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Categorified Quantum Groups and Braid Group Actions

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ABSTRACT OF THE DISSERTATION

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Let U be the quantum group associated to a Cartan matrix indexed by a set I. There is a remarkable family $(\theta_i)_{i\in I}$ of symmetries of U giving rise to the action of a braid group. The goal of this dissertation is to explain how to categorify these symmetries. The quantum group is categorified by a certain 2-category $\mathcal{U}$ introduced by Khovanov-Lauda and Rouquier. We propose two constructions providing a categorification of the symmetry θ_i on $\mathcal{U}$.

First, there is a complex Θ_i in $\mathcal{U}$. We prove that Θ_i is invertible in the homotopy category of $\mathcal{U}$, and that there are homotopy equivalences $\Theta_i E_i \simeq F_i\Theta_i[-1]$, where E_i, F_i denote the Chevalley generators of $\mathcal{U}$. Hence, conjugation by Θ_i provides an auto-equivalence of the homotopy category of $\mathcal{U}$. This auto-equivalence decategorifies to the symmetry θ_i. To obtain these results, we prove a faithfulness property for the 2-representations of $\mathcal{U}$.

Second, we construct a monoidal functor T_i by defining it on generators and checking relations. We prove that this functor decategorifies to the symmetry θ_i on a certain subalgebra of the positive part of U. Our approach is based on categorifying the formulas defining the symmetry θ_i on U. These formulas involve the adjoint action of U, of which we also construct a categorification.

The dissertation of Laurent Vera is approved.

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CHAPTER 1

Introduction

1.1 Categorification and Higher Representation Theory

The term *categorification* can be broadly defined as the process of finding higher levels of structure, usually categorical in nature. *Decategorification* refers to the inverse process of forgetting higher levels of structure. While these terms are recent, the ideas behind categorification have long been used in many areas of mathematics. For instance, the category of finite sets can be seen as a categorification of the set of non-negative integers. In this example, decategorification consists of taking the cardinality of a finite set, and the higher structure is given by functions between sets. Certain equalities between integers can be proved using bijections between sets: this is the most elementary application of the idea of categorification. Another classical example of topological nature is homology. The homology groups of a manifold can be thought of as a categorification of its Euler characteristic, which is recovered by taking the ranks of homology groups. Continuous maps between manifolds induce linear maps in homology. This provides a higher level of structure, resulting in a finer topological invariant.

This dissertation will be focused on categorification in the context of representation theory.

Representation theory is the study of actions of algebras, or more generally 1-categories, on vector spaces by linear maps. With that perspective, *higher representation theory* or *2-representation theory* can be described as the study of actions of monoidal categories, or more generally 2-categories, on additive categories by additive functors. The higher level of structure is given by natural transformations between functors. In this context, decategorifying means taking split Grothendieck groups (that is, taking isomorphism classes of objects).

The motivations behind higher representation theory are more than mere abstraction. A first one is that higher levels of structure shed a new light on the categorified objects. For instance, some classical algebras and their representations have canonical bases. In these bases, the structure constants are usually non-negative integers. Then, it is expected that these canonical bases arise from isomorphism classes of indecomposable objects in a suitable categorification. This can provide an explanation, or even a proof, for the integrality and positivity properties. For example, this strategy was employed by Elias and Williamson [EW14] to prove Soergel's conjecture about Kazdhan-Luzstig bases of Hecke algebras.

A second motivation for higher representation theory is that many of the categorical structures appearing in concrete examples are of inherent interest, independently of the objects that they categorify. For instance, categories of modular representations of symmetric groups are endowed with induction and restriction functors. These functors give rise to actions of type A affine Lie algebras on Grothendieck groups (see [Ari96]). This fact motivated the seminal work of Chuang and Rouquier on categorification of representations of $\mathfrak{sl}_2$ [CR08]. Chuang and Rouquier introduce a notion of 2-representations of $\mathfrak{sl}_2$ and prove strong structural properties about them. Applying these results to symmetric groups leads to a proof of Broué's abelian defect group conjecture for symmetric groups, a long-standing problem in modular representation theory.

This last example also illustrates an important subtlety in categorification. A categorification of an object, if it exists, has no reason to be unique. In general, the difficulty in pinning down how to categorify an object is to find the appropriate higher structure to ensure that the resulting theory has desirable properties. The key insight in [CR08] is to control the higher structure with actions of Hecke algebras.

Following the work of Chuang and Rouquier, higher representation theory developed quickly in multiple directions. Our work will mainly focus on the higher representation theory of quantum groups. Other related directions include finitary 2-categories and their 2-representations ([MM11], [MM16b], [MM16a]), Heisenberg categorifications ([Kho14], [BSW20]) and higher representations of Soergel bimodules ([MT19], [MMM$^+$21]).

1.2 Quantum Groups

Generalizing the work of Chuang and Rouquier on $\mathfrak{sl}_2$, the ideas of categorification were applied to quantum groups of arbitrary types. Let us start by giving some motivation for why it is expected that these objects should have an interesting categorification.

- Quantum groups are one of the main examples of quasi-triangular Hopf algebras which are neither commutative nor cocommutative. As a result, their categories of representations inherit the structure of braided monoidal categories. This structure can be used to construct 3-dimensional topological quantum field theories [RT91]. Crane and Frenkel [CF94] conjectured that there should exist a categorification of quantum groups and their representation theory leading to the construction of 4-dimensional topological quantum field theories.

- Several categories of interest are known to be equipped with endofunctors inducing actions of quantum groups on their Grothendick groups. The classical example seen above is the category of modular representations of symmetric groups. Other examples of such phenomena include the category $\mathcal{O}$ of the general linear Lie algebra, which categorifies tensor powers of the standard representation of $\mathfrak{sl}_2$ (see [BFK99]), or categories of representations of zig-zag algebras, which categorify adjoint representations of quantum groups (see [HK01]). These various examples hint at a unified theory of actions of quantum groups on categories.

- Quantum groups and their finite dimensional representations have canonical bases. These can be constructed via the geometry of quiver varities (see [Lus10, Part II]), or using Kashiwara's crystals (see [Kas91]). The structure constants in these bases are non-negative integers. It is then expected that these canonical bases should arise as isomorphism classes of indecomposable objects in a suitable categorification.

These ideas became a concrete program with the work of Khovanov-Lauda [KL09], [KL11], [KL10] and Rouquier [Rou08], [Rou12]. More precisely, let U be the quantum group associated to a symmetrizable Cartan matrix. In [KL11] and [Rou08], an algebra H depending on U and a family of coefficients is introduced. This algebra is known as the *quiver Hecke algebra*, or *Khovanov-Lauda-Rouquier (KLR) algebra*. The most remarkable property of H is that its category of projective modules is monoidal and categorifies the positive part of U. More precisely, let $\mathcal{U}^+$ be the category of finitely projective H-modules. It is shown in [KL11] that there is an isomorphism of algebras

$$K_0\left(\mathcal{U}^+\right) \otimes_{\mathbb{Z}} \mathbb{C} \simeq U^+,$$

where K_0 denotes the split Grothendieck group. A lot of the structure of U^+ can then be recovered from H. For instance, the crystal structure can be described with in terms of H-modules (see [LV11]). In types ADE, it was also proved in [Rou12], [VV11] that the indecomposable projective H-modules recover the canonical basis of U^+.

Furthermore, Khovanov-Lauda and Rouquier introduce a 2-category $\mathcal{U}$ which categorifies Beilinson-Lusztig-MacPherson's idempotent version of the quantum group $\dot{U}$. While the definitions of the 2-categories in [KL10] and [Rou08] differ on a few points, it was shown by Brundan that they are actually isomorphic (see [Bru16]). Hence, there is a unique *2-quantum group* or *categorified quantum group* $\mathcal{U}$ associated to a symmetrizable Cartan matrix. This 2-category has the remarkable property of decategorifying to the idempotent form of the quantum group. That is, there is an isomorphism of categories

$$K_0(\mathcal{U}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \dot{U}$$

proved by Khovanov-Lauda [KL10] and Dupont [Dup21]. The higher representation theory of $\mathcal{U}$ is very rich and categorifies much of the representation theory of $\dot{U}$. For instance, the simple representations of $\dot{U}$ arise from 2-representations of $\mathcal{U}$ on cyclotomic quotients of H (see [KK12], [Web17]).

1.3 Braid Group Actions

Quantum groups and their representations have remarkable symmetry properties encoded by actions of braid groups. Let us start by describing the case of $\mathfrak{sl}_2$. The standard Chevalley generators of this Lie algebra are $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. If V is an *integrable* representation of $\mathfrak{sl}_2$, we obtain an action of the Lie group SL_2 on V via the exponential map $\exp : \mathfrak{sl}_2 \to \mathrm{SL}_2$. In particular, the reflection

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp(-e)\exp(f)\exp(-e)$$

induces an automorphism of V. This fact generalizes to the quantum case. If V is an integrable representation of the quantum group U associated with $\mathfrak{sl}_2$, there is a quantum deformation of θ providing an automorphism of V. We will also denote this quantum deformation by θ for simplicity.

For a quantum group of general type, there is a set I of *simple roots* (in the case of $\mathfrak{sl}_2$, this set is just a singleton). For each simple root $i \in I$, there is a subalgebra U_i of U generated by Chevalley generators e_i, f_i. This subalgebra is isomorphic to the quantum group associated with $\mathfrak{sl}_2$. Thus, the construction described above yields an automorphism θ_i on every integrable representation of U. Let us summarize the main facts about these automorphisms.

- For V an integrable representation of U and λ a weight, the automorphism θ_i induces an isomorphism between weight subspaces $V_\lambda \xrightarrow{\sim} V_{s_i(\lambda)}$. Here s_i denotes the action of the Weyl group of U on the weight lattice.

- The action of θ_i on integrable representations lifts to a compatible algebra automorphism of U. This means that there exists an algebra automorphism θ_i of U, and that for an integrable representation V of U, we have $\theta_i(uv) = \theta_i(u)\theta_i(v)$ for all $u \in U$ and $v \in V$. These automorphisms are sometimes referred to as *Lusztig's symmetries.*

- The collection $(\theta_i)_{i \in I}$ of automorphisms satisfies the *braid relations.* This means that for all simple roots i, j, we have a relation of the form

$$\theta_i \theta_j \cdots = \theta_j \theta_i \cdots ,$$

  where the number of terms on each side of the equality can be deduced from the Cartan matrix of U. Hence the *braid group* associated with the Cartan matrix acts compatibly on U and its integrable representations.

One of the most remarkable features of the work of Chuang and Rouquier is that there is a categorification of the automorphisms θ_i acting on integrable representations. There is a complex Θ_i of 1-morphisms of the 2-category $\mathcal{U}$ whose action on integrable 2-representations decategorifies to θ_i. Furthermore, if $\mathcal{V}$ is an integrable 2-representation of $\mathcal{U}$, then Θ_i induces an auto-equivalence of the homotopy category $K(\mathcal{V})$. Just as representations of U decompose into weight subspaces, 2-representations of $\mathcal{U}$ decompose into *weight subcategories* $\mathcal{V}_\lambda$. Then the auto-equivalence Θ_i restricts to equivalences $K(\mathcal{V}_\lambda) \xrightarrow{\sim} K(\mathcal{V}_{s_i(\lambda)})$. For simply-laced quantum groups, Cautis and Kamnitzer show that the complexes $(\Theta_i)_{i\in I}$ satisfy the braid relations up to homotopy [CK12].

The equivalences provided by the complexes Θ_i are the main ingredient in the proof of the abelian defect group conjecture for symmetric groups by Chuang and Rouquier [CR08]. Another important application is the construction of braid group actions on derived categories of coherent sheaves on Nakajima quiver varietes [CKL13], [CK12].

However, a categorification of the compatible action of the braid group on U is still missing. In other words, we would like to have 2-auto-equivalences of the 2-quantum group $\mathcal{U}$ (or its homotopy category) decategorifying to the automorphism θ_i. These auto-equivalences should satisfy a certain compatibility with the complexes Θ_i, which would categorify the compatibility of the actions of the braid group on U and its representations. The main goal of this dissertation is to provide answers to this problem.

1.4 Summary of Results

Before we can explain our approach to solving this question, we need to give more details about the construction of braid group actions on quantum groups. For a simple root i, the automorphism θ_i can be constructed in two ways. The first one is to work in a completion of U. Then one can express θ_i as an invertible element of the completion, and the braid group action can be obtained by conjugation with θ_i (see [KS98, Chapter 4]). The second way is to define the action of θ_i explicitly on the generators of U. The formulas appearing involve the adjoint action of U. While this approach is more explicit, it requires lengthy computations to check that all desired relations hold (see [Lus10, Part VI], [Jan96, Chapter 8]). We will show that both of these constructions can be categorified.

In Chapter 3, we explain a categorification of the first construction. The complex Θ_i is a natural categorical lift of the element θ_i of the completion of U. Thus, our goal is to explain to what extent it is possible to conjugate by Θ_i. Our main result is the following.

Theorem 1.4.1 (Theorem 3.2.10)**.** *The complex Θ_i is invertible in the homotopy category of $\mathcal{U}$.*

This means that there exists a complex $\Theta_i^\vee$ of 1-morphisms of $\mathcal{U}$ and homotopy equivalences $\Theta_i\Theta_i^\vee \simeq 1 \simeq \Theta_i^\vee\Theta_i$. Hence, conjugation by Θ_i gives an auto-equivalence of the homotopy category of $\mathcal{U}$, which descends to the action θ_i when decategorifying. Note that by construction, this auto-equivalence is compatible with the action of Θ_i on integrable 2-representations.

The proof of Theorem 1.4.1 is based on faithfulness properties for the 2-representations of $\mathcal{U}$. There is a remarkable family of 2-representations of $\mathcal{U}$ called *simple 2-representations*. These categorify the finite dimensional simple representations of U. For the 2-category $\mathcal{U}$ associated with $\mathfrak{sl}_2$, we prove that the simple 2-representations satisfy the following faithfulness property.

Theorem 1.4.2 (Theorem 3.1.3)**.** *Let $\mathcal{U}$ be the 2-category associated to $\mathfrak{sl}_2$.*

1. *Let C be a complex of 1-morphisms of $\mathcal{U}$. If the image of C in every simple 2-representation is null-homotopic, then C is hull-homotopic.*

2. *Let f be a morphism between complexes of 1-morphisms of $\mathcal{U}$. If the image of f in every simple 2-representation is a homotopy equivalence, then f is a homotopy equivalence.*

This gives a categorification of the well-known faithfulness of simple representations of quantum groups: if $z \in U$ acts by zero on every simple representation of U, then $z = 0$.

Using this property, we can prove the invertibility of the complexes Θ_i by checking it in the simple 2-representations, where explicit computations are manageable. As an another application of Theorem 1.4.2, we prove a categorification of the relation $\theta e = -f\theta$ in $\mathfrak{sl}_2$.

Theorem 1.4.3 (Theorem 3.2.14). *Let $\mathcal{U}$ be the 2-category associated to $\mathfrak{sl}_2$. There is a homotopy equivalence $\Theta E \simeq F\Theta[-1]$, where E, F denote the 1-morphisms of $\mathcal{U}$ lifting the Chevalley generators e, f.*

The results of Chapter 3 also appear in our preprint [Ver20b].

In Chapters 4 and 5, we propose a second construction for the braid group action on the 2-category $\mathcal{U}$ based on explicit formulas. In the decategorified case, the automorphism θ_i can be defined by

$$\theta_i(e_j) = \mathrm{ad}_{e_i}^{(-c_{i,j})}(e_j) \tag{1.4.1}$$

for a simple root $j \neq i$. Here, ad denotes the adjoint action of U on itself coming from the Hopf algebra structure, and $c_{i,j}$ denotes the entry (i, j) of the Cartan matrix. To be able to categorify this formula, we must first explain how to categorify the operator ad_{e_i}. This is the object of Chapter 4. We define a quotient H^i of the quiver Hecke algebra H and an endofunctor ad_{E_i} of the category $H^i-\mathrm{mod}$. We show the following results regarding these objects.

Theorem 1.4.4 (Theorems 4.2.1, 4.3.1 and 4.4.13). *1. For all $M \in H^i-\mathrm{mod}$, there is a short exact sequence*

$$0 \to ME_i \to E_iM \to \mathrm{ad}_{E_i}(M) \to 0.$$

The concatenation denotes the monoidal structure in $H-\mathrm{mod}$. This categorifies the relation $\mathrm{ad}_{e_i}(y) = e_iy - ye_i$ in U.

2. The category $H^i-\mathrm{mod}$ is monoidal, and for all $M, N \in H^i-\mathrm{mod}$ we have a short exact *sequence*

$$0 \to M\,\mathrm{ad}_{E_i}(N) \to \mathrm{ad}_{E_i}(MN) \to \mathrm{ad}_{E_i}(M)N \to 0.$$

This categorifies the relation $\mathrm{ad}_{e_i}(yz) = \mathrm{ad}_{e_i}(y)z + y\mathrm{ad}_{e_i}(z)$.

3. *The functor* ad_{E_i} *sends projective modules to projective modules and induces a structure of 2-representation of* $\mathfrak{sl}_2$ *on* $H^i-\mathrm{proj}$.

4. *The algebra* $K_0(H^i-\mathrm{proj})$ *is isomorphic to a subalgebra of* U^+ *which is integrable for the action of* ad_{e_i} *and maximal for this property.*

The necessity to restrict to a subalgebra of U^+ comes from finiteness constraints. Indeed, the higher representation theory of quantum groups is only well-suited for integrable representations. Despite these restrictions, this subalgebra contains the Chevalley generators e_j for $j \neq i$. Thus, our results are enough to categorify formula (1.4.1).

A key feature of our work is the compatibility of the structure of 2-representation and the monoidal structure, expressed in statement 2 of Theorem 1.4.4. It is used in a crucial way to simplify computations. To our knowledge, this is the first example of a 2-representation with such structure.

The results of Chapter 4 also appear in our preprint [Ver20a].

In Chapter 5, we apply these results to construct a partial categorification of Lusztig's symmetry θ_i. We start by resolving the H^i-modules $\mathrm{ad}_{E_i}^{(n)}(E_j)$ in terms of H-modules. This yields complexes $\mathrm{Ad}_{E_i}^{(n)}(E_j)$ of objects of $\mathcal{U}^+$. Then, we prove the following result.

Theorem 1.4.5 (Theorem 5.2.1)**.** *There is a monoidal functor* $T_i : \mathcal{U}^+_{I\setminus\{i\}} \to K^{\mathrm{b}}(\mathcal{U}^+)$ *such that*

$$T_i\left(E_j\right) = \mathrm{Ad}_{E_i}^{(-c_{i,j})}\left(E_j\right).$$

This functor decategorifies to Lusztig's symmetry θ_i.

In this theorem, $\mathcal{U}^+_{I\setminus\{i\}}$ denotes the subcategory of $\mathcal{U}^+$ generated by the E_j for $j \neq i$. It is necessary to restrict to this subcategory for technical homological reasons. Indeed, proving that T_i is a well-defined monoidal functor involves checking many relations in the homotopy category. Using the fact that the complexes $\mathrm{Ad}_{E_i}^{(-c_{i,j})}(E_j)$ are projective resolutions, computations simplify significantly. We can only use this trick when restricting to $\mathcal{U}^+_{I\setminus\{i\}}$.

Theorem 1.4.5 holds for quantum groups of arbitrary types. In the simply-laced case, a functor $T_i : \mathcal{U} \to K(\mathcal{U})$ extending ours is constructed in work in preparation of Abram, Lamberto-Egan, Lauda and Rose [ALELR]. Their approach is different than ours; it is based on checking relations by explicitly giving out homotopies. Consequently, the computations involved are rather long and difficult, and it seems unlikely that they could be generalized as such to arbitrary quantum groups.

Both approaches to categorifying θ_i have their own advantage: the first one provides the compatibility with the action on 2-representations automatically, while the second one provides explicit formula. We expect that they coincide up to homotopy. More precisely, we make the following conjecture.

Conjecture 1. *For every simple root $j \neq i$, there is a homotopy equivalence*

$$\Theta_i E_j \simeq \mathrm{Ad}_{E_i}^{(-c_{i,j})}(E_j)\,\Theta_i.$$

CHAPTER 2

Categorified Quantum Groups

In this chapter, we introduce the main objects of interest for this dissertation.

2.1 Basic Notation and Defintions

For the rest of this dissertation, we fix a field K. We use the symbol $\otimes$ to mean $\otimes_K$.

Graded Categories

A *graded category* is a K-linear category $\mathcal{C}$ together with an auto-equivalence $\mathcal{C} \to \mathcal{C}$ called the *shift functor*. We denote by qM the shift of an object M of $\mathcal{C}$. An example of graded category is the category gVect_K of finitely generated graded K-vector spaces with homogeneous K-linear maps. In that case, the shift functor defined by $(qV)_k = V_{k-1}$ for $V = \oplus_{k\in\mathbb{Z}} V_k \in \mathrm{gVect}_K$. More generally, given a graded K-algebra A, the category $A-$mod of finitely generated graded A-modules and its full subcategory $A-$proj of graded projective modules are graded. Their shift functor is the restriction of that of gVect_K. Given a Laurent polynomial $p = \sum_{\ell\in\mathbb{Z}} p_\ell q^\ell \in \mathbb{N}[q, q^{-1}]$ and M an object of a graded category $\mathcal{C}$, we put

$$pM = \bigoplus_{\ell\in\mathbb{Z}} q^\ell M^{\oplus p_\ell}.$$

For two objects M, N of a graded category $\mathcal{C}$ and $k \in \mathbb{Z}$, we define

$$\mathrm{Hom}_{\mathcal{C}}^{k}(M, N) = \mathrm{Hom}_{\mathcal{C}}\left(M, q^{-k}N\right).$$

The elements of $\mathrm{Hom}_{\mathcal{C}}^{k}(M, N)$ are said to be *morphisms of degree* k from M to N. We can then define a graded K-vector space $\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, N)$ by

$$\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, N) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}^{k}(M, N).$$

Then, as graded vector spaces we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, qN) &= q\,\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, N), \\
\mathrm{Hom}_{\mathcal{C}}^{\bullet}(qM, N) &= q^{-1}\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, N).
\end{aligned}$$

Graded Dimensions

A graded K-vector space $V = \oplus_{k \in \mathbb{Z}} V_k$ is said to be *locally finite* if each V_k is finite dimensional. In this case, its *graded dimension* is the formal series defined by

$$\mathrm{grdim}(V) = \sum_{k \in \mathbb{Z}} \dim(V_k) q^k \in \mathbb{N}((q, q^{-1})).$$

We have $\mathrm{grdim}(qV) = q\,\mathrm{grdim}(V)$. If M, N are two objects of a graded category $\mathcal{C}$ and if $\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, N)$ is locally finite, we put

$$\langle M, N \rangle = \mathrm{grdim}\left(\mathrm{Hom}_{\mathcal{C}}^{\bullet}(M, N)\right).$$

Then we have $\langle M, qN \rangle = q \langle M, N \rangle$ and $\langle qM, N \rangle = q^{-1} \langle M, N \rangle$. In the case $M = N$, we use the notation $|M| = \langle M, M \rangle$.

2-Categories

We refer to [Lei98] for a concise introduction to 2-categories. We recall here the definitions of the basic notions that will be used throughout this dissertation. A *(strict) 2-category* $\mathfrak{C}$ is the data of

- a class O of *objects*,
- for all $a, b \in O$, a category $\mathrm{Hom}_{\mathfrak{C}}(a, b)$ whose objects and morphisms are called *1-morphisms* and *2-morphisms* of $\mathfrak{C}$ respectively,
- composition functors $\mathrm{Hom}_{\mathfrak{C}}(b, c) \times \mathrm{Hom}_{\mathfrak{C}}(a, b) \to \mathrm{Hom}_{\mathfrak{C}}(a, c), (G, H) \mapsto GH$ for all $a, b, c \in O$,
- 1-morphisms $1_a \in \mathrm{End}_{\mathfrak{C}}(a)$ called *identity 1-morphisms* for all $a \in O$,

such that

1. $G_1(G_2G_3) = (G_1)G_2G_3$ for all $a, b, c, d \in O$, $G_1 \in \mathrm{Hom}_{\mathfrak{C}}(c, d)$, $G_2 \in \mathrm{Hom}_{\mathfrak{C}}(b, c)$ and $G_3 \in \mathrm{Hom}_{\mathfrak{C}}(a, b)$,
2. $1_bG = G = G1_a$ for all $a, b \in O$ and $G \in \mathrm{Hom}_{\mathfrak{C}}(a, b)$.

As an example, the strict 2-category $\mathfrak{Cat}$ is defined as having categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms. A strict 2-category with a singleton as set of objects is simply a strict monoidal category.

Given two 1-morphisms G, H of a 2-category with same source and target, we will denote by $\mathrm{Hom}_{\mathfrak{C}}(G, H)$ the set of 2-morphisms $G \to H$. The identity 2-morphism of a 1-morphism G will be simply denoted by G when no confusion is possible. We will call a 2-category K-linear (resp. graded, resp. idempotent complete) when its morphism categories are K-linear (resp. graded, resp. idempotent complete) and the compositions functors respect that structure. For instance, the 2-category $\mathfrak{Lin}_K$ of K-linear graded categories, with additive graded functors as 1-morphisms, is K-linear and graded.

Given two strict 2-categories $\mathfrak{C}, \mathfrak{D}$ with respective classes of objects $O_{\mathfrak{C}}, O_{\mathfrak{D}}$, a strict 2-functor $\Phi : \mathfrak{C} \to \mathfrak{D}$ is the data of

- a map $\Phi : O_{\mathfrak{C}} \to O_{\mathfrak{D}}$,
- functors $\Phi : \mathrm{Hom}_{\mathfrak{C}}(a, b) \to \mathrm{Hom}_{\mathfrak{D}}(\Phi(a), \Phi(b))$ for all $a, b \in O_{\mathfrak{C}}$,

such that $\Phi(GH) = \Phi(G)\Phi(H)$ for any composable 1-morphisms G, H of $\mathfrak{C}$ and $\Phi(1_a) = 1_{\Phi(a)}$ for any object a of $\mathfrak{C}$.

Homological Algebra

Let $\mathcal{C}$ be a K-linear category. We denote by $\mathrm{Comp}(\mathcal{C})$ the category of (cochain) complexes of $\mathcal{C}$. That is, the objects of $\mathrm{Comp}(\mathcal{C})$ are of the form

$$M = (\cdots \to M^r \xrightarrow{d_M^r} M^{r+1} \xrightarrow{d_M^{r+1}} M^{r+2} \to \cdots),$$

with $d_M^{r+1} d_M^r = 0$ for all $r \in \mathbb{Z}$. For $M \in \mathrm{Comp}(\mathcal{C})$, its homological shift $M[1]$ is defined by $(M[1])^r = M^{r-1}$ with differential $d_{M[1]}^r = -d_M^{r-1}$ for all $r \in \mathbb{Z}$. Note that when $\mathcal{C}$ is graded, $\mathrm{Comp}(\mathcal{C})$ has two compatible gradings: the one coming from $\mathcal{C}$ and the homological grading.

Let $M, N \in \mathrm{Comp}(\mathcal{C})$ and let $f : M \to N$ be a morphism of complexes. The *cone* of f is the object $\mathrm{Cone}(f)$ of $\mathrm{Comp}(\mathcal{C})$ defined by $\mathrm{Cone}(f)^r = M^{r+1} \oplus N^r$ with differential given by

$$d_{\mathrm{Cone}(f)}^r = \begin{bmatrix} -d_M^{r+1} & 0 \\ f & d_N^r \end{bmatrix}.$$

We denote by $K(\mathcal{C})$ the homotopy category of $\mathcal{C}$. If $\mathcal{C}$ is abelian, we denote by $D(\mathcal{C})$ its derived category. In that case, we denote by $H^r(M)$ the r^{th} cohomology object of a complex

M.

2.2 Quantum Groups

We recall the essential definitions and facts about quantum groups. We refer to [Lus10] or [Jan96] for a more detailed introduction to the subject.

2.2.1 Root Datum

A *Cartan datum* $(I, \cdot)$ is the data of a non empty set I, whose elements are referred to as *simple roots*, and a symmetric bilinear form $\cdot : \mathbb{Z}I \otimes_{\mathbb{Z}} \mathbb{Z}I \to \mathbb{Z}$ such that

- for all $i \in I$, $i \cdot i \in 2\mathbb{Z}_{>0}$,
- for all $i, j \in I$, $2\dfrac{i \cdot j}{i \cdot i} \in \mathbb{Z}_{\leqslant 0}$.

A *root datum* (X, Y) of type $(I, \cdot)$ is the data of finitely generated free abelian groups X (the *weight latice*) and Y (the *dual weight latice*), together with a perfect pairing $\langle \cdot, \cdot \rangle : Y \otimes_{\mathbb{Z}} X \to \mathbb{Z}$ and injective set maps $(I \hookrightarrow X, i \mapsto i)$ and $(I \hookrightarrow Y, i \mapsto i^\vee)$ satisfying the condition

$$\langle i^\vee, j \rangle = 2\frac{i \cdot j}{i \cdot i}$$

for all $i, j \in I$. Let $c_{i,j} = \langle i^\vee, j \rangle$ and $d_i = \frac{i \cdot i}{2}$. For all $i \in I$, there is an automorphism s_i of the lattice X defined by $s_i(\lambda) = \lambda - \langle i^\vee, \lambda \rangle\, i$ for $\lambda \in X$. The *root lattice* is $Q = \oplus_{i \in I} \mathbb{Z}i$, and we let $Q^+ = \oplus_{i \in I} \mathbb{Z}_{\geqslant 0} i$. Given an element $\beta = \sum_{i \in I} n_i i$ of Q^+, the integer $n = \sum_{i \in I} n_i$ is called the *height* of β, and denoted $|\beta|$. We also let

$$I^\beta = \Big\{ (j_1, \dots, j_n) \in I^n,\ \sum_{k=1}^{n} j_k = \beta \Big\}.$$

2.2.2 Quantum Groups

Let us start by introducing some notation in $\mathbb{Q}(q)$. For all $i \in I$, we let $q_i = q^{d_i}$. For $k \in \mathbb{Z}$, the quantum integer $[k]_i$ is defined by

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}.$$

If $k \geqslant 0$, we define the quantum factorial $[k]_i!$ by

$$[k]_i! = \prod_{\ell=1}^{k} [\ell]_i \,.$$

For $0 \leqslant \ell \leqslant k$, we define the quantum binomial coefficient by

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_i = \frac{[k]_i!}{[\ell]_i![k-\ell]_i!}.$$

It can be checked these are actually all elements of $\mathbb{N}\left[q, q^{-1}\right]$ (see [KC02] for a proof).

Definition 2.2.1. The *quantum group U* associated to the above root datum is the unital $\mathbb{Q}(q)$-algebra on the generators e_i, f_i for $i \in I$, k_μ for $\mu \in Y$, subject to the following relations.

1. For all $\mu, \mu' \in Y$, $k_0 = 1$ and $k_\mu k_{\mu'} = k_{\mu+\mu'}$.

2. For all $i \in I$ and $\mu \in Y$, $k_\mu e_i = q^{\langle \mu, i \rangle} e_i k_\mu$ and $k_\mu f_i = q^{-\langle \mu, i \rangle} f_i k_\mu$.

3. For all $i, j \in I$,
$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
where $k_i = k_{d_i i^\vee}$.

4. For all $i \neq j \in I$,

$$\sum_{\ell=0}^{-c_{i,j}+1} (-1)^\ell e_i^{(\ell)} e_j e_i^{(-c_{i,j}+1-\ell)} = 0, \quad \sum_{\ell=0}^{-c_{i,j}+1} (-1)^\ell f_i^{(\ell)} f_j f_i^{(-c_{i,j}+1-\ell)} = 0,$$

where the elements $e_i^{(\ell)}$ and $f_i^{(\ell)}$ are called the *divided powers* and are defined by

$$e_i^{(\ell)} = \frac{1}{[\ell]_i!} e_i^\ell, \quad f_i^{(\ell)} = \frac{1}{[\ell]_i!} f_i^\ell.$$

The algebra U is Q-graded with e_i in degree i and f_i in degree $-i$ for all $i \in I$, and k_μ in degree 0 for all $\mu \in Y$. Furthermore, U has a structure of Q-graded Hopf algebra, with coproduct Δ, counit ϵ and antipode S defined by the following formulas

$$\begin{aligned}
&\Delta(e_i) = e_i \otimes 1 + k_i^{-1} \otimes e_i, && \Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i, && \Delta(k_\mu) = k_\mu \otimes k_\mu, \\
&\epsilon(e_i) = 0, && \epsilon(f_i) = 0, && \epsilon(k_\mu) = 1, \\
&S(e_i) = -k_i e_i, && S(f_i) = -f_i k_i^{-1}, && S(k_\mu) = k_\mu^{-1},
\end{aligned}$$

for all $i \in I$ and $\mu \in Y$.

For categorification purposes, we are only interested in *type 1 weight representations* of U. These are the representations V of U such that

$$V = \bigoplus_{\lambda \in X} V_\lambda,$$

where

$$V_\lambda = \left\{ v \in V,\ \forall \mu \in Y,\ k_\mu v = q^{\langle \mu, \lambda \rangle} v \right\}.$$

In this situation, it is more convenient to introduce the *idempotent form* of U.

Definition 2.2.2. The idempotent form of the quantum group is the $\mathbb{Q}(q)$-linear category $\dot{U}$ defined by the following data.

- The set of objects of $\dot{U}$ is X.

- Given $\lambda, \lambda' \in X$, the morphisms $\lambda \to \lambda'$ are $\mathbb{Q}(q)$-linear combinations of words of the form $e_{i_1}^{n_1} f_{j_1}^{m_1} \dots e_{i_r}^{n_r} f_{j_r}^{m_r}$, with $r, n_k, m_k \in \mathbb{N}$ and $i_k, j_k \in I$ such that

$$\lambda' - \lambda = \sum_{k=1}^{r} (n_k i_k - m_k j_k).$$

  Composition of morphisms is given by concatenation of words, and the identity morphism of λ is the empty word, denoted 1_λ. When we want to specify that a word y has source λ (resp. target λ'), we will write $y1_\lambda$ (resp. $1_{\lambda'}y$). On these morphisms, we impose the following relations.

  1. For all $i, j \in I$ and $\lambda \in X$, $(e_i f_j - f_j e_i)\, 1_\lambda = \delta_{i,j} \left[\langle i^\vee, \lambda\rangle\right]_i 1_\lambda$.

  2. For all $i \neq j \in I$,

$$\sum_{\ell=0}^{-c_{i,j}+1} (-1)^\ell e_i^{(\ell)} e_j e_i^{(-c_{i,j}+1-\ell)} = 0, \qquad \sum_{\ell=0}^{-c_{i,j}+1} (-1)^\ell f_i^{(\ell)} f_j f_i^{(-c_{i,j}+1-\ell)} = 0.$$

With this definition, type 1 weight representations of U are in one-to-one correspondence with functors $\dot{U} \to \mathrm{Vect}_{\mathbb{Q}(q)}$.

2.3 Quiver Hecke Algebras

2.3.1 Symmetric Groups and Polynomial Rings

Let $n \in \mathbb{N}$. We denote by $\mathfrak{S}_n$ the symmetric group on the set $\{1, \dots, n\}$. Given two distinct integers $k, m \in \{1, \dots, n\}$, we denote by $s_{k,m}$ the permutation $(k\ m)$ of $\mathfrak{S}_n$. We put $s_k = s_{k,k+1}$. For $k \leqslant m$, we denote by $\mathfrak{S}_{[k,m]}$ the subgroup of $\mathfrak{S}_n$ generated by $s_k, \dots, s_{m-1}$.

Given a sequence $\underline{k} = (k_1, \dots, k_r)$ of elements of $\{1, \dots, n-1\}$, we put $s_{\underline{k}} = s_{k_1} \dots s_{k_r}$.

If $k \leqslant \ell$, we denote by $[k \uparrow \ell]$ the sequence $(k, k+1, \dots, \ell)$ and by $[\ell \downarrow k]$ the sequence $(\ell, \ell-1, \dots k)$. If $k > \ell$, $[k \uparrow \ell]$ and $[\ell \downarrow k]$ are understood to be the empty sequence $\varnothing$, in which case $s_\varnothing = 1$.

Let $\omega \in \mathfrak{S}_n$. A *reduced expression* of ω is a finite sequence $\underline{k} = (k_1, \dots, k_r)$ such that $\omega = s_{\underline{k}}$ and r is minimal. The integer r is then called the *length* of ω and denoted $l(\omega)$. The longest element of $\mathfrak{S}_{[k,m]}$ is denoted $\omega_0[k, m]$; it has length $\frac{(m-k)(m-k+1)}{2}$.

We denote by P_n the K-algebra $K[x_1, \dots, x_n]$. The group $\mathfrak{S}_n$ acts on P_n by permuting the variables $x_1, \dots, x_n$. For $k \neq m$ two integers in $\{1, \dots, n\}$, the *Demazure operator* or *divided difference operator* is the $P_n^{\mathfrak{S}_n}$-linear operator $\partial_{k,m}$ on P_n defined by

$$\partial_{k,m}(f) = \frac{f - s_{k,m}(f)}{x_m - x_k}.$$

The Demazure operators are conjugate under the action of $\mathfrak{S}_n$. This means that for $k \neq m$ and $\omega \in \mathfrak{S}_n$, we have $\omega\, \partial_{k,m}\, \omega^{-1} = \partial_{\omega(k),\omega(m)}$. We put $\partial_k = \partial_{k,k+1}$. Given a sequence $\underline{k} = (k_1, \dots, k_r)$, we put $\partial_{\underline{k}} = \partial_{k_1} \dots \partial_{k_r}$. The Demazure operators satisfy the following relations:

$$\partial_{k,m}(fg) = \partial_{k,m}(f)g + s_{k,m}(f)\partial_{k,m}(g), \tag{2.3.1}$$

$$\partial_{k,m}^2 = 0, \tag{2.3.2}$$

$$\partial_k \partial_\ell = \partial_\ell \partial_k \quad \text{ if } |k - \ell| > 1, \tag{2.3.3}$$

$$\partial_k \partial_{k+1} \partial_k = \partial_{k+1} \partial_k \partial_{k+1} \quad \text{ if } k < n-1. \tag{2.3.4}$$

For all $\omega \in \mathfrak{S}_n$, we can define an operator ∂_ω in the following way: take $\underline{k}$ a reduced expression of ω, and let $\partial_\omega = \partial_{\underline{k}}$. Since the Demazure operators satisfy relations (2.3.3) and (2.3.4), this does not depend on the choice of the reduced expression. As a graded $P_n^{\mathfrak{S}_n}$-module, P_n

is free of graded rank $q^{\frac{n(n-1)}{2}}[n]!$. The following sets are bases

$$\{x_1^{a_1}\dots x_n^{a_n},\ 0\leqslant a_i\leqslant n-i\},\quad \left\{\partial_\omega(x_2x_3^2\dots x_n^{n-1}),\ \omega\in\mathfrak{S}_n\right\}.$$

2.3.2 Quiver Hecke Algebras

Definitions

We follow [Rou08, Subsection 3.1] (see also [Rou12]). We fix a root datum (X,Y) of type $(I,\cdot)$. For all $i,j\in I$, we fix a polynomial $Q_{i,j}(u,v)\in K[u,v]$. We assume that this data satisfies the following conditions.

- For all $i\in I$, $Q_{i,i}(u,v)=0$.
- For all $i,j\in I$, $Q_{i,j}(u,v)=Q_{j,i}(v,u)$.
- For all $i\neq j\in I$, there are invertible elements $t_{i,j}$ and $t_{j,i}$ of K such that $Q_{i,j}(u,v)$ has the form

  $$Q_{i,j}(u,v)\in t_{i,j}u^{-c_{i,j}}+t_{j,i}v^{-c_{j,i}}+\sum_{\substack{s,t>0\\ d_is+d_jt=-i\cdot j}}Ku^sv^t.$$

  This last condition ensures that $Q_{i,j}(u,v)$ is homogeneous of degree $-2(i\cdot j)$ when u,v are variables of respective degrees $i\cdot i$, $j\cdot j$.

The quiver Hecke algebras are a family of K-algebras attached to the data of these polynomials.

Definition 2.3.1. Let β be an element of Q^+ of height n. The *quiver Hecke algebra H_β* is the unital K-algebra with generators 1_ν for $\nu\in I^\beta$, $x_1,\dots,x_n$ and $\tau_1,\dots,\tau_{n-1}$ subject to the following relations.

1. For all $\nu,\nu'\in I^\beta$, $1_\nu 1_{\nu'}=\delta_{\nu,\nu'}1_\nu$, and $\sum_{\nu\in I^\beta}1_\nu=1_\beta$, where 1_β is the unit of H_β.

2. For all $\nu \in I^\beta$ and $k \in \{1, \dots, n\}$, $x_k 1_\nu = 1_\nu x_k$.

3. For all $\nu \in I^\beta$ and $k \in \{1, \dots, n-1\}$, $\tau_k 1_\nu = 1_{s_k(\nu)} \tau_k$.

4. For all $k, \ell \in \{1, \dots, n\}$, $x_k x_\ell = x_\ell x_k$.

5. For all $\nu \in I^\beta$ and $k \in \{1, \dots, n-1\}$, $\tau_k^2 1_\nu = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) 1_\nu$.

6. For all $\nu \in I^\beta$, $\ell \in \{1, \dots n\}$ and $k \in \{1, \dots, n-1\}$,

$$\left(\tau_k x_\ell - x_{s_k(\ell)} \tau_k\right) 1_\nu = \begin{cases} 1_\nu & \text{if } \ell = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ -1_\nu & \text{if } \ell = k \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

7. For all $k, \ell \in \{1, \dots, n-1\}$ such that $|k - \ell| > 1$, $\tau_k \tau_\ell = \tau_\ell \tau_k$.

8. For all $k \in \{1, \dots, n-2\}$ and $\nu \in I^\beta$,

$$\left(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k\right) 1_\nu = \delta_{\nu_k, \nu_{k+2}} \partial_{k,k+2} \left(Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1})\right) 1_\nu.$$

Convention. We will number components of tuples $\nu \in I^n$ from the right to the left. Namely, we will write $\nu = (\nu_n, \dots, \nu_1)$ for $\nu \in I^n$. We let the symmetric group $\mathfrak{S}_n$ act on I^n accordingly. For instance, if $\nu = (c, b, a) \in I^3$, we have $s_1(\nu) = (c, a, b)$ and $s_2(\nu) = (b, c, a)$.

For a sequence $\underline{k} = (k_1, \dots, k_r)$ of elements of $\{1, \dots, n-1\}$, we put $\tau_{\underline{k}} = \tau_{k_1} \dots \tau_{k_r}$. We also define K-algebras H_n and H by

$$H_n = \bigoplus_{\substack{\beta \in Q^+ \\ |\beta| = n}} H_\beta,$$

$$H = \bigoplus_{n \in \mathbb{N}} H_n.$$

The algebra H_n is unital only when I is finite, while the algebra H is not unital. There is a grading on H_n, defined as follows.

- For all $\nu \in I^n$, 1_ν is in degree 0.
- For all $\nu \in I^n$ and $k \in \{1, \dots, n\}$, $x_k 1_\nu$ is in degree $\nu_k \cdot \nu_k$.
- For all $\nu \in I^n$ and $k \in \{1, \dots, n-1\}$, $\tau_k 1_\nu$ is in degree $-\nu_k \cdot \nu_{k+1}$.

Affine Nil Hecke Algebra

A case of particular importance is that of root datum of type A_1 (that is, when I is a singleton). The quiver Hecke algebras are then known as the *affine nil Hecke algebras*, and we will denote them by H_n^0. For more clarity, we give the definition in that case.

Definition 2.3.2. The *affine nil Hecke algebra* H_n^0 is the K-algebra on the generators $x_1, \dots, x_n$ and $\tau_1, \dots, \tau_{n-1}$, subject to the following relations.

1. For all $k, \ell \in \{1, \dots, n\}$, $x_k x_\ell = x_\ell x_k$.
2. For all $k \in \{1, \dots, n-1\}$, $\tau_k^2 = 0$.
3. For all $k \in \{1, \dots, n-1\}$ and $\ell \in \{1, \dots n\}$,

$$\tau_k x_\ell - x_{s_k(\ell)} \tau_k = \begin{cases} 1 & \text{if } \ell = k+1, \\ -1 & \text{if } \ell = k, \\ 0 & \text{otherwise.} \end{cases}$$

4. For all $k, \ell \in \{1, \dots, n-1\}$ such that $|k - \ell| > 1$, $\tau_k \tau_\ell = \tau_\ell \tau_k$.
5. For all $k \in \{1, \dots, n-2\}$,

$$\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k = 0.$$

The algebra H_n^0 is graded, with x_ℓ in degree 2 for all $\ell \in \{1,\dots,n\}$, and τ_k in degree -2 for all $k \in \{1,\dots,n-1\}$. For all $\omega \in \mathfrak{S}_n$, we can define an element $\tau_\omega \in H_n^0$ in the following way: choose $\underline{k}$ a reduced expression of ω, and let $\tau_\omega = \tau_{\underline{k}}$. By relations (4) and (5) in Definition 2.3.2, this does not depend on the choice of the reduced expression.

The algebra H_n^0 can be given a explicit realization as an algebra of operators on P_n. There is an isomorphism of graded K-algebras ([Rou08, Proposition 3.4])

$$\begin{cases} H_n^0 & \xrightarrow{\sim} & \mathrm{End}^\bullet_{P_n^{\mathfrak{S}_n}}(P_n), \\ x_k & \mapsto & x_k, \\ \tau_k & \mapsto & \partial_k. \end{cases} \tag{2.3.5}$$

In particular, H_n^0 is isomorphic the algebra of $(n!)\times(n!)$ matrices with coefficients in $P_n^{\mathfrak{S}_n}$. The H_n^0-module P_n with action given by (2.3.5) is the unique indecomposable graded projective H_n-module up to isomorphism and grading shift. Define

$$\begin{aligned} e_n &= x_2 x_3^2 \dots x_n^{n-1} \tau_{\omega_0[1,n]}, \\ e'_n &= (-1)^{\frac{n(n-1)}{2}} \tau_{\omega_0[1,n]} x_1^{n-1} x_2^{n-2} \dots x_{n-1}. \end{aligned}$$

These are orthogonal primitive idempotents of H_n^0. We have isomorphisms of graded $(H_n, P_n^{\mathfrak{S}_n})$-bimodules:

$$\begin{cases} P_n & \xrightarrow{\sim} & q^{n(n-1)} H_n e_n, \\ 1 & \mapsto & \tau_{\omega_0[1,n]}, \end{cases} \quad \text{and} \quad \begin{cases} P_n & \xrightarrow{\sim} & H_n e'_n, \\ 1 & \mapsto & e'_n. \end{cases}$$

Hence we have Morita equivalences

$$\begin{cases} H_n^0\text{−mod} & \xrightarrow{\sim} & P_n^{\mathfrak{S}_n}\text{−mod}, \\ M & \mapsto & e_n M, \end{cases} \quad \text{and} \quad \begin{cases} H_n^0\text{−mod} & \xrightarrow{\sim} & P_n^{\mathfrak{S}_n}\text{−mod}, \\ M & \mapsto & e'_n M. \end{cases} \tag{2.3.6}$$

In the following chapters, we will often have to make computations in affine nil Hecke algebras. The following lemma records the basic facts that we will use.

Lemma 2.3.3. *Let $n \in \mathbb{N}$.*

1. *For all $k \in \{1, \dots, n-1\}$ and $P \in P_n$ we have*

$$\tau_k P - s_k(P)\tau_k = \partial_k(P).$$

2. *For all $\omega_1, \omega_2 \in \mathfrak{S}_n$ we have*

$$\tau_{\omega_1}\tau_{\omega_2} = \begin{cases} \tau_{\omega_1\omega_2} & \text{if } l(\omega_1\omega_2) = l(\omega_1) + l(\omega_2), \\ 0 & \text{otherwise.} \end{cases}$$

3. *Let $\underline{k}$ a finite sequence of elements of $\{1, \dots, n-1\}$. For all $P \in P_n$ we have*

$$\tau_{\underline{k}} P \tau_{\omega_0[1,n]} = \partial_{\underline{k}}(P)\tau_{\omega_0[1,n]}.$$

Proof. For the first statement, consider the maps

$$\begin{cases} P_n & \to & H_n^0, \\ P & \mapsto & P\tau_k - s_k(P)\tau_k, \end{cases} \quad \text{and} \quad \begin{cases} P_n & \to & H_n^0, \\ P & \mapsto & \partial_k(P). \end{cases}.$$

They are both skew derivations, in the sense that they satisfy relation (2.3.1) with respect to the automorphism s_k. Furthermore, by relation (3) in Definition 2.3.2, they coincide on $x_1, \dots, x_n$, which are algebra generators of P_n. Hence the two maps are equal, proving the statement.

For the second statement, the case $l(\omega_1\omega_2) = l(\omega_1) + l(\omega_2)$ follows from the definition of τ_ω. If $l(\omega_1\omega_2) < l(\omega_1) + l(\omega_2)$, then we can decompose ω_1 in the form $\omega_1 = \omega s_k \omega'$ where $l(\omega'\omega_2) = l(\omega') + l(\omega_2)$ and ω' has maximal length for this property. Since $s_k\omega' < \omega'$, there is a reduced expression of ω' of the form $(k, \ldots)$. Hence, $\tau_{\omega'\omega_2} \in \tau_k H_n^0$, so $\tau_k \tau_{\omega'\omega_2} = 0$ by relation (2) in Definition 2.3.2. The result follows.

For the last statement, by induction on the length of the finite sequence $\underline{k}$, it suffices to treat the case where $\underline{k}$ has length 1. If $k \in \{1, \ldots, n-1\}$, then for all $P \in P_n$ we have

$$\tau_k P \tau_{\omega_0[1,n]} = s_k(P) \tau_k \tau_{\omega_0[1,n]} + \partial_k(P) \tau_{\omega_0[1,n]}$$

by statement (1) of the Lemma. However, $\tau_k \tau_{\omega_0[1,n]} = 0$ by statement (2) of the Lemma. Hence we have $\tau_k P \tau_{\omega_0[1,n]} = \partial_k(P) \tau_{\omega_0[1,n]}$, which completes the proof. □

Polynomial Representation

We now go back to quiver Hecke algebras of arbitrary types. The action of H_n^0 on P_n has an analogue in that general case. Let $\beta \in Q^+$ and consider the commutative algebra

$$P_\beta = \bigoplus_{\nu \in I^\beta} P_{|\beta|} 1_\nu,$$

where $(1_\nu)_{\nu \in I^\beta}$ forms a system of orthogonal idempotents. Also choose polynomials $P_{i,j}(u,v)$ for all $i, j \in I$ such that $Q_{i,j}(u,v) = P_{i,j}(u,v) P_{j,i}(v,u)$. One possible choice is to fix a total order $<$ on I and let

$$P_{i,j}(u,v) = \begin{cases} Q_{i,j}(u,v) & \text{if } i \geqslant j, \\ 1 & \text{otherwise.} \end{cases}$$

Associated to this data is an action of H_β on P_β defined by the following formulas:

$$
\begin{aligned}
&1_\mu \cdot f1_\nu = \delta_{\nu,\mu} f1_\nu, \\
&x_k \cdot f1_\nu = x_k f1_\nu, \\
&\tau_k \cdot f1_\nu = \begin{cases} \partial_k(f)1_{s_k(\nu)} & \text{if } \nu_k = \nu_{k+1}, \\ P_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k)s_k(f)1_{s_k(\nu)} & \text{otherwise.} \end{cases}
\end{aligned}
$$

This representation is faithful ([Rou08, Propostion 3.12]). As a consequence, we can determine the center and bases of H_β.

Theorem 2.3.4 ([Rou08, Theorem 3.7])**.** *The K-algebra H_β is free as a K-module. Let $n = |\beta|$ and let T be a complete set of reduced decompositions of elements of $\mathfrak{S}_n$. Then the following sets are bases over K for H_β:*

$$
\begin{aligned}
&\left\{ \tau_{\underline{\omega}} x_1^{a_1} \dots x_n^{a_n} 1_\nu \middle|\, \nu \in I^\beta,\, a_1, \dots, a_n \geqslant 0,\, \underline{\omega} \in T \right\}, \\
&\left\{ x_1^{a_1} \dots x_n^{a_n} \tau_{\underline{\omega}} 1_\nu \middle|\, \nu \in I^\beta,\, a_1, \dots, a_n \geqslant 0,\, \underline{\omega} \in T \right\}.
\end{aligned}
$$

This theorem is known as the *PBW Theorem* for quiver Hecke algebras. In particular, it implies that the commutative subalgebra of H_β generated by $x_1, \dots, x_n$ and 1_ν for $\nu \in I^\beta$ is isomorphic to P_β. Furthermore, H_β is free of rank $n!$ as a left (or right) module over P_β. A basis is given by $\{\tau_{\underline{\omega}} | \, \underline{\omega} \in T\}$, for T any set of reduced expressions for the elements of $\mathfrak{S}_n$.

Proposition 2.3.5 ([Rou08, Proposition 3.9])**.** *The center of H_β is the subalgebra $P_\beta^{\mathfrak{S}_n}$, where the action of $\mathfrak{S}_n$ on P_β is defined by $\omega(f1_\nu) = \omega(f)1_{\omega(\nu)}$.*

2.4 Categorified Quantum Groups

2.4.1 Positive Part

Recall that we have fixed a root datum (X, Y) of type $(I, \cdot)$, and polynomials $(Q_{i,j})_{i,j\in I}$ satisfying certain conditions. We define now a graded monoidal category $\mathcal{U}^+$ attached to this data, following [Rou08].

Definition 2.4.1. Let $\mathcal{U}^+$ be the K-linear, graded and idempotent complete strict monoidal category defined as follows. The objects of $\mathcal{U}^+$ are generated by E_i for $i \in I$. The morphisms of $\mathcal{U}^+$ are generated by $x_i : E_i \to E_i$ of degree $(i \cdot i)$ and $\tau_{i,j} : E_iE_j \to E_jE_i$ of degree $-(i \cdot j)$ for all $i, j \in I$. On these generating morphisms we impose the following relations.

1. $\tau_{i,j} \circ x_iE_j - E_jx_i \circ \tau_{i,j} = \delta_{i,j} = x_jE_i \circ \tau_{i,j} - \tau_{i,j} \circ E_ix_j$.

2. $\tau_{j,i} \circ \tau_{i,j} = Q_{i,j}(x_i, x_j)$.

3. $\tau_{j,k}E_i \circ E_j\tau_{i,k} \circ \tau_{i,j}E_k - E_k\tau_{i,j} \circ \tau_{i,k}E_j \circ E_i\tau_{j,k} = \delta_{i,k}\partial_{1,3}(Q_{i,j}(x_i, x_j))$.

A concise way to restate these relations is to say that the generating morphisms of $\mathcal{U}^+$ induce a graded action of quiver Hecke algebras on the objects of $\mathcal{U}^+$. More precisely, given $\beta \in Q^+$ of height r, consider the object E_β of $\mathcal{U}^+$ defined by

$$E_\beta = \bigoplus_{\nu\in I^\beta} E_\nu, \quad \text{where} \quad E_\nu = E_{\nu_r} \dots E_{\nu_1}.$$

Then there is an isomorphism of graded K-algebras:

$$\begin{cases} H_\beta^{\mathrm{op}} & \to & \mathrm{End}^\bullet_{\mathcal{U}^+}(E_\beta), \\ x_k & \mapsto & \bigoplus_{\nu\in I^\beta} E_{\nu_r} \dots E_{\nu_{k+1}} x_{\nu_k} E_{\nu_{k-1}} \dots E_{\nu_1}, \\ \tau_k & \mapsto & \bigoplus_{\nu\in I^\beta} E_{\nu_r} \dots E_{\nu_{k+2}} \tau_{\nu_k,\nu_{k+1}} E_{\nu_{k-1}} \dots E_{\nu_1}. \end{cases}$$

It follows that there is an equivalence of categories

$$\mathcal{U}^+ \simeq \bigoplus_{\beta \in Q^+} H_\beta\text{-proj}.$$

Under this equivalence, the generating object E_i corresponds to the free H_i-module of rank 1. The monoidal structure on $\mathcal{U}^+$ corresponds to an induction product for quiver Hecke algebras that we describe now. For $\beta, \gamma \in Q^+$, there is a (non-unital) morphism of K-algebras $r_\beta^{\beta+\gamma} : H_\beta \to H_{\beta+\gamma}$ (called *right inclusion*) defined by

$$r_\beta^{\beta+\gamma}(1_\nu) = \sum_{\mu \in I^\gamma} 1_{\mu\nu}, \quad r_\beta^{\beta+\gamma}(x_k) = x_k, \quad r_\beta^{\beta+\gamma}(\tau_\ell) = \tau_\ell$$

for all $\nu \in I^\beta$, $k \in \{1, \dots, |\beta|\}$ and $\ell \in \{1, \dots, |\beta|-1\}$. Here, $\mu\nu$ denotes the concatenation of $\mu \in I^\gamma$ with $\nu \in I^\beta$. We also have a (non-unital) morphism of K-algebras $l_\beta^{\beta+\gamma} : H_\beta \to H_{\beta+\gamma}$ (called *left inclusion*) defined by

$$l_\beta^{\beta+\gamma}(1_\nu) = \sum_{\mu \in I^\gamma} 1_{\nu\mu}, \quad l_\beta^{\beta+\gamma}(x_k) = x_{|\gamma|+k}, \quad l_\beta^{\beta+\gamma}(\tau_\ell) = \tau_{|\gamma|+\ell}$$

for all $\nu \in I^\beta$, $k \in \{1, \dots, |\beta|\}$ and $\ell \in \{1, \dots, |\beta|-1\}$. By Theorem 2.3.4, right and left inclusions are injective. There is an injective morphism of K-algebras:

$$\begin{cases} H_\beta \otimes H_\gamma & \to & H_{\beta+\gamma}, \\ y \otimes z & \mapsto & y \diamond z = l_\gamma^{\beta+\gamma}(y) r_\beta^{\beta+\gamma}(z). \end{cases}$$

This defines a binary operation $\diamond$ on H, which is simply the tensor product operation in $\mathcal{U}^+$. We let $1_{\beta,\gamma} = 1_\beta \diamond 1_\gamma$ and $H_{\beta,\gamma} = H_\beta \otimes H_\gamma$. Note that this morphism is not unital in general, since $1_{\beta,\gamma} \neq 1_{\beta+\gamma}$ unless β and γ are both multiples of the same simple root. Still, this endows $H_{\beta+\gamma}1_{\beta,\gamma}$ with a natural structure of $(H_{\beta+\gamma}, H_{\beta,\gamma})$-bimodule. This bimodule is free as a right $H_{\beta,\gamma}$-module, and by Theorem 2.3.4 it decomposes as

$$H_{\beta+\gamma}1_{\beta,\gamma} = \bigoplus_{\omega \in T} \tau_{\underline{\omega}} H_{\beta,\gamma}, \tag{2.4.1}$$

where T denotes a set of minimal length representatives of left cosets of $\mathfrak{S}_{|\gamma|} \times \mathfrak{S}_{|\beta|}$ in $\mathfrak{S}_{|\beta+\gamma|}$. Given M an H_β-module and N an H_γ-module, we can then define

$$MN = H_{\beta+\gamma}1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes N).$$

If M and N are projective, then so is MN. This defines the monoidal structure of $\mathcal{U}^+$ in terms of representations of quiver Hecke algebras. As a consequence of (2.4.1), we have the following lemma which will be useful in the subsequent chapters.

Lemma 2.4.2. *Let $m, n \in \mathbb{N}$ and let T be a complete set of reduced expressions of minimal length representatives of left cosets of $\mathfrak{S}_m \times \mathfrak{S}_n$ in $\mathfrak{S}_{m+n}$. Let $\underline{\omega}_0$ be the longest element in T. Then for all $y \in H_n$ and $z \in H_m$ we have*

$$(y \diamond z)\tau_{\underline{\omega}_0} - \tau_{\underline{\omega}_0}(z \diamond y) \in \sum_{\underline{\omega} \in T \setminus \{\underline{\omega}_0\}} \tau_{\underline{\omega}} H_{m,n}.$$

As an important consequence of the action of quiver Hecke algebras on the objects of $\mathcal{U}^+$, we can define the *divided powers* $E_i^{(n)}$ for all $i \in I$ and $n \in \mathbb{N}$. First, remark that we have an isomorphism of K-algebras $H_n^0 = H_{ni} \simeq \mathrm{End}^\bullet_{\mathcal{U}^+}(E_i^n)^{\mathrm{op}}$. Thus, the idempotent $e_n \in H_n^0$ acts on E_i^n, and we can define

$$E_i^{(n)} = q_i^{\frac{n(n-1)}{2}}(E_i^n)e_n,$$

where we write the action of H_n^{opp} on E_i^n on the right. When seen as a graded projective H_{ni}-module, we have simply $E_i^{(n)} \simeq q_i^{-\frac{n(n-1)}{2}} P_{ni}$, where the action on P_{ni} is given by the polynomial representation. Then we have isomorphisms

$$E_i^n \simeq [n]_i! E_i^{(n)}, \tag{2.4.2}$$

$$E_i^{(a)} E_i^{(b)} \simeq \begin{bmatrix} a+b \\ a \end{bmatrix}_i E_i^{(a+b)}. \tag{2.4.3}$$

2.4.2 2-Categories Associated to Root Data

We now define a 2-category $\mathcal{U}$ categorifying the integral idempotent version of the quantum group. The 2-category $\mathcal{U}$ will be defined as the idempotent completion of a 2-category $\mathcal{U}'$ that we define first.

Definition 2.4.3. Define a strict 2-category $\mathcal{U}'$ by the following data.

- The set of objects of $\mathcal{U}'$ is X.
- Given $\lambda, \lambda' \in X$, the 1-morphisms $\lambda \to \lambda'$ are direct sums of shifts of words of the form $E_{i_1}^{n_1} F_{j_1}^{m_1} \dots E_{i_r}^{n_r} F_{j_r}^{m_r}$, with $r, n_i, m_i \in \mathbb{N}$ and $i_1, j_1, \dots, i_r, j_r \in I$ such that

  $$\lambda' - \lambda = \sum_{k=1}^{r} (n_k i_k - m_k j_k).$$

  Composition of of 1-morphisms is given by concatenation of words, and the identity 1-morphism of λ is the empty word, denoted 1_λ. When we want to specify that a word X has source λ (resp. target λ'), we will write $X 1_\lambda$ (resp. $1_{\lambda'} X$).

- The 2-morphisms of $\mathcal{U}'$ are generated by $x_i : E_i \to E_i$ of degree $(i \cdot i)$, $\tau_{i,j} : E_i E_j \to E_j E_i$ of degree $-(i \cdot j)$, $\eta_i : 1_\lambda \to F_i E_i 1_\lambda$ of degree $d_i(1 + \langle i^\vee, \lambda \rangle)$, and $\varepsilon_i : E_i F_i 1_\lambda \to 1_\lambda$ of degree $d_i(1 - \langle i^\vee, \lambda \rangle)$ for all $i, j \in I$ and $\lambda \in X$. On these 2-morphisms we impose the following conditions.

  1. The morphisms x_i and $\tau_{i,j}$ satisfy the relations of Definition 2.4.1.

2. We have the relations $E_i = \varepsilon_i E_i \circ E_i \eta_i$ and $F_i = F_i \varepsilon_i \circ \eta_i F_i$.

3. For all $i \neq j$ and $\lambda \in X$, we add formal inverses to certain 2-morphisms $\sigma_{i,j}$ and $\rho_{i,\lambda}$. These 2-morphisms are defined as follows. Given $i, j \in I$, we define $\sigma_{i,j}$ by

$$\sigma_{i,j} = F_j E_i \varepsilon_j \circ F_j \tau_{j,i} F_j \circ \eta_j E_i F_j : E_i F_j \to F_j E_i. \tag{2.4.4}$$

If $\langle i^\vee, \lambda \rangle \geqslant 0$, we define $\rho_{i,\lambda}$ by

$$\rho_{i,\lambda} = \begin{bmatrix} \sigma_{i,i} \\ \varepsilon_i \\ \varepsilon_i \circ x_i F_i \\ \vdots \\ \varepsilon_i \circ x_i^{\langle i^\vee, \lambda \rangle - 1} F_i \end{bmatrix} : E_i F_i 1_\lambda \to F_i E_i 1_\lambda \oplus [\langle i^\vee, \lambda \rangle]_i \, 1_\lambda. \tag{2.4.5}$$

If $\langle i^\vee, \lambda \rangle \leqslant 0$, we define $\rho_{i,\lambda}$ by

$$\rho_{i,\lambda} = \begin{bmatrix} \sigma_{i,i} & \eta_i & F_i x_i \circ \eta_i & \dots & F_i x_i^{1-\langle i^\vee, \lambda \rangle} \circ \eta_i \end{bmatrix} : E_i F_i 1_\lambda \oplus [- \langle i^\vee, \lambda \rangle]_i 1_\lambda \to F_i E_i 1_\lambda. \tag{2.4.6}$$

Definition 2.4.4. The 2-category $\mathcal{U}$ is the idempotent completion of $\mathcal{U}'$. This means that $\mathcal{U}$ has the same set of objects as $\mathcal{U}'$, and that for all $\lambda, \lambda' \in X$, the category $\mathrm{Hom}_{\mathcal{U}}(\lambda, \lambda')$ is the idempotent completion of the category $\mathrm{Hom}_{\mathcal{U}'}(\lambda, \lambda')$.

In particular, the divided powers $E_i^{(n)}$ of $\mathcal{U}^+$ induce 1-morphisms $1_{\lambda+ni} E_i^{(n)} 1_\lambda$ in $\mathcal{U}$. Condition (2) in Definition 2.4.3 means that we have adjoint pairs $\left(E_i 1_\lambda, q_i^{-1-\langle i^\vee, \lambda \rangle} F_i 1_{\lambda+i}\right)$ with unit and counit of adjunction given by η_i and ε_i. Using these adjunctions, we obtain an action of quiver Hecke algebras on products of F_i's. We can then define the divided power $F_i^{(n)}$ by

$$F_i^{(n)} = q_i^{\frac{n(n-1)}{2}} e_n'(F^n).$$

Then, there are adjoint pairs $\left(E_i^{(n)}1_\lambda, q_i^{-n(\langle i^\vee,\lambda\rangle+n)}F_i^{(n)}1_{\lambda+ni}\right)$.

Remark 2.4.5. This definition of $\mathcal{U}$ is due to Rouquier. Khovanov and Lauda define specializations of the quiver Hecke algebras introduced here and a different 2-category, see [KL09], [KL11], [KL10]. The main differences are that Khovanov-Lauda's version requires adjoint pairs $\left(q_i^{1+\langle i^\vee,\lambda\rangle}F_i1_{\lambda+i}, E_i1_\lambda\right)$ and the inverses of the maps $\sigma_{i,j}$ and $\rho_{i,\lambda}$ are given explicitly in terms of the unit and counit of this second adjunction. However, Brundan proved in [Bru16] that the 2-categories of Khovanov-Lauda and Rouquier are isomorphic. In particular, in the 2-category $\mathcal{U}$ that we defined, the 2-morphisms E_i1_λ and F_i1_λ have both left and right adjoints.

Given $\lambda \in \mathbb{Z}$ we define

$$\mathcal{U}1_\lambda = \bigoplus_{\lambda' \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{U}}(\lambda, \lambda').$$

This is a K-linear, graded and idempotent complete category. In $\mathcal{U}1_\lambda$, there are isomorphisms

$$E_i^{(a)}F_i^{(b)}1_\lambda \simeq \bigoplus_{k=0}^{\min\{a,b\}} \left[\begin{smallmatrix}\langle i^\vee,\lambda\rangle+a-b\\ k\end{smallmatrix}\right]_i F_i^{(b-k)}E_i^{(a-k)}1_\lambda \quad \text{if } \langle i^\vee,\lambda\rangle \geqslant b-a, \tag{2.4.7}$$

$$F_i^{(b)}E_i^{(a)}1_\lambda \simeq \bigoplus_{k=0}^{\min\{a,b\}} \left[\begin{smallmatrix}-\langle i^\vee,\lambda\rangle-a+b\\ k\end{smallmatrix}\right]_i E_i^{(a-k)}F_i^{(b-k)}1_\lambda \quad \text{if } \langle i^\vee,\lambda\rangle \leqslant b-a. \tag{2.4.8}$$

They can be constructed using the isomorphisms $\rho_{i,\lambda}$ (see [Rou08, Lemma 4.14] for details).

2.4.3 Categorification Theorems

We now relate the monoidal category $\mathcal{U}^+$ and the 2-category $\mathcal{U}$ to the quantum group U. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. We denote by $\mathbf{f}$ the $\mathbb{Q}(q)$-subalgebra of U generated by e_i for $i \in I$, and by $\mathbf{f}_{\mathcal{A}}$ the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by $e_i^{(n)}$ for $i \in I$ and $n \in \mathbb{N}$.

Consider the *split Grothendieck group* $K_0(\mathcal{U}^+)$. This is abelian group generated by isomorphism classes $[M]$ of objects $M \in \mathcal{U}^+$ modulo the relations $[M] + [N] = [M \oplus N]$ for all $M, N \in \mathcal{U}^+$. Since $\mathcal{U}^+$ is graded and monoidal, $K_0(\mathcal{U}^+)$ inherits the structure of an $\mathcal{A}$-algebra, via the formulas $q[M] = [qM]$, $[M][N] = [MN]$ for all M, N in $\mathcal{U}^+$.

Theorem 2.4.6 ([KL11, Theorem 8]). *There exists a unique isomorphism of $\mathcal{A}$-algebras*

$$\gamma : \mathbf{f}_{\mathcal{A}} \xrightarrow{\sim} K_0(\mathcal{U}^+)$$

such that $\gamma\left(e_i^{(n)}\right) = \left[E_i^{(n)}\right]$ for all $i \in I$ and $n \in \mathbb{N}$.

Similarly, we denote by $\dot{U}_{\mathcal{A}}$ the $\mathcal{A}$-linear subcategory of $\dot{U}$ whose morphisms are generated by $e_i^{(n)}$ and $f_i^{(n)}$ for $i \in I$ and $n \in \mathbb{N}$. The split Grothendieck group $K_0(\mathcal{U})$ is the $\mathcal{A}$-linear category with same set of objects as $\mathcal{U}$, and with $\mathrm{Hom}_{K_0(\mathcal{U})}(\lambda, \mu) = K_0\left(\mathrm{Hom}_{\mathcal{U}}(\lambda, \mu)\right)$ for all objects λ and μ.

Theorem 2.4.7 ([KL10], [Dup21]). *There exists a unique isomorphism of $\mathcal{A}$-linear categories*

$$\dot{\gamma} : \dot{U}_{\mathcal{A}} \xrightarrow{\sim} K_0(\mathcal{U})$$

such that $\dot{\gamma}\left(e_i^{(n)} 1_\lambda\right) = \left[E_i^{(n)} 1_\lambda\right]$ and $\dot{\gamma}\left(f_i^{(n)} 1_\lambda\right) = \left[F_i^{(n)} 1_\lambda\right]$ for all $i \in I$, $n \in \mathbb{N}$ and $\lambda \in X$.

2.4.4 2-Representations

Definition 2.4.8. A *2-representation* of $\mathcal{U}$ is a strict 2-functor $\mathcal{U} \to \mathfrak{Lin}_K$, where $\mathfrak{Lin}_K$ denotes the strict 2-category of graded K-linear categories. More explicitly, the data of a 2-representation $\mathcal{V}$ of $\mathcal{U}$ is equivalent to the following.

- Categories $\mathcal{V}_\lambda \in \mathfrak{Lin}_K$ for each $\lambda \in X$.

- Functors $E_i : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+i}$ and $F_i : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-i}$ for each $\lambda \in X$ and $i \in I$.

- Natural transformations $x_i : E_i \to E_i$ of degree $(i \cdot i)$, $\tau : E_i E_j \to E_j E_i$ of degree

$-(i \cdot j)$, $\varepsilon_i : E_i F_i 1_\lambda \to 1_\lambda$ of degree $d_i(1 - \langle i^\vee, \lambda \rangle)$, and $\eta_i : 1_\lambda \to F_i E_i 1_\lambda$ of degree $d_i(1 + \langle i^\vee, \lambda \rangle)$, such that the conditions of Definition 2.4.3 are satisfied.

An *abelian 2-representation* is a 2-representation $\mathcal{V}$ for which the category $\mathcal{V}_\lambda$ is abelian for all $\lambda \in X$. In that case, the functors E_i, F_i are automatically exact on $\mathcal{V}$ since they are both left and right adjoints. A 2-representation $\mathcal{V}$ is said to be *integrable* if for every $M \in \mathcal{V}$ and $i \in I$, we have $E_i^k(M) = F_i^k(M) = 0$ for some $k \in \mathbb{N}$.

Given 2-representations $\mathcal{V}, \mathcal{W}$, a *morphism of 2-representations* $D : \mathcal{V} \to \mathcal{W}$ is the data of:

- functors $D : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ for all $\lambda \in X$,
- natural isomorphisms $\alpha_i : DE_i \xrightarrow{\sim} E_i D$ and $\beta_i : DF_i \xrightarrow{\sim} F_i D$ for all $i \in I$,

such that α_i and β_i are compatible with the 2-morphisms $x_i, \tau_{i,j}, \varepsilon_i$ and η_i of $\mathcal{U}$. This compatibility condition means that the following diagrams commute:

$$\begin{tikzcd}
DE_i \arrow[r, "Dx_i"] \arrow[d, "\alpha_i"'] & DF \arrow[d, "\alpha_i"] \\
E_i D \arrow[r, "x_i D"'] & E_i D
\end{tikzcd}
\qquad
\begin{tikzcd}
DE_iE_j \arrow[r, "D\tau_{i,j}"] \arrow[d, "E_i\alpha_j \circ \alpha_i E_j"'] & DE_jE_i \arrow[d, "E_j\alpha_i \circ \alpha_j E_i"] \\
E_iE_jD \arrow[r, "\tau_{i,j}D"'] & E_jE_iD
\end{tikzcd}$$

$$\begin{tikzcd}
DE_iF_i \arrow[r, "D\varepsilon_i"] \arrow[d, "E_i\beta_i \circ \alpha_i F_i"'] & D \\
E_iF_iD \arrow[ur, "\varepsilon_i D"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "D\eta_i"] \arrow[dr, "\eta_i D"'] & DF_iE_i \arrow[d, "F_i\alpha_i \circ \beta_i E_i"] \\
& F_iE_iD
\end{tikzcd}$$

Then, given D a morphism of 2-representations and C a complex of 1-morphisms of $\mathcal{U}$, we have a canonical isomorphism of complexes of functors $DC \simeq CD$.

2.5 Diagrammatic Calculus

Morphisms in monoidal categories have naturally two compatible multiplicative operations: composition and tensor product. It is sometimes convenient to represent morphisms using planar diagrams to take advantage of the inherent two-dimensionality of the algebraic structure. Then, composition is then given by vertical stacking, and the monoidal structure is given by horizontal stacking. We refer to [Sav18] for a general introduction to diagrammatic techniques for categorification.

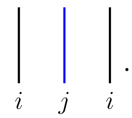
2.5.1 Diagrammatic Presentation of $\mathcal{U}^+$

The general idea of representing morphisms by planar diagrams can be applied to the monoidal category $\mathcal{U}^+$. In what follows, we define a diagrammatic notation to represent the elements of the quiver Hecke algebra H, which act on the objects of $\mathcal{U}^+$ by right multiplication. Given two elements h, h' of H, the diagram corresponding to hh' will be obtained by stacking the diagram of h on top of that of h'. The diagram corresponding to $h \diamond h'$ will be obtained by concatenating the diagram of h to the left of that of h'.This principle is illustrated by the following diagrams:

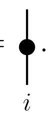
$$\boxed{h} \circ \boxed{h'} = \begin{array}{|c|} \hline h \\ \hline h' \\ \hline \end{array}, \qquad\qquad \boxed{h} \otimes \boxed{h'} = \begin{array}{|c|c|} \hline h & h' \\ \hline \end{array}.$$

We now define the diagrams corresponding to the generating morphisms of $\mathcal{U}^+$. Given $i \in I$, we represent the identity morphism of E_i as a vertical strand labeled by i:

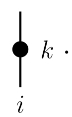
When multiple simple roots are involved, we sometimes draw their respective strands with different colors for the sake of clarity. For example, the identity morphism of $E_iE_jE_i$ is represented by the diagram



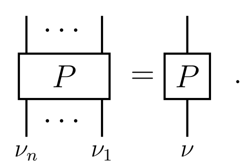
where we have assigned the color black to i and blue to j. The morphism $x_i \in \mathrm{End}^{\bullet}_{\mathcal{U}^+}(E_i)$ is represented by decorating the strand corresponding to the identity with a dot:

$x_i =$ 

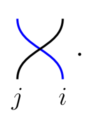
Powers of x_i are simply indicated by labeling the dot with an integer:

$x_i^k =$ 

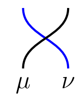
If $P \in P_n$ and $\nu \in I^n$, we represent the morphism $P(x_{\nu_1}, \ldots, x_{\nu_n})1_\nu$ by the following diagrams



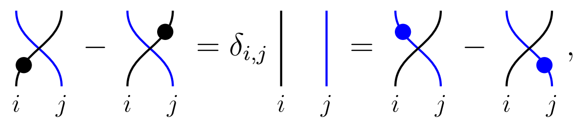
The second one will be a shorthand for the first one. The morphism $\tau_{i,j} \in \mathrm{Hom}^{\bullet}_{\mathcal{U}^+}(E_iE_j, E_jE_i)$ is represented by a crossing of the strands:

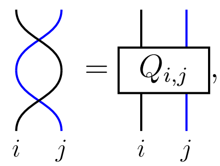
$\tau_{i,j} =$ 

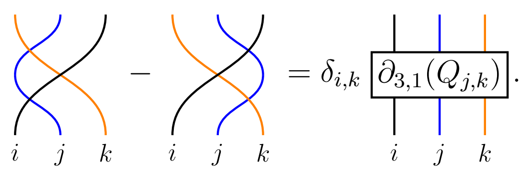
Similarly, given $\nu \in I^n$ and $\mu \in I^m$, we have a morphism $\tau_{\nu,\mu} : E_\nu E_\mu \to E_\mu E_\nu$ obtained by composing the morphisms $\tau_{i,j}$ to permute the factors to E_ν to the right of E_μ. This morphism is represented by

$\tau_{\nu,\mu} =$  .

With these notations, the defining relations of $\mathcal{U}^+$ take the following form:





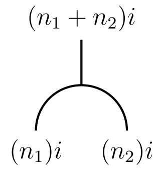


2.5.2 Thick Diagrammatic Calculus

It is convenient to further develop this diagrammatic to include strands labeled by divided powers. This extension is known as *thick calculus* and was originally introduced in [KLMS12] and [Sto19]. Given $i \in I$ and $n \in \mathbb{N}$, the identity morphism of $E_i^{(n)}$ is represented by a vertical strand labeled by $(n)i$:

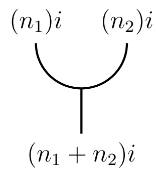
 .

Given $n_1, n_2 \in \mathbb{N}$, the inclusion $E_i^{(n_1+n_2)} \hookrightarrow E_i^{(n_1)} E_i^{(n_2)}$ is represented by



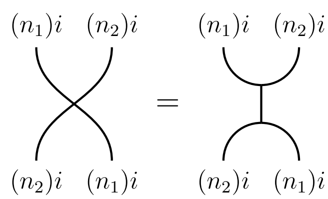
.

The morphism $\tau_{\omega_0[1,n_1+n_2]\omega_0[1,n_2]^{-1}\omega_0[n_2+1,n_1+n_2]^{-1}} : E_i^{(n_1)} E_i^{(n_2)} \twoheadrightarrow E_i^{(n_1+n_2)}$ is represented by



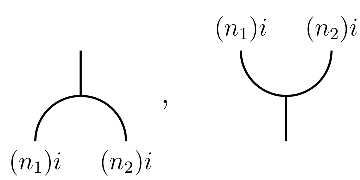
.

We can then define the crossing $E_i^{(n_1)} E_i^{(n_2)} \to E_i^{(n_2)} E_i^{(n_1)}$ by



.

Remark 2.5.1. When no confusion is possible, we omit certain labels on the strands. For instance, we often use the following shorthands:



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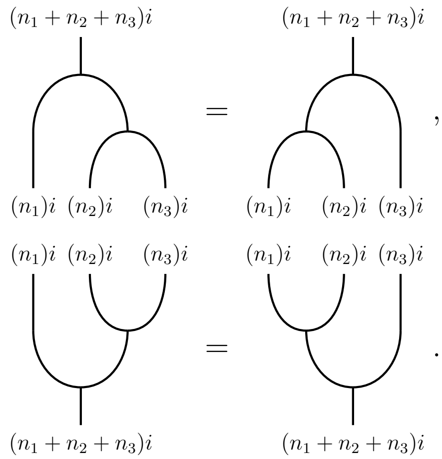
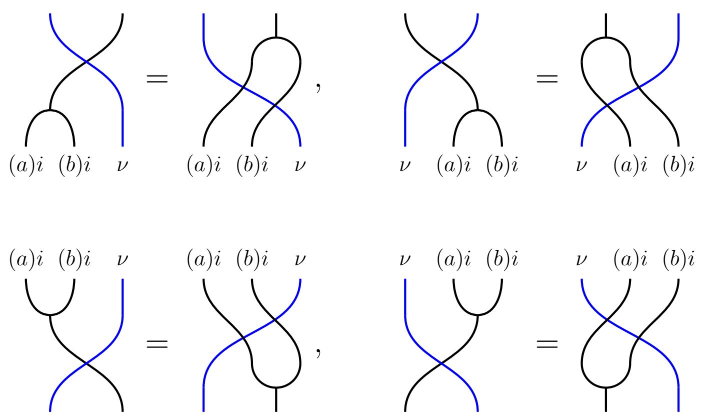
2.5.3 Basic Results

In this subsection, we give some elementary results about diagrammatic calculus which will be used in subsequent chapters.

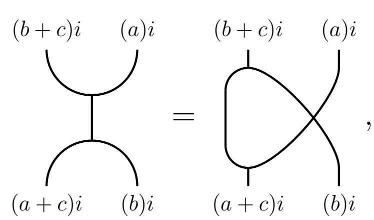
Lemma 2.5.2 ([KLMS12, Proposition 2.4]). *Let $n_1, n_2, n_3 \in \mathbb{N}$ and let $i \in I$. The following* *equalities hold:*

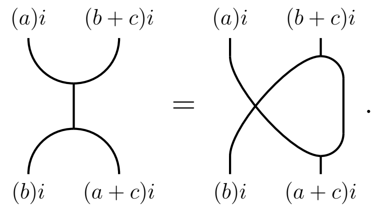


Lemma 2.5.3 ([Sto19, Proposition 3])**.** *For $i \in I$, $\nu \in I^n$ and $a, b \in \mathbb{N}$ we have*



Lemma 2.5.4 ([KLMS12, Proposition 2.5])**.** *For $i \in I$ and $a, b, c \in \mathbb{N}$, the following equalities hold:*



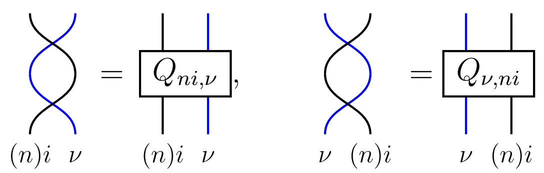


We now state multi-strands versions of the quadratic and braid relations from Definition 2.4.1. For these, we need generalizations of the polynomials $Q_{i,j}$. Let $i \in I$, $n \in \mathbb{N}$ and $\nu \in I^r$ for some $r \in \mathbb{N}$. We define a polynomial $Q_{ni,\nu} \in P_{n+r}$ by

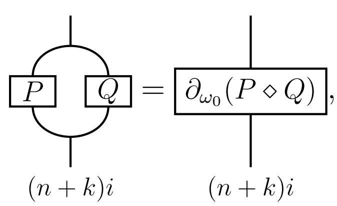
$$Q_{ni,\nu}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+n}) = \prod_{\substack{1 \leqslant \ell \leqslant n \\ 1 \leqslant k \leqslant r \\ \nu_k \neq i}} Q_{i,\nu_k}(x_{r+\ell}, x_k). \tag{2.5.1}$$

There is a symmetrically defined polynomial $Q_{\nu,ni}$.

Lemma 2.5.5. *1. Let $i \in I$, $n \in \mathbb{N}$ and $\nu \in (I \setminus \{i\})^r$ for some $r \in \mathbb{N}$. The following relations hold:*

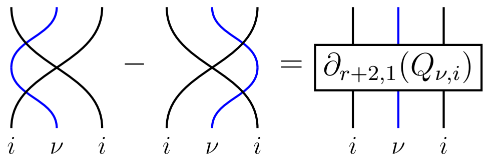


2. Let $i \in I$ and let $n, k \in \mathbb{N}$. Then for all $P \in P_k^{\mathfrak{S}_k}$ and $Q \in P_n^{\mathfrak{S}_n}$ we have

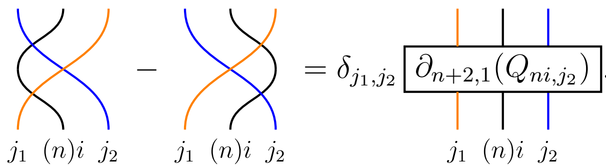


where $\omega_0 = \omega_0[1, n+k]\omega_0[1, n]^{-1}\omega_0[n+1, n+k]^{-1}$.

Lemma 2.5.6. *1. Let $i \in I$ and $\nu \in (I \setminus \{i\})^r$ for some $r \in \mathbb{N}$. The following relation* *holds:*



2. *Let $i \in I$, let j_1, j_2 be elements of $I \setminus \{i\}$ and let and let $n \in \mathbb{N}$. The following relation holds:*



CHAPTER 3

Invertiblity of the Categorified Reflection

In this chapter, we work with the root datum of type A_1. This means that $X = Y = \mathbb{Z}$, the simple root is 2, and the simple coroot is 1. The paring between X and Y is simply the multiplication. This root datum corresponds to the Lie algebra $\mathfrak{sl}_2$. Recall that if V is an integrable weight representation of $\mathfrak{sl}_2$, the reflection

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp(-e)\exp(f)\exp(-e)$$

induces an automorphism of V restricting to isomorphisms $V_\lambda \xrightarrow{\sim} V_{-\lambda}$ between weight spaces of opposite weights (see [Hum78, 21.2]). Furthermore, there is an explicit formula for the action of θ of V. For all $v \in V_\lambda$, we have (see [CR08, Lemma 4.2])

$$\theta(v) = \sum_{r \geqslant \max(0,-\lambda)} (-1)^r f^{(\lambda+r)} e^{(r)}(v).$$

These facts generalize to the quantum case. If V is an integrable representation of the integral quantum group $\dot{U}$ associated with $\mathfrak{sl}_2$, there is a quantum deformation of θ providing isomorphisms $V_\lambda \xrightarrow{\sim} V_{-\lambda}$. Explicitly, for $v \in V_\lambda$, we have (see [Lus09, Lemma 2.2] or [CKL13, Proposition 2.3])

$$\theta(v) = \sum_{r \geqslant \max(0,-\lambda)} \left(-q^{-1}\right)^r f^{(\lambda+r)} e^{(r)}(v).$$

At the categorical level, the reflection θ lifts to a complex Θ of 1-morphisms of $\mathcal{U}$. The necessity to work with complexes is expected given the alternating sums appearing in the formulas above. We now define the complex Θ.

Definition 3.0.1. Given $\lambda \in \mathbb{Z}$, the complex $\Theta 1_\lambda$ is the complex of objects of $1_{-\lambda}\mathcal{U}1_\lambda$ defined as follows.

- For all $r \in \mathbb{Z}$, the r^{th} component of $\Theta 1_\lambda$ is $\Theta^r 1_\lambda = q^{-r} F^{(\lambda+r)} E^{(r)}$. (We put $F^{(\ell)} = E^{(\ell)} = 0$ if $\ell < 0$.)

- The differential of $\Theta 1_\lambda$ is the composition

$$F^{(\lambda+r)} E^{(r)} \xrightarrow{F^{(\lambda+r)}\eta E^{(r)}} F^{(\lambda+r)} F E E^{(r)} \xrightarrow{\tau_{[1\uparrow\lambda+r]}\otimes\tau_{[r\downarrow 1]}} F^{(\lambda+r+1)} E^{(r+1)}.$$

  The fact that the differential squares to zero follows from the commutativity of the following diagram:

$$\begin{tikzcd} 1_\lambda \arrow[r, "F\eta E\circ\eta"] \arrow[d, "F\eta E\circ\eta"'] & F^2E^2 1_\lambda \arrow[d, "\tau\otimes\tau"] \\ F^2E^2 1_\lambda \arrow[r, "F^2\tau^2=0"'] & F^2E^2 1_\lambda. \end{tikzcd}$$

The goal of this chapter is to prove that Θ is invertible in the homotopy category of $\mathcal{U}$. In Section 3.1, we introduce a family $(\mathcal{L}(n))_{n\in\mathbb{N}}$. of 2-representations of $\mathcal{U}$. These categorify the finite dimensional simple representations of $\mathfrak{sl}_2$. We prove that this family satisfies certain faithfulness properties. In Section 3.2, we apply these properties to prove the invertibility of Θ up to homotopy and a compatibility property with Chevalley generators.

3.1 Faithfulness of Simple 2-Representations

3.1.1 Simple 2-Representations

We now define the simple 2-representations of $\mathcal{U}$, following [Rou08, Subsection 5.2]. For $n \in \mathbb{N}$, and $k \in \{0, \ldots, n\}$, we denote by $H^0_{k,n}$ the subalgebra of H^0_n generated by $P_n^{\mathfrak{S}_n}$, $\tau_1, \ldots, \tau_{k-1}$ and $x_1, \ldots, x_k$. We have an isomorphism of algebras $H^0_{k,n} \simeq H^0_k \otimes P_{n-k}^{\mathfrak{S}_{n-k}}$, and a tower of algebras

$$P_n^{\mathfrak{S}_n} = H^0_{0,n} \subseteq H^0_{1,n} \subseteq \cdots \subseteq H^0_{n,n} = H^0_n.$$

The 2-representation $\mathcal{L}(n)$ of $\mathcal{U}$ is defined as follows.

- For $k \in \{0, \ldots, n\}$, we define $\mathcal{L}(n)_{-n+2k} = H^0_{k,n}-\mathrm{proj}$, the category of finitely generated, graded and projective $H^0_{k,n}$-modules.
- The functor $E : \mathcal{L}(n)_{-n+2k} \to \mathcal{L}(n)_{-n+2(k+1)}$ is $\mathrm{ind}_{H^0_{k,n}}^{H^0_{k+1,n}}$ and the functor $F : \mathcal{L}(n)_{-n+2k} \to \mathcal{L}(n)_{-n+2(k-1)}$ is $q^{2k-n-1}\mathrm{res}_{H^0_{k-1,n}}^{H^0_{k,n}}$.
- The morphism $x : E \to E$ is right multiplication by x_{k+1} on $\mathrm{ind}_{H^0_{k,n}}^{H^0_{k+1,n}}$, and the morphism $\tau : E^2 \to E^2$ is right multiplication by τ_{k+1} on $\mathrm{ind}_{H^0_{k,n}}^{H^0_{k+2,n}}$.
- The morphisms ε and η are the counit and unit of the canonical adjunction between induction and restriction.

Proposition 3.1.1 ([Rou08, 5.2.3])**.** *The above data defines a 2-representation of $\mathcal{U}$ on $\mathcal{L}(n)$.*

Define

$$\mathcal{L}(n)-\mathrm{bim} = \bigoplus_{0 \leqslant k, \ell \leqslant n} \left(H^0_{k,n}, H^0_{\ell,n}\right)-\mathrm{bim},$$

where $\left(H^0_{k,n}, H^0_{\ell,n}\right)-\mathrm{bim}$ is the category of finitely generated graded $\left(H^0_{k,n}, H^0_{\ell,n}\right)$-bimodules. The structure of 2-representation of $\mathcal{U}$ on $\mathcal{L}(n)$ induces functors $\Phi_n : \mathcal{U}1_\lambda \to \mathcal{L}(n)-\mathrm{bim}$ for all $\lambda \in \mathbb{Z}$. For $k \in \{0, \ldots, n\}$ and $a \in \{0, \ldots, k\}$ we have

$$\Phi_n \left(E^a 1_{-n+2(k-a)}\right) = H^0_{k,n} \text{ as } \left(H^0_{k,n}, H^0_{k-a,n}\right)\text{-bimodules,}$$
$$\Phi_n\Big(F^a 1_{-n+2k}\Big) = q^{a(2k-n-a)} H^0_{k,n} \text{ as } \left(H^0_{k-a,n}, H^0_{k,n}\right)\text{-bimodules.}$$

Let us now describe explicitly the images of the divided powers under Φ_n. Given two integers $r < \ell \in \{1, \ldots, n\}$, we put

$$x_{[r,\ell]} = x_{r+1} x^2_{r+2} \ldots x^{\ell-r}_\ell \in P_n,$$
$$x'_{[r,\ell]} = (-1)^{\frac{(\ell-r)(\ell-r-1)}{2}} x^{\ell-r}_r x^{\ell-r-1}_{r+1} \ldots x_{\ell-1} \in P_n,$$
$$e_{[r,\ell]} = x_{[r,\ell]} \tau_{\omega_0[r,\ell]} \in H_{\ell,n},$$
$$e'_{[r,\ell]} = \tau_{\omega_0[r,\ell]} x'_{[r,\ell]} \in H_{\ell,n}.$$

Note that $e_{[r,\ell]}$ and $e'_{[r,\ell]}$ are orthogonal idempotents of $H_{\ell,n}$. Then we have

$$\Phi_n \left(E^{(a)} 1_{-n+2(k-a)}\right) = q^{\frac{a(a-1)}{2}} H^0_{k,n} e_{[k-a+1,k]} \text{ as } \left(H^0_{k,n}, H^0_{k-a,n}\right)\text{-bimodules,}$$
$$\Phi_n \left(F^{(a)} 1_{-n+2k}\right) = q^{a(2k-n-a)+\frac{a(a-1)}{2}} e'_{[k-a+1,k]} H^0_{k,n} \text{ as } \left(H^0_{k-a,n}, H^0_{k,n}\right)\text{-bimodules.}$$

A crucial fact about the simple 2-representations is the following universal property.

Theorem 3.1.2 ([Rou08, Proposition 5.15])**.** *Let $\mathcal{V}$ be an integrable 2-representation of $\mathcal{U}$, and let $M \in \mathcal{V}_n$ be such that $E(M) = 0$. Then there is a unique morphism of 2-representations $R_M : \mathcal{L}(n) \to \mathcal{V}$ such that $R_M(P_n) = M$, where P_n is seen as an object of $\mathcal{L}(n)_n$ with action of H_n given by the polynomial representation.*

3.1.2 Faithfulness

The goal of this subsection is to prove the following faithfulness result.

Theorem 3.1.3. *1. Let C be a complex of 1-morphisms of $\mathcal{U}$. If $\Phi_n(C)$ is null-homotopic for all $n \in \mathbb{N}$, then C is null-homotopic.*

2. Let f be a morphism between complexes of 1-morphisms of $\mathcal{U}$. If $\Phi_n(f)$ is a homotopy equivalence for all $n \in \mathbb{N}$, then f is a homotopy equivalence.

Generalities About Krull-Schmidt Categories

Theorem 3.1.3 will be a consequence of a general result about representations of Krull-Schmidt categories. Let us start by recalling some basic facts. A K-linear category $\mathcal{C}$ is called *Krull-Schmidt* if every object of $\mathcal{C}$ is isomorphic to a finite direct sum of objects whose endomorphism algebra is local. If $\mathcal{C}$ is Krull-Schmidt, then the indecomposable objects are precisely the objects whose endomorphism algebra is local, and every object decomposes uniquely as a finite direct sum of indecomposable objects up to reordering of the terms. We will also say that a 2-category $\mathfrak{C}$ is Krull-Schmidt if for all objects λ, μ of $\mathfrak{C}$ the category $\mathrm{Hom}_{\mathfrak{C}}(\lambda, \mu)$ is Krull-Schmidt. In other words, $\mathfrak{C}$ is Krull-Schmidt if every 1-morphism of $\mathfrak{C}$ decomposes as a finite direct sum of 1-morphisms whose 2-endomorphism algebra is local.

The general theorem that we will prove regarding representations of Krull-Schmidt categories is the following.

Theorem 3.1.4. *Let A be a non-empty set. Let $\mathcal{C}, (\mathcal{D}_a)_{a\in A}$ be Krull-Schmidt categories, and let $(\Phi_a : \mathcal{C} \to \mathcal{D}_a)_{a\in A}$ be linear functors. Assume that for any finite collection $M_1, \dots, M_n$ of indecomposable and pairwise non-isomorphic objects of $\mathcal{C}$, there exists $a \in A$ such that:*

1. *$\Phi_a(M_1), \dots, \Phi_a(M_n)$ are indecomposable and pairwise non-isomorphic,*

2. *for all* $k \in \{1, \ldots, n\}$, *the morphism of* K*-algebras* $\mathrm{End}_{\mathcal{C}}(M_k) \to \mathrm{End}_{\mathcal{D}_a}(\Phi_a(M_k))$ *induced by* Φ_a *is local.*

Then for all $C \in \mathrm{Comp}(\mathcal{C})$, C *is null-homotopic if and only if for all* $a \in A$, $\Phi_a(C)$ *is null-homotopic.*

Let us introduce some terminology that will be used throughout the proof. With the notations of Theorem 3.1.4, given a finite collection $M_1, \ldots, M_n$ of indecomposable and pairwise non-isomorphic objects of $\mathcal{C}$, we say that an element $a \in A$ is *adapted* to $M_1, \ldots, M_n$ if conditions (1) and (2) of Theorem 3.1.4 are satisfied. Similarly, given an object M of $\mathcal{C}$, we say that $a \in A$ is adapted to M if it is adapted to the indecomposable summands of M. Given an arrow $f : M \to N$ of $\mathcal{C}$, we say that $a \in A$ is adapted to f if it is adapted to $M \oplus N$.

The first step to prove Theorem 3.1.4 is to prove a lifting result for split injections and split surjections. This will be based on the following lemma.

Lemma 3.1.5. *Let* $\mathcal{C}$ *be a Krull-Schmidt category, and let* $f : M \to N$, $g : N \to L$ *be morphisms in* $\mathcal{C}$. *Assume given a decomposition* $N = \oplus_j N_j$ *into indecomposable objects. Denote by* $i_j : N_j \to N, p_j : N \to N_j$ *the inclusion and projection morphisms associated to this decomposition.*

1. *Assume that* L *is indecomposable and that* gf *is a split surjection. Then, there exists* j *such that* gi_j *is an isomorphism and* $p_j f$ *is a split surjection.*

2. *Assume that* M *is indecomposable and that* gf *is a split injection. Then, there exists* j *such that* gi_j *is a split injection and* $p_j f$ *is an isomorphism.*

Proof. We prove the first statement, the second one being similar. Fix a right inverse u of gf. Then we have

$$\sum_j g i_j p_j f u = \mathrm{id}_L.$$

Since $\mathrm{End}_{\mathcal{C}}(L)$ is a local K-algebra, there exists j such that gi_jp_jfu is invertible. Thus gi_j is a split surjection. Since N_j is indecomposable and $L \neq 0$, it follows that gi_j is an isomorphism. Hence p_jfu is invertible, so p_jf is a split surjection. $\square$

Proposition 3.1.6. *Let $\mathcal{C}, \mathcal{D}$ be Krull-Schmidt categories, and let $\Phi : \mathcal{C} \to \mathcal{D}$ be a functor such that:*

1. *for any indecomposable object $X \in \mathcal{C}$, $\Phi(X)$ is indecomposable and the morphism of K-algebras $\mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(\Phi(X))$ induced by Φ is local,*

2. *for any indecomposable objects $X, Y \in \mathcal{C}$, $X \simeq Y$ if and only if $\Phi(X) \simeq \Phi(Y)$.*

Let $f : M \to N$ be a morphism in $\mathcal{C}$. Then f is a split surjection (resp. split injection) if and only if $\Phi(f)$ is a split surjection (resp. split injection).

Proof. We treat the case of split surjections, the case of split injections being similar. If f is a split surjection, it is clear that $\Phi(f)$ is a split surjection. Conversely, assume that $\Phi(f)$ is a split surjection. We proceed by induction on the number of indecomposable summands of N. The result is clear if $N = 0$.

Assume that $N \neq 0$ and fix a decomposition $N = N_1 \oplus N'$ with N_1 indecomposable. Denote by $p : N \to N_1$ the projection associated with this decomposition. We also fix a decomposition $M = \oplus_{j=1}^m M_j$ into indecomposable objects, and we denote by $i_j : M_j \to M, p_j : M \to M_j$ the inclusion and projection morphisms associated with this decomposition.

The composition $\Phi(M) \xrightarrow{\mathrm{id}} \Phi(M) \xrightarrow{\Phi(pf)} \Phi(N_1)$ is a split surjection and $\Phi(N_1), (\Phi(M_j))_j$ are indecomposable by assumption (1). Thus by Lemma 3.1.5, there exists j such that $\Phi(pfi_j)$ is an isomorphism. Without loss of generality, we can assume that $j = 1$. Then $\Phi(N_1)$ and $\Phi(M_1)$ are isomorphic, so by assumption (2) N_1 and M_1 are isomorphic. Fix an isomorphism $g : N_1 \xrightarrow{\sim} M_1$. The morphism $\mathrm{End}_{\mathcal{C}}(N_1) \to \mathrm{End}_{\mathcal{D}}(\Phi(N_1))$ induced by Φ is local and $\Phi(pfi_1g)$ is invertible. Thus pfi_1g is invertible, and pfi_1 is invertible as well.

Hence, if we let $M' = \oplus_{j>1} M_j$, in the decompositions $M = M_1 \oplus M'$ and $N = N_1 \oplus N'$ we can write f as a matrix

$$f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with $\alpha = pfi_1$. Since α is an isomorphism, there exist automorphisms u of N and v of M such that

$$ufv = \begin{pmatrix} \alpha & 0 \\ 0 & f' \end{pmatrix}$$

for some morphism $f' : M' \to N'$. Since $\Phi(f)$ is a split surjection and u, v are invertible, $\Phi(ufv)$ is a split surjection as well. Thus, $\Phi(f')$ is a split surjection. The number of indecomposable summands of N' is one less than that of N. By induction, we deduce that f' is a split surjection. It follows that ufv is a split surjection. So f is a split surjection. □

Corollary 3.1.7. *Let $\mathcal{C}, (\mathcal{D}_a)_{a\in A}$ be Krull-Schmidt categories, and let $(\Phi_a : \mathcal{C} \to \mathcal{D}_a)_{a\in A}$ be functors satisfying the same assumptions as in Theorem 3.1.4. Let $f : M \to N$ be a morphism in $\mathcal{C}$. Then, the following assertions are equivalent.*

1. *f is a split surjection (resp. injection).*

2. *There exists $a \in A$ adapted to f such that $\Phi_a(f)$ is a split surjection (resp. injection).*

3. *For all $a \in A$, $\Phi_a(f)$ is a split surjection (resp. injection).*

Proof. Denote by $\mathcal{C}_f$ the smallest full subcategory of $\mathcal{C}$ containing $M \oplus N$ and closed under direct sums and direct summands. This is a Krull-Schmidt category with finitely many isomorphism classes of indecomposable objects. By assumption, we can fix $a \in A$ adapted to f. Then the functor $\mathcal{C}_f \to \mathcal{D}_a$ induced by Φ_a satisfies the assumptions of Proposition 3.1.6. The result follows from Propostion 3.1.6. □

We can now prove Theorem 3.1.4 for complexes that are bounded above or below.

Proposition 3.1.8. *Let $\mathcal{C}, (\mathcal{D}_a)_{a\in A}$ be Krull-Schmidt categories, and let $(\Phi_a : \mathcal{C} \to \mathcal{D}_a)_{a\in A}$ be functors satisfying the same assumptions as in Theorem 3.1.4. Let $C \in \mathrm{Comp}(\mathcal{C})$ be bounded above or below. Then C is null-homotopic if and only if for all $a \in A$, $\Phi_a(C)$ is null-homotopic.*

Proof. We treat the case where C is bounded above, the bounded below case being similar. If C is null-homotopic, it is clear that for all $a \in A$, $\Phi_a(C)$ is null-homotopic. Conversely, assume that for all $a \in A$, $\Phi_a(C)$ is null-homotopic. Since C is bounded above, it has the form

$$\cdots \to C^j \xrightarrow{d^j} C^{j+1} \to \cdots \to C^{\ell-1} \xrightarrow{d^{\ell-1}} C^\ell \to 0,$$

with ℓ such that $C^j = 0$ for all $j > \ell$. We inductively construct maps $h_j \in \mathrm{Hom}_{\mathcal{C}}(C^j, C^{j-1})$ for $j \leqslant \ell$, such that $h_{j+1}d^j + d^{j-1}h_j = \mathrm{id}_{C^j}$.

By assumption, $\Phi_a(d^{\ell-1})$ is a split surjection for all $a \in A$. Hence $d^{\ell-1}$ is a split surjection by Corollary 3.1.7. We define h_ℓ to be a right inverse to $d^{\ell-1}$. Assume that the h_k are constructed for $k \geqslant j$. From $h_{j+1}d^j + d^{j-1}h_j = \mathrm{id}_{C^j}$ and $d^j d^{j-1} = 0$, we get that $e = h_j d^{j-1}$ is an idempotent of C^{j-1}. Hence there is a decomposition $C^{j-1} = \mathrm{im}(e) \oplus \mathrm{im}(1-e)$. We have $ed^{j-2} = 0$ and $d^{j-1}(1-e) = 0$. Thus C has the form

$$\cdots \to C^{j-2} \xrightarrow{d^{j-2} = \left(\begin{smallmatrix} 0 \\ \delta \end{smallmatrix}\right)} \mathrm{im}(e) \oplus \mathrm{im}(1-e) \xrightarrow{d^{j-1} = \left(\begin{smallmatrix} \star & 0 \end{smallmatrix}\right)} C^j \to \cdots$$

for some morphism $\delta : C^{j-2} \to \mathrm{im}(1-e)$. For all $a \in A$, $\Phi_a(\delta)$ is a split surjection since $\Phi_a(C)$ is null-homotopic. Thus, δ is also a split surjection by Corollary 3.1.7. Let δ' be a right inverse to δ, and define $h_{j-1} = \left(\begin{smallmatrix} 0 & \delta' \end{smallmatrix}\right) : \mathrm{im}(e) \oplus \mathrm{im}(1-e) \to C^{j-2}$. By construction, we have $h_j d^{j-1} + d^{j-2}h_{j-1} = \mathrm{id}_{C^{j-1}}$, which completes the induction. □

To finish the proof of Theorem 3.1.4, we show how to extend this result to unbounded complexes. We will need the following standard lemma.

Lemma 3.1.9 (Gaussian elimination). *1. Let $\mathcal{C}$ be an additive category, and let $C \in \mathrm{Comp}(\mathcal{C})$. Assume that C has the form*

$$C = \cdots \to C^{i-1} \to X \oplus Y \xrightarrow{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)} Z \oplus W \to C^{i+2} \to \cdots,$$

with α an isomorphism. Then C is homotopy equivalent to a complex of the form

$$\cdots \to C^{i-1} \to Y \to W \to C^{i+2} \to \cdots.$$

2. Let $\mathcal{C}$ be an idempotent complete category, and let $C \in \mathrm{Comp}(\mathcal{C})$. Assume that C has the form

$$C = \cdots \to C^{i-1} \to C^{i} \xrightarrow{\left(\begin{smallmatrix} \alpha \\ \gamma \end{smallmatrix}\right)} Z \oplus W \to C^{i+2} \to \cdots,$$

with α a split surjection. Then C is homotopy equivalent to a complex of the form

$$\cdots \to C^{i-1} \to Y \to W \to C^{i+2} \to \cdots.$$

where Y is an object of $\mathcal{C}$ such that $C^{i} \simeq Y \oplus Z$.

3. Let $\mathcal{C}$ be an idempotent complete category, and let $C \in \mathrm{Comp}(\mathcal{C})$. Assume that C has the form

$$C = \cdots \to C^{i-1} \to X \oplus Y \xrightarrow{(\alpha\ \beta)} C^{i+1} \to C^{i+2} \to \cdots,$$

with α a split injection. Then C is homotopy equivalent to a complex of the form

$$\cdots \to C^{i-1} \to Y \to W \to C^{i+2} \to \cdots.$$

where W is an object of $\mathcal{C}$ such that $C^{i+1} \simeq X \oplus W$.

Proof. 1. Since α is an isomorphism, there exists an automorphism p of $X \oplus Y$, an automorphim q of $Y \oplus Z$ and a morphism $\delta' : Y \to W$ such that

$$q \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) p = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta' \end{smallmatrix}\right).$$

Hence C is isomorphic to

$$\cdots \to C^{i-1} \to X \oplus Y \xrightarrow{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta' \end{smallmatrix}\right)} Z \oplus W \to C^{i+2} \to \cdots.$$

Since α is a isomorphism, the compositions $C^{i-1} \to X \oplus Y \twoheadrightarrow X$ and $X \hookrightarrow X \oplus Y \to C^{i+1}$ are zero. Hence C is isomorphic to a direct sum

$$\left(\cdots \to C^{i-1} \to Y \xrightarrow{\delta'} W \to C^{i+2} \to \cdots\right) \bigoplus \left(0 \to X \xrightarrow{\alpha} Z \to 0\right).$$

The complex $0 \to X \xrightarrow{\alpha} Z \to 0$ is homotopy equivalent to zero since α is an isomorphism. The result follows.

2. Fix a right inverse α' of α. Then $\alpha' a$ is an idempotent of $\mathrm{End}_{\mathcal{C}}(C^i)$. Put $X = \mathrm{im}(\alpha'\alpha)$ and $Y = \mathrm{im}(1 - \alpha'\alpha)$. We have a decomposition $C^i = X \oplus Y$ and α induces an isomorphism $X \xrightarrow{\sim} Z$. The result then follows from (1).

3. The proof is similar to (2).

□

Proof of Theorem 3.1.4. For $C \in \mathrm{Comp}(\mathcal{C})$, we denote by $n(C)$ the number of summands in a decomposition of C^0 into indecomposable components. Assume that for all $a \in A$, $\Phi_a(C)$ is null-homotopic. We start by proving that $d_C^0 = 0$ or C is homotopy equivalent to a complex C' with $n(C') = n(C) - 1$.

Fix $a \in A$ adapted to d_C^{-1} and d_C^0. If $\Phi_a(d_C^0) = 0$, then $\Phi_a(d_C^{-1})$ is a split surjection since $\Phi_a(C)$ is null-homotopic. By Corollary 3.1.7, we deduce that d_C^{-1} is a split surjection. Thus,

$d_C^0 = 0$. Assume now that $\Phi_a(d_C^0) \neq 0$. Fix a decomposition into indecomposable summands $C^0 = \oplus_j M_j$. Since $\Phi_a(C)$ is null-homotopic, there exists a split injection $u : M \to \Phi_a(C^0)$ with $M \in \mathcal{D}_a$ indecomposable such that $\Phi_a(d_C^0)u$ is a split injection. By Lemma 3.1.5, there exists j such that $\Phi_a(d_C^0 i_j)$ is a split injection, where $i_j : M_j \to C^0$ is the inclusion associated with the decomposition of C^0. By Corollary 3.1.7, we deduce that $d_C^0 i_j$ is a split injection. Thus, by Lemma 3.1.9, C is homotopy equivalent to a complex C' with $n(C') = n(C) - 1$.

Hence, if $\Phi_a(C)$ is null-homotopic for all $a \in A$, we have proved that $d_C^0 = 0$, or C is homotopy equivalent to a complex C' with $n(C') = n(C) - 1$. By induction, it follows that C is homotopy equivalent to a complex C' such that $d_{C'}^0 = 0$. Hence, C is homotopy equivalent to a direct sum $C_1 \oplus C_2$, with C_1 bounded above and C_2 bounded below. For all $a \in A$, $\Phi_a(C_1)$ and $\Phi_a(C_2)$ are null-homotopic since $\Phi_a(C)$ is as well. So by Proposition 3.1.8, C_1 and C_2 are null-homotopic. Hence, C is null-homotopic. Conversely, if C is null-homotopic, it is clear that for all $a \in A$, $\Phi_a(C)$ is null-homotopic. □

Corollary 3.1.10. *Let $\mathcal{C}, (\mathcal{D}_a)_{a\in A}$ be Krull-Schmidt categories, and let $(\Phi_a : \mathcal{C} \to \mathcal{D}_a)_{a\in A}$ be functors satisfying the same assumptions as in Theorem 3.1.4. Let $f : C \to D$ be a morphism between complexes of objects of $\mathcal{C}$. Then f is a homotopy equivalence if and only if for all $a \in A$, $\Phi_a(f)$ is a homotopy equivalence.*

Proof. It suffices to apply Theorem 3.1.4 to the complex $\mathrm{Cone}(f)$. □

Application to Simple 2-Representations

To prove our faithfulness result, we show that the collection of simple 2-representations of $\mathcal{U}$ satisfies the assumptions of Theorem 3.1.4. In particular, we need to know the indecomposable 1-morphisms of $\mathcal{U}$ and their 2-endomorphism algebra. These have been determined by Lauda in [Lau10].

Theorem 3.1.11 ([Lau10, Propositions 9.8 and 9.10]). *The 2-category $\mathcal{U}$ is Krull-Schmidt. A complete set of pairwise non-isomorphic indecomposable 1-morphisms of $\mathcal{U}$ is given by*

$$\left\{q^s E^{(a)} F^{(b)} 1_\lambda,\ \lambda \leqslant b-a,\ s \in \mathbb{Z}\right\} \cup \left\{q^s F^{(b)} E^{(a)} 1_\lambda,\ \lambda > b-a,\ s \in \mathbb{Z}\right\}.$$

Furthermore, for X an indecomposable 1-morphism of $\mathcal{U}$ we have $|X| \in 1 + q\mathbb{N}[q]$.

The first step to check the assumptions of Theorem 3.1.4 is to study the images of the indecomposable 1-morphisms of $\mathcal{U}$ in the simple 2-representations. We start by proving some preliminary results.

Lemma 3.1.12. *Let $k, \ell, r \in \mathbb{N}$ and put $n = k + \ell + r$. Then the multiplication map*

$$P_n^{\mathfrak{S}_{k+\ell} \times \mathfrak{S}_r} \otimes P_n^{\mathfrak{S}_k \times \mathfrak{S}_{\ell+r}} \to P_n^{\mathfrak{S}_k \times \mathfrak{S}_\ell \times \mathfrak{S}_r}$$

is surjective.

Proof. Let B be the image of the multiplication map. We denote by ϵ_j the elementary symmetric polynomial of degree j. As a K-algebra, $P_n^{\mathfrak{S}_k \times \mathfrak{S}_\ell \times \mathfrak{S}_r}$ is generated by $\epsilon_j(x_1, \dots, x_k)$, $\epsilon_j(x_{k+1}, \dots, x_{k+\ell})$ and $\epsilon_j(x_{k+\ell+1}, \dots, x_n)$ for $j \in \mathbb{N}$. Since the multiplication map is an algebra map, it suffices to prove that $\epsilon_j(x_1, \dots, x_k) \in B$, $\epsilon_j(x_{k+1}, \dots, x_{k+\ell}) \in B$ and $\epsilon(x_{k+\ell+1}, \dots, x_n) \in B$ for all $j \in \mathbb{N}$. It is clear that $\epsilon_j(x_1, \dots, x_k) \in B$ and $\epsilon(x_{k+\ell+1}, \dots, x_n) \in B$ for all $j \in \mathbb{N}$, so let us now prove that $\epsilon_j(x_{k+1}, \dots, x_{k+\ell}) \in B$ for all $j \geqslant 1$. For $j = 1$, we have

$$\epsilon_1(x_{k+1}, \dots, x_{k+\ell}) = \epsilon_1(x_1, \dots, x_{k+\ell}) - \epsilon_1(x_1, \dots, x_k) \in B.$$

In general, we have:

$$\epsilon_{j+1}(x_{k+1} \dots, x_{k+\ell}) = \epsilon_{j+1}(x_1 \dots, x_{k+\ell}) - \sum_{i=1}^{j+1} \epsilon_i(x_1, \dots, x_k)\epsilon_{j+1-i}(x_{k+1}, \dots, x_{k+\ell}),$$

and the result follows by induction on j. $\square$

Proposition 3.1.13 ([Lau10, Proposition 9.8]). *Let $\mathcal{V}$ be a 2-representation of $\mathcal{U}$. Denote by $\mathcal{W}$ the category of endofunctors of $\mathcal{V}$. Let $\lambda \in \mathbb{Z}$ and denote by $\Phi : \mathcal{U}1_\lambda \to \mathcal{W}$ the functor induced by the structure of 2-representation on $\mathcal{V}$. Let $a, b, c, d \in \mathbb{N}$ be such that $a-b = c-d$. If $\lambda \geqslant b-a$, there is an isomorphism*

$$\begin{aligned}&\operatorname{Hom}^\bullet_{\mathcal{W}}\left(\Phi\left(F^{(b)}E^{(a)}1_\lambda\right), \Phi\left(F^{(d)}E^{(c)}1_\lambda\right)\right) \simeq \\ &\qquad \bigoplus_{k=0}^{\min\{a,c\}} q^{(a+c-k)(\lambda+a+c-k)} \begin{bmatrix} \lambda+a+c \\ k \end{bmatrix} \begin{bmatrix} b+c-k \\ b \end{bmatrix} \begin{bmatrix} a+d-k \\ d \end{bmatrix} \operatorname{End}^\bullet_{\mathcal{W}}\left(\Phi\left(F^{(a+d-k)}1_{\lambda+2(a+c-k)}\right)\right).\end{aligned}$$

If $\lambda \leqslant b-a$, there is an isomorphism

$$\begin{aligned}&\operatorname{Hom}^\bullet_{W}\left(\Phi\left(E^{(a)}F^{(b)}1_\lambda\right), \Phi\left(E^{(c)}F^{(d)}1_\lambda\right)\right) \simeq \\ &\qquad \bigoplus_{k=0}^{\min\{b,d\}} q^{(b+d-k)(b+d-k-\lambda)} \begin{bmatrix} b+d-\lambda \\ k \end{bmatrix} \begin{bmatrix} a+d-k \\ a \end{bmatrix} \begin{bmatrix} b+c-k \\ c \end{bmatrix} \operatorname{End}^\bullet_{\mathcal{W}}\left(\Phi\left(E^{(a+d-k)}1_{\lambda-2(b+d-k)}\right)\right).\end{aligned}$$

Using the previous two results, we can now study the images of the indecomposable 1-morphisms of $\mathcal{U}$ in the simple 2-representations.

Proposition 3.1.14. *Let M be an indecomposable object of $\mathcal{U}1_\lambda$ for some $\lambda \in \mathbb{Z}$. If n is large enough and of the same parity as λ, then $|\Phi_n(M)| \in 1 + q\mathbb{N}[q]$.*

Proof. We start by proving the result for $M = E^{(a)}1_\lambda$ for some $a \in \mathbb{N}$. Let $n > |\lambda| + 2a$ be an integer of the same parity as λ, and let $k = \frac{\lambda+n}{2}$. Then $\Phi_n\left(E^{(a)}1_\lambda\right) = q^{\frac{a(a-1)}{2}} H^0_{k+a,n} e_{[k+1,k+a]}$ as a graded $\left(H^0_{k+a,n}, H^0_{k,n}\right)$-bimodule. By the Morita equivalences from (2.3.6), we have an isomorphism of graded K-algebras

$$\operatorname{End}^\bullet_{\left(H^0_{k+a,n}, H^0_{k,n}\right)}\left(H^0_{k+a,n}e_{[k+1,k+a]}\right) \simeq \operatorname{End}^\bullet_{\left(P_n^{\mathfrak{S}_{k+a}\times\mathfrak{S}_{n-k-a}}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)}\left(e'_{[1,k+a]}H^0_{k+a,n}e_{[1,k]}e_{[k+1,k+a]}\right).$$

There is an isomorphism of $\left(P_n^{\mathfrak{S}_{k+a}\times\mathfrak{S}_{n-k-a}}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$-bimodules

$$\begin{cases} P_n^{\mathfrak{S}_k\times\mathfrak{S}_a\times\mathfrak{S}_{n-k-a}} & \simeq & e'_{[1,k+a]}H^0_{k+a,n}e_{[1,k]}e_{[k+1,k+a]}, \\ f & \mapsto & \tau_{\omega_0[1,k+a]}f. \end{cases}$$

By Lemma 3.1.12, $P_n^{\mathfrak{S}_k\times\mathfrak{S}_a\times\mathfrak{S}_{n-k-a}}$ is cyclic generated by 1 as a $\left(P_n^{\mathfrak{S}_{k+a}\times\mathfrak{S}_{n-k-a}}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$-bimodule. It follows that

$$\mathrm{End}^\bullet_{\mathcal{L}(n)-\mathrm{bim}}\left(\Phi_n\left(E^{(a)}1_\lambda\right)\right) = P_n^{\mathfrak{S}_k\times\mathfrak{S}_a\times\mathfrak{S}_{n-k-a}}.$$

Thus, the result holds for $M = E^{(a)}1_\lambda$. The case $M = F^{(a)}1_\lambda$ is similar.

Assume now that $M = q^sF^{(b)}E^{(a)}1_\lambda$ for $s \in \mathbb{Z}$ and $\lambda \geqslant b-a$. By Proposition 3.1.13, we have

$$\left|\Phi_n\left(F^{(b)}E^{(a)}1_\lambda\right)\right| = \sum_{\ell=0}^{a} q^{(2a-\ell)(\lambda+2a-\ell)} \begin{bmatrix} \lambda+2a \\ \ell \end{bmatrix} \begin{bmatrix} a+b-\ell \\ b \end{bmatrix}^2 \left|\Phi_n\left(F^{(a+b-\ell)}1_{\lambda+2(2a-\ell)}\right)\right|.$$

Let us find the lowest degree term in that expression. The lowest degree term of $\begin{bmatrix} \lambda+2a \\ \ell \end{bmatrix}$ is $q^{-\ell(\lambda+2a-\ell)}$, and that of $\begin{bmatrix} a+b-\ell \\ b \end{bmatrix}^2$ is $q^{-2b(a-\ell)}$. It follows that the lowest degree term of

$$q^{(2a-\ell)(\lambda+2a-\ell)} \begin{bmatrix} \lambda+2a \\ \ell \end{bmatrix} \begin{bmatrix} a+b-\ell \\ b \end{bmatrix}^2$$

is $q^{2(a-\ell)(\lambda+2a-b-\ell)}$. For $\ell = a$, this term is 1. For $\ell < a$, we have $(a-\ell)(\lambda+2a-b-\ell) > 0$ since $\lambda \geqslant b-a$. Now using the first part of the proof, we know that for n large enough and of the same parity as λ, the lowest degree term of $\left|\Phi_n\left(F^{(a+b-\ell)}1_{\lambda+2(2a-\ell)}\right)\right|$ is 1 for all $\ell \in \{0,\dots,a\}$. Putting all this together, we indeed have

$$\left|\Phi_n\left(F^{(b)}E^{(a)}1_\lambda\right)\right| \in 1 + q\mathbb{N}[q],$$

when n is large enough and of the same parity as λ. The case $M = q^s E^{(a)} F^{(b)} 1_\lambda$ with $\lambda \leqslant b-a$ and $s \in \mathbb{Z}$ is proved similarly. $\square$

In particular, if M is an indecomposable object of $\mathcal{U}1_\lambda$ and n is large enough and of the same parity as λ, then $\Phi_n(M)$ is indecomposable and the morphism $\mathrm{End}_{\mathcal{U}}(M) \to \mathrm{End}_{\mathcal{L}(n)-\mathrm{bim}}(\Phi_n(M))$ induced by Φ_n is local, since both sides are equal to K. The next step is to check that the simple 2-representations distinguish non-isomorphic indecomposable 1-morphisms of $\mathcal{U}$.

Proposition 3.1.15. *Let M, N be indecomposable objects of $\mathcal{U}1_\lambda$ for some $\lambda \in \mathbb{Z}$. Assume that M and N are not isomorphic. Then, for $n \in \mathbb{N}$ large enough and of the same parity as λ, $\Phi_n(M)$ and $\Phi_n(N)$ are not isomorphic.*

Proof. Assume $M = F^{(b)}E^{(a)}1_\lambda$ and $N = F^{(d)}E^{(c)}1_\lambda$ with $\lambda \geqslant b-a = d-c$, and $a \neq c$. By Proposition 3.1.13 we have

$$\langle \Phi_n(M), \Phi_n(N) \rangle = \sum_{k=0}^{\min\{a,c\}} q^{(a+c-k)(\lambda+a+c-k)} \begin{bmatrix} \lambda+a+c \\ k \end{bmatrix} \begin{bmatrix} b+c-k \\ b \end{bmatrix} \begin{bmatrix} a+d-k \\ d \end{bmatrix} \left| \Phi_n \left(F^{(a+d-k)} 1_{\lambda+2(a+c-k)} \right) \right| .$$

The lowest degree term of of $q^{(a+c-k)(\lambda+a+c-k)} \begin{bmatrix} \lambda+a+c \\ k \end{bmatrix} \begin{bmatrix} b+c-k \\ b \end{bmatrix} \begin{bmatrix} a+d-k \\ d \end{bmatrix}$ is

$$q^{(a+c-k)(\lambda+a+c-k)-k(\lambda+a+c-k)-b(c-k)-d(a-k)}.$$

Let us rewrite the exponent. We have:

$$\begin{aligned}
&(a+c-k)(\lambda+a+c-k) - k(\lambda+a+c-k) - b(c-k) - d(a-k) \\
&= (a-k)(\lambda+a+c-k) + (c-k)(\lambda+a+c-k) - b(c-k) - d(a-k) \\
&= (a-k)(\lambda+a+c-d-k) + (c-k)(\lambda+a-b+c-k) \\
&= (a-k)(\lambda+2a-b-k) + (c-k)(\lambda+2c-d-k).
\end{aligned}$$

Since the roles of (a,b) and (c,d) are symmetric in this last expression, we can assume without loss of generality that $a > c$. Then using $k \leqslant c < a$ and $\lambda \geqslant b-a = d-c$ we see that $(a-k)(\lambda+2a-b-k) > 0$ and $(c-k)(\lambda+2c-d-k) \geqslant 0$. By Proposition 3.1.14, we have $\left|\Phi_n\left(F^{(a+d-k)}1_{\lambda+2(a+c-k)}\right)\right| \in \mathbb{N}[q]$ for all $k \in \{0,\ldots,\min(a,c)\}$ if n is large enough and of the same parity as λ. Hence, $\langle\Phi_n(M),\Phi_n(N)\rangle \in q\mathbb{N}[q]$ for such an n. By symmetry, we also have $\langle\Phi_n(M),\Phi_n(N)\rangle \in q\mathbb{N}[q]$ if n is large enough and of the same parity as λ.

Thus, we can fix $n \in \mathbb{N}$ such that $\langle\Phi_n(M),\Phi_n(N)\rangle, \langle\Phi_n(N),\Phi_n(M)\rangle \in q\mathbb{N}[q]$ and $\Phi_n(M)$, $\Phi_n(N)$ are indecomposable. Then, a composition $\Phi_n(M) \to \Phi_n(N) \to \Phi_n(M)$ of homogeneous morphisms of arbitrary degrees is zero or of positive degree. In particular, it is not invertible. Hence, $\Phi_n(M)$ and $\Phi_n(N)$ are not isomorphic. The same argument holds if $M = q^s F^{(b)}E^{(a)}1_\lambda$ and $N = q^r F^{(d)}E^{(c)}1_\lambda$ for some $r,s \in \mathbb{Z}$. The case of $M = q^s E^{(a)}F^{(b)}1_\lambda$ and $N = q^r E^{(c)}F^{(d)}1_\lambda$ is proved similarly. $\square$

Proof of Theorem 3.1.3. Note that every complex of 1-morphisms of $\mathcal{U}$ can be written as a direct sum of complexes of objects of $\mathcal{U}1_\lambda$, for λ ranging over $\mathbb{Z}$. Hence, it suffices to prove the result for complex of objects of $\mathcal{U}1_\lambda$. By Propositions 3.1.14 and 3.1.15, the family of functors $(\Phi_n : \mathcal{U}1_\lambda \to \mathcal{L}(n)\text{−bim})_{n\in\mathbb{N}}$ satisfies the assumptions of Theorem 3.1.4. Thus, the result follows from Theorem 3.1.4 and Corollary 3.1.10. $\square$

3.1.3 Extension to the Derived Category

The goal of this subsection is to show that the conclusions of Theorem 3.1.3 still hold if the null-homotopic assumption and the homotopy equivalence assumption are weakened to acyclic and quasi-isomorphism respectively, modulo some finiteness conditions.

Theorem 3.1.16. *1. Let C be a complex of 1-morphisms of $\mathcal{U}$ such that for every integrable 2-representation $\mathcal{V}$ and $M \in \mathcal{V}$, the complex $C(M)$ is bounded. Assume that for all $n \in \mathbb{N}$, $\Phi_n(C)$ is acyclic. Then C is null-homotopic.*

2. *Let f be a morphism between complexes of 1-morphisms of $\mathcal{U}$ that satisfy the same condition as in (1). Assume that for all $n \in \mathbb{N}$, $\Phi_n(f)$ is a quasi-isomorphism. Then f is a homotopy equivalence.*

The first step is the following result from [CR08]. Since this result is not explicitly singled out in [CR08], we provide a proof. However all of the arguments are taken from [CR08], with slight modifications to change from the abelian setup of minimal 2-representations to simple 2-representations.

Proposition 3.1.17. *Let C be a complex of 1-morphisms of $\mathcal{U}$ such that for every integrable 2-representation $\mathcal{V}$ and $M \in \mathcal{V}$, the complex $C(M)$ is bounded. Assume that for all $n \in \mathbb{N}$, $\Phi_n(C)$ is acyclic. Then for any abelian integrable 2-representation $\mathcal{V}$ and $M \in \mathcal{V}$, the complex $C(M)$ is null-homotopic.*

Proof. Let $\mathcal{V}$ be an abelian integrable 2-representation of $\mathcal{U}$. The first step is to show that for every highest weight object $N \in \mathcal{V}$ and $i \geqslant 0$, the complex $C(F^i N)$ is null-homotopic. Denote by n the weight of N. By Proposition 3.1.2, we have a morphism of 2-representations $R_N : \mathcal{L}(n) \to \mathcal{V}$ sending P_n to N. The complex of projective modules $C(F^i P_n)$ is bounded and acyclic by assumption. Hence $C(F^i P_n)$ is null-homotopic. It follows that $C(F^i N) \simeq R_N(C(F^i P_n))$ is null-homotopic as well.

The second step is to show that for any $M \in \mathcal{V}$, $C(M)$ is acyclic. Let $X = C^\vee C(M)$, where $C^\vee$ denotes the right dual of C. Since $\mathrm{End}_{D(\mathcal{V})}(C(M)) \simeq \mathrm{Hom}_{D(\mathcal{V})}(M, X)$, it suffices to show that X is acyclic. Assume it is not, and denote by j the smallest integer such that $H^j(X) \neq 0$. Let k be the maximal integer such that $E^k H^j(X) \neq 0$. Put $N = E^k H^j(X) \simeq H^j(E^k X)$, a highest weight object of $\mathcal{V}$. Then we have

$$\mathrm{Hom}_{D(\mathcal{V})}\left(F^k N, X[-j]\right) \simeq \mathrm{Hom}_{D(\mathcal{V})}\left(N, E^k X[-j]\right) \neq 0,$$

the isomorphism begin up to a degree shift. However, by the first step we have

$$\mathrm{Hom}_{D(\mathcal{V})}\left(F^k N, X[-j]\right) \simeq \mathrm{Hom}_{D(\mathcal{V})}\left(C(F^k N), C(M)[-j]\right) = 0,$$

which is a contradiction. Thus $C(M)$ is acyclic.

Finally, we prove that $C(M)$ is null-homotopic. By [CR08, Corollary 5.33], there exists an algebra A, a 2-representation of $\mathcal{U}$ on $A-$mod restricting to a 2-representation on $A-$proj, and a morphism of 2-representations $S : A-\mathrm{mod} \to \mathcal{V}$ such that M is a direct summand of $S(A)$. Applying the previous step to the 2-representation $A-$mod and the object A, we obtain that $C(A)$ is acyclic. Since it is also a bounded complex of projective A-modules, we conclude that $C(A)$ is null-homotopic. Hence, $C(S(A)) \simeq S(C(A))$ is null-homotopic, from which we deduce that $C(M)$ is null-homotopic. □

Proof of Theorem 3.1.16. Let $n \in \mathbb{N}$. The 2-representation of $\mathcal{U}$ on $\mathcal{L}(n)$ induces a 2-representation of $\mathcal{U}$ on $\mathcal{L}(n)-$bim by tensoring on the left. Consider the object

$$M_n = \bigoplus_{k=0}^{n} H^0_{k,n} \in \mathcal{L}(n)-\mathrm{bim},$$

where each summand $H^0_{k,n}$ is seen as a $\left(H^0_{k,n}, H^0_{k,n}\right)$-bimodule with left and right actions given by multiplication. If C is a complex of 1-morphisms of $\mathcal{U}$ then we have $C(M_n) = \Phi_n(C)$ as objects of $\mathcal{L}(n)-$bim.

Assume now that for every integrable 2-representation $\mathcal{V}$ and $M \in \mathcal{V}$, the complex $C(M)$ is bounded, and that $\Phi_n(C)$ is acyclic for all $n \in \mathbb{N}$. By Proposition 3.1.17, the complex $C(M_n)$ is null-homotopic. Thus $\Phi_n(C)$ is null-homotopic. The first statement then follows from Theorem 3.1.3. We deduce the second statement by applying the first one to $C = \mathrm{Cone}(f)$. □

3.2 Application to the Complex Θ

Recall that for $\lambda \in \mathbb{Z}$, the complex $\Theta 1_\lambda$ is defined by

$$\Theta 1_\lambda = 0 \to F^{(\lambda)} \to \cdots \to q^{-r} F^{(\lambda+r)} E^{(r)} \to q^{-r-1} F^{(\lambda+r+1)} E^{(r+1)} \to \cdots .$$

The differential is given by the composition of $F^{(\lambda+r)} \eta E^{(r)} : F^{(\lambda+r)} E^{(r)} \to F^{(\lambda+r)} F E E^{(r)}$ with the projection on $F^{(\lambda+r+1)} E^{(r+1)}$ given by the elements $\tau_{[1\uparrow\lambda+r]}$ and $\tau_{[r\downarrow 1]}$.

We now describe the images of these complexes in the simple 2-representations. Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, \frac{n}{2}\}$. Then, as complexes of $\left(H^0_{n-k,n}, H^0_{k,n}\right)$-bimodules we have

$$\begin{aligned} &\Phi_n \left(\Theta 1_{-n+2k}\right) = \\ &\qquad 0 \to H^0_{n-k,n} e_{[k+1,n-k]} \to \cdots \to e'_{[n-k+1,r]} H^0_{r,n} e_{[k+1,r]} \to \cdots \to e'_{[n-k+1,n]} H^0_n e_{[k+1,n]} \to 0. \end{aligned} \tag{3.2.1}$$

This complex has $k+1$ non-zero terms, and the last non-zero term is in cohomological degree $n-k$. Now let $k \in \{\frac{n}{2}, \ldots, n\}$. Then, as complexes of $\left(H^0_{n-k,n}, H^0_{k,n}\right)$-bimodules we have

$$\begin{aligned} &\Phi_n \left(\Theta 1_{-n+2k}\right) = \\ &0 \to e'_{[n-k+1,k]} H^0_{k,n} \to \cdots \to e'_{[n-k+1,r]} H^0_{r,n} e_{[k+1,r]} \to \cdots \to e'_{[n-k+1,n]} H^0_n e_{[k+1,n]} \to 0. \end{aligned} \tag{3.2.2}$$

This complex has $n-k+1$ non-zero term, the last one being in cohomological degree $n-k$. In both cases, the differential is the composition of the inclusion with multiplication by suitable elements τ_ω. The main result regarding these complexes is the following.

Theorem 3.2.1 ([CR08, Theorem 6.6]). *Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, n\}$. The cohomology of $\Phi_n \left(\Theta 1_{-n+2k}\right)$ is concentrated in top degree $n-k$, and $H^{n-k}\left(\Phi_n \left(\Theta 1_{-n+2k}\right)\right)$ induces an equivalence of categories $\mathcal{L}(n)_{-n+2k} \xrightarrow{\sim} \mathcal{L}(n)_{n-2k}$.*

In [CR08], Chuang and Rouquier prove this theorem for the *minimal 2-representations*, rather than the simple 2-representations. The minimal 2-representations are abelian versions of the simple 2-representations. They use it to prove that Θ provides derived equivalences on integrable 2-representations of $\mathcal{U}$. In Subsection 3.2.1, we give another proof of Theorem 3.2.1 in the setting of simple 2-representations. Our proof is based on finding explicit bases for the terms of $\Phi_n(\Theta 1_{-n+2k})$ and computing the action of the differential in these bases. In Subsection 3.2.2 and Subsection 3.2.3, we apply our main result on faithfulness of simple 2-representations, Theorem 3.1.16, to prove that Θ is invertible up to homotopy and that there are homotopy equivalences $\Theta E 1_\lambda \simeq q^{\lambda+2} F \Theta 1_\lambda[-1]$ for all $\lambda \in \mathbb{Z}$.

3.2.1 Action of Θ on Simple 2-Representations

In this subsection, we study the complexes (3.2.1) and (3.2.2) and give a proof of Theorem 3.2.1. To do so, we will need to compute various graded dimensions. We start by introducing notation and recalling some elementary results.

The graded K-algebras P_n and $P_n^{\mathfrak{S}_n}$ are locally finite and have graded dimensions given by

$$\operatorname{grdim}(P_n) = \frac{1}{(1-q^2)^n}, \quad \operatorname{grdim}\left(P_n^{\mathfrak{S}_n}\right) = \frac{1}{(1-q^2)\dots(1-q^{2n})}.$$

The following notation will be convenient: given $k \geqslant 0$, we define a variant of the quantum integer $\{k\}$ and the quantum factorial $\{k\}!$ by

$$\{k\} = \frac{q^{2k}-1}{q^2-1}, \quad \{k\}! = \prod_{\ell=1}^{k} \{\ell\}.$$

With these definitions, we have $\operatorname{grdim}(P_n) = \operatorname{grdim}\left(P_n^{\mathfrak{S}_n}\right)\{n\}!$. More generally, given two integers $k < \ell \in \{1, \dots, n\}$, we have

$$\operatorname{grdim}(P_n) = \operatorname{grdim}\left(P_n^{\mathfrak{S}_{[k,\ell]}}\right)\{\ell - k + 1\}!. \tag{3.2.3}$$

From $H_n^0 \simeq \operatorname{End}^\bullet_{P_n^{\mathfrak{S}_n}}(P_n)$, we deduce that the graded dimension of H_n is given by

$$\operatorname{grdim}\left(H_n^0\right) = \operatorname{grdim}(P_n^{\mathfrak{S}_n})\{n\}!\overline{\{n\}!}, \tag{3.2.4}$$

where $\overline{\cdot}$ refers to the automorphism switching q and q^{-1} on formal series. If M is a finitely generated graded H_n^0-module, then M is locally finite as a K-vector space, and we have

$$M \simeq \overline{\{n\}!}\; e_n(M) \simeq \{n\}!\; e_n'(M)$$

as graded $P_n^{\mathfrak{S}_n}$-modules. In particular, we have

$$\operatorname{grdim}(M) = \overline{\{n\}!}\,\operatorname{grdim}(e_n M) = \{n\}!\,\operatorname{grdim}(e_n' M). \tag{3.2.5}$$

Finally, we will need the two following formulas:

$$\sum_{\omega \in \mathfrak{S}_n} q^{2l(\omega)} = \{n\}!, \tag{3.2.6}$$

$$\sum_{0 \leqslant u_1 < \ldots < u_r \leqslant n} q^{2(u_1 + \ldots + u_r)} = q^{r(r-1)} \frac{\{n\}!}{\{r\}!\{n-r\}!}. \tag{3.2.7}$$

The first one can be proved easily by induction on n, see [KC02, Theorem 6.1] for a proof of the second one.

Bases

Fix an integer $n \in \mathbb{N}$. Let k, ℓ, m be integers such that $k, \ell \leqslant m \leqslant n$. We construct a basis for $e'_{[\ell,m]} H_{m,n} e_{[k,m]}$ as a left $P_n^{\mathfrak{S}[\ell,n]}$-module, which is the general form of a term of $\Phi_n(\Theta)$. Put

$$X_{\ell,m} = \{(a_\ell, \dots, a_m),\, 0 \leqslant a_i \leqslant n-i\},$$
$$Y_{\ell,m} = \{(a_\ell, \dots, a_m) \in X_{\ell,m},\, a_\ell > \dots > a_m\}.$$

Note that these sets actually depend on n. Since we assume n fixed for the rest of this subsection, we do not include the dependence on n in the notation. We denote by $S_{k,m}$ the set of minimal length representatives of left cosets of $\mathfrak{S}_{[k,m]}$ in $\mathfrak{S}_m$. For $a \in Y_{\ell,m}$ and $\omega \in S_{k,m}$, we define

$$b_m(a, \omega) = \tau_{\omega_0[\ell,m]} x^a \tau_\omega \tau_{\omega_0[k,m]} \in e'_{[\ell,m]} H^0_{m,n} e_{[k,m]},$$

where $x^a = x_\ell^{a_l} \dots x_m^{a_m}$. Our goal is to prove the following result.

Theorem 3.2.2. *The set* $\{b_m(a,\omega),\ a \in Y_{\ell,m},\ \omega \in S_{k,m}\}$ *is a basis of* $e'_{[\ell,m]} H^0_{m,n} e_{[k,m]}$ *as a left* $P_n^{\mathfrak{S}[\ell,n]}$*-module.*

We will need the following lemma in the proof of Theorem 3.2.2.

Lemma 3.2.3. *The set* $\left\{\partial_{\omega_0[\ell,m]}(x^a),\ a \in Y_{\ell,m}\right\}$ *is a basis of* $P_n^{\mathfrak{S}_{[\ell,m]} \times \mathfrak{S}_{[m+1,n]}}$ *as a* $P_n^{\mathfrak{S}_{[\ell,n]}}$*-module.*

Proof. The set $\{x^a,\ a \in X_{\ell,m}\}$ is a basis of $P_n^{\mathfrak{S}_{[m+1,n]}}$ over $P_n^{\mathfrak{S}_{[\ell,n]}}$. Since the $P_n^{\mathfrak{S}_{[\ell,n]}}$-linear map $\partial_{\omega_0[\ell,m]} : P_n^{\mathfrak{S}_{[m+1,n]}} \to P_n^{\mathfrak{S}_{[\ell,m]} \times \mathfrak{S}_{[m+1,n]}}$ is surjective, the set $\left\{\partial_{\omega_0[\ell,m]}(x^a),\ a \in X_{\ell,m}\right\}$ generates $P_n^{\mathfrak{S}_{[\ell,m]} \times \mathfrak{S}_{[m+1,n]}}$ as a $P_n^{\mathfrak{S}_{[\ell,n]}}$-module. Furthermore, we have $\partial_{\omega_0[\ell,m]} s_i = -\partial_{\omega_0[\ell,m]}$ for $i \in \{\ell, \dots, m-1\}$ and $\partial_{\omega_0[\ell,m]}(x^a) = 0$ if there exists two indexes $i \neq j$ such that $a_i = a_j$. Thus if $a \in X_{\ell,m}$, the element $\partial_{\omega_0[\ell,m]}(x^a)$ is a multiple of $\partial_{\omega_0[\ell,m]}(x^b)$ for some $b \in Y_{\ell,m}$. It follows that the set $\left\{\partial_{\omega_0[\ell,m]}(x^a),\ a \in Y_{\ell,m}\right\}$ generates $P_n^{\mathfrak{S}_{[\ell,m]}\times\mathfrak{S}_{[m+1,n]}}$ as a $P_n^{\mathfrak{S}_{[\ell,n]}}$-module.

We conclude by computing graded dimensions. We have

$$\begin{aligned}\sum_{a\in Y_{\ell,m}} \operatorname{grdim}\left(P_n^{\mathfrak{S}_{[\ell,n]}}\partial_{\omega_0[\ell,m]}(x^a)\right) &= \operatorname{grdim}\left(P_n^{\mathfrak{S}_{[\ell,n]}}\right)\sum_{a\in Y_{\ell,m}} q^{2(a_\ell+\ldots+a_m)-(m-\ell)(m-\ell+1)} \\ &= \operatorname{grdim}\left(P_n^{\mathfrak{S}_{[\ell,n]}}\right)\frac{\{n-\ell+1\}!}{\{m-\ell+1\}!\{n-m\}!} \\ &= \operatorname{grdim}\left(P_n^{\mathfrak{S}_{[\ell,m]}\times\mathfrak{S}_{[m+1,n]}}\right),\end{aligned}$$

the second equality following from (3.2.7) and the third one from (3.2.3). Hence, the result is proved. □

Proof of Theorem 3.2.2. Let us start by proving that the set $\{b_m(a,\omega),\ a \in Y_{\ell,m},\ \omega \in S_{k,m}\}$ is free over $P_n^{\mathfrak{S}[\ell,n]}$. Let $a \in Y_{\ell,m}$. In $H^0_{m,n}$ we have a decomposition of the form

$$\tau_{\omega_0[\ell,m]}x^a \in \partial_{\omega_0[\ell,m]}(x^a) + \sum_{z>1} P_n^{\mathfrak{S}_{[m+1,n]}}\tau_z.$$

Hence, for $\omega \in S_{k,m}$ we have

$$\tau_{\omega_0[\ell,m]}x^a\tau_{\omega\omega_0[k,m]} \in \partial_{\omega_0[\ell,m]}(x^a)\tau_{\omega\omega_0[k,m]} + \sum_{z>\omega} P_n^{\mathfrak{S}_{[m+1,n]}}\tau_z\tau_{\omega_0[k,m]}.$$

Using Lemma 3.2.3 and the fact that the set $\left\{\tau_z\tau_{\omega_0[k,m]},\ z \in S_{k,m}\right\}$ is free over P_n, we obtain that the set $\{b_m(a,\omega),\ a \in Y_{\ell,m},\ \omega \in S_{k,m}\}$ is free over $P_n^{\mathfrak{S}[\ell,n]}$. To conclude, it suffices to compute graded dimensions. On the one hand, by (3.2.4) and (3.2.5) we have

$$\operatorname{grdim}\left(e'_{[\ell,m]}H^0_{m,n}e_{[k,m]}\right) = \operatorname{grdim}\left(P_n^{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}\right)\frac{\{m\}!\overline{\{m\}!}}{\{m-\ell+1\}!\ \{m-k+1\}!}.$$

On the other hand, the left $P_n^{\mathfrak{S}[\ell,n]}$-submodule spanned by the set $\{b_m(a,\omega),\ a \in Y_{\ell,m},\ \omega \in S_{k,m}\}$

has graded dimension

$$\operatorname{grdim}\left(P_n^{\mathfrak{S}_{[\ell,n]}}\right)\left(\sum_{a\in Y_{\ell,m}} q^{2(a_\ell+\ldots+a_m)}\right)\left(\sum_{\omega\in S_{k,m}} q^{-2l(\omega)}\right) q^{-(m-k)(m-k+1)-(m-\ell)(m-\ell+1)}. \quad (3.2.8)$$

By (3.2.7) we have

$$q^{-(m-\ell)(m-\ell-1)}\sum_{a\in Y_{\ell,m}} q^{2(a_\ell+\ldots+a_m)} = \frac{\{n-\ell+1\}!}{\{m-\ell+1\}!\ \{n-m\}!}.$$

By (3.2.6) we have

$$\sum_{\omega\in S_{k,m}} q^{-2l(\omega)} = \frac{\overline{\{m\}!}}{\overline{\{m-k+1\}!}} = q^{(m-k)(m-k+1)}\frac{\overline{\{m\}!}}{\{m-k+1\}!}.$$

Hence the graded dimension in equation (3.2.8) is equal to

$$\begin{aligned}
&\operatorname{grdim}\left(P_n^{\mathfrak{S}_{[\ell,n]}}\right)\frac{\{n-\ell+1\}!}{\{m-\ell+1\}!\ \{n-m\}!}\ \frac{\overline{\{m\}!}}{\{m-k+1\}!}\\
&=\operatorname{grdim}\left(P_n^{\mathfrak{S}_{[1,m]}\times\mathfrak{S}_{[m+1,n]}}\right)\frac{\{m\}!\overline{\{m\}!}}{\{m-\ell+1\}!\ \{m-k+1\}!}\\
&=\operatorname{grdim}\left(e'_{[\ell,m]}H^0_{m,n}e_{[k,m]}\right),
\end{aligned}$$

and the proof is complete. □

Combinatorics of the Differential

We now study the effect of the differential of $\Phi_n(\Theta)$ on the bases given by Theorem 3.2.2. On a general term of $\Phi_n(\Theta)$, the differential has the form

$$d_m : \begin{cases} e'_{[\ell,m]}H^0_{m,n}e_{[k,m]} & \to & e'_{[\ell,m+1]}H^0_{m+1,n}e_{[k,m+1]}, \\ h & \mapsto & \tau_{[\ell\uparrow m]}h\tau_{[m\downarrow k]}, \end{cases}$$

where the integers k, ℓ, m depend on the weight and the cohomological degree. Remark that d_m is a $P_n^{\mathfrak{S}_{[\ell,n]}}$-linear map. We will explicitly determine the kernel and image of d_m in terms of the bases of Theorem 3.2.2. To do so, we prove that either $d_m(b_m(a,\omega)) = 0$ or $d_m(b_m(a,\omega)) = b_{m+1}(\varphi(a,\omega))$, where φ is an injective map that we define below. Hence the study of the kernel and images of the map d_m can be done by studying the combinatorics of the sets $Y_{\ell,m}$, $S_{k,m}$ and the map φ.

We now introduce the necessary notation to define the map φ. Given $r \in \{0, \dots, m-\ell\}$, we define

$$Y^r_{m,\ell} = \{a \in Y_{m,\ell} \,|\, \forall\, i \leqslant r,\, a_{m-i} = i \text{ and } a_{m-r-1} > r+1\}\,.$$

The subsets $\left(Y^r_{m,\ell}\right)_r$ of $Y_{m,\ell}$ are disjoint and we denote by $Y^+_{\ell,m}$ the complement of their union. More explicitly, the elements of $Y^+_{\ell,m}$ are the sequences $a \in Y_{m,\ell}$ such that $a_m > 0$. Given $u \in \{1, \dots, m\}$, we denote by $S^u_{k,m}$ (resp. $S^{\geqslant u}_{k,m}$, $S^{<u}_{k,m}$) the subset of $S_{k,m}$ containing the elements ω such that $\omega(m) = u$ (resp. $\omega(m) \geqslant u$, $\omega(m) < u$).

We now define a map $\varphi_Y : Y_{\ell,m} \to Y_{\ell,m+1}$. If $a \in Y^+_{\ell,m}$ we put $\varphi_Y(a) = (a, 0)$. If $a \in Y^r_{\ell,m}$ we put $\varphi_Y(a) = (a_\ell, \dots, a_{m-r-1}, r+1, r, \dots, 0)$ (so we have increased the entries $a_{m-r}, \dots, a_m$ by 1 and added a 0 as the last entry). We can now define the map φ as follows

$$\varphi : \begin{cases} \left(Y^+_{\ell,m} \times S_{k,m}\right) \cup \left(\bigcup_{r=0}^{m-\ell} \left(Y^r_{\ell,m} \times S^{<m-r}_{k,m}\right)\right) & \to \quad Y_{\ell,m+1} \times S_{k,m+1}, \\ \qquad\qquad\qquad\qquad (a,\omega) & \mapsto \quad \begin{cases} (\varphi_Y(a), \omega) & \text{if } a \in Y^+_{\ell,m}, \\ (\varphi_Y(a), s_{m-r} \dots s_m \omega) & \text{if } a \in Y^r_{\ell,m}. \end{cases} \end{cases}$$

Proposition 3.2.4. *The map φ is injective and has image*

$$\operatorname{im}(\varphi) = \bigcup_{r=0}^{m+1-\ell} Y^r_{\ell,m+1} \times S^{\geqslant m+1-r}_{k,m+1}.$$

Proof. The map φ sends $Y^+_{\ell,m} \times S_{k,m}$ to $Y_{\ell,m+1} \times S^{m+1}_{k,m+1}$ and $Y^r_{\ell,m} \times S^{<m-r}_{k,m}$ to $Y_{\ell,m+1} \times S^{m-r}_{k,m+1}$.

Since the $S^u_{k,m+1}$ are disjoint for distinct values of u, it suffices to prove that $\varphi_{|Y^+_{\ell,m}\times S_{k,m}}$ and $\varphi_{|Y^r_{\ell,m}\times S^{<m-r}_{k,m}}$ are injective. However it is clear that the maps $(S_{k,m} \to S_{k,m+1}, \omega \mapsto \omega)$ and $(S^{<m-r}_{k,m} \to S_{k,m+1}, \omega \mapsto s_{m-r} \dots s_m\omega)$ are injective. Hence φ is injective.

We now compute $\mathrm{im}(\varphi)$. We have

$$\varphi\left(Y^+_{\ell,m} \times S_{k,m}\right) = \bigcup_{r'=0}^{m+1-\ell} Y^{r'}_{\ell,m+1} \times S^{m+1}_{k,m+1}$$

and

$$\varphi\left(Y^r_{\ell,m} \times S^{<m-r}_{k,m}\right) = \bigcup_{r'=r+1}^{m+1-\ell} Y^{r'}_{\ell,m+1} \times S^{m-r}_{k,m+1}.$$

Hence,

$$\mathrm{im}(\varphi) = \left(\bigcup_{r'=0}^{m+1-\ell} Y^{r'}_{\ell,m+1} \times S^{m+1}_{k,m+1}\right) \cup \left(\bigcup_{r=0}^{m-\ell}\left(\bigcup_{r'=r+1}^{m+1-\ell} Y^{r'}_{\ell,m+1} \times S^{m-r}_{k,m+1}\right)\right).$$

Switching the order of the two unions in the second term gives

$$\mathrm{im}(\varphi) = \left(\bigcup_{r'=0}^{m+1-\ell} Y^{r'}_{\ell,m+1} \times S^{m+1}_{k,m+1}\right) \cup \left(\bigcup_{r'=1}^{m+1-\ell}\left(\bigcup_{r=0}^{r'-1} Y^{r'}_{\ell,m+1} \times S^{m-r}_{k,m+1}\right)\right).$$

We can isolate the term $r' = 0$ in the first union and merge the two remaining unions to obtain

$$\begin{aligned}\mathrm{im}(\varphi) &= \left(Y^0_{\ell,m+1} \times S^{m+1}_{k,m+1}\right) \cup \left(\bigcup_{r'=1}^{m+1-\ell}\left(\bigcup_{r=0}^{r'} Y^{r'}_{\ell,m+1} \times S^{m+1-r}_{k,m+1}\right)\right)\\ &= \bigcup_{r'=0}^{m+1-\ell} Y^{r'}_{\ell,m+1} \times S^{\geqslant m+1-r'}_{k,m+1}.\end{aligned}$$

$\square$

Proposition 3.2.5. *Let $(a,\omega) \in Y_{\ell,m} \times S_{k,m}$. Then we have*

$$d_m(b_m(a,\omega)) = \begin{cases} 0 & \text{if } (a,\omega) \in \bigcup_{r=0}^{m-\ell} Y^r_{\ell,m} \times S^{\geqslant m-r}_{k,m}, \\ b_{m+1}(\varphi(a,\omega)) & \text{otherwise.} \end{cases}$$

Proof. We have $d_m(b_m(a,\omega)) = \tau_{\omega_0[\ell,m+1]} x^a \tau_\omega \tau_{\omega_0[k,m+1]}$. When $a \in Y^+_{\ell,m}$, it is clear that $d_m(b_m(a,\omega)) = b_{m+1}(\varphi(a,\omega))$. Assume now that $a \in Y^r_{\ell,m}$. In P_n, we have

$$\partial_m \ldots \partial_{m-r} \left(x^{r+1}_{m-r} x^r_{m-r+1} \ldots x_m \right) = (-1)^{r+1} x^r_{m-r} x^{r-1}_{m-r+1} \ldots x_{m-1}.$$

Furthermore, if $w \in \mathfrak{S}_{[\ell,m+1]}$ and $P \in P_n$, we have $\tau_{\omega_0[\ell,m+1]} P \tau_w = (-1)^{l(w)} \tau_{\omega_0[\ell,m+1]} \partial_{w^{-1}}(P)$. It follows that

$$\begin{aligned} \tau_{\omega_0[\ell,m+1]} x^a &= \tau_{\omega_0[\ell,m+1]} x^a x_{m-r} x_{m-r-1} \ldots x_m \tau_{m-r} \tau_{m-r+1} \ldots \tau_m \\ &= \tau_{\omega_0[\ell,m+1]} x^{\varphi_Y(a)} \tau_{m-r} \ldots \tau_m. \end{aligned}$$

Hence

$$d_m(b_m(a,\omega)) = \tau_{\omega_0[\ell,m+1]} x^{\varphi_Y(a)} \tau_{m-r} \ldots \tau_m \tau_\omega \tau_{\omega_0[k,m+1]}.$$

If $\omega(m) \geqslant m-r$, then $s_{m-r} \ldots s_m \omega$ does not have minimal length in its left coset modulo $\mathfrak{S}_{[k+1,m]}$. In that case, $\tau_{m-r} \ldots \tau_m \tau_\omega \tau_{\omega_0[k,m+1]} = 0$ by Lemma 2.3.3, and we conclude that $d_m(b_m(a,\omega)) = 0$. If $\omega(m) < m-r$, then $s_{m-r} \ldots s_m \omega$ has minimal length in its left coset modulo $\mathfrak{S}_{[k+1,m+1]}$ and $\tau_{m-r} \ldots \tau_m \tau_\omega = \tau_{s_{m-r} \ldots s_m \omega}$. If follows that $d_m(b_m(a,\omega)) = b_{m+1}(\varphi(a,\omega))$. ∎

From Propositions 3.2.4 and 3.2.5, we obtain bases for $\ker(d_m)$ and $\operatorname{im}(d_m)$.

Corollary 3.2.6. *The sets*

$$\left\{d_m(a,\omega),\ (a,\omega)\in\bigcup_{r=0}^{m-\ell}Y^r_{\ell,m}\times S^{\geqslant m-r}_{k,m}\right\},$$
$$\left\{d_{m+1}(a,\omega),\ (a,\omega)\in\bigcup_{r=0}^{m+1-\ell}Y^r_{\ell,m+1}\times S^{\geqslant m+1-r}_{k,m+1}\right\}$$

are bases of $\ker(d_m)$, $\mathrm{im}(d_m)$ *respectively, as left* $P_n^{\mathfrak{S}_{[\ell,n]}}$*-modules.*

Corollary 3.2.7. *For all* $n\in\mathbb{N}$, $k\in\{0,\dots,n\}$ *and* $\ell\neq n-k$ *we have:*

$$H^\ell\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)=0.$$

Proof. By Corollary 3.2.6 we have $\mathrm{im}(d_m)=\ker(d_{m+1})$ if $m+1<n$. The result follows. □

3.2.2 Invertibility of Θ

Top Cohomology

Let $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$. We give a description of $H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)$ as $\left(H^0_{n-k,n},H^0_{k,n}\right)$-bimodule. Using Corollary 3.2.6, we can give a basis for $H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)$ as a left $P_n^{\mathfrak{S}_{[n-k+1,n]}}$-module. With the notations introduced previously, we have $Y_{n-k+1,n}=\{(k-1,\dots,0)\}$ and $S^{<n-k+1}_{k,n}=\{\sigma_k\omega,\ \omega\in\mathfrak{S}_k\}$, where σ_k the element of $\mathfrak{S}_n$ sending i to $i+n-k$ if $i\leqslant k$, and to $i-k$ otherwise. Then there is a decomposition as left $P_n^{\mathfrak{S}_{[n-k+1,n]}}$-modules:

$$H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)=\bigoplus_{\omega\in\mathfrak{S}_k}P_n^{\mathfrak{S}_{[n-k+1,n]}}\tau_{\omega_0[n+1-k,n]}x_{n+1-k}^{k-1}\dots x_{n-1}\tau_{\sigma_k\omega\omega_0[k+1,n]}. \tag{3.2.9}$$

To describe the bimodule structure, it is easier to work in terms of $\left(P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k},P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$-bimodules using the Morita equivalences from (2.3.6).

Theorem 3.2.8. *There is an isomorphism of graded* $\left(P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$*-bimodules*

$$e'_{[1,n-k]}H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)e_{[1,k]} \simeq q^{n(n-1)}P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k},$$

where the left action of $P_n^{\mathfrak{S}_{n-k}\times s_k}$ *is given by multiplication and the right action of* $P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}$ *is given by* σ_k.

Proof. We view $e'_{[1,n-k]}H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)e_{[1,k]}$ as a quotient of $e'_{[1,n-k]}e'_{[n-k+1,n]}H_n^0 e_{[1,k]}e_{[k+1,n]}$. We denote by b the class of $\tau_{\omega_0[1,n]}$ in $e'_{[1,n-k]}H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)e_{[1,k]}$. Consider the $P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}$-linear map defined by

$$f:\begin{cases} P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k} & \to & e'_{[1,n-k]}H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)e_{[1,k]}, \\ 1 & \mapsto & b. \end{cases}$$

Let us show that f is surjective. By decomposition (3.2.9), we have

$$\begin{aligned} e'_{[1,n-k]}H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)e_{[1,k]} &= \tau_{\omega_0[1,n-k]}P_n^{\mathfrak{S}_{[n-k+1,n]}}\tau_{\omega_0[n+1-k,n]}x_{n+1-k}^{k-1}\dots x_{n-1}b \\ &= P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}b. \end{aligned}$$

Thus f is surjective. To conclude that f is an isomorphism, we compute graded dimensions. By decomposition (3.2.9), we have

$$\operatorname{grdim}\left(H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)\right) = \operatorname{grdim}\left(P_n^{\mathfrak{S}_{[n-k+1,n]}}\right)q^{-(n-k)(n-k-1)-2k(n-k)}\overline{\{k\}!}.$$

Hence we deduce

$$\begin{aligned}\operatorname{grdim}\left(e'_{[1,n-k]}H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)e_{[1,k]}\right) &= \frac{\operatorname{grdim}\left(P_n^{\mathfrak{S}_{[n-k+1,n]}}\right)q^{-(n-k)(n-k-1)-2k(n-k)}\overline{\{k\}!}}{\{k\}!\ \{n-k\}!}\\ &= \operatorname{grdim}\left(P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}\right)q^{-(n-k)(n-k-1)-k(k-1)-2k(n-k)}\\ &= \operatorname{grdim}\left(P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}\right)q^{-n(n-1)}.\end{aligned}$$

Since $\deg(b) = -n(n-1)$, we deduce that f is an isomorphism of left $P_n^{\mathfrak{S}_{n-k}\times\mathfrak{S}_k}$-modules. All that remains to prove is the compatibility of f with the right $P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}$-module structure, which we do in Lemma 3.2.9 below. □

Lemma 3.2.9. *Keep the notations of the proof of Theorem 3.2.8. For all $P \in P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}$, we have $bP = \sigma_k(P)b$.*

Proof. We prove that $\tau_{\omega_0[1,n]}P - \sigma_k(P)\tau_{\omega_0[1,n]}$ is a coboundary of $\Phi_n\left(\Theta 1_{-n+2k}\right)$. Using the relations of the affine nil Hecke algebra, we have

$$\tau_{\omega_0[1,n]}P - \sigma_k(P)\tau_{\omega_0[1,n]} \in \sum_{\substack{\omega\in\mathfrak{S}_n/\mathfrak{S}_k\times\mathfrak{S}_{n-k}\\ \omega<\sigma_k}} P_n\tau_{\omega\omega_0[1,k]\omega_0[k+1,n]}.$$

Since $\tau_{\omega_0[1,n]} = e'_{[n-k+1,n]}\tau_{\omega_0[1,n]}$, we actually have

$$\tau_{\omega_0[1,n]}P - \sigma_k(P)\tau_{\omega_0[1,n]} \in \sum_{\substack{\omega\in\mathfrak{S}_n/\mathfrak{S}_k\times\mathfrak{S}_{n-k}\\ \omega<\sigma_k}} \tau_{\omega_0[n-k+1,n]}P_n\tau_{\omega\omega_0[1,k]\omega_0[k+1,n]}. \tag{3.2.10}$$

Let us explain why the right-hand side of (3.2.10) is contained in the image of the differential of $\Phi_n\left(\Theta 1_{-n+2k}\right)$. Since $\left\{\partial_\omega\left(x_{n-k+1}^{k-1}\dots x_{n-1}\right),\ \omega\in\mathfrak{S}_{[n-k+1,n]}\right\}$ is a basis for P_n as a $P_n^{\mathfrak{S}_{[n-k+1,n]}}$-module, we have:

$$\tau_{\omega_0[n-k+1,n]}P_n = \bigoplus_{\omega\in\mathfrak{S}_{[n-k+1,n]}} P_n^{\mathfrak{S}_{[n-k+1,n]}}\tau_{\omega_0[n-k+1,n]}x_{n-k+1}^{k-1}\dots x_{n-1}\tau_\omega.$$

Hence the right-hand side of (3.2.10) has the form

$$\sum_{\substack{\omega\in\mathfrak{S}_n/\mathfrak{S}_k\times\mathfrak{S}_{n-k},\ \omega'\in\mathfrak{S}_{[n-k+1,n]}\\ \omega<\sigma_k}} P_n^{\mathfrak{S}_{[n-k+1,n]}}\tau_{\omega_0[n-k+1,n]}x_{n-k+1}^{k-1}\cdots x_{n-1}\tau_{\omega'}\tau_{\omega\omega_0[1,k]\omega_0[k+1,n]}.$$

For $\omega\in\mathfrak{S}_n/\mathfrak{S}_k\times\mathfrak{S}_{n-k}$, the condition $\omega<\sigma_k$ is equivalent to $\omega(n)>n-k$. If furthermore $\omega'\in\mathfrak{S}_{[n-k+1,n]}$, we have $\omega'\omega(n)>n-k$. Hence on the basis of Theorem 3.2.2, the right-hand side of (3.2.10) decomposes as

$$\bigoplus_{\substack{\omega\in\mathfrak{S}_n/\mathfrak{S}_k\times\mathfrak{S}_{n-k}\\ \omega<\sigma_k}} P_n^{\mathfrak{S}_{[n-k+1,n]}}\tau_{\omega_0[n-k+1,n]}x_{n-k+1}^{k-1}\cdots x_{n-1}\tau_{\omega\omega_0[1,k]\omega_0[k+1,n]}.$$

By Corollary 3.2.6, this is contained in the image of the differential of $\Phi_n\left(\Theta 1_{-n+2k}\right)$. Hence $\tau_{\omega_0[1,n]}P-\sigma_k(P)\tau_{\omega_0[1,n]}$ is a coboundary, and the result follows. $\square$

Theorem 3.2.8 implies in particular that $H^{n-k}\left(\Phi_n\left(\Theta 1_{-n+2k}\right)\right)$ is an invertible bimodule, which proves Theorem 3.2.1.

Invertibility of Θ

We denote by $\Theta^\vee$ the right adjoint of Θ in the 2-category $\mathcal{U}$.

Theorem 3.2.10. *For all $\lambda\in\mathbb{Z}$, the unit $u_\lambda : 1_\lambda\to\Theta^\vee\Theta 1_\lambda$ and counit $v_\lambda:\Theta\Theta^\vee 1_{-\lambda}\to 1_{-\lambda}$ of adjunction are homotopy equivalences. In other words, $\Theta 1_\lambda$ is invertible in $K(\mathcal{U})$.*

Proof. If M is an object of an integrable 2-representation of $\mathcal{U}$, we have $E^i(M)=F^i(M)=0$ for i large enough. Hence the complexes $\Theta\Theta^\vee(M)$ and $\Theta^\vee\Theta(M)$ are bounded. So by Theorem 3.1.16, it suffices to check that for all $n\in\mathbb{N}$, $\Phi_n(u_\lambda)$ and $\Phi_n(v_\lambda)$ are quasi-isomorphisms.

Let $n\in\mathbb{N}$. If $\lambda\notin\{-n+2k,\ k\in\{0,\dots,n\}\}$, then $\Phi_n(1_\lambda)=\Phi_n(1_{-\lambda})=0$, so the result is clear. If $k \in \{0, \dots, n\}$, then by Corollary 3.2.7 there is a quasi-isomorphism

$$\Phi_n(\Theta 1_{-n+2k}) \simeq H^{n-k}(\Phi_n(\Theta 1_{-n+2k}))[n-k].$$

By Theorem 3.2.8, $H^{n-k}(\Phi_n(\Theta 1_{-n+2k}))$ is invertible as a $\left(H^0_{n-k,n}, H^0_{k,n}\right)$-bimodule. It follows that the complex $\Phi_n(\Theta 1_{-n+2k})$ is invertible in $D(\mathcal{L}(n){-}\mathrm{bim})$. Thus $\Phi_n(u_{-n+2k})$ and $\Phi_n(v_{-n+2k})$ are quasi-isomorphisms, which ends the proof. $\square$

3.2.3 Compatibility with Chevalley Generators

The goal of this subsection is to prove that for all $\lambda \in \mathbb{Z}$, there is a homotopy equivalence $\Theta E 1_\lambda \simeq q^{\lambda+2} F \Theta 1_\lambda[-1]$. We start by constructing a map $G_\lambda : \Theta E 1_\lambda \to q^{\lambda+2} F \Theta 1_\lambda[-1]$. We will then prove that G_λ is a homotopy equivalence using Theorem 3.1.16.

Given $k, \ell \in \mathbb{N}$, we define an element $G_{k,\ell} \in H^0_k \otimes (H^0_\ell)^{\mathrm{op}}$ of degree 0 by

$$G_{k,\ell} = \sum_{r=0}^{\ell-1} (-1)^r x_1^r e'_k \otimes \epsilon_{\ell-1-r}(x_2, \dots, x_\ell) x_{[2,\ell]} \tau_{\omega_0[1,\ell]},$$

where ϵ_j denotes the elementary symmetric polynomial of degree j. In $H^0_k \otimes (H^0_\ell)^{\mathrm{op}}$, we have $G_{k,\ell} = G_{k,\ell}(e'_k \otimes e_{[2,\ell]}) = (e'_{[2,k]} \otimes e_\ell) G_{k,\ell}$. Hence $G_{k,\ell}$ defines a map

$$G_{k,\ell} : F^{(k)} E^{(\ell-1)} E \to q^{k-\ell} F F^{(k-1)} E^{(\ell)}.$$

Lemma 3.2.11. *For all $k, \ell \in \mathbb{N}$, the following diagram commutes:*

$$\begin{array}{ccc} F^{(k)}E^{(\ell-1)}E & \xrightarrow{d_{k,\ell-1}E} & F^{(k+1)}E^{(\ell)}E \\ {\scriptstyle G_{k,\ell}}\big\downarrow & & \big\downarrow{\scriptstyle G_{k+1,\ell+1}} \\ FF^{(k-1)}E^{(\ell)} & \xrightarrow[Fd_{k-1,\ell}]{} & FF^{(k)}E^{(\ell+1)}. \end{array}$$

In this diagram, $d_{k,\ell} : F^{(k)}E^{(\ell)} \to F^{(k+1)}E^{(\ell+1)}$ is the composition of the unit of adjunction *with the projection* $\tau_{[1\uparrow k]}\otimes\tau_{[\ell\downarrow 1]}$.

Proof. This is a straightforward computation. On the one hand, we have

$$\begin{aligned}
Fd_{k-1,\ell}\circ G_{k,\ell} &= \big(\tau_{[2\uparrow k]}\otimes\tau_{[\ell\downarrow 1]}\big)\circ F^{(k)}\eta E^{(\ell-1)}E\circ\sum_{r=0}^{\ell-1}(-1)^r x_1^r e'_{[1,k]}\otimes\epsilon_{\ell-1-r}(x_2,\ldots,x_\ell)x_{[2,\ell]}\tau_{\omega_0[1,\ell]}\\
&= \left(\sum_{r=0}^{\ell-1}(-1)^r x_1^r\tau_{[2\uparrow k]}e'_{[1,k]}\otimes\epsilon_{\ell-1-r}(x_2,\ldots,x_\ell)x_{[2,\ell]}\tau_{\omega_0[1,\ell+1]}\right)\circ F^{(k)}\eta E^{(\ell-1)}E.
\end{aligned}$$

On the other hand, we have

$$G_{k+1,\ell+1}\circ d_{k,\ell-1}E=\left(\sum_{r=0}^{\ell}(-1)^r x_1^r\tau_{[1\downarrow k]}e'_{[1,k]}\otimes\epsilon_{\ell-r}(x_2,\ldots x_{\ell+1})x_{[2,\ell]}\tau_{\omega_0[2,\ell+1]}\right)\circ F^{(k)}\eta E^{(\ell-1)}E.$$

The elementary symmetric polynomials satisfy the relation

$$\epsilon_{\ell-r}(x_2,\ldots x_{\ell+1})=\epsilon_{\ell-r}(x_2,\ldots x_\ell)+x_{\ell+1}\epsilon_{\ell-r-1}(x_2\ldots,x_\ell),$$

and we can rewrite the above as

$$\begin{aligned}
G_{k+1,\ell+1}\circ d_{k,\ell-1}E=&\Bigg(\sum_{r=0}^{\ell-1}(-1)^r x_1^r\tau_{[1\downarrow k]}e'_{[1,k]}\otimes x_{\ell+1}\epsilon_{\ell-r-1}(x_2,\ldots x_\ell)x_{[2,\ell]}\tau_{\omega_0[2,\ell+1]}\\
&+\sum_{r=1}^{\ell}(-1)^r x_1^r\tau_{[1\downarrow k]}e'_{[1,k]}\otimes\epsilon_{\ell-r}(x_2,\ldots x_\ell)x_{[2,\ell]}\tau_{\omega_0[2,\ell+1]}\Bigg)\circ F^{(k)}\eta E^{(\ell-1)}E.
\end{aligned}$$

We have $Fx\circ\eta=xE\circ\eta$. Using this, we can take the $x_{\ell+1}$ in the first sum to the left side of the tensor. We obtain

$$
\begin{aligned}
&G_{k+1,\ell+1} \circ d_{k,\ell-1}E \\
=&\Bigg(\sum_{r=0}^{\ell-1} (-1)^r x_1^r \tau_{[1\downarrow k]} e'_{[1,k]} x_{k+1} \otimes \epsilon_{\ell-r-1}(x_2, \ldots x_\ell) x_{[2,\ell]} \tau_{\omega_0[2,\ell+1]} \\
&- \sum_{r=0}^{\ell-1} (-1)^r x_1^{r+1} \tau_{[1\downarrow k]} e'_{[1,k]} \otimes \epsilon_{\ell-r-1}(x_2, \ldots x_\ell) x_{[2,\ell]} \tau_{\omega_0[2,\ell+1]} \Bigg) \circ F^{(k)} \eta E^{(\ell-1)} E \\
=&\Bigg(\sum_{r=0}^{\ell-1} (-1)^r x_1^r \left(\tau_{[1\uparrow k]} x_{k+1} - x_1 \tau_{[1\uparrow k]} \right) e'_{[1,k]} \otimes \epsilon_{\ell-r-1}(x_2, \ldots x_\ell) x_{[2,\ell]} \tau_{\omega_0[2,\ell+1]} \Bigg) \circ F^{(k)} \eta E^{(\ell-1)} E.
\end{aligned}
$$

In H_{k+1}, we have the relation $\left(\tau_{[1\uparrow k]} x_{k+1} - x_1 \tau_{[1\uparrow k]}\right) e'_{[1,k]} = \tau_{[2\uparrow k]} e'_{[1,k]}$, and this completes the proof. $\square$

We can now define the morphism $G_\lambda : \Theta E 1_\lambda \to q^{\lambda+2} F \Theta 1_\lambda[-1]$. On the r^{th} component of $\Theta E 1_\lambda$, G_λ is defined to be the map $G_{\lambda+r+2,r+1} : q^{-r} F^{(\lambda+r+2)} E^{(r)} E 1_\lambda \to q^{\lambda+1-r} F F^{(\lambda+r+1)} E^{(r+1)} 1_\lambda$. This gives a morphism of complexes by Lemma 3.2.11. Similarly, we define a morphism of complexes $T_\lambda : q^{\lambda+2} F \Theta 1_\lambda[-1] \to \Theta E 1_\lambda$ using the elements

$$
T_{k,\ell} = \sum_{r=0}^{k-1} (-1)^r \tau_{\omega_0[1,k]} \epsilon_{k-1-r}(x_2, \ldots, x_k) x'_{[2,k]} \otimes e_\ell x_1^r \in H_k^0 \otimes \left(H_\ell^0\right)^{\text{op}}.
$$

The fact that these give a morphism of complexes is proved as for G_λ.

Lemma 3.2.12. *Let $k, \ell \in \mathbb{N}$. If $\ell \leqslant k$, then $T_{k,\ell} G_{k,\ell} = e'_k \otimes e_{[2,\ell]}$. If $k \leqslant \ell$, then $G_{k,\ell} T_{k,\ell} = e'_{[2,k]} \otimes e_\ell$.*

Proof. We prove the result in the case $\ell \leqslant k$, the other one being similar. We have

$$
T_{k,\ell} G_{k,\ell} = \sum_{\substack{0 \leqslant r \leqslant k-1 \\ 0 \leqslant u \leqslant \ell-1}} (-1)^{r+u} \tau_{\omega_0[1,k]} \epsilon_{k-1-r}(x_2, \ldots, x_k) x_1^u x'_{[2,k]} e'_k \otimes \epsilon_{\ell-1-u}(x_2, \ldots, x_\ell) e_\ell x_{[2,\ell]} \tau_{\omega_0[1,\ell]} x_1^r.
$$

In H_k^0 we have

$$\begin{aligned}\tau_{\omega_0[1,k]}\epsilon_{k-1-r}(x_2,\ldots,x_k)x_1^u x'_{[2,k]}e'_k &= \partial_{[k-1\downarrow 1]}\left(\epsilon_{k-1-r}(x_2,\ldots,x_k)x_1^u\right)e'_k \\ &= (-1)^u\delta_{u,r}e'_k.\end{aligned}$$

Hence

$$T_{k,\ell}G_{k,\ell} = e'_k \otimes \sum_{u=0}^{\ell-1}(-1)^u e_{[2,\ell]}\epsilon_{\ell-1-u}(x_2,\ldots,x_\ell)\tau_{[1\uparrow\ell-1]}x_1^u.$$

Using the polynomial representation, it is easy to check that

$$\sum_{u=0}^{\ell-1}(-1)^u e_{[2,\ell]}\epsilon_{\ell-1-u}(x_2,\ldots,x_\ell)\tau_{[1\uparrow\ell-1]}x_1^u = e_{[2,\ell]},$$

and the result follows. □

Corollary 3.2.13. *If $\lambda \geqslant 0$, then $T_\lambda G_\lambda = \mathrm{id}_{\Theta E1_\lambda}$. If $\lambda \leqslant 0$, then $G_\lambda T_\lambda = \mathrm{id}_{F\Theta 1_\lambda[-1]}$.*

Theorem 3.2.14. *For all $\lambda \in \mathbb{Z}$, $G_\lambda : \Theta E1_\lambda \to q^{\lambda+2}F\Theta 1_\lambda[-1]$ is a homotopy equivalence.*

Proof. If M is an object of an integrable 2-representation of $\mathcal{U}$, the complexes $\Theta E(M)$ and $F\Theta(M)$ are bounded. Thus, by Theorem 3.1.16, it suffices to prove that for all $n \in \mathbb{N}$, $\Phi_n(G_\lambda)$ is a quasi-isomorphism.

Let $n \in \mathbb{N}$ and $k \in \{0,\ldots,n\}$. There is an isomorphism of $\left(P_n^{\mathfrak{S}_{k+1}\times\mathfrak{S}_{n-k-1}}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$-bimodules

$$e'_{[1,k+1]}\Phi_n\left(E1_{-n+2k}\right)e_{[1,k]} \simeq P_n^{\mathfrak{S}_{[1,k]}\times\mathfrak{S}_{[k+2,n]}}.$$

Hence, by Corollary 3.2.7 and Theorem 3.2.8, there is an isomorphism of $\left(P_n^{\mathfrak{S}_{n-k-1}\times\mathfrak{S}_{k+1}}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$-

bimodules

$$e'_{[1,n-k-1]}H^{\ell}\left(\Phi_n\left(\Theta E 1_{-n+2k}\right)\right)e_{[1,k]} = \begin{cases} P_n^{\mathfrak{S}_{[1,k]}\times\mathfrak{S}_{[k+2,n]}} & \text{if } \ell = n-k-1, \\ 0 & \text{otherwise,} \end{cases}$$

where the left action is given by σ_{k+1}^{-1} in the first case. Similarly, there is an isomorphism

$$e'_{[1,n-k-1]}H^{\ell}\left(\Phi_n\left(F\Theta 1_{-n+2k}\right)\right)e_{[1,k]} = \begin{cases} P_n^{\mathfrak{S}_{[1,n-k-1]}\times\mathfrak{S}_{[n-k+1,n]}} & \text{if } \ell = n-k, \\ 0 & \text{otherwise,} \end{cases}$$

where the right action is given by σ_k in the first case. Now remark that there is an isomorphism of $\left(P_n^{\mathfrak{S}_{n-k-1}\times\mathfrak{S}_{k+1}}, P_n^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}\right)$-bimodules

$$s_{n-k-1}\dots s_1\sigma_k : P_n^{\mathfrak{S}_{[1,k]}\times\mathfrak{S}_{[k+2,n]}} \xrightarrow{\sim} P_n^{\mathfrak{S}_{[1,n-k-1]}\times\mathfrak{S}_{[n-k+1,n]}}.$$

Hence the cohomology bimodules of $\Phi_n\left(\Theta E 1_\lambda\right)$ and $\Phi_n\left(F\Theta 1_\lambda[-1]\right)$ are isomorphic as graded K-vector spaces. Furthermore by Lemma 3.2.13, for all ℓ, $H^{\ell}\left(\Phi_n(G_\lambda)\right)$ is either a split surjection or a split injection. Since the source and target of $H^{\ell}\left(\Phi_n(G_\lambda)\right)$ are locally finite and have the same graded dimension, we conclude that $H^{\ell}(\Phi_n(G_\lambda))$ is an isomorphism. Hence $\Phi_n(G_\lambda)$ is a quasi-isomorphism, and the proof is complete. □

CHAPTER 4

Categorification of the Adjoint Action

We now return to the case of a general root datum (X, Y) of type $(I, \cdot)$, and the associated quantum group U and 2-quantum group $\mathcal{U}$. For each $i \in I$, the subalgebra U_i of U generated by e_i, f_i and k_i is isomorphic to the quantum group $U_{\mathfrak{sl}_2}$ of type A_1 (up to changing the parameter from q to q_i in the definition of $U_{\mathfrak{sl}_2}$). In particular, if V is an integrable representation of $\dot{U}$, we obtain a collection $(\theta_i)_{i\in I}$ of automorphisms of V. These automorphisms induce isomorphisms between weight spaces $\theta_i : V_\lambda \xrightarrow{\sim} V_{s_i(\lambda)}$ for all $i \in I$ and $\lambda \in X$. Furthermore, Lusztig [Lus10, Chapter 37] also defined automorphisms $(\theta_i)_{i\in I}$ of U compatible with the action on integrable representations. These automorphisms, commonly known as *Lusztig's symmetries* are defined by the formulas

$$\theta_i(e_j) = \begin{cases} -k_i f_i & \text{if } j = i, \\ \mathrm{ad}_{e_i}^{(-c_{i,j})}(e_j) & \text{otherwise.} \end{cases} \tag{4.0.1}$$

In this equation, ad denotes the adjoint action of U coming from the Hopf algebra structure.

For a Hopf algebra A with coproduct Δ and antipode S, the left adjoint representation of A on itself is defined by

$$\mathrm{ad}_z(y) = \sum z_{(1)} y S(z_{(2)})$$

for all $y, z \in A$. Here we use Sweedler's notation for the coproduct

$$\Delta(z) = \sum z_{(1)} \otimes z_{(2)}.$$

In the case of the quantum group U, there are simple formulas for the adjoint action of the algebra generators of U. For $y \in U$ homogeneous of degree $\beta \in Q^+$, $i \in I$ and $\mu \in Y$ we have

$$\mathrm{ad}_{e_i}(y) = e_i y - q_i^{-\langle i^\vee, \beta\rangle} y e_i, \quad \mathrm{ad}_{f_i}(y) = (f_i y - y f_i) k_i^{-1}, \quad \mathrm{ad}_{k_\mu}(y) = q^{\langle \mu, \beta\rangle} y. \tag{4.0.2}$$

The object of this chapter is to explain what the elements $\mathrm{ad}_{e_i}^{(-c_{i,j})}(e_j)$ appearing in the right-hand side of equation (4.0.1) should be at the categorified level. To achieve this, we undertake the task of categorifying the adjoint action of U. Stated at this level of generality, this task is unreasonable. Indeed, our setting for categorification is only well-suited for integrable weight representations of U, while the adjoint action of U on itself is not integrable. To see this, observe for that for all $m \in \mathbb{N}$ we have

$$\mathrm{ad}_{e_i}^m(e_i) = \left(\prod_{k=1}^{m}(1 - q_i^{2m})\right) e_i^{m+1} \neq 0. \tag{4.0.3}$$

Therefore, it is necessary to make some restrictions in order to obtain an integrable representation from the adjoint action. The first restriction we will make is to fix a simple root $i \in I$ and consider the adjoint action of U_i only. The second restriction is to consider the adjoint action of U_i on a subalgebra $\mathbf{f}[i]$ of $\mathbf{f}$ that is integrable for the adjoint action of U_i and maximal for this property. The subalgebra $\mathbf{f}[i]$ is studied by Lusztig in [Lus10, Chapter 38] and can be described explicitly: it is generated by the elements $\mathrm{ad}_{e_i}^{(n)}(e_j)$ for $n \in \mathbb{N}$ and $j \in I \setminus \{i\}$. In this chapter, we categorify the adjoint action of U_i on $\mathbf{f}[i]$.

The subalgebra $\mathbf{f}[i]$ will be categorified by projective modules over an algebra H^i. In

Section 4.1, we introduce this algebra and define an endofunctor $\mathrm{ad}_{E_i} : H^i\text{−mod} \to H^i\text{−mod}$. Then, in Section 4.2, we prove that there are natural short exact sequences

$$0 \to q_i^{-\langle i^\vee,\beta\rangle} M E_i \to E_i M \to \mathrm{ad}_{E_i}(M) \to 0.$$

These categorify the equation $\mathrm{ad}_{e_i}(y) = e_i y - q_i^{-\langle i^\vee,\beta\rangle} y e_i$ and imply strong structural results. One of the consequences of these sequences is that ad_{E_i} induces a structure of 2-representation of $\mathfrak{sl}_2$ on $H^i\text{−proj}$. Section 4.3 is devoted to proving that result. Our approach is similar to that of Kang-Kashiwara [KK12] for categorification of highest weight modules. The algebra H^i that we introduce can be seen as an analogue of the cyclotomic quiver Hecke algebras for the adjoint action. However, it behaves quite differently: for instance, it is infinite dimensional, while the cyclotomic quiver Hecke algebras are always finite dimensional. Finally, in Section 4.4 we identify the Grothendieck group of this 2-representation and prove that it recovers the adjoint action of U_i on $\mathbf{f}[i]$.

4.1 Adjoint Action of E_i

4.1.1 The Algebras H^i_β

Definitions

Definition 4.1.1. For $\beta \in Q^+$, let H^i_β be the graded K-algebra defined by $H^i_\beta = H_\beta/(1_{\beta-i,i})$, where $(1_{\beta-i,i})$ is the two-sided ideal of H_β generated by $1_{\beta-i,i}$ (when $\beta - i \notin Q^+$, we put $1_{\beta-i,i} = 0$). For $n \geqslant 0$, let H^i_n be the graded K-algebra defined by

$$H^i_n = \bigoplus_{\substack{\beta\in Q^+ \\ |\beta|=n}} H^i_\beta.$$

We also let

$$H^i = \bigoplus_{n\in\mathbb{N}} H_n^i.$$

Equivalently, for $n \geqslant 1$ we can define H_n^i as the quotient of H_n by the two sided ideal generated by the elements $1_{\nu i}$ for $\nu \in I^{n-1}$. If I is finite and $\beta, \gamma \in Q^+$, then the right inclusion $r_\beta^{\beta+\gamma} : H_\beta \to H_{\beta+\gamma}$ satisfies

$$r_\beta^{\beta+\gamma}(1_{\nu i}) = \sum_{\mu\in I^\gamma} 1_{\mu\nu i} \in (1_{\rho i})_{\rho\in I^{\beta+\gamma-i}}.$$

Hence, we have an induced morphism $r_\beta^{\beta+\gamma} : H_\beta^i \to H_{\beta+\gamma}^i$. However, the left inclusions do not induce morphisms at the level of the algebras H_n^i.

We will use the following notation to denote the categories of modules over H and H^i:

$$\mathcal{H} = \bigoplus_{\beta\in Q^+} H_\beta\text{−mod},$$
$$\mathcal{H}^i = \bigoplus_{\beta\in Q^+} H_\beta^i\text{−mod}.$$

We view $\mathcal{H}^i$ as a full subcategory of $\mathcal{H}$. With this perspective, if $M \in H_\beta$−mod, then M is an object of $\mathcal{H}^i$ if and only if $1_{\beta-i,i}M = 0$.

Example 4.1.2. If $j \in I \setminus \{i\}$, we have $H_{i+j}^i \simeq K[x_1, x_2]/Q_{i,j}(x_2, x_1)$. In particular, H_{i+j}^i is either zero (in the case $i \cdot j = 0$) or not finite dimensional.

We say that a subcategory of a graded abelian category is a *Serre subcategory* if it is non-empty, full and closed under subquotients, extensions and degree shifts.

Proposition 4.1.3. *The category $\mathcal{H}^i$ is a Serre and monoidal subcategory of $\mathcal{H}$.*

Proof. Let $\beta \in Q^+$. The functor

$$\begin{cases} H_\beta-\text{mod} & \mapsto & 1_{\beta-i,i}H_\beta 1_{\beta-i,i}-\text{mod}, \\ M & \mapsto & 1_{\beta-i,i}M. \end{cases}$$

is exact. Hence its kernel is a Serre subcategory of $H_\beta-$mod. If follows that $\mathcal{H}^i$ is a Serre subcategory of $\mathcal{H}$.

Let $M \in H^i_\beta-$mod and $N \in H^i_\gamma-$mod, where $\beta, \gamma \in Q^+$ have respective heights m, n. Then by (2.4.1) we have a decomposition

$$MN = \bigoplus_{\underline{\omega} \in T} \tau_{\underline{\omega}} 1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes N),$$

where T is a complete set of reduced decompositions for minimal length representatives of left cosets of $\mathfrak{S}_m \times \mathfrak{S}_n$ in $\mathfrak{S}_{m+n}$. If $\underline{\omega} \in T$, then $s_{\underline{\omega}}^{-1}(1) \in \{1, n\}$. Hence we have

$$1_{\beta+\gamma-i,i}\tau_{\underline{\omega}}1_{\beta,\gamma} = \begin{cases} \tau_{\underline{\omega}}1_{\beta-i,i,\gamma} & \text{if } s_{\underline{\omega}}^{-1}(1) = n, \\ \tau_{\underline{\omega}}1_{\beta,\gamma-i,i} & \text{if } s_{\underline{\omega}}^{-1}(1) = 1. \end{cases}$$

Since both $1_{\beta-i,i,\gamma}$ and $1_{\beta,\gamma-i,i}$ act by zero on $M \otimes N$, we conclude that $1_{\beta+\gamma-i,i}MN = 0$, and $MN \in \mathcal{H}^i$. Thus $\mathcal{H}^i$ is a monoidal subcategory of $\mathcal{H}$. □

Proposition 4.1.4. *Let $M \in \mathcal{H}$. Then, $M \in \mathcal{H}^i$ if and only if for all $N \in \mathcal{H}$,*

$$\mathrm{Hom}_{\mathcal{H}}(NE_i, M) = 0.$$

Proof. Assume that M is an H_β-module for $\beta \in Q^+$. Then the condition $1_{\beta-i,i}M = 0$ is equivalent to $\mathrm{Hom}_{H_{\beta-i}}(N, 1_{\beta-i,i}M) = 0$ for all $N \in H_{\beta-i}-$mod. The functor $(-)E_i$ is left adjoint to the functor $1_{\beta-i,i}(-)$. Thus, this last condition is equivalent to $\mathrm{Hom}_{H_\beta}(NE_i, M) = 0$ for all $N \in H_{\beta-i}-$mod. □

The inclusion functor $\mathcal{H}^i \hookrightarrow \mathcal{H}$ has a left adjoint $\pi_i : \mathcal{H} \to \mathcal{H}^i$, which will play an important role in the constructions of this chapter. Explicitly, for a module $M \in H_\beta - \mathrm{mod}$, we have

$$\pi_i(M) = M/(1_{\beta-i,i}M),$$

where $(1_{\beta-i,i}M)$ denotes the H_β-submodule of M generated by $1_{\beta-i,i}M$. We denote by p_M the canonical quotient morphism $M \twoheadrightarrow \pi_i(M)$. This is the unit of the adjunction between π_i and the inclusion of $\mathcal{H}^i$ in $\mathcal{H}$. There is a canonical isomorphism of H^i_β-modules

$$\left\{\begin{array}{rcl} H^i_\beta \otimes_{H_\beta} M & \xrightarrow{\sim} & \pi_i(M), \\ 1_\beta \otimes m & \mapsto & m + (1_{\beta-i,i}M). \end{array}\right.$$

Note that $\pi_i(M) = M$ if and only if $M \in \mathcal{H}^i$. The following lemma gives compatibilities between the functor π_i and the monoidal structure. It will be used repeatedly below.

Lemma 4.1.5. *Let $M, N \in \mathcal{H}$. Then the canonical quotient morphism*

$$\pi_i(Mp_N) : \pi_i(MN) \to \pi_i(M\pi_i(N))$$

is an isomorphism. If $M \in \mathcal{H}^i$, then this provides an isomorphism $\pi_i(MN) \simeq M\pi_i(N)$.

Proof. Assume that $M \in H_\beta$−mod and $N \in H_\gamma$−mod for $\beta, \gamma \in Q^+$. We have canonical isomorphisms

$$\pi_i(MN) \simeq H^i_{\beta+\gamma} 1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes N),$$
$$\pi_i(M\pi_i(N)) \simeq H^i_{\beta+\gamma} 1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes \pi_i(N)).$$

Under these identifications, the morphism $\pi_i(Mp_N) : \pi_i(MN) \to \pi_i(M\pi_i(N))$ is given by

$$\begin{cases} H^i_{\beta+\gamma}1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes N) & \to & H^i_{\beta+\gamma}1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes \pi_i(N)), \\ 1_{\beta,\gamma} \otimes (m \otimes n) & \mapsto & 1_{\beta,\gamma} \otimes (m \otimes (n + (1_{\gamma-i,i}N))). \end{cases}$$

Since $H^i_{\beta+\gamma}1_{\beta,\gamma-i,i} = 0$, this morphism has a well-defined inverse, so it is an isomorphism. This proves the first statement. If $M \in \mathcal{H}^i$, then we have $M\pi_i(N) \in \mathcal{H}[i]$. So $\pi_i(M\pi_i(N)) = M\pi_i(N)$, and the second statement follows. $\square$

Polynomial Annihilators

Let $\beta \in Q^+$. Since H^i_β is a quotient of H_β, it is naturally an H_β-bimodule, and in particular a P_β-bimodule. Contrary to H_β,H^i_β is not a faithful P_β-module. The goal of the following results is to determine explicitly some elements of P_β which act by zero on H^i_β. This will be an important ingredient in the proofs of the main theorems of this chapter.

We start with some general results about affine nil Hecke algebras. For a K-algebra A, consider the K-algebra $A \otimes H^0_n$. It contains the K-algebra $A[x_1, \ldots, x_n] = A \otimes P_n$ as a subalgebra. The action of H^0_n on P_n given by the isomorphism (2.3.5) induces actions of H^0_n and $A \otimes H^0_n$ on $A[x_1, \ldots, x_n]$.

Lemma 4.1.6. *Let J be a two-sided ideal of $A \otimes H^0_n$.*

1. *As a left $A[x_1, \ldots, x_n]$-module, J decomposes as*

$$J = \bigoplus_{\omega \in \mathfrak{S}_n} (J \cap A[x_1, \ldots, x_n])\, \tau_\omega.$$

2. *$J \cap A[x_1, \ldots, x_n]$ is an H^0_n-submodule of $A[x_1, \ldots, x_n]$.*

Proof. We start by proving the first statement. Let $h \in J$. We have $h = \sum_{\omega \in \mathfrak{S}_n} h_\omega \tau_\omega$ for some $h_\omega \in A[x_1 \ldots, x_n]$. We prove that for all $\omega \in \mathfrak{S}_n$, $h_\omega \in J$. We proceed by induction on the length of ω. For the base case, notice that $h\tau_{\omega_0[1,n]} = h_1\tau_{\omega_0[1,n]}$. Thus $h_1\tau_{\omega_0[1,n]} \in J$. However, in H_n^0 we have $1 \in P_n\tau_{\omega_0[1,n]}P_n$. Hence $h_1 \in P_n h_1\tau_{\omega_0[1,n]}P_n$, from which we deduce that $h_1 \in J$. Assume now that $\omega \in \mathfrak{S}_n$ and that $h_y \in J$ for all $y \in \mathfrak{S}_n$ such that $l(y) < l(\omega)$. Then we have

$$\sum_{\substack{y\in\mathfrak{S}_n\\ l(y)\geqslant l(\omega)}} h_y\tau_y \in J.$$

We have $\tau_y\tau_{\omega^{-1}\omega_0[1,n]} = 0$ unless $y \leqslant \omega$. Hence, by multiplying the previous equation by $\tau_{\omega^{-1}\omega_0[1,n]}$ we obtain $h_\omega\tau_{\omega_0[1,n]} \in J$. Then, the same argument as in the case $\omega = 1$ implies that $h_\omega \in J$. This completes the induction and proves the first statement.

For the second statement, consider the isomorphism of H_n^0-modules

$$\begin{cases} A[x_1,\ldots,x_n] & \xrightarrow{\sim} & (A\otimes H_n^0)\,\tau_{\omega_0[1,n]}, \\ P & \mapsto & P\tau_{\omega_0[1,n]}. \end{cases} \tag{4.1.1}$$

It induces an injection $J \cap A[x_1,\ldots,x_n] \hookrightarrow J\tau_{\omega_0[1,n]}$. By the first statement, we also have $J\tau_{\omega_0[1,n]} = (J\cap A[x_1,\ldots,x_n])\tau_{\omega_0[1,n]}$. Hence, the isomorphism of H_n^0-modules (4.1.1) maps $J\cap A[x_1,\ldots,x_n]$ bijectively to $J\tau_{\omega_0[1,n]}$. Since $J\tau_{\omega_0[1,n]}$ is an H_n^0-submodule of $(A\otimes H_n^0)\tau_{\omega_0[1,n]}$, it follows that $J\cap A[x_1,\ldots,x_n]$ is an H_n^0-submodule of $A[x_1,\ldots,x_n]$. □

Corollary 4.1.7. 1. *Let $p \in Z(A)[x_1,\ldots,x_n]$, and let J be the two-sided ideal of $A\otimes H_n^0$ generated by p. Then we have*

$$J\cap A[x_1,\ldots,x_n] = (A\otimes H_n^0)\cdot p,$$

where $(A\otimes H_n^0)\cdot p$ denotes the $(A\otimes H_n^0)$-submodule of $A[x_1,\ldots,x_n]$ generated by p.

2. *Let M be a left (resp. right) $(A\otimes H_n^0)$-module, and let $p\in A[x_1,\ldots,x_n]$ be such that $pM = 0$ (resp. $Mp = 0$). Then $(H_n^0\cdot p)M = 0$ (resp. $M(H_n^0\cdot p) = 0$), where $H_n^0\cdot p$ denotes the H_n^0 submodule of $A[x_1,\ldots,x_n]$ generated by p.*

Proof. For the first statement, by Lemma 4.1.6 we have

$$\begin{aligned}(J \cap A[x_1, \ldots, x_n])\, \tau_{\omega_0[1,n]} &= J\tau_{\omega_0[1,n]} \\ &= (A \otimes H_n^0)p(A \otimes H_n^0)\tau_{\omega_0[1,n]}.\end{aligned}$$

We have $H_n^0\tau_{\omega_0[1,n]} = P_n\tau_{\omega_0[1,n]}$, and p is central in $A[x_1, \ldots, x_n]$ by assumption. Hence $(A \otimes H_n^0)p(A \otimes H_n^0)\tau_{\omega_0[1,n]} = (A \otimes H_n^0)p\tau_{\omega_0[1,n]}$. However for all $h \in A \otimes H_n^0$ we have $hp\tau_{\omega_{[}1,n]} = (h \cdot p)\tau_{\omega_0[1,n]}$. Hence we have proved that

$$(J \cap A[x_1, \ldots, x_n])\tau_{\omega_{[}1,n]} = ((A \otimes H_n^0) \cdot p)\tau_{\omega_{[}1,n]}$$

and the first statement follows.

For the second statement, let J be the annihilator of M in $A \otimes H_n^0$. Then J is a two-sided ideal of $A \otimes H_n^0$. Hence $J \cap A[x_1, \ldots, x_n]$ is an H_n^0-submodule of $A[x_1, \ldots, x_n]$ by Lemma 4.1.6. The result follows. □

We now return to our problem of constructing elements of P_β which act by zero on H_β^i. Recall that for $\nu \in I^n$, we have defined in equation (2.5.1) a polynomial

$$Q_{i,\nu}(u, v_1, \ldots, v_n) = \prod_{\substack{1 \leqslant k \leqslant n \\ \nu_k \neq i}} Q_{i,\nu_k}(u, v_k).$$

Proposition 4.1.8. *Let $n \geqslant 1$, let $\nu \in I^n$ and let $a \in \{1, \ldots, n\}$ be such that $\nu_a = i$. Then $Q_{i,\nu}(x_a, x_1, \ldots, x_n)1_\nu$ acts by zero on H_n^i.*

Proof. The proof is by induction on n. Assume that $n = 1$. If $\nu_1 = i$, 1_i is zero in H_1^i, so the result holds. If $\nu_1 \neq i$, there is no a such that $\nu_a = i$, and the result holds too.

Assume that the result is proved for $n \geqslant 1$. Let $\nu \in I^{n+1}$. We write ν in the form $\nu = (\nu_{n+1}, \nu')$ with $\nu' \in I^n$. We consider two cases depending on whether $\nu_{n+1} \neq i$ or

$\nu_{n+1} = i$.

Case 1: if $\nu_{n+1} \neq i$, then we have $a < n+1$ and

$$\begin{aligned} Q_{i,\nu}(x_a, x_1, \ldots, x_{n+1})1_\nu &= Q_{i,\nu_{n+1}}(x_a, x_{n+1}) Q_{i,\nu'}(x_a, x_1, \ldots, x_n)1_\nu \\ &= Q_{i,\nu_{n+1}}(x_a, x_{n+1}) r_n^{n+1}(Q_{i,\nu'}(x_a, x_1, \ldots, x_n)1_{\nu'})1_\nu. \end{aligned}$$

By induction, $Q_{i,\nu'}(x_a, x_1, \ldots, x_n)1_{\nu'}$ is zero in H_n^i. Hence $Q_{i,\nu}(x_a, x_1, \ldots, x_{n+1})1_\nu$ is zero in H_{n+1}^i.

Case 2: If $\nu_{n+1} = i$, then we consider three subcases: (i) $a < n+1$, (ii) $a = n+1$ and $\nu_n = i$, and (iii) $a = n+1$ and $\nu_n \neq i$.

- Sub-case (i): if $a < n+1$, then we have

$$\begin{aligned} Q_{i,\nu}(x_a, x_1, \ldots, x_{n+1})1_\nu &= Q_{i,\nu'}(x_a, x_1, \ldots, x_n)1_\nu \\ &= r_n^{n+1}(Q_{i,\nu'}(x_a, x_1, \ldots, x_n)1_{\nu'})1_\nu. \end{aligned}$$

  By induction, $Q_{i,\nu'}(x_a, x_1, \ldots, x_n)1_{\nu'}$ is zero in H_n^i so $Q_{i,\nu}(x_a, x_1, \ldots, x_{n+1})1_\nu$ is zero in H_{n+1}^i.

- Sub-case (ii): if $a = n+1$ and $\nu_n = i$, then we have

$$\begin{aligned} Q_{i,\nu}(x_{n+1}, x_1, \ldots, x_{n+1})1_\nu &= Q_{i,\nu'}(x_{n+1}, x_1, \ldots, x_{n-1}, x_{n+1})1_\nu \\ &= s_n(Q_{i,\nu'}(x_n, x_1, \ldots, x_n))1_\nu. \end{aligned}$$

  By induction $Q_{i,\nu'}(x_n, x_1, \ldots, x_n)1_\nu = r_n^{n+1}(Q_{i,\nu'}(x_n, x_1, \ldots, x_n)1_{\nu'})1_\nu$ is zero in H_{n+1}^i. Since $\nu_{n+1} = \nu_n = i$, there is a natural structure of left $(H_2^0 \otimes K[x_1, \ldots, x_{n-1}])$-module on $1_\nu H_{n+1}^i$, for which the polynomial $Q_{i,\nu'}(x_n, x_1, \ldots, x_n)$ acts by zero. By Corollary 4.1.7, $s_n(Q_{i,\nu'}(x_n, x_1, \ldots, x_n))$ also acts by zero, thus $Q_{i,\nu}(x_{n+1}, x_1, \ldots, x_{n+1})1_\nu$ is zero in H^i_{n+1}.

- Sub-case (iii): if $a = n+1$ and $\nu_n \neq i$, let $\nu'' = (i, \nu_{n-1}, \dots, \nu_1)$. Then we have

$$\begin{aligned}
Q_{i,\nu}(x_{n+1}, x_1, \dots, x_{n+1})1_\nu &= Q_{i,\nu_n}(x_{n+1}, x_n) Q_{i,\nu''}(x_{n+1}, x_1, \dots, x_{n-1}, x_{n+1})1_\nu \\
&= \tau_n^2 Q_{i,\nu''}(x_{n+1}, x_1, \dots, x_{n-1}, x_{n+1})1_\nu \\
&= \tau_n Q_{i,\nu''}(x_n, x_1, \dots, x_{n-1}, x_n)\tau_n 1_\nu \\
&= \tau_n r_n^{n+1}(Q_{i,\nu''}(x_n, x_1, \dots, x_n)1_{\nu''})\tau_n 1_\nu.
\end{aligned}$$

By induction, $Q_{i,\nu''}(x_n, x_1, \dots, x_n)1_{\nu''}$ is zero in H^i_n, hence $Q_{i,\nu}(x_{n+1}, x_1, \dots, x_{n+1})1_\nu$ is zero in H^i_{n+1}.

□

The polynomials $Q_{i,\nu}(u, v_1, \dots, v_n)$ have important symmetry properties. For $\beta \in Q^+$ of height n, let

$$Q_{i,\beta}(u, x_1, \dots, x_n) = \sum_{\nu \in I^\beta} Q_{i,\nu}(u, x_1, \dots, x_n)1_\nu \in P_\beta[u].$$

Lemma 4.1.9. *Let $\beta \in Q^+$ of height n. As an element of $P_\beta[u]$, the polynomial $Q_{i,\beta}$ has coefficients in $P_\beta^{\mathfrak{S}_n}$.*

Proof. Let $\omega \in \mathfrak{S}_n$. We have

$$\begin{aligned}
\omega(Q_{i,\beta}(u, x_1, \dots, x_n)) &= \sum_{\nu \in I^\beta} \Big(\prod_{\substack{1 \leqslant k \leqslant n \\ \nu_k \neq i}} \omega(Q_{i,\nu_k}(u, x_k)) \Big) \omega(1_\nu) \\
&= \sum_{\nu \in I^\beta} \Big(\prod_{\substack{1 \leqslant k \leqslant n \\ \nu_k \neq i}} Q_{i,\nu_k}(u, x_{\omega(k)}) \Big) 1_{\omega(\nu)}.
\end{aligned}$$

Doing the re-indexation $\nu' = \omega(\nu)$ in this last sum yields

$$\begin{aligned}\omega(Q_{i,\beta}(u, x_1, \ldots, x_n)) &= \sum_{\nu' \in I^\beta} \Bigg(\prod_{\substack{1 \leqslant k \leqslant n \\ \nu'_{\omega(k)} \neq i}} Q_{i,\nu'_{\omega(k)}}(u, x_{\omega(k)}) \Bigg) 1_{\nu'} \\ &= \sum_{\nu' \in I^\beta} \Bigg(\prod_{\substack{1 \leqslant \ell \leqslant n \\ \nu'_\ell \neq i}} Q_{i,\nu'_\ell}(u, x_\ell) \Bigg) 1_{\nu'},\end{aligned}$$

where the second equality follows from doing the re-indexation $\ell = \omega(k)$ in the product. Hence $Q_{i,\beta}(u, x_1, \ldots, x_n)$ is invariant under the action of $\mathfrak{S}_n$ and the result is proved. □

4.1.2 The Endofunctor ad_{E_i}

Definition

Consider the functor $\mathrm{ad}_{E_i} : \mathcal{H}^i \to \mathcal{H}^i$ defined by:

$$\mathrm{ad}_{E_i}(M) = E_i M / (1_{\beta,i} E_i M) = \pi_i(E_i M).$$

for $M \in H^i_\beta$−Mod. So ad_{E_i} is the composition of the left i-induction functor with the functor π_i. Left i-induction is an exact endofunctor of $\mathcal{H}$, and π_i is right exact since it is a left adjoint. Hence, the functor ad_{E_i} is right exact. We will prove below that it is actually exact.

Proposition 4.1.10. *1. For $n \geqslant 0$, there is a natural isomorphism in $M \in \mathcal{H}^i$*

$$\mathrm{ad}^n_{E_i}(M) \simeq \pi_i\left(E_i^n M\right).$$

2. For all $n \geqslant 0$, there is a morphism of graded algebras

$$H^{\mathrm{op}}_{ni} \to \mathrm{End}^\bullet_{\mathcal{H}^i}\left(\mathrm{ad}^n_{E_i}\right).$$

Hence, the affine nil Hecke algebra H_{ni} acts on $\mathrm{ad}^n_{E_i}$.

Proof. We construct the isomorphism in (1) by induction on n. For the case $n = 0$, we have $\pi_i(M) = M$ for $M \in \mathcal{H}[i]$, and the canonical isomorphism is just the identity. Assume that we have constructed the natural isomorphism $\mathrm{ad}^n_{E_i}(M) \xrightarrow{\sim} \pi_i(E_i^n M)$ for some $n \geqslant 0$. Then we have an isomorphism

$$\mathrm{ad}^{n+1}_{E_i}(M) \xrightarrow{\sim} \mathrm{ad}_{E_i}(\pi_i(E_i^n M)) = \pi_i(E_i \pi_i(E_i^n M)).$$

By Lemma 4.1.5, there is a canonical isomorphism $\pi_i(E_i\pi_i(E_i^n M)) \xrightarrow{\sim} \pi_i(E_i^{n+1} M)$, which completes the construction of (1).

For (2), notice that we have an action of H_{ni}^{op} on E_i^n by right multiplication, so we get an action of H_{ni}^{op} on the functor $M \mapsto E_i^n M$. By horizontal composition, we get an action of H_{ni}^{op} on the functor $M \mapsto \pi_i(E_i^n M)$, which is canonically isomorphic to $\mathrm{ad}^n_{E_i}$ by (1), whence the result. $\square$

For the rest of this dissertation, we fix the action of H_{ni}^{op} on $\mathrm{ad}^n_{E_i}$ to be the one constructed in the proof of Proposition 4.1.10 via the canonical isomorphism $\mathrm{ad}^n_{E_i}(M) \simeq \pi_i(E_i^n M)$. In particular, one can define the *divided power* $\mathrm{ad}^{(n)}_{E_i}$. Recall that the element $e_n = x_2 \dots x_n^{n-1}\tau_{\omega_0[1,n]}$ is a primitive idempotent of H_{ni}. Then the functor $\mathrm{ad}^{(n)}_{E_i}$ is defined by

$$\mathrm{ad}^{(n)}_{E_i} = q_i^{-\frac{n(n-1)}{2}} e_n\big(\mathrm{ad}^n_{E_i}\big).$$

By construction, for all $M \in \mathcal{H}^i$ there is an isomorphism

$$\mathrm{ad}^{(n)}_{E_i}(M) \simeq \pi_i\big(E_i^{(n)} M\big)$$

which is natural in M.

Induction

Let us now give another description of the functor ad_{E_i} as an induction functor. For $\beta \in Q^+$, the right inclusion induces a (possibly non-unital) morphism $H^i_\beta \to H^i_{\beta+i}$. This endows $H^i_{\beta+i}1_{i,\beta}$ with a structure of $(H^i_{\beta+i}, H^i_\beta)$-bimodule, and we get an induction functor

$$\mathrm{ind}_{H^i_\beta}^{H^i_{\beta+i}} : \begin{cases} H^i_\beta\text{−mod} & \to & H^i_{\beta+i}\text{−mod}, \\ M & \mapsto & H^i_{\beta+i}1_{i,\beta} \otimes_{H^i_\beta} M. \end{cases}$$

Proposition 4.1.11. *There is an isomorphism*

$$\mathrm{ad}_{E_i} \simeq \bigoplus_{\beta \in Q^+} \mathrm{ind}_{H^i_\beta}^{H^i_{\beta+i}}.$$

Proof. Let $\beta \in Q^+$ and let $N \in H_{\beta+i}$−mod. We have isomorphisms

$$\begin{aligned} \mathrm{ad}_{E_i}(M) &= \pi_i(E_i M) \\ &\simeq H^i_{\beta+i} \otimes_{H_{\beta+i}} (E_i M) \\ &\simeq H^i_{\beta+i} \otimes_{H_{\beta+i}} H_{\beta+i}1_{i,\beta} \otimes_{H_\beta} M \\ &\simeq H^i_{\beta+i}1_{i,\beta} \otimes_{H_\beta} M. \end{aligned}$$

Since the left (resp. right) action of H_β on M (resp. $H^i_{\beta+i}1_{i,\beta}$) factors through H^i_β, we have

$$H^i_{\beta+i}1_{i,\beta} \otimes_{H_\beta} M \simeq H^i_{\beta+i}1_{i,\beta} \otimes_{H^i_\beta} M,$$

which concludes the proof. $\square$

The Morphism $\tau_{E_i,M}$

We now describe of the functor ad_{E_i} as the cokernel of a natural transformation. For $\beta \in Q^+$ of height r and $M \in H^i_\beta - \mathrm{mod}$, we define a morphism $\tau_{E_i,M} : ME_i \to E_iM$ as follows:

$$\tau_{E_i,M} : \begin{cases} ME_i & \to & E_iM, \\ h1_{\beta,i} \otimes_{H_\beta} m & \mapsto & h\tau_{[1\uparrow r]}1_{i,\beta} \otimes_{H_\beta} m. \end{cases}$$

This morphism $\tau_{E_i,M}$ is an analogue of the morphism P in [KK12, Theorem 4.7]. Similar morphisms also appear in [BKM14].

Proposition 4.1.12. *The map $\tau_{E_i,M}$ is a well-defined morphism of $H_{\beta+i}$-modules of degree $-i \cdot \beta$.*

Proof. By adjunction, it suffices to check that the map

$$\begin{cases} M & \to & 1_{\beta,i}H_{\beta+i}1_{i,\beta} \otimes_{H_\beta} M, \\ m & \mapsto & \tau_{[1\uparrow r]}1_{i,\beta} \otimes_{H_\beta} m, \end{cases}$$

is a morphism of H_β-modules. To check this, we need to prove that for all $y \in H_\beta$ we have

$$\big((y \diamond 1_i)\tau_{[1\uparrow r]} - \tau_{[1\uparrow r]}(1_i \diamond y)\big)\, 1_{i,\beta} \otimes_{H_\beta} M = 0.$$

Let $Z = \big((y \diamond 1_i)\tau_{[1\uparrow r]} - \tau_{[1\uparrow r]}(1_i \diamond y)\big)\, 1_{i,\beta}$. By Lemma 2.4.2, the element Z can be written in the form

$$Z = \sum_{k=2}^{r+1} \tau_{[k\uparrow r]} z_k$$

for some $z_k \in H_{i,\beta}$. However, $Z = 1_{\beta,i} Z 1_{i,\beta}$ and $1_{\beta,i}\tau_{[k\uparrow r]} = \tau_{[k\uparrow r]} 1_{\beta,i}$ when $k > 1$. So

$$\begin{aligned} Z &= 1_{\beta,i} Z 1_{i,\beta} \\ &= \sum_{k=2}^{n+1} \tau_{[k\uparrow r]} 1_{\beta,i} 1_{i,\beta} z_k \\ &= \sum_{k=2}^{n+1} \tau_{[k\uparrow r]} 1_{i,\beta-i,i} z_k. \end{aligned}$$

Since $1_{\beta-i,i}M = 0$, we deduce that $Z \otimes_{H_\beta} M = 0$. Hence $\tau_{E_i,M}$ is well-defined. The statement about the degree follows from the definition of the grading on quiver Hecke algebras. □

It is clear that the morphism $\tau_{E_i,M}$ is natural in M. Hence, $\tau_{E_i,(-)} : (-)E_i \to E_i(-)$ is a natural transformation of functors $\mathcal{H}^i \to \mathcal{H}$. We now relate $\tau_{E_i,(-)}$ and ad_{E_i}.

Proposition 4.1.13. *For $\beta \in Q^+$ and $M \in H^i_\beta{-}\mathrm{mod}$, we have $\mathrm{ad}_{E_i}(M) = \mathrm{coker}(\tau_{E_i,M})$. Hence, we have an exact sequence which is natural in M*

$$q_i^{-\langle i^\vee,\beta\rangle} M E_i \xrightarrow{\tau_{E_i,M}} E_i M \to \mathrm{ad}_{E_i}(M) \to 0.$$

Proof. Denote by r the height of β. We show that $\mathrm{im}(\tau_{E_i,M}) = H_{\beta+i} 1_{\beta,i}(E_i M)$. Let $h \in H_{\beta+i}$ and $m \in M$. We have

$$\tau_{E_i,M}(h 1_{\beta,i} \otimes_{H_\beta} m) = h\tau_{[1\uparrow r]} 1_{i,\beta} \otimes_{H_\beta} m \in H_{\beta+i} 1_{\beta,i}(E_i M).$$

So $\mathrm{im}(\tau_{E_i,M}) \subseteq H_{\beta+i} 1_{i,\beta}(E_i M)$. Conversely, by (2.4.1) we have

$$E_i M = \bigoplus_{k=1}^{r+1} \tau_{[k\uparrow r]} 1_{i,\beta} \otimes_{H_{i,\beta}} (E_i \otimes M).$$

Note that

$$1_{\beta,i}\tau_{[k\uparrow r]}1_{i,\beta} = \begin{cases} \tau_{[k\uparrow r]}1_{i,\beta-i,i} & \text{if } k>1, \\ \tau_{[1\uparrow r]}1_{\beta,i} & \text{if } k=1. \end{cases}$$

Since $1_{\beta-i,i}M = 0$, we deduce that

$$1_{\beta,i}(ME_i) = \tau_{[1\uparrow r]}1_{i,\beta} \otimes_{H_{i,\beta}} (E_i \otimes M) \subseteq \operatorname{im}(\tau_{E_i,M}).$$

Hence $H_{\beta+i}1_{\beta,i}(E_iM) \subseteq \operatorname{im}(\tau_{E_i,M})$. So we have proved that $\operatorname{im}(\tau_{E_i,M}) = H_{\beta+i}1_{\beta,i}(E_iM)$, from which we deduce that $\operatorname{ad}_{E_i}(M) = \operatorname{coker}(\tau_{E_i,M})$. □

A key property of $\tau_{E_i,(-)}$ is that it is compatible with the monoidal structure of $\mathcal{H}^i$, in the following sense.

Proposition 4.1.14. *For all $M, N \in \mathcal{H}[i]$, we have $\tau_{E_i,MN} = \tau_{E_i,M}N \circ M\tau_{E_i,N}$.*

Proof. Assume that $M \in H^i_\beta-$mod with β of height m, and that $N \in H^i_\gamma-$mod with γ of height n. Let

$$y = h1_{\beta,\gamma} \otimes_{H_{\beta,\gamma}} (a \otimes b) \in MN,$$

where $h \in H_{\beta+\gamma}$, $a \in M$ and $b \in N$. Let

$$z = 1_{\beta+\gamma,i} \otimes_{H_{\beta+\gamma}} y \in MNE_i.$$

Such elements z generate MNE_i as an $H_{\beta+\gamma+i}$-module. Hence, it suffices to prove that $\tau_{E_i,MN}(z) = \tau_{E_i,M}N(M\tau_{E_i,N}(z))$. On the one hand, we have

$$\begin{aligned} \tau_{E_i,MN}(z) &= \tau_{[1\uparrow m+n]}1_{i,\beta+\gamma} \otimes_{H_{\beta+\gamma}} y \\ &= \tau_{[1\uparrow m+n]}(1_i \diamond h)1_{i,\beta,\gamma} \otimes_{H_{\beta,\gamma}} (a \otimes b). \end{aligned}$$

On the other hand, we have

$$M\tau_{E_i,N}(z) = M\tau_{E_i,N}\left((h \diamond 1_i)1_{\beta,\gamma,i} \otimes_{H_{\beta,\gamma}} (a \otimes b)\right)$$
$$= (h \diamond 1_i)\tau_{[1\uparrow n]}1_{\beta,i,\gamma} \otimes_{H_{\beta,\gamma}} (a \otimes b).$$

So

$$\tau_{E_i,M}N(M\tau_{E_i,N}(z)) = (h \diamond 1_i)\tau_{[1\uparrow s]}\tau_{[n+1\uparrow m+n]}1_{i,\beta,\gamma} \otimes_{H_{\beta,\gamma}} (a \otimes b)$$
$$= (h \diamond 1_i)\tau_{[1\uparrow m+n]}1_{i,\beta,\gamma} \otimes_{H_{\beta,\gamma}} (a \otimes b).$$

Hence, it suffices to prove that

$$\left((h \diamond 1_i)\tau_{[1\uparrow m+n]} - \tau_{[1\uparrow m+n]}(1_i \diamond h)\right) 1_{i,\beta,\gamma} \otimes_{H_{\beta,\gamma}} (M \otimes N) = 0.$$

Let $Z = \left((h \diamond 1_i)\tau_{[1\uparrow m+n]} - \tau_{[1\uparrow m+n]}(1_i \diamond h)\right) 1_{i,\beta,\gamma}$. By Lemma 2.4.2, we have

$$Z \in \sum_{k=2}^{m+n+1} \tau_{[k\uparrow m+n]}H_{i,\beta+\gamma}1_{i,\beta,\gamma}.$$

Let S be a complete set of reduced expressions for minimal length representatives of left cosets of $\mathfrak{S}_m \times \mathfrak{S}_n$ in $\mathfrak{S}_{m+n}$. By decomposition (2.4.1) we have

$$H_{\beta+\gamma}1_{\beta,\gamma} = \bigoplus_{\underline{\omega}\in S} \tau_{\underline{\omega}}H_{\beta,\gamma}.$$

Hence Z can be written in the form

$$Z = \sum_{\substack{1<k\leqslant m+n+1\\ \underline{\omega}\in S}} \tau_{[k\uparrow m+n]}(1_i \diamond \tau_{\underline{\omega}})h_{k,\underline{\omega}}$$

for some $h_{k,\underline{\omega}} \in H_{i,\beta,\gamma}$. However $Z = 1_{\beta+\gamma,i}Z$, so

$$\begin{aligned}
Z &= 1_{\beta+\gamma,i}Z \\
&= \sum_{\substack{1<k\leqslant m+n+1\\ \underline{\omega}\in S}} 1_{\beta+\gamma,i}\tau_{[k\uparrow m+n]}(1_i\diamond\tau_{\underline{\omega}})h_{k,\underline{\omega}} \\
&= \sum_{\substack{1<k\leqslant m+n+1\\ \underline{\omega}\in S}} \tau_{[k\uparrow m+n]}1_{\beta+\gamma,i}(1_i\diamond\tau_{\underline{\omega}})h_{k,\underline{\omega}}.
\end{aligned}$$

For $\underline{\omega} \in S$, we have $s_{\underline{\omega}}^{-1}(1) \in \{1, n+1\}$. It follows that

$$Z = \sum_{\substack{1<k\leqslant m+n+1\\ \underline{\omega}\in S,\, s_{\underline{\omega}}^{-1}(1)=1}} \tau_{[k\uparrow m+n]}(1_i\diamond\tau_{\underline{\omega}})h_{k,\underline{\omega}}1_{i,\beta,\gamma-i,i} + \sum_{\substack{1<k\leqslant m+n+1\\ \underline{\omega}\in S,\, s_{\underline{\omega}}^{-1}(1)=n+1}} \tau_{[k\uparrow m+n]}(1_i\diamond\tau_{\underline{\omega}})h_{k,\underline{\omega}}1_{i,\beta-i,i,\gamma}.$$

Since both $1_{\beta-i,i,\gamma}$ and $1_{\beta,\gamma-i,i}$ act by zero on $M\otimes N$, we deduce $Z\otimes_{H_{\beta,\gamma}}(M\otimes N) = 0$, which completes the proof. □

Adjoint Action on Chevalley Generators

In this example, we describe the modules $\mathrm{ad}_{E_i}^{(n)}(E_j)$ for $n \geqslant 0$ and $j \in I\setminus\{i\}$.

Proposition 4.1.15. *As left graded P_{n+1}-modules, we have*

$$H_{j+ni}1_{j+(n-1)i,i}(E_i^nE_j) = \left(\bigoplus_{\omega\in\mathfrak{S}_n} J_{n+1}(\tau_\omega\diamond 1_j)1_{ni,j}\right) \oplus \left(\bigoplus_{\substack{\omega\in\mathfrak{S}_n\\ 1\leqslant k\leqslant n}} P_{n+1}\tau_{[k\downarrow 1]}(\tau_\omega\diamond 1_j)1_{ni,j}\right),$$

where J_{n+1} is the ideal of P_{n+1} generated by the $\partial_{[k\downarrow 2]}(Q_{i,j}(x_2,x_1))$ for $k\in\{1,\ldots,n\}$.

Proof. Call L the left hand-side of the equality, and L' the right-hand side. Let us start by proving that L' is an H_{j+ni}-submodule of $H_{j+ni}1_{ni,j}$. It is clear that L' is stable by the left action of P_{n+1}. Furthermore, note that J_{n+1} is the $(H_n^0\otimes K[x_1])$-submodule of P_{n+1} generated by $Q_{i,j}(x_2,x_1)$. Using this and the fact that $\tau_1^2 1_{ni,j} = Q_{i,j}(x_2,x_1)$, we see easily that L' is also stable by left multiplication by $\tau_1, \ldots, \tau_{n-1}$. Hence L' is an H_{j+ni}-submodule.

By the same argument as in the proof of Proposition 4.1.13, L is the H_{j+ni}-submodule of $H_{j+ni}1_{ni,j}$ generated by the $\tau_{[1\uparrow k]}1_{ni,j}$ for $k \in \{1, \ldots, n\}$. Since $\tau_{[1\uparrow k]}1_{ni,j} \in L'$ for $k \in \{1, \ldots, n\}$, we have $L \subseteq L'$.

Conversely, it is clear that

$$\bigoplus_{\substack{\omega \in \mathfrak{S}_n \\ 1 \leqslant k \leqslant n}} P_{n+1}\tau_{[k\downarrow 1]}(\tau_\omega \diamond 1_j)1_{ni,j} \subseteq L.$$

Furthermore, $Q_{i,j}(x_2, x_1)1_{ni,j} = \tau_1^2 1_{ni,j} \in L$. By Corollary 4.1.7, we deduce that $J_{n+1}1_{ni,j} \subseteq L$, and it follows that $L' \subseteq L$. $\square$

Corollary 4.1.16. *As left graded P_{n+1}-modules, we have*

$$\begin{aligned}
\mathrm{ad}^n_{E_i}(E_j) &= \bigoplus_{\omega \in \mathfrak{S}_n} (P_{n+1}/J_{n+1})\,(\tau_\omega \diamond 1_j)1_{ni,j}, \\
\mathrm{ad}^{(n)}_{E_i}(E_j) &= q_i^{-\frac{n(n-1)}{2}}\,(P_{n+1}/J_{n+1})\,(\tau_{\omega_0[1,n]} \diamond 1_j)1_{ni,j}.
\end{aligned}$$

In particular, we have a simple formula for the graded dimensions of $\mathrm{ad}^n_{E_i}(E_j)$ and $\mathrm{ad}^{(n)}_{E_i}(E_j)$. They are given by

$$\begin{aligned}
\mathrm{grdim}(\mathrm{ad}^n_{E_i}(E_j)) &= \frac{1}{(1-q_i^2)^n(1-q_j^2)}\left(\prod_{k=0}^{n-1}\left(1-q_i^{2(-c_{i,j}-k)}\right)\right)\left(\sum_{\omega \in S_n} q_i^{-2l(\omega)}\right), \\
\mathrm{grdim}(\mathrm{ad}^{(n)}_{E_i}(E_j)) &= \frac{q_i^{\frac{n(n-1)}{2}}}{(1-q_i^2)^n(1-q_j^2)}\left(\prod_{k=0}^{n-1}\left(1-q_i^{2(-c_{i,j}-k)}\right)\right).
\end{aligned}$$

A consequence of these formulas is the following vanishing criterion:

$$\mathrm{ad}^n_{E_i}(E_j) = 0 \Leftrightarrow n > -c_{i,j}. \tag{4.1.2}$$

4.2 Short Exact Sequences Associated to the Adjoint Action

The goal of this section is to prove the following theorem.

Theorem 4.2.1. *For all $\beta \in Q^+$ and $M \in H^i_\beta\text{-mod}$, the morphism $\tau_{E_i,M}$ is injective. Hence, we have a short exact sequence which is natural in M*

$$0 \to q_i^{-\langle i^\vee,\beta\rangle} ME_i \xrightarrow{\tau_{E_i,M}} E_iM \to \mathrm{ad}_{E_i}(M) \to 0.$$

In particular, this theorem gives a categorification of the relation $\mathrm{ad}_{e_i}(y) = e_iy - q_i^{-\langle i^\vee,\beta\rangle} ye_i$ for $y \in U^+$ of degree $\beta \in Q^+$. Before proving Theorem 4.2.1, we give some corollaries.

4.2.1 Exactness of ad_{E_i}

Theorem 4.2.1 is the analogue of [KK12, Theorem 4.7] for cyclotomic quiver Hecke algebras. It can be used formally in the same way to prove that the functor ad_{E_i} is exact.

Corollary 4.2.2. *Let $\beta \in Q^+$. The right H^i_β-module $H^i_{\beta+i}1_{i,\beta}$ is projective.*

Proof. As in [KK12], we use the following lemma.

Lemma ([KK12, Lemma 4.18])**.** *Let A be a K-algebra and let N be an $A[x]$-module. Assume that the following properties hold.*

1. *The projective dimension of N as an $A[x]$-module is at most 1.*

2. *There exists $p(x) \in Z(A)[x]$ with invertible leading coefficient such that $p(x)N = 0$.*

Then N is projective as an A-module.

We use this lemma with $A = H^i_\beta$ and $N = H^i_{\beta+i}1_{i,\beta}$. By Theorem 4.2.1 applied to $M = H^i_\beta$, we have a short exact sequence of $(H_{\beta+i}, H_i \otimes H^i_\beta)$-bimodules:

$$0 \to H_{\beta+i}1_{\beta,i} \otimes_{H_\beta} H^i_\beta \to H_{\beta+i}1_{i,\beta} \otimes_{H_\beta} H^i_\beta \to H^i_{\beta+i}1_{i,\beta} \to 0.$$

Note that the compatibility with the right $\left(H_i \otimes H^i_\beta\right)$-module structure comes from the naturality. Since $H_i \otimes H^i_\beta \simeq H^i_\beta[x]$, we can view this sequence as an exact sequence of right $H^i_\beta[x]$-modules.

Since $H_{\beta+i}1_{\beta,i}$ is free as a right $H_{\beta,i}$-module, $H_{\beta+\alpha_i}1_{\beta,i} \otimes_{H_\beta} H^i_\beta$ is free as a right $H^i_\beta[x]$-module. Similarly, $H_{\beta+i}1_{i,\beta} \otimes_{H_\beta} H^i_\beta$ is free as a right $H^i_\beta[x]$-module. Hence, $H^i_{\beta+i}1_{i,\beta}$ has projective dimension at most 1 as a right $H^i_\beta[x]$-module. Let

$$p(x_{n+1}) = Q_{i,\beta}(x_{n+1}, x_1, \ldots, x_n)1_{i,\beta} = \sum_{\nu \in I^\beta} \left(\prod_{\substack{1 \leqslant k \leqslant n \\ \nu_k \neq i}} Q_{i,\nu_k}(x_{n+1}, x_k) \right) 1_{i,\nu} \in P_{\beta+i}.$$

When considered as an element in $H_\beta[x_{n+1}]$, $p(x_{n+1})$ has coefficients in $Z(H_\beta)$ by Proposition 4.1.9. Furthermore, $H^i_{\beta+i}p(x_{n+1}) = 0$ by Proposition 4.1.8. Hence the lemma applies and the result follows. □

Corollary 4.2.3. *The functor* ad_{E_i} *is exact and sends projective modules to projective modules.*

Proof. This follows immediately from Proposition 4.1.11 and Corollary 4.2.2. □

4.2.2 Compatibility with the Monoidal Structure

In general, the adjoint action of a Hopf algebra A on itself is compatible with the multiplication, in the following sense: for all $y, y', z \in A$ we have

$$\mathrm{ad}_z(yy') = \sum \mathrm{ad}_{z_{(1)}}(y)\mathrm{ad}_{z_{(2)}}(y').$$

In the case of the quantum group, the compatibility with the product takes the form of the following "q-Leibniz rule"

$$\begin{aligned} \mathrm{ad}_{e_i}(yz) &= \mathrm{ad}_{e_i}(y)z + q_i^{-\langle i^\vee,\beta\rangle} y\mathrm{ad}_{e_i}(z), \\ \mathrm{ad}_{f_i}(yz) &= q_i^{-\langle i^\vee,\gamma\rangle}\mathrm{ad}_{f_i}(y)z + y\mathrm{ad}_{e_i}(z) \end{aligned} \tag{4.2.1}$$

for all $i \in I$ and $y, z \in U$ homogeneous of respective degrees $\beta, \gamma \in Q^+$. Hence, ad_{e_i} and ad_{f_i} can be thought of as "q-derivations" of the algebra U. We prove that this property categorifies in the form of short exact sequences.

Corollary 4.2.4. *Let $\beta, \gamma \in Q^+$ and let $M \in H^i_\beta-\mathrm{mod}$ and $N \in H^i_\gamma-\mathrm{mod}$. There is a short exact sequence which is natural in M, N*

$$0 \to q_i^{-\langle i^\vee,\beta\rangle} M\ \mathrm{ad}_{E_i}(N) \to \mathrm{ad}_{E_i}(MN) \to \mathrm{ad}_{E_i}(M)N \to 0.$$

Proof. This a consequence of the following elementary statement: given morphisms $f = f_2 \circ f_1$ in an abelian category, we have an exact sequence

$$\mathrm{coker}(f_1) \to \mathrm{coker}(f) \to \mathrm{coker}(f_2) \to 0.$$

The first morphism is induced by f_2, and the second morphism is the quotient map. Furthermore, if f_2 is injective, then the morphism $\mathrm{coker}(f_1) \to \mathrm{coker}(f)$ is injective as well. By Proposition 4.1.14 and Theorem 4.2.1, the statement applies to $f = \tau_{E_i,MN}$, $f_1 = M\tau_{E_i,N}$ and $f_2 = \tau_{E_i,M}N$, yielding the result. □

We can think of this result as expressing the fact that ad_{E_i} is a "categorical derivation". This feature does not appear for cyclotomic quiver Hecke algebras, as there is no monoidal structure in that case.

More generally, Corollary 4.2.4 can be applied repeatedly to construct an interesting filtration of $\mathrm{ad}_{E_i}^n(MN)$. To describe it, let us set up some notation. Given an integer $k \in \mathbb{N}$ and its binary expansion

$$k = \sum_{\ell \geqslant 0} k_\ell 2^\ell$$

where $k_\ell \in \{0, 1\}$ are almost all 0, we define its *binary weight* $\zeta(k)$ as the number of k_ℓ that are equal to 1. There are inductive relations for the binary weight function: for all $k \geqslant 0$ we have

$$\begin{cases} \zeta(2k) = \zeta(k), \\ \zeta(2k+1) = \zeta(k) + 1. \end{cases} \tag{4.2.2}$$

We also define a function σ by

$$\sigma(k) = \sum_{\ell \geqslant 0} \ell k_\ell - \frac{\zeta(k)(\zeta(k) - 1)}{2}.$$

There are simple recursive relations for the function σ: for all $k \geqslant 0$ we have

$$\begin{cases} \sigma(2k) = \sigma(k) + \zeta(k), \\ \sigma(2k+1) = \sigma(k). \end{cases} \tag{4.2.3}$$

With these notations, we can now state and prove the result.

Proposition 4.2.5. *Let $M \in H_\beta^i-\mathrm{mod}$, $N \in H_\gamma^i-\mathrm{mod}$ with $\beta, \gamma \in Q^+$, and let $n \in \mathbb{N}$. Then, there is a filtration of* $\mathrm{ad}_{E_i}^n(MN)$

$$0 = V_{-1} \subseteq V_0 \subseteq \cdots \subseteq V_{2^n-1} = \mathrm{ad}_{E_i}^n(MN),$$

with quotients given by

$$V_k/V_{k-1} \simeq q_i^{(\zeta(k)-n)\langle i^\vee, \beta\rangle - 2\sigma(k)} \mathrm{ad}_{E_i}^{\zeta(k)}(M) \mathrm{ad}_{E_i}^{n-\zeta(k)}(N).$$

Proof. We proceed inductively on n. The case $n = 0$ is clear. Assume that for some n we have a filtration of $\mathrm{ad}^n_{E_i}(MN)$

$$0 = V_{-1} \subseteq V_0 \subseteq \cdots \subseteq V_{2^n-1} = \mathrm{ad}^n_{E_i}(MN),$$

with quotients given by

$$V_k/V_{k-1} \simeq q_i^{(\zeta(k)-n)\langle i^\vee,\beta\rangle - 2\sigma(k)} \mathrm{ad}^{\zeta(k)}_{E_i}(M)\mathrm{ad}^{n-\zeta(k)}_{E_i}(N).$$

Since ad_{E_i} is exact by Corollary 4.2.3, we can apply ad_{E_i} to the filtration above to get a filtration of $\mathrm{ad}^{n+1}_{E_i}(MN)$

$$0 = \mathrm{ad}_{E_i}(V_{-1}) \subseteq \mathrm{ad}_{E_i}(V_0) \subseteq \cdots \subseteq \mathrm{ad}_{E_i}(V_{2^n-1}) = \mathrm{ad}^{n+1}_{E_i}(MN),$$

with quotients given by

$$\mathrm{ad}_{E_i}(V_k)/\mathrm{ad}_{E_i}(V_{k-1}) \underset{p_k}{\xrightarrow{\sim}} q_i^{(\zeta(k)-n)\langle i^\vee,\beta\rangle - 2\sigma(k)} \mathrm{ad}_{E_i}\left(\mathrm{ad}^{\zeta(k)}_{E_i}(M)\mathrm{ad}^{n-\zeta(k)}_{E_i}(N)\right).$$

By Corollary 4.2.4, there are short exact sequences

$$\begin{aligned} 0 \to q_i^{-\langle i^\vee,\beta\rangle - 2\zeta(k)} \mathrm{ad}^{\zeta(k)}_{E_i}(M)\mathrm{ad}^{n+1-\zeta(k)}_{E_i}(N) \xrightarrow{f_k} \mathrm{ad}_{E_i}&\left(\mathrm{ad}^{\zeta(k)}_{E_i}(M)\mathrm{ad}^{n-\zeta(k)}_{E_i}(N)\right) \\ &\to \mathrm{ad}^{\zeta(k)+1}_{E_i}(M)\mathrm{ad}^{\zeta(k)}_{E_i}(N) \to 0. \end{aligned}$$

We can use them to refine the filtration of $\mathrm{ad}^{n+1}_{E_i}(MN)$. More precisely, for $k \in \{-1, \ldots 2^{n+1} - 1\}$ let

$$W_k = \begin{cases} \mathrm{ad}_{E_i}(V_{(k-1)/2}) & \text{if } k \text{ is odd,} \\ p^{-1}_{k/2}(\mathrm{im}(f_{k/2})) & \text{if } k \text{ is even.} \end{cases}$$

We get a filtration of $\mathrm{ad}_{E_i}^{n+1}(MN)$ of the form

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2^{n+1}-1} = \mathrm{ad}_{E_i}^{n+1}(XY),$$

with quotients

$$W_k/W_{k-1} \simeq \begin{cases} q_i^{(\zeta((k-1)/2)-n)\langle i^\vee,\beta\rangle - 2\sigma((k-1)/2)} \mathrm{ad}_{E_i}^{\zeta((k-1)/2)+1}(M)\mathrm{ad}_{E_i}^{n-\zeta((k-1)/2)}(N) & \text{if } k \text{ is odd,} \\ \\ q_i^{(\zeta(k/2)-n)\langle i^\vee,\beta\rangle - 2\sigma(k/2) - \langle i^\vee,\beta\rangle - 2\zeta(k/2)} \mathrm{ad}_{E_i}^{\zeta(k/2)}(M)\mathrm{ad}_{E_i}^{n+1-\zeta(k/2)}(N) & \text{if } k \text{ is even.} \end{cases}$$

The conclusion follows from the recursive relations (4.2.2) for ζ and (4.2.3) for σ. □

4.2.3 Integrability

Proposition 4.2.6. *Let $\beta \in Q^+$. If $s_i(\beta) \notin Q^+$, then $H^i_\beta = 0$.*

Proof. We write β in the form $\beta = \gamma + ni$, with $\gamma \in \bigoplus_{j\neq i} \mathbb{N}j$ and $n \geqslant 0$. Since $s_i(\beta) \notin Q^+$, we have $n > -\langle i^\vee, \gamma\rangle$. Consider an idempotent of H^i_β of the form $1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k}$ with $n_1 + \ldots + n_k = n$, and $\nu_1 \ldots \nu_k \in I^\gamma$. Then

$$\begin{aligned} H^i_\beta 1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k} &= \pi_i(E_i^{n_1} E_{\nu_1} \ldots E_i^{n_k} E_{\nu_k}) \\ &\simeq \pi_i\left(E_i^{n_1} \pi_i(E_{\nu_1} \ldots E_i^{n_k} E_{\nu_k})\right) \end{aligned}$$

by Proposition 4.1.5. Furthermore by Proposition 4.1.10, we have isomorphisms

$$\begin{aligned} \pi_i\left(E_i^{n_1} \pi_i(E_{\nu_1} \ldots E_i^{n_k} E_{\nu_k})\right) &\simeq \mathrm{ad}_{E_i}^{n_1}\left(\pi_i\left(E_{\nu_1} \ldots E_i^{n_k} E_{\nu_k}\right)\right) \\ &\simeq \mathrm{ad}_{E_i}^{n_1}\left(E_{\nu_1} \pi_i\left(E_i^{n_2} E_{\nu_2} \ldots E_i^{n_k} E_{\nu_k}\right)\right) \end{aligned}$$

where the second isomorphism is obtained by applying Proposition 4.1.5. Hence there is an isomorphism

$$H^i_\beta 1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k} \simeq \mathrm{ad}^{n_1}_{E_i}\left(E_{\nu_1}\pi_i\left(E_i^{n_2}E_{\nu_2}\ldots E_i^{n_k}E_{\nu_k}\right)\right).$$

This procedure can be repeated inductively and we obtain an isomorphism

$$H^i_\beta 1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k} \simeq \mathrm{ad}^{n_1}_{E_i}\left(E_{\nu_1}\mathrm{ad}^{n_2}_{E_i}\left(E_{\nu_2}\ldots\mathrm{ad}^{n_k}_{E_i}(E_{\nu_k})\ldots\right)\right).$$

Using Proposition 4.2.5 repeatedly, we see that $H^i_\beta 1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k}$ has a filtration whose quotients are shifts of modules of the form $\mathrm{ad}^{m_r}_{E_i}(E_{j_r})\ldots\mathrm{ad}^{m_1}_{E_i}(E_{j_1})$ for some $m_1,\ldots,m_r \in \mathbb{N}$ such that $m_1+\ldots+m_r = n$ and $(j_r,\ldots,j_1) \in I^\gamma$. Let us prove that such a quotient is zero. Since

$$m_1+\ldots+m_r = n > -\langle i^\vee,\gamma\rangle = -\langle i^\vee, j_1\rangle - \ldots - -\langle i^\vee, j_r\rangle,$$

there exists an index ℓ such that $m_\ell > -\langle i^\vee, j_\ell\rangle$. But then by the vanishing criterion (4.1.2), we have $\mathrm{ad}^{m_\ell}_{E_i}(E_{j_\ell}) = 0$, and $\mathrm{ad}^{m_r}_{E_i}(E_{j_r})\ldots\mathrm{ad}^{m_1}_{E_i}(E_{j_1}) = 0$. Hence, all the quotients in the filtration of $H^i_\beta 1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k}$ are zero. So $H^i_\beta 1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k} = 0$. Since 1_β is the sum of all idempotents of the form $1_{n_1 i,\nu_1,\ldots,n_k i,\nu_k}$, we conclude that $H^i_\beta = 0$. □

Corollary 4.2.7. *The functor* ad_{E_i} *is locally nilpotent on* $\mathcal{H}^i$.

Proof. Let $\beta \in Q^+$. For $n \in \mathbb{N}$, we have $s_i(\beta+ni) = s_i(\beta) - ni$. In particular, if n is large enough, then $s_i(\beta+ni) \notin Q^+$, so $H^i_{\beta+ni} = 0$. Thus, for such an n and for $M \in H^i_\beta$−mod, we have $\mathrm{ad}^n_{E_i}(M)$. The result follows. □

4.2.4 Proof of Theorem 4.2.1

We start by introducing the notation that we will use throughout the proof. Let $\mathcal{I}$ be the full subcategory of $\mathcal{H}^i$ whose objects are the modules M for which $\tau_{E_i,M}$ is injective. Our goal is to show that $\mathcal{I} = \mathcal{H}^i$. We denote by $\mathcal{S}$ the Serre and monoidal subcategory of $\mathcal{H}^i$ generated by the modules $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$ for $\nu \in (I \setminus \{i\})^r$ and $n, r \in \mathbb{N}$. To prove that $\mathcal{H}^i = \mathcal{I}$, we will first prove that $\mathcal{S} \subseteq \mathcal{I}$ and then that $\mathcal{H}^i = \mathcal{S}$. The first step is to prove formal properties of $\mathcal{I}$.

Lemma 4.2.8. *The category $\mathcal{I}$ is a monoidal subcategory of $\mathcal{H}^i$ and is closed under extensions. The restriction of ad_{E_i} to $\mathcal{I}$ is exact and is a categorical derivation in the sense of Proposition 4.2.4.*

Proof. Let $M, N \in \mathcal{I}$. By Proposition 4.1.14, we have $\tau_{E_i,MN} = M\tau_{E_i,N} \circ \tau_{E_i,M}N$. Since the monoidal structure is bi-exact, $M\tau_{E_i,N}$ and $\tau_{E_i,M}N$ are injective. So $\tau_{E_i,MN}$ is injective as well. Hence $\mathcal{I}$ is a monoidal subcategory of $\mathcal{H}^i$.

Consider now a short exact sequence $0 \to L \to M \to N \to 0$ with $L, N \in \mathcal{I}$. We obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & LE_i \arrow[r] \arrow[d, "\tau_{E_i,L}"] & ME_i \arrow[r] \arrow[d, "\tau_{E_i,M}"] & NE_i \arrow[r] \arrow[d, "\tau_{E_i,N}"] & 0 \\
0 \arrow[r] & E_iL \arrow[r] & E_iM \arrow[r] & E_iN \arrow[r] & 0.
\end{tikzcd}$$

Since $\tau_{E_i,L}$ and $\tau_{E_i,N}$ are injective, $\tau_{E_i,M}$ is also injective by the Four Lemma. Hence $\mathcal{I}$ is closed under extensions. Now remark that the previous commutative diagram extends to a commutative square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & LE_i \arrow[r] \arrow[d, "\tau_{E_i,L}"] & ME_i \arrow[r] \arrow[d, "\tau_{E_i,M}"] & NE_i \arrow[r] \arrow[d, "\tau_{E_i,N}"] & 0 \\
0 \arrow[r] & E_iL \arrow[r] \arrow[d] & E_iM \arrow[r] \arrow[d] & E_iN \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{ad}_{E_i}(L) \arrow[r] \arrow[d] & \mathrm{ad}_{E_i}(M) \arrow[r] \arrow[d] & \mathrm{ad}_{E_i}(N) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

in which the first two rows are exact and the three columns are exact. By the Nine Lemma, the last row is also exact. This proves that the restriction of ad_{E_i} to $\mathcal{I}$ is exact. The fact it is also a derivation follows from the same formal proof as that of Proposition 4.2.4. $\square$

To prove that $\mathcal{S} \subseteq \mathcal{I}$, we construct a quasi-left-inverse $\tau_{(-),E_i}$ to $\tau_{E_i,(-)}$ for the objects of $\mathcal{S}$.

Definition 4.2.9. For $\beta \in Q^+$ of height r and $M \in H^i_\beta-\mathrm{mod}$, we define

$$\tau_{M,E_i} : \begin{cases} E_iM & \to & ME_i, \\ h1_{i,\beta} \otimes_{H_\beta} m & \mapsto & hQ_{i,\beta}(x_{r+1}, x_1, \ldots, x_r)\tau_{[r\downarrow 1]}1_{\beta,i}. \otimes_{H_\beta} m. \end{cases}$$

Lemma 4.2.10. *The map τ_{M,E_i} is a well-defined morphism of $H_{\beta+i}$-modules and is natural in M.*

Proof. Let $P = Q_{i,\beta}(x_{r+1}, x_1, \ldots, x_r)$. To check that τ_{M,E_i} is well-defined, we need to prove that for all $y \in H_\beta$, we have

$$\big((1_i \diamond y)P\tau_{[r\downarrow 1]} - P\tau_{[r\downarrow 1]}(y \diamond 1_i)\big)\, 1_{\beta,i} \otimes_{H_\beta} M = 0.$$

Let $Z = \big((1_i \diamond y)P\tau_{[r\downarrow 1]} - P\tau_{[r\downarrow 1]}(y \diamond 1_i)\big)\, 1_{\beta,i}$. The element P is central in $H_{i,\beta}$ by Proposition 4.1.9, so we have $Z = P\,\big((1_i \diamond y)\tau_{[r\downarrow 1]} - \tau_{[r\downarrow 1]}(y \diamond 1_i)\big)\, 1_{\beta,i}$. By Proposition 2.4.2, there are elements $z_k \in H_{\beta,i}$ such that

$$Z = P\sum_{k=0}^{r-1} \tau_{[k\downarrow 1]} z_k 1_{\beta,i} = \sum_{k=0}^{r-1} \tau_{[k\downarrow 1]} P z_k 1_{\beta,i}.$$

However

$$\begin{aligned}
P1_{\beta,i} &= \sum_{\nu\in I^\beta}\Bigg(\prod_{\substack{1\leqslant k\leqslant r\\ \nu_k\neq i}} Q_{i,\nu_k}(x_{r+1},x_k)\Bigg)1_{i,\nu}1_{\beta,i}\\
&= \sum_{\mu\in I^{\beta-i}}\Bigg(\prod_{\substack{1\leqslant k\leqslant r-1\\ \mu_k\neq i}} Q_{i,\mu_k}(x_{r+1},x_{k+1})\Bigg)1_{i,\mu,i}\\
&= \sum_{\substack{\rho\in I^\beta\\ \rho_r=i}} Q_{i,\rho}(x_{r+1},x_2,\dots,x_{r+1})1_{\rho,i}\\
&= \Bigg(\sum_{\substack{\rho\in I^\beta\\ \rho_r=i}} Q_{i,\rho}(x_r,x_1,\dots,x_r)1_\rho\Bigg)\diamond 1_i.
\end{aligned}$$

By Proposition 4.1.8, $Q_{i,\rho}(x_r,x_1,\dots,x_r)1_\rho$ acts by zero on M for all $\rho\in I^\beta$ such that $\rho_r=i$. So $Z\otimes_{H_\beta}M=0$, and the morphism τ_{M,E_i} is well defined. The naturality of τ_{M,E_i} follows immediately from its definition. □

Proposition 4.2.11. *Let $n\in\mathbb{N}$ and let $\nu\in(I\setminus\{i\})^r$ for some $r\in\mathbb{N}$. Let $M=\mathrm{ad}_{E_i}^{(n)}(E_\nu)$. Then $\tau_{E_i,M}$ is injective.*

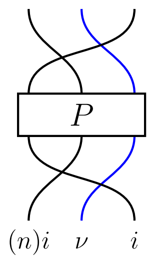
Proof. Let β be the element of Q^+ such that M is an H_β^i-module. We compute the composition $\tau_{M,E_i}\circ\tau_{E_i,M}$ and prove that it is injective. It is given by

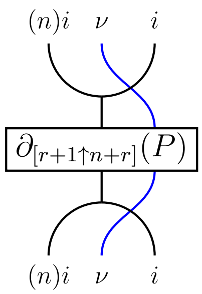
$$\tau_{M,E_i}\circ\tau_{E_i,M}:\left\{\begin{array}{rcl} ME_i & \to & ME_i,\\ h1_{\beta,i}\otimes_{H_\beta}m & \mapsto & h\tau_{[1\uparrow n+r]}P\tau_{[n+r\downarrow 1]}1_{\beta,i}\otimes_{H_\beta}m,\end{array}\right.$$

where $P=Q_{i,\beta}(x_{n+r+1},x_1,\dots,x_{n+r})1_{i,\beta}$. The $H_{\beta+i}$-module $E_i^{(n)}E_\nu E_i$ is cyclic and generated by the element $\tau_{\omega_0[r+2,n+r+1]}1_{ni,\nu,i}$. Since ME_i is a quotient of $E_i^{(n)}E_\nu E_i$, it is also cyclic. The class c of $\tau_{\omega_0[r+2,n+r+1]}1_{ni,\nu,i}$ is a generator of ME_i. We start by computing $\tau_{M,E_i}\circ\tau_{E_i,M}(c)$. Let

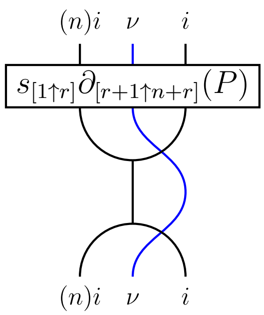
$$z=\tau_{[1\uparrow n+r]}P\tau_{[n+r\downarrow 1]}e_{[r+2,n+r+1]}1_{ni,\nu,i},$$

so that $\tau_{M,E_i} \circ \tau_{E_i,M}(c) = \tau_{\omega_0[r+2,n+r+1]} z x_{[r+2,n+r+1]} c$. We compute a new expression of z using diagrammatic calculus. We view z as an endomorphism of $E_i^{(n)} E_\nu E_i$. Then we have

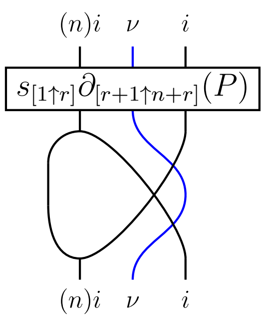
$z =$ 

$=$ 

(Lemma 2.5.5)

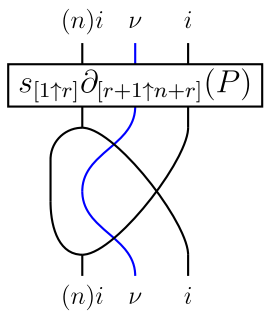
$=$ 

(relation (3) in Definition 2.3.1)

$=$ 

(Lemma 2.5.4)

$$= s_{[1\uparrow r]} \left(\partial_{[r+1\uparrow n+r]}(P)\right) \partial_{[n+r\downarrow r+2]} \partial_{r+2,1} \left(Q_{\nu,i}\right) e_{[r+2,n+r+1]} 1_{ni,\nu,i}$$

$+$ 

(Lemma 2.5.6).

This last diagram lies in $H_{\beta+i}1_{(n-1)i,\nu,2i}H_{\beta,i}$. In particular, it acts by zero on ME_i. Hence we have proved that $\tau_{M,E_i} \circ \tau_{E_i,M}(c) = z'c$ where

$$z' = s_{[1\uparrow r]}\left(\partial_{[r+1\uparrow n+r]}(P)\right)\partial_{[n+r\downarrow r+2]}\partial_{r+2,1}\left(Q_{i,\nu}\right)1_{ni,\nu,i} \in P_{\beta+i}1_{ni,\nu,i}.$$

The polynomials P and $Q_{i,\nu}$ are symmetric in the variables labeled by ν by definition. Furthermore, the Demazure operators in the definition of z' ensure that z' is also symmetric in the variables labeled by i. Hence z' is fixed by the stabilizer $\mathrm{Stab}(ni,\nu,i)$ of the sequence (ni,ν,i) in $\mathfrak{S}_{n+r+1}$. Now let

$$Z = \sum_{\omega \in \mathfrak{S}_{n+r+1}/\mathrm{Stab}(ni,\nu,i)} \omega(z') \in P_{\beta+i}.$$

Since z' is fixed by $\mathrm{Stab}(ni,\nu,i)$, Z is fixed by $\mathfrak{S}_{n+r+1}$. By Proposition 2.3.5, this implies that Z is a central element of $H_{\beta+i}$. Thus, multiplication by Z is an endomorphism of $H_{\beta+i}$-module of ME_i. Furthermore, we have

$$Zc = Z1_{ni,\nu,i}c = z'c = \tau_{M,E_i} \circ \tau_{E_i,M}(c).$$

Hence $\tau_{M,E_i} \circ \tau_{E_i,M}$ and multiplication by Z agree on c. Since c generates ME_i, we conclude that $\tau_{M,E_i} \circ \tau_{E_i,M}(x) = Zx$ for all $x \in ME_i$. Now remark that $Z1_{\beta,i}$ has the form

$$Z1_{\beta,i} \in tx_1^k 1_{\beta,i} + \sum_{\ell<k} \left(P_\beta \diamond x_1^\ell\right) 1_{\beta,i}$$

for some integer k and $t \in K^\times$. It follows that multiplication by $Z1_{\beta,i}$ is an injective endomorphism of $M \otimes E_i$. Furthermore, there is a decomposition

$$ME_i = \bigoplus_{k=0}^{n+r} \tau_{[k\downarrow 1]} 1_{\beta,i} \otimes_{H_{\beta,i}} (M \otimes E_i).$$

Since Z is central, multiplication by Z acts diagonally along this decomposition. The action of Z on each summand $M \otimes E_i$ is given by multiplication by $Z1_{\beta,i}$. We conclude that multiplication by Z is an injective endomorphism of ME_i. Hence $\tau_{E_i,M}$ is injective. ☐

Corollary 4.2.12. *Let $n \in \mathbb{N}$ and let $\nu \in (I \setminus \{i\})^r$ for some $r \in \mathbb{N}$. Let N be a subquotient of $M = \mathrm{ad}_{E_i}^{(n)}(E_\nu)$. Then $\tau_{E_i,N}$ is injective.*

Proof. We can fit N in a diagram

$$N \hookrightarrow N' \twoheadleftarrow M$$

for some H_β^i-module N'. By naturality, $\tau_{N',E_i} \circ \tau_{E_i,N'}$ is multiplication by Z, where Z is the central element from the previous proof. By the same argument as above, multiplication by Z is injective on $N'E_i$. Hence $\tau_{E_i,N'}$ is injective. By naturality, $\tau_{E_i,N}$ is the restriction of $\tau_{E_i,N'}$ to NE_i. Thus, it is injective as well. ☐

Proposition 4.2.13. *We have $\mathcal{S} \subseteq \mathcal{I}$ and $\mathcal{S}$ is stable under ad_{E_i}.*

Proof. Denote by $\mathcal{G}$ the class of subquotients of modules of the form $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$ for $\nu \in (I \setminus \{i\})^r$ and $n, r \in \mathbb{N}$. By definition, the subcategory $\mathcal{S}$ is generated by $\mathcal{G}$ under extensions and tensor products. Furthermore, the elements of $\mathcal{G}$ are objects of $\mathcal{I}$ by Corollary 4.2.12, and $\mathcal{I}$ is closed under extensions and tensor products by Lemma 4.2.8. It follows that $\mathcal{S} \subseteq \mathcal{I}$.

Hence, ad_{E_i} restricts to an exact categorical derivation $\mathcal{S} \to \mathcal{H}^i$. Denote by $\mathcal{G}'$ the class of modules of the form $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$ for $\nu \in (I \setminus \{i\})^r$ and $n, r \in \mathbb{N}$. We have $\mathrm{ad}_{E_i}(\mathcal{G}') \subseteq \mathcal{S}$ since $\mathrm{ad}_{E_i}\left(\mathrm{ad}_{E_i}^{(n)}(E_\nu)\right) \simeq [n+1]_i \, \mathrm{ad}_{E_i}^{(n+1)}(E_\nu)$. Furthermore, $\mathcal{S}$ is generated by $\mathcal{G}'$ as a Serre and monoidal subcategory. The fact that the restriction of ad_{E_i} to $\mathcal{S}$ is an exact derivation then implies that $\mathrm{ad}_{E_i}(\mathcal{S}) \subseteq \mathcal{S}$. □

Proposition 4.2.14. *We have $\mathcal{H}^i = \mathcal{S}$.*

Proof. Let $M \in \mathcal{H}^i$ and let N be its projective cover in $\mathcal{H}$. Since π_i is right exact and $\pi_i(M) = M$, M is a quotient of $\pi_i(N)$. Furthermore, every projective indecomposable module of $\mathcal{H}$ is a summand of a module of the form E_ν for $\nu \in I^r$ and $r \in \mathbb{N}$. Hence, to prove that $M \in \mathcal{S}$, it suffices to prove that for all $r \in \mathbb{N}$ and $\nu \in I^r$, we have $\pi_i(E_\nu) \in \mathcal{S}$. We proceed by induction on r.

For $r = 1$, we have $\pi_i(E_j) = E_j$ if $j \neq i$ and $\pi_i(E_i) = 0$. In all cases, we have $\pi_i(E_j) \in \mathcal{S}$. Assume now that $\pi_i(E_\nu) \in \mathcal{S}$ and let $j \in I$. We prove that $\pi_i(E_j E_\nu) \in \mathcal{S}$. Consider two cases.

- If $j \neq i$, then $\pi_i(E_j E_\nu) \simeq E_j \pi_i(E_\nu)$ by Lemma 4.1.5. Since $\mathcal{S}$ is a monoidal subcategory of $\mathcal{H}^i$, we have $\pi_i(E_j E_\nu) \in \mathcal{S}$.

- If $j = i$, we have $\pi_i(E_i E_\nu) \simeq \mathrm{ad}_{E_i}(\pi_i(E_\nu))$ by Lemma 4.1.5. Since $\mathcal{S}$ is stable by ad_{E_i} by Proposition 4.2.13, we deduce that $\pi_i(E_j E_\nu) \in \mathcal{S}$.

This completes the induction and proves that $\mathcal{H}^i = \mathcal{S}$. □

Now Proposition 4.2.13 and Proposition 4.2.14 imply Theorem 4.2.1.

4.2.5 Generating Objects

As an immediate consequence of Proposition 4.2.14 and Proposition 4.2.5, we get the following generating result for $\mathcal{H}^i$.

Corollary 4.2.15. *As a Serre and monoidal full subcategory of $\mathcal{H}$, $\mathcal{H}^i$ is generated by the objects* $\mathrm{ad}_{E_i}^{(n)}(E_j)$ *for $n \geqslant 0$ and $j \in I \setminus \{i\}$.*

Corollary 4.2.15 is a categorification of the definition of $\mathbf{f}[i]$ as the subalgebra generated by the $\mathrm{ad}_{e_i}^{(n)}(e_j)$ for $n \geqslant 0$ and $j \in I \setminus \{i\}$. Lusztig also proves that $\mathbf{f} = \sum_{n\geqslant 0} \mathbf{f}[i] e_i^{(n)}$ [Lus10, Lemma 38.1.2]. We finish this section by proving a similar result for $\mathcal{H}$.

Proposition 4.2.16. *The Serre subcategory of $\mathcal{H}$ generated by the subcategories $\mathcal{H}^i E_i^{(n)}$ for $n \geqslant 0$ is equal to $\mathcal{H}$.*

Proof. Let $\mathcal{H}'$ be the Serre subcategory of $\mathcal{H}$ generated by the subcategories $\mathcal{H}^i E_i^n$ for $n \geqslant 0$. Since $\mathcal{H}$ is generated as an abelian category by the modules of the form E_ν for $\nu \in I^m$, it suffices to prove that $E_\nu \in \mathcal{H}'$ for all $m \in \mathbb{N}$ and $\nu \in I^m$. Since $1 \in \mathcal{H}'$, it suffices to prove that $\mathcal{H}'$ is stable under the multiplication functors $E_j(-)$ for all $j \in I$. Since these functors are exact, it suffices to prove that they send $\mathcal{H}^i E_i^n$ to $\mathcal{H}'$ for all $n \in \mathbb{N}$.

Assume that $j \neq i$. Then $E_j \in \mathcal{H}^i$, so $E_j \mathcal{H}^i \subseteq \mathcal{H}^i$ and $E_j \mathcal{H}^i E_i^n \subseteq \mathcal{H}'$. For $j = i$ and $M \in \mathcal{H}[i]$, we have an inclusion $E_i M E_i^n \hookrightarrow M E_i^{n+1}$ by Theorem 4.2.1. Hence $E_i M E_i^n \in \mathcal{H}'$. This proves that $\mathcal{H}' = \mathcal{H}$. $\square$

4.3 Structure of 2-Representation

4.3.1 Data

The goal of this section is to prove that the endofunctor ad_{E_i} induces a 2-representation of $\mathfrak{sl}_2$ on $\mathcal{H}^i$. Let us start by defining the data giving rise to this structure. The weight subcategories of $\mathcal{H}^i$ are the subcategories $\mathcal{H}^i_\lambda$, $\lambda \in \mathbb{Z}$, defined by

$$\mathcal{H}^i_\lambda = \bigoplus_{\substack{\beta \in Q^+ \\ \langle i^\vee, \beta \rangle = \lambda}} H^i_\beta\text{-mod}.$$

We consider a pair $(\mathrm{ad}_{E_i}, \mathrm{ad}_{F_i})$ of endofunctors of $\mathcal{H}^i$. The functor ad_{E_i} is the one we studied in Section 4.1. The functor ad_{F_i} is defined by

$$\mathrm{ad}_{F_i} = \bigoplus_{\beta \in Q^+} q_i^{\langle i^\vee, \beta\rangle - 1} \mathrm{res}_{H^i_{\beta-i}}^{H^i_\beta}.$$

More precisely, if $M \in H^i_\beta\text{−mod}$, we have $\mathrm{ad}_{F_i}(M) = q_i^{\langle i^\vee,\beta\rangle-1} 1_{i,\beta-i}M$, viewed as an $H^i_{\beta-i}$-module via the right inclusion $H^i_{\beta-i} \to 1_{i,\beta-i}H^i_\beta 1_{i,\beta-i}$. Finally, there are natural transformations $x \in \mathrm{End}^\bullet(\mathrm{ad}_{E_i})$ and $\tau \in \mathrm{End}^\bullet(\mathrm{ad}^2_{E_i})$ defined by Proposition 4.1.10.

Theorem 4.3.1. *This data defines a 2-representation of $\mathfrak{sl}_2$ on $\mathcal{H}^i$.*

Proof. We must prove that the conditions in Definition 2.4.8 are satisfied. The action of the affine nil Hecke algebra on powers of ad_{E_i} was proved in Proposition 4.1.10. The adjunction between ad_{E_i} and ad_{F_i} follows from their definition. Hence, all that remains to be proved is the invertibility of the maps $(\rho_\lambda)_{\lambda\in\mathbb{Z}}$ defined in equations (2.4.5) and (2.4.6). We do this in Theorem 4.3.4 below. □

4.3.2 Mackey Formulas

The Quiver Hecke Algebra Case

We follow [KK12, Section 3]. Let $\beta \in Q^+$ and $i \in I$. The left i-induction is the functor defined by

$$\begin{cases} H_\beta\text{−mod} & \to & H_{\beta+i}\text{−mod}, \\ M & \mapsto & E_i M \simeq H_{\beta+i}1_{i,\beta} \otimes_{H_\beta} M. \end{cases}$$

We define the left i-restriction functor F_i as its right adjoint. It is given by

$$\begin{cases} H_{\beta+i}\text{−mod} & \to & H_\beta\text{−mod}, \\ N & \mapsto & F_i(N) = 1_{i,\beta}N. \end{cases}$$

There is a Mackey type result for left i-induction and left i-restriction.

Proposition 4.3.2 ([KK12, Theorem 3.6]). *Let $\beta \in Q^+$. There is an isomorphism of graded H_β-bimodules*

$$\begin{cases} q_i^{-2}\left(H_\beta 1_{i,\beta-i} \otimes_{H_{\beta-i}} 1_{i,\beta-i} H_\beta\right) \oplus H_{i,\beta} & \to \quad 1_{i,\beta} H_{\beta+i} 1_{i,\beta}, \\ ((y \otimes y'), z) & \mapsto \quad (1_i \diamond y)\tau_{|\beta|}(1_i \diamond y') + z. \end{cases}$$

It induces a natural isomorphism in $M \in H_\beta$ $-$mod

$$q_i^{-2} E_i F_i(M) \oplus M[X_i] \xrightarrow{\sim} F_i(E_i M),$$

where X_i is a formal variable of degree $2d_i$. More explicitly,

$$M[X_i] = M \otimes K[X_i] \simeq \bigoplus_{k \in \mathbb{N}} q_i^{2k} M.$$

Right i-induction and right i-restriction functors are defined similarly and satisfy an analogous Mackey type decomposition. Finally, there is a Mackey type result for right i-induction and left i-restriction.

Proposition 4.3.3 ([KK12, Theorem 3.9]). *Let $\beta \in Q^+$. There is a short exact sequence of graded H_β-bimodules*

$$0 \to H_\beta 1_{\beta-i,i} \otimes_{H_{\beta-i}} 1_{i,\beta-i} H_\beta \xrightarrow{R} 1_{i,\beta} H_{\beta+i} 1_{\beta,i} \xrightarrow{S} q_i^{-\langle i^\vee, \beta \rangle} H_{\beta,i} \to 0.$$

It induces a short exact sequence which is natural in $M \in H_\beta$ $-$mod

$$0 \to F_i(M)E_i \xrightarrow{R_M} F_i(M E_i) \xrightarrow{S_M} q_i^{-\langle i^\vee, \beta \rangle} M[X_i] \to 0.$$

Let us describe explicitly the morphisms R and S. The morphism R is given by

$$\begin{cases} H_\beta 1_{\beta-i,i} \otimes_{H_{\beta-i}} 1_{i,\beta-i} H_\beta & \to \quad 1_{i,\beta} H_{\beta+i} 1_{\beta,i}, \\ \qquad\qquad\qquad y \otimes y' & \mapsto \quad (1_i \diamond y)(y' \diamond 1_i). \end{cases}$$

To describe the morphism S, we use decomposition (2.4.1) to write

$$H_{\beta+i} 1_{\beta,i} = \bigoplus_{k=0}^{|\beta|} \tau_{[k\downarrow 1]} H_{\beta,i}.$$

If $z \in 1_{i,\beta} H_{\beta+i} 1_{\beta,i}$, then we can thus decompose z as

$$z = \sum_{k=0}^{|\beta|} \tau_{[k\downarrow 1]} z_k$$

for some unique $z_0, \dots, z_{|\beta|} \in H_{\beta,i}$. Then $S(z) = z_{|\beta|}$. Equivalently, we can also compute $S(z)$ by decomposing along

$$1_{i,\beta} H_{\beta+i} = \bigoplus_{k=1}^{|\beta|+1} H_{i,\beta} \tau_{[|\beta|\downarrow k]}$$

and taking the coefficient of $\tau_{[|\beta|\downarrow 1]}$ (see [KK12, Corollary 3.8]).

Mackey Formulas in $\mathcal{H}^i$

We can now prove the invertibility of the maps ρ_λ, $\lambda \in \mathbb{Z}$.

Theorem 4.3.4. *Let $M \in H^i_\beta-\mathrm{mod}$ with $\beta \in Q^+$, and let $\lambda = \langle i^\vee, \beta \rangle$. Then the morphism ρ_λ gives an isomorphism*

$$\begin{cases} \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \xrightarrow[\sim]{\rho_\lambda} \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M) \oplus [\lambda]_i M & \text{if } \lambda \geqslant 0, \\ \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \oplus [-\lambda]_i M \xrightarrow[\sim]{\rho_\lambda} \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M) & \text{if } \lambda \leqslant 0. \end{cases}$$

The rest of this section is devoted to the proof of this Theorem. It mirrors that of [KK12, Theorem 5.2] for cyclotomic quiver Hecke algebras and is based on two ingredients: the short exact sequence of Theorem 4.2.1 and the Mackey decompositions for quiver Hecke algebras. One difference, as in the proof of Theorem 4.2.1, is that we only do explicit computations for modules of the form $\mathrm{ad}^n_{E_i}(E_\nu)$, which is enough since they generate $\mathcal{H}^i$ by Theorem 4.2.14.

Lemma 4.3.5. *Let $M \in H^i_\beta$−mod with $\beta \in Q^+$, and let $\lambda = \langle i^\vee, \beta\rangle$. There is a commutative diagram with exact rows which is natural in M:*

$$\begin{tikzcd}
0 \arrow[r] & q_iF_i(M)E_i \arrow[r, "R_M"] \arrow[d, "\tau_{E_i,F_i(M)}"'] & q_iF_i(ME_i) \arrow[r, "S_M"] \arrow[d, "F_i(\tau_{E_i,M})"'] & q_i^{1-\lambda}M[X_i] \arrow[r] \arrow[d, "\Phi_M"'] & 0 \\
0 \arrow[r] & q_i^{\lambda-1}E_iF_i(M) \arrow[r, "T_M"'] & q_i^{\lambda+1}F_i(E_iM) \arrow[r, "W_M"'] & q_i^{\lambda+1}M[X_i] \arrow[r] & 0.
\end{tikzcd} \tag{4.3.1}$$

Proof. The maps R_M and S_M are those of Proposition 4.3.3, and the maps T_M and W_M are those induced by the isomorphism of Proposition 4.3.2. The exactness of the two rows also follows from Propositions 4.3.2 and 4.3.3. The map Φ_M is defined by commutativity of the diagram. So the only statement to prove is that the leftmost square of the diagram commutes.

Let $y = 1_{\beta-i,i} \otimes_{H_{\beta-i}} 1_{i,\beta-i}m \in F_i(M)E_i$, with $m \in M$. We have

$$\begin{aligned}
\tau_{E_i,F_i(M)}(y) &= \tau_{[1\uparrow|\beta|-1]}1_{i,\beta-i} \otimes_{H_{\beta-i}} 1_{i,\beta-i}m,\\
T_M(\tau_{E_i,F_i(M)}(y)) &= \tau_{[1\uparrow|\beta|]}1_{2i,\beta-i} \otimes_{H_\beta} m,
\end{aligned}$$

and

$$\begin{aligned}
R_M(y) &= 1_{i,\beta-i,i} \otimes_{H_\beta} m,\\
F_i(\tau_{E_i,M})(f_M(y)) &= \tau_{[1\uparrow|\beta|]}1_{2i,\beta} \otimes_{H_\beta} m.
\end{aligned}$$

Hence $T_M(\tau_{E_i,F_i(M)}(y)) = F_i(\tau_{E_i,M})(R_M(y))$. Since such elements $y \in F_i(M)E_i$ generate

$F_i(M)E_i$ as an H_β-module, we deduce that $T_M \circ \tau_{E_i,F_i(M)} = F_i(\tau_{E_i,M}) \circ R_M$. $\square$

Keep the notation of the previous lemma. The morphism $\tau_{E_i,F_i(M)}$ is injective by Theorem 4.2.1, and its cokernel is $\mathrm{ad}_{E_i}(F_i(M)) = q_i^{1-\lambda}\mathrm{ad}_{E_i}(\mathrm{ad}_{F_i}(M))$. Since the left i-restriction functor F_i is exact, we also know that $F_i(\tau_{E_i,M})$ is injective, and its cokernel is $F_i(\mathrm{ad}_{E_i}(M)) = q_i^{-\lambda-1}\mathrm{ad}_{F_i}(\mathrm{ad}_{E_i}(M))$. Hence, by the Snake Lemma, there is a short exact sequence which is natural in $M \in H^i_\beta-\mathrm{mod}$:

$$0 \to \ker(\Phi_M) \to \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \xrightarrow{\overline{T}_M} \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M) \to \mathrm{coker}(\Phi_M) \to 0. \tag{4.3.2}$$

Lemma 4.3.6. *For all $M \in H^i_\beta-\mathrm{mod}$ with $\beta \in Q^+$, the map $\overline{T}_M : \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \to \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M)$ in the sequence (4.3.2) is equal to the map σ defined in equation (2.4.4).*

Proof. We prove this by chasing the diagram giving rise to the sequence (4.3.2). Let y be an element of $\mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M)$ of the form $y = 1_{i,\beta-i} \otimes_{H^i_{\beta-i}} 1_{i,\beta-i}m$, for $m \in M$. We can write y as the image of the element $z = 1_{i,\beta-i} \otimes_{H_{\beta-i}} 1_{i,\beta-i}m \in E_iF_i(M)$ by the canonical quotient morphism. Then we have

$$T_M(z) = \tau_{|\beta|}1_{2i,\beta-i} \otimes_{H_\beta} m.$$

Hence

$$\overline{T}_M(y) = \tau_{|\beta|}1_{2i,\beta-i} \otimes_{H^i_\beta} m = \sigma(y).$$

So σ and $\overline{T}_M$ coincide on elements y of the form given above. Since these generate $\mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M)$ as an H_β-module, we deduce that $\sigma = \overline{T}_M$. $\square$

We now want to compute explicitly the kernel and cokernel of the map Φ_M. An element y of $M[X_i]$ will be written formally in the form

$$y = \sum_{k=0}^{\ell} m_k X_i^k,$$

where $\ell \in \mathbb{N}$, and $m_0, \ldots, m_\ell \in M$. Our discussion up to this point is valid for any module in $\mathcal{H}^i$, but we now restrict to the generators of $\mathcal{H}^i$ to simplify the computations.

Proposition 4.3.7. *Let $n \in \mathbb{N}$ and let $\nu \in (I \setminus \{i\})^r$ for some $r \in \mathbb{N}$. Let $M = \mathrm{ad}_{E_i}^{(n)}(E_\nu)$. Denote by β the element of Q^+ such that M is an H^i_β-module and let $\lambda = \langle i^\vee, \beta\rangle$. Then there exists an invertible element b of K such that for all $m \in M$ and $k \geqslant 0$ we have*

$$\Phi_M\left(mX_i^k\right) \in bmX_i^{k-\lambda} + \sum_{\ell < k-\lambda} MX_i^\ell,$$

where we put $X_i^\ell = 0$ when $\ell < 0$.

Proof. The module M is cyclic, generated by the class c of $\tau_{\omega_0[r+1,n+r]}1_{ni,\nu} \in E_i^{(n)}E_\nu$. So it suffices to prove the result for $m = c$. Let $c_k = x^k_{n+r+1}\tau_{[n+r\downarrow 1]}1_{\beta,i} \otimes_{H_\beta} c \in F_i(ME_i)$. Then $cX_i^k = S_M(c_k)$ by definition of S. Since the rightmost square in the diagram (4.3.1) commutes, $\Phi_M(cX_i^k) = W_M(\tau_{E_i,M}(c_k))$. We have

$$\begin{aligned}\tau_{E_i,M}(c_k) &= x^k_{n+r+1}\tau_{[n+r\downarrow 1]}\tau_{[1\uparrow n+r]}1_{i,\beta} \otimes_{H_\beta} c \\ &= \tau_{\omega_0[r+1,n+r]}x^k_{n+r+1}\tau_{[n+r\downarrow 1]}\tau_{[1\uparrow n+r]}e_{[r+1,n+r]}1_{(n+1)i,\nu} \otimes_{H_\beta} x_{[r+1,n+r]}c,\end{aligned}$$

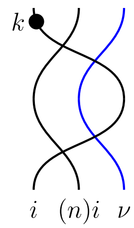
the second equality coming from the fact that $c = \tau_{\omega_0[r+1,n+r]}e_{[r+1,n+r]}x_{[r+1,n+r]}c$. Let

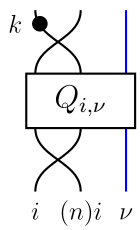
$$c'_k = x^k_{n+r+1}\tau_{[n+r\downarrow 1]}\tau_{[1\uparrow n+r]}e_{[r+1,n+r]}1_{(n+1)i,\nu},$$

so that $\tau_{E_i,M}(c_k) = \tau_{\omega_0[r+1,n+r]}c'_k \otimes_{H_\beta} x_{[r+1,n+r]}c$. To compute $W_M(\tau_{E_i,M}(c_k))$, we must find the component of c'_k on $H_{i,\beta}$ in the direct sum decomposition

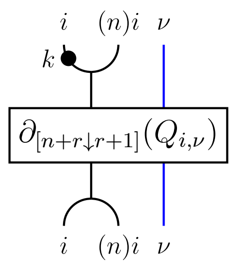
$$1_{i,\beta}H_{\beta+i}1_{i,\beta} = H_{i,\beta} \oplus (1_i \diamond H_\beta)\tau_{n+r}(1_i \diamond H_\beta)$$

given by Proposition 4.3.2. We do this using the diagrammatic calculus for quiver Hecke algebras. We view c'_k as a endomorphism of $E_i E_i^{(n)} E_\nu$. Then we have

$c'_k =$ 

$=$ 

(Lemma 2.5.5)

$=$ 

(Lemma 2.5.5).

Before we continue the computation, we write $\partial_{[n+r\downarrow r+1]}(Q_{i,\nu})$ in a more explicit way. First, given the assumptions on the form of the polynomials $Q_{i,j}$, we can write $Q_{i,\nu}$ in the form

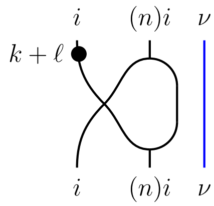
$$Q_{i,\nu} \in t x_{r+1}^{2n-\lambda} + \sum_{\ell < 2n-\lambda} x_{r+1}^{\ell} K[x_1, \ldots, x_r],$$

where t is an invertible element of K. Hence

$$\partial_{[n+r\downarrow r+1]}(Q_{i,\nu}) = \sum_{\ell \leqslant n-\lambda} x_{n+r+1}^{\ell} p_\ell$$

for some $p_\ell \in P_\beta 1_{ni,\nu}$, with $p_{n-\lambda} = (-1)^n t 1_{ni,\nu}$ an invertible element of K. Hence we have

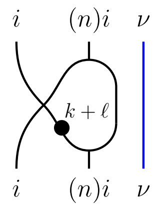
$$c'_k = \sum_{\ell \leqslant n-\lambda} p_\ell \cdot$$



(Lemma 2.5.4)

$$= \sum_{\ell \leqslant n-\lambda} p_\ell \partial_{[r+1 \uparrow n+r]} \left(x_{n+r+1}^{k+\ell}\right) e_{[r+1,n+r]} 1_{(n+1)i,\nu}$$

$$+ \sum_{\ell \leqslant n-\lambda} p_\ell \cdot$$



(Lemma 2.3.3).

In this last expression, the second term lies in $(1_i \diamond H_\beta)\tau_{n+r}(1_i \diamond H_\beta)$, while the first one lies in $H_{i,\beta}$. Hence, the component of c'_k on $H_{i,\beta}$ is given by the first term, which has the form

$$p_{n-\lambda} x_{n+r+1}^{k-\lambda} e_{[r+1,n+r]} 1_{(n+1)i,\nu} + \sum_{\ell < k-\lambda} x_{n+r+1}^{\ell} (1_i \diamond H_\beta).$$

It follows that

$$\begin{aligned} \Phi_M(cX_i^k) &= W_M(\tau_{\omega_0[r+1,r+n]} c'_k \otimes_{H_\beta} x_{[r+1,n+r]} c) \\ &\in p_{n-\lambda} c X_i^{k-\lambda} + \sum_{\ell < k-\lambda} M X_i^\ell, \end{aligned}$$

which proves the result. □

Remark 4.3.8. Since the map Φ_M is natural in M, Proposition 4.3.7 actually holds for M any subquotient of a module of the form $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$.

As an immediate consequence of Proposition 4.3.7, we get an explicit description of the kernel and cokernel of Φ_M.

Corollary 4.3.9. *Let M be a subquotient of $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$ for some $\nu \in (I \setminus \{i\})^r$ and $n, r \in \mathbb{N}$. Denote by $\beta \in Q^+$ the weight of M and let $\lambda = \langle i^\vee, \beta \rangle$. Then we have*

$$\begin{cases} \ker(\Phi_M) = q_i^{1-\lambda} \left(\bigoplus_{k=0}^{\lambda-1} M X_i^k \right) = [\lambda]_i M \quad \textit{and} \quad \mathrm{coker}(\Phi_M) = 0 & \textit{if } \lambda \geqslant 0, \\ \\ \ker(\Phi_M) = 0 \quad \textit{and} \quad \mathrm{coker}(\Phi_M) = q_i^{1+\lambda} \left(\bigoplus_{k=0}^{-\lambda-1} M X_i^k \right) = [-\lambda]_i M & \textit{if } \lambda \leqslant 0. \end{cases}$$

With this proved, we can now show that the maps ρ_λ are isomorphisms. We separate two cases depending on whether the weight is positive or negative.

Proposition 4.3.10. *Let $n \in \mathbb{N}$ and let $\nu \in (I \setminus \{i\})^r$ for some $r \in \mathbb{N}$. Let M be a subquotient of $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$. Denote by $\beta \in Q^+$ the weight of M and let $\lambda = \langle i^\vee, \beta \rangle$. Assume that $\lambda \leqslant 0$. Then ρ_λ is an isomorphism.*

Proof. Given Lemma 4.3.6 and Corollary 4.3.9, the short exact sequence (4.3.2) becomes

$$0 \to \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \xrightarrow{\sigma} \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M) \to [-\lambda]_i M \to 0. \tag{4.3.3}$$

We also have a commutative diagram:

$$\begin{tikzcd} q_i^{1+\lambda} F_i(E_i M) \arrow[r, "W_M"] \arrow[d] & q_i^{1+\lambda} M[X_i] & {[-\lambda]_i M} \arrow[l, hook', "\supseteq"'] \arrow[dl, "\sim"] \\ \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M) \arrow[r] & \mathrm{coker}(\Phi_M) & \end{tikzcd}.$$

The map W_M admits a right inverse A_M given by the direct sum decomposition

$$1_{i,\beta} H_{i+\beta} 1_{i,\beta} = H_{i,\beta} \oplus (1_i \diamond H_\beta)\tau_{n+r}(1_i \diamond H_\beta).$$

This right inverse provides a map $\overline{A}_M : [-\lambda]_i M \to \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M)$ that splits the short exact sequence (4.3.3) above. Let us compute the map $\overline{A}_M$. Let $y \in [-\lambda]_i M$. We view y as an element of $q_i^{1+\lambda} M[X_i]$ that we write in the form

$$y = \sum_{k=0}^{-\lambda-1} m_k X_i^k,$$

where $m_0, \ldots, m_{-\lambda-1} \in M$. Then we have

$$A_M(y) = \sum_{k=0}^{-\lambda-1} x_{n+r+1}^k 1_{i,\beta} \otimes_{H_\beta} m_k.$$

So

$$\begin{aligned}\overline{A}_M(y) &= \sum_{k=0}^{-\lambda-1} x_{n+r+1}^k 1_{i,\beta} \otimes_{H_\beta^i} m_k \\ &= \sum_{k=0}^{-\lambda-1} x_{n+r+1}^k \eta_M(m_k).\end{aligned}$$

Hence, $\overline{A}_M$ is exactly the component of ρ_λ on $[-\lambda]_i M$. So ρ_λ is an isomorphism. $\square$

Proposition 4.3.11. *Let $n \in \mathbb{N}$ and let $\nu \in (I \setminus \{i\})^r$ for some $r \in \mathbb{N}$. Let M be a subquotient of $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$. Denote by $\beta \in Q^+$ the weight of M and let $\lambda = \langle i^\vee, \beta \rangle$. Assume that $\lambda \geqslant 0$. Then ρ_λ is an isomorphism.*

Proof. Given Lemma 4.3.6 and Corollary 4.3.9, the short exact sequence (4.3.2) becomes

$$0 \to [\lambda]_i M \xrightarrow{\delta} \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \xrightarrow{\sigma} \mathrm{ad}_{F_i}\mathrm{ad}_{E_i}(M) \to 0,$$

where δ is the connecting map given by the Snake Lemma. Let $\Gamma : \mathrm{ad}_{E_i}\mathrm{ad}_{F_i}(M) \to [\lambda]_i M$ be the component of ρ_λ on $[\lambda]_i M$. To prove that ρ_λ is an isomorphism, it suffices to prove that $\Gamma\delta$ is an isomorphism. To this end, we prove that there exists an invertible element $b \in K$

such that for all $m \in M$ and $k \in \{0, \ldots, \lambda-1\}$, we have

$$\Gamma\delta\left(mX_i^k\right) = bmX_i^{\lambda-1-k} + \sum_{\lambda-1-k<\ell\leqslant\lambda-1} MX_i^\ell. \tag{4.3.4}$$

By naturality, it suffices to prove this for $M = \mathrm{ad}_{E_i}^{(n)}(E_\nu)$. In this case, M is cyclic, generated by the class c of $\tau_{\omega_0[r+1,n+r]}1_{ni,\nu} \in E_i^{(n)}E_\nu$. Thus, it suffices to treat the case $m = c$. We start by computing $\delta(cX_i^k)$, which we do by chasing the diagram (4.3.1). We resume the computations done in the proof of Proposition 4.3.7. Recall that $cX_i^k = S_M(c_k)$ with $c_k = x_{n+r+1}^k\tau_{[n+r\downarrow 1]}1_{\beta,i}\otimes_{H_\beta} c$. We proved that $\tau_{E_i,M}(c_k) = \tau_{\omega_0[r+1,r+n]}c'_k\otimes_{H_\beta} x_{[r+1,r+n]}c$, where

$$\begin{aligned}
c'_k = &\sum_{\ell\leqslant n-\lambda} p_\ell\tau_{[r+1\uparrow n+r]}x_{n+r}^{k+\ell}e_{[r+1,n+r]}1_{(n+1)i,\nu}\\
&+\sum_{\ell\leqslant n-\lambda} p_\ell\partial_{[r+1\uparrow n+r]}\left(x_{n+r+1}^{k+\ell}\right)e_{[r+1,n+r]}1_{(n+1)i,\nu},
\end{aligned}$$

with $p_\ell \in K[x_1,\ldots,x_{n+r}]$ and $p_{n-\lambda}\in K^\times$. Since $cX_i^k \in \ker(\Phi_M)$, the second sum vanishes (this is also a straightforward consequence of the fact that $k \leqslant \lambda-1$). Thus we have

$$\tau_{E_i,M}(c_k) = T_M\left(\sum_{\ell\leqslant n-\lambda} p_\ell\tau_{[r+1\uparrow n+r-1]}1_{i,\beta-i}\otimes_{H_{\beta-i}} 1_{i,\beta-i}x_{n+r}^{k+\ell}c\right)$$

and

$$\delta(cX_i^k) = \sum_{\ell\leqslant n-\lambda} p_\ell\tau_{[r+1\uparrow n+r-1]}1_{i,\beta-i}\otimes_{H_{\beta-i}^i} 1_{i,\beta-i}x_{n+r}^{k+\ell}c.$$

So

$$\begin{aligned}
\Gamma\delta(cX_i^k) &= \sum_{\substack{\ell\leqslant n-\lambda\\ u\leqslant\lambda-1}} p_\ell\tau_{[r+1\uparrow n+r-1]}x_{n+r}^{k+\ell+u}cX_i^u\\
&= \sum_{\substack{\ell\leqslant n-\lambda\\ u\leqslant\lambda-1}} p_\ell\partial_{[r+1\uparrow n+r-1]}(x_{n+r}^{k+\ell+u})cX_i^u.
\end{aligned}$$

We have $\partial_{[r+1\uparrow n+r-1]}(x_{n+r}^{k+\ell+u}) = 0$ unless $u \geqslant \lambda - 1 - k$. This implies

$$\Gamma\delta(cX_i^k) = p_{n-\lambda}cX_i^{\lambda-1-k} + \sum_{\lambda-1-k<\ell\leqslant\lambda-1} MX_i^\ell.$$

Since $p_{n-\lambda} \in K^\times$, we have established equation (4.3.4). So $\Gamma\delta$ is an isomorphism and the proof is complete. $\square$

Hence, we have proved that Theorem 4.3.4 holds for any subquotient of a module of the form $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$. To conclude that Theorem 4.3.4 holds for every module in $\mathcal{H}^i$, we apply the following elementary lemma.

Lemma 4.3.12. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, let $G, G' : \mathcal{C} \to \mathcal{D}$ be exact functors, and let $\alpha : G \to G'$ be a natural transformation. Assume that we have a collection $\mathcal{M}$ of objects of $\mathcal{C}$ such that $\mathcal{C}$ is generated by $\mathcal{M}$ under extensions and that α_M is an isomorphism for all objects of $\mathcal{M}$. Then α is an isomorphism.*

By Theorem 4.2.14, subquotients of modules of the form $\mathrm{ad}_{E_i}^{(n)}(E_\nu)$ generate $\mathcal{H}^i$ under extensions. Since Theorem 4.3.4 holds such objects, we conclude that it holds for every module in $\mathcal{H}^i$. This finishes the proof of Theorem 4.3.4.

4.4 Categorification Theorem

4.4.1 Adjoint Action on the Positive Half

Before we state and prove a categorification theorem, we explain the decategorified case. Recall that $\mathbf{f}$ denotes the positive part of the quantum group U, and $\mathbf{f}_{\mathcal{A}}$ denotes the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by $e_j^{(n)}$ for $j \in I$ and $n \in \mathbb{N}$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$.

Definition 4.4.1. Let $\mathbf{f}[i]$ be the subalgebra of $\mathbf{f}$ generated by $\mathrm{ad}_{e_i}^{(n)}(e_j)$ for $j \in I \setminus \{i\}$ and $n \in \mathbb{N}$.

Proposition 4.4.2 ([Lus10, Lemma 38.1.2 and Proposition 38.1.6]). *The subalgebra* $\mathbf{f}[i]$ *contains precisely the elements* $y \in \mathbf{f}$ *such that* $(ze_i, y) = 0$ *for all* $z \in \mathbf{f}$. *Furthermore, we have a decomposition*

$$\mathbf{f} = \bigoplus_{n \in \mathbb{N}} \mathbf{f}[i] e_i^{(n)}.$$

Intuitively, $\mathcal{H}^i$ is a categorical analogue of $\mathbf{f}[i]$. In that light, we have categorified the definition of $\mathbf{f}[i]$ and its properties in Propositions 4.1.4, 4.2.15 and 4.2.16. However, to state a precise categorification result about Grothendieck groups, it is necessary to introduce an integral version of $\mathbf{f}[i]$.

Definition 4.4.3. We denote by $\mathbf{f}_{\mathcal{A}}[i]$ the $\mathcal{A}$-subalgebra of $\mathbf{f}_{\mathcal{A}}$ generated by the elements $\mathrm{ad}_{e_i}^{(n)}\left(e_j^{(m)}\right)$ for all $j \in I \setminus \{i\}$ and $m, n \in \mathbb{N}$.

Lemma 4.4.4. *We have a decomposition*

$$\mathbf{f}_{\mathcal{A}} = \bigoplus_{n \in \mathbb{N}} \mathbf{f}_{\mathcal{A}}[i] e_i^{(n)}.$$

Proof. Denote by $\mathbf{s}$ the right-hand side. It is clear that $\mathbf{s} \subseteq \mathbf{f}_{\mathcal{A}}$. Let us prove the reverse inclusion. We have $e_j^{(n)} \in \mathbf{s}$ for all $j \in I$ and $n \in \mathbb{N}$. Hence it suffices to prove that $\mathbf{s}$ is stable under multiplication, which is equivalent to showing that $e_i^{(n)} \mathbf{f}[i]_{\mathcal{A}} \subseteq \mathbf{s}$ for all $n \in \mathbb{N}$. We prove this by induction on n. The case $n = 0$ is clear. Now observe that

$$e_i^{(n+1)} \mathbf{f}_{\mathcal{A}}[i] \subseteq \mathrm{ad}_{e_i}^{(n+1)}\left(\mathbf{f}_{\mathcal{A}}[i]\right) + \sum_{k \leqslant n} e_i^{(k)} \mathbf{f}_{\mathcal{A}} e_i^{(n+1-k)}.$$

Since $\mathbf{f}_{\mathcal{A}}[i]$ is stable under $\mathrm{ad}_{e_i}^{(n+1)}$ by definition, the result follows. □

From Proposition 4.4.2 and Lemma 4.4.4, we obtain the following alternative description of $\mathbf{f}_{\mathcal{A}}[i]$.

Corollary 4.4.5. *We have* $\mathbf{f}_{\mathcal{A}}[i] = \mathbf{f}_{\mathcal{A}} \cap \mathbf{f}[i]$.

We now relate $\mathbf{f}_{\mathcal{A}}[i]$ to the adjoint action.

Proposition 4.4.6. *The subalgebra $\mathbf{f}_{\mathcal{A}}[i]$ is stable under the adjoint action of $\dot{U}_{i,\mathcal{A}}$. The adjoint action of $\dot{U}_{i,\mathcal{A}}$ on $\mathbf{f}_{\mathcal{A}}[i]$ is integrable and $\mathbf{f}_{\mathcal{A}}[i]$ is maximal for this property.*

Proof. We need to prove that $\mathbf{f}_{\mathcal{A}}[i]$ is stable under $\mathrm{ad}_{e_i}^{(n)}$ and $\mathrm{ad}_{f_i}^{(n)}$. Since ad_{e_i} and ad_{f_i} satisfy formulas (4.2.1), it suffices to prove that $\mathrm{ad}_{e_i}^{(n)}(y), \mathrm{ad}_{f_i}^{(n)}(y) \in \mathbf{f}_{\mathcal{A}}[i]$ for y an algebra generator of $\mathbf{f}_{\mathcal{A}}[i]$.

Let $y = \mathrm{ad}_{e_i}^{(m)}\left(e_j^{(\ell)}\right)$ for $m, \ell \in \mathbb{N}$ and $j \in I \setminus \{i\}$. Then

$$\mathrm{ad}_{e_i}^{(n)}(y) = \left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right]_i \mathrm{ad}_{e_i}^{(n+m)}\left(e_j^{(\ell)}\right) \in \mathbf{f}_{\mathcal{A}}[i].$$

Hence $\mathbf{f}_{\mathcal{A}}[i]$ is stable under $\mathrm{ad}_{e_i}^{(n)}$. To prove that $\mathrm{ad}_{f_i}^{(n)}(y) \in \mathbf{f}_{\mathcal{A}}[i]$, we remark that

$$\mathrm{ad}_{f_i}(e_j) = 0.$$

The commutation relation between $e_i^{(m)}$ and $f_i^{(n)}$ then implies that $\mathrm{ad}_{f_i}^{(n)}(y) \in \mathbf{f}_{\mathcal{A}}[i]$.

Hence we have a representation of $\dot{U}_{i,\mathcal{A}}$ on $\mathbf{f}_{\mathcal{A}}[i]$. Let us check that this representation is integrable. The set of elements $y \in \mathbf{f}_{\mathcal{A}}[i]$ on which ad_{e_i} and ad_{f_i} act nilpotently is an $\dot{U}_{i,\mathcal{A}}$-submodule and also a subalgebra by relations (4.2.1). By relation (4) in Definition 2.2.1, ad_{e_i} and ad_{f_i} act nilpotently on every $e_j^{(m)}$ for $j \in I \setminus \{i\}$ and $m \in \mathbb{N}$. Since these generate $\mathbf{f}_{\mathcal{A}}[i]$ as an algebra and $\dot{U}_{i,\mathcal{A}}$-module, we conclude that the action of $\dot{U}_{i,\mathcal{A}}$ on $\mathbf{f}_{\mathcal{A}}[i]$ is integrable.

Finally, we prove the maximality of $\mathbf{f}_{\mathcal{A}}[i]$ with respect to these properties. Let $y \in \mathbf{f}_{\mathcal{A}}$ be such that $\mathrm{ad}_{e_i}^{m}(y) = 0$ for some $m \in \mathbb{N}$. By Proposition 4.4.4, we can write y in the form

$$y = \sum_{\ell=0}^{N} y_\ell e_i^{(\ell)}$$

for some elements $y_0, \ldots, y_N$ of $\mathbf{f}_{\mathcal{A}}[i]$ with $y_N \neq 0$. Assume $N > 0$. Using equations (4.2.1) and (4.0.3), we see that for all $\ell > 0$ we have

$$\mathrm{ad}_{e_i}^m(y_\ell e_i^{(\ell)}) \in a_\ell y_\ell e_i^{(\ell+m)} + \sum_{k<\ell+m} \mathbf{f}_{\mathcal{A}}[i] e_i^{(k)}$$

for some $a_\ell \in \mathbb{Q}(q)$ non-zero. Hence

$$0 = \mathrm{ad}_{e_i}^m(y) \in a_N y_N e_i^{(N+m)} + \sum_{k<N+m} \mathbf{f}_{\mathcal{A}}[i] e_i^k.$$

By the uniqueness part of Proposition 4.4.4, we have $y_N = 0$, which is a contradiction. So $N = 0$, and the result follows. □

4.4.2 Functorial Resolutions

Recall that there is an isomorphism of $\mathcal{A}$-algebras $\gamma : \mathbf{f}_{\mathcal{A}} \xrightarrow{\sim} K_0(\mathcal{U}^+)$. The functor π_i induces a surjective morphism of $\mathcal{A}$-modules $K_0(\mathcal{U}^+) \twoheadrightarrow K_0(H^i\text{−proj})$. Hence we obtain a surjective morphism of $\mathcal{A}$-modules $\mathbf{f}_{\mathcal{A}} \twoheadrightarrow K_0(H^i\text{−proj})$. We want to find a right inverse of this surjection and identify its image. To do so, we functorially resolve the objects of H^i−proj as H-modules. We start by introducing the category in which our resolutions will lie.

Definition 4.4.7. Let $\mathcal{U}^+[i]$ be the full subcategory of $K^{\mathrm{b}}(\mathcal{U}^+)$ whose objects are the complexes C for which the canonical projection $p_C : C \to \pi_i(C)$ is a quasi-isomorphism.

Proposition 4.4.8. *The subcategory $\mathcal{U}^+[i]$ is thick and stable under left multiplication by E_j for all $j \in I \setminus \{i\}$.*

Proof. It is clear that $\mathcal{U}^+[i]$ is closed under shifts and direct summands. Let $C, D \in \mathcal{U}^+[i]$

and let $f : C \to D$ be a morphism of complexes. We obtain a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "p_C"] & D \arrow[r] \arrow[d, "p_D"] & \mathrm{Cone}(f) \arrow[d, "p_{\mathrm{Cone}(f)}"] \\ \pi_i(C) \arrow[r] & \pi_i(D) \arrow[r] & \mathrm{Cone}(\pi_i(f)) \end{tikzcd}$$

where the two rows give distinguished triangles in $K^{\mathrm{b}}(\mathcal{U}^+)$. Taking the long exact sequences in cohomology, we obtain a commutative diagram of H-modules:

$$\begin{tikzcd} H^k(C) \arrow[r] \arrow[d, "H^k(p_C)"] & H^k(D) \arrow[r] \arrow[d, "H^k(p_D)"] & H^k\left(\mathrm{Cone}(f)\right) \arrow[r] \arrow[d, "H^k\left(p_{\mathrm{Cone}(f)}\right)"] & H^{k+1}(C) \arrow[r] \arrow[d, "H^{k+1}(p_C)"] & H^{k+1}(D) \arrow[d, "H^{k+1}(p_D)"] \\ H^k(\pi_i(C)) \arrow[r] & H^k(\pi_i(D)) \arrow[r] & H^k\left(\mathrm{Cone}(\pi_i(f))\right) \arrow[r] & H^{k+1}(\pi_i(C)) \arrow[r] & H^{k+1}(\pi_i(D)). \end{tikzcd}$$

Since $C, D \in \mathcal{U}^+$, the two leftmost and rightmost vertical arrows are isomorphisms. By the Five Lemma, we deduce that $H^k\left(p_{\mathrm{Cone}(f)}\right)$ is an isomorphism. This proves that $\mathcal{U}^+[i]$ is a thick subcategory of $K^{\mathrm{b}}\left(\mathcal{U}^+\right)$.

Let $j \in I \setminus \{i\}$ and let $C \in \mathcal{U}^+[i]$. Consider the following commutative diagram:

$$\begin{tikzcd} E_jC \arrow[r, "p_{E_jC}"] \arrow[d, "E_jp_C"'] & \pi_i(E_jC) \arrow[d, "\pi_i(E_jp_C)"] \\ E_j\pi_i(C) \arrow[r, "="] & \pi_i(E_j\pi_i(C)). \end{tikzcd}$$

By Lemma 4.1.5, $\pi_i\left(p_{E_j\pi_i(C)}\right)$ is an isomorphism of complexes, and E_jp_C is a quasi-isomorphism since $C \in \mathcal{U}^+[i]$. It follows that p_{E_jC} is a quasi-isomorphism. Hence $\mathcal{U}^+[i]$ is stable under left multiplication by E_j. $\square$

The key ingredient to construct projective resolutions is to define a variant of the adjoint action on $\mathcal{U}^+[i]$.

Definition 4.4.9. For $C \in \mathcal{U}^+[i]$, let $\tilde{\tau}_{E_i,C}$ be the unique element of $\mathrm{End}^\bullet_{K(\mathcal{U}^+)}(CE_i, E_iC)$

such that the following diagram commutes:

$$
\begin{tikzcd}
CE_i \arrow[r, "\tilde{\tau}_{E_i,C}"] \arrow[d, "p_C E_i"] & E_iC \arrow[d, "E_i p_C"] \\
\pi_i(C)E_i \arrow[r, "\tau_{E_i,\pi_i(C)}"'] & E_i\pi_i(C).
\end{tikzcd}
$$

The definition of $\tilde{\tau}_{E_i,C}$ is valid because the vertical arrows in the above diagram are quasi-isomorphisms. For each $C \in \mathcal{U}^+[i]$, we choose a morphism of complexes $\tau_{E_i,C} : CE_i \to E_iC$ lifting $\tilde{\tau}_{E_i,C}$ and fix this choice for the rest of this chapter. Then we define

$$\mathrm{Ad}_{E_i}(C) = \mathrm{Cone}\left(\tau_{E_i,C}\right).$$

Proposition 4.4.10. *Definition 4.4.9 yields a well-defined endofunctor* Ad_{E_i} *of* $\mathcal{U}^+[i]$*. Furthermore, for all* $C \in \mathcal{U}^+[i]$*, there is a quasi-isomorphism* $\mathrm{Ad}_{E_i}(C) \simeq \mathrm{ad}_{E_i}(\pi_i(C))$*.*

Proof. As a complex of projective H-modules, we have $\mathrm{Ad}_{E_i}(C) = CE_i[-1] \oplus E_iC$ with differential $\begin{pmatrix} -d_CE_i & 0 \\ \tau_{E_i,C} & E_id_C \end{pmatrix}$. There is a morphism of complexes $g_C : \mathrm{Ad}_{E_i}(C) \to \mathrm{ad}_{E_i}(\pi_i(C))$ defined by

$$g_C : CE_i[-1] \oplus E_iC \xrightarrow{\begin{bmatrix} 0 & p_{E_i\pi_i(C)} \circ E_ip_C \end{bmatrix}} \pi_i(E_i\pi_i(C)).$$

If fits into a commutative diagram in $K(\mathcal{U}^+)$:

$$
\begin{tikzcd}
CE_i \arrow[r] \arrow[d, "p_C E_i"] & E_iC \arrow[r] \arrow[d, "E_i p_C"] & \mathrm{Ad}_{E_i}(C) \arrow[d, "g_C"] \\
\pi_i(C)E_i \arrow[r] & E_i\pi_i(C) \arrow[r] & \mathrm{ad}_{E_i}(\pi_i(C)).
\end{tikzcd}
$$

The first row is a distinguished triangle by definition of $\mathrm{Ad}_{E_i}(C)$, and the second one is too by Theorem 4.2.1. Since p_CE_i and E_ip_C are quasi-isomorphisms, g_C is a quasi-isomorphism. This proves the second claim of the Proposition.

To prove the first claim, we start by proving that $\mathrm{Ad}_{E_i}(C)$ is indeed an object of $\mathcal{U}^+[i]$.

Observe that there is a commutative diagram:

$$\begin{tikzcd}
& \mathrm{Ad}_{E_i}(C) \arrow[dl, "p_{\mathrm{Ad}_{E_i}(C)}"'] \arrow[dr, "g_C"] & \\
\pi_i(\mathrm{Ad}_{E_i}(C)) \arrow[d, "="'] & & \mathrm{ad}_{E_i}(\pi_i(C)) \arrow[d, "="] \\
\pi_i(E_iC) \arrow[rr, "\pi_i(E_i p_C)"] & & \pi_i(E_i\pi_i(C)).
\end{tikzcd}$$

By Lemma 4.1.5, $\pi_i(E_ip_C)$ is an isomorphism of complexes. Since g_C is a quasi-isomorphism, we deduce that $p_{\mathrm{Ad}_{E_i}(C)}$ is a quasi-isomorphism as well. Hence $\mathrm{Ad}_{E_i}(C) \in \mathcal{U}^+[i]$.

We now define Ad_{E_i} on morphisms. Given a morphism $f : C \to D$ in $\mathcal{U}^+[i]$, we define $\mathrm{Ad}_{E_i}(f)$ to be the unique morphism of $\mathcal{U}^+[i]$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Ad}_{E_i}(C) \arrow[r, "\mathrm{Ad}_{E_i}(f)"] \arrow[d, "g_C"'] & \mathrm{Ad}_{E_i}(D) \arrow[d, "g_D"] \\
\mathrm{ad}_{E_i}(\pi_i(C)) \arrow[r, "\mathrm{ad}_{E_i}(\pi_i(f))"'] & \mathrm{ad}_{E_i}(\pi_i(D)).
\end{tikzcd}$$

This is valid definition since g_C, g_D are quasi-isomorphisms. This concludes the proof. $\square$

We can now define projective resolutions. We define an object $R_\nu \in \mathcal{U}^+[i]$ for all $n \in \mathbb{N}$ and $\nu \in I^n$ by induction on n. We let $R_\varnothing = 1_\varnothing$. If R_ν is defined and $j \in I$, then we define

$$R_{j\nu} = \begin{cases} \mathrm{Ad}_{E_i}(R_\nu) & \text{if } j = i, \\ E_jR_\nu & \text{otherwise.} \end{cases}$$

Proposition 4.4.11. *For all $n \in \mathbb{N}$ and $\nu \in I^n$, $\pi_i(R_\nu) = \pi_i(E_\nu)$ as complexes of H-modules. In particular, R_ν is a resolution of $\pi_i(E_\nu)$ by projective H-modules.*

Proof. We proceed by induction on n. The result is clear if $n = 0$. Assume now that $\pi(R_\nu) = \pi_i(E_\nu)$. Let $j \in I$. If $j \neq i$, then by Lemma 4.1.5 we have isomorphisms of complexes

$$
\begin{aligned}
\pi_i(R_{j\nu}) &\simeq E_j\pi_i(R_\nu) \\
&= E_j\pi_i(E_\nu) \\
&= \pi_i(E_{j\nu}).
\end{aligned}
$$

Assume now that $j = i$. Then $R_{i\nu} = \mathrm{Ad}_{E_i}(R_\nu) = R_\nu E_i[-1] \oplus E_i R_\nu$ as H-modules. It follows that $\pi_i(R_{i\nu}) = \pi_i(E_i R_\nu)$. By Lemma 4.1.5, we deduce that there is an isomorphism of complexes $\pi_i(R_{i\nu}) \simeq \mathrm{ad}_{E_i}(\pi_i(R_\nu))$. By induction, $\pi_i(R_\nu)$ is a complex concentrated in degree 0 and equal to $\pi_i(E_\nu)$. The result follows. □

We can now define a projective resolution functor $\Psi : H^i\text{−proj} \to \mathcal{U}^+[i]$. The objects of H^i−proj are direct sums of direct summands of objects of the form $\pi_i(E_\nu)$ for $\nu \in I^n$ and $n \in \mathbb{N}$. Hence, the value of the additive functor Ψ on objects is uniquely determined by the complexes $\Psi(\pi_i(E_\nu))$. We define

$$\Psi(\pi_i(E_\nu)) = R_\nu.$$

By Lemma 4.4.11, there are canonical isomorphisms

$$\mathrm{Hom}_H\left(\pi_i(E_\nu), \pi_i(E_{\nu'})\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}^+[i]}\left(R_\nu, R_{\nu'}\right).$$

This defines Ψ on morphisms.

Proposition 4.4.12. *The functor Ψ satisfies the following properties.*

1. *For all $M \in H^i$−proj, $\Psi(M)$ is a resolution of M by projective H-modules.*

2. *For all $M \in H^i$−proj, $\Psi\left(\mathrm{ad}_{E_i}(M)\right) = \mathrm{Ad}_{E_i}(\Psi(M))$.*

3. *For all $M, N \in H^i$−proj, there is a homotopy equivalence $\Psi(MN) \simeq \Psi(M)\Psi(N)$.*

Proof. By construction, the first two properties hold for $M = \pi_i(E_\nu)$ for all $\nu \in I^n$. Since such objects generate H^i-proj under direct sums and direct summands, the result follows. For the last property, observe that $\Psi(MN)$ and $\Psi(M)\Psi(N)$ are both projective resolutions of M. Hence they are homotopy equivalent. $\square$

We are now ready to prove a categorification result for the adjoint action.

Theorem 4.4.13. *There is a unique isomorphism of $\mathcal{A}$-algebras*

$$\gamma_i : K_0\left(H^i\text{−proj}\right) \xrightarrow{\sim} \mathbf{f}_{\mathcal{A}}[i]$$

such that

- *for all $j \in I \setminus \{i\}$ and $n \in \mathbb{N}$, $\gamma_i\left(\left[E_j^{(n)}\right]\right) = e_j^{(n)}$,*
- *for all $z \in K_0\left(H^i\text{−proj}\right)$, $\gamma_i\left(\left[\mathrm{ad}_{E_i}\right](z)\right) = \mathrm{ad}_{e_i}(\gamma_i(z))$.*

Proof. Recall that there exists a unique isomorphism of $\mathcal{A}$-algebras $\gamma : \mathbf{f}_{\mathcal{A}} \xrightarrow{\sim} K_0\left(\mathcal{U}^+\right)$ such that $\gamma\left(E_j^{(n)}\right) = \left[E_j^{(n)}\right]$ for all $j \in I$ and $n \in \mathbb{N}$. This induces a surjection

$$[\pi_i] \circ \gamma : \mathbf{f}_{\mathcal{A}} \twoheadrightarrow K_0\left(H^i\text{−proj}\right).$$

We define a right inverse γ_i and prove that its image is $\mathbf{f}_{\mathcal{A}}[i]$. We define γ_i as the composition

$$K_0\left(H^i\text{−proj}\right) \xrightarrow{[\Psi]} K_0^{\Delta}\left(\mathcal{U}^+[i]\right) \to K_0\left(\mathcal{U}^+\right) \xrightarrow{\gamma^{-1}} \mathbf{f}_{\mathcal{A}},$$

where K_0^{Δ} denotes the triangulated Grothendieck group. The morphism $K_0^{\Delta}\left(\mathcal{U}^+[i]\right) \to K_0\left(\mathcal{U}^+\right)$ is the natural morphism of $\mathcal{A}$-algebras taking the class of a complex to the alternating sum of the classes of its components. By Proposition 4.4.12, γ_i is a morphism of $\mathcal{A}$-algebras. By construction of Ψ and Ad_{E_i}, is it clear that γ_i satisfy the two properties of the theorem and that $[\pi_i] \circ \gamma \circ \gamma_i = 1$.

We now prove that the image of γ_i is $\mathbf{f}_{\mathcal{A}}[i]$. Notice first that $\operatorname{im}(\gamma_i)$ is stable under the adjoint action of $U_{i,\mathcal{A}}$. Furthermore, $\operatorname{im}(\gamma_i)$ is integrable for this action by Corollary 4.2.7. By Proposition 4.4.6, this proves that $\operatorname{im}(\gamma_i) \subseteq \mathbf{f}_{\mathcal{A}}[i]$. For the converse inclusion, we have

$$\mathrm{ad}_{e_i}^{(n)}\left(e_j^{(m)}\right) = \gamma_i\left(\left[\mathrm{ad}_{E_i}^{(n)}\left(E_j^{(m)}\right)\right]\right)$$

for all $j \in I \setminus \{i\}$ and $m, n \in \mathbb{N}$. Since such elements generate $\mathbf{f}_{\mathcal{A}}[i]$ as an $\mathcal{A}$-algebra, we obtain $\operatorname{im}(\gamma_i) = \mathbf{f}_{\mathcal{A}}[i]$. $\square$

CHAPTER 5

Categorification of Lusztig's Symmetries

Recall that for a simple root $i \in I$, there is an automorphism θ_i of the quantum group U. For $j \neq i$, this automorphism is defined by the formula

$$\theta_i(e_j) = \mathrm{ad}_{e_i}^{(-c_{i,j})}(e_j)\,.$$

In this chapter, we propose a partial categorification of θ_i based on our work on the adjoint action in Chapter 4. Recall that in Section 4.4, we defined complexes of objects of $\mathcal{U}^+$ denoted $\mathrm{Ad}_{E_i}^{(m)}(E_j)$ and decategorifying to $\mathrm{ad}_{e_i}^{(m)}(e_j)$. The goal of this chapter is to construct a monoidal functor T_i such that $T_i(E_j) = \mathrm{Ad}_{E_i}^{(-c_{i,j})}(E_j)$.

The complexes $\mathrm{Ad}_{E_i}^{m}(E_j)$ were constructed abstractly by taking projective resolutions of the objects of H^i-proj. In Section 5.1, we first give an explicit construction of the complexes $\mathrm{Ad}_{E_i}^{m}(E_j)$ and $\mathrm{Ad}_{E_i}^{(m)}(E_j)$. In Section 5.2, we use these complexes to define the functor T_i. Proving that T_i is a well-defined monoidal functor involves checking several relations, which we do in Section 5.3.

5.1 Projective Resolutions

In U, there are simple formulas for $\mathrm{ad}_{e_i}^n$ and $\mathrm{ad}_{e_i}^{(n)}$ in the form of alternating sums. For all $y \in U$ of weight $\beta \in Q^+$, we have

$$\begin{aligned}
\mathrm{ad}_{e_i}^n(y) &= \sum_{k=0}^{n} (-1)^k q_i^{-k(n-1+\langle i^\vee,\beta\rangle)} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_i e_i^{n-k} y e_i^k, \\
\mathrm{ad}_{e_i}^{(n)}(y) &= \sum_{k=0}^{n} (-1)^k q_i^{-k(n-1+\langle i^\vee,\beta\rangle)} e_i^{(n-k)} y e_i^{(k)}.
\end{aligned} \tag{5.1.1}$$

These formulas can be easily obtained by induction. Therefore, it is natural to expect that for $M \in H_\beta^i-$proj, the complexes $\mathrm{Ad}_{E_i}^n(M)$ and $\mathrm{Ad}_{E_i}^{(n)}(M)$ have the form

$$\mathrm{Ad}_{E_i}^n(M) =$$

$$0 \to q_i^{-n(n-1+\langle i^\vee,\beta\rangle)} M E_i^n \to \cdots \to q_i^{-k(n-1+\langle i^\vee,\beta\rangle)} \begin{bmatrix} n \\ k \end{bmatrix}_i E_i^{n-k} M E_i^k \to \cdots \to E_i^n M \to 0,$$

$$\mathrm{Ad}_{E_i}^{(n)}(M) =$$

$$0 \to q_i^{-n(n-1+\langle i^\vee,\beta\rangle)} M E_i^{(n)} \to \cdots \to q_i^{-k(n-1+\langle i^\vee,\beta\rangle)} E_i^{(n-k)} M E_i^{(k)} \to \cdots \to E_i^{(n)} M \to 0.$$

In this section, we define such complexes $\mathrm{Ad}_{E_i}^n(M)$ and $\mathrm{Ad}_{E_i}^{(n)}(M)$. We prove that their cohomology is concentrated in top degree, and is equal to $\mathrm{ad}_{E_i}^n(M)$ and $\mathrm{ad}_{E_i}^{(n)}(M)$ respectively. This proves that up to homotopy, they coincide with the abstract construction done in Section 4.4.

5.1.1 Adjoint Action as a Complex

We start by constructing the complex $\mathrm{Ad}_{E_i}^n(M)$ for $M \in \mathcal{H}^i$. To do this, we need to express the q_i-binomial coefficients in a combinatorial way. Let $\mathcal{P}_n^k$ be the set of subsets of $\{1, \ldots, n\}$ containing k elements. For $S \in \mathcal{P}_n^k$, we let $\Sigma(S)$ be the sum of the elements of S. Then we have (see [KC02, Theorem 6.1])

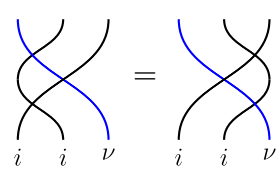
$$\begin{bmatrix} n \\ k \end{bmatrix}_i = q_i^{k(n+1)} \sum_{S \in \mathcal{P}_n^k} q_i^{-2\Sigma(S)}.$$

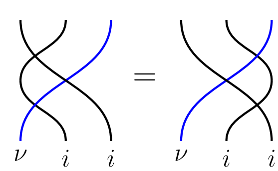
With this expression, we can rewrite $\mathrm{ad}_{e_i}^n(y)$ as

$$\mathrm{ad}_{e_i}^n(y) = \sum_{k=0}^{n} (-1)^k q_i^{-k\langle i^\vee, \beta\rangle} \left(\sum_{S \in \mathcal{P}_n^k} q_i^{-2(\Sigma(S)-k)} \right) e_i^{n-k} y e_i^k. \tag{5.1.2}$$

We construct a complex which categorifies this alternating sum. The construction is based on the following elementary lemma, which follows immediately from the relations of the quiver Hecke algebra.

Lemma 5.1.1. *Let $i \in I$ and let $\nu \in I^m$ for some $m \in \mathbb{N}$. Then in H_{m+2} we have the relations*





Definition 5.1.2. Let $M \in H_\beta^i$–mod, let $n \in \mathbb{N}$ and put $\lambda = \langle i^\vee, \beta\rangle$. We define a complex

$\mathrm{Ad}^n_{E_i}(M)$ of $H_{\beta+ni}$-modules of the form

$$0 \to q_i^{-n(\lambda+n-1)} M E_i^n \to \cdots \to \bigoplus_{S \in \mathcal{P}_n^k} q_i^{-k\lambda-2(\Sigma(S)-k)} E_i^{n-k} M E_i^k \to \cdots \to E_i^n M \to 0$$

where $E_i^n M$ is in cohomological degree 0. To define the differential of $\mathrm{Ad}^n_{E_i}(M)$, we first define a map

$$d_{S,S'} : q_i^{-k\lambda-2(\Sigma(S)-k)} E_i^{n-k} M E_i^k \to q_i^{-(k-1)\lambda-2(\Sigma(S')-k+1)} E_i^{n+1-k} M E_i^{k-1}$$

for any $S \in \mathcal{P}_n^k$ and $S' \in \mathcal{P}_n^{k-1}$. The definition of $d_{S,S'}$ is as follows.

- If S' is not a subset of S, then $d_{S,S'} = 0$.
- If $S' = S \setminus \{\ell\}$, let a be the number of elements of S which are greater than ℓ. Then we define

$$\begin{aligned} d_{S,S'} &= (-1)^a \left(\tau_{[1\uparrow \ell-1-a]} M E_i^{k-1}\right) \circ \left(E_i^{n-k} \tau_{E_i,M} E_i^{k-1}\right) \circ \left(E_i^{n-k} M \tau_{[k-a\uparrow k-1]}\right) \\ &= \begin{cases} E_i^{n-k} M E_i^k & \to \quad E_i^{n-k+1} M E_i^{k-1}, \\ 1_{(n-k)i,\beta,ki} \otimes_{H_\beta} m & \mapsto \quad (-1)^a \tau_{[k-a\uparrow k-a+\ell+|\beta|]} 1_{(n-k+1)i,\beta,(k-1)i} \otimes_{H_\beta} m. \end{cases} \end{aligned}$$

  Notice that $d_{S,S'}$ has indeed the required degree $-d_i(\lambda + 2(\ell - 1))$.

Then, the differential of $\mathrm{Ad}^n_{E_i}(M)$ is defined to be the direct sum of all maps $d_{S,S'}$. This ends the definition of $\mathrm{Ad}^n_{E_i}(M)$.

It can be checked directly using Lemma 5.1.1 that the differential we defined on $\mathrm{Ad}^n_{E_i}(M)$ indeed squares to 0. Alternatively, we can also notice that $\mathrm{Ad}^n_{E_i}(M)$ can be constructed inductively, as the following proposition explains.

Proposition 5.1.3. *Let $M \in H^i_\beta\text{-mod}$ and let $n \in \mathbb{N}$. There is a morphism of complexes*

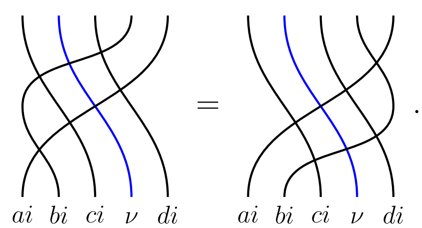
$$\tau_{E_i,\mathrm{Ad}^n_{E_i}(M)} : q_i^{-\langle i^\vee,\beta\rangle-2n}\mathrm{Ad}^n_{E_i}(M)E_i \to E_i\mathrm{Ad}^n_{E_i}(M)$$

defined on the component of cohomological degree $-k$ as the direct sum over $S \in \mathcal{P}^k_n$ of the maps

$$\begin{aligned}\tau_S &= \left(\tau_{[1\uparrow n-k]}ME_i^k\right) \circ \left(E_i^{n-k}\tau_{E_i,M}E_i^k\right) \circ \left(E_i^{n-k}M\tau_{[1\uparrow k]}\right)\\ &= \begin{cases} E_i^{n-k}ME_i^{k+1} & \to \quad E_i^{n-k+1}ME_i^k,\\ 1_{(n-k)i,\beta,(k+1)i} \otimes_{H_\beta} m & \mapsto \quad \tau_{[1\uparrow n+|\beta|]}1_{(n-k+1)i,\beta,ki} \otimes_{H_\beta} m.\end{cases}\end{aligned}$$

The cone of this morphism is $\mathrm{Ad}^{n+1}_{E_i}(M)$.

Proof. By Lemma 5.1.1, for all $a,b,c,d \in \mathbb{N}$ and $\nu \in I^m$ we have



It follows immediately that $\tau_{E_i,\mathrm{Ad}^n_{E_i}(M)}$ is a morphism of complexes. We now need to check that the cone of this morphism is $\mathrm{Ad}^{n+1}_{E_i}(M)$. Let $\lambda = \langle i^\vee, \beta\rangle$. The term of the cone in cohomological degree $-k$ is given by

$$\begin{aligned}&q_i^{-\lambda-2n}\left(\bigoplus_{S\in\mathcal{P}^{k-1}_n} q_i^{-(k-1)\lambda-2(\Sigma(S)-k+1)}E_i^{n+1-k}ME_i^k\right) \oplus \left(\bigoplus_{S\in\mathcal{P}^k_n} q_i^{-k\lambda-2(\Sigma(S)-k)}E_i^{n-k}ME_i^k\right)\\ &= \left(\bigoplus_{S\in\mathcal{P}^{k-1}_n} q_i^{-k\lambda-2(\Sigma(S)+n+1-k)}E_i^{n+1-k}ME_i^k\right) \oplus \left(\bigoplus_{S\in\mathcal{P}^k_n} q_i^{-k\lambda-2(\Sigma(S)-k)}E_i^{n-k}ME_i^k\right).\end{aligned}$$

The terms of the first sum can be indexed by the $S \in \mathcal{P}^k_{n+1}$ such that $n+1 \in S$, and the terms of the second sum by the $S \in \mathcal{P}^k_{n+1}$ such that $n+1 \notin S$. With this indexation, we see that we indeed get the term of cohomological degree $-k$ of $\mathrm{Ad}^{n+1}_{E_i}(M)$. Finally, we check that the differential of the cone matches that of $\mathrm{Ad}^{n+1}_{E_i}(M)$. Let $S \in \mathcal{P}^k_{n+1}$ and $S' \in \mathcal{P}^{k-1}_{n+1}$. With the parametrization of the terms of the cone given above, we let $d'_{S,S'}$ be the differential of the cone from the term indexed by S to the term indexed by S'. By definition of the cone, we have

$$d'_{S,S'} = \begin{cases} \tau_S = d_{S,S\setminus\{n+1\}} & \text{if } n+1 \in S \text{ and } S' = S \setminus \{n+1\}, \\ 0 & \text{if } n+1 \in S \text{ and } S' \neq S \setminus \{n+1\}, \\ E_i d_{S,S'} & \text{if } n+1 \notin S, S', \\ -d_{S\setminus\{n+1\},S'\setminus\{n+1\}} E_i & \text{if } n+1 \in S, S'. \end{cases}$$

From this, we see easily that $d'_{S,S'} = d_{S,S'}$. This concludes the proof. $\square$

We can now prove the main result regarding the cohomology of $\mathrm{Ad}^n_{E_i}$.

Theorem 5.1.4. *Let $M \in H^i_\beta\mathrm{-mod}$ and let $n \in \mathbb{N}$. Then*

$$H^k(\mathrm{Ad}^n_{E_i}(M)) \simeq \begin{cases} \mathrm{ad}^n_{E_i}(M) & \textit{if } k = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We proceed by induction on n. The result is clear if $n = 0$. Assume that the result is established for $\mathrm{Ad}^n_{E_i}(M)$. By Proposition 5.1.3 there is a distinguished triangle

$$q_i^{-\langle i^\vee,\beta\rangle-2n}\mathrm{Ad}^n_{E_i}(M)E_i \to E_i\mathrm{Ad}^n_{E_i}(M) \to \mathrm{Ad}^{n+1}_{E_i}(M) \xrightarrow{[1]} .$$

It gives rise to a long exact sequence in cohomology

$$\cdots \to q_i^{-\langle i^\vee,\beta\rangle-2n}H^k(\mathrm{Ad}^n_{E_i}(M))E_i \to E_iH^k(\mathrm{Ad}^n_{E_i}(M)) \to H^k(\mathrm{Ad}^{n+1}_{E_i}(M)) \to \cdots .$$

We deduce from this exact sequence that $H^k(\mathrm{Ad}^{n+1}_{E_i}(M)) = 0$ if $k \notin \{0, -1\}$, and that there is an exact sequence

$$0 \to H^{-1}(\mathrm{Ad}^{n+1}_{E_i}(M)) \to q_i^{-\langle i^\vee,\beta\rangle - 2n}\mathrm{ad}^n_{E_i}(M)E_i \to E_i\mathrm{ad}^n_{E_i}(M) \to H^0(\mathrm{Ad}^{n+1}_{E_i}(M)) \to 0.$$

From the definition of the map $q_i^{-\langle i^\vee,\beta\rangle-2n}\mathrm{Ad}^n_{E_i}(M)E_i \to E_i\mathrm{Ad}^n_{E_i}(M)$ in Proposition 5.1.3, we see that the induced map $q_i^{-\langle i^\vee,\beta\rangle-2n}\mathrm{ad}^n_{E_i}(M)E_i \to E_i\mathrm{ad}^n_{E_i}(M)$ is simply $\tau_{E_i,\mathrm{ad}^n_{E_i}(M)}$. By Theorem 4.2.1, this map is injective with cokernel $\mathrm{ad}^{n+1}_{E_i}(M)$. The result follows. $\square$

5.1.2 Divided Powers Complex

We now introduce a similar complex for the divided powers.

Definition 5.1.5. Let $M \in H^i_\beta-$mod with β of height r, and let $n \in \mathbb{N}$. We define a complex $\mathrm{Ad}^{(n)}_{E_i}(M)$ of the form

$$0 \to q_i^{-n(n+\lambda-1)}ME_i^{(n)} \to \cdots \to q_i^{-k(n+\lambda-1)}E_i^{(n-k)}ME_i^{(k)} \to \cdots \to E_i^{(n)}M \to 0$$

where $E_i^{(n)}M$ is in cohomological degree 0 and $\lambda = \langle i^\vee, \beta\rangle$. The differential of $\mathrm{Ad}^{(n)}_{E_i}(M)$ is defined by

$$d_k = (-1)^{k-1}\left(\tau_{[1\uparrow n-k]}ME_i^{k-1}\right) \circ \left(E_i^{n-k}\tau_{E_i,M}E_i^{k-1}\right).$$

More explicitly, given $m \in M$, put

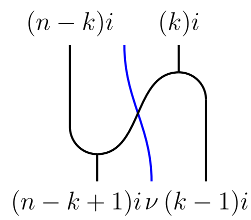
$$c^k_m = \tau_{\omega_0[k+r,n+r]}\tau_{\omega_0[1,k]}1_{(n-k)i,\beta,ki} \otimes_{H_\beta} m \in E_i^{(n-k)}ME_i^{(k)}.$$

Such elements c^k_m generate $E_i^{(n-k)}ME_i^{(k)}$ when m ranges in M. Then the morphism d_k is given by

$$d_k : \begin{cases} q_i^{-k(n+\lambda-1)} E_i^{(n-k)} M E_i^{(k)} & \to \quad q_i^{-(k-1)(n+\lambda-1)} E_i^{(n-k+1)} M E_i^{(k-1)}, \\ c_m^k & \mapsto \quad (-1)^{k-1} \tau_{[1\uparrow k+r]} c_m^{k+1}. \end{cases}$$

With thick diagrammatic calculus notation, the differential is given by

$$d_k = (-1)^{k-1}$$



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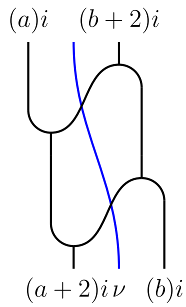
Let us explain why this is well-defined. First, by Lemma 5.1.1, we have

$$\tau_{\omega_0[k+r,n+r]}\tau_{\omega_0[1,k]}\tau_{[k\uparrow n+r]}1_{(n-k+1)i,\beta,(k-1)i} = \tau_{[1\uparrow k+r-1]}\tau_{\omega_0[k-1+r,n+n]}\tau_{\omega_0[1,k-1]}1_{(n-k+1)i,\beta,(k-1)i},$$

which proves that d_k indeed takes values in the summand $E_i^{(n-k+1)}ME_i^{(k-1)}$ of $E_i^{n-k+1}ME_i^{k-1}$. Furthermore, in $H_{\beta+ni}1_{(n-k+2)i,\beta,(k-2)i}$ we have

$$\begin{aligned} \tau_{\omega_0[k+r,n+r]}\tau_{\omega_0[1,k]}\tau_{[k\uparrow n+r]}\tau_{[k-1\uparrow n+r]} &= \tau_{\omega_0[k+r,n+r]}\tau_{\omega_0[1,k]}\tau_{k-1}\tau_{[k\uparrow n+r]}\tau_{[k-1\uparrow n+r-1]} \\ &= 0, \end{aligned}$$

where the first equality is obtained by using Lemma 5.1.1. Hence $d_{k-1}d_k = 0$, and we have indeed defined a complex. Alternatively, it is possible to check this directly using diagrammatic calculus by remarking that for all $a, b \in \mathbb{N}$ and $\nu \in I^m$ we have



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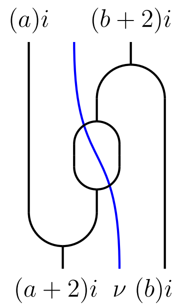
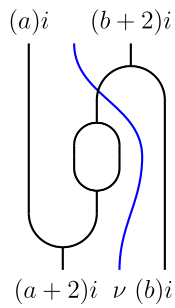


(Lemma 2.5.2)

=



(Lemma 2.5.3)

$= 0$ (Lemma 2.5.5).

We now show, as in Theorem 5.1.4, that the cohomology of $\mathrm{Ad}_{E_i}^{(n)}(M)$ is concentrated in degree zero.

Theorem 5.1.6. *Let $M \in \mathcal{H}^i$ and let $n \in \mathbb{N}$. Then*

$$H^k(\mathrm{Ad}_{E_i}^{(n)}(M)) = \begin{cases} \mathrm{ad}_{E_i}^{(n)}(M) & \textit{if } k = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Assume that $M \in H_\beta$−mod with $\beta \in Q^+$ of height r. The proof is by induction on n. The result is clear if $n = 0$. Assume the result proved for some $n \in \mathbb{N}$. There is a morphism of complexes

$$\tau_{E_i, \mathrm{Ad}_{E_i}^{(n)}(M)} : q_i^{\langle i^\vee, \beta \rangle + 2n} \mathrm{Ad}_{E_i}^{(n)}(M) E_i \to E_i \mathrm{Ad}_{E_i}^{(n)}(M)$$

defined as in Proposition 5.1.3. Let Υ be the cone of $\tau_{E_i, \mathrm{Ad}_{E_i}^{(n)}(M)}$. By the same argument as the proof of Theorem 5.1.4, the cohomology of Υ is concentrated in degree zero and equal to $\mathrm{ad}_{E_i}(\mathrm{ad}^{(n)}_{E_i}(M))$.

We now prove that $\mathrm{Ad}^{(n+1)}_{E_i}(M)$ is a direct summand of Υ. First, we define a morphism of complexes $V : \mathrm{Ad}^{(n+1)}_{E_i}(M) \to \Upsilon$. On the component in cohomological degree $-k$, V is defined by

$$V^{-k} = \begin{bmatrix} \mathrm{incl} \\ 0 \end{bmatrix} : E_i^{(n+1-k)} M E_i^{(k)} \to E_i^{(n+1-k)} M E_i^{(k-1)} E_i \oplus E_i E_i^{(n-k)} M E_i^{(k)}.$$

The restriction of $\tau_{E_i, \mathrm{Ad}^{(n)}_{E_i}(M)}$ to $E_i^{(n+1-k)} M E_i^{(k)}$ is zero, and it follows from that fact that V is a morphism of complexes.

We define a left inverse to V. We will denote by h_d the complete symmetric polynomial of degree d. Recall the idempotent $e_n = x_2 \dots x_n^{n-1} \tau_{\omega_0[1,n]} \in H_{ni}$. We define new idempotents f_k, f'_k in $H_{\beta+(n+1)i}$ by

$$\begin{aligned}
f_k &= \left(e_{n+1-k} \diamond 1_\beta \diamond e_k\right), \\
f'_k &= (-1)^{k-1} h_{k-1}(x_1, x_{k+r+1}, \dots, x_{n+r+1}) \left(e_{n+1-k} \diamond 1_\beta \diamond \left(e_{[2,k]} \tau_{[1\uparrow k-1]}\right)\right).
\end{aligned}$$

Note that $f_k f'_k = f_k$ and $f'_k f_k = f'_k$. We have a morphism

$$\begin{cases} E_i^{n+1-k} M E_i^k & \to \quad E_i^{(n+1-k)} M E_i^{(k)}, \\ 1_{(n+1-k)i,\beta,ki} \otimes_{H_\beta} m & \mapsto \quad f_k \otimes_{H_\beta} m, \end{cases}$$

that we will just denote f_k, and similarly for f'_k. Using these idempotents, we define a morphism $W : \Upsilon \to \mathrm{Ad}^{(n+1)}_{E_i}(M)$. On the component in cohomological degree $-k$, W is defined by

$$W^{-k} = \begin{bmatrix} f'_k & x^k_{n+r+1} f_k \end{bmatrix} : E_i^{(n+1-k)} M E_i^{(k-1)} E_i \oplus E_i E_i^{(n-k)} M E_i^{(k)} \to E_i^{(n+1-k)} M E_i^{(k)}.$$

It is clear that $WV = 1$, so we just need to check that W is indeed a morphism of complexes. This amounts to checking the following relations in $H_{\beta+(n+1)i}1_{(n+2-k)i,\beta,(k-1)i} \otimes_{H_\beta} H^i_\beta$:

$$x^k_{n+r+1} f_k \tau_{[k\uparrow n+r]} = (1_i \diamond e_{n-k} \diamond 1_\beta \diamond e_k)\, \tau_{[k\uparrow n+r-1]} x^{k-1}_{n+r+1} f_{k-1},$$
$$f'_k \tau_{[k\uparrow n+r]} = (e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i) \left(\tau_{[1\uparrow n+r]} x^{k-1}_{n+r+1} f_{k-1} - \tau_{[k\uparrow n+r]} f'_{k-1}\right).$$

The first relation follows from the following computation

$$\begin{aligned}
&(1_i \diamond e_{n-k} \diamond 1_\beta \diamond e_k)\, \tau_{[k\uparrow n+r-1]} x^{k-1}_{n+r+1} f_{k-1} \\
&= x^{k-1}_{n+r+1} x_{[k+r+1,n+r]} e_{[1,k]} \tau_{\omega_0[k+r+1,n+r]} \tau_{[k\uparrow n+r-1]} (e_{n+2-k} \diamond 1_\beta \diamond e_{k-1}) \\
&= x^{k-1}_{n+r+1} x_{[k+r+1,n+r]} e_{[1,k]} \tau_{[k\uparrow k+r-1]} \tau_{\omega_0[k+r,n+r]} (e_{n+2-k} \diamond 1_\beta \diamond e_{k-1}) \\
&= x^{k-1}_{n+r+1} x_{[k+r+1,n+r]} e_{[1,k]} \tau_{[k\uparrow k+r-1]} x^{n+1-k}_{n+r+1} \left(\tau_{\omega_0[1,n+2-k]} \diamond 1_\beta \diamond e_{k-1}\right) \\
&= x^k_{n+r+1} x_{[k+r+1,n+r+1]} \left(\tau_{\omega_0[1,n+1-k]} \diamond 1_\beta \diamond e_k\right) \tau_{[k\uparrow n+r]} \\
&= x^k_{n+r+1} (e_{n+1-k} \diamond 1_\beta \diamond e_k)\, \tau_{[k\uparrow n+r]}.
\end{aligned}$$

Note that this computation actually holds in $H_{\beta+(n+1)i}1_{(n+2-k)i,\beta,(k-1)i}$, without needing to tensor by H^i_β on the right.

For the second equality, we start by computing $(e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\, \tau_{[1\uparrow n+r]} x^{k-1}_{n+r+1} f_{k-1}$ and $(e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\, \tau_{[k\uparrow n+r]} f'_{k-1}$ separately. We have

$$\begin{aligned}
&(e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\, \tau_{[1\uparrow n+r]} x^{k-1}_{n+r+1} f_{k-1} \\
&= (e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\, \tau_{[1\uparrow n+r]} x^{k-1}_{n+r+1} (e_{n+2-k} \diamond 1_\beta \diamond e_{k-1}) \\
&= \left(x_{[1,n+1-k]} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i\right) \tau_{[1\uparrow k+r-1]} h_{k-1}(x_{k+r}, \ldots, x_{r+n+1}) \left(\tau_{\omega_0[1,n+2-k]} \diamond 1_\beta \diamond e_{k-1}\right) \\
&= (e_{n+1-k} \diamond 1_\beta \diamond e_k \diamond 1_i)\, \tau_{[1\uparrow k+r-1]} h_{k-1}(x_{k+r}, \ldots, x_{r+n+1}) \tau_{[k+r\uparrow n+r]}.
\end{aligned}$$

Now in $H_{\beta+(n+1)i}1_{(n+2-k)i,\beta,(k-1)i} \otimes_{H_\beta} H^i_\beta$ we have

$$\tau_{[k\uparrow k+r-1]}x_{k+r} = x_k\tau_{[k\uparrow k+r-1]}$$

because $\left(\tau_{[k\uparrow k+r-1]}x_{k+r} - x_k\tau_{[k\uparrow k+r-1]}\right)1_{i,\beta,(k-1)i} \in \left(1_{i,\beta-i,i,(k-1)i}\right)$ (this is the same argument that we used to prove that $\tau_{E_i,M}$ is well-defined in Proposition 4.1.12). Hence we conclude that

$$\begin{aligned}&(e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\,\tau_{[1\uparrow n+r]}x^{k-1}_{n+r+1}f_{k-1} \\ &\qquad = (e_{n+1-k} \diamond 1_\beta \diamond e_k \diamond 1_i)\,\tau_{[1\uparrow k-1]}h_{k-1}(x_k, x_{k+r+1} \dots, x_{n+r+1})\tau_{[k\uparrow n+r]}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&(e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\,\tau_{[k\uparrow n+r]}f'_{k-1} \\ &\qquad = (e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i)\,h_{k-2}(x_1, x_k, x_{k+r+1}, \dots, x_{n+r+1})\tau_{[1\uparrow k-2]}\tau_{[k\uparrow n+r]}.\end{aligned}$$

To conclude, we use the following formula in the affine nil Hecke algebra H^0_k:

$$\tau_{\omega_0[2,k]}\tau_{[1\uparrow k-1]}x^a_k = \tau_{\omega_0[2,k]}x^a_1\tau_{[1\uparrow k-1]} + \tau_{\omega_0[2,k]}h_{a-1}(x_1, x_k)\tau_{[1\uparrow k-2]}.$$

This is easily derived by induction on a using the relation $h_a(x_1, x_k) = x^a_1 + x_k h_{a-1}(x_1, x_k)$.

Then we have

$$
\begin{aligned}
&(e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i) \left(\tau_{[1\uparrow n+r]} x_{n+r+1}^{k-1} f_{k-1} - \tau_{[k\uparrow n+r]} f'_{k-1}\right) \\
&= (e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \diamond 1_i) \Big(\tau_{[1\uparrow k-1]} h_{k-1}(x_k, x_{k+r+1} \ldots, x_{n+r+1}) \\
&\quad - h_{k-2}(x_1, x_k, x_{k+r+1}, \ldots, x_{n+r+1}) \tau_{[1\uparrow k-2]}\Big) \tau_{[k\uparrow n+r]} \\
&= (e_{n+1-k} \diamond 1_\beta \diamond e_{k-1} \circ 1_i)\, h_{k-1}(x_1, x_{k+r+1} \ldots, x_{n+r+1}) \tau_{[1\uparrow n+r]} \\
&= f'_k \tau_{[k\uparrow n+r]}.
\end{aligned}
$$

This finishes proving that W is a morphism of complexes.

Thus, $\mathrm{Ad}_{E_i}^{(n+1)}(M)$ is a direct summand of Υ. Hence, its cohomology is concentrated in degree 0. Furthermore, by definition, $H^0(VW)$ is the idempotent projecting on the summand $\mathrm{ad}_{E_i}^{(n+1)}(M)$ of $\mathrm{ad}_{E_i}\left(\mathrm{ad}_{E_i}^{(n)}(M)\right)$. Hence $H^0\left(\mathrm{Ad}_{E_i}^{(n+1)}(M)\right) = \mathrm{ad}_{E_i}^{(n+1)}(M)$. $\square$

5.2 The Functor T_i

Denote by $\mathcal{U}_{I\setminus\{i\}}^+$ the monoidal category from Definition 2.4.1 corresponding to the Cartan datum on the set $I \setminus \{i\}$ induced by the Cartan datum structure on I. For the rest of this chapter, we make the assumption that the invertible scalars t_{j_1,j_2} appearing in the definition of the polynomials Q_{j_1,j_2} are all equal to 1. That is, the polynomials Q_{j_1,j_2} are now assumed to have the form

$$Q_{j_1,j_2}(u,v) \in u^{-c_{j_1,j_2}} + v^{-c_{j_2,j_1}} + \sum_{\substack{s,t>0 \\ d_{j_1}s + d_{j_2}t = -j_1\cdot j_2}} K u^s v^t.$$

Theorem 5.2.1. *There is a monoidal functor $T_i : \mathcal{U}_{I\setminus\{i\}}^+ \to K^{\mathrm{b}}\left(\mathcal{U}^+\right)$ such that*

$$T_i\left(E_j\right) = \mathrm{Ad}_{E_i}^{(-c_{i,j})}\left(E_j\right).$$

This functor decategorifies to Lusztig's symmetry θ_i.

The rest of this section is devoted to defining the functor T_i on the generating morphisms of $\mathcal{U}^+_{I\setminus\{i\}}$. Before doing so, we need to prove some results that we will use in the construction.

5.2.1 Preliminaries

Our construction is based on the following elementary result.

Lemma 5.2.2. *Let A be a K-algebra, and let $C, D \in K^{\leqslant 0}(A{-}\mathrm{proj})$ be such that $H^k(C) = 0$ and $H^k(D) = 0$ if $k < 0$. Then the 0^{th} cohomology functor induces an isomorphism*

$$\mathrm{Hom}_{K(A)}(C, D) \xrightarrow{\sim} \mathrm{Hom}_A\left(H^0(C), H^0(D)\right).$$

In particular, if $f \in \mathrm{Hom}_A\left(C^0, D^0\right)$ is such that $f\left(\mathrm{im}\left(d_C^{-1}\right)\right) \subseteq \mathrm{im}\left(d_D^{-1}\right)$, then there exists a morphism of complexes $\tilde{f} : C \to D$ extending f and unique up to homotopy.

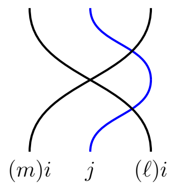
By Theorem 5.1.6, this result applies to any complexes C, D belonging to the additive and monoidal subcategory of $K(\mathcal{U}^+)$ generated by the complexes $\mathrm{Ad}^{(m)}_{E_i}(E_j)$ for $m \in \mathbb{N}$ and $j \in I \setminus \{i\}$. The strategy to define morphisms and check relations is then clear.

- To define a morphism in $\mathrm{Hom}_{K(\mathcal{U}^+)}(C, D)$, it suffices to prescribe a morphism in $\mathrm{Hom}_{\mathcal{U}^+}(C^0, D^0)$ (which we will do using diagrams) and check that this morphism sends $\mathrm{im}\left(d_C^{-1}\right)$ to $\mathrm{im}\left(d_D^{-1}\right)$.

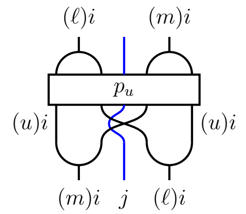
- To prove that two morphisms defined by such a procedure are equal, it suffices to prove the difference between these two has image lying in $\mathrm{im}\left(d_D^{-1}\right)$.

The following lemma will be useful to perform the calculations necessary to implement this strategy.

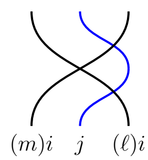
Lemma 5.2.3. *Let $i, j \in I$ and let $m, \ell \in \mathbb{N}$. Then we have*



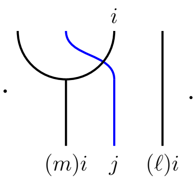
$$= \sum_{u=0}^{\min(m,\ell)}$$



where $p_u \in P_{(m+\ell)i+j}$ has degree $u(i \cdot i)(-c_{i,j} - u)$ and $p_0 = 1$. In particular, if $m > \ell$ then we have

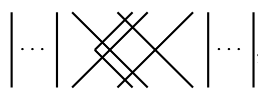


$$\in H \cdot$$

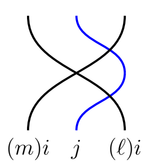


Proof. The minimal length representatives of double cosets $\mathfrak{S}_\ell \times 1 \times \mathfrak{S}_m \backslash \mathfrak{S}_{m+\ell+1} / \mathfrak{S}_m \times 1 \times \mathfrak{S}_\ell$ sending $\ell + 1$ to $m + 1$ have the form:

$$\omega_u =$$



where there are u straight strands on each side, and $u \in \{0, \dots, \min(m, \ell)\}$. Denote by $\underline{\omega_u}$ the reduced decomposition of ω_u determined by the above strand diagram. Notice that



$$\in 1_{\ell i,j,mi} e_{[1,m]} e_{[m+2,m+\ell+1]} H_{(m+\ell)i+j} \tau_{\omega_0[1,\ell]\omega_0[\ell+2,m+\ell+1]} 1_{mi,j,\ell i}$$

$$= \bigoplus_{u=0}^{\min(m,\ell)} 1_{\ell i,j,mi} e_{[1,m]} e_{[m+2,m+\ell+1]} P_{(m+\ell)i+j} \tau_{\underline{\omega_u}} \tau_{\omega_0[1,\ell]\omega_0[\ell+2,m+\ell+1]}.$$

Hence, the first statement follows. The statement about the case $m > \ell$ is an immediate consequence of the first statement. $\square$

5.2.2 Images of generating morphisms

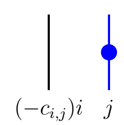
We are now ready to define the images of the generating morphisms of $\mathcal{U}^+$ under T_i.

Proposition 5.2.4. *Let $j \in I \setminus \{i\}$. There exists a unique morphism*

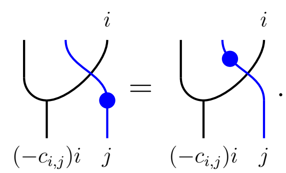
$$T_i(x_j) \in \mathrm{End}^{\bullet}_{K(\mathcal{U}^+)}\left(\mathrm{Ad}^{(-c_{i,j})}_{E_i}(E_j)\right)$$

whose component in cohomological degree 0 is given by

$$T_i(x_j)^0 =$$



Proof. Since $i \neq j$, we have



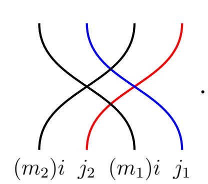
By Lemma 5.2.2, $T_i(x_j)^0$ extends uniquely to an element of $\mathrm{End}^{\bullet}_{K(\mathcal{U}^+)}\left(\mathrm{Ad}^{(-c_{i,j})}_{E_i}(E_j)\right)$. ☐

Proposition 5.2.5. *Let $j_1, j_2 \in I \setminus \{i\}$, and put $m_1 = -c_{i,j_1}$ and $m_2 = -c_{i,j_2}$. Then there exists a unique morphism*

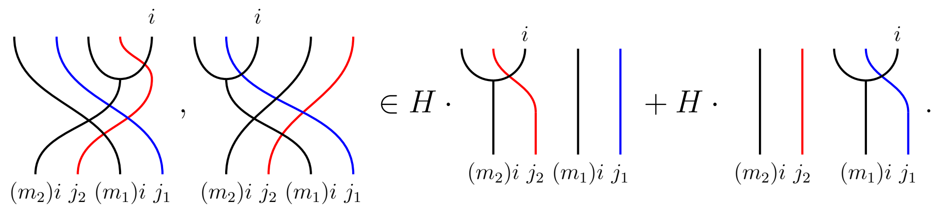
$$T_i(\tau_{j_1,j_2}) \in \mathrm{Hom}^{\bullet}_{K(\mathcal{U}^+)}\left(\mathrm{Ad}^{(m_1)}_{E_i}(E_{j_1})\mathrm{Ad}^{(m_2)}_{E_i}(E_{j_2}),\ \mathrm{Ad}^{(m_2)}_{E_i}(E_{j_2})\mathrm{Ad}^{(m_1)}_{E_i}(E_{j_1})\right).$$

whose component in cohomological degree 0 is given by

$$T_i(\tau_{j_1,j_2})^0 =$$

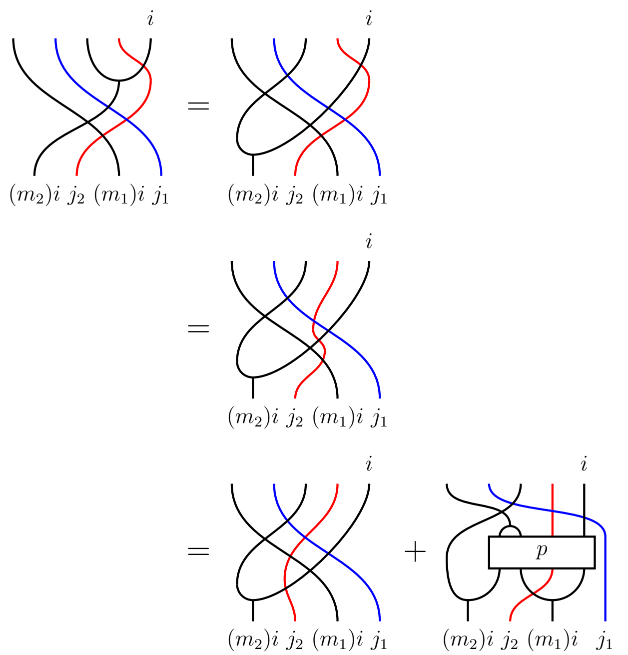


Proof. By Lemma 5.2.2, it suffices to check that



(5.2.1)

Let us start with the first diagram. We have

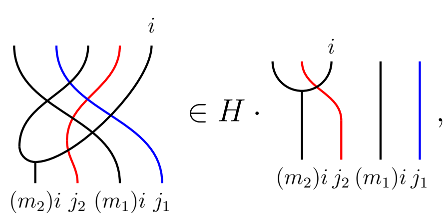


(Lemma 2.5.3)

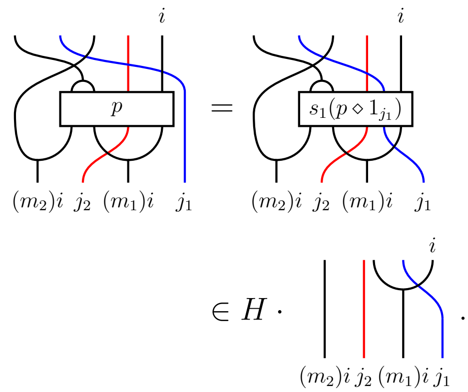
(Lemma 2.5.6)

(Lemma 5.2.3),

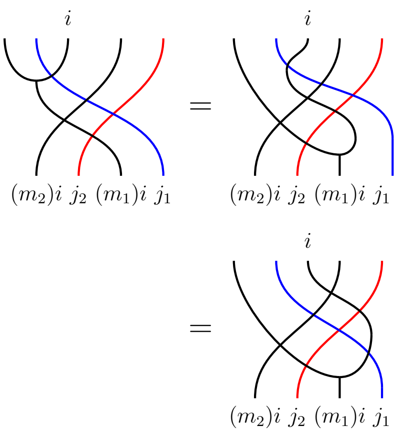
where $p \in P_{(m_1+1)i+j_2}$. We have



and



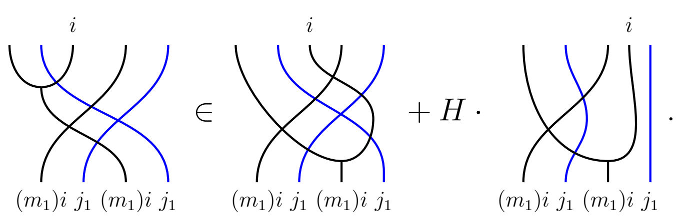
This proves relation (5.2.1) for the first diagram. For the second diagram, we split the proof into two cases. Assume first that $j_1 \neq j_2$. Then we have



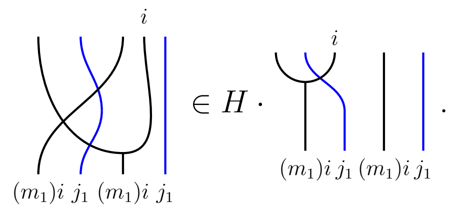
(Lemma 2.5.3)

(relation (8) in Definition 2.3.1).

In the case where $j_1 = j_2$, there is an extra term that appears when applying relation (8) from Definition 2.3.1. This yields



By Lemma 5.2.3, we have



This completes the proof that relation (5.2.1) holds. □

Remark 5.2.6. The morphisms $T_i\left(x_j\right)$ and $T_i\left(\tau_{j_1,j_2}\right)$ have degree $j \cdot j$ and $-j_1 \cdot j_2$ respectively. Hence, T_i will be a graded functor once we finish proving its well-definition.

5.3 Proof of the Relations

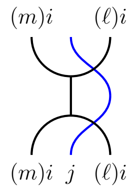
The goal of this section is to prove the following theorem.

Theorem 5.3.1. *The morphisms $\left(T_i(x_j)\right)_{j\neq i}$ and $\left(T_i\left(\tau_{j_1,j_2}\right)\right)_{j_1,j_2\neq i}$ satisfy the defining relations of $\mathcal{U}^+_{I\setminus\{i\}}$, that is:*

1. $T_i\left(\tau_{j_1,j_2}\right)\circ T_i\left(x_{j_1}E_{j_2}\right) - T_i\left(E_{j_2}x_{j_1}\right)\circ T_i\left(\tau_{j_1,j_2}\right) = \delta_{j_1,j_2}$
   $T_i\left(x_{j_2}E_{j_1}\right)\circ T_i(\tau_{j_1,j_2}) - T_i(\tau_{j_1,j_2})\circ T_i\left(E_{j_1}x_{j_2}\right) = \delta_{j_1,j_2}$,
2. $T_i(\tau_{j_2,j_1})\circ T_i(\tau_{j_1,j_2}) = Q_{j_1,j_2}\left(T_i(x_{j_1}), T_i(x_{j_2})\right)$,
3. $T_i\left(\tau_{j_2,j_3}E_{j_1}\right)\circ T_i\left(E_{j_2}\tau_{j_1,j_3}\right)\circ T_i\left(\tau_{j_1,j_2}E_{j_3}\right) - T_i\left(E_{j_3}\tau_{j_1,j_2}\right)\circ T_i\left(\tau_{j_1,j_3}E_{j_2}\right)\circ T_i\left(E_{j_1}\tau_{j_2,j_3}\right)$
   $= \delta_{j_1,j_3}\partial_{1,3}\left(Q_{j_1,j_2}\left(T_i\left(x_{j_1}\right), T_i\left(x_{j_2}\right)\right)\right)$.

The strategy of the proof is to use Lemma 5.2.2. For each relation, we check that the diagrams defining both sides of the equation in cohomological degree 0 are equal modulo diagrams factoring through the differential. We start by proving a preliminary lemma that will be used for the proof of all three relations.

Lemma 5.3.2. *For all $m, \ell \in \mathbb{N}$ and $i, j \in I$ we have*



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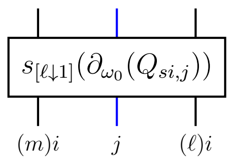
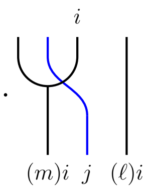


$+ H \cdot$



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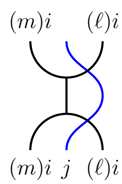
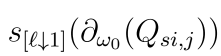
Proof. Let $\omega \in \mathfrak{S}_{m+\ell+1}/\mathfrak{S}_\ell \times 1 \times \mathfrak{S}_m$. We can decompose ω in the form $\omega = \omega_1\omega_2$ with $\omega_1 \in \mathfrak{S}_{m+\ell+1}/\mathfrak{S}_\ell \times \mathfrak{S}_{m+1}$ and $\omega_2 \in \mathfrak{S}_{[\ell+1,m+\ell+1]}/\mathfrak{S}_{[\ell+2,m+\ell+1]}$. Pick a reduced expression $\underline{\omega}$ of ω obtained by concatenating reduced expressions of ω_1 and ω_2. Assume that $\omega(\ell+1) = \ell+1$. If $\omega_2 = 1$, then $\omega_1(\ell+1) = \ell+1$, so $\omega_1 = 1$. Hence, either $\omega = 1$ or $\underline{\omega}$ has the form $\underline{\omega} = (\ldots, s_{\ell+1})$. Now observe that



$$\in \bigoplus_{\substack{\omega \in \mathfrak{S}_{m+\ell+1}/\mathfrak{S}_\ell \times 1 \times \mathfrak{S}_m \\ \omega(\ell+1)=\ell+1}} P_{(m+\ell)i+j}\tau_{\underline{\omega}}\tau_{\omega_0[1,\ell]\omega_0[\ell+2,\ell+m+1]}1_{mi,j,\ell i}.$$

Using the polynomial representation, it is easily seen that the coefficient of $\underline{\omega} = 1$ in this decomposition is



. The result follows. $\square$

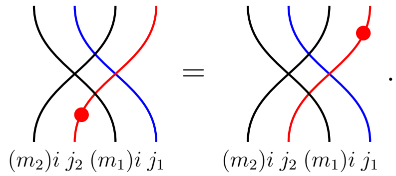
Proposition 5.3.3. *Let $j_1, j_2 \in I \setminus \{i\}$. Then we have*

$$T_i\left(x_{j_2}E_{j_1}\right) \circ T_i(\tau_{j_1,j_2}) - T_i(\tau_{j_1,j_2}) \circ T_i\left(E_{j_2}x_{j_1}\right) = \delta_{j_1,j_2},$$

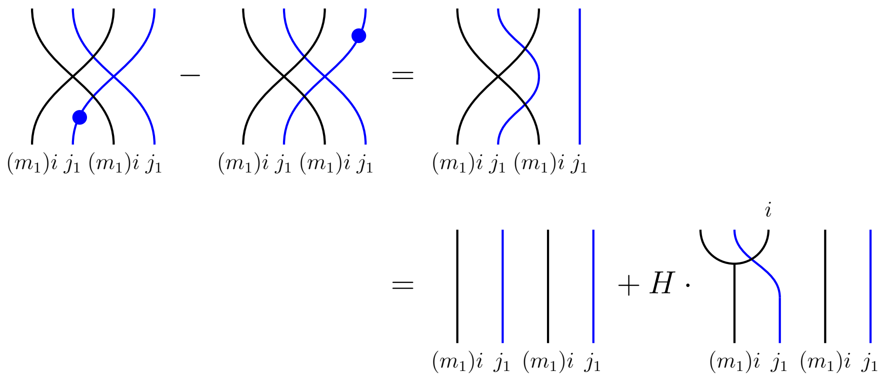
$$T_i\left(\tau_{j_1,j_2}\right) \circ T_i\left(x_{j_1}E_{j_2}\right) - T_i\left(E_{j_2}x_{j_1}\right) \circ T_i\left(\tau_{j_1,j_2}\right) = \delta_{j_1,j_2}.$$

Proof. We give the proof of the first relation, the second one being proved similarly. Let

$m_1 = -c_{i,j_1}$ and $m_2 = -c_{i,j_2}$. Assume first that $j_1 \neq j_2$. Then we have



In the case where $j_1 = j_2$, we have

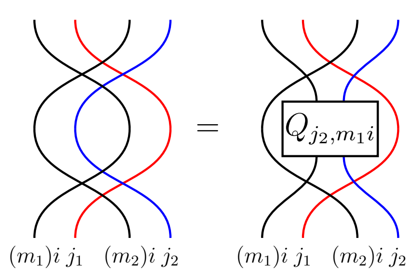


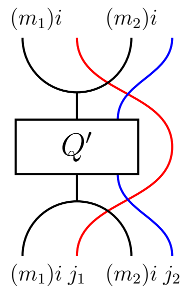
where the second equality follows from Lemma 5.3.2. For both cases, the result then follows from Lemma 5.2.2. □

Proposition 5.3.4. *Let $j_1, j_2 \in I \setminus \{i\}$. Then we have*

$$T_i(\tau_{j_2,j_1}) \circ T_i(\tau_{j_1,j_2}) = Q_{j_1,j_2}\left(T_i(x_{j_1}), T_i(x_{j_2})\right).$$

Proof. Let $m_1 = -c_{i,j_1}$ and $m_2 = -c_{i,j_2}$. By Lemma 2.5.6, we have



$=$ 

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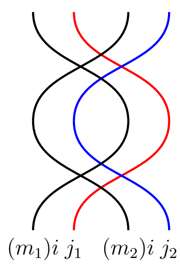
where

$$Q' = \partial_{\omega_0[2,m_2+m_1+1]\omega_0[2,m_1+1]^{-1}\omega_0[m_1+2,m_2+m_1+1]^{-1}}(Q_{j_2,m_1 i}).$$

Given the form of $Q_{j_2,m_1 i}$, we have $Q' = 1$. Thus we have



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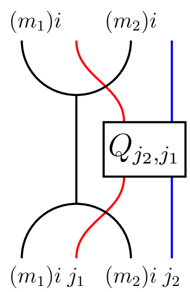
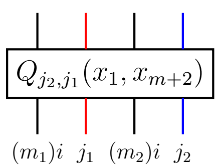
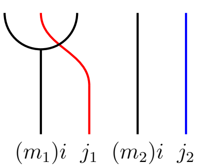


$\in$ 

$+ H \cdot$ 

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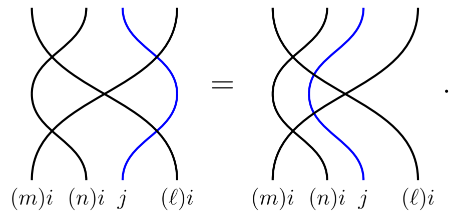
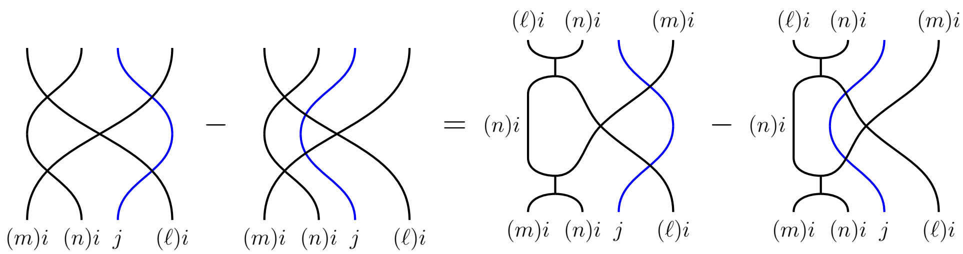
where the last equality follows from Lemma 5.3.2. By Lemma 5.2.2, the result follows. $\square$

To finish proving Theorem 5.3.1, it suffices to prove the braid relations. This will be based on the following lemma.

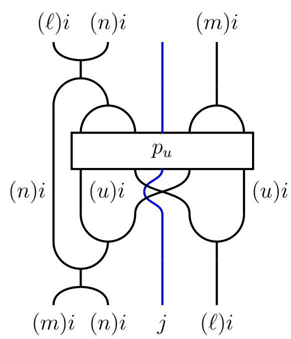
Lemma 5.3.5. *Let $i, j \in I$, let $m, \ell \in \mathbb{N}$, and let $n \geqslant -c_{i,j}$. Then we have*



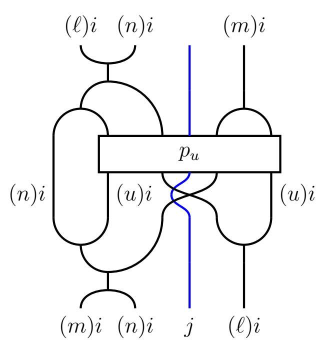
Proof. By Lemma 5.2.3, there exists polynomials $p_u \in P_{(\ell+n)i+j}$ such that



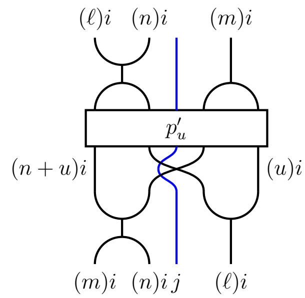
$$= \sum_{u=1}^{\min(m,\ell)}$$



$$= \sum_{u=1}^{\min(m,\ell)}$$



(Lemma 2.5.2)

$$= \sum_{u=1}^{\min(m,\ell)}$$ 

(Lemma 2.5.5),

where $p'_u = \partial_{\omega_u}(p_u)$ with

$$\omega_u = \omega_0[\ell+m+2, \ell+m+n+1]\omega_0[\ell+m-u+2, \ell+m+1]^{-1}\omega_0[\ell+m+2, \ell+m+n+1]^{-1}.$$

For $u \geqslant 1$, the polynomial p'_u has degree

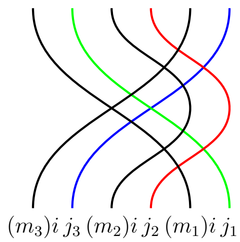
$$\begin{aligned}
\deg(p'_u) &= \deg(p_u) - (i \cdot i)nu \\
&= (i \cdot i)u(-c_{i,j} - u) - (i \cdot i)nu \\
&= u(i \cdot i)(-c_{i,j} - n - u) \\
&\leqslant -(i \cdot i)u^2 \\
&< 0.
\end{aligned}$$

Hence $p'_u = 0$, and the result is proved. $\square$

Proposition 5.3.6. *Let $j_1, j_2, j_3 \in I \setminus \{i\}$. Then we have*

$$T_i\left(\tau_{j_2,j_3}E_{j_1}\right) \circ T_i\left(E_{j_2}\tau_{j_1,j_3}\right) \circ T_i\left(\tau_{j_1,j_2}E_{j_3}\right) - T_i\left(E_{j_3}\tau_{j_1,j_2}\right) \circ T_i\left(\tau_{j_1,j_3}E_{j_2}\right) \circ T_i\left(E_{j_1}\tau_{j_2,j_3}\right)$$
$$= \delta_{j_1,j_3}\partial_{1,3}\left(Q_{j_1,j_2}\right)\left(T_i\left(x_{j_1}\right), T_i\left(x_{j_2}\right)\right).$$

Proof. Put $m_k = -c_{i,j_k}$ for $k \in \{1,2,3\}$. We start by treating the case $j_1 \neq j_3$. Then we have



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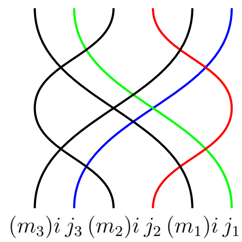

(Lemma 2.5.6)

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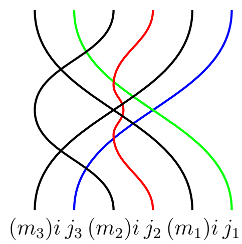

(Lemma 2.5.6)

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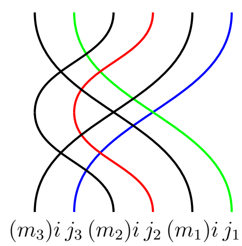
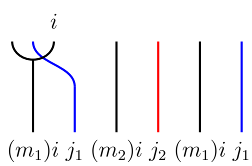


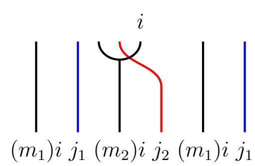
(Lemma 5.3.5).

Then the relation follows from Lemma 5.2.2. For the case $j_1 = j_3$, to simplify the exposition, we will denote by $\sim$ the equivalence relation on H corresponding to the subspace

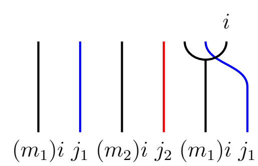
$H \cdot$



$+ H \cdot$

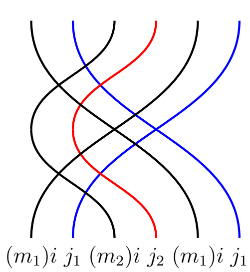


$+ H \cdot$



.

We also put $B(x_1, x_2, x_3) = \partial_{1,3}\left(Q_{j_1,j_2}(x_3, x_2)\right)$. With these notations, we want to prove that



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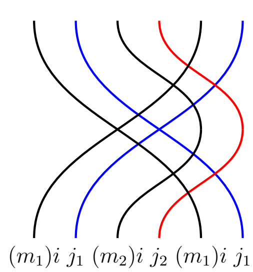
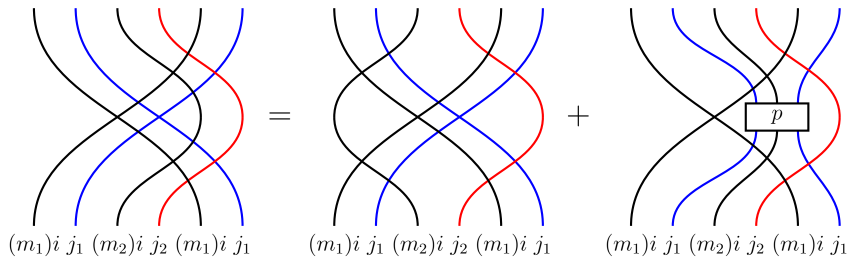
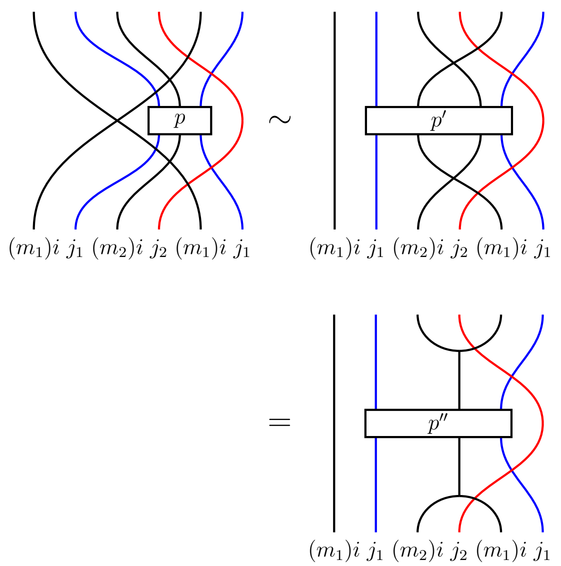


$\sim B(x_1, x_{m_1+2}, x_{m_1+m_2+3}) 1_{m_1 i, j_1, m_2 i, j_2, m_1 i, j_1}.$

By Lemma 2.5.6, we have



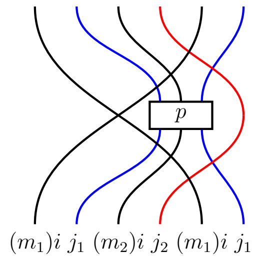
where $p = s_{[m_2+1\downarrow 2]}(\partial_1(Q_{j_1,m_2 i}(x_2,\ldots,x_{m_2+2})))$. By Lemma 5.3.2, we have



where $p' = s_{[m_1+m_2+1\downarrow m_2+2]}(p)$ and $p'' = \partial_{\omega_0[2,m_1+m_2+2]\omega_0[2,m_2+1]^{-1}\omega_0[m_2+2,m_1+m_2+1]^{-1}}(p')$. Notice that p'' has degree:

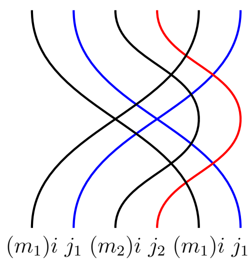
$$\begin{aligned}
\deg(p'') &= \deg(Q_{m_2 i,j_1}) - (j_1\cdot j_1) - m_1 m_2 (i\cdot i)\\
&= -2m_2(i\cdot j_1) - (j_1\cdot j_1) + c_{i,j_1} m_2 (i\cdot i)\\
&= -(j_1\cdot j_1)\\
&< 0.
\end{aligned}$$

Hence $p'' = 0$, and we have

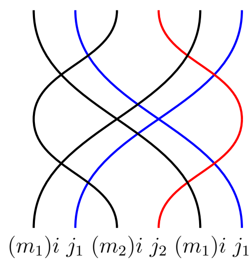


$\sim 0.$

Thus we have



$\sim$



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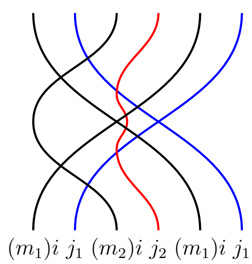
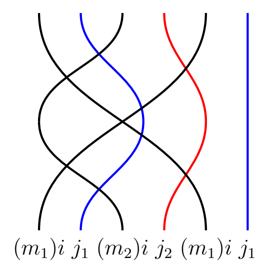


$- B(x_1, x_{m_1+2}, x_{m_1+m_2+3}) \cdot$



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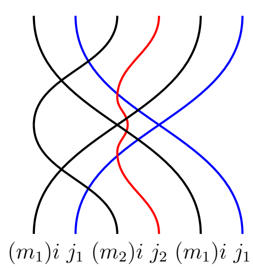
Let us compute these last two summands separately. By Lemma 5.3.5, we have



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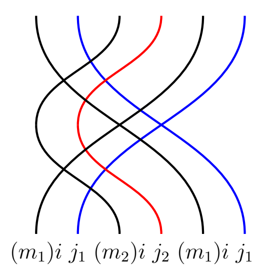

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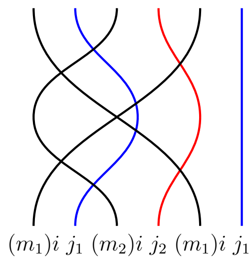
For the other term, we have



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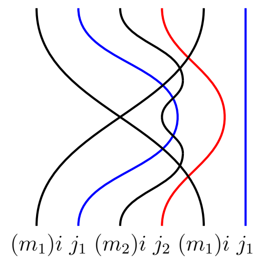

(Lemma 2.5.6)

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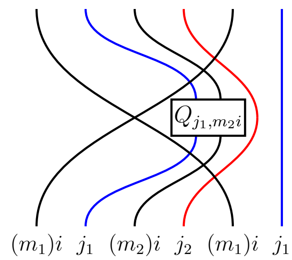

(Lemma 2.5.5)

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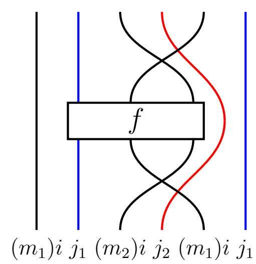

(Lemma 5.3.2)

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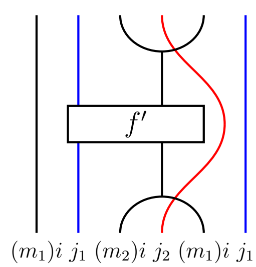
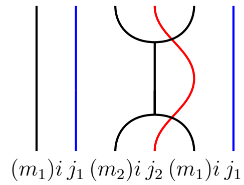


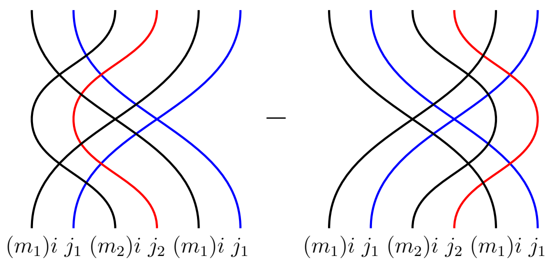
(Lemma 2.5.6),

with $f = s_{[m_1+m_2+1\downarrow m_2+1]}(Q_{j_1,m_2i})$ and $f' = \partial_{\omega_0[1,m_1+m_2]\omega_0[1,m_2]^{-1}\omega_0[m_2+1,m_1+m_2]^{-1}}(f')$. Given the form of Q_{j_1,m_2i}, we have $f' = 1$. Now by Lemma 5.3.2, we have



$$\sim 1.$$

Putting everything together, we have proved that



$$\sim B(x_1, x_{m_1+2}, x_{m_1+m_2+3})1_{m_1i,j_1,m_2i,j_2,m_1i,j_1},$$

which is the relation we wanted. □

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