the cyclic group order p^n, and let $N \leqslant G$ be a subgroup of order p^ℓ where $\ell \geq 2$. Then N does not satisfy Definition 9.3. Indeed, take $H \leqslant G$ to be the unique subgroup of order p, and take $w = 1$. Let us write $[m]$ for the m-dimensional indecomposable kG-module so that

$$kG \cong [p^n], \qquad k(G/H) \cong [p^{n-1}], \qquad \text{and} \qquad \mathrm{rad}_N(kG) \cong [p^n - p^{n-\ell}].$$

We also have

$$\mathrm{soc}_N(k(G/H)) \cong [p^{n-\ell}] \qquad \text{and} \qquad k(G/H)/\,\mathrm{soc}_N(k(G/H)) \cong [p^{n-1} - p^{n-\ell}] \neq 0$$

using that $\ell \geq 2$. Take $h\colon\ \mathrm{rad}_N(kG) \to k(G/H)$ to be a surjection:

$$\begin{tikzcd}
{[p^n]} \arrow[dr, dotted, "\widetilde{h}"] & & \\
 & {[p^{n-1}]} \arrow[dr, two heads] & \\
{[p^n - p^{n-\ell}]} \arrow[uu, tail] \arrow[r, two heads, "h"] & {[p^{n-1}]} \arrow[r, two heads] & {[p^{n-1} - p^{n-\ell}]}.
\end{tikzcd}$$

There does not exist $\widetilde{h}$ as required in Definition 9.3 because the composite along the bottom row is a surjection whereas the other composite cannot be.

9.19 Lemma. *If P is a projective object in* $\mathsf{pfmod}(G)$, *then its underlying object* $\mathrm{unf}(P)$ *is a projective object in* $\mathsf{mod}(G)$.

Proof. Suppose $P \in \mathsf{pfmod}(G)$ is projective, and consider the following lifting problem in $\mathsf{mod}(G)$:

$$\begin{tikzcd} & C_1 \arrow[d, two heads, "p"] \\ \mathrm{unf}(P) \arrow[r, "h"'] \arrow[ur, dotted] & C_2. \end{tikzcd} \tag{9.20}$$

Let w be large enough so that unf induces isomorphisms (Lemma 3.15)

$$\mathrm{Hom}_{\mathsf{pfmod}(G)}(P, \mathrm{radf}(C_1)(w)) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{mod}(G)}(\mathrm{unf}(P), C_1)$$

and

$$\mathrm{Hom}_{\mathsf{pfmod}(G)}(P, \mathrm{radf}(C_2)(w)) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{mod}(G)}(\mathrm{unf}(P), C_2).$$

Since p induces an admissible epi $p\colon \mathrm{radf}(C_1) \twoheadrightarrow \mathrm{radf}(C_2)$ in $\mathsf{pfmod}(G)$, we have a lift

$$\begin{tikzcd} & \mathrm{radf}(C_1)(w) \arrow[d, two heads, "p"] \\ P \arrow[r, "h"'] \arrow[ur, dotted, "f"] & \mathrm{radf}(C_2)(w). \end{tikzcd}$$

in $\mathsf{pfmod}(G)$. Now $\mathrm{unf}(f)$ solves the lifting problem (9.20). $\square$

9.21 Example. Let

$$n \geq \begin{cases} 2 & p > 2 \\ 3 & p = 2, \end{cases}$$

and let $G = C_{p^n}$. Suppose we have a projective object $P \in \mathsf{pfmod}(G)$ that admits a projective pre-cover $P \twoheadrightarrow \mathbb{1}$ of the unit. By Lemma 9.19, we know that $\mathrm{unf}(P) \in \mathsf{mod}(G)$ must be projective, but unfortunately $\mathrm{unf}(P)$ cannot be kG itself. Indeed, consider the filtration $A \in \mathsf{pfmod}(G)$ of kG such that the sequence of $\mathrm{gr}(A)^w$ for $w = 0, 1, \ldots, n(p-1)$ is the sequence of indecomposable permutation modules

$$[1], \overbrace{[p^{n-1}], \ldots, [p^{n-1}]}^{p-1 \text{ times}}, \overbrace{[p^{n-2}], \ldots, [p^{n-2}]}^{p-1 \text{ times}}, \ldots, \overbrace{[p], \ldots, [p]}^{p-1 \text{ times}}, \overbrace{[1], \ldots, [1]}^{p-1 \text{ times}}.$$

This exists because $kG \simeq k[X]/(X^{p^n})$ by picking a generator x of G (Notation 4.5). Let $N \leqslant G$ be the order p subgroup. We have a morphism $f\colon A \to \mathrm{fz}\, k(G/N)(1)$ in $\mathsf{pfmod}(G)$ such that $\mathrm{unf}(f)$ is the augmentation, but there does not exist a morphism $\widetilde{f}\colon A \to \mathrm{radf}_N(kG)(1)$ in $\mathsf{pfmod}(G)$ such that $\mathrm{unf}(\widetilde{f})$ is the identity because

$$A^{n(p-1)} = [1] \neq 0 = \mathrm{radf}_N(kG)(1)^{n(p-1)},$$

where $n(p-1)-1 \geq p$ by our assumption on n. Thus $A \in \mathsf{pfmod}(G)$ is not projective because it fails to lift

$$\begin{tikzcd} & \mathrm{radf}_N(kG)(1) \arrow[d, two heads] \\ A \arrow[ur, dotted] \arrow[r, "f"'] & \mathrm{fz}\, k(G/N)(1). \end{tikzcd}$$

However, by testing

$$\begin{tikzcd} & A \arrow[d, two heads] \\ P \arrow[ur, dotted] \arrow[r, two heads] & \mathbb{1}, \end{tikzcd}$$

we may argue that in order for $P \in \mathsf{pfmod}(G)$ to be projective with $\mathrm{unf}(P) = kG$, we must have $P = A$.

10 The vanishing locus of $\mathrm{cone}(\beta)$

10.1 Notation. For this section, we assume our p-group G (Notation 2.1) is such that $\mathsf{pfmod}(G)$ has enough projectives. For example, we may take $G = E$ to be elementary abelian (Corollary 9.14).

In this section we study the vanishing locus of $\mathrm{cone}(\beta) \in \mathrm{D_b}(\mathsf{pfmod}(G))$ in the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$. In particular, we show in Proposition 10.9 that it is homeomorphic to $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G))$ (Recollection 6.29) via unfiltering.

Let us isolate how we use our assumption in Notation 10.1 that $\mathsf{pfmod}(G)$ has enough projectives.

10.2 Lemma. *Let $X, Y \in \mathrm{Ch_b}(\mathsf{pfmod}(G))$. For $w \geq 0$ sufficiently large, the tt-functor $\mathrm{unf}\colon \mathrm{D_b}(\mathsf{pfmod}(G)) \to \mathrm{D_b}(\mathsf{mod}(G))$ induces an isomorphism*

$$\mathrm{unf}\colon \mathrm{Hom}_{\mathrm{D_b}(\mathsf{pfmod}(G))}(X, Y(w)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{D_b}(\mathsf{mod}(G))}(\mathrm{unf}(X), \mathrm{unf}(Y(w))).$$

Proof. Let $\pi\colon P \to X$ be a projective resolution in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$ (Recollection 7.4). We claim that for w sufficiently large, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D_b}(\mathsf{pfmod}(G))}(X, Y(w)) &\cong \mathrm{Hom}_{\mathrm{K_+}(\mathsf{pfmod}(G))}(P, Y(w))\\ &\cong \mathrm{Hom}_{\mathrm{K_+}(\mathsf{mod}(G))}(\mathrm{unf}(P), \mathrm{unf}(Y))\\ &\cong \mathrm{Hom}_{\mathrm{D_b}(\mathsf{mod}(G))}(\mathrm{unf}(X), \mathrm{unf}(Y)).\end{aligned}$$

The first isomorphism is (7.5). The second isomorphism is induced by unf for $w \geq 0$ sufficiently large: it is a surjection by applying Lemma 3.15, and it is an injection by applying Lemma 3.15 on a nullhomotopy. The third isomorphism is again (7.5), where $\mathrm{unf}(P)$ is a complex of projectives by Lemma 9.19 and where $\mathrm{unf}(\pi)\colon \mathrm{unf}(P) \to \mathrm{unf}(X)$ is a quasi-isomorphism in $\mathrm{Ch_b}(\mathsf{mod}(G))$ by exactness of unf (Remark 5.23). $\square$

The limiting behavior in Lemma 10.2 arises in the colimit of the abelian groups

$$\mathrm{Hom}_{\mathrm{D_b}(\mathsf{pfmod}(G))}(X, Y(w))$$

where the transition maps are given by β. These are the homs in the central localization at β.

10.3 Recollection. Let $\mathcal{K}$ be a tt-category, let $u \in \mathcal{K}$ be a $\otimes$-invertible object with trivial central switch $u^{\otimes 2} \to u^{\otimes 2}$, and let $\beta\colon \mathbb{1} \to u$ be a morphism from the unit. We denote by $\mathcal{K}[\beta^{-1}]$ the *central localization at* β. It is the tt-category whose objects are the objects in $\mathcal{K}$, whose morphisms are given by the colimits

$$\mathrm{Hom}_{\mathcal{K}[\beta^{-1}]}(X, Y) = \operatorname*{colim}_{n\geq 0} \mathrm{Hom}_{\mathcal{K}}(X, Y \otimes u^{\otimes n}) \tag{10.4}$$

of abelian groups with transition maps $f \mapsto \beta f$, and whose tt-structure makes $\mathcal{K} \to \mathcal{K}[\beta^{-1}]$ a tt-functor. There exists an equivalence $\mathcal{K}[\beta^{-1}] \to \mathcal{K}/\langle \mathrm{cone}(\beta)\rangle$ making the following diagram of tt-categories and tt-functors commute:

$$\begin{tikzcd} \mathcal{K} \arrow[d] \arrow[dr, two heads] & \\ \mathcal{K}[\beta^{-1}] \arrow[r, "\cong"'] & \dfrac{\mathcal{K}}{\langle \mathrm{cone}(\beta)\rangle}. \end{tikzcd}$$

Indeed, this is [Bal10a, Theorem 3.6] together with [Bal18, Proposition 2.10]; see also [BG22b, Proposition 6.16].

10.5 Lemma. *We have*

$$\mathrm{Hom}_{\mathrm{D_b}(\mathsf{pfmod}(G))/\langle \mathrm{cone}(\beta)\rangle}(X,Y) \cong \operatorname*{colim}_{w\geq 0} \mathrm{Hom}_{\mathrm{D_b}(\mathsf{pfmod}(G))}(X,Y(w)) \tag{10.6}$$

where the transition maps in the colimit are given by $f \mapsto \beta f$.

Proof. Observe that

$$\begin{tikzcd}
\mathbb{1}\otimes\mathrm{cone}(\beta)\colon \arrow[d, "\beta\otimes\mathrm{cone}(\beta)"'] & \mathbb{1} \arrow[r] \arrow[d, "\beta"] & \mathbb{1}(1) \arrow[d, "\beta"] \arrow[dl, dotted, "1"] \\
\mathbb{1}(1)\otimes\mathrm{cone}(\beta)\colon & \mathbb{1}(1) \arrow[r] & \mathbb{1}(2)
\end{tikzcd}$$

exhibits a nullhomotopy of $\beta\otimes\mathrm{cone}(\beta)$. Thus (Recollection 10.3) the Verdier quotient $\mathrm{D_b}(\mathsf{pfmod}(G))/\langle\mathrm{cone}(\beta)\rangle$ is the central localization of $\mathrm{D_b}(\mathsf{pfmod}(G))$ at β, so (10.6) is given by (10.4). ∎

10.7 Remark. The transition maps in the colimit in (10.6) are not necessarily injective. For example, for the nonzero morphism

$$\begin{tikzcd}
\mathbb{1}(1)\colon \arrow[d, "f"'] & & \mathbb{1}(1) \arrow[d, "1"] \\
\mathrm{cone}(\beta)\colon & \mathbb{1} \arrow[r] & \mathbb{1}(1)
\end{tikzcd}$$

in $\mathrm{D_b}(\mathsf{pfmod}(G))$, we have $\beta f = 0$ in $\mathrm{D_b}(\mathsf{pfmod}(G))$.

We need one final ingredient.

10.8 Lemma. *The tt-functor* $\mathrm{unf}\colon \mathrm{D_b}(\mathsf{pfmod}(G)) \to \mathrm{D_b}(\mathsf{mod}(G))$ *is essentially surjective.*

Proof. This is straightforward using Lemma 3.15 and Example 3.44. Briefly, given a complex $Z \in \mathrm{Ch_b}(\mathsf{mod}(G))$, we define $X \in \mathrm{Ch_b}(\mathsf{pfmod}(G))$ to have objects of the form $X_i = \mathrm{radf}(Z_i)(w_i) \in \mathsf{pfmod}(G)$, where the twists w_i are chosen so that each differential $Z_i \to Z_{i+1}$ in $\mathsf{mod}(G)$ induces a morphism $X_i \to X_{i+1}$ in $\mathsf{pfmod}(G)$. Then $\mathrm{unf}(X) = Z$. ∎

10.9 Proposition. *Unfiltering* $\mathrm{unf}\colon \mathsf{pfmod}(G) \to \mathsf{mod}(G)$ *induces an equivalence of tt-categories*

$$\overline{\mathrm{unf}}\colon \frac{\mathrm{D_b}(\mathsf{pfmod}(G))}{\langle \mathrm{cone}(\beta)\rangle} \longrightarrow \mathrm{D_b}(\mathsf{mod}(G)). \tag{10.10}$$

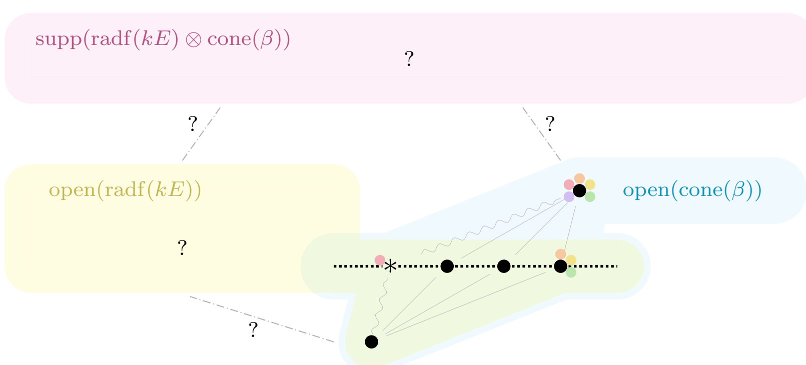
Geometrically, $\mathrm{Spc}(\overline{\mathrm{unf}})$ *induces a homeomorphism*

$$\mathrm{Spc}(\overline{\mathrm{unf}})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G)) \xrightarrow{\ \cong\ } \mathrm{open}(\mathrm{cone}(\beta)) \ \subset\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G)).$$

Proof. Note that $\mathrm{unf}(\mathrm{cone}(\beta)) = 0$ because $\mathrm{unf}(\beta) = 1$. The geometric statement is immediate from the algebraic one (Method 6.10(*ii*)), so let us explain why $\overline{\mathrm{unf}}$ is an equivalence. It is essentially surjective by Lemma 10.8. To see that it is fully faithful, consider an arbitrary hom-set $\mathrm{colim}_{w\geq 0}\,\mathrm{Hom}_{\mathrm{D_b}(\mathsf{pfmod}(G))}(X, Y(w))$ (Lemma 10.5). Applying unf, the transition maps stabilize (Lemma 10.2), so the colimit is the stabilizing object $\mathrm{Hom}_{\mathrm{D_b}(\mathsf{mod}(G))}(\mathrm{unf}(X), \mathrm{unf}(Y(w)))$. □

10.11 Running Example. We return to our Running Example 1.13. The following illustrates the region of the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E))$ that Proposition 10.9 determines.

$\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E))$



● $\mathrm{supp}(\mathrm{radf}(C_{[a:b]}))$ ● $\mathrm{supp}(\mathrm{fz}\,k(E/N_{[0:1]}))$ ● $\mathrm{supp}(\mathrm{radf}\,k(E/N_{[0:1]}))$

● $\mathrm{supp}(\mathrm{gap}^1\,\mathrm{radf}(C_{[a:b]}))$ ● $\mathrm{supp}(\mathrm{fz}\,\mathrm{kos}_E(1))$ ● $\mathrm{supp}(\mathrm{gap}^1\,\mathrm{radf}\,k(E/N_{[0:1]}))$.

The highlighted regions illustrate our overall strategy (see (1.21)) to divide and conquer the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$ using the objects P and $\operatorname{cone}(\beta)$; in this case $P = \operatorname{radf}(kE)$ (Corollary 9.14). The question marks denote unknown points and specializations. In particular, here there are no specializations between the purely yellow and purely cyan region regions because the yellow and cyan regions are both open sets. The cyan region is the region that we have just determined, and it distinguishes between objects that have different underlying objects in $\mathrm{D_b}(\mathsf{mod}(E))$. For example, it can distinguish between $\operatorname{radf}(C_{[a:b]})$ and $\operatorname{radf} k(E/N_{[0:1]})$. However, it cannot distinguish between the various filtrations $\operatorname{fz} k(E/N_{[0:1]})$, $\operatorname{radf} k(E/N_{[0:1]})$, and $\operatorname{gap}^1 \operatorname{radf} k(E/N_{[0:1]})$ (recall Example 3.26) of the indecomposable permutation module $k(E/N_{[0:1]})$, and it cannot distinguish $\operatorname{radf}(C_{[a:b]})$ from $\operatorname{gap}^1 \operatorname{radf}(C_{[a:b]})$ (note $\operatorname{fz} C_{[a:b]} \in \mathsf{fmod}(E)$ is not an object in $\mathsf{pfmod}(E)$). It also cannot distinguish $\operatorname{fz} \operatorname{kos}_E(1)$ from the zero object 0.

11 The support of $P \otimes \operatorname{cone}(\beta)$

11.1 Notation. In this section, we assume that our p-group G (Notation 2.1) is such that $\mathsf{pfmod}(G)$ has an effective comonoid P whose counit $\varepsilon\colon P \twoheadrightarrow \mathbb{1}$ is a projective pre-cover of the unit. We emphasize that P is a comonoid in $\mathsf{pfmod}(G)$, not just in our ambient category fvect. For example, we may take $G = E$ to be elementary abelian (Corollary 9.14 and Example 3.45).

In this section we study the support of $P \otimes \operatorname{cone}(\beta)$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(G))$. In particular, we show in Theorem 11.5 that it is homeomorphic to the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(G))$ of the homotopy category of permutation kG-modules, namely by the map on spectra induced by $\operatorname{ung} \operatorname{gr}$. The proof of Theorem 11.5 is the most intricate proof in this work.

Let us unravel what it means for a morphism in $\mathrm{D_b}(\mathsf{pfmod}(G))$ to vanish when tensored with $\operatorname{cone}(\beta)$. Compare the following to Lemma 3.12.

11.2 Lemma. *Let $\mathcal{K}$ be a tt-category, and suppose $\beta\colon u \to v$ is a morphism where v is $\otimes$-invertible and where $\beta \otimes \operatorname{cone}(\beta) = 0$. A morphism $f\colon X \to Y$ in $\mathcal{K}$ satisfies $f \otimes \operatorname{cone}(\beta) = 0$*

if and only if there exists γ such that

$$\begin{tikzcd} & X \otimes v \arrow[dl, dotted, "\gamma"'] \arrow[d, "f\otimes v"] \\ Y \otimes u \arrow[r, "Y\otimes\beta"'] & Y \otimes v. \end{tikzcd}$$

Proof. Consider the following diagram in $\mathcal{K}$ in which the top and bottom rows are distinguished triangles:

$$\begin{tikzcd} X \otimes u \arrow[r, "X\otimes\beta"] \arrow[d, "f\otimes u"'] & X \otimes v \arrow[r] \arrow[d, "f\otimes v"] \arrow[dl, dotted, "\gamma"] & X \otimes \mathrm{cone}(\beta) \arrow[r] \arrow[d, "f\otimes \mathrm{cone}(\beta)"] & X \otimes u[1] \arrow[d, "f\otimes u[1]"] \\ Y \otimes u \arrow[r, "Y\otimes\beta"'] & Y \otimes v \arrow[r] & Y \otimes \mathrm{cone}(\beta) \arrow[r] & Y \otimes u[1]. \end{tikzcd}$$

If $f \otimes \mathrm{cone}(\beta) = 0$, then there exists γ as described because $Y \otimes \beta$ is a weak kernel of $Y \otimes v \to Y \otimes \mathrm{cone}(\beta)$. For the converse, observe that

$$f \otimes \mathrm{cone}(\beta) = f \otimes (v \otimes v^{\otimes -1}) \otimes \mathrm{cone}(\beta) = ((Y \otimes \beta)\gamma) \otimes v^{\otimes -1} \otimes \mathrm{cone}(\beta) = 0,$$

where we have used that $\beta \otimes \mathrm{cone}(\beta) = 0$ and that v is $\otimes$-invertible. $\square$

We will use iterates of the comultiplication on P.

11.3 Notation. For $m \geq 1$, we denote by $\Delta_m\colon P \to P^{\otimes m}$ the $(m-1)$st iterate of the comultiplication $\Delta\colon P \to P^{\otimes 2}$.

11.4 Lemma. *For any morphism $f\colon \mathbb{1} \to A$ in $\mathsf{pfmod}(G)$ and any $m \geq 1$, we have*

$$f^{\otimes m} \otimes 1 = ((f\varepsilon)^{\otimes m} \otimes 1)\Delta_{m+1}\colon P \longrightarrow A^{\otimes m} \otimes P.$$

Proof. Simply recall that $(\varepsilon \otimes 1)\Delta = 1$ for any comonoid. $\square$

We now give the most intricate proof in this work.

11.5 Theorem. *The tt-functor* $\mathrm{ung\,gr}\colon \mathrm{D_b}(\mathsf{pfmod}(G)) \to \mathrm{D_b}(\mathsf{pmod}(G))$ *detects nilpotence on $P \otimes \mathrm{cone}(\beta)$. Geometrically,* $\mathrm{Spc}(\mathrm{ung\,gr})$ *induces a homeomorphism*

$$\mathrm{Spc}(\mathrm{ung\,gr})\colon \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G)) \xrightarrow{\cong} \mathrm{supp}(P \otimes \mathrm{cone}(\beta)) \subset \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G)).$$

Proof. The algebraic statement implies the geometric one (Method 6.10(i)). Indeed, the counit $\varepsilon\colon P \twoheadrightarrow \mathbb{1}$ being an admissible epi implies $\mathbb{1} \mid \operatorname{ung}\operatorname{gr}(P)$, so

$$\operatorname{ung}\operatorname{gr}(P \otimes \operatorname{cone}(\beta)) = \operatorname{ung}\operatorname{gr}(P) \otimes (\mathbb{1} \oplus \mathbb{1}[1]) \tag{11.6}$$

has full support. Moreover $\operatorname{fz}\colon \mathrm{D_b}(\mathsf{pmod}(G)) \to \mathrm{D_b}(\mathsf{pfmod}(G))$ is a section of $\operatorname{ung}\operatorname{gr}$.

Let $f\colon Y \to X$ be a morphism in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$ and $s\colon \mathbb{1} \to X$ be a quasi-isomorphism in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$ so that the fraction $s^{-1}f\colon \mathbb{1} \to X \leftarrow Y$ represents an arbitrary morphism $\mathbb{1} \to Y$ in $\mathrm{D_b}(\mathsf{pfmod}(G))$. Let us assume $\operatorname{ung}\operatorname{gr}(s^{-1}f) = 0$ in $\mathrm{D_b}(\mathsf{pmod}(G))$. It suffices (Lemma 6.9(i)) to show $(s^{-1}f)^{\otimes m} \otimes P \otimes \operatorname{cone}(\beta) = 0$ in $\mathrm{D_b}(\mathsf{pfmod}(G))$ for sufficiently large m.

In the remainder of this proof, we will show that for sufficiently large m, there exists a morphism $t\colon P \to X^{\otimes m}(-1)$ in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$ such that the following diagram commutes in $\mathrm{K_b}(\mathsf{pfmod}(G))$:

$$\begin{tikzcd} & P \arrow[dl, "(t\otimes 1)\Delta"'] \arrow[d, "((f\varepsilon)^{\otimes m}\otimes 1)\Delta_{m+1}"] \\ X^{\otimes m} \otimes P(-1) \arrow[r, "\beta"'] & X^{\otimes m} \otimes P. \end{tikzcd} \tag{11.7}$$

To see that this suffices, note that $((f\varepsilon)^{\otimes m} \otimes 1)\Delta_{m+1} = f^{\otimes m} \otimes P$ (Lemma 11.4), so the existence of $(t \otimes 1)\Delta$ in (11.7) is equivalent to $f^{\otimes m} \otimes P \otimes \operatorname{cone}(\beta) = 0$ in $\mathrm{K_b}(\mathsf{pfmod}(G))$ (Lemma 11.2). Then certainly $(s^{-1}f)^{\otimes m} \otimes P \otimes \operatorname{cone}(\beta) = 0$ in $\mathrm{D_b}(\mathsf{pfmod}(G))$.

We begin by giving the obvious names to our morphisms:

$$\begin{tikzcd} P\colon \arrow[d, "\varepsilon"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, two heads, "\varepsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\ \mathbb{1}\colon \arrow[d, "f"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{1} \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] \arrow[d] & \cdots \\ X\colon & \cdots \arrow[r] & X_1 \arrow[r, "d_1"'] & X_0 \arrow[r, "d_0"'] & X_{-1} \arrow[r] & \cdots . \end{tikzcd} \tag{11.8}$$

Now we construct a morphism $h'\colon P \to X_1$ in $\mathsf{pfmod}(G)$. Our hypothesis that the morphism $\operatorname{ung}\operatorname{gr}(s^{-1}f)$ vanishes in $\mathrm{D_b}(\mathsf{pmod}(G))$ implies $\operatorname{ung}\operatorname{gr}(f)\colon \mathbb{1} \to \operatorname{ung}\operatorname{gr}(X)$ vanishes in $\mathrm{K_b}(\mathsf{pmod}(G))$ because $\mathsf{pmod}(G)$ is equipped with the split exact structure. Since the unit

$\mathbb{1}$ is concentrated in grading zero, in fact $\mathrm{gr}(f)^0 \colon \mathbb{1} \to \mathrm{gr}(X)^0$ vanishes in $\mathrm{K_b}(\mathsf{pmod}(G))$, so there exists a nullhomotopy $\overline{h} \colon \mathbb{1} \to \mathrm{gr}(X_1)^0$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{1} \arrow[r] \arrow[d, "\mathrm{gr}(f_0)^0"] \arrow[dl, dotted, "\overline{h}"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{gr}(X_1)^0 \arrow[r, "\mathrm{gr}(d_1)^0"'] & \mathrm{gr}(X_0)^0 \arrow[r, "\mathrm{gr}(d_0)^0"'] & \mathrm{gr}(X_{-1})^0 \arrow[r] & \cdots
\end{tikzcd} \tag{11.9}$$

in $\mathsf{pmod}(G)$ commute. We construct $h' \colon P \to X_1$ in $\mathsf{pfmod}(G)$ using the following diagram in $\mathsf{pfmod}(G)$ and projectivity of P:

$$\begin{tikzcd}
 & & X_1 \\
 & & X_1^{\geq 0} \arrow[u, hook] \arrow[d, two heads] \\
P \arrow[r, two heads, "\varepsilon"'] \arrow[uurr, dotted, "h'"] \arrow[urr, dotted] & \mathbb{1} \arrow[r, "\mathrm{fz}\,\overline{h}"'] & \mathrm{fz}\,\mathrm{gr}(X_1^0).
\end{tikzcd}$$

By applying $\mathrm{gr}(-)^0$, we see that

$$\mathrm{gr}(h')^0 = \overline{h}\,\mathrm{gr}(\varepsilon)^0. \tag{11.10}$$

We now construct the morphism $t \colon P \to X^{\otimes m}(-1)$ in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$, which is the crux of our goal (11.7). Consider the morphism $t' \colon P \to X$ in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$ defined by

$$t'_0 = f_0\varepsilon - d_1 h' \colon P \longrightarrow X_0 \tag{11.11}$$

in $\mathsf{pfmod}(G)$. This is well-defined because

$$d_0 f_0 \varepsilon = 0 \tag{11.12}$$

in $\mathsf{pfmod}(G)$ by (11.8) and hence

$$d_0 t'_0 = d_0 f_0 \varepsilon - d_0 d_1 h = 0 \tag{11.13}$$

in $\mathsf{pfmod}(G)$. Observe that

$$\begin{aligned}
\mathrm{gr}(t'_0)^0 &= \mathrm{gr}(f_0\varepsilon - d_1 h')^0 && \text{(by definition (11.12))} \\
&= \mathrm{gr}(f_0)^0\,\mathrm{gr}(\varepsilon)^0 - \mathrm{gr}(d_1)^0\overline{h}\,\mathrm{gr}(\varepsilon)^0 && \text{(by (11.10))} \\
&= 0. && \text{(by (11.9))}
\end{aligned}$$

Thus $\mathrm{gr}(t_0'^{\otimes m})^w \colon \mathrm{gr}(P)^{\otimes m} \to X_0^{\otimes m}$ is zero for gradings $w = 0, \ldots, m-1$ because P is effective (Notation 11.1). We therefore see that for sufficiently large $m \geq 1$, the morphism

$$t_0'^{\otimes m}\Delta_m \colon P \xrightarrow{\Delta_m} P^{\otimes m} \xrightarrow{t_0'^{\otimes m}} X_0^{\otimes m}$$

in $\mathsf{pfmod}(G)$ is such that $\mathrm{gr}(t_0'^{\otimes m}\Delta_m) = 0$; it follows (Lemma 3.12) that $t_0'^{\otimes m}\Delta_m$ factors through β in $\mathsf{pfmod}(G)$, and hence so does $t'^{\otimes m}\Delta_m$ in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$. We define t to be the factoring morphism in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$:

$$\begin{tikzcd} & P \arrow[dl, dotted, "t"'] \arrow[d, "t'^{\otimes m}\Delta_m"] \\ X^{\otimes m}(-1) \arrow[r, "\beta"'] & X^{\otimes m}. \end{tikzcd} \tag{11.14}$$

Finally, having constructed $t \colon P \to X^{\otimes m}(-1)$, we now show that (11.7) commutes in $\mathrm{K_b}(\mathsf{pfmod}(G))$, which simply means that the morphism

$$((f\varepsilon)^{\otimes m} \otimes 1)\Delta_{m+1} - \beta(t \otimes 1)\Delta \colon P \longrightarrow X^{\otimes m} \otimes P$$

in $\mathrm{Ch_b}(\mathsf{pfmod}(G))$ is nullhomotopic. We need to find a morphism $h \colon P \to (X^{\otimes m} \otimes P)_1$ in $\mathsf{pfmod}(G)$ that defines a nullhomotopy

$$\begin{tikzcd} P\colon \arrow[d, "((f\varepsilon)^{\otimes m}\otimes 1)\Delta_{m+1}-\beta(t\otimes 1)\Delta"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[dl, dotted, "h"'] \arrow[d, "((f_0\varepsilon_0)^{\otimes m}\otimes 1)\Delta_{m+1}-\beta(t_0\otimes 1)\Delta"] & 0 \arrow[r] & \cdots \\ X^{\otimes m}\otimes P\colon & \cdots \arrow[r] & (X^{\otimes m}\otimes P)_1 \arrow[r, "d_1"'] & (X^{\otimes m}\otimes P)_1 \arrow[r, "d_0"'] & (X^{\otimes m}\otimes P)_{-1} \arrow[r] & \cdots . \end{tikzcd}$$

Behold:

$$h = \sum_{i=0}^{m} ((f_0\varepsilon)^{\otimes i} \otimes h' \otimes t_0'^{\otimes(m-i)} \otimes 1)\Delta_{m+1} \colon P \longrightarrow (X^{\otimes m} \otimes P)_1.$$

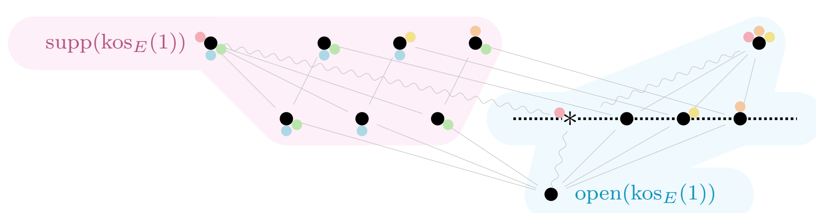
This is well-defined because h' is a morphism $P \to X_1$ in $\mathsf{pfmod}(G)$ whereas $f_0\varepsilon$ and t_0' are morphisms $P \to X_0$ in $\mathsf{pfmod}(G)$. It is indeed a nullhomotopy by the following computation, where in the second line there are no signs because $f_0\varepsilon$ is a morphism $P \to X_0$:

$$
\begin{aligned}
d_1 h &= d_1 \sum_{i=0}^{m} ((f_0\varepsilon)^{\otimes i} \otimes h' \otimes t_0'^{\otimes(m-i)} \otimes 1)\Delta_{m+1} \\
&= \sum_{i=0}^{m} ((f_0\varepsilon)^{\otimes i} \otimes (d_1 h') \otimes t_0'^{\otimes(m-i)} \otimes 1)\Delta_{m+1} && \text{(by (11.12) and (11.13))} \\
&= \sum_{i=0}^{m} ((f_0\varepsilon)^{\otimes i} \otimes (f_0\varepsilon - t_0') \otimes t_0'^{\otimes(m-i)} \otimes 1)\Delta_{m+1} && \text{(by (11.11))} \\
&= ((f_0\varepsilon)^{\otimes m} \otimes 1 - t_0'^{\otimes m} \otimes 1)\Delta_{m+1} && \text{(telescoping sum)} \\
&= ((f_0\varepsilon)^{\otimes m} \otimes 1)\Delta_{m+1} - \beta(t_0 \otimes 1)\Delta. && \text{(by (11.14))}
\end{aligned}
$$

This concludes the proof. □

11.15 Running Example. We return to our Running Example 1.13 of the Klein-four group $E = C_2^{\times 2}$. The spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(E))$ of the derived category of permutation kE-modules was one of the main examples of [BG25a], and we depicted the spectrum in (1.4) in our introduction. Here is a more detailed picture:

$\operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(E))$

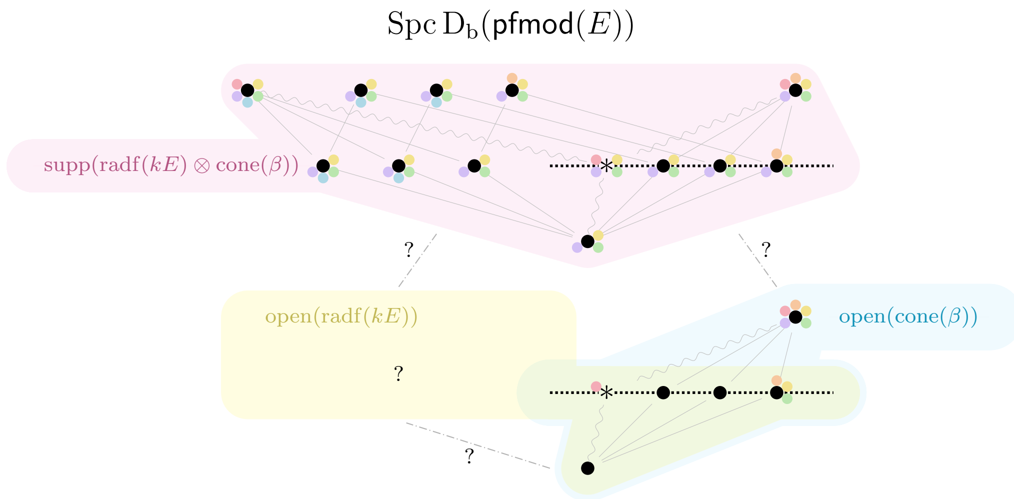


• $\operatorname{supp}(\text{resol. of } C_{[a:b]})$ • $\operatorname{supp}(k(E/N_{[0:1]}))$ • $\operatorname{supp}(k(E/N_{[1:1]}))$

• $\operatorname{supp}(\operatorname{kos}_E(N_{[0:1]}))$ • $\operatorname{supp}(\operatorname{kos}_E(N_{[1:1]}))$.

Note that the red markers depict the support of a permutation resolution of $C_{[a:b]}$, by which we mean an object in $\mathrm{D_b}(\mathsf{pmod}(E))$ that is isomorphic in $\mathrm{D_b}(\mathsf{mod}(E))$ to $C_{[a:b]}$ (see Example 7.14); we assume it is minimal so that it has the depicted support.

Thus, the following illustrates the region of the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$ that Theorem 11.5 determines, namely the magenta one:



● $\operatorname{supp}(\mathrm{fz}(\text{resol. of } C_{[a:b]}))$ ● $\operatorname{supp}(\mathrm{fz}\, k(E/N_{[0:1]}))$ ● $\operatorname{supp}(\mathrm{radf}\, k(E/N_{[0:1]}))$

● $\operatorname{supp}(\mathrm{gap}^1 \mathrm{radf}(kE))$ ● $\operatorname{supp}(\mathrm{fz}\, \mathrm{kos}_E(N_{[0:1]}))$ ● $\operatorname{supp}(\mathrm{gap}^1 \mathrm{radf}\, k(E/N_{[0:1]}))$.

It distinguishes between objects that have non-tt-equivalent associated graded objects in $\mathrm{D_b}(\mathsf{pmod}(E))$. Together with the cyan region, these regions can distinguish between, for example, the objects $\mathrm{fz}(\text{resol. of } C_{[a:b]})$, $\mathrm{fz}\, k(E/N_{[0:1]})$, $\mathrm{radf}\, k(E/N_{[0:1]})$, $\mathrm{fz}\, \mathrm{kos}_E(N_{[0:1]})$, and 0. However, they still cannot distinguish $\mathrm{radf}\, k(E/N_{[0:1]})$ from $\mathrm{gap}^1 \mathrm{radf}\, k(E/N_{[0:1]})$ or even $\mathrm{radf}(kE)$ from $\mathrm{gap}^1 \mathrm{radf}(kE)$.

We wrap up this section by observing that the points in the image of $\operatorname{Spc}(\mathrm{ung}\,\mathrm{gr})$ are generated in filtration weight zero.

11.16 Proposition. *For* $\mathcal{P} \in \operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(G))$, *we have* $\operatorname{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P}) = \langle \mathrm{fz}\, \mathcal{P} \rangle$.

Proof. Certainly $\operatorname{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P}) \supset \mathrm{fz}\, \mathcal{P}$ because fz is a section of ung gr. Conversely, let $X \in \mathrm{D_b}(\mathsf{pfmod}(G))$ be such that $\mathrm{ung}\,\mathrm{gr}(X) \in \mathcal{P}$. We may assume without loss of generality that X is effective, and we proceed by induction on the filtration length of X. If X has filtration length 1, then $X = \mathrm{fz}\,\mathrm{ung}\,\mathrm{gr}(X) \in \langle \mathrm{fz}\, \mathcal{P} \rangle$ because $\mathrm{ung}\,\mathrm{gr}(X) \in \mathcal{P}$. To induct, we use the distinguished triangle (6.40) where $\mathrm{ung}\,\mathrm{gr}(X^{\leq w-1}) = \mathrm{ung}\,\mathrm{gr}(X)^{\leq w-1} \in \mathcal{P}$ and the fact that $\langle \mathrm{fz}\, \mathcal{P} \rangle$ is triangulated and closed under twists. $\square$

12 Filtered $\mathrm{fz}\,kG$-modules

In this section we study the derived category $\mathrm{D_b}(\mathsf{fmod}(G))$ of plain filtered kG-modules. In particular, we determine its spectrum in Proposition 12.11 by dividing and conquering (Method 6.10) using the object $\mathrm{cone}(\beta) \in \mathrm{D_b}(\mathsf{fmod}(G))$ and using a filtered version of π-points to determine specializations. We recall that G denotes a p-group (Notation 2.1).

We begin by determining $\mathrm{open}(\mathrm{cone}(\beta))$ and $\mathrm{supp}(\mathrm{cone}(\beta))$ in $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(G))$.

12.1 Proposition. *Unfiltering* $\mathrm{unf}\colon \mathsf{fmod}(G) \to \mathsf{mod}(G)$ *induces an equivalence of tt-categories*

$$\overline{\mathrm{unf}}\colon \frac{\mathrm{D_b}(\mathsf{fmod}(G))}{\langle \mathrm{cone}(\beta)\rangle} \longrightarrow \mathrm{D_b}(\mathsf{mod}(G)).$$

Geometrically, $\mathrm{Spc}(\overline{\mathrm{unf}})$ *induces a homeomorphism*

$$\mathrm{Spc}(\overline{\mathrm{unf}})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G)) \xrightarrow{\ \cong\ } \mathrm{open}(\mathrm{cone}(\beta)) \quad\subset\quad \mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(G)).$$

Proof. The statement of the proposition is almost identical to that of Proposition 10.9, and we can prove it in almost exactly the same way. An alternative proof will also drop out of the proof of Proposition 13.10 (see (13.11)). □

12.2 Proposition. *The tt-functor* $\mathrm{ung}\,\mathrm{gr}\colon \mathrm{D_b}(\mathsf{fmod}(G)) \to \mathrm{D_b}(\mathsf{mod}(G))$ *is faithful on the object* $\mathrm{cone}(\beta)$*. Geometrically,* $\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr})$ *induces a homeomorphism*

$$\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G)) \xrightarrow{\ \cong\ } \mathrm{supp}(\mathrm{cone}(\beta)) \subset \mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(G)).$$

Proof. Since $\mathrm{ung}\,\mathrm{gr}(\mathrm{cone}(\beta)) = \mathbb{1}[1] \oplus \mathbb{1}$ has full support and fz is a section of $\mathrm{ung}\,\mathrm{gr}$, the geometric statement follows from the algebraic statement (Method 6.10(i)). Let $f\colon \mathbb{1} \to X$ be a morphism in $\mathrm{D_b}(\mathsf{fmod}(G))$ such that $\mathrm{ung}\,\mathrm{gr}(f) = 0$ in $\mathrm{D_b}(\mathsf{mod}(G))$. Since $\mathrm{gr}(\mathbb{1})$ is concentrated in grading zero, this simply means $\mathrm{gr}(f)^0\colon \mathbb{1} \to \mathrm{gr}(X)^0$ is zero in $\mathrm{D_b}(\mathsf{mod}(G))$. Consider the following diagram in $\mathrm{D_b}(\mathsf{fmod}(G))$, where the bottom triangle is the distinguished triangle induced by the short exact sequence $A^{\geq 1} \rightarrowtail A^{\geq 0} \twoheadrightarrow \mathrm{gr}(A)^0$ in $\mathsf{fmod}(G)$

degree-wise:

$$\begin{tikzcd} & & \mathbb{1} \arrow[dll, dotted] \arrow[d, "f^0"] \arrow[dr, "\mathrm{fz}\,\mathrm{gr}(f)^0=0"] & & \\ X^{\geq 1} \arrow[rr] & & X^{\geq 0} \arrow[r] & \mathrm{fz}(\mathrm{gr}(X)^0) \arrow[r] & X^{\geq 1}[1]. \end{tikzcd}$$

Since $\mathrm{fz}\,\mathrm{gr}(f)^0 = 0$, we see that f^0 factors through $X^{\geq 1} \to X^{\geq 0}$. In other words, f factors through $\beta\colon X(-1) \to X$, which is equivalent (Lemma 11.2) to $f \otimes \mathrm{cone}(\beta) = 0$, as we wanted. □

12.3 Remark. If we assume Notation 11.1, then we can deduce the geometric part of Proposition 12.2 from the geometric part of Theorem 11.5.

So far, we have seen that the regions $\mathrm{open}(\mathrm{cone}(\beta))$ and $\mathrm{supp}(\mathrm{cone}(\beta))$ in the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(G))$ are both homeomorphic to $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G))$. Before discussing specializations, let us push our understanding of these points. First, we can use π-points to produce residue functors.

12.4 Lemma. *The points of* $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(G))$ *are precisely the kernels of the tt-functors*

$$\mathrm{D_b}(\mathsf{fmod}(G)) \xrightarrow{\mathrm{ung}\,\mathrm{gr}} \mathrm{D_b}(\mathsf{mod}(G)) \xrightarrow{\mathrm{Res}^G_1} \mathrm{D_b}(\mathsf{vect})$$
$$\mathrm{D_b}(\mathsf{fmod}(G)) \xrightarrow{\mathrm{unf}} \mathrm{D_b}(\mathsf{mod}(G)) \xrightarrow{\mathrm{Res}^G_1} \mathrm{D_b}(\mathsf{vect})$$

and of the triangulated functors

$$\mathrm{D_b}(\mathsf{fmod}(G)) \xrightarrow{\mathrm{ung}\,\mathrm{gr}} \mathrm{D_b}(\mathsf{mod}(G)) \twoheadrightarrow \mathrm{st}(\mathsf{mod}(G)) \xrightarrow{(\alpha,K)^*} \mathrm{st}(\mathsf{mod}_K(C_p))$$
$$\mathrm{D_b}(\mathsf{fmod}(G)) \xrightarrow{\mathrm{unf}} \mathrm{D_b}(\mathsf{mod}(G)) \twoheadrightarrow \mathrm{st}(\mathsf{mod}(G)) \xrightarrow{(\alpha,K)^*} \mathrm{st}(\mathsf{mod}_K(C_p))$$

over the π-points (α, K) up to equivalence.

Proof. This is Proposition 12.1, Proposition 12.2, and Recollection 6.19. □

In fact we can pair up these points using a filtered version of π-points, and this produces for each pair a specialization from the lower point to the upper point.

12.5 Lemma. *Let K/k be a field extension. The spectrum of* $\mathrm{st}(\mathsf{fmod}_K(C_p))$ *is*

$$\operatorname{Spc}\mathrm{st}(\mathsf{fmod}_K(C_p)) = \begin{array}{c} 0 \\ | \\ \langle \mathrm{cone}(\beta)\rangle. \end{array}$$

The closed point 0 *has residue field functor* $\mathrm{gr}\colon \mathrm{st}(\mathsf{fmod}_K(C_p)) \to \mathrm{st}(\mathsf{mod}_K(C_p))$, *and the non-closed point* $\langle \mathrm{cone}(\beta)\rangle$ *has residue field functor* $\mathrm{unf}\colon \mathrm{st}(\mathsf{fmod}_K(C_p)) \to \mathrm{st}(\mathsf{mod}_K(C_p))$.

Proof. Note that $\mathrm{st}(\mathsf{fmod}(C_p)) \cong \mathrm{D_b}(\mathsf{fmod}(C_p))/\langle \mathrm{fz}\, kC_p\rangle$ (Lemma 6.38 and Example 5.31), and of the four points in $\mathrm{D_b}(\mathsf{fmod}(C_p))$ described in Lemma 12.4, only two of them contain $\mathrm{fz}\, kC_p$. The residue field functors are clear from Lemma 12.4. □

12.6 Construction. Given a π-point (α, K) of G, we construct the triangulated functor

$$(\alpha,K)^*\colon\ \mathrm{st}(\mathsf{fmod}(G)) \xrightarrow{-\otimes_k K} \mathrm{st}(\mathsf{fmod}_K(G)) \xrightarrow{\mathrm{fz}\,\alpha^*} \mathrm{st}(\mathsf{fmod}_K(C_p)).$$

This is well-defined by Example 5.34. It is compatible with the triangulated functor

$$(\alpha,K)^*\colon\ \mathrm{st}(\mathsf{mod}(G)) \to \mathrm{st}(\mathsf{mod}_K(C_p))$$

from (6.20) in the sense that

$$\begin{array}{ccc} \mathrm{st}(\mathsf{fmod}(G)) & \xrightarrow{(\alpha,K)^*} & \mathrm{st}(\mathsf{fmod}_K(C_p)) \\ \big\downarrow{\scriptstyle \mathrm{ung\,gr}} & & \big\downarrow{\scriptstyle \mathrm{ung\,gr}} \\ \mathrm{st}(\mathsf{mod}(G)) & \xrightarrow{(\alpha,K)^*} & \mathrm{st}(\mathsf{mod}_K(C_p)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{st}(\mathsf{fmod}(G)) & \xrightarrow{(\alpha,K)^*} & \mathrm{st}(\mathsf{fmod}_K(C_p)) \\ \big\downarrow{\scriptstyle \mathrm{unf}} & & \big\downarrow{\scriptstyle \mathrm{unf}} \\ \mathrm{st}(\mathsf{mod}(G)) & \xrightarrow{(\alpha,K)^*} & \mathrm{st}(\mathsf{mod}_K(C_p)) \end{array}$$

commute. It follows from Lemma 12.5 that the triangulated functor $(\alpha, K)^*$ induces a continuous map on spectra, which we denote by

$$(\alpha,K)^{**}\colon\ \operatorname{Spc}\mathrm{st}(\mathsf{fmod}_K(C_p)) \longrightarrow \operatorname{Spc}\mathrm{st}(\mathsf{fmod}(G)).$$

12.7 Proposition. *The points of* $\operatorname{Spc}\mathrm{D_b}(\mathsf{fmod}(G))$ *are precisely the images of*

$$\operatorname{Spc}(\mathrm{Res}^G_1)\colon\ \mathrm{D_b}(\mathsf{fmod}(1)) \longrightarrow \mathrm{D_b}(\mathsf{fmod}(G))$$

and

$$(\alpha, K)^{**} \colon \operatorname{Spc} \operatorname{st}(\mathsf{fmod}_K(C_p)) \longrightarrow \operatorname{Spc} \operatorname{st}(\mathsf{fmod}(G)) \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{fmod}(G))$$

for the π-points (α, K) of G up to equivalence. Moreover, for every point $\mathcal{P}$ in the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(G))$, *the point* $\operatorname{Spc}(\mathrm{unf})(\mathcal{P})$ *in* $\operatorname{Spc} \mathrm{D_b}(\mathsf{fmod}(G))$ *specializes to the point* $\operatorname{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P})$ *in* $\operatorname{Spc} \mathrm{D_b}(\mathsf{fmod}(G))$.

Proof. Immediate from Lemma 12.4, Construction 12.6, and the fact that continuous maps preserve specializations. □

Conversely, these are the only specializations.

12.8 Terminology. The *Sierpiński space* is the following two-point topological space:

$$\begin{tikzcd} \bullet \arrow[d, dash, dotted] \\ \bullet \end{tikzcd}$$

A *doubling* of a topological space X is the product of X with the Sierpiński space.

12.9 Remark. In the following (Lemma 12.10), we will take $M = \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(G))$, $F = \operatorname{Spc} \mathrm{D_b}(\mathsf{fmod}(G))$, $U = \operatorname{open}(\operatorname{cone}(\beta))$, $g = \operatorname{Spc}(\mathrm{gr})$, $u = \operatorname{Spc}(\mathrm{unf})$, and $z = \operatorname{Spc}(\mathrm{fz})$. Note that then $F \setminus U = \operatorname{supp}(\operatorname{cone}(\beta))$.

12.10 Lemma. *Let M and F be topological spaces. Suppose there exists an open set $U \subset F$ and continuous maps $g, u \colon M \to F$ that induce homeomorphisms $g \colon M \to F \backslash U$ and $u \colon M \to U$, and suppose $u(m)$ specializes to $g(m)$ for every $m \in M$. If there exists a continuous map $z \colon F \to M$ such that g and u are both sections of z, then F is a doubling of M.*

Proof. Let $\{0, 1\}$ be the Sierpiński space, where 0 is the closed point. Then

$$\begin{aligned} \chi \colon F &\longrightarrow \{0,1\} \\ f &\longmapsto \begin{cases} 0 & f \notin U \\ 1 & f \in U \end{cases} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varphi \colon M \times \{0,1\} &\longrightarrow F \\ (m, 0) &\longmapsto g(m) \\ (m, 1) &\longmapsto u(m) \end{aligned}$$

are continuous. Indeed, χ is continuous because $\chi^{-1}(\{0\}) = F \setminus U$ is closed. To see that φ is continuous, note that by our assumptions on g and u, any closed subset of F is of the form $g(T) \sqcup u(S)$ for closed sets $S \subset T \subset M$. Thus

$$\varphi^{-1}(g(T) \sqcup u(S)) = (T \times \{0\}) \sqcup (S \times \{1\}),$$

which is closed in $M \times \{0,1\}$ because $S \subset T$ and 1 specializes to 0. Thus

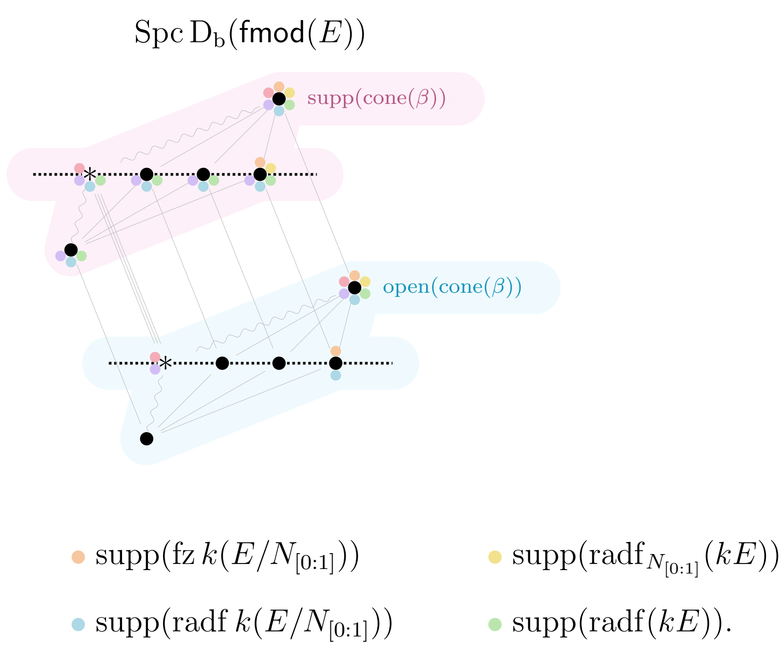
$$F \underset{\varphi}{\overset{z\times\chi}{\rightleftarrows}} M \times \{0,1\}$$

are inverse continuous maps. □

12.11 Proposition. *The spectrum of* $\mathrm{D_b}(\mathsf{fmod}(G))$ *is a doubling of* $\mathrm{D_b}(\mathsf{mod}(G))$.

Proof. Proposition 12.1 and Proposition 12.2 describe open(cone(β)) and supp(cone(β)) respectively, and Proposition 12.7 and Lemma 12.10 (see also Remark 12.9) determine the specializations from open(cone(β)) to supp(cone(β)). See Method 6.10 if necessary. □

12.12 Running Example. We return to our Running Example 1.13. The following illustrates the spectrum $\operatorname{Spc}\mathrm{D_b}(\mathsf{fmod}(E))$ as described by Proposition 12.11:



Recall that a $*$ point represents an arbitrary non-$\mathbb{F}_2$-rational point $[a:b]$ of $\mathbb{P}^1_k$. The tripled line between the $*$ points indicates a doubling between the non-$\mathbb{F}_2$-rational points; given such a point $[a:b]$ in the lower copy, the only such point to which it specializes in the higher copy is $[a:b]$. Note that $\mathrm{fz}\, C_{[c]}$ is an object in $\mathsf{fmod}(E)$ but not in $\mathsf{pfmod}(E)$.

13 The vanishing locus of $\mathsf{fmod}(G)$-acyclics

In this section, we study the $\mathsf{fmod}(G)$-acyclic objects in $\mathrm{D_b}(\mathsf{pfmod}(G))$. In particular, in Proposition 13.8 we use the theory of finite resolutions from Section 7 to show that the vanishing locus of $\mathsf{fmod}(G)$-acyclic objects is homeomorphic to the spectrum of $\mathrm{D_b}(\mathsf{fmod}(G))$. We will determine $\operatorname{Spc}\mathrm{D_b}(\mathsf{fmod}(G))$ in Section 12.

13.1 Notation. We recall that G denotes a p-group (Notation 2.1). In this section, we let Υ denote many things, such as the fully faithful exact tensor functors $\Upsilon\colon \mathsf{pfmod}(G) \rightarrowtail \mathsf{fmod}(G)$ and $\Upsilon\colon \mathsf{pmod}(G) \rightarrowtail \mathsf{mod}(G)$, as well as their induced tt-functors on derived categories. We depict them in the following commutative diagrams:

$$\begin{array}{ccc} \mathsf{pfmod}(G) & \xrightarrow{\text{ung gr}} & \mathsf{pmod}(G) \\ \downarrow \Upsilon & & \downarrow \Upsilon \\ \mathsf{fmod}(G) & \xrightarrow{\text{ung gr}} & \mathsf{mod}(G) \end{array} \qquad \begin{array}{ccc} \mathrm{D_b}(\mathsf{pfmod}(G)) & \xrightarrow{\text{ung gr}} & \mathrm{D_b}(\mathsf{pmod}(G)) \\ \downarrow \Upsilon & & \downarrow \Upsilon \\ \mathrm{D_b}(\mathsf{fmod}(G)) & \xrightarrow{\text{ung gr}} & \mathrm{D_b}(\mathsf{mod}(G)). \end{array} \tag{13.2}$$

Note that we have prematurely decorated $\Upsilon\colon \mathrm{D_b}(\mathsf{pfmod}(G)) \to \mathrm{D_b}(\mathsf{fmod}(G))$ as a Verdier quotient; this will be shown in Proposition 13.8.

13.3 Recollection. The $\mathsf{fmod}(G)$-acyclic objects (Terminology 7.2) are the objects in the kernel $\ker(\Upsilon) \subset \mathrm{D_b}(\mathsf{pfmod}(G))$. Since both of the tt-functors ung gr are conservative (Example 5.13 and Lemma 6.37), in fact this kernel can be written $\ker(\Upsilon) = \ker(\Upsilon\,\text{ung gr}) = \ker(\text{ung gr}\,\Upsilon)$.

We begin by introducing the Koszul object $\mathrm{kos}_G(1) \in \mathrm{D_b}(\mathsf{pmod}(G))$, which generates the $\mathsf{mod}(G)$-acyclics in $\mathrm{D_b}(\mathsf{pmod}(G))$.

13.4 Recollection. For each subgroup $H \leqslant G$, we have a Koszul object $\mathrm{kos}_G(H)$ in the derived category $\mathrm{D_b}(\mathsf{pmod}(G))$ [BG25a, Construction 3.15]. They are of the form

$$\begin{array}{c} \mathrm{kos}_G(H)\colon \quad 0 \to \bigwedge^d k(G/H) \to \cdots \to \bigwedge^1 k(G/H) \to \bigwedge^0 k(G/H) \to 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k(G/H) \xrightarrow[\varepsilon]{\quad\quad} \mathbb{1} \end{array} \tag{13.5}$$

where these exterior products are permutation by [BG23, Lemma 3.8]. Every Koszul object is $\mathsf{mod}(G)$-acyclic, but in fact $\mathrm{kos}_G(1)$ alone generates the tt-ideal of $\mathsf{mod}(G)$-acyclics [BG25a, Proposition 3.22]:

$$\ker(\Upsilon) = \langle \mathrm{kos}_G(1) \rangle \subset \mathrm{D_b}(\mathsf{pmod}(G)).$$

If $N \trianglelefteq G$ is normal, then $\mathrm{Infl}^G_{G/N} \mathrm{kos}_{G/N}(1) \cong \mathrm{kos}_G(N)$.

The obvious guess for a generator of $\ker(\Upsilon) \subset \mathrm{D_b}(\mathsf{pfmod}(G))$ works.

13.6 Lemma. *The Koszul object* $\mathrm{fz}\,\mathrm{kos}_G(1) \in \mathrm{D_b}(\mathsf{pfmod}(G))$ *in filtration zero generates the tt-ideal of* $\mathsf{fmod}(G)$*-acyclics:*

$$\ker(\Upsilon) = \langle \mathrm{fz}\,\mathrm{kos}_G(1) \rangle \subset \mathrm{D_b}(\mathsf{pfmod}(G)).$$

Proof. Certainly $\ker(\Upsilon) \supset \langle \mathrm{fz}\,\mathrm{kos}_G(1) \rangle$. Conversely, let $X \in \ker(\Upsilon)$, which is equivalent to having $\Upsilon\,\mathrm{ung}\,\mathrm{gr}(X) = 0$ in $\mathrm{D_b}(\mathsf{mod}(G))$ (Recollection 13.3). In order to show $X \in \langle \mathrm{fz}\,\mathrm{kos}_G(1) \rangle$, we proceed by induction on the filtration length of X, and we may assume without losing generality that X is effective. If X has filtration length 1, then this follows from the observation $X = \mathrm{fz}\,\mathrm{ung}\,\mathrm{gr}(X)$ and Recollection 13.4. To induct, we use the distinguished triangle (6.40) where $\Upsilon\,\mathrm{ung}\,\mathrm{gr}(X^{\leq w-1}) = \Upsilon\,\mathrm{ung}(\mathrm{gr}(X)^{\leq w-1}) = 0$ since $\Upsilon\,\mathrm{ung}\,\mathrm{gr}(X) = 0$, and we use the fact that $\langle \mathrm{fz}\,\mathrm{kos}_G(1) \rangle$ is triangulated and closed under twists. □

We now invoke Proposition 7.13.

13.7 Lemma. *For all* $m \geq 0$*, there exists an* m*-projective* $\mathsf{pfmod}(G)$*-resolution of the unit* $\mathbb{1} \in \mathsf{fmod}(G)$.

Proof. Let $m \geq 0$. Recall that there exists an m-projective $\mathsf{pmod}(G)$-resolution $\pi\colon \Upsilon(X) \to \mathbb{1}$ of the unit in $\mathrm{Ch_b}(\mathsf{mod}(G))$, where $X \in \mathrm{Ch_b}(\mathsf{pmod}(G))$ (Example 7.14). Then

$$\mathrm{fz}\,\pi\colon\ \mathrm{fz}\,\Upsilon(X) \to \mathbb{1}$$

is an m-projective $\mathsf{pfmod}(G)$-resolution of the unit in $\mathrm{Ch_b}(\mathsf{fmod}(G))$. It is m-projective because $\mathrm{fz}\,kG$ is projective in $\mathsf{fmod}(G)$ (Example 5.31), and it is a quasi-isomorphism in $\mathrm{Ch_b}(\mathsf{fmod}(G))$ because the exact structure on $\mathsf{fmod}(G)$ is pulled back along $\mathrm{ung}\,\mathrm{gr}$, of which fz is a section. $\square$

13.8 Proposition. *The tt-functor Υ induces an equivalence*

$$\overline{\Upsilon}\colon\ \frac{\mathrm{D_b}(\mathsf{pfmod}(G))}{\langle \mathrm{fz}\,\mathrm{kos}_G(1)\rangle} \longrightarrow \mathrm{D_b}(\mathsf{fmod}(G)) \tag{13.9}$$

of tt-categories. Geometrically, $\mathrm{Spc}(\overline{\Upsilon})$ *induces a homeomorphism*

$$\mathrm{Spc}(\overline{\Upsilon})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{fmod}(G)) \xrightarrow{\ \cong\ } \mathrm{open}(\mathrm{fz}\,\mathrm{kos}_G(1)) \quad \subset \quad \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G)).$$

Proof. The twists of the unit generate $\mathrm{D_b}(\mathsf{fmod}(G))$ as a triangulated subcategory (Example 6.43), and twisting the resolutions from Lemma 13.7 produces m-projective $\mathsf{fmod}(G)$-resolutions of the twists of the unit. Thus we may invoke Proposition 7.13. Finally, we identify $\ker(\Upsilon) = \langle \mathrm{fz}\,\mathrm{kos}_G(1)\rangle$ using Lemma 13.6. $\square$

We wrap up this section by showing that in the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$, the vanishing locus $\mathrm{open}(\mathrm{fz}\,\mathrm{kos}_G(1))$ of $\mathsf{fmod}(G)$-acyclics covers the vanishing locus $\mathrm{open}(\mathrm{cone}(\beta))$ embedded by unfiltering.

13.10 Proposition. *We have* $\mathrm{fz}\,\mathrm{kos}_G(1) \in \langle \mathrm{cone}(\beta)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(G))$. *Geometrically, in* $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(G))$ *we have* $\mathrm{open}(\mathrm{cone}(\beta)) \subset \mathrm{open}(\mathrm{fz}\,\mathrm{kos}_G(1))$.

Proof. This follows from chasing equivalences in the following diagram:

$$\begin{tikzcd}
\mathrm{D_b}(\mathsf{pfmod}(G)) \arrow[r, two heads] \arrow[d, two heads] & \dfrac{\mathrm{D_b}(\mathsf{pfmod}(G))}{\langle \mathrm{fz}\,\mathrm{kos}_G(1)\rangle} \arrow[r, "\overline{\Upsilon}", "\cong"'] \arrow[d, two heads] & \mathrm{D_b}(\mathsf{fmod}(G)) \arrow[d, two heads] \\
\dfrac{\mathrm{D_b}(\mathsf{pfmod}(G))}{\langle \mathrm{cone}(\beta)\rangle} \arrow[r, two heads, "\cong"'] \arrow[drr, bend right, "\cong", "\overline{\mathrm{unf}}"'] & \dfrac{\mathrm{D_b}(\mathsf{pfmod}(G))}{\langle \mathrm{cone}(\beta), \mathrm{fz}\,\mathrm{kos}_G(1)\rangle} \arrow[r, "\cong"'] \arrow[dr, bend right, "\cong"'] & \dfrac{\mathrm{D_b}(\mathsf{fmod}(G))}{\langle \mathrm{cone}(\beta)\rangle} \arrow[d, "\cong"] \\
& & \mathrm{D_b}(\mathsf{mod}(G)).
\end{tikzcd} \tag{13.11}$$

The top-right square arises from localizing both sides of the equivalence (13.9) at $\mathrm{cone}(\beta)$. The three tt-functors pointing to $\mathrm{D_b}(\mathsf{mod}(G))$ are all induced by unf. The left-most one is the equivalence (10.10); since $\mathrm{unf}(\mathrm{fz}\,\mathrm{kos}_G(1)) = 0$ in $\mathrm{D_b}(\mathsf{mod}(G))$, the middle one is also an equivalence, hence so is the one on the right. Thus, we deduce that the bottom tt-functor in the top-left square is an equivalence. □

To establish our second main result from Part II, we will need to know that there are no other specializations from $\mathrm{open}(\mathrm{cone}(\beta))$ to $\mathrm{supp}(\mathrm{fz}\,\mathrm{kos}_G(1) \otimes \mathrm{cone}(\beta))$.

13.12 Proposition. *Let* $\mathcal{P} \in \mathrm{supp}(\mathrm{kos}_G(1)) \subset \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G))$ *and* $\mathcal{Q} \in \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G))$. *Then* $\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr}\,\Upsilon)(\mathcal{Q})$ *specializes to* $\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P})$ *if and only if* $\mathrm{Spc}(\mathrm{unf}\,\Upsilon)(\mathcal{Q})$ *does.*

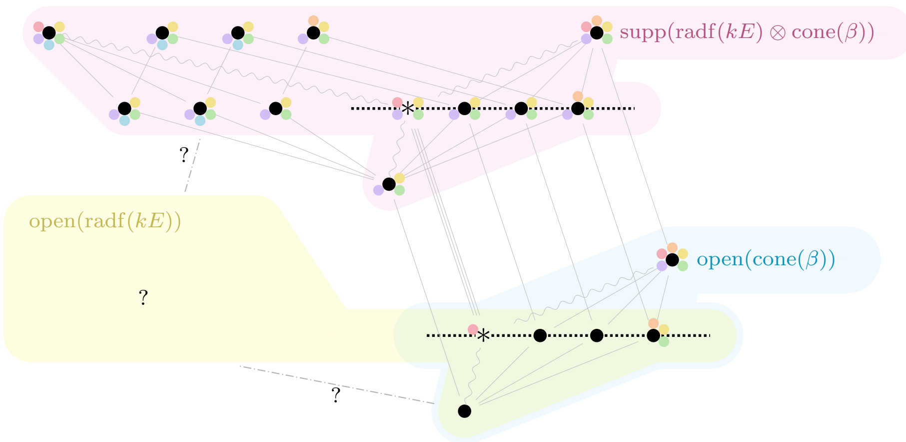
Proof. The forward implication is immediate from Proposition 12.11. Conversely, suppose $\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P}) \subset \mathrm{Spc}(\mathrm{ung}\,\mathrm{gr}\,\Upsilon)(\mathcal{Q})$. Recall (Proposition 11.16) that $\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P}) = \langle \mathrm{fz}\,\mathcal{P}\rangle$, so $\mathrm{fz}\,\mathcal{P} \subset \mathrm{Spc}(\mathrm{ung}\,\mathrm{gr}\,\Upsilon)(\mathcal{Q})$. Thus

$$\mathrm{unf}\,\Upsilon\,\mathrm{fz}(\mathcal{P}) = \Upsilon(\mathcal{P}) = \mathrm{ung}\,\mathrm{gr}\,\Upsilon\,\mathrm{fz}(\mathcal{P}) \subset \mathcal{Q},$$

or in other words $\mathrm{fz}\,\mathcal{P} \subset \mathrm{Spc}(\mathrm{unf}\,\Upsilon)(\mathcal{Q})$. Again using Proposition 11.16, this means that $\mathrm{Spc}(\mathrm{ung}\,\mathrm{gr})(\mathcal{P}) \subset \mathrm{Spc}(\mathrm{unf}\,\Upsilon)(\mathcal{Q})$. □

13.13 Running Example. We return to our Running Example 1.13. The following illustrates the information in $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E))$ that we obtain from Proposition 12.11, Proposition 13.8, and Proposition 13.12 together:

$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$



- supp(fz(resol. of $C_{[a:b]}$))
- supp(fz $k(E/N_{[0:1]})$)
- supp(radf $k(E/N_{[0:1]})$)
- supp(gap^1 radf(kE))
- supp(fz $\mathrm{kos}_E(N_{[0:1]})$)
- supp(gap^1 radf $k(E/N_{[0:1]})$).

Namely, we have a doubling downward from the cohomological open in $\operatorname{supp}(\operatorname{cone}(\beta) \otimes \operatorname{cone}(\beta))$ to the vanishing locus $\operatorname{open}(\operatorname{cone}(\beta))$, and there are no other specializations from the cyan region to the magenta region. Note that the progress we have made since Running Example 11.15 was finding specializations. In particular, we have not found any new points, so we still have the problem of distinguishing between, for example, $\mathrm{gap}^1 \operatorname{radf}(kE)$ and $\operatorname{radf}(kE)$.

Part III

The spectrum of the stable category

In this part we study the vanishing locus $\mathrm{open}(P) \subset \mathrm{Spc}\, \mathrm{D_b}(\mathsf{pfmod}(G))$ of the projectives, in the language of Notation 1.20. We will quickly specialize to the elementary abelian case.

14 The relative subcategories

In this section we introduce relative versions of our Frobenius exact rigid tensor categories (5.1). The main construction is summarized in Proposition 14.9. Importantly, we will see that the relative versions $\mathsf{pfmod}(G, -)$ of $\mathsf{pfmod}(G)$ form a poset of subcategories of $\mathsf{pfmod}(G)$ (Proposition 14.13). We recall that G denotes a p-group (Notation 2.1).

For intuition, recall the filtrations (1.14) of kE in our running example as well as the example computations (1.17) and (1.18) of associated graded modules. Roughly speaking, for a normal subgroup $N \trianglelefteqslant G$, the relative subcategory $\mathsf{pfmod}(G, N) \subset \mathsf{pfmod}(G)$ consists of the objects in $\mathsf{pfmod}(G)$ where the radical elements $X_n = n - 1$ for $n \in N$ act with filtration weight 1, and the associated graded functor $\mathrm{gr}\colon \mathsf{pfmod}(G, N) \to \mathsf{pgmod}(G, N)$ on this subcategory takes values in a category of graded modules where the elements $X_n = n-1$ for $n \in N$ act with grading 1.

14.1 Terminology. A *normal complement* (N, σ) *in* G is a normal subgroup $N \trianglelefteqslant G$ such that $G \simeq N \rtimes G/N$ along with a choice of section

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, tail] & G \arrow[r, two heads] & G/N \arrow[l, bend right, "\sigma"'] \arrow[r] & 1. \end{tikzcd} \tag{14.2}$$

14.3 Terminology. We denote by $\mathrm{nc}(G)$ the poset category of normal complements in G, constructed as follows. If (N, σ_N) and (L, σ_L) are normal complements in G where $N \leqslant L \leqslant G$, then a morphism $\sigma_L^N\colon (N, \sigma_N) \to (L, \sigma_L)$ is the data of a section $\sigma_L^N\colon G/L \to G/N$ of the quotient map $G/N \twoheadrightarrow G/L$ making the sections in the following diagram commute:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, tail] \arrow[d, hook] & G \arrow[r, two heads] \arrow[d, equal] & G/N \arrow[l, bend right, "\sigma_N"'] \arrow[r, two heads] \arrow[d, shift right] & 1 \\ 1 \arrow[r] & L \arrow[r, tail] & G \arrow[r, two heads] & G/L \arrow[l, bend right, "\sigma_L"'] \arrow[r] \arrow[u, hook, shift right, "\sigma_L^N"'] & 1. \end{tikzcd}$$

14.4 Remark. Let us mention two related notions. One is the standard notion, which we will not use at all, of a normal p-complement of a finite group G, which is a normal subgroup $N \trianglelefteqslant G$ of order coprime to p such that G/N is a p-group. The other is the poset category $\mathrm{cs}(G)$ of complementary subgroups in G whose objects are pairs (N, Q_N) of complementary subgroups in G and whose morphisms $(N, Q_N) \to (L, Q_L)$ consist of inclusions $N \leqslant L$ and $Q_L \leqslant Q_N$. We have forgetful functors

$$\begin{array}{rcl} \mathrm{nc}(G) & \longrightarrow & \mathrm{sub}(G) \\ (N,\sigma) & \longmapsto & N \end{array} \qquad \begin{array}{rcl} \mathrm{cs}(G) & \longrightarrow & \mathrm{sub}(G) \\ (N,Q_N) & \longmapsto & N \end{array} \qquad \begin{array}{rcl} \mathrm{cs}(G) & \longrightarrow & \mathrm{sub}(G) \\ (N,Q_N) & \longmapsto & Q_N \end{array}$$

to the category $\mathrm{sub}(G)$ of subgroups of G whose morphisms are the inclusions of subgroups. For finite abelian groups G the categories $\mathrm{nc}(G)$ and $\mathrm{cs}(G)$ are equivalent because then the conjugation action of G/N on N is trivial, but certainly $\mathrm{nc}(G)$ and $\mathrm{cs}(G)$ are not equivalent in general.

14.5 Remark. Given a normal subgroup $N \trianglelefteqslant G$, the following augmentation morphisms ε all admit sections, but the first two depend on a choice of a normal complement $(N, \sigma) \in \mathrm{nc}(G)$:

$$\begin{tikzcd} kG \arrow[d, shift right, two heads, "\varepsilon"'] & \mathrm{radf}_N(kG) \arrow[l, mapsto, "\mathrm{unf}"'] \arrow[r, mapsto, "\mathrm{gr}"] \arrow[d, shift right, two heads, "\varepsilon"'] & \mathrm{gr}\,\mathrm{radf}_N(kG) \arrow[r, mapsto, "\mathrm{ung}"] \arrow[d, shift right, two heads, "\varepsilon"'] & \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kG) \arrow[d, shift right, two heads, "\varepsilon"'] \\ k(G/N) \arrow[u, shift right, tail, "\sigma"'] & \mathrm{fz}\,k(G/N) \arrow[u, shift right, tail, "\sigma"'] & \mathrm{gz}\,k(G/N) \arrow[u, shift right, tail] & k(G/N) \arrow[u, shift right, tail] \end{tikzcd} \tag{14.6}$$

On the one hand, the section $\sigma\colon G/N \to G$ defines a non-canonical inclusion $\sigma\colon k(G/N) \to kG$ of Hopf monoids in vect that is a section of the augmentation morphism $\varepsilon\colon kG \to k(G/N)$. This in turn induces a non-canonical inclusion $\sigma\colon \mathrm{fz}\,k(G/N) \rightarrowtail \mathrm{radf}_N(kG)$ of Hopf monoids in fvect that is a section of the augmentation morphism $\varepsilon\colon \mathrm{radf}_N(kG) \twoheadrightarrow \mathrm{fz}\,k(G/N)$. On the other hand, since $\mathrm{gr}(\mathrm{radf}_N(kG))^0 = k(G/N)$, there is a canonical inclusion $\mathrm{gz}\,k(G/N) \rightarrowtail \mathrm{gr}\,\mathrm{radf}_N(kG)$ of Hopf monoids in gvect that is a section of the augmentation morphism $\varepsilon\colon \mathrm{gr}\,\mathrm{radf}_N(kG) \twoheadrightarrow \mathrm{gz}\,k(G/N)$. This canonical inclusion can be realized as $\mathrm{gr}(\sigma\colon \mathrm{fz}\,k(G/N) \rightarrowtail \mathrm{radf}_N(kG))$, independent of the choice of σ.

We now explain the constructions.

14.7 Recollection. For a normal subgroup $N \trianglelefteq G$, we denote by $\operatorname{radf}_N(kG)$ the filtration of kG by powers of the ideal $\operatorname{rad}_N(kG) = \operatorname{rad}(kN)kG$. This is a Hopf monoid in fvect (Recollection 3.46) because G is a p-group. So $\operatorname{gr}\operatorname{radf}_N(kG)$ is a Hopf monoid in gvect and $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kG)$ is a Hopf monoid in vect.

14.8 Definition. We define

$$\begin{aligned} \mathsf{fmod}(G,N) &= \operatorname{mod}_{\mathsf{fvect}}(\operatorname{radf}_N(kG)) \\ \mathsf{gmod}(G,N) &= \operatorname{mod}_{\mathsf{gvect}}(\operatorname{gr}\operatorname{radf}_N(kG)) \\ \mathsf{mod}(G,N) &= \operatorname{mod}_{\mathsf{vect}}(\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kG)) \end{aligned}$$

to be the exact (Lemma 5.12) rigid tensor categories of modules over these Hopf monoids (Recollection 14.7). For a normal complement $(N,\sigma) \in \operatorname{nc}(G)$ in G, we write

$$\begin{aligned} \operatorname{Res}^{\sigma}_{G/N} &\colon \mathsf{fmod}(G,N) \twoheadrightarrow \mathsf{fmod}(G/N) \\ \operatorname{Res}_{G/N} &\colon \mathsf{gmod}(G,N) \twoheadrightarrow \mathsf{gmod}(G/N) \\ \operatorname{Res}_{G/N} &\colon \mathsf{mod}(G,N) \twoheadrightarrow \mathsf{mod}(G/N) \end{aligned}$$

for the restrictions of scalars along the inclusions in (14.6); they are essentially surjective because the inclusions are sections (Remark 14.5).

Note that in the following proposition, the horizontal functors in the front face are essentially surjective whereas the ones in the back face are not necessarily. This is explained by Example 3.21(i) and Example 3.21(iii).

14.9 Proposition. *Let $(N,\sigma) \in \operatorname{nc}(G)$ be a normal complement in G. We have the following diagram of exact rigid tensor categories, where each one of the seven upright squares is a pullback of exact rigid tensor categories (Lemma 2.10):*

$$\begin{tikzcd}
\mathsf{pfmod}(G,N) \arrow[rr, "\operatorname{gr}"] \arrow[dr, two heads, "\operatorname{Res}^{\sigma}_{G/N}"'] \arrow[dd, tail] & & \mathsf{pgmod}(G,N) \arrow[rr, "\operatorname{ung}"] \arrow[dr, two heads, "\operatorname{Res}_{G/N}"'] \arrow[dd, tail] & & \mathsf{pmod}(G,N) \arrow[dr, two heads, "\operatorname{Res}_{G/N}"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(G/N) \arrow[rr, two heads, "\operatorname{gr}"] \arrow[dd, tail] & & \mathsf{pgmod}(G/N) \arrow[rr, two heads, "\operatorname{ung}"] \arrow[dd, tail] & & \mathsf{pmod}(G/N) \arrow[dd, tail] \\
\mathsf{fmod}(G,N) \arrow[rr, "\operatorname{gr}"] \arrow[dr, two heads, "\operatorname{Res}^{\sigma}_{G/N}"'] & & \mathsf{gmod}(G,N) \arrow[rr, "\operatorname{ung}"] \arrow[dr, two heads, "\operatorname{Res}_{G/N}"'] & & \mathsf{mod}(G,N) \arrow[dr, two heads, "\operatorname{Res}_{G/N}"'] & \\
& \mathsf{fmod}(G/N) \arrow[rr, two heads, "\operatorname{gr}"'] & & \mathsf{gmod}(G/N) \arrow[rr, two heads, "\operatorname{ung}"'] & & \mathsf{mod}(G/N).
\end{tikzcd} \tag{14.10}$$

Proof. The two squares in the front face

$$\begin{tikzcd}
\mathsf{pfmod}(G/N) \arrow[r, two heads, "\mathrm{gr}"] \arrow[d, tail] & \mathsf{pgmod}(G/N) \arrow[r, two heads, "\mathrm{ung}"] \arrow[d, tail] & \mathsf{pmod}(G/N) \arrow[d, tail] \\
\mathsf{fmod}(G/N) \arrow[r, two heads, "\mathrm{gr}"'] & \mathsf{gmod}(G/N) \arrow[r, two heads, "\mathrm{ung}"'] & \mathsf{mod}(G/N)
\end{tikzcd}$$

are given in (5.22). We recall that the exact structures along the bottom are pulled back and that both of these squares are pullbacks of exact rigid tensor categories (Definition 5.21). Thus the exact structures along the top are also pulled back (Proposition 5.19(ii)).

We construct the bottom face

$$\begin{tikzcd}
& \mathsf{pfmod}(G/N) \arrow[rr, two heads, "\mathrm{gr}"] \arrow[dd, tail] & & \mathsf{pgmod}(G/N) \arrow[rr, two heads, "\mathrm{ung}"] \arrow[dd, tail] & & \mathsf{pmod}(G/N) \arrow[dd, tail] \\
\mathsf{fmod}(G,N) \arrow[rr, "\mathrm{gr}"] \arrow[dr, two heads, "\mathrm{Res}^{\sigma}_{G/N}"'] & & \mathsf{gmod}(G,N) \arrow[rr, "\mathrm{ung}"] \arrow[dr, two heads, "\mathrm{Res}_{G/N}"'] & & \mathsf{mod}(G,N) \arrow[dr, two heads, "\mathrm{Res}_{G/N}"'] & \\
& \mathsf{fmod}(G/N) \arrow[rr, two heads, "\mathrm{gr}"'] & & \mathsf{gmod}(G/N) \arrow[rr, two heads, "\mathrm{ung}"'] & & \mathsf{mod}(G/N).
\end{tikzcd}$$

The restrictions were defined in Definition 14.8, and we may construct

$$\mathsf{fmod}(G,N) \xrightarrow{\mathrm{gr}} \mathsf{gmod}(G,N) \xrightarrow{\mathrm{ung}} \mathsf{mod}(G,N)$$

using Lemma 3.20 and the following diagram (Remark 14.5):

$$\begin{tikzcd}
\mathrm{radf}_N(kG) \arrow[rr, mapsto, "\mathrm{gr}"'] & & \mathrm{gr}\,\mathrm{radf}_N(kG) \arrow[rr, mapsto, "\mathrm{ung}"'] & & \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kG) & \\
& \mathrm{fz}(k(G/N)) \arrow[ul, tail, "\sigma"] & & \mathrm{gz}(k(G/N)) \arrow[ul, tail] & & k(G/N). \arrow[ul, tail]
\end{tikzcd}$$

Note that in the bottom face, all exact structures are all pulled back because the exact structures are pulled back from the ambient categories (Lemma 5.12); if necessary, recall the two-out-of-three property for pulled back exact structures (Lemma 5.6).

We now construct the exact rigid tensor categories $\mathsf{pfmod}(G,N)$, $\mathsf{pgmod}(G,N)$, and $\mathsf{pmod}(G,N)$ by using Proposition 5.19(i) to pull back along the restrictions $\mathrm{Res}^{\sigma}_{G/N}$ and

$\mathrm{Res}_{G/N}$:

$$\begin{tikzcd}
\mathsf{pfmod}(G,N) \arrow[rr, dotted, "\mathrm{gr}"] \arrow[dr, two heads, "\mathrm{Res}^{\sigma}_{G/N}"] \arrow[dd, tail] & & \mathsf{pgmod}(G,N) \arrow[rr, dotted, "\mathrm{ung}"] \arrow[dr, two heads, "\mathrm{Res}_{G/N}"] \arrow[dd, tail] & & \mathsf{pmod}(G,N) \arrow[dr, two heads, "\mathrm{Res}_{G/N}"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(G/N) \arrow[rr, two heads, "\mathrm{gr}"] \arrow[dd, tail] & & \mathsf{pgmod}(G/N) \arrow[rr, two heads, "\mathrm{ung}"] \arrow[dd, tail] & & \mathsf{pmod}(G/N) \arrow[dd, tail] \\
\mathsf{fmod}(G,N) \arrow[rr, "\mathrm{gr}"] \arrow[dr, two heads, "\mathrm{Res}^{\sigma}_{G/N}"'] & & \mathsf{gmod}(G,N) \arrow[rr, "\mathrm{ung}"] \arrow[dr, two heads, "\mathrm{Res}_{G/N}"'] & & \mathsf{mod}(G,N) \arrow[dr, two heads, "\mathrm{Res}_{G/N}"'] & \\
& \mathsf{fmod}(G/N) \arrow[rr, two heads, "\mathrm{gr}"'] & & \mathsf{gmod}(G/N) \arrow[rr, two heads, "\mathrm{ung}"'] & & \mathsf{mod}(G/N).
\end{tikzcd}$$

Because they are pullbacks, we obtain the exact tensor functors

$$\mathsf{pfmod}(G,N) \xrightarrow{\mathrm{gr}} \mathsf{pgmod}(G,N) \xrightarrow{\mathrm{ung}} \mathsf{pmod}(G,N) \tag{14.11}$$

making the diagram commute. We observe that as rigid tensor categories, each square in the back face is a pullback because the other three upright faces (in its cube) are pullbacks. Moreover, the exact structures are pulled back in (14.11) by applying two-out-of-three (Lemma 5.6) to the top faces. We conclude that the squares in the back face are therefore pullbacks as exact rigid tensor categories, using our characterization of pullbacks of exact categories (Proposition 5.19(*ii*)). □

The relative categories $\mathsf{pfmod}(G,N)$ form a poset of subcategories of $\mathsf{pfmod}(G)$, where $\mathsf{pfmod}(G,1) = \mathsf{pfmod}(G)$ recovers our original category.

14.12 Notation. A morphism of normal complements $\sigma^N_L\colon (N,\sigma_N) \to (L,\sigma_L)$ in $\mathrm{nc}(G)$ induces a morphism

$$\sigma^N_L\colon\ \mathrm{radf}_N(kG) \longrightarrow \mathrm{radf}_L(kG)$$

of Hopf monoids in fvect given by the identity on underlying monoids. We denote by

$$\mathrm{incl}^N_L = (\sigma^N_L)^*\colon\ \mathsf{fmod}(G,L) \longrightarrow \mathsf{fmod}(G,N)$$

the restriction of scalars along σ^N_L.

14.13 Proposition. *Let* $\sigma_L^N\colon (N,\sigma_N) \to (L,\sigma_L)$ *be a morphism of normal complements in* $\mathrm{nc}(G)$*. The exact rigid tensor category* $\mathsf{pfmod}(G,L)$ *is a fully exact rigid tensor subcategory of* $\mathsf{pfmod}(G,N)$*:*

$$\mathrm{incl}_L^N = (\sigma_L^N)^*\colon\ \mathsf{pfmod}(G,L) \rightarrowtail \mathsf{pfmod}(G,N). \tag{14.14}$$

The same is true for $\mathsf{fmod}(G,-) \subset \mathsf{fmod}(G)$*, and we have the following commutative diagram:*

$$\begin{tikzcd}
\mathsf{pfmod}(G,L) \arrow[dr, tail, "\mathrm{incl}_L^N"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(G,N) \arrow[dd, tail] \\
\mathsf{fmod}(G,L) \arrow[dr, tail, "\mathrm{incl}_L^N"'] & \\
& \mathsf{fmod}(G,N).
\end{tikzcd} \tag{14.15}$$

Proof. Note that σ_L^N is an epi in fvect since $\mathrm{unf}(\sigma_L^N) = 1$ is an epi (Remark 3.11), so incl_L^N is fully faithful (Lemma 3.22). Consider the following commutative diagram, where the front and back faces are pullbacks of exact rigid tensor categories (see (14.10)):

$$\begin{tikzcd}
\mathsf{pfmod}(G,L) \arrow[rr, "\mathrm{Res}_{G/L}\,\mathrm{ung}\,\mathrm{gr}"] \arrow[dr, tail, dotted, "\mathrm{incl}_L^N"'] \arrow[dd, tail] & & \mathsf{pmod}(G/L) \arrow[dr, tail, "\mathrm{Infl}_{G/L}^{G/N}"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(G,N) \arrow[rr, "\mathrm{Res}_{G/N}\,\mathrm{ung}\,\mathrm{gr}" near start] \arrow[dd, tail] & & \mathsf{pmod}(G/N) \arrow[dd, tail] \\
\mathsf{fmod}(G,L) \arrow[rr, crossing over] \arrow[dr, tail, "\mathrm{incl}_L^N"'] & & \mathsf{mod}(G/L) \arrow[dr, tail, "\mathrm{Infl}_{G/L}^{G/N}"] & \\
& \mathsf{fmod}(G,N) \arrow[rr, "\mathrm{Res}_{G/N}\,\mathrm{ung}\,\mathrm{gr}"'] & & \mathsf{mod}(G/N).
\end{tikzcd} \tag{14.16}$$

This shows that (14.14) is well-defined, and (14.15) is the left face. $\square$

14.17 Remark. The restrictions of scalars along the morphisms of monoids $\mathrm{gr}(\sigma_L^N)$ and $\mathrm{ung}\,\mathrm{gr}(\sigma_L^N)$ are not necessarily fully faithful because these morphisms are not epis in gvect

and vect in general, but we do obtain the following commutative diagram:

$$
\begin{tikzcd}
\mathsf{pfmod}(G,L) \arrow[rr, "\mathrm{gr}"] \arrow[dr, tail, "\mathrm{incl}_L^N"'] \arrow[dd, tail] & & \mathsf{pgmod}(G,L) \arrow[rr, "\mathrm{ung}"] \arrow[dr, "\mathrm{gr}(\sigma_L^N)^*"'] \arrow[dd, tail] & & \mathsf{pmod}(G,L) \arrow[dr, "\mathrm{ung}\,\mathrm{gr}(\sigma_L^N)^*"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(G,N) \arrow[rr, "\mathrm{gr}"] \arrow[dd, tail] & & \mathsf{pgmod}(G,N) \arrow[rr, "\mathrm{ung}"] \arrow[dd, tail] & & \mathsf{pmod}(G,N) \arrow[dd, tail] \\
\mathsf{fmod}(G,L) \arrow[rr, "\mathrm{gr}"] \arrow[dr, tail, "\mathrm{incl}_L^N"'] & & \mathsf{gmod}(G,L) \arrow[rr, "\mathrm{ung}"] \arrow[dr, "\mathrm{gr}(\sigma_L^N)^*"'] & & \mathsf{mod}(G,L) \arrow[dr, "\mathrm{ung}\,\mathrm{gr}(\sigma_L^N)^*"'] & \\
& \mathsf{fmod}(G,N) \arrow[rr, "\mathrm{gr}"'] & & \mathsf{gmod}(G,N) \arrow[rr, "\mathrm{ung}"'] & & \mathsf{mod}(G,N).
\end{tikzcd}
\tag{14.18}
$$

14.19 Remark. Let us offer some intuition about these relative subcategories $\mathsf{pfmod}(G,N)$. In the Hopf monoid $\mathrm{radf}_N(kG)$, for any $n \in N$, the element

$$X_n = n - 1 \in \mathrm{rad}_N(kG) = \mathrm{radf}_N(kG)^1$$

is in filtration weight 1. Thus, for a module $A \in \mathsf{pfmod}(G,N)$, if $a \in A^w$ for some $w \in \mathbb{Z}$, then $X_n a \in A^{w+1}$. Verbally, we say that X_n increases the filtration weight of a. If we view $A \in \mathsf{fmod}(G)$, then we can alternatively phrase this in terms of the associated graded by saying that N acts trivially on $\mathrm{ung}\,\mathrm{gr}(A) \in \mathsf{pmod}(G)$ or, in other words, that $\mathrm{ung}\,\mathrm{gr}(A)$ is inflated from $\mathsf{pmod}(G/N)$. This is what the diagram (14.16) expresses.

We wrap up this section by discussing generators and acyclics. We will use our distinguished triangles (6.40) and (6.41) that let us induct on filtration and grading length, respectively. They work for any effective monoid in fvect and gvect, hence in particular for our $\mathrm{radf}_N(kG)$ and $\mathrm{gr}\,\mathrm{radf}_N(kG)$.

14.20 Example. We have the following generators as triangulated subcategories:

$$
\begin{aligned}
\mathrm{D_b}(\mathsf{pfmod}(G,N)) &: \text{twists of } \ \mathrm{fz}\,k(G/H) \text{ for } N \leqslant H \leqslant G \\
\mathrm{D_b}(\mathsf{pgmod}(G,N)) &: \text{twists of } \ \mathrm{gz}\,k(G/H) \text{ for } N \leqslant H \leqslant G \\
\mathrm{D_b}(\mathsf{pmod}(G,N)) &: k(G/H) \text{ for } N \leqslant H \leqslant G \\
\mathrm{D_b}(\mathsf{fmod}(G,N)) &: \text{twists of the unit } \mathbb{1} \\
\mathrm{D_b}(\mathsf{gmod}(G,N)) &: \text{twists of the unit } \mathbb{1} \\
\mathrm{D_b}(\mathsf{mod}(G,N)) &: \text{the unit } \mathbb{1}.
\end{aligned}
$$

Indeed, this is analogous to Example 6.43.

14.21 Lemma.

(i) The tt-ideal of the $\mathsf{fmod}(G,N)$*-acyclics (Terminology 7.2) in* $\mathrm{D_b}(\mathsf{pfmod}(G,N))$ *is generated by the object* $\mathrm{fz}\,\mathrm{kos}_G(N) \in \mathrm{D_b}(\mathsf{pfmod}(G,N))$.

(ii) The tt-ideal of the $\mathsf{gmod}(G,N)$*-acyclics (Terminology 7.2) in* $\mathrm{D_b}(\mathsf{pgmod}(G,N))$ *is generated by the object* $\mathrm{gz}\,\mathrm{kos}_G(N) \in \mathrm{D_b}(\mathsf{pgmod}(G,N))$.

Proof. For (i), this is similar to Lemma 13.6, but let us spell it out. Let us denote

$$\Upsilon\colon\ \mathrm{D_b}(\mathsf{pmod}(G/N)) \to \mathrm{D_b}(\mathsf{mod}(G/N)).$$

The tt-ideal of $\mathsf{fmod}(G,N)$-acyclics contains $\langle \mathrm{fz}\,\mathrm{kos}_G(N)\rangle$ (Recollection 13.4). Conversely, suppose $X \in \mathrm{D_b}(\mathsf{pfmod}(G,N))$ is $\mathsf{fmod}(G,N)$-acyclic, which means $\Upsilon\,\mathrm{Res}_{G/N}\,\mathrm{ung}\,\mathrm{gr}(X) = 0$ in $\mathrm{D_b}(\mathsf{mod}(G/N))$ (Proposition 14.9). In order to show $X \in \langle \mathrm{fz}\,\mathrm{kos}_G(N)\rangle$, we proceed by induction on the filtration length of X, and we may assume without losing generality that X is effective. If X has filtration length 1, then this follows from the observation $X = \mathrm{fz}\,\mathrm{ung}\,\mathrm{gr}(X)$. To induct, we use the distinguished triangle (6.40) where

$$\Upsilon\,\mathrm{Res}_{G/N}\,\mathrm{ung}\,\mathrm{gr}(X^{\le w-1}) = \Upsilon\,\mathrm{Res}_{G/N}\,\mathrm{ung}(\mathrm{gr}(X)^{\le w-1}) = 0$$

since $\Upsilon\,\mathrm{Res}_{G/N}\,\mathrm{ung}\,\mathrm{gr}(X) = 0$, and we use the fact that $\langle \mathrm{fz}\,\mathrm{kos}_G(N)\rangle$ is triangulated and closed under twists.

For (ii), this is entirely analogous to (i). □

15 The elementary abelian case

In this section we specialize to the elementary abelian case $G = E$; we recall that E denotes an elementary abelian p-group (Notation 2.1). In particular, we choose complementary subgroups in a compatible way (Notation 15.2) so that our Hopf monoids decompose into tensor products of Hopf monoids (Lemma 15.6). This will allow us to show that all of our categories are Frobenius exact (Proposition 15.17).

We begin by choosing a complementary subgroup $Q_N \leqslant E$ for each subgroup $N \leqslant E$ in a compatible way. Recall that $\mathrm{nc}(E) \cong \mathrm{cs}(E)$ (Remark 14.4) and that we have fixed a minimal set of generators $z_1, \ldots, z_t \in E$ (Notation 6.21).

15.1 Lemma. *There exists a section*

$$\begin{tikzcd} \mathrm{nc}(E) \arrow[r, phantom, "\cong"] & \mathrm{cs}(E) \arrow[d, two heads, shift right] \\ & \mathrm{sub}(E) \arrow[u, hook, shift right, "{(-,Q_{(-)})}"'] \end{tikzcd}$$

of the forgetful functor $\mathrm{cs}(E) \twoheadrightarrow \mathrm{sub}(E)$ *such that* Q_N *is generated by a subset of* $z_1, \ldots, z_t$*.*

Proof. Our minimal generating set $z_1, \ldots, z_t$ of E gives rise to a complete flag

$$1 \leqslant E_1 \leqslant E_2 \leqslant \cdots \leqslant E_t = E$$

of E defined by $E_i = \langle z_1, \ldots, z_i \rangle$. Let us dictate what $Q_{(-)}$ does to objects $N \in \mathrm{sub}(E)$. Given a subgroup $N \leqslant E$, we construct the flag

$$1 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_t = N$$

of N defined by $N_i = N \cap E_i$ and also the flag

$$1 \leqslant Q_1 \leqslant Q_2 \leqslant \cdots \leqslant Q_t = Q_N$$

where $Q_i \leqslant E$ is the subgroup of E defined as follows. We set $Q_0 = 1$. If $N_{i-1} = N_i$ so that $z_i \notin N$, then $Q_i \leqslant E$ is the subgroup generated by Q_{i-1} and z_i, and otherwise $Q_i = Q_{i-1}$. Observe that $Q_i \cap N_i = 1$ for all i and $Q_i N_i = E_i$. Thus $Q_t N = E$, and we may put $Q_N = Q_t$. For the morphisms, we see that if $N \leqslant L$, then $Q_L \leqslant Q_N$. $\square$

15.2 Notation. Once and for all, we fix a choice of the data in Lemma 15.1. In particular, if we have $N \leqslant E$, then we let s denote the rank of N. We then have a complementary subgroup $Q_N \leqslant E$ of rank $r = t - s$ that is generated by a subset $x_1, \ldots, x_r$ of our fixed generators $z_1, \ldots, z_t$ (Notation 6.21). When $N \leqslant L \leqslant E$, the following diagram describes the compatibility morphisms, where the sections commute and where the retracts commute:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, tail, "i_N^E"'] \arrow[d, shift right, "i_N^L"'] & E \arrow[l, bend right, two heads, "r_N^E"'] \arrow[r, two heads, "q_1^N"'] \arrow[d, equal] & Q_N \arrow[r, two heads] \arrow[l, bend right, tail, "\sigma_1^N"'] \arrow[d, shift left, two heads, "q_N^L"'] & 1 \\
1 \arrow[r] & L \arrow[r, tail, "i_L^E"'] \arrow[u, shift right, bend right, two heads, "r_N^L"'] & E \arrow[l, bend right, two heads, "r_L^E"'] \arrow[r, two heads, "q_1^L"'] & Q_L \arrow[r] \arrow[l, bend right, tail, "\sigma_1^L"'] \arrow[u, bend right, tail, "\sigma_N^L"'] & 1.
\end{tikzcd} \tag{15.3}$$

15.4 Running Example. We return to our Running Example 1.13. Our choice of complementary subgroups, following the proof of Lemma 15.1, are

$$\begin{tikzcd}[column sep=small]
 & E \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
N_{[1:0]} = \langle x \rangle \arrow[dr, dash] & N_{[1:1]} = \langle xy \rangle \arrow[d, dash] & N_{[0:1]} = \langle y \rangle \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & Q_E = 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
Q_{[1:0]} = \langle y \rangle \arrow[dr, dash] & Q_{[1:1]} = \langle x \rangle \arrow[d, dash] & Q_{[0:1]} = \langle x \rangle \arrow[dl, dash] \\
 & Q_1 = E &
\end{tikzcd}$$

where we abbreviate $Q_{[1:0]} = Q_{N_{[1:0]}}$, $Q_{[0:1]} = Q_{N_{[0:1]}}$, and $Q_{[1:1]} = Q_{N_{[1:1]}}$.

We now study the ramifications this has on our categories.

15.5 Notation. For the rest of this section, we fix a subgroup $N \leqslant E$.

Each of our Hopf monoids $\mathrm{radf}_N(kE)$, $\mathrm{gr}\,\mathrm{radf}_N(kE)$, and $\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kE)$ (Recollection 14.7) decomposes into a Q_N-factor and an N-factor.

15.6 Lemma. *We have the following decompositions of Hopf monoids, which are induced* *by our sections σ_1^N and inclusions i_N^E in (15.3):*

$$\begin{aligned} \operatorname{radf}_N(kE) &\cong \operatorname{fz} kQ_N \otimes \operatorname{radf}(kN) && \textit{in } \mathsf{fvect} \\ \operatorname{gr}\operatorname{radf}_N(kE) &\cong \operatorname{gz} kQ_N \otimes \operatorname{gr}\operatorname{radf}(kN) && \textit{in } \mathsf{gvect} \\ \operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE) &\cong kQ_N \otimes \operatorname{ung}\operatorname{gr}\operatorname{radf}(kN) && \textit{in } \mathsf{vect}. \end{aligned} \tag{15.7}$$

Let $y_1, \ldots, y_s$ be a minimal set of generators of N. Then on the one hand

$$\operatorname{radf}_N(kE) \simeq (Y_j)\text{-}\textit{adic filtration of } k[X_i, Y_j]/(X_i^p, Y_j^p) \qquad \textit{in } \mathsf{fvect}, \tag{15.8}$$

where the comultiplication is given by

$$\begin{aligned} \Delta(X_i) &= X_i \otimes 1 + 1 \otimes X_i + X_i \otimes X_i \\ \Delta(Y_j) &= Y_j \otimes 1 + 1 \otimes Y_j + Y_j \otimes Y_j. \end{aligned} \tag{15.9}$$

And on the other hand

$$\begin{aligned} \operatorname{gr}\operatorname{radf}_N(kE) &\simeq k[X_i, Y_j]/(X_i^p, Y_j^p) \textit{ where } |X_i| = 0, |Y_j| = 1 && \textit{in } \mathsf{gvect} \\ \operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE) &\simeq k[X_i, Y_j]/(X_i^p, Y_j^p) && \textit{in } \mathsf{vect} \end{aligned} \tag{15.10}$$

where the comultiplications are given by

$$\begin{aligned} \Delta(X_i) &= X_i \otimes 1 + 1 \otimes X_i + X_i \otimes X_i \\ \Delta(Y_j) &= Y_j \otimes 1 + 1 \otimes Y_j. \end{aligned} \tag{15.11}$$

In particular, as monoids in fvect *we have (Notation 3.5)*

$$\operatorname{radf}_N(kE) \simeq \operatorname{trf}\operatorname{gr}\operatorname{radf}_N(kE). \tag{15.12}$$

Proof. Indeed, the isomorphism

$$kQ_N \otimes kN \xrightarrow{\sigma_1^N \otimes i_N^E} kE^{\otimes 2} \xrightarrow{\nabla} kE$$

in vect identifies the Hopf ideal $kQ_N \otimes \mathrm{rad}(kN) \subset kQ_N \otimes kN$ with $\mathrm{rad}_N(kE) \subset kE$. Thus it induces an isomorphism

$$\mathrm{fz}\, kQ_N \otimes \mathrm{radf}(kN) \xrightarrow{\sigma_1^N \otimes i_N^E} \mathrm{radf}_N(kE)^{\otimes 2} \xrightarrow{\nabla} \mathrm{radf}_N(kE)$$

in fvect, and applying gr and then ung produces two more isomorphisms. These are the decompositions in (15.7).

With the generators $y_1, \ldots, y_s \in N$, the decompositions (15.8) and (15.10) follow from Example 4.7. Let us discuss the comultiplications. The comultiplication $\Delta\colon \mathrm{radf}_N(kE) \to \mathrm{radf}_N(kE)^{\otimes 2}$ is induced by the comultiplication $\Delta\colon kE \to kE^{\otimes 2}$, which is given by (15.9). Importantly, the comultiplication on $\mathrm{gr}\,\mathrm{radf}_N(kE)$ is given by

$$\mathrm{gr}(\Delta)\colon\ \mathrm{gr}\,\mathrm{radf}_N(kE) \to \mathrm{gr}\,\mathrm{radf}_N(kE)^{\otimes 2}$$

and $\mathrm{gr}(\Delta)(Y_j)$ is the class of $\Delta(Y_j) = Y_j \otimes 1 + 1 \otimes Y_j + Y_j \otimes Y_j$ in $\mathrm{gr}\,\mathrm{radf}_N(kE)^1$ because Y_j has grading 1. But since $Y_j \in \mathrm{radf}_N(kE)^1$, the class of $Y_j \otimes Y_j$ in $\mathrm{gr}(\mathrm{radf}_N(kE))^1$ is zero, so

$$\mathrm{gr}(\Delta)(Y_j) = Y_j \otimes 1 + 1 \otimes Y_j \in \mathrm{gr}\,\mathrm{radf}_N(kE)^1.$$

This explains (15.11).

Finally, (15.12) is clear from (15.8) and (15.10). $\square$

15.13 Convention. We implicitly inflate objects from the Q_N-factor or from the N-factor. Let us elaborate. The morphisms

$$\mathrm{fz}\, kQ_N \xleftarrow{q_1^N} \mathrm{fz}\, kQ_N \otimes \mathrm{radf}(kN) \xrightarrow{r_N^E} \mathrm{radf}(kN)$$

of Hopf monoids in fvect induces the following inflations

$$\begin{array}{ccccc}
\mathsf{fmod}(Q_N, 1) & \xrightarrow{(q_1^N)^*} & \mathsf{fmod}(E, N) & \xleftarrow{(r_N^E)^*} & \mathsf{fmod}(N, N), \\
\mathrm{fz}\, kQ_N & \longmapsto & \mathrm{fz}\, kQ_N & & \\
 & & \mathrm{radf}(kN) & \longleftarrow & \mathrm{radf}(kN)
\end{array}$$

and we simply write $\mathrm{radf}(kN), \mathrm{fz}\, kQ_N \in \mathsf{fmod}(E,N)$ as depicted. We similarly simply write $\mathrm{gz}\, kQ_N, \mathrm{gr}\,\mathrm{radf}(kN) \in \mathsf{gmod}(E,N)$ and $kQ_N, \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN) \in \mathsf{mod}(E,N)$. Note that in fact $\mathrm{gz}\, kQ_N \in \mathsf{gmod}(E,N)$ and $kQ_N \in \mathsf{mod}(E,N)$ are already canonical (Remark 14.5).

15.14 Lemma. *The restrictions of scalars* Res_{Q_N} *along the various inclusions* σ_1^N *of the* Q_N*-factors in (15.7) induce the following adjoint pairs of functors:*

$$\begin{tikzcd}[column sep=small]
\mathsf{pfmod}(E,N) \arrow[dr, shift left, two heads, "\mathrm{Res}_{Q_N}"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(Q_N) \arrow[ul, shift left, "-\otimes \mathrm{radf}(kN)"] \arrow[dd, tail] \\
\mathsf{fmod}(E,N) \arrow[dr, shift left, two heads, "\mathrm{Res}_{Q_N}"] & \\
& \mathsf{fmod}(Q_N) \arrow[ul, shift left, "-\otimes \mathrm{radf}(kN)"]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathsf{pgmod}(E,N) \arrow[dr, shift left, two heads, "\mathrm{Res}_{Q_N}"] \arrow[dd, tail] & \\
& \mathsf{pgmod}(Q_N) \arrow[ul, shift left, "-\otimes \mathrm{gr}\,\mathrm{radf}(kN)"] \arrow[dd, tail] \\
\mathsf{gmod}(E,N) \arrow[dr, shift left, two heads, "\mathrm{Res}_{Q_N}"] & \\
& \mathsf{gmod}(Q_N) \arrow[ul, shift left, "-\otimes \mathrm{gr}\,\mathrm{radf}(kN)"]
\end{tikzcd}
\tag{15.15}$$

$$\begin{tikzcd}[column sep=small]
\mathsf{pmod}(E,N) \arrow[dr, shift left, two heads, "\mathrm{Res}_{Q_N}"] \arrow[dd, tail] & \\
& \mathsf{pmod}(Q_N) \arrow[ul, shift left, "-\otimes \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN)"] \arrow[dd, tail] \\
\mathsf{mod}(E,N) \arrow[dr, shift left, two heads, "\mathrm{Res}_{Q_N}"] & \\
& \mathsf{mod}(Q_N). \arrow[ul, shift left, "-\otimes \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN)"]
\end{tikzcd}$$

Proof. On the module categories $\mathsf{fmod}(E,N)$, $\mathsf{gmod}(E,N)$, and $\mathsf{mod}(E,N)$, the restrictions of scalars Res_{Q_N} are well-defined and right adjoint to the extensions of scalars $- \otimes_{\mathrm{fz}\, Q_N} \mathrm{radf}_N(kE)$, $- \otimes_{\mathrm{gz}\, Q_N} \mathrm{gr}\,\mathrm{radf}_N(kE)$, and $- \otimes_{Q_N} \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kE)$ because our monoids are commutative (see [Bra14, Remark 4.1.11]). Using our decompositions (15.7), we may write these extensions of scalars in the forms depicted in (15.15). Moreover, since the functors Res_{Q_N} are restricting scalars along the inclusions σ_1^N defined in (15.3) as group homomorphisms, they induce functors on the permutation categories $\mathsf{pfmod}(E,N)$, $\mathsf{pgmod}(E,N)$, and $\mathsf{pmod}(E,N)$, as depicted. It follows immediately that their right adjoints on the module categories restrict to right adjoints on the permutation subcategories. $\square$

We now state our setup for the elementary abelian case.

15.16 Notation. Having chosen a section $\sigma_1^N\colon E/N \to E$ in Notation 15.2, we now write $\mathrm{Res}_{Q_N}\colon \mathsf{pfmod}(E,N) \to \mathsf{pfmod}(Q_N)$.

15.17 Proposition. *Our diagram (14.10) simplifies to*

$$
\begin{tikzcd}
\mathsf{pfmod}(E,N) \arrow[rr, two heads, "\mathrm{gr}"] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] \arrow[dd, tail] & & \mathsf{pgmod}(E,N) \arrow[rr, "\mathrm{ung}"] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] \arrow[dd, tail] & & \mathsf{pmod}(E,N) \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] \arrow[dd, tail] & \\
& \mathsf{pfmod}(Q_N) \arrow[rr, two heads, crossing over, "\mathrm{gr}"] \arrow[dd, tail] & & \mathsf{pgmod}(Q_N) \arrow[rr, two heads, crossing over, "\mathrm{ung}"] \arrow[dd, tail] & & \mathsf{pmod}(Q_N) \arrow[dd, tail] \\
\mathsf{fmod}(E,N) \arrow[rr, two heads, "\mathrm{gr}", crossing over] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] & & \mathsf{gmod}(E,N) \arrow[rr, "\mathrm{ung}", crossing over] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] & & \mathsf{mod}(E,N) \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] & \\
& \mathsf{fmod}(Q_N) \arrow[rr, two heads, "\mathrm{gr}"'] & & \mathsf{gmod}(Q_N) \arrow[rr, two heads, "\mathrm{ung}"'] & & \mathsf{mod}(Q_N).
\end{tikzcd}
\tag{15.18}
$$

The exact rigid tensor categories in this diagram are all Frobenius exact. In fact, we have the following projective pre-covers of units (see Convention 15.13):

$\varepsilon\colon\ \mathrm{radf}(kE) \twoheadrightarrow \mathbb{1}$	*in* $\mathsf{pfmod}(E,N)$
$\varepsilon\colon\ \mathrm{gr}\,\mathrm{radf}(kN) \twoheadrightarrow \mathbb{1}$	*in* $\mathsf{pgmod}(E,N)$
$\varepsilon\colon\ \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN) \twoheadrightarrow \mathbb{1}$	*in* $\mathsf{pmod}(E,N)$
$\varepsilon\colon\ \mathrm{radf}_N(kE) \twoheadrightarrow \mathbb{1}$	*in* $\mathsf{fmod}(E,N)$
$\varepsilon\colon\ \mathrm{gr}\,\mathrm{radf}_N(kE) \twoheadrightarrow \mathbb{1}$	*in* $\mathsf{gmod}(E,N)$
$\varepsilon\colon\ \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kE) \twoheadrightarrow \mathbb{1}$	*in* $\mathsf{mod}(E,N)$.

Proof. Note that now the gr functors in the back face are essentially surjective due to (15.12) and Example 3.21(*ii*). To see that these categories are all Frobenius exact, by Proposition 5.29(*ii*) we only need to verify that the depicted pre-covers are indeed projective. For the module categories $\mathsf{fmod}(E,N)$, $\mathsf{gmod}(E,N)$, and $\mathsf{mod}(E,N)$, this is immediate by Example 5.31. For the permutation subcategories, we use the adjoint pairs of functors in (15.15), recalling that a functor with an exact (Proposition 14.9 and Terminology 2.2) right adjoint preserves projectives. In $\mathsf{pgmod}(Q_N)$ and $\mathsf{pmod}(Q_N)$ the units are projective, so $\mathrm{gr}\,\mathrm{radf}(kN)$ and $\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN)$ are projective in $\mathsf{pgmod}(E,N)$ and $\mathsf{pmod}(E,N)$ respectively. In $\mathsf{pfmod}(Q_N)$, the object $\mathrm{radf}(kQ_N)$ is projective by 9.13, so $\mathrm{radf}(kE) \cong \mathrm{radf}(kQ_N) \otimes \mathrm{radf}(kN)$ (Lemma 4.10) is projective in $\mathsf{pfmod}(E,N)$. $\square$

15.19 Remark. Alternatively, for $\mathsf{pfmod}(Q_N)$ we may simply use the inclusion

$$\mathrm{incl}^1_N \colon \mathsf{pfmod}(E, N) \rightarrowtail \mathsf{pfmod}(E)$$

from (14.14) to argue that $\mathrm{radf}(kE) \in \mathsf{pfmod}(E)$ is automatically projective.

15.20 Remark. The decompositions (15.7) permit us to define our categories in terms of the Q_N and N factors, without reference to E. We introduce the following alternative notation for (15.18):

$$\begin{tikzcd}
\mathsf{pfmod}[Q_N, N] \arrow[rr, two heads, "\mathrm{gr}"] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"] \arrow[dd, tail] & & \mathsf{pgmod}[Q_N, N] \arrow[rr, "\mathrm{ung}"] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"] \arrow[dd, tail] & & \mathsf{pmod}[Q_N, N] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"] \arrow[dd, tail] & \\
& \mathsf{pfmod}(Q_N) \arrow[rr, two heads, "\mathrm{gr}"] \arrow[dd, tail] & & \mathsf{pgmod}(Q_N) \arrow[rr, two heads, "\mathrm{ung}"] \arrow[dd, tail] & & \mathsf{pmod}(Q_N) \arrow[dd, tail] \\
\mathsf{fmod}[Q_N, N] \arrow[rr, two heads, "\mathrm{gr}"] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] & & \mathsf{gmod}[Q_N, N] \arrow[rr, "\mathrm{ung}"] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] & & \mathsf{mod}[Q_N, N] \arrow[dr, two heads, "\mathrm{Res}_{Q_N}"'] & \\
& \mathsf{fmod}(Q_N) \arrow[rr, two heads, "\mathrm{gr}"'] & & \mathsf{gmod}(Q_N) \arrow[rr, two heads, "\mathrm{ung}"'] & & \mathsf{mod}(Q_N).
\end{tikzcd}$$

15.21 Proposition. *We have the following equivalences of Frobenius exact rigid tensor categories:*

$$\begin{aligned}
\mathsf{fmod}[Q_N, N] &\cong \mathrm{mod}_{\mathsf{fmod}(Q_N)}(\mathrm{radf}(kN)) \\
\mathsf{gmod}[Q_N, N] &\cong \mathrm{mod}_{\mathsf{gmod}(Q_N)}(\mathrm{gr}\,\mathrm{radf}(kN)) \\
\mathsf{mod}[Q_N, N] &\cong \mathrm{mod}_{\mathsf{mod}(Q_N)}(\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN)) \\
\mathsf{pfmod}[Q_N, N] &\cong \mathrm{mod}_{\mathsf{pfmod}(Q_N)}(\mathrm{radf}(kN)) \\
\mathsf{pgmod}[Q_N, N] &\cong \mathrm{mod}_{\mathsf{pgmod}(Q_N)}(\mathrm{gr}\,\mathrm{radf}(kN)) \\
\mathsf{pmod}[Q_N, N] &\cong \mathrm{mod}_{\mathsf{pmod}(Q_N)}(\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}(kN)).
\end{aligned}$$

Proof. At the level of rigid tensor categories, the first three are immediate from Lemma 3.32 and our decompositions (15.7) of our Hopf monoids, and the last three follow from the first three. For the exact structures, note that on the left-hand side they are pulled back along the restrictions Res_{Q_N} and on the right-hand side they are pulled back along the forgetful functors, so they agree. $\square$

We wrap up this section with some concrete examples and discussions that we will use later. The following is a corollary of Proposition 15.17.

15.22 Corollary. *For each of our relative categories, its projective objects are precisely as follows:*

$$\begin{aligned}
\mathsf{pfmod}(E,N)&\colon \textit{sums of twists of } \operatorname{radf}_H(kQ_N)\otimes\operatorname{radf}(kN) \textit{ where } H\leqslant Q_N\\
\mathsf{pgmod}(E,N)&\colon \textit{sums of twists of } \operatorname{gz}k(Q_N/H)\otimes\operatorname{gr}\operatorname{radf}(kN) \textit{ where } H\leqslant Q_N\\
\mathsf{pmod}(E,N)&\colon \textit{sums of } k(Q_N/H)\otimes\operatorname{ung}\operatorname{gr}\operatorname{radf}(kN), \textit{ and possibly more}\\
\mathsf{fmod}(E,N)&\colon \textit{sums of twists of } \operatorname{radf}_N(kE)\\
\mathsf{gmod}(E,N)&\colon \textit{sums of twists of } \operatorname{gr}\operatorname{radf}_N(kE)\\
\mathsf{mod}(E,N)&\colon \textit{sums of } \operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE).
\end{aligned}$$

Consequently if $P\in\mathsf{pfmod}(E,N)$ is projective, then $\operatorname{gr}(P)\in\mathsf{pgmod}(E,N)$ is projective.

Proof. The proofs for $\mathsf{pfmod}(E,N)$ and $\mathsf{pgmod}(E,N)$ are very similar to the proof of Corollary 9.15 because we may use our short exact sequences (5.17) and (5.18). For $\mathsf{pmod}(E,N)$, we do not have an obvious analogous short exact sequence, so there may be more projective objects, as we have stated. The others are clear. □

15.23 Remark. The converse of the consequence of Corollary 15.22 is false when $N<E$. Indeed, if $N<E$, then $\operatorname{radf}(kN)\in\mathsf{pfmod}(E,N)$ is not projective because $kN=\operatorname{unf}\operatorname{radf}(kN)\in\mathsf{mod}(E)$ is not projective (Corollary 9.15), whereas the object $\operatorname{gr}\operatorname{radf}(kN)\in\mathsf{pgmod}(E,N)$ is. In fact, we claim without proof (we will not use it) that the converse is true when $Q_N=1$.

15.24 Example. Let $H,L,K\leqslant E$. Then $\operatorname{radf}_L k(E/H)\in\mathsf{pfmod}(E,K)$ if and only if $K\leqslant HL$. In particular, we always have $\operatorname{radf}k(E/H)\in\mathsf{pfmod}(E,K)$, whereas $\operatorname{fz}k(E/H)\in\mathsf{pfmod}(E,K)$ if and only if $K\leqslant H$.

15.25 Proposition. *Let $N \leqslant L \leqslant E$. The inclusion* $\mathrm{incl}_L^N \colon \mathsf{pfmod}(E, L) \rightarrowtail \mathsf{pfmod}(E, N)$ *induces a fully faithful tt-functor*

$$\mathrm{incl}_L^N \colon \mathrm{D_b}(\mathsf{pfmod}(E, L)) \rightarrowtail \mathrm{D_b}(\mathsf{pfmod}(E, N)).$$

Proof. This is Lemma 7.17, using Corollary 15.22. $\square$

15.26 Lemma. *Let $m \geq 0$. Recall Definition 7.9.*

(i) There is an m-projective $\mathsf{pfmod}(E, N)$*-resolution of the unit in* $\mathsf{fmod}(E, N)$*.*

(ii) There is an m-projective $\mathsf{pgmod}(E, N)$*-resolution of the unit in* $\mathsf{gmod}(E, N)$*.*

(iii) There is an m-projective $\mathsf{pmod}(E, N)$*-resolution of the unit in* $\mathsf{mod}(E, N)$*.*

Proof. For (i), in $\mathsf{fmod}(C_p, 1)$ we have the 1-projective $\mathsf{pfmod}(C_p, 1)$-resolution

$$\begin{tikzcd} R\colon \arrow[d] & \mathbb{1} \arrow[r] & \mathrm{fz}\, kC_p \arrow[r] & \mathrm{fz}\, kC_p \arrow[d] \\ \mathbb{1} & & & \mathbb{1} \end{tikzcd} \tag{15.27}$$

of the unit, so $R^{\otimes m}$ is an m-projective $\mathsf{pfmod}(C_p, 1)$-resolution of the unit. Similarly, in $\mathsf{fmod}(C_p, C_p)$, we have the 1-projective $\mathsf{pfmod}(C_p, C_p)$-resolution

$$\begin{tikzcd} S\colon \arrow[d] & \mathbb{1}(p) \arrow[r] & \mathrm{radf}(kC_p)(1) \arrow[r] & \mathrm{radf}(kC_p) \arrow[d] \\ \mathbb{1} & & & \mathbb{1} \end{tikzcd} \tag{15.28}$$

of the unit, so $S^{\otimes m}$ is an m-projective $\mathsf{pfmod}(C_p, C_p)$-resolution of the unit.

We now inflate these from the C_p factors of N and Q_N and tensor them together. Spelling it out, for each generator x_i in our minimal set of generators $x_1, \ldots, x_r \in Q_N$, let $R_i \to \mathbb{1}$ be the inflation in $\mathsf{pfmod}(E, N)$ of $R \to \mathbb{1}$ along the quotient $E \twoheadrightarrow \langle x_i \rangle \cong C_p$ (by the subgroup generated by N and x_j for $j \neq i$). Similarly, let $y_1, \ldots, y_s$ be a minimal set of generators of N. For each generator y_j, let $S_j \to \mathbb{1}$ be the inflation in $\mathsf{pfmod}(E, N)$ of $S \to \mathbb{1}$ along the quotient $E \twoheadrightarrow \langle y_j \rangle \cong C_p$ (by the subgroup generated by Q_N and y_i for $i \neq j$). Then

$$\bigotimes_{i=1}^{r} R_i^{\otimes m} \otimes \bigotimes_{j=1}^{s} S_j^{\otimes m} \to \mathbb{1}$$

is an m-projective $\mathsf{pfmod}(E,N)$-resolution of the unit in $\mathsf{fmod}(E,N)$ (Lemma 15.6 and Proposition 15.17). In particular, the projective terms are all sums of twists of $\operatorname{radf}_N(kE)$.

For (ii), apply $\operatorname{gr}\colon \mathsf{pfmod}(E,N)\to\mathsf{pgmod}(E,N)$ to the resolution in (i). Since the object $\operatorname{gr}\operatorname{radf}_N(kE)$ is projective in $\mathsf{gmod}(E,N)$ (Proposition 15.17), it becomes an m-projective $\mathsf{pgmod}(E,N)$-resolution of the unit in $\mathsf{gmod}(E,N)$.

For (iii), similarly apply $\operatorname{ung}\operatorname{gr}\colon \mathsf{pfmod}(E,N)\to\mathsf{pmod}(E,N)$ to the resolution in (i). Since $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$ is projective in $\mathsf{mod}(E,N)$ (Proposition 15.17), this produces an m-projective $\mathsf{pmod}(E,N)$-resolution of the unit in $\mathsf{mod}(E,N)$. $\square$

15.29 Remark. We have the following exact functors (compare with Remark 5.23):

$$\begin{tikzcd} & \mathsf{pfmod}(E,N) \arrow[dl, two heads, "\operatorname{unf}"'] \arrow[r, two heads, shift left, "\operatorname{gr}"] \arrow[d, tail] & \mathsf{pgmod}(E,N) \arrow[l, shift left, "\operatorname{trf}"] \arrow[r, "\operatorname{ung}"] \arrow[d, tail] & \mathsf{pmod}(E,N) \arrow[d, tail] \\ \mathsf{mod}(E) & \mathsf{fmod}(E,N) \arrow[l, two heads, "\operatorname{unf}"] \arrow[r, two heads, shift left, "\operatorname{gr}"] & \mathsf{gmod}(E,N) \arrow[l, shift left, "\operatorname{trf}"] \arrow[r, "\operatorname{ung}"] & \mathsf{mod}(E,N). \end{tikzcd}$$

They are all exact tensor except for the trf functors. The unf functors are tensor because the comultiplication on $\operatorname{radf}_N(kE)$ is induced by the comultiplication on kE, and they are essentially surjective because as usual $\operatorname{radf}(C)\in\mathsf{pfmod}(E,N)$ and $\operatorname{unf}\operatorname{radf}(C)=C$ for any $C\in\mathsf{mod}(E)$.

15.30 Example. Let us compare our relative categories to our original categories by discussing some specific objects and morphisms. Consider the following diagram extracted from (14.18):

$$\begin{tikzcd} \mathsf{pfmod}[Q_N,N] \arrow[r, "\operatorname{gr}"] \arrow[d, tail, "\operatorname{incl}^1_N"] & \mathsf{pgmod}[Q_N,N] \arrow[d, "\operatorname{gr}(\sigma^N_1)^*"] \\ \mathsf{pfmod}(E) \arrow[r, "\operatorname{gr}"] & \mathsf{pgmod}(E). \end{tikzcd}$$

(i) Consider the object $\operatorname{radf}(kN)\in\mathsf{pfmod}[Q_N,N]$ in our relative subcategory. The object $\operatorname{gr}\operatorname{radf}(kN)\in\mathsf{pgmod}[Q_N,N]$ is indecomposable and generated as a module over $\operatorname{gz} kQ_N\otimes\operatorname{gr}\operatorname{radf}(kN)$ by the element $1\in\operatorname{gr}\operatorname{radf}(kN)^0$ since Y_j has grading one in $\operatorname{gr}\operatorname{radf}_N(kE)$. Up to twists on both sides and scalar multiples, the only nontrivial morphisms in $\mathsf{pgmod}[Q_N,N]$ that $\operatorname{gr}\operatorname{radf}(kN)$ admits to and from the unit $\mathbb{1}$ are respectively (recall Notation 9.16)

$$\varepsilon\colon\ \operatorname{gr}\operatorname{radf}(kN)\twoheadrightarrow\mathbb{1}\qquad\text{and}\qquad\eta\colon\mathbb{1}\rightarrowtail\operatorname{gr}\operatorname{radf}(kN)(-\ell_N+1).$$

(ii) Consider the object $\mathrm{incl}^1_N \operatorname{radf}(kN) \in \mathsf{pfmod}(E)$ in our original category, which we simply denote by $\operatorname{radf}(kN) \in \mathsf{pfmod}(E)$. The object $\operatorname{gr}\operatorname{radf}(kN) \in \mathsf{pgmod}(E)$ is a direct sum of twists of the unit since Y_j has grading zero in $\operatorname{gz} kE$. Thus, even up to twists on both sides and scalar multiples, there are many nontrivial morphisms in $\mathsf{pgmod}(E)$ that $\operatorname{gr}\operatorname{radf}(kN)$ admits to and from the unit $\mathbb{1}$. We denote by

$$\pi\colon\ \operatorname{gr}\operatorname{radf}(kN) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{1} \qquad \text{and} \qquad \iota\colon \mathbb{1} \rightarrowtail\longrightarrow \operatorname{gr}\operatorname{radf}(kN)(-\ell_N+1).$$

the extreme ones. These are respectively

$$\operatorname{gr}(\varepsilon\colon\ \operatorname{radf}(kN) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{1}) \qquad \text{and} \qquad \operatorname{gr}(\eta\colon \mathbb{1} \rightarrowtail\longrightarrow \operatorname{radf}(kN)(-\ell_N+1)).$$

(iii) Let $H \leqslant Q_N$, and let us further decompose $kQ_N \simeq k(Q_N/H) \otimes kH$ so that $kE \simeq k(Q_N/H) \otimes kH \otimes kN$ (see Lemma 15.6). The objects $\operatorname{fz} k(Q_N/H)$ and $\operatorname{radf} k(Q_N/H)$ behave the same way in both categories. Indeed, in both categories $\mathsf{pgmod}[Q_N,N]$ and $\mathsf{pgmod}(E)$, the object $\operatorname{gz} k(Q_N/H)$ is indecomposable and generated as a module over the monoid $\operatorname{gz} kQ_N \otimes \operatorname{gr}\operatorname{radf}(kN)$ by the element $1 \in \operatorname{gz} k(Q_N/H)^0$ and the object $\operatorname{gr}\operatorname{radf} k(Q_N/H)$ is a sum of twists of the unit. We denote by

$$\pi\colon\ \operatorname{gr}\operatorname{radf} k(Q_N/H) \twoheadrightarrow \mathbb{1} \qquad \text{and} \qquad \iota\colon \mathbb{1} \rightarrowtail \operatorname{gr}\operatorname{radf} k(Q_N/H)(-\ell_{Q_N/H}+1).$$

the extreme morphisms. These are respectively

$$\operatorname{gr}(\varepsilon\colon\ \operatorname{radf} k(Q_N/H) \twoheadrightarrow \mathbb{1}) \qquad \text{and} \qquad \operatorname{gr}(\eta\colon \mathbb{1} \rightarrowtail \operatorname{radf} k(Q_N/H)(-\ell_{Q_N/H}+1)).$$

(iv) Let $H \leqslant Q_N$, and consider the object $\operatorname{fz} k(Q_N/H) \otimes \operatorname{radf}(kN) \in \mathsf{pfmod}[Q_N,N]$. Combining the arguments in (i) and (iii), we deduce that

$$\operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kN) \in \mathsf{pgmod}[Q_N,N]$$

is indecomposable and generated as a module over $\operatorname{gz} kQ_N \otimes \operatorname{gr}\operatorname{radf}(kN)$ by the element $1 \otimes 1 \in (\operatorname{gz} k(Q_N/H) \otimes \operatorname{radf}(kN))^0$ and that, up to twists on both sides and scalar multiples, the only nontrivial morphisms in $\mathsf{pgmod}[Q_N, N]$ that it admits to and from the unit $\mathbb{1}$ are respectively

$$\varepsilon\colon\ \operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kN) \twoheadrightarrow \mathbb{1}$$

and

$$\eta\colon \mathbb{1} \rightarrowtail \operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kN)(-\ell_N + 1). \tag{15.31}$$

16 π-points for $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$

16.1 Notation. For this section, we fix a subgroup $N \leqslant E$ of our elementary abelian p-group E (Notation 2.1).

In this short section we describe the spectrum of $\mathrm{D_b}(\mathsf{mod}(E, N))$. In particular, with a careful choice of generators, we show in Proposition 16.5 that it is homeomorphic to $\mathbb{P}_k^{r+s-1}$ and that we can describe the points using π-points.

Recall our description from Section 6 of π-points in the elementary abelian case, which we will now extend.

16.2 Notation. For this section, we fix a minimal set of generators $y_1, \ldots, y_s$ of N. Recall (Notation 15.2 and Lemma 15.6) that we then have a complementary subgroup $Q_N \leqslant E$, a minimal set of generators $x_1, \ldots, x_r$ of Q_N, an isomorphism of monoids

$$\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE) \simeq \frac{k[X_1, \ldots, X_r, Y_1, \ldots, Y_s]}{(X_1^p, \ldots, X_r^p, Y_1^p, \ldots, Y_s^p)} \tag{16.3}$$

in vect, and an isomorphism of monoids

$$\operatorname{gr}\operatorname{radf}_N(kE) \simeq \frac{k[X_1, \ldots, X_r, Y_1, \ldots, Y_s]}{(X_1^p, \ldots, X_r^p, Y_1^p, \ldots, Y_s^p)} \text{ where } |X_i| = 0 \text{ and } |Y_j| = 1 \tag{16.4}$$

in gvect.

We denote by $[a : b]_{K/k}$ an arbitrary point of $\mathbb{P}_k^{r+s-1}$, where K/k is a field extension and

$(a,b) \in K^{r+s} \setminus \{0\}$, so that the following diagram commutes (Recollection 6.26):

$$\begin{tikzcd}[row sep=small]
{[a:b]} \arrow[r, mapsto] & {[a:b]_{K/k}} \\
\mathbb{P}_K^{r+s-1} \arrow[r] \arrow[d] & \mathbb{P}_k^{r+s-1} \arrow[d] \\
\operatorname{Spec}(K) \arrow[r] & \operatorname{Spec}(k).
\end{tikzcd}$$

Given such a point $[a:b]_{K/k}$, we will often do the following things:

(*i*) We may write $a \in K^r$ and $b \in K^s$, in which case we are implicitly choosing a representative $(a,b) \in K^{r+s} \setminus \{0\}$ of $[a:b]$ and asserting that the choice of representative does not matter in the context of the discussion.

(*ii*) We may split the point $[a:b] \in \mathbb{P}_K^{r+s-1}$ into points $[a] \in \dot{\mathbb{P}}_K^{r-1}$ and $[b] \in \dot{\mathbb{P}}_K^{s-1}$ of the one-point compactifications (Notation 6.15).

(*iii*) We may abbreviate $aX = a_1X_1 + \cdots + a_rX_r$ and $bY = b_1Y_1 + \cdots + b_sY_s$, as elements in either $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$ or $\operatorname{gr}\operatorname{radf}_N(kE)$.

16.5 Proposition. *We have the following commutative diagram of spaces:*

$$\begin{tikzcd}
\dot{\mathbb{P}}_k^{r+s-1} \arrow[r, "\cong"] & \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E,N)) \\
\mathbb{P}_k^{r+s-1} \arrow[r, "\mathcal{P}", "\cong"'] \arrow[u, hook] & \operatorname{Spc} \operatorname{st}(\mathsf{mod}(E,N)). \arrow[u, hook]
\end{tikzcd}$$

Given $[a:b]_{K/k} \in \mathbb{P}_k^{r+s-1}$, the prime tt-ideal $\mathcal{P}([a:b]_{K/k}) \subset \operatorname{st}(\mathsf{mod}(E,N))$ consists of the objects $C \in \mathsf{mod}(E,N)$ such that the K-linear map on C_K given by multiplication by $aX+bY$ has rank precisely

$$\operatorname{rank}_K(aX + bY \colon C_K \to C_K) = \frac{p-1}{p}\dim_K(C_K). \tag{16.6}$$

Proof. In (16.3) we may further identify (Example 4.6)

$$\frac{k[X_1,\ldots,X_r,Y_1,\ldots,Y_s]}{(X_1^p,\ldots,X_r^p,Y_1^p,\ldots,Y_s^p)} \simeq kE.$$

Let $h\colon \operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE) \to kE$ denote the composite isomorphism. Then the restriction of scalars $h^*\colon \mathrm{D_b}(\mathsf{mod}(E)) \to \mathrm{D_b}(\mathsf{mod}(E,N))$ is an equivalence of triangulated categories that identifies the units in these tt-categories. Since both tt-categories are rigid and generated by their units, h^* induces a homeomorphism on spectra (Lemma 6.14). The description in terms of $\mathcal{P}$ is a matter of chasing these identifications and recalling the concrete description of the prime tt-ideals given in Recollection 6.23. $\square$

16.7 Remark. Let us clarify the role of the comultiplication on $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$ in vect. We described it explicitly in (15.9), and it determines the tensor structure in $\mathsf{mod}(E,N)$ (Recollection 3.30). However, the proof of Proposition 16.5 argues that the spectrum of $\mathrm{st}(\mathsf{mod}(E,N))$ only depends on the monoid structure of $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$, so the statement of Proposition 16.5 hides the comonoid structure on $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$ once and for all. Even though we are hiding it, we would like to remark that the Hopf monoid $\operatorname{ung}\operatorname{gr}\operatorname{radf}_N(kE)$ is not completely artificial. It is isomorphic as a Hopf algebra to the group algebra of the finite group scheme

$$Q_N \times (\mathbb{G}_{\mathrm{a}(1)})^{\times s},$$

where we recall that Q_N is elementary abelian and where $\mathbb{G}_{\mathrm{a}(n)}$ denotes the nth Frobenius kernel of the additive group $\mathbb{G}_{\mathrm{a}}$ over k. This is quite similar to a quasi-elementary group scheme, which is one of the form

$$Q \times \mathbb{G}_{\mathrm{a}(s)}$$

for Q elementary abelian and which play an important role in the theory of π-points for general finite group schemes, for example in [FP07].

17 Graded $\operatorname{gr}\operatorname{radf}_N(kE)$-modules

17.1 Notation. For this section, we fix a subgroup $N \leqslant E$ of our elementary abelian p-group E (Notation 2.1), and we adopt Notation 16.2.

In this section we study the spectrum of $\mathrm{D_b}(\mathsf{gmod}(E,N))$. In particular, we summarize in Corollary 17.16 that ungrading induces a surjection

$$\dot{\mathbb{P}}_k^{r+s-1} \xrightarrow[\cong]{\mathcal{P}} \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E,N)) \xrightarrow{\operatorname{Spc}(\mathrm{ung})} \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E,N))$$

and that we can describe the behavior of this surjection on closed points when k is algebraically closed.

To study $\operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E,N))$, we divide and conquer (Method 6.10) using $\operatorname{gr}\operatorname{radf}_N(kE)$. In studying the support, we deduce that $\mathrm{D_b}(\mathsf{gmod}(E,N))$ is local, which means its spectrum has a unique closed point.

17.2 Lemma. *The tt-category* $\mathrm{D_b}(\mathsf{gmod}(E,N))$ *is local. The unique closed point* 0 *is the only point in the support of* $\operatorname{gr}\operatorname{radf}_N(kE)$*, and it is the kernel of*

$$\operatorname{Res}_1\colon\ \mathrm{D_b}(\mathsf{gmod}(E,N)) \longrightarrow \mathrm{D_b}(\mathsf{gvect}). \tag{17.3}$$

Proof. The tt-category $\mathrm{D_b}(\mathsf{gmod}(E,N))$ is indeed local (Recollection 6.8(*ii*)) because Res_1 (in (17.3)) is conservative (Lemma 6.37) and $\operatorname{Spc} \mathrm{D_b}(\mathsf{gvect}) = \{0\}$. To see that there are no other points in the support of $\operatorname{gr}\operatorname{radf}_N(kE)$, we will show that any nonzero tt-ideal in $\mathrm{D_b}(\mathsf{gmod}(E,N))$ contains $\operatorname{gr}\operatorname{radf}_N(kE)$. Observe first that for $B \in \mathsf{gmod}(E,N)$, we have (Proposition 15.17 and Corollary 15.22)

$$\operatorname{gr}\operatorname{radf}_N(kE) \otimes B \cong \operatorname{gr}\operatorname{radf}_N(kE) \otimes_k \operatorname{Res}_1(B),$$

where $\operatorname{gr}\operatorname{radf}_N(kE) \otimes_k -\colon \mathsf{gvect} \to \mathsf{gmod}(E,N)$ is extension of scalars (see [Bra14, Remark 4.1.11]). Thus we have a commutative diagram

$$\begin{tikzcd} \mathsf{gmod}(E,N) \arrow[rr, "\operatorname{gr}\operatorname{radf}_N(kE)\otimes -"] \arrow[dr, "\operatorname{Res}_1"'] & & \mathsf{gmod}(E,N) \\ & \mathsf{gvect} \arrow[ur, "\operatorname{gr}\operatorname{radf}_N(kE)\otimes_k -"'] & \end{tikzcd}$$

of exact categories and hence a commutative diagram

$$\begin{tikzcd} \mathrm{D_b}(\mathsf{gmod}(E,N)) \arrow[rr, "\operatorname{gr}\operatorname{radf}_N(kE)\otimes -"] \arrow[dr, "\operatorname{Res}_1"'] & & \mathrm{D_b}(\mathsf{gmod}(E,N)) \\ & \mathrm{D_b}(\mathsf{gvect}) \arrow[ur, "\operatorname{gr}\operatorname{radf}_N(kE)\otimes_k -"'] & \end{tikzcd}$$

of triangulated categories. Now, suppose $Y \in \mathrm{Ch}_{\mathrm{b}}(\mathsf{gmod}(E,N))$ is a nonzero object in $\mathrm{D}_{\mathrm{b}}(\mathsf{gmod}(E,N))$; since we have seen that Res_1 (in (17.3)) is conservative, this is equivalent to saying $\mathrm{Res}_1(Y)$ is isomorphic to a nontrivial sum of shifts of twists of the unit in $\mathrm{D}_{\mathrm{b}}(\mathsf{gvect})$. Thus

$$\mathrm{gr}\,\mathrm{radf}_N(kE) \otimes Y \cong \mathrm{gr}\,\mathrm{radf}_N(kE) \otimes_k \mathrm{Res}_1(Y)$$

in $\mathrm{D}_{\mathrm{b}}(\mathsf{gmod}(E,N))$ is isomorphic to a sum of shifts of twists of the object $\mathrm{gr}\,\mathrm{radf}_N(kE)$. We conclude that $\mathrm{gr}\,\mathrm{radf}_N(kE) \in \langle Y \rangle$. □

Thus by Proposition 15.17, the vanishing locus of $\mathrm{gr}\,\mathrm{radf}_N(kE)$ is the spectrum of the stable category $\mathrm{st}(\mathsf{gmod}(E,N))$. To capture it, we use ung.

17.4 Proposition. *The tt-functor* $\mathrm{ung}\colon \mathrm{st}(\mathsf{gmod}(E,N)) \to \mathrm{st}(\mathsf{mod}(E,N))$ *is faithful. Geometrically,*

$$\mathbb{P}_k^{r+s-1} \xrightarrow[\simeq]{\mathcal{P}} \mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(E,N)) \xrightarrow{\mathrm{Spc}(\mathrm{ung})} \mathrm{Spc}\,\mathrm{st}(\mathsf{gmod}(E,N)) \tag{17.5}$$

is a surjection.

Proof. The geometric statement follows from the algebraic one (Recollection 6.8(i)). Let $f\colon \mathbb{1} \to X$ be a morphism in $\mathsf{gmod}(E,N)$ such that $\mathrm{ung}(f) = 0$ in the stable category $\mathrm{st}(\mathsf{mod}(E,N))$. It suffices by rigidity (Lemma 6.9(i)) to show that $f = 0$ in $\mathrm{st}(\mathsf{gmod}(E,N))$. Our hypothesis that $\mathrm{ung}(f) = 0$ in $\mathrm{st}(\mathsf{mod}(E,N))$ means $\mathrm{ung}(f)$ factors through a projective in $\mathsf{mod}(E,N)$. Such a projective is a sum of copies of $\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kE)$ (Corollary 15.22), but we may assume by linearity that it is simply $\mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kE)$ so that we have the following commutative diagram in $\mathsf{mod}(E,N)$:

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\mathrm{ung}(f)"] \arrow[dr, "\eta"'] & & \mathrm{ung}(X). \\ & \mathrm{ung}\,\mathrm{gr}\,\mathrm{radf}_N(kE) \arrow[ur, "h"'] & \end{tikzcd}$$

By Lemma 3.28, there exists $g\colon \operatorname{gr}\operatorname{radf}_N(kE) \to X$ such that we have the following commutative diagram in $\mathsf{gmod}(E, N)$:

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "f"] \arrow[dr, "\eta"'] & & X. \\ & \operatorname{gr}\operatorname{radf}_N(kE) \arrow[ur, "g"'] & \end{tikzcd}$$

Since $\operatorname{gr}\operatorname{radf}_N(kE)$ is projective in $\mathsf{gmod}(E, N)$, we conclude that $f = 0$ in $\operatorname{st}(\mathsf{gmod}(E, N))$.

□

We are now ready to study the surjection (17.5). Much of the remainder of this section is dedicated to proving the following proposition.

17.6 Proposition. *Assume k is algebraically closed, let K/k be a field extension, and let $[a : b]_{K/k}$ and $[c : d]_{K/k}$ be points in $\mathbb{P}_k^{r+s-1}$ both from K/k. Consider the following statements:*

$$\text{(A)} \qquad \begin{array}{c} \operatorname{Spc}(\mathrm{ung})(\mathcal{P}([a : b]_{K/k})) \\ | \\ \operatorname{Spc}(\mathrm{ung})(\mathcal{P}([c : d]_{K/k})) \end{array} \quad \textit{in } \operatorname{Spc}\operatorname{st}(\mathsf{gmod}(E, N)),$$

$$\text{(B)} \qquad \begin{array}{c} [a] \\ | \\ [c] \end{array} \ \textit{in } \dot{\mathbb{P}}_K^{r-1} \qquad \textit{and} \qquad \begin{array}{c} [b] \\ | \\ [d] \end{array} \ \textit{in } \dot{\mathbb{P}}_K^{s-1}.$$

The reverse implication (A) $\Leftarrow$ (B) *is true for any field extension K/k, and the forward implication* (A) $\Rightarrow$ (B) *is true for $K = k$.*

We will deal with the two implications in Proposition 17.6 separately, starting with the reverse implication.

17.7 Remark. Let us isolate the structure that $\mathsf{gmod}(E, N)$ affords us. For simplicity, suppose we are given $B \in \mathsf{gmod}(E, N)$ and a rational closed point $[a : b] \in \mathbb{P}_k^{r+s-1}$. Then the k-linear maps $aX\colon B \to B$ and $bY\colon B \to B$, which (16.6) leads us to consider, are severely constrained by the graded structure of B. Indeed, recall that $aX \in \operatorname{gr}\operatorname{radf}_N(kE)^0$ whereas $bY \in \operatorname{gr}\operatorname{radf}_N(kE)^1$. If we decompose $B = \bigoplus_{w\in\mathbb{Z}} B^w$ as k-vector spaces according to its grading, then our k-linear maps restrict to

$$\operatorname{ung}(aX)\colon B^w \longrightarrow B^w \qquad \text{and} \qquad \operatorname{ung}(bY)\colon B^w \longrightarrow B^{w+1}$$

for all $w \in \mathbb{Z}$. In other words, they induce morphisms $aX\colon B \to B$ and $bY\colon B \to B(-1)$ in gvect.

17.8 Lemma. *Let $B_1, B_2 \in \mathsf{gvect}_K$, and let $X\colon B_1 \to B_2$ and $Y\colon B_1 \to B_2(-1)$ be morphisms in gvect_K. The following hold:*

(i) $\operatorname{rank}_K(\operatorname{ung}(X)) \leq \operatorname{rank}_K(\operatorname{ung}(X) + \operatorname{ung}(Y))$

(ii) $\operatorname{rank}_K(\operatorname{ung}(Y)) \leq \operatorname{rank}_K(\operatorname{ung}(X) + \operatorname{ung}(Y))$

(iii) $\operatorname{rank}_K(u\operatorname{ung}(X) + v\operatorname{ung}(Y)) = \operatorname{rank}_K(\operatorname{ung}(X) + \operatorname{ung}(Y))$ *for any $u, v \in K^\times$.*

Proof. For (i), observe that for any morphism f in fvect_K, we have

$$\operatorname{rank}_K(\operatorname{ung}\operatorname{gr}(f)) \leq \operatorname{rank}_K(\operatorname{unf}(f)).$$

Applying this to $\operatorname{trf}(X) + \beta\operatorname{trf}(Y)\colon \operatorname{trf}(B_1) \to \operatorname{trf}(B_2)$ gives

$$\begin{aligned}
\operatorname{rank}_K(\operatorname{ung}(X)) &= \operatorname{rank}_K(\operatorname{ung}\operatorname{gr}\operatorname{trf}(X)) \\
&= \operatorname{rank}_K(\operatorname{ung}\operatorname{gr}(\operatorname{trf}(X) + \beta\operatorname{trf}(Y))) \\
&\leq \operatorname{rank}_K(\operatorname{unf}(\operatorname{trf}(X) + \beta\operatorname{trf}(Y))) \\
&= \operatorname{rank}_K(\operatorname{ung}(X) + \operatorname{ung}(Y)).
\end{aligned}$$

For (ii), set $B_3 = B_2(-1)$ so that $Y\colon B_1 \to B_3$ and $X\colon B_1 \to B_3(1)$. Now flip the gradings, and apply (i).

For (iii), by dividing by $u \neq 0$, it suffices to show that

$$\operatorname{rank}_K(\operatorname{ung}(X) + t\operatorname{ung}(Y)) = \operatorname{rank}_K(\operatorname{ung}(X) + \operatorname{ung}(Y))$$

for any nonzero $t \in K$. Let L be the invertible linear operator on $\mathrm{ung}(B_2)$ that scales B_2^w by t^w, and let R be the invertible linear operator on $\mathrm{ung}(B_1)$ that scales B_1^w by t^{-w}. Then

$$\mathrm{ung}(X) + t\,\mathrm{ung}(Y) = L(\mathrm{ung}(X) + \mathrm{ung}(Y))R$$

has the same rank as $\mathrm{ung}(X) + \mathrm{ung}(Y)$, as claimed. $\square$

Proof of (A) $\Leftarrow$ (B) *in Proposition 17.6.* Suppose $[c]$ specializes to $[a]$ in $\dot{\mathbb{P}}_K^{r-1}(K)$ and $[d]$ specializes to $[b]$ in $\dot{\mathbb{P}}_K^{s-1}(K)$. We have three cases:

(1) $[b] = 0$, in which case $[a] = [c] \in \mathbb{P}_K^{s-1}$

(2) $[a] = 0$, in which case $[b] = [d] \in \mathbb{P}_K^{r-1}$

(3) $[a]$ and $[b]$ are both nonzero, in which case $[a] = [c] \in \mathbb{P}_K^{r-1}$ and $[b] = [d] \in \mathbb{P}_K^{s-1}$.

Recall that we want to show $\mathrm{Spc}(\mathrm{ung})(\mathcal{P}([a:b]_{K/k})) \subset \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([c:d]_{K/k}))$. Thus, in case (1), let $B \in \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([a:0]_{K/k}))$. Then

$$\begin{aligned}
\frac{p-1}{p}\dim_K(\mathrm{ung}(B_K)) &= \mathrm{rank}_K(\mathrm{ung}(aX)) && \text{(by (16.6))}\\
&\le \mathrm{rank}_K(\mathrm{ung}(aX+dY)) && \text{(Lemma 17.8}(i))\\
&= \mathrm{rank}_K(\mathrm{ung}(cX+dY)), && \text{(Lemma 17.8}(iii)\text{ and Notation 16.2}(i))
\end{aligned}$$

so $B \in \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([c:d]_{K/k}))$. Case (2) is the same up to symmetry, and case (3) is similar and easier. $\square$

For the forward implication, we construct objects with certain prescribed supports. To do this, we carry over Example 6.32 to the setting of $\mathsf{gmod}(E, N)$, with the caveat that we must be careful to not mix the X_i and Y_j. This construction is where we use the hypothesis that k is algebraically closed.

17.9 Notation. For the remainder of this section, we assume k is algebraically closed, and we freely describe closed subsets of $\mathbb{P}_k^{r+s-1}$ using their closed points (Lemma 6.31).

17.10 Notation. For closed subsets $X \subset \mathbb{P}_k^{r-1}$ and $Y \subset \mathbb{P}_k^{s-1}$, we denote by

$$X * Y \subset \mathbb{P}_k^{r+s-1}$$

the *ruled join of X and Y* [FP07, Example 8.4.5] which has the closed points

$$(X * Y)(k) = \{[ux : vy] \mid [u : v] \in \mathbb{P}_k^1,\ x \in X(k),\ y \in Y(k)\} \subset \mathbb{P}_k^{r+s-1}. \tag{17.11}$$

17.12 Remark. Intuitively, the ruled join $X * Y$ is the union of all lines in $\mathbb{P}_k^{r+s-1}$ from points in X to points in Y, where we view X as living in the first r coordinates and Y in the last s coordinates. Except for the points in X and Y, every point in $X * Y$ is on a unique such line, so using (17.11) we may verify that intersections and ruled joins commute: for closed subsets $X_1, X_2 \subset \mathbb{P}_k^{r-1}$ and $Y_1, Y_2 \subset \mathbb{P}_k^{s-1}$, we have

$$(X_1 * Y_1) \cap (X_2 * Y_2) = (X_1 \cap X_2) * (Y_1 \cap Y_2). \tag{17.13}$$

17.14 Proposition. *For any irreducible subvariety $P \subset \mathbb{P}_k^{r-1}$ and linear subspace $Q \subset \mathbb{P}_k^{s-1}$, there exists an object $B_{P,Q} \in \mathsf{gmod}(E, N)$ such that*

$$\operatorname{supp}(\operatorname{ung}(B_{P,Q})) = \mathcal{P}(P * Q) \subset \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E, N)).$$

Proof. The proof proceeds in four steps. Step 1 is for the Q_N-direction, steps 2 and 3 are for the N-direction, and step 4 combines the two directions.

Step 1 (Q_N-direction): There exists an object $B_{P,\mathbb{P}_k^{s-1}} \in \mathsf{gmod}(E, N)$ such that

$$\operatorname{supp}(\operatorname{ung}(B_{P,\mathbb{P}_k^{s-1}})) = \mathcal{P}(P * \mathbb{P}_k^{s-1}) \subset \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E, N)).$$

Indeed, $\zeta \in \mathbb{P}_k^{r-1}$ be the generic point of P, and let $L_\zeta \in \mathsf{mod}(Q_N)$ be the Carlson module for ζ so that $\operatorname{supp}(L_\zeta) = P$. Then we may take $B_{P,\mathbb{P}_k^{s-1}} = \operatorname{gz} L_\zeta$.

Step 2 (N-direction): For any closed point $[b] \in \mathbb{P}_k^{s-1}$, there exists a p-dimensional module $B_{\mathbb{P}_k^{r-1},[b]^\perp} \in \mathsf{gmod}(E, N)$ such that

$$\operatorname{supp}(\operatorname{ung}(B_{\mathbb{P}_k^{r-1},[b]^\perp})) = \mathcal{P}(\mathbb{P}_k^{r-1} * [b]^\perp) \subset \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E, N)).$$

Indeed, take $B_{\mathbb{P}_k^{r-1},[b]^\perp} = \operatorname{gr}\operatorname{radf}(C_{[b]^\perp})$ where $C_{[b]^\perp} \in \mathsf{mod}(N)$ is our p-dimensional module from Example 6.32. Using (16.6), we see that $\operatorname{ung}(B_{\mathbb{P}_k^{r-1},[b]^\perp}) = C_{[b]^\perp}$ has the claimed support.

Step 3 (N-direction): For any linear subspace $P \subset \mathbb{P}_k^{r-1}$, there exists an object $B_{P,\mathbb{P}_k^{s-1}} \in \mathsf{gmod}(E, N)$ such that

$$\operatorname{supp}(\operatorname{ung}(B_{P,\mathbb{P}_k^{s-1}})) = \mathcal{P}(P * \mathbb{P}_k^{s-1}) \subset \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E, N)).$$

Indeed, as in Example 6.34, we tensor together objects of the form $B_{[a]^\perp,\mathbb{P}_k^{s-1}} \in \mathsf{gmod}(E, N)$, and we invoke the formula (17.13) for intersections of ruled joins.

Step 4 (combining directions): The object

$$B_{P,Q} = B_{P,\mathbb{P}_k^{s-1}} \otimes B_{Q,\mathbb{P}_k^{r-1}} \in \mathsf{gmod}(E, N)$$

is such that

$$\begin{aligned}
\operatorname{supp}(\operatorname{ung}(B_{P,Q})) &= \operatorname{supp}(\operatorname{ung}(B_{P,\mathbb{P}_k^{s-1}})) \cap \operatorname{supp}(\operatorname{ung}(B_{\mathbb{P}_k^{r-1}})) \\
&= \mathcal{P}((P * \mathbb{P}_k^{s-1}) \cap (\mathbb{P}_k^{r-1} * Q)) && (\mathcal{P} \text{ is a homeomorphism}) \\
&= \mathcal{P}(P * Q) && (\text{by } (17.13)) \\
&\subset \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E, N)),
\end{aligned}$$

as we wanted. □

The forward implication in Proposition 17.6 is now straightforward.

Proof of (A) $\Rightarrow$ (B) *in Proposition 17.6.* Suppose

$$\begin{array}{c} \operatorname{Spc}(\operatorname{ung})(\mathcal{P}([a:b])) \\ | \\ \operatorname{Spc}(\operatorname{ung})(\mathcal{P}([c:d])) \end{array} \quad \text{in } \operatorname{Spc}\operatorname{st}(\mathsf{gmod}(E, N)). \tag{17.15}$$

Suppose for now that $c \neq 0$ and $d \neq 0$ so that we may set $P = \{[c]\} \subset \mathbb{P}_k^{r-1}$ and $Q = \{[d]\} \subset$

$\mathbb{P}_k^{s-1}$. For $B_{P,Q} \in \mathsf{gmod}(E,N)$ as in Proposition 17.14, we have

$$\begin{aligned}\operatorname{Spc}(\mathrm{ung})^{-1}(\operatorname{supp}(B_{P,Q})) &= \operatorname{supp}(\mathrm{ung}(B_{P,Q})) && \text{(Recollection 6.3)}\\ &= \mathcal{P}(\{[c]\} * \{[d]\}) && \text{(definition of } B_{P,Q})\\ &\subset \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E,N)).\end{aligned}$$

Observe that since $\operatorname{Spc}(\mathrm{ung})(\mathcal{P}([c:d]))$ is in the closed set $\operatorname{supp}(\mathrm{ung}(B_{P,Q}))$, the point $\operatorname{Spc}(\mathrm{ung})(\mathcal{P}([a:b]))$ is as well by our hypothesis (17.15). Thus $\mathcal{P}([a:b]) \in \mathcal{P}(\{[c]\} * \{[d]\})$, whence $[a:b] = [uc:vd]$ for some $[u:v] \in \mathbb{P}_k^1$ because $\mathcal{P}$ is a homeomorphism. We conclude that $[c]$ specializes to $[a]$ in $\dot{\mathbb{P}}_k^{r-1}$ and $[d]$ specializes to $[b]$ in $\dot{\mathbb{P}}_k^{s-1}$.

Let us now discuss the degenerate cases. Recall (16.3). If $c = 0$, then $d \neq 0$, so we set $Q = \{[d]\}$ and use the object

$$B_{0,Q} = \frac{k[X_1,\ldots,X_r]}{(X_1^p,\ldots,X_r^p)} \otimes B_{\mathbb{P}_k^{r-1},Q} \in \mathsf{gmod}(E,N)$$

which has $\operatorname{supp}(\mathrm{ung}(B_{0,Q})) = \{\mathcal{P}([0:d])\}$. Similarly, if $d = 0$, then $c \neq 0$, so we set $P = \{[c]\}$ and use the object

$$B_{P,0} = B_{P,\mathbb{P}_k^{s-1}} \otimes \frac{k[Y_1,\ldots,Y_s]}{(Y_1^p,\ldots,Y_s^p)} \in \mathsf{gmod}(E,N)$$

which has $\operatorname{supp}(\mathrm{ung}(B_{P,0})) = \{\mathcal{P}([c:0])\}$. In both cases, we argue as in the previous paragraph. $\square$

Let us describe our answer in a more geometric way.

17.16 Corollary. *The map*

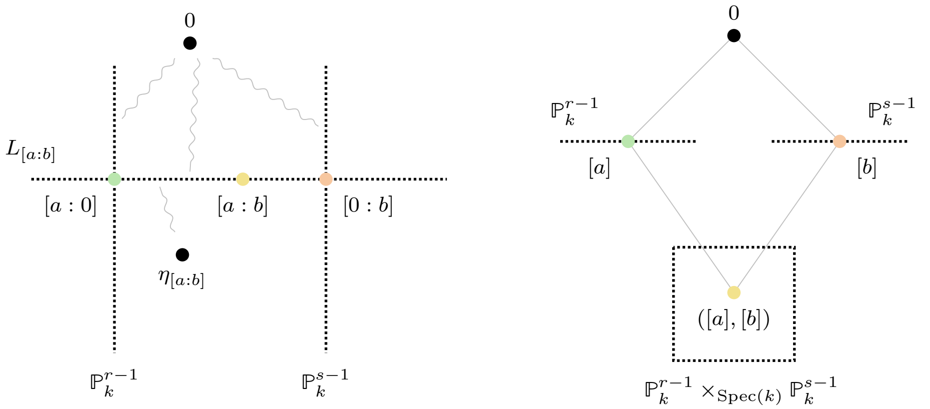
$$\dot{\mathbb{P}}_k^{r+s-1} \xrightarrow[\cong]{\mathcal{P}} \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E,N)) \xrightarrow{\operatorname{Spc}(\mathrm{ung})} \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E,N)) \tag{17.17}$$

is a surjection, and only 0 *is mapped to* 0*. When* k *is algebraically closed, for closed points* $[a:b], [c:d] \in \mathbb{P}_k^{r+s-1}$*, we have* $\operatorname{Spc}(\mathrm{ung})(\mathcal{P}([a:b])) = \operatorname{Spc}(\mathrm{ung})(\mathcal{P}([c:d]))$ *if and only if* $[a] = [c]$ *in* $\dot{\mathbb{P}}_k^{r-1}$ *and* $[b] = [d]$ *in* $\dot{\mathbb{P}}_k^{s-1}$*.*

Proof. Follows from Proposition 17.6 and Lemma 17.2. $\square$

The following non-rigorous picture illustrates the following notation and lemma:

$$\dot{\mathbb{P}}_k^{r+s-1} \xrightarrow[\cong]{\mathcal{P}} \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E,N)) \xrightarrow{\operatorname{Spc}(\mathrm{ung})} \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E,N))$$



17.18 Notation. For this discussion, let $V \subset \mathbb{P}_k^{r+s-1}$ be the closed subset with closed points $\mathbb{P}_k^{r-1}(k) \sqcup \mathbb{P}_k^{s-1}(k) \subset \mathbb{P}_k^{r+s-1}$, where we view $\mathbb{P}_k^{r-1}(k)$ and $\mathbb{P}_k^{s-1}(k)$ as respectively living in the first r and last s coordinates. Let $U \subset \mathbb{P}_k^{r+s-1}$ denote the open complement of V, which has closed points

$$U(k) = \{[a:b] \mid [a] \in \mathbb{P}_k^{r-1}, [b] \in \mathbb{P}_k^{s-1}\}.$$

Let us fix $[a:b] \in U$. The orbit

$$\{[ua:vb] \in \mathbb{P}_k^{r+s-1} \mid [u:v] \in \mathbb{P}_k^1\}$$

of $[a:b] \in U$ gives rise to a copy of $\mathbb{P}_k^1 \subset \mathbb{P}_k^{r+s-1}$ which intersects V in precisely two points: $[a:0]$ and $[0:b]$. We denote this copy of $\mathbb{P}_k^1$ by $L_{[a:b]}$, and we denote its generic point by $\eta_{[a:b]}$.

17.19 Lemma. *Let $U, V \subset \mathbb{P}_k^{r+s-1}$ be as in Notation 17.18, and let f denote the map (17.17). The images $f(U(k))$ and $f(V(k))$ are disjoint. On $V(k)$, the map f is injective, and on $U(k)$, the map f factors through the $k^\times$-action $\lambda[a:b] = [\lambda a : b]$. Thus for any $[a:b] \in U$, the image $f(L_{[a:b]})$ consists of precisely three points: $f([a:0]), f([0:b]) \in f(V(k))$ and $f(\eta_{[a:b]}) \in f(U(k))$.*

Proof. Indeed we have $f([\lambda a : b]) = f([a : b])$ (Corollary 17.16). The claim about $f(L_{[a:b]})$ then follows from the following observation: if X is a spectral space and $g\colon \mathbb{P}^1_k \to X$ is a continuous map that is constant on the closed points $U(k)$ of a nonempty open subset $U \subset \mathbb{P}^1_k$, then g is constant on U. To prove the observation, let $g(U(k)) = \{x\}$ for some $x \in X$, and let η denote the generic point of $\mathbb{P}^1_k$. Since U is dense in $\mathbb{P}^1_k$, we have $g(\eta) \in \overline{\{x\}}$. Conversely $x \in \overline{\{g(\eta)\}}$ because continuous maps preserve specializations. Since X is spectral, we conclude that $g(\eta) = x$. $\square$

Before we return to our running example, we make one quick observation that is a corollary of Proposition 17.6.

17.20 Corollary. *Assume k is algebraically closed. The surjection (17.5) takes closed points to closed points.*

Proof. Let $[c : d] \in \mathbb{P}^{r+s-1}_k$ be a closed point, let $[a : b]_{K/k} \in \mathbb{P}^{r+s-1}_k$ be an arbitrary point, and suppose we have a specialization

$$\begin{array}{c} \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([a : b]_{K/k})) \\ | \\ \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([c : d])) \end{array} \qquad \text{in} \qquad \mathrm{Spc}\,\mathrm{st}(\mathsf{gmod}(E, N)).$$

Since $\mathrm{Spc}(\mathrm{ung})\mathcal{P}$ is a surjection, it is enough to show that these are in fact equal. Let $[a' : b'] \in \mathbb{P}^{r+s-1}_k$ be a closed point to which $[a : b]_{K/k}$ specializes. Then

$$\begin{array}{c} \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([a' : b'])) \\ | \\ \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([a : b]_{K/k})) \\ | \\ \mathrm{Spc}(\mathrm{ung})(\mathcal{P}([c : d])) \end{array} \qquad \text{in} \qquad \mathrm{Spc}\,\mathrm{st}(\mathsf{gmod}(E, N)). \tag{17.21}$$

By the forward implication in Proposition 17.6, we see that

$$\begin{array}{c} [a'] \\ | \\ [c] \end{array} \text{ in } \dot{\mathbb{P}}^{r-1}_k \qquad \text{and} \qquad \begin{array}{c} [b'] \\ | \\ [d] \end{array} \text{ in } \dot{\mathbb{P}}^{s-1}_k.$$

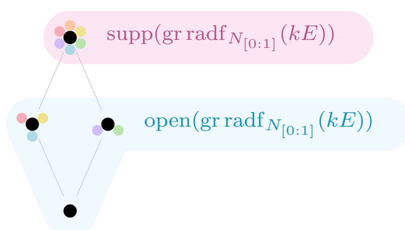
In other words $[a' : b'] \in L_{[c:d]}$, and we deduce that $[a : b]_{K/k}$ is either $\eta_{[c:d]}$ or $[c : d]$. In either case, we may plug $[c : d]$ for $[a' : b']$ into (17.21) and read off that $\mathrm{Spc}(\mathrm{ung})(\mathcal{P}([c : d]))$ and $\mathrm{Spc}(\mathrm{ung})(\mathcal{P}([a : b])_{K/k})$ specialize to each other, so they are indeed equal. $\square$

17.22 Running Example. We return to our Running Example 1.13. When $N = 1$, we have $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}(E, 1)) = \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(E))$, so nothing is new. For N with rank 1, let us just consider $N = N_{[0:1]} = \langle y \rangle$ so that $Q_{[0:1]} = \langle x \rangle$, as the other two are similar. Our surjection (17.5) is

$$\mathbb{P}^1_k \xrightarrow[\cong]{\mathcal{P}} \mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(E, N_{[0:1]})) \xrightarrow{\mathrm{Spc}(\mathrm{ung})} \mathrm{Spc}\,\mathrm{st}(\mathsf{gmod}(E, N_{[0:1]})),$$

and moreover $L_{[1:1]} = \mathbb{P}^1_k$. Thus we get the following picture:

$$\mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}(E, N_{[0:1]}))$$



$\mathrm{supp}(\mathrm{gr\,radf}(C_{[a:b]}))$ $\quad$ $\mathrm{supp}(\mathrm{gr\,radf}(kN_{[0:1]}))$

$\mathrm{supp}(\mathrm{gr}(A^{\mathrm{horz}}_{[a:b]}))$ $\quad$ $\mathrm{supp}(\mathrm{gr}(A^{\mathrm{vert}}_{[a:b]}))$ $\quad$ $\mathrm{supp}(\mathrm{gz}\,kQ_{[0:1]})$.

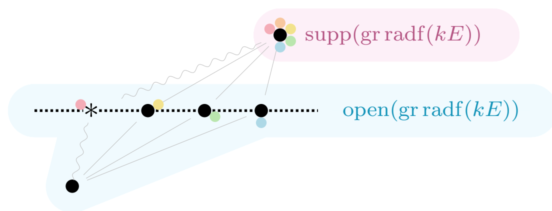
Following our convention of putting Q before N, note that our coordinates $[a : b] \in \mathbb{P}^{r+s-1}_k = \mathbb{P}^1_k$ are with respect to the generators x, y. Thus the $\mathbb{P}^{r-1}_k = \mathbb{P}^0_k$ is on the left-hand side, and the $\mathbb{P}^{s-1}_k = \mathbb{P}^0_k$ is on the right-hand side. In other words, the point on the left-hand side is $\mathrm{Spc}(\mathrm{ung})(\mathcal{P}([1 : 0]))$, and the point on the right-hand side is $\mathrm{Spc}(\mathrm{ung})(\mathcal{P}([0 : 1]))$. The point on the bottom is the image under $\mathrm{Spc}(\mathrm{ung})$ of any other point in $\mathbb{P}^1_k$, including the generic point (Lemma 17.19). The objects $A^{\mathrm{horz}}_{[a:b]}$ and $A^{\mathrm{vert}}_{[a:b]}$ were defined in (1.15).

When $N = E$, our surjection (17.5) is

$$\mathbb{P}^1_k \xrightarrow[\cong]{\mathcal{P}} \mathrm{Spc}\,\mathrm{st}(\mathsf{mod}(E, E)) \xrightarrow{\mathrm{Spc}(\mathrm{ung})} \mathrm{Spc}\,\mathrm{st}(\mathsf{gmod}(E, E)),$$

and it is not only an injection on closed points (Corollary 17.16) but also takes closed points to closed points (Corollary 17.20). We deduce that it is a homeomorphism. (When E has rank > 2 it is unclear whether it is still a homeomorphism). This gives the following picture:

$$\operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E, E))$$



● $\operatorname{supp}(\operatorname{gr}\operatorname{radf}(C_{[a:b]}))$ ● $\operatorname{supp}(\operatorname{gr}\operatorname{radf} k(E/N_{[1:0]}))$

● $\operatorname{supp}(\operatorname{gr}\operatorname{radf} k(E/N_{[0:1]}))$ ● $\operatorname{supp}(\operatorname{gr}\operatorname{radf} k(E/N_{[1:1]}))$.

We conclude this section with a conjecture.

17.23 Conjecture. For simplicity, we assume here that p is odd. Consider the $\otimes$-invertible objects $u = \mathbb{1}[2]$ and $v = \mathbb{1}[2](p)$ in $\mathrm{D_b}(\mathsf{gmod}(E, N))$ and the bi-graded cohomology ring

$$\mathrm{H}^{\bullet\bullet}(E, N) = \bigoplus_{a,b\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{D_b}(\mathsf{gmod}(E,N))}(\mathbb{1}, u^{\otimes a} \otimes v^{\otimes b}).$$

For each $i = 1, \ldots, r$, we have an element $U_i \colon \mathbb{1} \to u$ with bidegree $(1, 0)$ from the usual group cohomology of Q_N arising from (15.27) (see also (6.25)). Similarly, for each $j = 1, \ldots, s$, we have a morphism $V_j \colon \mathbb{1} \to v$ with bi-degree $(0, 1)$ arising from (15.28). We conjecture that this cohomology ring is

$$\mathrm{H}^{\bullet\bullet}(E, N) \cong k[U_1, \ldots, U_r, V_1, \ldots, V_s]$$

and that there exists a comparison map

$$\mathrm{comp}\colon \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E, N)) \xrightarrow{\cong} \operatorname{Spec}^{\mathrm{h}} \mathrm{H}^{\bullet\bullet}(E, N)$$

that is a homeomorphism. The homogeneous spectrum of this bi-graded ring is some sort of product of $\dot{\mathbb{P}}_k^{r-1}$ and $\dot{\mathbb{P}}_k^{s-1}$.

18 Graded modular fixed points

18.1 Notation. For this section, we fix a subgroup $N \leqslant E$ of our elementary abelian p-group E (Notation 2.1).

In this section we study $\mathrm{D_b}(\mathsf{pgmod}(E,N))$. In particular, in this special case of an elementary abelian p-group, we see that the story is analogous to [BG25a]. Each spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}(E,N))$ contains a vanishing locus of acyclics homeomorphic to the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}(E,N))$ (Proposition 18.2), and a graded version (Definition 18.8) of modular fixed points captures the support of the acyclics.

Crucially, we will work in terms of our complementary subgroup Q_N (Notation 15.2) and N rather than in terms of E and N by writing $\mathsf{pgmod}(E,N) \cong \mathsf{pgmod}[Q_N,N]$ (Remark 15.20). To study $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}[Q_N,N])$, we divide and conquer (Method 6.10) using the object $\mathrm{gz}\,\mathrm{kos}_{Q_N}(1)$ (Recollection 13.4). The following proposition shows that the vanishing locus of $\mathrm{gz}\,\mathrm{kos}_{Q_N}(1)$ is the spectrum of $\mathrm{D_b}(\mathsf{gmod}[Q_N,N])$, which we studied in Section 17.

18.2 Proposition. *The tt-functor* $\Upsilon\colon\ \mathrm{D_b}(\mathsf{pgmod}[Q_N,N]) \to \mathrm{D_b}(\mathsf{gmod}[Q_N,N])$ *induced by the inclusion* $\Upsilon\colon\ \mathsf{pgmod}[Q_N,N] \rightarrowtail \mathsf{gmod}[Q_N,N]$ *induces an equivalence*

$$\overline{\Upsilon}\colon\ \frac{\mathrm{D_b}(\mathsf{pgmod}[Q_N,N])}{\langle \mathrm{gz}\,\mathrm{kos}_{Q_N}(1)\rangle} \longrightarrow \mathrm{D_b}(\mathsf{gmod}[Q_N,N])$$

of tt-categories. Geometrically, $\mathrm{Spc}(\overline{\Upsilon})$ *induces a homeomorphism*

$$\mathrm{Spc}(\overline{\Upsilon})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}[Q_N,N]) \xrightarrow{\cong} \mathrm{open}(\mathrm{gz}\,\mathrm{kos}_{Q_N}(1)) \subset \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}[Q_N,N]).$$

Proof. We invoke the theory of finite resolutions (Proposition 7.13). The twists of the unit generate $\mathrm{D_b}(\mathsf{gmod}[Q_N,N])$ as a triangulated subcategory (Example 14.20), and twisting the resolutions in Lemma 15.26(*ii*) exhibits the required resolutions. Note that we may identify the $\mathsf{gmod}[Q_N,N]$-acyclics $\ker(\Upsilon) \subset \mathrm{D_b}(\mathsf{pgmod}[Q_N,N])$ with $\langle \mathrm{gz}\,\mathrm{kos}_{Q_N}(1)\rangle$ using Lemma 14.21(*ii*). ∎

We now introduce a graded version of the modular fixed points functors from [BG25a, Section 5].

18.3 Recollection. In this recollection, we allow G to be any finite group. Let $H \leqslant G$ be a p-subgroup [BG25a, Warning 5.1]. We have the *modular H-fixed points functor*

$$\Psi^H\colon\ \mathrm{D_b}(\mathsf{pmod}(G)) \longrightarrow \mathrm{D_b}(\mathsf{pmod}(G/\!\!/H)), \tag{18.4}$$

where $G/\!\!/H$ denotes the Weyl group $\mathrm{N}_G(H)/H$, and its composite

$$\check{\Psi}^H\colon\ \mathrm{D_b}(\mathsf{pmod}(G)) \xrightarrow{\Psi^H} \mathrm{D_b}(\mathsf{pmod}(G/\!\!/H)) \twoheadrightarrow \mathrm{D_b}(\mathsf{mod}(G/\!\!/H)) \tag{18.5}$$

with the Verdier quotient (7.15) at the $\mathsf{mod}(G/\!\!/H)$-acyclics. The functor (18.4) is in fact defined at the level of exact rigid tensor categories

$$\Psi^H\colon\ \mathsf{pmod}(G) \longrightarrow \mathsf{pmod}(G/\!\!/H) \tag{18.6}$$

in order to satisfy $\Psi^H(kX) = k(X^H)$ for every finite G-set X. For any subgroup $K \leqslant G$, the map $\mathrm{Spc}(\check{\Psi}^H)\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G/\!\!/H)) \to \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G))$ on spectra interacts with the Koszul object $\mathrm{kos}_G(K) \in \mathrm{D_b}(\mathsf{pmod}(G))$ (Recollection 13.4) as follows:

(i) If no conjugate of H is contained in K, then $\mathrm{im}\,\mathrm{Spc}(\check{\Psi}^H) \subset \mathrm{supp}(\mathrm{kos}_G(K))$.

(ii) If some conjugate of H is contained in K, then $\mathrm{im}\,\mathrm{Spc}(\check{\Psi}^H) \subset \mathrm{open}(\mathrm{kos}_G(K))$

Indeed these follow respectively from [BG25a, Lemma 5.21(a) and (b)] (see Recollection 6.5 if necessary, and recall (13.5) for intuition). One deduces, much like we will in Proposition 18.13, that over the subgroups $H \leqslant G$ up to conjugacy, the maps $\mathrm{Spc}(\check{\Psi}^H)$ are injective with disjoint images. In fact, these images partition the spectrum as sets

$$\bigsqcup_{H\in \mathrm{sub}(G)/G} \mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(G/\!\!/H)) \xrightarrow{\cong} \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(G)) \tag{18.7}$$

because the functors $\check{\Psi}^H$ collectively detect nilpotence [BG25a, Corollary 7.2].

18.8 Definition. Let $H \leqslant Q_N$ be a (p-)subgroup of our elementary abelian p-group Q_N. We will define the *N-graded modular H-fixed points functor*

$$\Psi^H_N\colon\ \mathrm{D_b}(\mathsf{pgmod}[Q_N, N]) \longrightarrow \mathrm{D_b}(\mathsf{pgmod}[Q_N /\!\!/ H, N]). \tag{18.9}$$

As in Recollection 18.3, we begin at the level of exact rigid tensor categories. Applying Ψ^H grading-wise in $\mathsf{pgmod}(Q_N)$ gives rise to an exact tensor functor

$$\Psi^H\colon\ \mathsf{pgmod}(Q_N) \longrightarrow \mathsf{pgmod}(Q_N /\!\!/ H)$$

and thus to an exact tensor functor (Lemma 3.32 and Lemma 5.12)

$$\begin{array}{ccc} \Psi^H\colon\ \mathsf{pgmod}[Q_N, N] & \longrightarrow & \mathsf{pgmod}[Q_N /\!\!/ H, N] \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{mod}_{\mathsf{pgmod}(Q_N)}(\mathrm{gr}\,\mathrm{radf}(kN)) & \longrightarrow & \mathrm{mod}_{\mathsf{pgmod}(Q_N/\!\!/ H)}(\mathrm{gr}\,\mathrm{radf}(kN)). \end{array}$$

Passing to derived categories gives (18.9). In line with (18.5), we denote by

$$\begin{array}{ccc} \check{\Psi}^H_N\colon\ \mathrm{D_b}(\mathsf{pgmod}[Q_N, N]) & \xrightarrow{\Psi^H_N} & \mathrm{D_b}(\mathsf{pgmod}[Q_N /\!\!/ H, N]) \\ & & \twoheaddownarrow \\ & & \mathrm{D_b}(\mathsf{gmod}[Q_N /\!\!/ H, N]) \end{array} \tag{18.10}$$

the composite with the Verdier quotient at the $\mathsf{gmod}[Q_N /\!\!/ H, N]$-acyclics.

On spectra, $\check{\Psi}^H_N$ induces a map

$$\mathrm{Spc}(\check{\Psi}^H_N)\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}[Q_N /\!\!/ H, N]) \longrightarrow \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}[Q_N, N]). \tag{18.11}$$

Note that we studied the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{gmod}[Q_N /\!\!/ H, N])$ in Section 17.

18.12 Lemma. *Let $H \leqslant Q_N$ be a (p-)subgroup, and let $K \leqslant Q_N$ be a subgroup. Consider the image* $\mathrm{im}\,\mathrm{Spc}(\check{\Psi}^H_N) \subset \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}[Q_N, N])$ *of* (18.11).

(i) If (every conjugate of) H is not contained in K, then

$$\mathrm{im}\,\mathrm{Spc}(\check{\Psi}^H_N) \subset \mathrm{supp}(\mathrm{gz}\,\mathrm{kos}_{Q_N}(K)).$$

(*ii*) *If (some conjugate of) H is contained in K, then*

$$\operatorname{im}\operatorname{Spc}(\check{\Psi}^H_N) \subset \operatorname{open}(\operatorname{gz}\operatorname{kos}_{Q_N}(K)).$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(Q_N)) & \operatorname{Spc}\mathrm{D_b}(\mathsf{mod}(Q_N/\!\!/H)) \arrow[l, "\operatorname{Spc}(\check{\Psi}^H_N)"'] \\
\operatorname{Spc}\mathrm{D_b}(\mathsf{pgmod}[Q_N, N]) \arrow[u, "\operatorname{Spc}(\mathrm{gz})"'] & \operatorname{Spc}\mathrm{D_b}(\mathsf{gmod}[Q_N/\!\!/H, N]) \arrow[l, "\operatorname{Spc}(\check{\Psi}^H_N)"'] \arrow[u, "\operatorname{Spc}(\mathrm{gz})"']
\end{tikzcd}$$

together with Recollection 18.3(i) and Recollection 18.3(ii) (see Recollection 6.5 if necessary).

□

18.13 Proposition. *The maps* $\operatorname{Spc}(\check{\Psi}^H_N)$, *for (p-)subgroups* $H \leqslant Q_N$, *enjoy the following properties:*

(*i*) *The map* $\operatorname{Spc}(\check{\Psi}^H_N)$ *is a homeomorphism onto its image.*

(*ii*) *A point in* $\operatorname{im}\operatorname{Spc}(\check{\Psi}^H_N)$ *specializes to a point in* $\operatorname{im}\operatorname{Spc}(\check{\Psi}^K_N)$ *only if (some conjugate of) H is contained in K.*

(*iii*) *The images* $\operatorname{im}\operatorname{Spc}(\check{\Psi}^H_N)$ *are disjoint.*

Proof. For (i), since inflation $\mathsf{pgmod}(Q_N) \to \mathsf{pgmod}[Q_N, N]$ is a section of Ψ^H_N, the map $\operatorname{Spc}(\Psi^H_N)$ is a homeomorphism onto its image (Recollection 6.6(ii)). From the definition (18.10), we observe that the map $\operatorname{Spc}(\check{\Psi}^H_N)$ is the inclusion of an open subset (Recollection 6.6 and Proposition 18.2) followed by this homeomorphism, so it is also a homeomorphism onto its image.

For (ii), suppose $\operatorname{Spc}(\check{\Psi}^K_N)(\mathcal{Q}) \subset \operatorname{Spc}(\check{\Psi}^H_N)(\mathcal{P})$ for some $\mathcal{P} \in \mathrm{D_b}(\mathsf{gmod}[Q_N/\!\!/H, N])$ and $\mathcal{Q} \in \mathrm{D_b}(\mathsf{gmod}[Q_N/\!\!/K, N])$. Note that we have $\operatorname{gz}\operatorname{kos}_{Q_N}(K) \in \operatorname{Spc}(\check{\Psi}^K_N)(\mathcal{Q})$ (Lemma 18.12(ii)), so $\operatorname{gz}\operatorname{kos}_{Q_N}(K) \in \operatorname{Spc}(\check{\Psi}^H_N)(\mathcal{P})$ by our assumption. Thus we deduce that $H \leqslant K$ using Lemma 18.12(i).

For (iii), if $\operatorname{Spc}(\check{\Psi}^H_N)(\mathcal{P})$ and $\operatorname{Spc}(\check{\Psi}^K_N)(\mathcal{Q})$ are equal, then they specialize to each other, so $H = K$ by (ii). $\square$

We now show that our graded modular fixed points functors surject the closed points of $\operatorname{Spc}\mathrm{D_b}(\mathsf{pgmod}[Q_N, N])$, and in analogy with (18.7), we conjecture that they surject the spectrum entirely.

18.14 Proposition. *The tt-functors $\check{\Psi}^H_N$ for (p-)subgroups $H \leqslant Q_N$ are collectively conservative. Geometrically, the maps $\operatorname{Spc}(\check{\Psi}^H_N)$ for (p-)subgroups $H \leqslant Q_N$ surject the closed points in $\operatorname{Spc}\mathrm{D_b}(\mathsf{pgmod}[Q_N, N])$.*

Proof. The geometric statement follows from the algebraic one by using Recollection 6.8(ii). To prove the algebraic statement, consider the following commutative diagram of tt-functors:

$$\begin{tikzcd}
\mathrm{D_b}(\mathsf{pgmod}[Q_N, N]) \arrow[r, two heads, "\operatorname{Res}_{Q_N}"] \arrow[d, "\check{\Psi}^H_N"'] & \mathrm{D_b}(\mathsf{pgmod}(Q_N)) \arrow[r, two heads, "\mathrm{ung}"] & \mathrm{D_b}(\mathsf{pmod}(Q_N)) \arrow[d, "\check{\Psi}^H"] \\
\mathrm{D_b}(\mathsf{gmod}[Q_N/\!\!/H, N]) \arrow[r, two heads, "\operatorname{Res}_{Q_N/\!\!/H}"'] & \mathrm{D_b}(\mathsf{gmod}(Q_N/\!\!/H)) \arrow[r, two heads, "\mathrm{ung}"'] & \mathrm{D_b}(\mathsf{mod}(Q_N/\!\!/H)).
\end{tikzcd}$$

Suppose $X \in \mathrm{D_b}(\mathsf{pgmod}[Q_N, N])$ is such that $\check{\Psi}^H_N(X) = 0$ in $\mathrm{D_b}(\mathsf{gmod}[Q_N/\!\!/H, N])$ for all (p-)subgroups $H \leqslant Q_N$. Then

$$\check{\Psi}^H \operatorname{ung}\operatorname{Res}_{Q_N}(X) = \operatorname{ung}\operatorname{Res}_{Q_N/\!\!/H} \check{\Psi}^H_N(X) = 0$$

for all $H \leqslant Q_N$ by commutativity of the diagram. Since the tt-functors $\check{\Psi}^H$ collectively detect nilpotence (recall (18.7)), in particular they are collectively conservative, so we have $\operatorname{ung}\operatorname{Res}_{Q_N}(X) = 0$. By conservativity of the horizontal tt-functors (Lemma 6.37), we conclude that $X = 0$. $\square$

18.15 Conjecture. The graded modular fixed points functors $\check{\Psi}^N_H$ for (p-)subgroups $H \leqslant Q_N$ collectively detect nilpotence. Geometrically, the images $\operatorname{im}\operatorname{Spc}(\check{\Psi}^H_N)$ for (p-)subgroups $H \leqslant Q_N$ (up to conjugacy) partition the spectrum as sets:

$$\bigsqcup_{H \in \mathrm{sub}_p(Q_N)/Q_N} \operatorname{Spc}\mathrm{D_b}(\mathsf{gmod}[Q_N/\!\!/H, N]) \xrightarrow{\cong} \operatorname{Spc}\mathrm{D_b}(\mathsf{pgmod}[Q_N, N]). \tag{18.16}$$

We assume this conjecture from now on. We now illustrate the spectrum for our running example.

18.17 Running Example. We return to our Running Example 1.13. When $N = 1$, we have $\mathsf{pgmod}(E, 1) = \mathsf{pgmod}(E)$, so we get the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E)) \cong \operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(E))$ of [BG25a] (see Running Example 11.15).

For N of rank 1, let us again just consider $N = N_{[0:1]} = \langle y \rangle$ so that $Q_{[0:1]} = \langle x \rangle$, as the other two are similar. Recall that we write

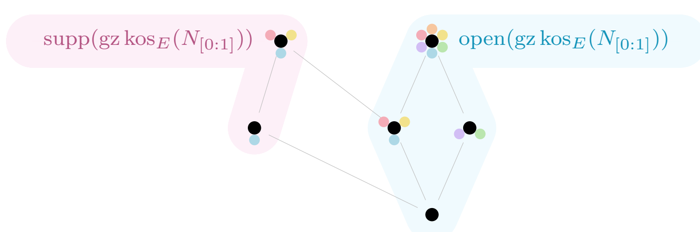
$$\mathsf{pgmod}(E, N_{[0:1]}) = \mathsf{pgmod}[Q_{[0:1]}, N_{[0:1]}]$$

and that the spectrum of $\mathrm{D_b}(\mathsf{pgmod}[Q_{[0:1]}, N_{[0:1]}])$ is partitioned (Conjecture 18.15) by the graded modular fixed points functors

$$\operatorname{Spc}(\check{\Psi}^{1}_{N_{[0:1]}})\colon\ \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}[Q_{[0:1]}, N_{[0:1]}]) \longrightarrow \operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}[Q_{[0:1]}, N_{[0:1]}])$$
$$\operatorname{Spc}(\check{\Psi}^{Q_{[0:1]}}_{N_{[0:1]}})\colon\ \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}[Q_{[0:1]}, N_{[0:1]}]) \longrightarrow \operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}[1, N_{[0:1]}]).$$

We obtain the following picture:

$$\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N_{[0:1]}))$$



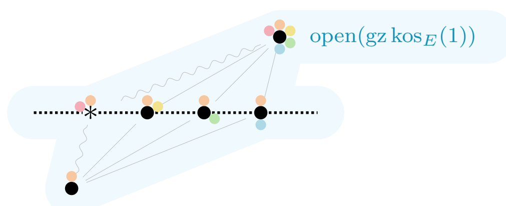
- $\mathrm{supp}(\mathrm{gr}\,\mathrm{radf}(C_{[a:b]}))$
- $\mathrm{supp}(\mathrm{gr}\,\mathrm{radf}_{N_{[0:1]}}(kE))$
- $\mathrm{supp}(\mathrm{gr}\,\mathrm{radf}(kN_{[0:1]}))$
- $\mathrm{supp}(\mathrm{gr}(A^{\mathrm{horz}}_{[a:b]}))$
- $\mathrm{supp}(\mathrm{gr}(A^{\mathrm{vert}}_{[a:b]}))$
- $\mathrm{supp}(\mathrm{gz}\,kQ_{[0:1]})$.

Note that the cyan region is the vanishing locus of the $\mathsf{gmod}[Q_{[0:1]}, N_{[0:1]}]$-acyclics and also the image of $\operatorname{Spc}(\check{\Psi}^{1}_{N_{[0:1]}})$, and note that the magenta region is the support of these acyclics and also the image of $\operatorname{Spc}(\check{\Psi}^{Q_{[0:1]}}_{N_{[0:1]}})$. The objects $A^{\mathrm{horz}}_{[a:b]}$ and $A^{\mathrm{vert}}_{[a:b]}$ were defined in (1.15).

We owe an explanation for the two specializations between the strata. Note that there can only be specializations from the cyan to the magenta region, and by the computation of $\operatorname{supp}(\operatorname{gr}(A^{\mathrm{horz}}_{[a:b]}))$, we are reduced to considering the left-hand point and the bottom point of the cyan region. From the homeomorphic copy of $\operatorname{Spc}\mathrm{D_b}(\mathsf{pmod}(Q_{[0:1]}))$ given by $\operatorname{Res}_{Q_{[0:1]}}$, we deduce the specialization for the left-hand point. For the bottom point, let us restrict our attention to the spectrum of the stable category $\operatorname{st}(\mathsf{pgmod}(E, N_{[0:1]}))$, which is $\operatorname{open}(\operatorname{gr}\operatorname{radf}(kN_{[0:1]}))$. Since it must be connected, there must be a specialization from one of the two points in the cyan region to the point in the magenta region. But the higher point in the cyan region is in $\operatorname{supp}(\operatorname{gz} kQ)$, hence admits no other specializations.

In the case $N = E$, we only have one graded modular fixed points functor, so we have the following picture:

$$\operatorname{Spc}\mathrm{D_b}(\mathsf{pgmod}(E, E))$$



- supp(gr radf($C_{[a:b]}$)) • supp(gr gap^1 radf(kE)) • supp(gr radf $k(E/N_{[1:0]})$)
- supp(gr radf $k(E/N_{[0:1]})$) • supp(gr radf $k(E/N_{[1:1]})$).

After all, this is not surprising at all because $\mathsf{pgmod}(E, E) \cong \mathsf{gmod}(E, E)$.

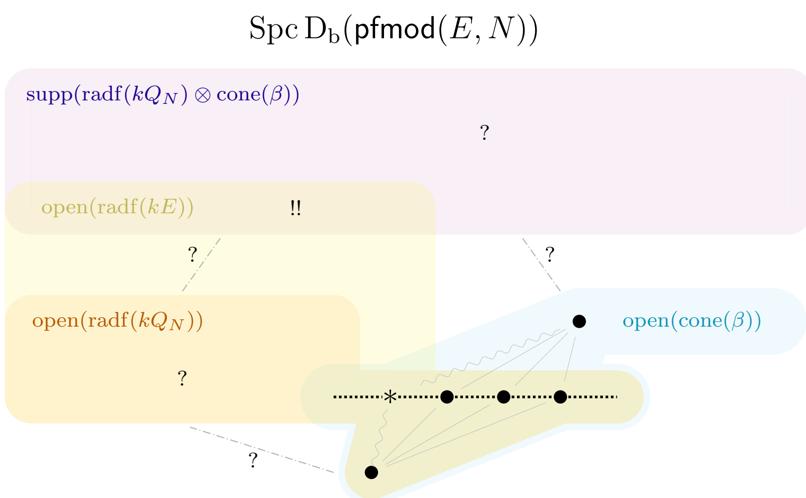
18.18 Remark. We can easily generalize this section to the category $\mathsf{pgmod}[G, N]$ of modules in gvect over the Hopf monoid $\operatorname{gz} kG \otimes \operatorname{gr}\operatorname{radf}(kN)$. Indeed, we still have a cohomological open by generalizing Proposition 18.2 and Lemma 15.26(ii) using Example 7.14.

19 The support of $\mathrm{radf}(kQ_N)$ in the relative subcategory

19.1 Notation. For this section, we fix a subgroup $N \leqslant E$ of our elementary abelian p-group E (Notation 2.1).

In this section we finally begin determining the spectrum of $\mathrm{D_b}(\mathsf{pfmod}(E, N))$. Our strategy is to divide and conquer (Method 6.11) using the object $\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)$ in $\mathrm{D_b}(\mathsf{pfmod}(E, N))$. By pushing our methods from Part II, we will see that the support of $\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)$ is a copy of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}(E, N))$ induced by $\mathrm{gr}\colon \mathrm{D_b}(\mathsf{pfmod}(E, N)) \to \mathrm{D_b}(\mathsf{pgmod}(E, N))$ (Theorem 19.13). We will also see that the vanishing locus of $\mathrm{cone}(\beta)$ in the spectrum $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E, N))$ is, as usual, a copy of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{mod}(E))$ induced by unf (Proposition 19.3).

19.2 Remark. We illustrate the significance of this section using the following rough picture, where we depict $\mathrm{open}(\mathrm{radf}(kQ_N))$ and $\mathrm{open}(\mathrm{cone}(\beta))$ separately:



Since we are considering $\mathrm{radf}(kQ_N)$ instead of $\mathrm{radf}(kE)$, we shift our colors somewhat. We keep the cyan $\mathrm{open}(\mathrm{cone}(\beta))$, but now we use orange for $\mathrm{open}(\mathrm{radf}(kQ_N))$ and purple for $\mathrm{supp}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta))$. In this picture, we have included the vanishing locus $\mathrm{open}(\mathrm{radf}(kE))$ in its usual yellow. Note that $\mathrm{open}(\mathrm{radf}(kQ_N)) \subset \mathrm{open}(\mathrm{radf}(kE))$ because $\mathrm{radf}(kQ_N) \otimes \mathrm{radf}(kN) \simeq \mathrm{radf}(kE)$.

Let us first discuss the extreme cases. When $N = E$ so that $Q_N = 1$, the orange and cyan regions agree, so this section entirely determines the points in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, E))$. However, when $N = 1$ so that $\operatorname{radf}(kQ_N) = \operatorname{radf}(kE)$, this section contains nothing new because in Part II we have already determined $\operatorname{open}(\operatorname{cone}(\beta))$ and $\operatorname{supp}(\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta))$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$. Recall that in this case we were unable to determine anything in $\operatorname{open}(\operatorname{radf}(kE))$ outside of the cyan region.

In the non-extreme cases where $1 < N < E$, the spectrum of the stable category $\operatorname{open}(\operatorname{radf}(kE))$ is both nonempty and is larger than $\operatorname{open}(\operatorname{radf}(kQ_N))$ because $\langle \operatorname{radf}(kE) \rangle \subsetneq \langle \operatorname{radf}(kQ_N) \rangle$. The significance of this section is that now we are able to determine something in $\operatorname{open}(\operatorname{radf}(kE))$ outside of the cyan region, namely the slice of $\operatorname{open}(\operatorname{radf}(kE))$ marked by a double-exclam that invades the purple region. This may seem like an empty accomplishment because these double-exclam slices only live in the spectra $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$ of the relative subcategories, but we will see in the next section that in fact these double-exclam slices leak into the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$ of our original tt-category.

We begin by deducing as much as we can from our results from Part II. This will let us determine the vanishing locus of $\operatorname{cone}(\beta)$ and the support of $\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta)$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$.

19.3 Proposition. *The tt-functor* $\operatorname{unf}\colon \mathrm{D_b}(\mathsf{pfmod}(E, N)) \twoheadrightarrow \mathrm{D_b}(\mathsf{mod}(E))$ *induces an equivalence*

$$\overline{\operatorname{unf}}\colon \frac{\mathrm{D_b}(\mathsf{pfmod}(E, N))}{\langle \operatorname{cone}(\beta) \rangle} \xrightarrow{\cong} \mathrm{D_b}(\mathsf{mod}(E)).$$

Geometrically, $\operatorname{Spc}(\overline{\operatorname{unf}})$ *induces a homeomorphism*

$$\operatorname{Spc}(\overline{\operatorname{unf}})\colon \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E)) \xrightarrow{\cong} \operatorname{open}(\operatorname{cone}(\beta)) \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N)).$$

Proof. The functor $\overline{\operatorname{unf}}$ is fully faithful because it is the composite

$$\overline{\operatorname{unf}}\colon \frac{\mathrm{D_b}(\mathsf{pfmod}(E, N))}{\langle \operatorname{cone}(\beta) \rangle} \overset{\overline{\operatorname{incl}_N^1}}{\rightarrowtail} \frac{\mathrm{D_b}(\mathsf{pfmod}(E))}{\langle \operatorname{cone}(\beta) \rangle} \xrightarrow{\cong} \mathrm{D_b}(\mathsf{mod}(E)).$$

Here $\overline{\mathrm{incl}_N^1}$ is the Verdier quotient of $\mathrm{incl}_N^1\colon\ \mathrm{D_b}(\mathsf{pfmod}(E,N)) \rightarrowtail \mathrm{D_b}(\mathsf{pfmod}(E))$ at $\mathrm{cone}(\beta)$. Since incl_N^1 is fully faithful (Proposition 15.25), so is the Verdier quotient at $\mathrm{cone}(\beta)$ (Proposition 8.9). The equivalence in the composite is (10.10).

It follows that it is essentially surjective: its essential image is triangulated because it is fully faithful, and the twists of the unit generate $\mathrm{D_b}(\mathsf{mod}(E))$ as a triangulated subcategory (Example 6.43). $\square$

19.4 Proposition. *The tt-functor*

$$\mathrm{ung\,gr\,Res}_{Q_N}\colon\ \mathrm{D_b}(\mathsf{pfmod}(E,N)) \xrightarrow{\mathrm{Res}_{Q_N}} \mathrm{D_b}(\mathsf{pfmod}(Q_N)) \xrightarrow{\mathrm{ung\,gr}} \mathrm{D_b}(\mathsf{pmod}(Q_N))$$

induces a homeomorphism

$$\mathrm{Spc}(\mathrm{ung\,gr\,Res}_{Q_N})\colon\ \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pmod}(Q_N)) \xrightarrow{\cong} \mathrm{supp}(\mathrm{radf}(kE)\otimes\mathrm{cone}(\beta)) \tag{19.5}$$

with the support of $\mathrm{radf}(kE)\otimes\mathrm{cone}(\beta)$ *in* $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,N))$.

Proof. We may easily reduce to showing (19.5) is a surjection. Indeed (recall (15.3)),

$$\begin{tikzcd} \mathrm{D_b}(\mathsf{pfmod}(E,N)) \arrow[r, two heads, "\mathrm{Res}_{Q_N}"'] & \mathrm{D_b}(\mathsf{pfmod}(Q_N)) \arrow[l, bend right, "(\sigma_1^N)^*"'] \arrow[r, two heads, "\mathrm{ung\,gr}"'] & \mathrm{D_b}(\mathsf{pmod}(Q_N)) \arrow[l, bend right, "\mathrm{fz}"'] \end{tikzcd}$$

exhibits a section of $\mathrm{ung\,gr\,Res}_{Q_N}$, and the object

$$\mathrm{ung\,gr\,Res}_{Q_N}(\mathrm{radf}(kE)\otimes\mathrm{cone}(\beta)) \in \mathrm{D_b}(\mathsf{pmod}(Q_N))$$

has full support, for example by (11.6). See Method 6.10(i) if necessary.

To see that (19.5) is a surjection, consider the following commutative diagram of tt-categories:

$$\begin{tikzcd} \mathrm{D_b}(\mathsf{pmod}(E)) & \mathrm{D_b}(\mathsf{pmod}(Q_N)) \arrow[l, tail, "(\sigma_1^N)^*"'] \\ \mathrm{D_b}(\mathsf{pfmod}(E)) \arrow[u, two heads, "\mathrm{ung\,gr}"] & \mathrm{D_b}(\mathsf{pfmod}(E,N)), \arrow[l, tail, "\mathrm{incl}_N^1"] \arrow[u, two heads, "\mathrm{ung\,gr\,Res}_{Q_N}"'] \end{tikzcd}$$

where incl_N^1 is fully faithful by Proposition 15.25. On spectra, we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(E)) \arrow[r, two heads, "\operatorname{Spc}((\sigma_1^N)^*)"] \arrow[d, "\operatorname{Spc}(\mathrm{ung\,gr})"', "\cong"] & \operatorname{Spc} \mathrm{D_b}(\mathsf{pmod}(Q_N)) \arrow[d, "\operatorname{Spc}(\mathrm{ung\,gr\,Res}_{Q_N})"] \\
\operatorname{supp}(\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta)) \arrow[r, two heads, "\operatorname{Spc}(\mathrm{incl}_N^1)"] \arrow[d, phantom, "\cap"] & \operatorname{supp}(\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta)) \arrow[d, phantom, "\cap"] \\
\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E)) \arrow[r, two heads, "\operatorname{Spc}(\mathrm{incl}_N^1)"] & \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,N)),
\end{tikzcd}$$

where $\operatorname{Spc}(\mathrm{ung\,gr})$ induces the depicted homeomorphism by Theorem 11.5. The top square shows that $\operatorname{Spc}(\mathrm{ung\,gr\,Res}_{Q_N})$ surjects $\operatorname{supp}(\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta))$ (recall Recollection 6.5 if necessary), as we wanted. □

19.6 Proposition. *The tt-functor* $\Upsilon\colon \mathrm{D_b}(\mathsf{pfmod}(E,N)) \to \mathrm{D_b}(\mathsf{fmod}(E,N))$ *induces an equivalence*

$$\overline{\Upsilon}\colon \frac{\mathrm{D_b}(\mathsf{pfmod}(E,N))}{\langle \mathrm{fz\,kos}_{Q_N}(1)\rangle} \longrightarrow \mathrm{D_b}(\mathsf{fmod}(E,N))$$

of tt-categories. Geometrically, $\operatorname{Spc}(\overline{\Upsilon})$ *induces a homeomorphism*

$$\operatorname{Spc}(\overline{\Upsilon})\colon \operatorname{Spc} \mathrm{D_b}(\mathsf{fmod}(E,N)) \xrightarrow{\cong} \operatorname{open}(\mathrm{fz\,kos}_{Q_N}(1)) \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,N)).$$

Proof. This is analogous to the proof of Proposition 18.2. Spelling it out, we invoke the theory of finite resolutions (Proposition 7.13). The twists of the unit generate $\mathrm{D_b}(\mathsf{fmod}(E,N))$ as a triangulated subcategory (Example 14.20), and in Lemma 15.26(i) we exhibited the required resolutions of the unit. Note that we may identify the $\mathsf{fmod}(E,N)$-acyclics $\ker(\Upsilon) \subset \mathrm{D_b}(\mathsf{pfmod}(E,N))$ with $\langle \mathrm{fz\,kos}_{Q_N}(1)\rangle$ using Lemma 14.21(i). □

Recall that our goals for this section are to determine $\operatorname{supp}(\operatorname{radf}(kQ_N) \otimes \operatorname{cone}(\beta))$ and $\operatorname{open}(\operatorname{cone}(\beta))$ in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E,N))$. We have already accomplished the latter (Proposition 19.3), and we have made progress on the former by determining $\operatorname{supp}(\operatorname{radf}(kE) \otimes \operatorname{cone}(\beta))$ (Proposition 19.4). Thus, it remains to determine the support of the object $\operatorname{radf}(kQ_N) \otimes \operatorname{cone}(\beta)$ in the spectrum of the stable category $\operatorname{st}(\mathsf{pfmod}(E,N))$, as illustrated by the double-exclam slice in Remark 19.2. We will do this now, starting with three baby steps in the next three lemmas. Recall our notations in Example 15.30, Convention 15.13, and Notation 3.29.

19.7 Lemma. *Let $A \in \mathsf{pfmod}(E)$, and let $H \leqslant Q_N$. Let*

$$g\colon\ \mathrm{gz}\,k(Q_N/H) \longrightarrow \mathrm{gr}(A)$$

be a morphism in $\mathsf{pgmod}(E)$. Then there exists a morphism

$$f\colon\ \mathrm{fz}\,k(Q_N/H) \otimes \mathrm{radf}(kH) \otimes \mathrm{radf}(kN) \longrightarrow A$$

in $\mathsf{pfmod}(E)$ such that $\mathrm{gr}(f)^0 = g^0$ or, in other words, such that the following diagram in $\mathsf{pgmod}(E)$ commutes:

$$\begin{tikzcd} \mathrm{gz}\,k(Q_N/H) \arrow[d, hook, "1\otimes\iota\otimes\iota"'] \arrow[dr, "g"] & \\ \mathrm{gz}\,k(Q_N/H) \otimes \mathrm{gr}\,\mathrm{radf}(kH) \otimes \mathrm{gr}\,\mathrm{radf}(kN) \arrow[r, "\mathrm{gr}(f)"'] & \mathrm{gr}(A). \end{tikzcd}$$

Proof. In the statement of Proposition 9.6, we take the group G and its normal subgroup $N \trianglelefteqslant G$ there to be, respectively, our E and its subgroup $HN \leqslant E$ here. □

19.8 Lemma. *Let $A \in \mathsf{pfmod}[Q_N, N]$, and let $H \leqslant Q_N$. Let*

$$g\colon\ \mathrm{gz}\,k(Q_N/H) \otimes \mathrm{gr}\,\mathrm{radf}(kN) \longrightarrow \mathrm{gr}(A)$$

be a morphism in $\mathsf{pgmod}[Q_N, N]$. Then there exists a morphism

$$f\colon\ \mathrm{fz}\,k(Q_N/H) \otimes \mathrm{radf}(kH) \otimes \mathrm{radf}(kN) \longrightarrow A$$

in $\mathsf{pfmod}[Q_N, N]$ such that the following diagram in $\mathsf{pgmod}[Q_N, N]$ commutes:

$$\begin{tikzcd} \mathrm{gz}\,k(Q_N/H) \otimes \mathrm{gr}\,\mathrm{radf}(kN) \arrow[d, hook, "1\otimes\iota\otimes 1"] \arrow[dr, "g"] & \\ \mathrm{gz}\,k(Q_N/H) \otimes \mathrm{gr}\,\mathrm{radf}(kH) \otimes \mathrm{gr}\,\mathrm{radf}(kN) \arrow[r, "\mathrm{gr}(f)"'] & \mathrm{gr}(A). \end{tikzcd} \tag{19.9}$$

Proof. Consider the morphism $g'\colon \operatorname{gz} k(Q_N/H) \to \operatorname{gr}(A)$ in $\mathsf{pgmod}(E)$ defined by $(g')^0 = g^0$. By Lemma 19.7, there exists a morphism $f\colon \operatorname{fz} k(Q_N/H) \otimes \operatorname{radf}(kH) \otimes \operatorname{radf}(kN) \to A$ in $\mathsf{pfmod}(E)$ such that $\operatorname{gr}(f)^0 = (g')^0$ in $\mathsf{pgmod}(E)$. Observe that in fact f is in the full subcategory $\mathsf{pfmod}[Q_N, N] \subset \mathsf{pfmod}(E)$. Thus, we only need to verify that f makes (19.9) commute. The indecomposable module $\operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kN)$ over the monoid $\operatorname{gz} kQ_N \otimes \operatorname{gr}\operatorname{radf}(kN)$ is generated by the element $1 \otimes 1$ in grading 0 (Example 15.30(iv)), so it suffices to verify that (19.9) commutes for this generator. This follows from $g^0 = (g')^0 = \operatorname{gr}(f)^0$. $\square$

19.10 Lemma. *Let $f\colon A_1 \to A_2$ be a morphism in $\mathsf{pfmod}[Q_N, N]$. Consider the following statements (compare with Lemma 3.12 with Lemma 11.2):*

(i) *We have $\operatorname{gr}(f) = 0$ in $\operatorname{st}(\mathsf{pgmod}[Q_N, N])$.*

(ii) *The morphism f factors through $\beta\colon A_2(-1) \to A_2$ in $\operatorname{st}(\mathsf{pfmod}[Q_N, N])$:*

$$\begin{tikzcd} & A_2(-1) \arrow[d, "\beta"] \\ A_1 \arrow[ur, dotted] \arrow[r, "f"'] & A_2. \end{tikzcd}$$

(iii) *The morphism f factors through $\beta\colon A_1 \to A_1(1)$ in $\operatorname{st}(\mathsf{pfmod}[Q_N, N])$:*

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "\beta"'] & A_2 \\ A_1(1). \arrow[ur, dotted] & \end{tikzcd}$$

The statements (ii) and (iii) are equivalent, so we simply say that f factors through β. Moreover, (ii) and (iii) each imply (i) but not conversely.

Proof. Suppose (ii). This means there exists a morphism $f'\colon A_1 \to A_2(-1)$ in $\mathsf{pfmod}[Q_N, N]$ such that $f - \beta f'$ factors through a projective $P \in \mathsf{pfmod}[Q_N, N]$. Consider the following commutative diagram in $\mathsf{pfmod}[Q_N, N]$, recalling if necessary that $\beta\colon A_2 \to A_2(1)$ is a mono (Remark 3.11):

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "\beta"'] & A_2 \arrow[d, "\beta"] \\ A_1(1) \arrow[ur, "f'(1)"] \arrow[r, "f(1)"'] & A_2(1). \end{tikzcd}$$

We deduce that $f - f'(1)\beta = (f(1) - \beta f'(1))(-1)$ factors through $P(-1)$, hence (iii). The reverse implication is almost identical. The rest follows from Corollary 15.22 and Remark 15.23. $\square$

19.11 Proposition. *The tt-functor*

$$\mathrm{gr}\colon \mathrm{st}(\mathsf{pfmod}[Q_N, N]) \longrightarrow \mathrm{st}(\mathsf{pgmod}[Q_N, N])$$

is faithful on $\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)$.

Proof. Let $f\colon \mathbb{1} \to A$ be a morphism in $\mathsf{pfmod}[Q_N, N]$ such that $\mathrm{gr}(f)\colon \mathbb{1} \to \mathrm{gr}(A)$ is zero in $\mathrm{st}(\mathsf{pgmod}[Q_N, N])$. It suffices (Lemma 6.9$(ii)$ and Lemma 11.2) to show that the morphism

$$\mathrm{radf}(kQ_N)^\vee \otimes \mathrm{radf}(kQ_N) \xrightarrow{\mathrm{ev}_{\mathrm{radf}(kQ_N)}} \mathbb{1} \xrightarrow{f} A$$

in $\mathsf{pfmod}[Q_N, N]$ factors through β (recall Lemma 19.10) in $\mathrm{st}(\mathsf{pfmod}[Q_N, N])$. Recalling (Corollary 9.17) that $\mathrm{radf}(kQ_N)^\vee = \mathrm{radf}(kQ_N)(-\ell_{Q_N} + 1)$, we deduce using Lemma 4.10 that $\mathrm{radf}(kQ_N)^\vee \otimes \mathrm{radf}(kQ_N)$ is a sum of twists of $\mathrm{radf}(kQ_N)$ where exactly one of the twists is 0 and the others are negative. The components of the morphism on the negative twists already factor through β because we are passing through the unit $\mathbb{1}$, so we are reduced to showing that the morphism

$$\mathrm{radf}(kQ_N) \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{f} A$$

in $\mathsf{pfmod}[Q_N, N]$ factors through β in $\mathrm{st}(\mathsf{pfmod}[Q_N, N])$. This is what we now set out to prove.

Let us unravel our hypothesis that $\mathrm{gr}(f)\colon \mathbb{1} \to \mathrm{gr}(A)$ is zero in $\mathrm{st}(\mathsf{pgmod}[Q_N, N])$. This means the morphism $\mathrm{gr}(f)\colon \mathbb{1} \to \mathrm{gr}(A)$ in $\mathsf{pgmod}[Q_N, N]$ factors through a projective P:

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\eta"'] \arrow[d, "\mathrm{gr}(f)"] \\ P \arrow[r, "g"'] & \mathrm{gr}(A). \end{tikzcd}$$

By Corollary 15.22, the projective P is a sum of twists of objects of the form $\operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kN)$ where $H \leqslant Q_N$. Thus, by linearity and by Example 15.30(iv), we may assume P is $\operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kN)(-\ell_N+1)$ for some subgroup $H \leqslant Q_N$ and that η is the morphism (15.31). This produces the triangle on the right-hand side of the following diagram in $\mathsf{pgmod}[Q_N, N]$, which extensively uses our notations in Example 15.30:

$$\begin{tikzcd}
\operatorname{gr}\operatorname{radf} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kH) \otimes \mathbb{1} \arrow[r, two heads, "\pi\otimes\pi\otimes 1"] \arrow[d, two heads, "\pi\otimes 1\otimes 1"'] & \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \arrow[ddl, tail, "\eta\otimes 1\otimes\eta"] \arrow[ddd, "\operatorname{gr}(f)"] \\
\mathbb{1} \otimes \operatorname{gr}\operatorname{radf}(kH) \otimes \mathbb{1} \arrow[d, "\eta\otimes\pi\otimes\eta"'] & \\
\operatorname{gz} k(Q_N/H) \otimes \mathbb{1} \otimes \operatorname{gr}\operatorname{radf}(kN)(-\ell_N+1) \arrow[d, hook, "1\otimes\iota\otimes 1"'] \arrow[dr, "g"] & \\
\operatorname{gz} k(Q_N/H) \otimes \operatorname{gr}\operatorname{radf}(kH) \otimes \operatorname{gr}\operatorname{radf}(kN)(-\ell_N+1) \arrow[r, two heads, "\operatorname{gr}(f')"'] & \operatorname{gr}(A).
\end{tikzcd} \tag{19.12}$$

By Lemma 19.8, there exists a morphism

$$f'\colon \operatorname{fz} k(Q_N/H) \otimes \operatorname{radf}(kH) \otimes \operatorname{radf}(kN)(-\ell_N+1) \longrightarrow A$$

in $\mathsf{pfmod}[Q_N, N]$ making the bottom triangle commute. We then easily fill in the upper-left triangle. We define f'' to be the composite

$$\begin{tikzcd}
\operatorname{radf} k(Q_N/H) \otimes \operatorname{radf}(kH) \otimes \mathbb{1} \arrow[d, two heads, "\varepsilon\otimes 1\otimes 1"] & \\
\mathbb{1} \otimes \operatorname{radf}(kH) \otimes \mathbb{1} \arrow[d, tail, "\eta\otimes 1\otimes\eta"] & \\
\operatorname{fz} k(Q_N/H) \otimes \operatorname{radf}(kH) \otimes \operatorname{radf}(kN)(-\ell_N) \arrow[r, "f'"'] & A
\end{tikzcd}$$

so that $\operatorname{gr}(f'')$ is the composite along the left and bottom edges in (19.12), using Example 15.30(i) and Example 15.30(ii). Crucially, our diagram is in $\mathsf{pfmod}[Q_N, N]$ (as opposed to the stable category), so we may appeal to Lemma 3.12 to see that $f\varepsilon - f''$ factors through β.

Finally, we define f''' to be the factoring morphism: $\beta f''' = f\varepsilon - f''$. Then $f\varepsilon - \beta f''' = f''$ factors through a projective, hence is zero in $\operatorname{st}(\mathsf{pfmod}[Q_N, N])$. We conclude that $f\varepsilon = \beta f'''$ in $\operatorname{st}(\mathsf{pfmod}[Q_N, N])$ factors through β, which is what we set out to prove. $\square$

Let us make sense of these two gr functors by merging them into one.

19.13 Theorem. *The tt-functor*

$$\mathrm{gr}\colon \mathrm{D_b}(\mathsf{pfmod}[Q_N, N]) \longrightarrow \mathrm{D_b}(\mathsf{pgmod}[Q_N, N])$$

on spectra induces a homeomorphism

$$\mathrm{Spc}(\mathrm{gr})\colon \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pgmod}[Q_N, N]) \xrightarrow{\cong} \mathrm{supp}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta))$$

with the support of $\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)$ *in* $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}[Q_N, N])$.

Proof. Recall (from (15.12)) that crucially gr admits trf as a section. Moreover, the object $\mathrm{gr}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta))$ has full support in $\mathrm{D_b}(\mathsf{pgmod}[Q_N, N])$, as we discussed in Example 15.30(*iii*). We are therefore reduced (Recollection 6.5(*i*) and Recollection 6.6(*ii*)) to showing that Spc(gr) surjects the support $\mathrm{supp}(\mathrm{radf}(Q_N) \otimes \mathrm{cone}(\beta))$.

We divide and conquer (Method 6.10) this support $\mathrm{supp}(\mathrm{radf}(Q_N) \otimes \mathrm{cone}(\beta))$ using the object $\mathrm{radf}(kE) \in \mathrm{D_b}(\mathsf{pfmod}[Q_N, N])$. On the one hand, we have

$$\mathrm{supp}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)) \cap \mathrm{supp}(\mathrm{radf}(kE)) = \mathrm{supp}(\mathrm{radf}(kE) \otimes \mathrm{cone}(\beta)),$$

and Proposition 19.4 ensures that Spc(gr) surjects this. On the other hand, we have

$$\mathrm{supp}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)) \cap \mathrm{open}(\mathrm{radf}(kE)),$$

which is the support of $\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)$ in the stable category, and Proposition 19.11 asserts that Spc(gr) surjects this. ∎

We have now determined $\mathrm{open}(\mathrm{cone}(\beta))$ and $\mathrm{supp}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta))$. Let us take a moment to discuss specializations in the extreme case $N = E$.

19.14 Remark. Consider the following diagram:

$$\begin{tikzcd}
\dot{\mathbb{P}}_k^{r+s-1} \arrow[ddd, "\mathcal{P}"', "\cong"] & & \dot{\mathbb{P}}_k^{r+s-1} \arrow[d, "\cong"', "\mathcal{P}"] \\
& & \operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E, N)) \arrow[d, two heads, "\operatorname{Spc}(\mathrm{ung})"] \\
& & \operatorname{Spc} \mathrm{D_b}(\mathsf{gmod}(E, N)) \arrow[d, tail, "\operatorname{Spc}(\Psi^1_N) = \operatorname{Spc}(\Upsilon)"] \\
\operatorname{Spc} \mathrm{D_b}(\mathsf{mod}(E)) \arrow[d, "\operatorname{Spc}(\mathrm{unf})"', "\cong"] & & \operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N)) \arrow[d, "\cong"', "\operatorname{Spc}(\mathrm{gr})"] \\
\operatorname{open}(\operatorname{cone}(\beta)) \arrow[dr, hook] & & \operatorname{supp}(\operatorname{radf}(kQ_N) \otimes \operatorname{cone}(\beta)) \arrow[dl, hook'] \\
& \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N)). &
\end{tikzcd}$$

Let us respectively denote by L and H (for lower and higher) the composite on the left-hand and right-hand side. Geometrically, L is simply the embedding of $\operatorname{open}(\operatorname{cone}(\beta))$ into $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$, and H produces points in the vanishing locus of $\mathsf{gmod}(E, N)$-acyclics in $\operatorname{Spc} \mathrm{D_b}(\mathsf{pgmod}(E, N))$ and then embeds them into $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$ via $\operatorname{Spc}(\mathrm{gr})$. For the unique closed point $0 \in \dot{\mathbb{P}}_k^{r+s-1}$, we see that $L(0)$ specializes to $H(0)$ because $H(0) = \ker(\Upsilon\, \mathrm{gr}) \subset \ker(\mathrm{unf}) = L(0)$ (Lemma 6.37). For the non-closed points $\mathbb{P}_k^{r+s-1} \subset \dot{\mathbb{P}}_k^{r+s-1}$, things are not so clear. On these points, $\mathcal{P}$ is defined in terms of π-points (Proposition 16.5) which lend themselves naturally to stable categories, but H does not factor through stable categories because the projectives in $\mathsf{pgmod}(E, N)$ are generated by $\operatorname{gr} \operatorname{radf}(kN)$ (Proposition 15.17) whereas $\operatorname{radf}(kN)$ is not projective in $\mathsf{pfmod}(E, N)$ (Corollary 15.22). For this reason we consider the extreme case $N = E$, where the composite H does factor through the stable categories:

$$\begin{tikzcd}
\mathbb{P}_k^{r+s-1} \arrow[ddd, "\mathcal{P}"', "\cong"] & & \mathbb{P}_k^{r+s-1} \arrow[d, "\cong"', "\mathcal{P}"] \\
& & \operatorname{Spc}\operatorname{st}(\mathsf{mod}(E,E)) \arrow[d, two heads, "\operatorname{Spc}(\mathrm{ung})"] \\
& & \operatorname{Spc}\operatorname{st}(\mathsf{gmod}(E,E)) \arrow[d, tail, "\operatorname{Spc}(\Psi_E^1)=\operatorname{st}(\Upsilon)"] \\
\operatorname{Spc}\operatorname{st}(\mathsf{mod}(E)) \arrow[d, "\operatorname{Spc}(\mathrm{unf})"', "\cong"] & & \operatorname{Spc}\operatorname{st}(\mathsf{pgmod}(E,E)) \arrow[d, "\cong"', "\operatorname{Spc}(\mathrm{gr})"] \\
\operatorname{open}(\operatorname{cone}(\beta)) \arrow[dr, tail] & & \operatorname{supp}(\operatorname{radf}(kE)\otimes\operatorname{cone}(\beta)) \arrow[dl, tail] \\
& \operatorname{Spc}\operatorname{st}(\mathsf{pfmod}(E,E)). &
\end{tikzcd}$$

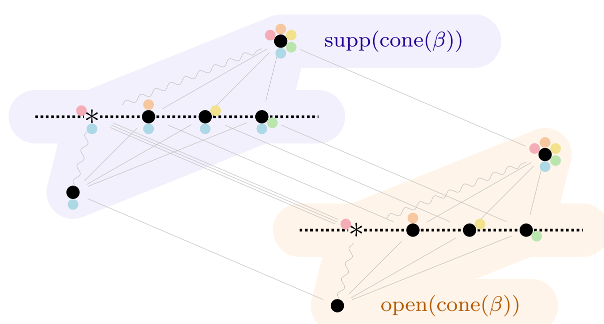
In this case we may argue that $L([a:b]_{K/k})$ specializes to $H([a:b]_{K/k})$.

We now illustrate the region we have determined for our running example.

19.15 Running Example. We return to our Running Example 1.13. Recall (Remark 19.2) that the case $N=1$ is what we did in Part II and that in the case $N=E$, we have found all of the points. We now apply Theorem 19.13 and Proposition 19.3.

For $N=E$, we obtain the following picture:

$\operatorname{Spc}\mathrm{D_b}(\mathsf{pfmod}(E,E))$



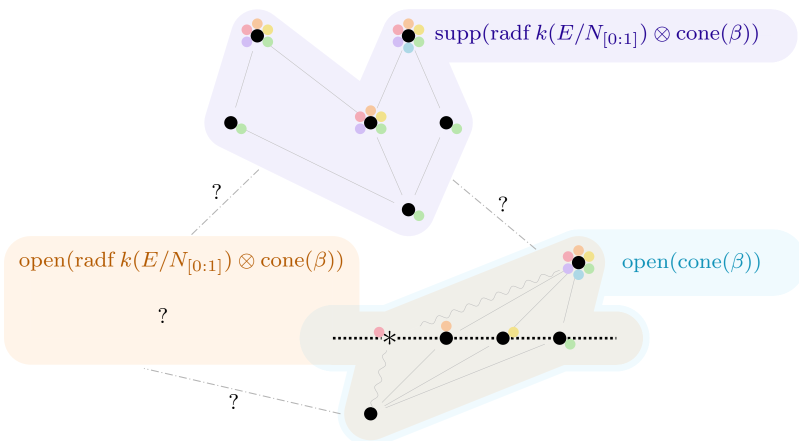
● $\operatorname{supp}(\operatorname{radf}(C_{[a:b]}))$ ● $\operatorname{supp}(\operatorname{radf} k(E/N_{[1:0]}))$ ● $\operatorname{supp}(\operatorname{radf} k(E/N_{[1:1]}))$

● $\operatorname{supp}(\operatorname{gap}^1\operatorname{radf}(kE))$ ● $\operatorname{supp}(\operatorname{radf} k(E/N_{[0:1]}))$.

The picture is claiming that we have a doubling of $\dot{\mathbb{P}}^1_k$. This is because Remark 19.14 shows that each of the lower points specializes to its corresponding higher point, and conversely the depicted supports show that it specializes nowhere else (besides the 0 points).

For N with rank 1, say $N = N_{[0:1]} = \langle y \rangle$, we obtain the following:

$$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N_{[0:1]}))$$



- $\mathrm{supp}(\mathrm{radf}(C_{[a:b]}))$
- $\mathrm{supp}(\mathrm{radf}\, k(E/N_{[1:0]}))$
- $\mathrm{supp}(\mathrm{radf}\, k(E/N_{[1:1]}))$
- $\mathrm{supp}(\mathrm{radf}\, kE)$
- $\mathrm{supp}(\mathrm{radf}_{N_{[0:1]}}(kE))$
- $\mathrm{supp}(\mathrm{radf}\, k(E/N_{[0:1]}))$.

20 Reduction to the relative subcategories

20.1 Notation. In this section, we let $N < E$ (Notation 2.1) be a proper subgroup.

In this section we determine the vanishing locus of $\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta)$ in the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$. In particular, we show that this vanishing locus $\mathrm{open}(\mathrm{radf}(kQ_N) \otimes \mathrm{cone}(\beta))$ is covered by smaller vanishing loci (Proposition 20.5), each of which is homeomorphic to a vanishing locus in a smaller relative subcategory (Corollary 20.16). Since we have already determined the spectrum $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, E))$ of the smallest relative subcategory (Remark 19.2), this lets us determine $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$ by induction.

We begin by introducing these smaller vanishing loci. Recall (Recollection 6.12) that in a rigid tt-category, an object is tt-equivalent to any tensor power of itself, for example $\langle \mathrm{cone}(\beta)\rangle = \langle \mathrm{cone}(\beta)^{\otimes m}\rangle$ for any $m \geq 1$. We will use this freely throughout this section.

20.2 Notation. Given a subgroup $L \in \mathrm{sub}(E)$ such that $N \leqslant L$, we denote by

$$\mathrm{sub}(E,N) \subset \mathrm{sub}(E) \text{ the sublattice of subgroups } H \in \mathrm{sub}(E) \text{ where } N \leqslant H$$
$$\mathrm{sub}(E,N,L) \subset \mathrm{sub}(E) \text{ the subset of subgroups } H \in \mathrm{sub}(E) \text{ where } N \leqslant H \not\geqslant L.$$

20.3 Notation. Let $L \in \mathrm{sub}(E,N)$. For $K \in \mathrm{sub}(E)$, which we will always take to be $K = N$ or $K = L$, we define the following finitely generated tt-ideal in $\mathrm{D_b}(\mathsf{pfmod}(E,K))$ (recall Example 15.24 if necessary):

$$\mathcal{U}_K(N,L) = \langle \mathrm{radf}\, k(E/H) \otimes \mathrm{cone}(\beta) \mid H \in \mathrm{sub}(E,N,L)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E,K)).$$

20.4 Running Example. We return to our Running Example 1.13. We have

$$\mathcal{U}_{N_{[1:0]}}(N_{[1:0]},E) = \langle \mathrm{radf}\, k(E/N_{[1:0]}) \otimes \mathrm{cone}(\beta)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E,N_{[1:0]}))$$
$$\mathcal{U}_{N_{[1:1]}}(N_{[1:1]},E) = \langle \mathrm{radf}\, k(E/N_{[1:1]}) \otimes \mathrm{cone}(\beta)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E,N_{[1:1]}))$$
$$\mathcal{U}_{N_{[0:1]}}(N_{[0:1]},E) = \langle \mathrm{radf}\, k(E/N_{[0:1]}) \otimes \mathrm{cone}(\beta)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E,N_{[0:1]}))$$

and

$$\mathcal{U}_1(1,N_{[1:0]}) = \left\langle \begin{matrix} \mathrm{radf}\, k(E/N_{[1:1]}) \otimes \mathrm{cone}(\beta), \\ \mathrm{radf}\, k(E/N_{[0:1]}) \otimes \mathrm{cone}(\beta) \end{matrix} \right\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E)),$$
$$\mathcal{U}_1(1,N_{[1:1]}) = \left\langle \begin{matrix} \mathrm{radf}\, k(E/N_{[1:0]}) \otimes \mathrm{cone}(\beta), \\ \mathrm{radf}\, k(E/N_{[0:1]}) \otimes \mathrm{cone}(\beta) \end{matrix} \right\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E)),$$
$$\mathcal{U}_1(1,N_{[0:1]}) = \left\langle \begin{matrix} \mathrm{radf}\, k(E/N_{[1:0]}) \otimes \mathrm{cone}(\beta), \\ \mathrm{radf}\, k(E/N_{[1:1]}) \otimes \mathrm{cone}(\beta) \end{matrix} \right\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E)).$$

We also have the simplified expression

$$\mathcal{U}_1(1,E) = \left\langle \begin{matrix} \mathrm{radf}\, k(E/N_{[1:0]}) \otimes \mathrm{cone}(\beta), \\ \mathrm{radf}\, k(E/N_{[1:1]}) \otimes \mathrm{cone}(\beta), \\ \mathrm{radf}\, k(E/N_{[0:1]}) \otimes \mathrm{cone}(\beta) \end{matrix} \right\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E))$$

because the tensor product of any two of the three objects $\operatorname{radf} k(E/N_{[1:0]})$, $\operatorname{radf} k(E/N_{[1:1]})$, and $\operatorname{radf} k(E/N_{[0:1]})$ is $\operatorname{radf}(kE)$.

20.5 Proposition. *Let $K \in \operatorname{sub}(E)$, and consider the functor (recall Notation 6.13)*

$$\mathcal{U}_K(N,-)\colon \operatorname{sub}(E,N) \longrightarrow \operatorname{ideals}(\mathrm{D_b}(\mathsf{pfmod}(E,K))).$$

(i) It preserves joins: if $L_1, L_2 \in \operatorname{sub}(E,N)$, then $\mathcal{U}_K(N, L_1L_2)$ is the tt-ideal generated by $\mathcal{U}_K(N,L_1)$ and $\mathcal{U}_K(N,L_2)$.

(ii) It may not necessarily preserve meets, but it does satisfy

$$\langle \operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)\rangle = \bigcap_{L>N} \mathcal{U}_K(N,L).$$

Consequently

$$\operatorname{open}(\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)) = \operatorname*{colim}_{L>N} \operatorname{open}(\mathcal{U}_K(N,L)).$$

Proof. Note (Recollection 6.4)

$$\mathcal{U}_K(N,L) = \left\langle \bigoplus_{H\in \operatorname{sub}(E,N,L)} \operatorname{radf} k(E/H) \otimes \operatorname{cone}(\beta) \right\rangle. \tag{20.6}$$

For (i), the direct sum

$$\bigoplus_{H_1\in \operatorname{sub}(E,N,L_1)} \operatorname{radf} k(E/H_1) \otimes \operatorname{cone}(\beta) \oplus \bigoplus_{H_2\in \operatorname{sub}(E,N,L_2)} \operatorname{radf} k(E/H_2) \otimes \operatorname{cone}(\beta)$$

generates the same tt-ideal as

$$\bigoplus_{H\in \operatorname{sub}(E,N,L_1L_2)} \operatorname{radf} k(E/H) \otimes \operatorname{cone}(\beta)$$

because $H \not\geq L_1L_2$ if and only if $H \not\geq L_1$ or $H \not\geq L_2$, using that E is elementary abelian. This proves (i) by (20.6).

For (ii), we compute

$$\bigcap_{L>N} \operatorname{supp}(\mathcal{U}_K(N,L)) = \bigcap_{L>N} \operatorname{supp}\left(\bigoplus_{H\in \mathrm{sub}(E,N,L)} \operatorname{radf} k(E/H) \otimes \operatorname{cone}(\beta)\right)$$
$$= \operatorname{supp}\left(\bigotimes_{L>N} \bigoplus_{H\in \mathrm{sub}(E,N,L)} \operatorname{radf} k(E/H) \otimes \operatorname{cone}(\beta)\right).$$

Consider expanding this tensor product of sums into a sum of tensor products. An arbitrary term in the expanded sum is of the form

$$\bigotimes_{L>N} \operatorname{radf} k(E/H_L) \otimes \operatorname{cone}(\beta) \tag{20.7}$$

where for each $L > N$, the subgroup $H_L \leqslant E$ is such that $H_L \in \mathrm{sub}(E, N, L)$. Since

$$\bigcap_{L>N} H_L = N \tag{20.8}$$

again using that E is elementary abelian, our arbitrary term (20.7) is the tensor product of a sum of twists of $\operatorname{radf} k(E/N)$ (Lemma 4.10) with a tensor power of $\operatorname{cone}(\beta)$. Thus

$$\bigcap_{L>N} \operatorname{supp}(\mathcal{U}_K(N,L)) = \operatorname{supp}(\operatorname{radf} k(E/N) \otimes \operatorname{cone}(\beta)),$$

as claimed. □

20.9 Remark. In Proposition 20.5(ii), it is enough to vary over the subgroups $L > N$ where L/N has rank 1. Indeed, (20.8) only requires such L.

We now show that each of these smaller vanishing loci come from the spectra of smaller subcategories. In fact we will work slightly more generally and consider smaller tt-ideals and hence larger vanishing loci. Compare the following to Notation 20.3.

20.10 Notation. Let $L \in \mathrm{sub}(E, N)$. For $K \leqslant L$, which we will always take to be $K = N$ or $K = L$, we define the following finitely generated tt-ideal in $\mathrm{D_b}(\mathsf{pfmod}(E, K))$ (recall Example 15.24 if necessary):

$$\mathcal{I}_K(N,L) = \langle \operatorname{radf}_L k(E/H) \otimes \operatorname{cone}(\beta) \mid H \in \mathrm{sub}(E,N,L)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E,K)).$$

20.11 Lemma. *Let* $L \in \operatorname{sub}(E,N)$*, and let* $H \in \operatorname{sub}(E,N,L)$*. In the Verdier quotient*

$$\frac{\mathrm{D_b}(\mathsf{pfmod}(E,N))}{\mathcal{I}_N(N,L)},$$

the objects $\operatorname{radf}_L k(E/H)$ *and* $\operatorname{fz} k(E/H)$ *are isomorphic.*

Proof. In this proof we forgo our convention in Notation 3.9 for abbreviating β and instead write our morphisms out fully. Let ℓ be sufficiently large so that the identity morphism on $k(E/H)$ induces a morphism $\operatorname{radf}_L k(E/H) \to \operatorname{fz} k(E/H)(\ell)$ in $\mathsf{pfmod}(E,N)$; for example we may take ℓ to be $\ell_{E/H}$ or, more conservatively, the filtration length of $\operatorname{radf}_L k(E/H)$. Consider the following diagram in the Verdier quotient $\mathrm{D_b}(\mathsf{pfmod}(E,N))/\mathcal{I}_N(N,L)$ where the underlying morphisms are all the identities:

$$\begin{tikzcd}
\operatorname{radf}_L k(E/H)(-\ell) \arrow[r] \arrow[rr, bend left, "\beta^{\otimes\ell}\otimes\operatorname{radf}_L k(E/H)"] & \operatorname{fz} k(E/H) \arrow[r] \arrow[rr, bend right, "\beta^{\otimes\ell}\otimes\operatorname{fz} k(E/H)"'] & \operatorname{radf}_L k(E/H) \arrow[r] & \operatorname{fz} k(E/H)(\ell).
\end{tikzcd} \tag{20.12}$$

Let us show that the labeled morphisms in (20.12) are invertible. We have

$$\operatorname{cone}(\beta^{\otimes\ell}) \otimes \operatorname{radf}_L k(E/H) \in \mathcal{I}_N(N,L) \tag{20.13}$$

since $\langle \operatorname{cone}(\beta^{\otimes\ell})\rangle = \langle \operatorname{cone}(\beta)\rangle$ [Bal18, Proposition 2.10]. From (20.13) we see that

$$\operatorname{cone}(\beta^{\otimes\ell}) \otimes \operatorname{fz} k(E/H) \otimes \operatorname{radf}_L k(E/H) \in \mathcal{I}_N(N,L).$$

This implies

$$\operatorname{cone}(\beta^{\otimes\ell}) \otimes \operatorname{fz} k(E/H) \in \mathcal{I}_N(N,L) \tag{20.14}$$

since Lemma 4.4 says $\operatorname{fz} k(E/H) \otimes \operatorname{radf}_L k(E/H)$ is a direct sum of twists of $\operatorname{fz} k(E/H)$. Now (20.13) and (20.14) show that the labeled morphisms are invertible, as we wanted.

We may now deduce that the middle morphism $\operatorname{fz} k(E/H) \to \operatorname{radf}_L k(E/H)$ in (20.12) is an isomorphism. Indeed, we easily use the labeled morphisms in (20.12) to construct respectively a left and right inverse of the middle morphism, and then of course these inverses are equal. □

20.15 Theorem. *Let $L \in \mathrm{sub}(E,N)$. The fully faithful (Proposition 15.25) tt-functor*

$$\mathrm{incl}_L^N\colon \mathrm{D_b}(\mathsf{pfmod}(E,L)) \rightarrowtail \mathrm{D_b}(\mathsf{pfmod}(E,N))$$

induces an equivalence

$$\overline{\mathrm{incl}_L^N}\colon \frac{\mathrm{D_b}(\mathsf{pfmod}(E,L))}{\mathcal{J}_L(N,L)} \xrightarrow{\cong} \frac{\mathrm{D_b}(\mathsf{pfmod}(E,N))}{\mathcal{J}_N(N,L)},$$

where $\mathcal{J}_N(N,L)$ is defined in Notation 20.10. Geometrically, $\mathrm{Spc}(\mathrm{incl}_L^N)$ *induces a homeomorphism*

$$\begin{tikzcd}
\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,N)) \arrow[r, "\mathrm{Spc}(\mathrm{incl}_L^N)"] & \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,L)) \\
\mathrm{open}(\mathcal{J}_N(N,L)) \arrow[u, hook] \arrow[r, "\cong"] & \mathrm{open}(\mathcal{J}_L(N,L)). \arrow[u, hook]
\end{tikzcd}$$

Proof. The geometric statement is immediate from the algebraic one (Recollection 6.6(*iii*)). The tt-functor incl_L^N is fully faithful by Proposition 15.25, so $\overline{\mathrm{incl}_L^N}$ is fully faithful by Proposition 8.9 (see Corollary 8.10 if necessary). It remains to show that $\overline{\mathrm{incl}_L^N}$ is essentially surjective. Recall (Example 14.20) that the twists of $\mathrm{fz}\,k(E/H)$ for $H \in \mathrm{sub}(E,N)$ generate the target as a triangulated subcategory. The essential image of $\overline{\mathrm{incl}_L^N}$ is a triangulated subcategory because we have seen that $\overline{\mathrm{incl}_L^N}$ is full, and it is also closed under twists because $\mathbb{1}(1)$ and $\mathbb{1}(-1)$ are in the essential image. Fix $H \in \mathrm{sub}(E,N)$; we are reduced to verifying that $\mathrm{fz}\,k(E/H)$ is in the essential image.

We consider the cases $H \geqslant L$ and $H \ngeqslant L$ separately. In the former case, we have $\mathrm{fz}\,k(E/H) \in \mathrm{D_b}(\mathsf{pfmod}(E,L))$ (Example 15.24), so certainly $\mathrm{fz}\,k(E/H)$ is in the essential image. In the latter case, we have $H \in \mathrm{sub}(E,N,L)$, and the objects $\mathrm{radf}_L\,k(E/H)$ and $\mathrm{fz}\,k(E/H)$ are isomorphic in the target of $\overline{\mathrm{incl}_L^N}$ by Lemma 20.11. Therefore, since $\mathrm{radf}_L\,k(E/H) \in \mathrm{D_b}(\mathsf{pfmod}(E,L))$ (Example 15.24) we see that $\mathrm{fz}\,k(E/H)$ is in the essential image. □

20.16 Corollary. *Let $L \in \mathrm{sub}(E,N)$. The fully faithful (Proposition 15.25) tt-functor*

$$\mathrm{incl}_L^N\colon \mathrm{D_b}(\mathsf{pfmod}(E,L)) \rightarrowtail \mathrm{D_b}(\mathsf{pfmod}(E,N))$$

induces an equivalence

$$\overline{\mathrm{incl}_L^N}\colon \frac{\mathrm{D_b}(\mathsf{pfmod}(E,L))}{\mathcal{U}_L(N,L)} \xrightarrow{\ \cong\ } \frac{\mathrm{D_b}(\mathsf{pfmod}(E,N))}{\mathcal{U}_N(N,L)},$$

where $\mathcal{U}_N(N,L)$ *is defined in Notation 20.3. Geometrically,* $\mathrm{Spc}(\mathrm{incl}_L^N)$ *induces a homeomorphism*

$$\begin{tikzcd}
\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,N)) \arrow[r, "\mathrm{Spc}(\mathrm{incl}_L^N)"] & \mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,L)) \\
\mathrm{open}(\mathcal{U}_N(N,L)) \arrow[u, hook] \arrow[r, "\mathrm{Spc}(\overline{\mathrm{incl}_L^N})", "\cong"'] & \mathrm{open}(\mathcal{U}_L(N,L)). \arrow[u, hook]
\end{tikzcd}$$

Proof. Observe that indeed $\mathcal{I}_K(N,L) \subset \mathcal{U}_K(N,L)$ by Lemma 4.8. Thus we may further quotient the equivalence in Theorem 20.15 to $\mathcal{U}_N(N,L)$ and $\mathcal{U}_L(N,L)$ respectively on each side, and this gives the corollary. □

20.17 Remark. Lemma 20.11 was inspired by [BG22b, proof of Lemma 6.11(b)], and Theorem 20.15 was inspired by [BG22b, proof of Theorem 6.18].

To illustrate Theorem 20.15, let us determine the rest of the points of the spectrum $\mathrm{D_b}(\mathsf{pfmod}(E))$.

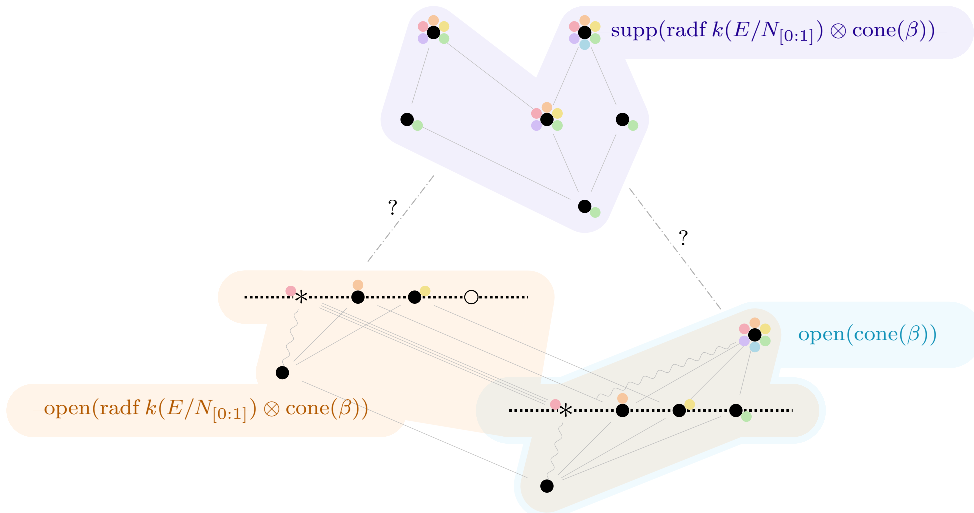
20.18 Running Example. We return to our Running Example 1.13. Let us put everything together. Recall that we have already determined $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,E))$ fully in Running Example 19.15, and recall that we are assuming Conjecture 17.23 and Conjecture 18.15.

So let us determine $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,N))$ for the rank 1 subgroups $N \leqslant E$. Starting with $N = N_{[0:1]}$, using Corollary 20.16 for the tt-ideal (see Running Example 20.4)

$$\mathcal{U}_{N_{[0:1]}}(N_{[0:1]},E) = \langle \mathrm{radf}\, k(E/N_{[0:1]}) \otimes \mathrm{cone}(\beta)\rangle \subset \mathrm{D_b}(\mathsf{pfmod}(E,N_{[0:1]}))$$

and our answer for $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,E))$ (Running Example 19.15), we obtain the following picture:

$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N_{[0:1]}))$



- $\operatorname{supp}(\operatorname{radf}(C_{[a:b]}))$
- $\operatorname{supp}(\operatorname{radf} k(E/N_{[1:0]}))$
- $\operatorname{supp}(\operatorname{radf} k(E/N_{[1:1]}))$
- $\operatorname{supp}(\operatorname{radf} kE)$
- $\operatorname{supp}(\operatorname{radf}_{N_{[0:1]}}(kE))$
- $\operatorname{supp}(\operatorname{radf} k(E/N_{[0:1]}))$.

We also do the same thing for $N_{[1:1]}$ and $N_{[0:1]}$, which give similar answers.

Finally, we determine $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$. We start by using Corollary 20.16 for the tt-ideals $\mathcal{U}_1(1, N_{[1:0]})$, $\mathcal{U}_1(1, N_{[1:1]})$, $\mathcal{U}_1(1, N_{[0:1]})$ in $\mathrm{D_b}(\mathsf{pfmod}(E))$ (see Running Example 20.4). For example, for $N = N_{[0:1]}$, this gives a copy of the vanishing locus

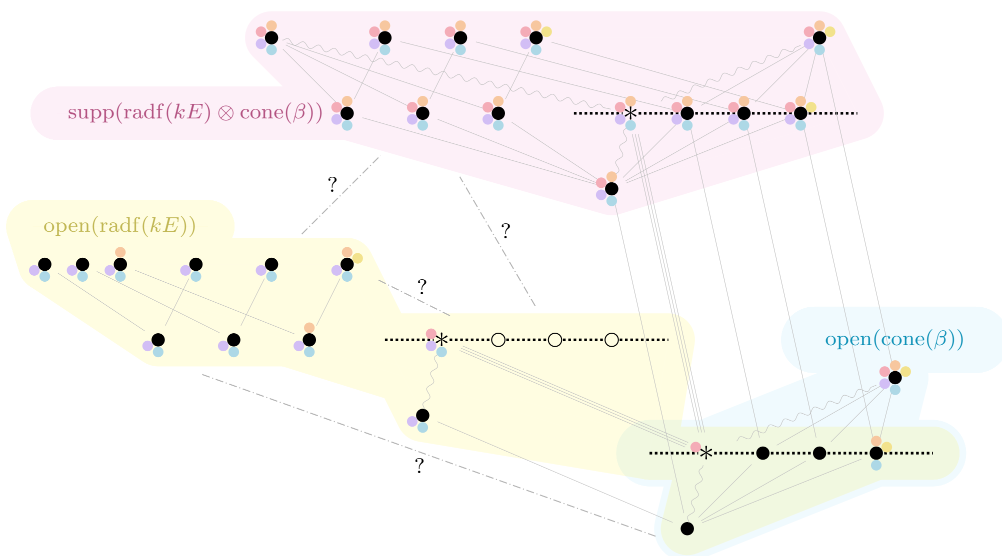
$$\begin{aligned}\operatorname{open}(\mathcal{U}_{N_{[0:1]}}(1, N_{[0:1]})) = \operatorname{open}\begin{pmatrix}\operatorname{radf} k(E/N_{[1:0]}) \otimes \operatorname{cone}(\beta), \\ \operatorname{radf} k(E/N_{[1:1]}) \otimes \operatorname{cone}(\beta)\end{pmatrix} \\ \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N_{[0:1]}))\end{aligned}$$

inside $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$. We get two similar copies from $N_{[1:0]}$ and $N_{[1:1]}$. Once we glue them along the vanishing locus of the tt-ideal

$$\operatorname{open}(\mathcal{U}_1(1, E)) = \operatorname{open}\begin{pmatrix}\operatorname{radf} k(E/N_{[1:0]}) \otimes \operatorname{cone}(\beta), \\ \operatorname{radf} k(E/N_{[1:1]}) \otimes \operatorname{cone}(\beta), \\ \operatorname{radf} k(E/N_{[0:1]}) \otimes \operatorname{cone}(\beta)\end{pmatrix} \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$$

we get open(radf$(kE)\otimes$cone$(\beta)) \subset \operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$ (Proposition 20.5). The picture on the front cover of this work depicts this; the yellow region in that picture is open$(\mathcal{U}_1(1, N_{[0:1]}))$, and the cyan region in that picture is open$(\mathcal{U}_1(1, E))$. Here is a more detailed picture of our spectrum:

$\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E))$



supp(radf$(C_{[a:b]})$) supp(radf $k(E/N_{[0:1]})$) supp(fz $k(E/N_{[0:1]})$)

supp(gap^1 radf(kE)) supp(gap^1 radf $k(E/N_{[0:1]})$).

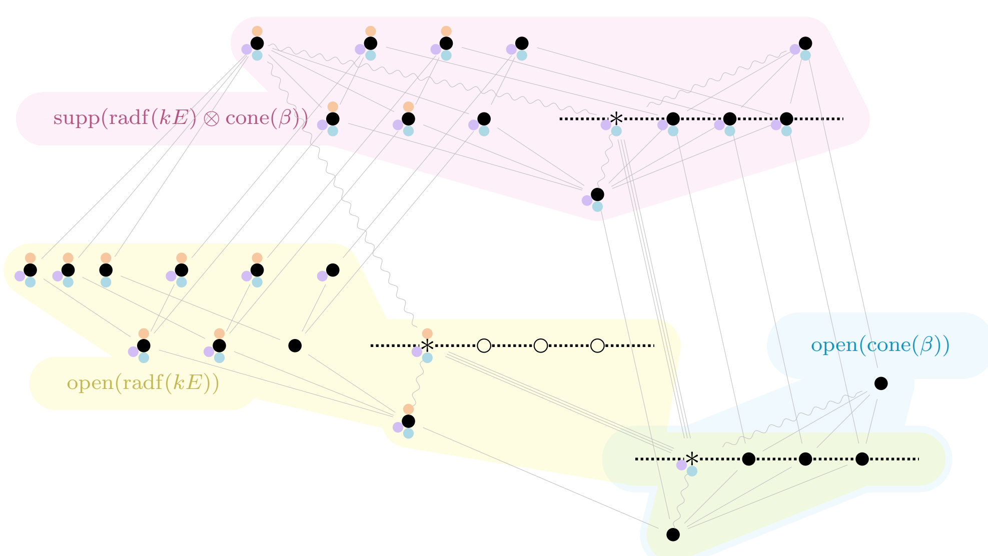
The yellow region has three distinct parts, in increasing order of height: the green part, the purely yellow part of open$(\mathcal{U}_1(1, E))$ which we call the st(pgmod) points of rank 2, and the remaining nine points which we call the st(pgmod) points of rank 1. Let us justify this discrimination by height. The st(pgmod) points of rank 2 are higher than the green part because it is doubled downward, according to the topology of $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, E))$. The st($\mathsf{pgmod}$) points of rank 1 are higher than the st(pgmod) points of rank 2 because the way in which we divided and conquered $\operatorname{Spc} \mathrm{D_b}(\mathsf{pfmod}(E, N))$ for rank 1 subgroups N forbade specializations the other way.

The dash-dotted lines and the question marks indicate missing specializations from these parts of the yellow region to other parts of the yellow region or to the support $\mathrm{supp}(\mathrm{fz}\,\mathrm{kos}_E(1))$ of the acyclics in the magenta region. These are the only missing specializations because we have determined the subspace consisting of the magenta and cyan regions, and as we just discussed, we know the specializations from the green part to the $\mathrm{st}(\mathsf{pgmod})$ points of rank 2. Note that the intra-yellow dash-dotted lines are essentially the same dash-dotted lines in our pictures of $\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E,N))$ for rank 1 subgroups N, whereas the dash-dotted lines from the yellow region to the magenta region are truly new.

We may easily compute the supports of radical filtrations of objects in $\mathsf{mod}(E)$ because they are in all of the relative subcategories. For $\mathrm{fz}\,k(E/N_{[0:1]})$, even though it is not in the relative subcategories $\mathsf{pfmod}(E,N_{[1:0]})$ and $\mathsf{pfmod}(E,N_{[1:1]})$, we may confirm its membership in primes coming from these relative subcategories by using that $\mathrm{radf}\,k(E/N_{[0:1]})\otimes\mathrm{fz}\,k(E/N_{[0:1]})$ is a sum of twists of $\mathrm{fz}\,k(E/N_{[0:1]})$ (Lemma 4.4).

20.19 Remark. We conjecture that the remaining specializations and the supports of the depicted objects (recall (1.15) if necessary) are as follows:

$\mathrm{Spc}\,\mathrm{D_b}(\mathsf{pfmod}(E))$



- $\mathrm{supp}(\mathrm{fz}\,\mathrm{kos}_E(N_{[0:1]}))$
- $\mathrm{supp}(A^{\mathrm{vert}}_{[a:b]})$
- $\mathrm{supp}(A^{\mathrm{horz}}_{[a:b]})$.

Note that it is unclear what the support of $\mathrm{fz}\,\mathrm{kos}_E(N_{[0:1]})$, say, is in the $\mathrm{st}(\mathsf{pgmod})$ points of rank 1 for the subgroups $N_{[1:0]}$ and $N_{[1:1]}$ and the $\mathrm{st}(\mathsf{pgmod})$ points of rank 2 because this object is not in any of the relative subcategories $\mathrm{D_b}(\mathsf{pfmod}(E, N_{[1:0]}))$, $\mathrm{D_b}(\mathsf{pfmod}(E, N_{[1:1]}))$, and $\mathrm{D_b}(\mathsf{pfmod}(E, E))$.

REFERENCES

[Aok23] Ko Aoki. Tensor triangular geometry of filtered objects and sheaves. *Math. Z.*, 303(62), 2023.

[Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[Bal10a] Paul Balmer. Spectra, spectra, spectra - tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebraic and Geometric Topology*, 10(3):1521–63, 2010.

[Bal13] Paul Balmer. Stacks of group representations. *Journal of the European Mathematical Society*, 17:189–228, 2013.

[Bal16] Paul Balmer. Separable extensions in tensor-triangular geometry and generalized Quillen stratification. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(4):907–925, 2016.

[Bal18] Paul Balmer. On the surjectivity of the map of spectra associated to a tensor-triangulated functor. *Bull. Lond. Math. Soc.*, 50(3):487–495, 2018.

[Bal21] Paul Balmer and James Cameron. Computing homological residue fields in algebra and topology. *Proceedings of the American Mathematical Society*, 149(8):3177–3185, 2021.

[BD24] Paul Balmer and Ivo Dell'Ambrogio. Cohomological Mackey 2-functors. *J. Inst. Math. Jussieu*, 23(1):279–309, 2024.

[BG23] Paul Balmer and Martin Gallauer. Finite permutation resolutions. *Duke Math. J.*, 172(2):201–229, 2023.

[BG22b] Paul Balmer and Martin Gallauer. Three real Artin-Tate motives. *Adv. Math.*, 406: paper no. 108535, 2022.

[BG25a] Paul Balmer and Martin Gallauer. The geometry of permutation modules. *Inventiones mathematicae.*, 241(3):841–928, 2025.

[BG25b] Paul Balmer and Martin Gallauer. The spectrum of Artin motives. *Trans. Amer. Math. Soc.*, 378(3): 1733–1754, 2025.

[Bra14] Martin Brandenburg. Tensor categorical foundations of algebraic geometry. *PhD thesis, University of Münster*. Available at https://arxiv.org/pdf/1410.1716, 2014.

[Buh10] Theo Bühler. Exact categories. *Expositiones Mathematicae*, 28(1):1–69, 2010.

[FP05] Eric M. Friedlander and Julia Pevtsova. Representation-theoretic support spaces for finite group schemes. *Amer. J. Math.*, 127(2):379–420, 2005

[FP07] Eric M. Friedlander and Julia Pevtsova. Π-supports for modules for finite group schemes. *Duke Math. J.*, 139(2): 317–368, 2007.

[FP07] William Fulton. Intersection theory. Springer-Verlag Berlin Heidelberg, 1998.

[Gal18] Martin Gallauer. Tensor triangular geometry of filtered modules. *Algebra Number Theory*, 12(8):1975–2003, 2018.

[Gal19] Martin Gallauer. tt-geometry of Tate motives over algebraically closed fields. *Compositio Mathematica*, 155(10):1888–1923, 2019.

[JS06] André Joyal and Ross Street. An introduction to Tannaka duality and quantum groups. In: Carboni, A., Pedicchio, M.C., Rosolini, G. (Eds) Category Theory. Volume 1488 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 2006.

[Kel18] Jack Kelly. Exact categories, Koszul duality, and derived analytic algebra. *Phd thesis, University of Oxford*, 2018.

[Kra21] Henning Krause. Homological theory of representations. Cambridge University Press, 2021.

[NO82] C. Nastasescu and F. van Oystaeyen. Graded ring theory. North-Holland Publishing Company, 1982.

[Nat15] Sonia Natale. Jordan-H older theorem for finite dimensional Hopf algebras *Proc. Amer. Math. Soc.*, 143:5195–5211, 2015.

[Nee01] Amnon Neeman. Triangulated Categories, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, 2001.

[Pas11] Donald S. Passman. The Algebraic Structure of Group Rings. *Dover Books on Mathematics Series*, 2011.

[Pos11] L. Positselski. Mixed Artin-Tate motives with finite coefficients. *Mosc. Math. J.*, 11(2):317–402, 2011.

[Rou06] Raphael Rouquier. The derived category as a quotient of the homotopy category of permutation modules. Available at https://www.math.ucla.edu/~rouquier/papers/perm.pdf, 2006.

[Sha23] Amit Shah. Krull-Remak-Schmidt decompositions in Hom-finite additive categories. *Expositiones Mathematicae*, 41(1):220–237 (2023).

[SS16] Pierre Schapira and Jean-Pierre Schneiders. Derived categories of filtered objects. *Astérisque*, 383:103–120, 2016.

[Voe00] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.