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Abstract

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Mikio Sato's fundamental idea of viewing objects, *a priori* defined on a space, as living on the cotangent bundle of that space led to the birth of the subject of microlocal analysis and spread to other fields of mathematics. It has been applied to and greatly enriched the theories of D-modules and constructible sheaves in the real or complex analytic context, with important applications to geometric representation theory and much more. In this dissertation, we study étale sheaves in positive characteristic from the microlocal point of view. The main results are: i) generically on a smooth surface, the vanishing cycle form a local system with respect to the variation of transverse test functions in high enough order terms; ii) the vanishing cycle of a tame simple normal crossing sheaf has the same stability as in the complex constructible case; iii) for a monodromic sheaf on a finite dimensional vector space, its characteristic cycle is canonically identified with that of the Fourier transform of the sheaf. In the Introduction, we also discuss the implications of these results in a broader context and an application of iii) to the study of character sheaves in positive characteristic.

To my father and mother

Contents

Acknowledgments

"知與愛永成正比。"
—木心《知與愛》[1]

I would like to express my sincere gratitude to my advisor David Nadler, for his generous and invaluable guidance for me in the world of mathematics through innumerable conversations across the years, and for his warm encouragement and continual support which have a direct effect on me beyond mathematics.

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Owen Barrett showed me Lemma 2.4.11. Bernd Ulrich helped me crucially in the proof of Proposition 2.3.18. Hélène Esnault corrected a statement in the appendix to Chapter 2. Sasha Beilinson pointed out a simpler way to deduce the general case of our main theorem in Chapter 3 from the trivial twist case.

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[1]"Knowing and loving are eternally proportional." This is Mu Xin (木心) paraphrasing the following words of Leonardo da Vinci: "...love of anything is the offspring of knowledge, love being more fervent in proportion as knowledge is more certain; and this certainty springs from a thorough knowledge of all those parts which united compose the whole of that thing which ought to be loved."

Chapter 1

Introduction

This dissertation is a combination of two research works finished during my PhD:

- Chapter 2: *On the stability of vanishing cycles of étale sheaves in positive characteristic* (arXiv:2307.00416)
- Chapter 3: *The Fourier transform and characteristic cycles of monodromic ℓ-adic sheaves* (arXiv:2404.01621)

Each chapter is self-contained. In this introduction, we provide background for these two works and summarise their contents.

In the 1960s, Mikio Sato introduced the microlocal point of view of viewing objects, *a priori* defined on a space, as living on the cotangent bundle of that space. This fundamental idea was initially used to study singularities of solutions of partial differential equations, it then led to the birth of the subject of microlocal analysis and spread to many other fields.

The theory of D-modules grew directly out of this in the 1970s. In the context of analytic geometry over $\mathbf{C}$, the microlocal point of view is concretised by constructions such as the vanishing cycle, singular support (SS), characteristic cycle (CC), the sheaf $\mathcal{E}$ of microdifferential operators, and microlocalisation. In the 1980s, Masaki Kashiwara and Pierre Schapira developed a parallel theory for constructible sheaves on real and complex manifolds (c.f. [KS90]). All the above constructions have their counterparts in the constructible world. Note that, strictly speaking, there is no analogue of the sheaf $\mathcal{E}$. Instead, it is the stack of $\mathcal{E}$-modules that has an exact analogue. We denote this analogue by μsh.

On the other hand, again during the 1970s, a strong analogy was observed between D-modules (over $\mathbf{C}$) and ℓ-adic sheaves (in positive characteristic) (c.f. [DMR07]). We list the analogies in the following table, also including the context of constructible sheaves on complex manifolds (see the appendix to Chapter 2 for more details):

D-modules	ℓ-adic	$\mathbf{C}$-constructible
6-functor formalism	6-functor formalism	6-functor formalism
nearby/vanishing cycles	nearby/vanishing cycles	nearby/vanishing cycles
regular/irregular singularities	tame/wild ramifications	all tame
Fourier transform	Fourier transform	Fourier transform
SS, CC, index formulae	?	SS, CC, index formulae
$\mathcal{E}$-modules, microlocalisation	?	μsh, microlocalisation

Table 1.1: Analogies

Naturally, we have the following

Question 1.0.1. *Is there a microlocal sheaf theory in the ℓ-adic context?*

Around 2015, Alexander Beilinson made a breakthrough by successfully defining the singular support of a $\mathbf{Z}/\ell^r$-sheaf ([Bei16]), subsequently Takeshi Saito defined the characteristic cycle, and index formulae were also proved ([Sai17b]). Their work was then extended to rational coefficients in [UYZ20; Bar23]. At this moment, we still do not know what is the correct analogue of $\mathcal{E}$-modules and microlocalisation in the ℓ-adic context.

We now summarise the content of each chapter. We refer to individual chapters for more detailed introduction, as well as precise definitions of terms used in the following.

Chapter 2 is devoted to studying a simpler question around the microlocalisation of a $\mathbf{Z}/\ell^r$-sheaf: instead of asking what is the microlocalisation, we ask what should be the stalk of the microlocalisation (the microstalk). In the $\mathbf{C}$-constructible context, the answer is the vanishing cycle. More precisely: for a complex analytic manifold X, $\mathcal{F}$ a $\mathbf{C}$-constructible sheaf on X, (x, ξ) a smooth point of $SS\mathcal{F}$, the microstalk of $\mathcal{F}$ at (x, ξ) is defined to be the vanishing cycle $\phi_f(\mathcal{F})_x$, where f is any transverse test function for $\mathcal{F}$ at (x, ξ). This is well-defined by a crucial stability theorem of Kashiwara and Schapira: for a family $\{f_s\}_{s\in T}$ of transverse test functions, $\phi_{f_s}(\mathcal{F})_x$ form a local system as s varies in T. In the positive characteristic context, we first observe that the analogous stability statement is false, due to wild ramifications. This leads to two expectations: i) the analogous statement should be true for tame sheaves constructible with respect to a "tame" stratification, ii) in general, the vanishing cycle should form a local system if $\{f_s\}_{s\in T}$ is a family of transverse test functions such that $f_s \equiv f_{s'} \bmod \mathfrak{m}_x^N$ for any s, s' in T and N large enough. The main theorems of Chapter 2 consist of the affirmation of i) in the case of tame simple normal crossing sheaves, and ii) in the case of a surface (generically), with an explicit bound on N (and we conjecture that the same holds in higher dimensions). Furthermore, we study those sheaves (still in positive characteristic) whose vanishing cycles have the strongest stability (i.e., for which the analogue of Kashiwara and Schapira's stability theorem holds for each smooth pullback,) (tame simple normal crossing sheaves are an example). We show that this class of sheaves has certain functorialities, most notably, they are preserved under the Radon transform. At the end of Chapter 2, we record some explicit computations involving the singular support, which highlight new phenomena in positive characteristic compared to the $\mathbf{C}$-constructible case.

The study in Chapter 2 suggests that, in contrast to the $\mathbf{C}$-constructible case, the microstalks in positive characteristic more naturally live in a higher jet bundle (instead of the cotangent bundle), and that one possible path for microlocalising ℓ-adic sheaves might be to work in higher jet bundles.

Chapter 3 is devoted to studying a specific question concerning the Fourier transform and characteristic cycles. The main theorem is that the characteristic cycle of a monodromic ℓ-adic sheaf on a finite dimensional vector space over an algebraically closed field of positive characteristic is canonically identified with that of the Fourier transform of the sheaf. The exact analogue of this theorem in the D-module case is a result of Jean-Luc Brylinski and Bernard Malgrange in 1986. In studying this question, apart from its own interest, we have in mind an application to geometric representation theory. Namely, in a subsequent paper ([Zho24]), this result will be applied to give the following microlocal characterisation of character (or admissible) sheaves on a reductive Lie algebra in (sufficiently large) positive characteristic: a perverse irreducible G-equivariant $\overline{\mathbf{Q}}_\ell$ sheaf on $\mathfrak{g}$ is a character sheaf if and only if it has nilpotent singular support and is quasi-admissible. This is the analogue of [MV88; Mir04; Psa23].

From another perspective, Chapter 3 is an initial step in understanding how the characteristic cycle changes under the Fourier transform. In the case of a curve, this has been thoroughly studied by Gérard Laumon in his beautiful work [Lau87]. It is very interesting and challenging to understand the higher dimensional situation.

Chapter 2

On the stability of vanishing cycles of étale sheaves in positive characteristic

2.1 Introduction

The microlocal point of view of viewing objects as living on the cotangent bundle rather than the base space was introduced by Sato in the field of partial differential equations. This idea led to the birth of microlocal analysis and spread out to other fields of mathematics. In [KS90] Kashiwara and Schapira systematically developed the theory of sheaves on real and complex analytic manifolds from this point of view.

Question: what does the theory look like for ℓ-adic sheaves on schemes in positive characteristic? Among many other works, we point out: in [Ver83] Verdier defined the specialisation[1], in [AS07] Abbes and Saito defined the characteristic class, and in [KS08] Kato and Saito defined the Swan class. A recent breakthrough is from Beilinson ([Bei16]) who defined singular supports (SS), and based on that Saito ([Sai17b]) defined characteristic cycles (CC).

The starting point of this paper is the following line of thought: apart from SS and CC, another key notion in microlocal sheaf theory is the microstalk, which is to microlocal sheaves as the stalk is to sheaves. In the complex analytic context[2], one definition of the microstalk is via the vanishing cycle functor: for (x, ξ) a *smooth* point of $SS\mathcal{F}$, the microstalk of $\mathcal{F}$ at (x, ξ) is defined to be $\phi_f(\mathcal{F})_x$, where f is any transverse test function at (x, ξ). Here, transverse test function means:

Definition 2.1.1 (transverse test function[3])**.** *A* <u>*transverse test function (ttfun)*</u> *of $\mathcal{F}$ at a* smooth *point (x, ξ) of $SS\mathcal{F}$ is a complex analytic* <u>*function f defined on an open*</u> *neighbour-*

[1] which, however, kills wild ramifications.

[2] We will focus on the complex case in this discussion. The same is true in the real case.

[3] This definition has its obvious analogue in the algebraic context: replace "complex analytic function" by "regular function", and "neighbourhood" means Zariski neighbourhood.

hood U of x such that
i) $f(x) = 0$*;*
ii) Γ_{df} *(the graph of the differential of* f*) intersects* $SS(\mathcal{F}|_U)$ *at* (x, ξ) *transversely.*

The microstalk is well-defined because of the crucial fact that, in the complex analytic context, vanishing cycles have strong stability with respect to the variation of the ttfun. More precisely:

Definition 2.1.2 (transverse test family[4]). *A transverse test family (ttfam) of* $\mathcal{F}$ *at a* smooth *point* (x, ξ) *of* $SS\mathcal{F}$*, denoted by* (T, U, V, f)*, is the following data (here* $\mathbb{A}^1 = \mathbb{C}$ *viewed as a complex manifold):*

$$\begin{tikzcd} U \times T & V \arrow[l, hook'] \arrow[d, "f"] & x_T := x \times T \arrow[l, hook'] \\ & \mathbb{A}^1_T := \mathbb{A}^1 \times T & \end{tikzcd}$$

where:
i) T *is a connected complex manifold, serving as the parameter space of the family. We will often identify* T *with* $0 \times T \subseteq \mathbb{A}^1 \times T$*, and occasionally with* x_T*;*
ii) U *is an open neighbourhood of* x*,* V *is an open of* $U \times T$ *containing* x_T*;*
iii) f *is a complex analytic map such that, for all* $s \in T$*, the s-slice* $f_s : V_s (:= V \times_{\mathbb{A}^1_T} \mathbb{A}^1_s) \to \mathbb{A}^1_s$ *is a ttfun with respect to* $\mathcal{F}$ *at* (x, ξ) *(in particular,* f_s *is* $SS\mathcal{F}$*-transversal except at* x*).*

Theorem 2.1.3. *([KS85, 7.2.4], Theorem 2.2.1) Let* X *be a complex analytic manifold,* $\mathcal{F} \in D(X)$*,* (x, ξ) *a smooth point of* $SS\mathcal{F}$*. Then:*
i) For two ttfun's f, g *of* $\mathcal{F}$ *at* (x, ξ)*, there exists a (noncanonical) isomorphism* $\phi_f(\mathcal{F})_x \cong \phi_g(\mathcal{F})_x$ *in* $D^b_c(\mathbb{C}[\mathbb{Z}])$*.*
ii) For any ttfam (T, U, V, f) *of* $\mathcal{F}$ *at* (x, ξ)*,* $\phi_{pf}(\mathcal{F}_V)$ *is a local system on* $x_T \cong T$*, with stalks at* s *canonically isomorphic to* $\phi_{f_s}(\mathcal{F})_x$*, for all* $s \in T$*. Here* p *is the projection* $\mathbb{A}^1 \times T \to \mathbb{A}^1$*,* $\mathcal{F}_V$ *is the pullback of* $\mathcal{F}$ *to* V*.*

i) says that the vanishing cycles, as vector spaces with monodromy actions, are independent of the choice of the ttfun. ii) is a (stronger) family version of i).

This totally fails in the (positive characteristic) algebraic context because of wild ramifications. Here is an example (see §2.2 for details): consider $\mathbb{A}^2$ over an algebraically closed field of characteristic $p > 3$. Let D be the y-axis and U be the complement. Let $\mathcal{F}$ be the !-extension to $\mathbb{A}^2$ of the Artin-Schreier sheaf on U determined by the equation $t^p - t = y/x^p$.

[4] For the definition of the ttfam in the algebraic context, see Definition 2.3.4.

One can show $SS\mathcal{F} = T^*_X X \cup \langle dy \rangle_D$, where $\langle dy \rangle_D$ denotes the subspace of $D \times_X T^*X$ consisting of covectors proportional to dy. Consider the vanishing cycles with respect to the following two functions: $f_0(x,y) = \frac{y}{1+x}$, $f_1(x,y) = \frac{y}{1+x} + x^3$. It is easily checked that f_0 and f_1 are ttfun's at (a, dy), where a is the origin. However, using a theorem of Deligne-Laumon (Theorem 2.2.5), one can compute: $\dim(\phi_{f_0}(\mathcal{F})_a) = -(p-1)$, while $\dim(\phi_{f_1}(\mathcal{F})_a) = -2$.

Question 2.1.4. *What stability do vanishing cycles have in the algebraic context?*

One expects that simple normal crossing tame sheaves have similar stability as in the complex analytic context. This turns out to be true (see the second theorem below). In general, one expects the dependence of vanishing cycles on the ttfun to be only up to a finite jet (as is suggested by a computation similar to the above with $f_1 = \frac{y}{1+x} + x^N$ for a big N). In the first part of this paper, we show that this is generically true on a smooth surface. This result is inspired by a result of Saito [Sai15, 2.14]. To state it precisely, we introduce the following notion:

Definition 2.1.5 (depth of $\mathcal{F}$)**.** *Let (x,ξ) be a smooth point of $SS\mathcal{F}$. The depth of $\mathcal{F}$ at $\overline{(x,\xi)}$ is the smallest $N \geq 2$ such that $\phi_f(\mathcal{F})$ is a local system for all ttfam $\overline{(T,U,V,f)}$ at (x,ξ) satisfying the following condition: $f_s \equiv f_{s'}$ mod $\mathfrak{m}_x^N$, for all closed points s, s' of T. If such an N does not exist, we say the depth is ∞.*

Here $\phi_f(\mathcal{F})$ is the analogue of $\phi_{pf}(\mathcal{F}_V)$ as in the previous theorem. Its precise definition involves vanishing cycle over general bases, and we refer to Definition 2.3.5 for details. Our main theorem in the first part of the paper is:

Theorem 2.1.6 (Theorems 2.3.10, 2.3.19)**.** *Let X be a smooth surface over an algebraically closed field k of characteristic $p > 2$, $\mathcal{F} \in D_{ctf}(X)$. Then, there exists a Zariski open dense $V = X - \{$finitely many closed points$\}$ and a Zariski open dense $S \subseteq SS(\mathcal{F}|_V)$ such that for any closed point $(x,\xi) \in S$, there exists an integer $N \geq 2$ such that the depth of $\mathcal{F}$ at (x,ξ) is $\leq N$. Moreover, we have an upper bound: if $\mathcal{F}$ is locally constant in some punctured neighbourhood of x, then $N = 2$; if x lies in a ramification divisor of $\mathcal{F}$ (still assuming $(x,\xi) \in S$), then $N \leq 2^{M-1}.i_x.|G| + (2p+1)^M.\max_{\sigma \neq id \in G}\{ep(I_{\sigma,\overline{X}})\}.i_x.|G|$. The terms are explained below.*

We briefly explain the notions in this theorem. See §2.3 for details. Let $\overline{U} \to U$ be the minimal étale Galois covering trivialising $\mathcal{F}$, with Galois group G. Let $\overline{X} \to X$ be the normalisation of X in $\overline{U}$. Then, $I_{\sigma,\overline{X}}$ is the ideal corresponding to the subscheme of fixed points of σ acting on $\overline{X}$; $ep(I_{\sigma,\overline{X}})$ is the smallest $a \in \mathbb{N}$ such that $(\sqrt{I_{\sigma,\overline{X}}})^a \subseteq I_{\sigma,\overline{X}}$; i_x is the intersection number at x of the zero locus of a ttfun at (x,ξ) with the ramification divisor of $\mathcal{F}$ (which is either 1 or 2 for a general $(x,\xi) \in SS\mathcal{F}$); M is a uniform bound for the number of blowups needed to resolve the singularities of the curve $f^{-1}(0) \times_X \overline{X} \hookrightarrow \overline{X}$, as f ranges through all ttfun's at (x,ξ). It is part of the claim that this bound exists, and it can be made explicit (see Proposition 2.3.18).

We outline the proof. By dévissage one reduces to the case of a local system !-extended along a divisor. Using a distinguished triangle of Saito, one can rephrase the local constancy of $\phi_f(\mathcal{F})$ as the pair $(f_T : V_T \to T, \mathcal{F}_T)$ being universally locally acyclic (ULA), where $V_T \to T$ is the family of zero loci of this ttfam, and $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ to V_T. Applying the theorem of Deligne-Laumon, we further translate the ULA condition to the Swan conductor on each fibre being constant along the family. This reduces the question to the stability of Swan conductors of restrictions to curves. Next, we recall the computation of Swan in terms of intersection numbers and representation-theoretic data and reduce the question to a purely geometric one. We analyse the geometric question by studying how various invariants change under blowups. The question is then reduced to bounding the blowup number with respect to the curve. Finally, an argument of Bernd Ulrich utilising Dedekind codifferents shows that this number is indeed bounded.

We conjecture that the theorem holds in greater generality:

Conjecture 2.1.7 (Conjectures 2.3.22, 2.3.23)**.** *Let X be a smooth variety over an algebraically closed field k of characteristic $p \neq 2$. Then $\mathcal{F} \in D(X)$ has finite depth at all smooth points of $SS\mathcal{F}$.*

The second part of this paper studies the class of sheaves whose vanishing cycles have the strongest stability, as well as certain functorialities of the depth. We call a sheaf μc if it has depth 2 at all smooth points in its SS, and μc^s if its smooth pullbacks are all μc (Definition 2.4.1). The stability of vanishing cycles for μc, μc^s sheaves is similar to that in the complex analytic context. More precisely, among other things, we show:

Theorem 2.1.8 (Proposition 2.4.7, Lemma 2.4.3 ii), Corollary 2.4.14)**.**
i) Let X be a smooth variety over an algebraically closed field of characteristic $p > 2$, $D \hookrightarrow X$ be a simple normal crossing divisor (allowed to be empty), $j : U \hookrightarrow X$ be its complement, $\mathcal{F}$ be a local system on U. Then $j_!\mathcal{F}$ is μc^s.
ii) Let X be a smooth variety over an algebraically closed field of characteristic $p > 2$, $\mathcal{F}$ be a μc sheaf on X, and (x, ξ) be a smooth point in $SS\mathcal{F}$. Then for any two ttfun's f, g at (x, ξ), there exists a (noncanonical) isomorphism $\phi_f(\mathcal{F})_x \cong \phi_g(\mathcal{F})_x$ as objects in $D^b_c(\mathbb{Z}/\ell^n)$. We call this the microstalk of $\mathcal{F}$ at (x, ξ).
iii) μc^s sheaves are preserved under the Radon transform. Moreover, their microstalks are invariant under the Radon transform.

We outline the proofs. For i), as for the previous theorem, we first rephrase the question as showing $(f_T : V_T \to T, \mathcal{F}_T)$ being ULA. We then need to understand the singularities of the intersections of the zero loci of ttfun's and the simple normal crossing divisors. We give explicit resolutions of such singularities. Then, $\mathcal{F}_T$ can be written as the pushforward of its pullback via the resolution map π. The map $f_T\pi$ will be transversal to $SS(\pi^*\mathcal{F}_T)$, from which the ULA statement follows easily. ii) follows from the definition of μc sheaves plus the lemma that any two ttfun's can be connected by some ttfam. For iii), the argument is similar to the complex analytic case and essentially reduces to a detailed understanding of the geometry of the Radon transform. Actually, one can show the stability of μc and μc^s sheaves for any proper pushforward which shares similar geometric properties as (the pushforward part of) the Radon transform (Proposition 2.4.13).

In the appendix, we list some analogies and contrasts among several sheaf theories from the microlocal point of view.

Conventions for Chapter 2

All derived categories are in the triangulated sense. All functors are derived. A "sheaf" means an object of $D(X)$ (see below). A "local system" means an object of $D(X)$ whose cohomology sheaves are locally constant constructible.

In the complex analytic context, $D(X)$ denotes $D^b_{\mathbb{C}-c}(X,\mathbb{C})$ in the sense of [KS90, 8.5]. We fix a generator of $\pi_1(\mathbb{C}^\times,1)$ and identify it with $\mathbb{Z}$.

In the algebraic context, we work with varieties (finite type reduced separated schemes over k) over an algebraically closed field k of characteristic $p\geq 0$. $D(X)$ denotes $D^b_c(X,\mathbb{Z}/\ell^n)$ for a fixed prime $l\neq p$. $D_{ctf}(X)\subseteq D(X)$ denotes the full subcategory of objects of finite tor-dimension. A "geometric point" means a map from the Spec of a separably closed field. G_η denotes $\pi_1(\mathbb{A}^1_{k,(0)}-\{0\},\overline{\eta})$, where $\mathbb{A}^1_{k,(0)}$ is the strict henselisation of $\mathbb{A}^1_k$ at the origin, $\overline{\eta}$ is a fixed geometric point over its generic point.

$D^b_c(\mathbb{C})$ (resp. $D^b_c(\mathbb{C}[\mathbb{Z}])$) denotes the triangulated category of bounded complexes of $\mathbb{C}$-vector spaces (resp. $\mathbb{C}[\mathbb{Z}]$-modules) with finite dimensional cohomologies. Similarly for $D^b_c(\mathbb{Z}/\ell^n)$, $D^b_c(\mathbb{Z}/\ell^n[G_\eta])$. For $M\in D^b_{ctf}(\mathbb{Z}/\ell^n[G_\eta])$, *swan* (resp. *dim*) means the swan conductor (resp. the dimension) over $\mathbb{F}_\ell$, after $-\otimes_{\mathbb{Z}/\ell^n}\mathbb{F}_\ell$ (derived tensor).

For f a complex analytic (resp. regular) function on a complex analytic manifold (resp. smooth scheme over k) X, df denotes its differential, Γ_{df} denotes the graph of df in the cotangent bundle T^*X (resp. $T^*X:=T^*(X/k)$). For $f:X\to Y$ a map of complex analytic manifolds (resp. smooth schemes over k), we have the correspondence $T^*X\leftarrow X\times_Y T^*Y\to T^*Y$. df denotes the first map. We refer to [Bei16] for the meaning of $f_\circ C$, $f^\circ C$ and C-transversality for a closed conical subset $C\subseteq T^*X$, and related terminologies. When speaking about points in C, we always mean closed points.

For two subvarieties C,D intersecting at finitely many points in some ambient variety, by $(C.D)$ we mean the intersection (a subscheme), or the intersection number, depending on the context.

2.2 Review

In this section, we review basic microlocal-sheaf-theoretic constructions in both complex analytic and algebraic contexts, and compare them. Except for the definitions of the ttfun and the Radon setup, §2.2 is logically independent of the rest of the paper, but serves as a motivation.

Complex analytic context

The basic reference is [KS90]. Let X be a complex analytic manifold, and $D(X)$ be the triangulated category of bounded $\mathbb{C}$-constructible complexes of sheaves of $\mathbb{C}$-vector spaces. The notion of the singular support (or the microsupport) $SS\mathcal{F}$ is defined for $\mathcal{F} \in D(X)$. It is a half-dimensional $\mathbb{C}^\times$-conic closed *Lagrangian* subset in T^*X which records the codirections in which $\mathcal{F}$ is not locally constant. More precisely, it equals the closure of all $(x, \xi) \in T^*X$ such that there exists some complex analytic function f on some open neighbourhood of x such that the vanishing cycle $\phi_f(\mathcal{F})_x$ is nonzero. $SS\mathcal{F}$ is the 0-th order invariant (the locus) of the "singularities" of $\mathcal{F}$. Clearly, the vanishing cycle, viewed as an object in $D^b_c(\mathbb{C}[\mathbb{Z}])$ (bounded complexes of $\mathbb{C}[\mathbb{Z}]$-modules with finite dimensional cohomologies), is a much finer measurement. However, it depends on the choice of the test function f. It is a crucial fact that, when restricted to transverse test functions, $\phi_f(\mathcal{F})_x$ is essentially independent of f, in the precise sense below.

Theorem 2.2.1. *Let X be a complex analytic manifold, $\mathcal{F} \in D(X)$, (x, ξ) a smooth point of $SS\mathcal{F}$. Then:*
i) For a ttfun f of $\mathcal{F}$ at (x, ξ), $\phi_f(\mathcal{F})_x \in D^b_c(\mathbb{C}[\mathbb{Z}])$ is abstractly independent of f, i.e. for any other ttfun g, there exists a (noncanonical) isomorphism $\phi_f(\mathcal{F})_x \cong \phi_g(\mathcal{F})_x$ in $D^b_c(\mathbb{C}[\mathbb{Z}])$.
ii) For any ttfam (T, U, V, f) of $\mathcal{F}$ at (x, ξ), $\phi_{pf}(\mathcal{F}_V)$ is a local system on $x_T \cong T$, with stalks at s canonically isomorphic to $\phi_{f_s}(\mathcal{F})_x$, for all $s \in T$. Here p is the projection $\mathbb{A}^1 \times T \to \mathbb{A}^1$, $\mathcal{F}_V$ is the pullback of $\mathcal{F}$ to V.

We refer to Definitions 2.1.1, 2.1.2 for the definitions of ttfun and ttfam. In the following discussion, we will need a variant of ttfam: in Definition 2.1.2, as s varies, instead of requiring f_s to be ttfun's at a fixed $\nu_0 = (x, \xi)$, we allow f_s to be ttfun's at $\nu(s) = (x(s), \xi(s))$ for varying smooth points $\nu(s)$ on $SS\mathcal{F}$, and require $\nu(s_0) = \nu_0$ for some s_0. We will call such families weak transverse test families (wttfam) at (x, ξ).

The real analytic counterpart of this result is contained in the statement and proof of [KS85, 7.2.4]. The complex case can be easily deduced from it. We include a proof for completeness.

We refer to [KS90] for details of this paragraph and the paragraph after the next proposition. For a real analytic manifold X, $D(X)$ denotes the triangulated category of bounded $\mathbb{R}$-constructible sheaves of $\mathbb{C}$-vector spaces. For f a real analytic function and $\mathcal{F} \in D(X)$, the vanishing cycle is defined as $\phi_f(\mathcal{F}) = R\Gamma_{\{f\geq 0\}}(\mathcal{F})|_H$, where $H = \{f = 0\}$. If H is smooth, it is also equal to $d_f^*\mu_Y(\mathcal{F})$,[5] where μ_Y is the microlocalisation along Y, d_f is the map $Y \to T_Y^*X$, $y \mapsto (y, df)$. The notions of ttfun, ttfam and wttfam have obvious analogues in the real analytic context.

Proposition 2.2.2. *[KS85, 7.2.4] Let X be a real analytic manifold, $\mathcal{F} \in D(X)$, (x,ξ) a smooth point of $SS\mathcal{F}$. Then for any wttfam (T,U,V,f) of $\mathcal{F}$ at (x,ξ), $\phi_{pf}(\mathcal{F}_V)$ is a local system on $x_T \cong T$, with stalks at s canonically isomorphic to $\phi_{f_s}(\mathcal{F})_x$, for all $s \in T$. Here p is the projection $\mathbb{A}^1 \times T \to \mathbb{A}^1$, $\mathcal{F}_V$ is the pullback of $\mathcal{F}$ to V.*

Proof. The proof in [KS85, 7.2.4] works for $\xi \neq 0$. We sketch the argument. Let $H = \{pf = 0\}, H_s = \{f_s = 0\}$. We have $\phi_{pf}(\mathcal{F}_V) = d_{pf}^*\mu_H(\mathcal{F}_V), \phi_{f_s}(\mathcal{F}) = d_{f_s}^*\mu_{H_s}(\mathcal{F})$. Let $W = SS\mathcal{F}_V \cap T_H^*V = \mathbb{R}_{>0}\{(s,x,d(f_s))\}_{s\in T}, W_s = SS\mathcal{F} \cap T_{H_s}^*V_s = \mathbb{R}_{>0}\{(s,x,d(f_s))\}$. By the estimate of SS of microlocalisations ([KS85, 5.2.1 ii)]), one checks that $SS(\mu_H(\mathcal{F}_V)) \subseteq T_W^*T_H^*X$. This implies $\mu_H(\mathcal{F}_V)|_W$ is locally constant. The first statement follows. By functoriality of the microlocalisation under noncharacteristic pullbacks ([KS85, 5.4.2]), we get $\mu_H(\mathcal{F}_V)|_{W_s} \cong \mu_{H_s}(\mathcal{F})$. The second statement follows.

For $\xi = 0$. Consider the embedding $i : X = X \times \{0\} \hookrightarrow X \times \mathbb{R}$. Let z be the standard coordinate on $\mathbb{R}$. One checks that the family of functions $\{z - f_s\}_{s\in T}$ gives a wttfam for $i_*\mathcal{F}$ at (x,ξ'), where ξ' is any nonzero conormal vector at x of X in $X \times \mathbb{R}$. Then the previous case applies, and the compatibility of vanishing cycles and proper pushforwards implies this case. □

For a complex analytic manifold Y, denote by $Y^{\mathbb{R}}$ the underlying real analytic manifold, there is a canonical identification $(T^*Y)^{\mathbb{R}} = T^*Y^{\mathbb{R}}$ (see, e.g., [KS90, 11.1]). For a complex analytic function h, $(\Gamma_{dh})^{\mathbb{R}} = \Gamma_{\mathrm{Re}(h)}$ under this identification. In particular, if h is a ttfun for some $\mathcal{F}$ then so is $\mathrm{Re}(h)$. Furthermore, by [KS90, 13], for a general h we have a canonical isomorphism $\phi_h \cong \phi_{\mathrm{Re}(h)}|_H$ in $D(H)$, where $H = \{\mathrm{Re}(h) = 0\}$.

Proof of Theorem 2.2.1. ii) is immediate from the above paragraph and Proposition 2.2.2: a ttfam on X induces a ttfam on $X^{\mathbb{R}}$ whose vanishing cycle is a local system with stalks isomorphic to the vanishing cycles on the slices. Transfer back to complex vanishing cycles, we get the result.

i) follows from the following observation: given any ttfam (T,U,V,f) on X, consider the family $((T\times\mathbb{C}^\times)^{\mathbb{R}}, U^{\mathbb{R}}, (V\times\mathbb{C}^\times)^{\mathbb{R}}, g)$ on $X^{\mathbb{R}}$, which on each slice $(s,\lambda) \in (T\times\mathbb{C}^\times)^{\mathbb{R}}$ is given by $g_{(s,\lambda)} = \mathrm{Re}(\lambda f_s)$. One checks this is a wttfam. By Proposition 2.2.2, $\phi_{pg}(\mathcal{F}_{(V\times\mathbb{C}^\times)^{\mathbb{R}}})$ is a local system on $(T\times\mathbb{C}^\times)^{\mathbb{R}}$ with stalks at (s,λ) isomorphic to $\phi_{\mathrm{Re}(\lambda f_s)}(\mathcal{F})_x$. Moreover, $\phi_{\mathrm{Re}(\lambda f_s)}(\mathcal{F})_x$ viewed as a local system with respect to λ (i.e. $\phi_{pg}(\mathcal{F}_{(V\times\mathbb{C}^\times)^{\mathbb{R}}})|_{s\times\mathbb{C}^\times}$) is exactly $\phi_{f_s}(\mathcal{F})_x$ viewed as a local system on $\mathbb{C}^\times$. This implies $\phi_{f_s}(\mathcal{F})_x \cong \phi_{f'_s}(\mathcal{F})_x$ (noncanonically)

[5] We use the notation "$*$" instead of "-1" (as in [KS90]) for the sheaf pullback.

for any $s, s' \in T$.

So, to show i), it suffices to show that any two ttfun's can be connected by a ttfam. This is a simple exercise: fix a coordinate, expand a ttfun in power series, cut off degree ≥ 3 terms with a ttfam[6], then observe that the space of all quadratic terms which makes the function a ttfun is a connected complex manifold. (See proof of Lemma 2.4.3 i) for a detailed argument in the algebraic context.) $\square$

As mentioned in the introduction, Theorem 2.2.1 is a fundamental fact underlying many microlocal-sheaf-theoretic constructions. In particular, to any smooth point (x, ξ) in $SS\mathcal{F}$, this allows us to define the microstalk($\mu stalk$) of $\mathcal{F}$ at (x, ξ): take $\phi_f(\mathcal{F})_x$ for any ttfun f at (x, ξ). It is an object in $D^b_c(\mathbb{C}[\mathbb{Z}])$, independent of f in the sense above.

Another (related) fundamental feature of real and complex analytic microlocal sheaf theory is its invariance under contact transformations, of which the Radon transform is the prototypical case. We will not discuss the full invariance, but focus on one aspect of it: how microstalks change under the Radon transform.

Radon setup 2.2.3 (for both complex analytic and algebraic contexts).

$$\begin{tikzcd} & Q \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P} & & \mathbb{P}^\vee \end{tikzcd}$$

Here $\mathbb{P}$ is the abbreviation for $\mathbb{P}^n$ (over the base field), $\mathbb{P}^\vee$ is its dual, Q is the universal incidence variety. Let $\mathcal{F} \in D(\mathbb{P})$. Its Radon transform is defined as $R\mathcal{F} = q_! p^*\mathcal{F}[n-1]$. Denote by $P(...)$ the projectivisation of $(...)$ after removing the zero section. We have the following facts (see, e.g., [Bry86; Bei16, 1.6, 3.3]):
i) $P(T^*\mathbb{P}) \cong Q \cong P(T^*\mathbb{P}^\vee)$;
ii) If z is a point in Q, $x = p(z), a = q(z)$, let ξ, α be nonzero covectors at x, a which are conormal to the hyperplanes represented by a, x, respectively. Then z is the codirection represented by ξ, α under the identifications in i). Furthermore, $T^*_z Q$ equals the pushout of $T^*_x\mathbb{P}$ and $T^*_a\mathbb{P}^\vee$ along $\langle\xi\rangle$ and $\langle\alpha\rangle$ (via dp_x and dq_a). We say (x, ξ) and (a, α) correspond to each other;
iii) $SS^+(R\mathcal{F}) = q_\circ SS^+(p^*\mathcal{F}) = q_\circ p^\circ SS^+\mathcal{F}$, where $^+$ means adding the zero section. $PSS\mathcal{F} = PSSR\mathcal{F}$ as subvarieties of Q.

Proposition 2.2.4. *Let ν be a smooth point in $PSS\mathcal{F} = PSSR\mathcal{F}$. Then, $\mu stalk(R\mathcal{F})_\nu \cong \mu stalk(\mathcal{F})_\nu$ if n is odd, and $\mu stalk(R\mathcal{F})_\nu \cong \mu stalk(\mathcal{F})_\nu \otimes \mathcal{K}_2$ if n is even. Here $\mathcal{K}_2 \in D^b_c(\mathbb{C}[\mathbb{Z}])$ is the vector space $\mathbb{C}$ concentrated in degree 0, with $1 \in \mathbb{Z}$ acting by multiplication by -1.*

[6] Note that being a ttfun only depends on degree ≤ 2 terms.

In particular, as vector spaces (i.e. as objects in $D^b_c(\mathbb{C})$), microstalks are invariant for all n.

Proof. We will prove a similar result in the algebraic context (Proposition 2.3.2). The same proof plus Theorem 2.2.1 imply the statement here. For Thom-Sebastiani in the complex analytic context, see, e.g., [Sch03, Theorem 1.0.1]. □

Algebraic context

We now consider the algebraic context: let X be a smooth variety over a field k algebraically closed of characteristic $p \geq 0$, and $D(X)$ be the triangulated category of bounded constructible complexes of étale sheaves of $\mathbb{Z}/\ell^n$-modules. The notion of the singular support $SS\mathcal{F}$ is defined for $\mathcal{F} \in D(X)$ ([Bei16]). It is a half-dimensional conic closed subset in T^*X. As in the analytic case, it records the non-locally-acyclic codirections of $\mathcal{F}$, and has a similar description in terms of test functions and vanishing cycles. The definition of the ttfun has its obvious analogue in this context (see the footnote to Definition 2.1.1). (For the ttfam, which is not used in this section, see Definition 2.3.4.)

In the positive characteristic world, in contrast to the above, singular supports need not be Lagrangian[7]. We will later record more new phenomena (§2.4). In this section, we discuss the failure of the analogue of Theorem 2.2.1.

We will use the following result of Deligne-Laumon to compute the dimensions of vanishing cycles ([Lau81, 2.1, 5.1], see [Sai17b, 2.12] for a more general version):

Theorem 2.2.5. *Let S be a Noetherian excellent scheme, $f : X \to S$ a separated smooth morphism of relative dimension 1, Z a closed subscheme of X finite flat over S with a single point in each fibre. Let $\mathcal{F} \in D(X)$ be a !-extension of a tor-finite locally constant sheaf concentrated in degree 0 on $U = X - Z$. Define an $\mathbb{N}$-valued function a_s on the points of S:*

$$a_s := \mathrm{dimtot}((\mathcal{F}|_{X_{\overline{s}}})_{\overline{\eta}_{\overline{z}}}) \tag{2.1}$$

where $\overline{s}$ is a geometric point over s with residue field an algebraic closure of the residue field of s, $\overline{z}$ is a geometric point of Z above $\overline{s}$, $\overline{\eta}_{\overline{z}}$ is a geometric point over the generic point of the strict henselisation of $X_{\overline{s}}$ at $\overline{z}$, dimtot *means* swan + dim *(see Conventions).*
Then:
i) a_s is constructible, and $a_s \leq a_\eta$ if η specialises to s.
ii) $(f,\mathcal{F})$ is universally locally acyclic (ULA) if and only if a_s is locally constant.
iii) If S is an excellent strict henselian trait, denote its closed and generic points by s, η respectively, then

$$a_s - a_\eta = \dim(\phi_f(\mathcal{F}_{\overline{z}})). \tag{2.2}$$

[7] Actually, Deligne ([Del15]) showed that on a smooth surface X, any half-dimensional conic closed subset in T^*X can be generically realised as a component of some $SS\mathcal{F}$.

We now come to examples showing the failure of the analogue of Theorem 2.2.1.

Example 2.2.6. *($p > 2$, $n = 1$ in $\mathbb{Z}/\ell^n$) Let $X = \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$. Fix a nontrivial character $\psi : \mathbb{F}_p \to (\mathbb{Z}/\ell)^\times$.*[8] *Let $\mathcal{F}$ be the Artin-Schreier sheaf determined by the equation $t^p - t = y/x^p$, !-extended along $D = \{x = 0\}$.*[9] *One can show $SS\mathcal{F} = T^*_X X \cup \langle dy\rangle_D$, where $\langle dy\rangle_D$ denotes the subspace of $D \times_X T^*X$ consisting of covectors proportional to dy (see, e.g., [Saito17, 3.6]). Consider the following family of ttfun's at $\nu = ((0, 0), dy)$: $f_s(x, y) := \frac{y}{1+x} + sx^N$, where N is some integer ≥ 3, $s \in k$. It is simple to check that for each fixed s, f_s (restricted to some Zariski neighbourhood of $(0, 0)$) is indeed a ttfun.*

For a fixed s, apply Deligne-Laumon to $f_s : U \to \mathbb{A}^1$, where U is some Zariski open neighbourhood of $(0, 0)$ on which f_s is defined. Let ρ be the standard coordinate on $\mathbb{A}^1$. The fibre $f_s^{-1}(\rho)$ is locally isomorphic to $\mathbb{A}^1$ with x as a coordinate. $\mathcal{F}|_{f_s^{-1}(\rho)}$ is the Artin-Schreier sheaf determined by $t^p - t = (\rho - sx^N)(1 + x)/x^p$ (!-extended at $x = 0$). In formula (2.1), $\dim(\mathcal{F}_{\overline{z}}) = 0, \mathrm{dimtot}((\mathcal{F}|_{X_{\overline{s}}})_{\overline{\eta}_{\overline{z}}}) = 1 + \mathrm{sw}((\mathcal{F}|_{X_{\overline{s}}})_{\overline{\eta}_{\overline{z}}})$. The Swan conductors are easily computed:

$\mathrm{sw}(\rho, s)$	s=0	s generic
ρ=0	0	p-N
ρ generic	p-1	p-1

$3 \leq N < p$

$\mathrm{sw}(\rho, s)$	s=0	s generic
ρ=0	0	0
ρ generic	p-1	p-1

$N \geq p$

By formula (2.2), the dimensions of $\phi_{f_s}(\mathcal{F})$ are as follows:

$\dim(\phi_{f_s}(\mathcal{F}))$	s=0	s generic
$3 \leq N < p$	-(p-1)	-(N-1)
$N \geq p$	-(p-1)	-(p-1)

We see that if $p > 3$ and $3 \leq N < p$, then $\dim(\phi_{f_s}(\mathcal{F}))$ depends on the parameter s. So the analogue of Theorem 2.2.1 is false. Nevertheless, if $N \geq p$, then $\dim(\phi_{f_s}(\mathcal{F}))$ does not depend on s (for s in a small neighbourhood of $0 \in \mathbb{A}^1$). This is a first indication that vanishing cycles depend on the ttfun only up to a finite jet. We will come back to this in §2.3.

In this example, $SS\mathcal{F}$ is not Lagrangian. Does the analogue of Theorem 2.2.1 hold if restricted to sheaves whose SS's are Lagrangian? The answer is no, as the next example shows:

[8] assuming it exists, i.e., $p|(l - 1)$.

[9] This equation determines a finite étale Galois covering of $U = X - \{x = 0\}$, with Galois group $\mathbb{F}_p$, corresponding to a surjection $\pi(U, \overline{\eta}_U) \twoheadrightarrow \mathbb{F}_p$. Compose with ψ gives a representation of $\pi(U, \overline{\eta}_U)$, which is the same thing as a local system on U.

Example 2.2.7. *Same setup and notations as above, but consider the Artin-Schreier sheaf determined by* $t^p - t = y/x^{p-1}$. *One can show* $SS\mathcal{F} = T^*_X X \cup T^*_{(0,0)}X \cup \langle dx \rangle_D$. *Consider the same* ν *and the same family of ttfun's as above.*

The computation is similar, the results are as follows:

$\mathrm{sw}(\rho, s)$	s=0	s generic
ρ=0	0	p-N-1
ρ generic	p-1	p-1

$3 \leq N < p-1$

$\mathrm{sw}(\rho, s)$	s=0	s generic
ρ=0	0	0
ρ generic	p-1	p-1

$N \geq p-1$

$\dim(\phi_{f_s}(\mathcal{F}))$	s=0	s generic
$3 \leq N < p-1$	-(p-1)	-N
$N \geq p-1$	-(p-1)	-(p-1)

We remark that the simplicity of π_1 (in particular, that it splits locally, and that all ramifications are tame) is one fundamental reason why in the analytic context vanishing cycles have strong stability, strong enough that they "live" on the cotangent bundle, leading to fundamental constructions in the microlocal sheaf theory. In the positive characteristic algebraic context, due to the complexity of π_1 (or wild ramifications), the (micro)local data of a sheaf is huge. This is analogous to the distinction between regular holonomic D-modules and general holonomic D-modules. In the appendix, we list some more analogies and distinctions.

2.3 The stability of vanishing cycles

This section is devoted to discussing the stability of vanishing cycles in the positive characteristic algebraic context. In §2.3 we discuss the independence of $\mathrm{dimtot}(\phi)$ with respect to the ttfun. In §2.3 we discuss the independence of ϕ of high jets of the ttfun. We fix the following setup:

X is a smooth variety over a field k algebraically closed of characteristic $p > 2$, $\mathcal{F} \in D(X)$, $(x, \xi) \in SS\mathcal{F}$ a *smooth* (closed) point. Note that in this setup ttfun's at (x, ξ) always exist ([Bei16, 4.12]).

The stability of dimtot(ϕ)

Proposition 2.3.1. *[Sai17b] With the above setup, assume further* $\mathcal{F} \in D_{ctf}(X)$, *then for a ttfun* f, $\mathrm{dimtot}(\phi_f(\mathcal{F})_x)$ *is independent of* f.

To see this is true, just apply the Milnor formula ([Sai17b, 5.9]): for a ttfun f, $\mathrm{dimtot}(\phi_f(\mathcal{F})_x)$ is equal to minus the coefficient of $CC\mathcal{F}$ at x. But logically, this proposition comes before the Milnor formula. Indeed, the very fact that dimtot of vanishing cycles "live" on the cotangent bundle allows one to define the characteristic cycle. See Remark 2.4.4 for a logically direct proof of this proposition.

Proposition 2.3.2. *[Sai17b]* $\mathrm{dimtot}(\phi)$ *is invariant under the Radon transform. More precisely, in the Radon setup 2.2.3, assume further $\mathcal{F} \in D_{ctf}(\mathbb{P})$, let (x, ξ) be a smooth point of $SS\mathcal{F}$ with $\xi \neq 0$. Denote by ν the image of (x, ξ) in $PSS\mathcal{F}$. Let (a, α) be any representative of ν in $T^*\mathbb{P}^\vee$. Let f (resp. g) be any ttfun for $\mathcal{F}$ (resp. $R\mathcal{F}$) at (x, ξ) (resp. (a, α)). Then* $\mathrm{dimtot}(\phi_f(\mathcal{F})_x) = \mathrm{dimtot}(\phi_g(R\mathcal{F})_a)$.

This is a consequence of [Sai17b, 6.4, 6.5]. For later purposes, we record a direct proof in our setting.

Proof. By the compatibility of vanishing cycles with proper pushforwards, $\phi_g(q_! p^*\mathcal{F})_a \cong q_*\phi_{gq}(p^*\mathcal{F})$. By the following Lemma 2.3.3, gq is a ttfun for $p^*\mathcal{F}$ at (z, ζ), where $z = (x, a), \zeta = dq(\alpha)$. Moreover, z is the only point in $q^{-1}(a)$ where $d(gq)$ and $SS(p^*\mathcal{F})$ intersects. So $q_*\phi_{gq}(p^*\mathcal{F}) \cong \phi_{gq}(p^*\mathcal{F})_z$ as objects in $D_c^b(\mathbb{Z}/\ell^n[G_\eta])$ (see Conventions for notations). By Proposition 2.3.1, $\mathrm{dimtot}\phi_{gq}(p^*\mathcal{F})_z$ can be computed by any ttfun for $p^*\mathcal{F}$ at (z, ζ). Use $f + h$ where h is a quadratic function in the fibre direction of p (in a local coordinate). It is an exercise to check that this is a ttfun. Apply Thom-Sebastiani [Ill17; Fu14], the assertion follows. Note the shift is computed as $(n - 1) + (-(n - 2)) + (-1)$, where the first term comes from the definition of the Radon transform, the second and third terms come from Thom-Sebastiani. □

Lemma 2.3.3. *In the Radon setup 2.2.3, let $\mathcal{F} \in D(\mathbb{P})$, (a, α) a smooth point in $SSR\mathcal{F}$, $\alpha \neq 0$, g a ttfun for $R\mathcal{F}$ at (a, α). Then*
i) on $q^{-1}(a)$, $\Gamma_{d(gq)}$ intersects $SS(p^*\mathcal{F})$ only at $(z, \zeta) \in T^*Q$, where z is the point in $Q \cong PT^*\mathbb{P}^\vee$ corresponding to (a, α), and $\zeta = dq(\alpha)$;
ii) the intersection of $\Gamma_{d(gq)}$ and $SS(p^*\mathcal{F})$ at (z, ζ) is transverse.
In particular, gq is a ttfun for $p^*\mathcal{F}$ at (z, ζ).

Proof. i) If (z', ζ') is in the intersection and $z' \in q^{-1}(a)$, because $SS^+(R\mathcal{F}) = q_\circ SS^+(p^*\mathcal{F}) = q_\circ p^\circ SS^+\mathcal{F}$, there must exist an $(x', \xi') \in T^*\mathbb{P}$ such that a) it lies in $SS^+\mathcal{F}$; b) it corresponds to (a, α). But a) forces $x' = p(z')$, $\xi' =$ the unique covector at x' which pulls back under $dp_{x'}$ to ζ'; b) forces $(z', \zeta') = (z, \zeta)$.

Note, actually more is true: if we restrict to a small Zariski neighbourhood V of a on which Γ_{dg} intersects $SS^+(R\mathcal{F})$ only at (a, α), then (z, ζ) is the only point of intersection of $\Gamma_{d(gq)}$ and $SS(p^*\mathcal{F})$ on $q^{-1}(V)$.

ii) Let V be a neighbourhood as above. Consider the correspondence:

$$\begin{tikzcd} & Q \times_V T^*V \arrow[dl, "u"'] \arrow[dr, "v"] & \\ T^*Q & & T^*V \end{tikzcd}$$

Abbreviate $SS(p^*\mathcal{F})$ as C. Abuse notation, denote the restriction of $\Gamma_{dg}, \Gamma_{d(gq)}$ to over V by $\Gamma_{dg}, \Gamma_{d(gq)}$ again. Let u_*, v^* denote the intersection theoretic pushforward and pullback. We claim $\Gamma_{d(gq)} = uv^{-1}\Gamma_{dg} = u_*v^*\Gamma_{dg}$ (i.e. no > 1 multiplicities are introduced in the intersection theoretic pull and push). Assume this for now. Note $\Gamma_{d(gq)}$ intersects C at the single point (z, ζ). We want to compute the intersection number. Since u is a closed immersion and v is proper smooth, u^*C and $v^*\Gamma_{dg}$ also intersect at a single point and, by the projection formula from intersection theory, $C.u_*v^*\Gamma_{dg} = (u^*C).(v^*\Gamma_{dg}) = (v_*u^*C).\Gamma_{dg}$. We claim that $v_*u^*C = vu^{-1}C$. Assuming this, then $vu^{-1}C^+ = q_\circ C^+ = SS^+(R\mathcal{F})$, so $C.u_*v^*\Gamma_{dg} = SSR\mathcal{F}.\Gamma_{d(gq)} = 1$.

It remains to show the two claims. The first claim follows from v being smooth and u being a closed immersion. For the second claim, we show separately below u^* and v_* introduce no > 1 multiplicities:

u^*: This is intersecting C with $Q \times_V T^*V$. Since (a, α) is a smooth point, C is also smooth near (z, ζ). By counting dimensions, it suffices to find $n - 1$ tangent vectors of C which are not tangent to $Q \times_V T^*V$. One verifies that the tangents of C in the p fibre direction work.

v_*: After removing the zero section, u^*C lies in the "diagonal" of $Q \times_V T^*V$, so is mapped isomorphically to its image. More precisely, consider $Q \times_{\mathbb{P}^\vee} T^*\mathbb{P}^\vee \to T^*\mathbb{P}^\vee$ ($Q \times_V T^*V \to T^*V$ is then its base change to V). Note $Q \times_{\mathbb{P}^\vee} (T^*\mathbb{P}^\vee - T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee) \to (T^*\mathbb{P}^\vee - T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee)$ admits a natural $\mathbb{G}_m$ action. Take the quotient (which does not change multiplicity computations), we get $Q \times_{\mathbb{P}^\vee} Q \to Q$ (identifying $(T^*\mathbb{P}^\vee - T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee)/\mathbb{G}_m$ with Q), where the map is just the projection to the second factor. Then, by $C = p^\circ SS\mathcal{F}$ and the description of $T^*_{z'}Q$ in the Radon setup 2.2.3, one checks that $(u^*C - \text{zero section})/\mathbb{G}_m$ lies in the diagonal of $Q \times_{\mathbb{P}^\vee} Q$, so is mapped isomorphically to its image. $\square$

The high-jet stability of ϕ

Return to Examples 2.2.6, 2.2.7: we noticed that when N is large enough, the Swan conductors are independent of s (for s in a small neighbourhood of $0 \in \mathbb{A}^1$), consequently $\dim(\phi)$'s are independent of s. This suggests that the dependence of vanishing cycles on the ttfun is only up to a high enough jet. In this section, we formulate precisely the notion of vanishing cycles being stable with respect to the variation of the ttfun in order $\geq N$-terms and prove such a result in a special case. At the end, we discuss the relation between our result and a result of Saito (whose formulation and proof inspired ours). We are in the setup of §2.3.

Transverse test families

The definition of ttfam in the algebraic context is as follows:

Definition 2.3.4 (transverse test family). *A transverse test family (ttfam) of $\mathcal{F}$ at a* smooth *point (x, ξ) of $SS\mathcal{F}$, denoted by (T, U, V, f), is the following data (here $\mathbb{A}^1 = \mathbb{A}^1_k$):*

$$\begin{tikzcd} U \times T & V \arrow[l, hook'] \arrow[d, "f"] & x_T := x \times T \arrow[l, hook'] \\ & \mathbb{A}^1_T := \mathbb{A}^1 \times T & \end{tikzcd}$$

where:
i) T is a connected smooth finite type scheme over k. We will often identify T with $0 \times T \subseteq \mathbb{A}^1 \times T$, and occasionally with x_T;
ii) U is an étale *neighbourhood of x (x is implicitly viewed as a point of U), V is a Zariski open of $U \times T$ containing x_T;*
iii) f is a morphism such that, for all closed points s of T, the s-slice $f_s : V_s(:= V \times_{\mathbb{A}^1_T} \mathbb{A}^1_s) \to \mathbb{A}^1_s$ is a ttfun with respect to $\mathcal{F}$ at (x, ξ) (base changed to over s) (in particular, f_s is $SS\mathcal{F}$-transversal except at x).

Definition 2.3.5 (the vanishing cycle associated to a ttfam). *Let $\mathcal{F} \in D(X)$ and (T, U, V, f) be a ttfam for $SS\mathcal{F}$ at (x, ξ). The vanishing cycle associated to this ttfam is the following sheaf on $\overleftarrow{T}T := x_T \overleftarrow{\times}_S (S - T)$:*

$$\phi_f(\mathcal{F}) := \Phi_f(\mathcal{F}_V)|_{\overleftarrow{T}T}$$

Here $\mathcal{F}_V$ is $\mathcal{F}$ pulled back to V, $\overleftarrow{\times}$ is the oriented product, Φ is the vanishing cycle over general bases for $f : V \to S$.[10]

Remark 2.3.6. *i) In a ttfam, the condition on f implies that f is $SS\mathcal{F}$-transversal outside x_T (essentially because for $s \hookrightarrow T$ the conormal bundle of $V_s \hookrightarrow V$ is isomorphic to the pullback of conormal bundle of $\mathbb{A}^1_s \hookrightarrow \mathbb{A}^1_T$, see [Sai17b, 2.9.1] for a proof). This implies it is ULA with respect to $\mathcal{F}$ outside x_T.*

ii) Directly from the definition of being $SS\mathcal{F}$-transversal, f is smooth in a neighbourhood of the base of $SS\mathcal{F}_V$ except possibly at points in x_T. If $\xi \neq 0$, then f is also smooth on x_T; if $\xi = 0$, then f is not smooth on x_T, nevertheless it is still flat on x_T because $V \to \mathbb{A}^1_T$ is always a family of hypersurfaces in a neighbourhood of x_T, hence flat there.

[10] We refer to [Org06; Ill17] for basics of oriented products and vanishing cycle over general bases.

iii) Apply [Org06, 6.1] and [Ill17, comments after 1.6.1] we see: $\Phi_f(\mathcal{F}_V)$ is constructible and commutes with any base change. In particular, it is supported on $x_T \overleftarrow{\times}_{\mathbb{A}^1_T} \mathbb{A}^1_T$ and its restriction to each slice equals the usual vanishing cycle, i.e. for any closed point s of T, $\Phi_f(\mathcal{F}_V)|_{V_s \overleftarrow{\times}_{\mathbb{A}^1_T} \mathbb{A}^1_s}$ is supported on $x \overleftarrow{\times}_{\mathbb{A}^1_s} (\mathbb{A}^1_s - 0) \cong \mathbb{A}^1_{s,(0)} - \{0\}$ and canonically isomorphic to $\phi_{f_s}(\mathcal{F})_x$.

iv) Apply [Sai17b, 2.8] to geometric points x of x_T, t of T (identified with $0 \times T \subseteq \mathbb{A}^1_T$) and u of $\mathbb{A}^1_T - T$, we get a distinguished triangle:

$$\Psi_f(\mathcal{F}_V)_{x\leftarrow t} \to \Psi_f(\mathcal{F}_V)_{x\leftarrow u} \to \Phi_f(\mathcal{F}_V)_{t\leftarrow u} \to$$

where the first map is the cospecialisation and the second is the composition of $\Psi_f(\mathcal{F}_V)_{x\leftarrow u} \to \Phi_f(\mathcal{F}_V)_{x\leftarrow u}$ and the cospecialisation $\Phi_f(\mathcal{F}_V)_{x\leftarrow u} \to \Phi_f(\mathcal{F}_V)_{t\leftarrow u}$. Compose this with

$$\mathcal{F}_x \;=\!=\; \mathcal{F}_x \longrightarrow 0 \longrightarrow$$

and take the cone ([BBDG, 1.1.11]) we get

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, equal] \arrow[d] & \mathcal{F}_x \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\Psi_f(\mathcal{F}_V)_{x\leftarrow t} \arrow[r] \arrow[d] & \Psi_f(\mathcal{F}_V)_{x\leftarrow u} \arrow[r] \arrow[d] & \Phi_f(\mathcal{F}_V)_{t\leftarrow u} \arrow[r] \arrow[d, equal] & {} \\
\Phi_f(\mathcal{F}_V)_{x\leftarrow t} \arrow[r] \arrow[d] & \Phi_f(\mathcal{F}_V)_{x\leftarrow u} \arrow[r] \arrow[d] & \Phi_f(\mathcal{F}_V)_{t\leftarrow u} \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where the two maps in the third row are cospecialisations.

v) Later we will often consider the condition of $\Phi_f(\mathcal{F}_V)|_{\overleftarrow{T}T}$ being a local system. Combine iii) and iv), we see: $\Phi_f(\mathcal{F}_V)|_{\overleftarrow{T}T}$ is a local system if and only if in the following diagram, $(f_T, \mathcal{F}_T)$ ($\mathcal{F}_T$ is the base change of $\mathcal{F}$ to V_T) is ULA, which is equivalent to $\Phi_{f_T}(\mathcal{F}_T) = 0$ since Φ commutes with base change by iii) (c.f. [Ill17, Example 1.7 (b)]):

$$\begin{tikzcd}
V_T := V \times_{\mathbb{A}^1_T} T \arrow[d, "f_T"] \\
T
\end{tikzcd} \tag{2.3}$$

If this is satisfied, $\Phi_f(\mathcal{F}_V)$ is automatically supported on $\overleftarrow{T}T$.

Generic finite depth

We are in the setup of §2.3. Recall the following definition in the introduction, which makes precise the notion of the vanishing cycle being dependent on the ttfun up to the N-th jet.

Definition 2.3.7 (depth of $\mathcal{F}$). *Let (x,ξ) be a smooth point of $SS\mathcal{F}$. The depth of $\mathcal{F}$ at (x,ξ) is the smallest $N \geq 2 \in \mathbb{N}$ such that $\phi_f(\mathcal{F})$ is a local system for all ttfam (T,U,V,f) at (x,ξ) satisfying the following condition: $f_s \equiv f_{s'}$ mod $\mathfrak{m}_x^N$, for all closed points s, s' of T. If such an N does not exist, we say the depth is ∞.*

Remark 2.3.8. *i) In the definition, one cannot allow for all functions having an isolated intersection with the SS at (x,ξ). Because the latter can have arbitrary intersection multiplicities*[11]*, and the depth thus defined would be infinity in general.*
ii) depth is étale local.
iii) In the next section, we will study functorialities of the depth as well as sheaves with depth 2.

We record a basic question which we do not know how to answer yet:

Question 2.3.9. *How does depth change under smooth pullbacks?*

Here is the first version of our main result.

Theorem 2.3.10. *Let X be a smooth surface over an algebraically closed field k of characteristic $p > 2$, $x \in D$ be a smooth point of a prime divisor, $U = X - D$, $\mathcal{F} = j_!\mathcal{F}_U$, where $\mathcal{F}_U$ is a tor-finite local system on U concentrated in degree 0, (x,ξ) be a smooth point of $SS\mathcal{F}$ with $\xi \neq 0$. Then $\mathcal{F}$ has finite depth at (x,ξ) if both of the following conditions are satisfied:*

i) $(x,\xi) \in SS\mathcal{F}$ is nonexceptional, in the sense that either it is not *conormal to D, or the component of $SS\mathcal{F}$ it lies in is the conormal of D.*

ii) Let $\overline{U} \to U$ be the minimal[12] *étale Galois covering trivialising $\mathcal{F}$, $\overline{X}$ be the normalisation of X in $\overline{U}$, $\overline{D} = D \times_X \overline{X}$. We require that $\overline{X}$ and $\overline{D}_{red}$ are smooth at points above x.*

The proof will give an explicit bound for the depth. Before giving the proof, we need to introduce some preliminary notions.

Terminology 2.3.11. *(blowup stages) We say a blowup sequence* at closed points *has r blowup stages if the longest sequence of successive blowups at points in* new *exceptional divisors has length r.*

[11] e.g. $f_s : \mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto (1-s)x^n + sx^m$.

[12] i.e., the covering corresponding to the quotient $\pi_1(X, \overline{\eta}_X) \twoheadrightarrow G$, where G is the image of $\pi_1(X, \overline{\eta}_X)$ in $\mathrm{Aut}_{\mathbb{Z}/\ell^n}(\mathcal{F}_{\overline{\eta}_X})$.

This is to be distinguished from the *number* of blowups. For example, the following blowup sequence has 2 blowup stages, while its number of blowups is 3.

Figure 2.1: A 2-stage blowup sequence on a surface

Definition 2.3.12 (ep(I)[13])**.** *Let A be a Noetherian ring, I be an ideal. $ep(I) :=$ the smallest $r \in \mathbb{N}$ such that $(\sqrt{I})^r \subseteq I$. Note $ep(I)$ exists because A is Noetherian.*

Lemma 2.3.13. *i) $ep(I) = \sup_x(ep(I_x))$, where x ranges through all closed points of $\mathrm{Spec}(A)$, and I_x denotes the localisation of I at x.*
ii) Assume A is Noetherian local excellent, then $ep(I) = ep(\hat{I})$, where $\hat{I}$ is the completion of I with respect to the maximal ideal.

$ep(I)$ measures of the "thickness" of I. Effectively, the lemma says $ep(I)$ can be computed locally, in the completion.

Proof. i) Follows from two standard commutative algebra facts: a) localisation commutes with taking radicals; b) inclusion relations of ideals can be checked by localisations at all closed points.

ii) First note that $\hat{A}$ is Noetherian ([Mat80, 23.K]), and A injects into $\hat{A}$ because its topology is Hausdorff ((11.D) in *loc. cit.*). $A/\sqrt{I}$ being reduced implies, by excellence of A, $\hat{A}/\hat{\sqrt{I}} = \widehat{A/\sqrt{I}}$ is reduced, so $\hat{\sqrt{I}}$ is radical. But $\hat{\sqrt{I}}$ is contained in $\sqrt{\hat{I}}$, because if $x_\infty \in \sqrt{\hat{I}}$, let $\{x_i\} \subseteq \sqrt{I}$ converge to it, then $\{x_i^{ep(I)}\} \subseteq I$ converges to $x_\infty^{ep(I)}$, so $x_\infty^{ep(I)}$ lies in $\hat{I}$. So $\hat{\sqrt{I}} = \sqrt{\hat{I}}$. This argument also shows $ep(\hat{I}) \leq ep(I)$. For the converse, notice $\sqrt{I}^{ep(\hat{I})} \subseteq (\hat{\sqrt{I}})^{ep(\hat{I})} = \sqrt{\hat{I}}^{ep(\hat{I})} \subseteq \hat{I}$, so $\sqrt{I}^{ep(\hat{I})} \subseteq \hat{I} \cap A = I$, where the last step is because $\hat{I} \cap A$ is the closure of I in A and ideals of A are closed in A ((24.A) in *loc. cit.*). □

Proof of Theorem 2.3.10. By Remark 2.3.6 v), $\phi_f(\mathcal{F})$ being a local system is equivalent to Diagram 2.3 being ULA. By Deligne-Laumon (Theorem 2.2.5), this is equivalent to the function a_s (Formula 2.1) being constant. By the constructibility of a_s, this is further equivalent to being constant for closed s. It is elementary to check that assumption i) ensures Deligne-Laumon is applicable in our situation, and a_s is just the Swan conductor of $\mathcal{F}$ restricted to

[13] ep stands for épaisseur.

the curve $C_s := \{f_s = 0\} \subseteq V_s$ at x. So it suffices to show:

In the setup of the theorem, there exists some $N \in \mathbb{N}$ such that for any test curves $C \equiv C'$ mod $\mathfrak{m}_x^N$ on a same étale open neighbourhood of x,[14] we have $sw(C) = sw(C')$, where sw means the Swan conductor at x of the restriction of $\mathcal{F}$.

We digress to recall a few facts about Swan conductors. For details, see, e.g., [Lau81, 1.1]. The notations here are independent of the rest of the proof. Let C be a strict henselian trait, $\mathcal{F}$ a tor-finite sheaf at its generic point η, concentrated in degree 0, given by the Galois representation $G_\eta \twoheadrightarrow G \hookrightarrow \mathrm{Aut}_{\mathbb{Z}/\ell^n}(\mathcal{F}_{\overline{\eta}})$. Let $C' \to C$ be the normalisation of C in the Galois cover of η corresponding to G. C' is a trait. To compute the Swan conductor of $\mathcal{F}$, one first form the filtration $G = G_0 \supseteq G_1 \supseteq ...$ induced by $i_G : G \to \mathbb{N} \cup \{\infty\}, \sigma \mapsto v'(\sigma(\pi') - \pi')$ if $\sigma \neq id$; ∞ if $\sigma = id$, where π' is any uniformiser of C', v' is the discrete valuation on C', and $G_i = \{\sigma \in G |\ i_G(\sigma) \geq i+1\}$. Then

$$sw(\mathcal{F}) = \sum_{i \geq 1} \frac{\dim(\mathcal{F}_{\overline{\eta}}/\mathcal{F}_{\overline{\eta}}^{G_i})}{[G : G_i]}$$

Important for us is the following geometric interpretation of i_G (see, e.g., [Ser79, VI.4]). Consider the G-action on C'. Then for $\sigma \neq id$, $i_G(\sigma) = (\Gamma_\sigma . \Delta_{C'})$, where the latter is the intersection number of the graph of σ and the diagonal. Denote by $I_{\sigma,C'}$ the ideal on C' corresponding to this intersection, then $(\Gamma_\sigma . \Delta_{C'}) = \lambda_{\mathcal{O}_{C'}}(\mathcal{O}_{C'}/I_{\sigma,C'})$ (λ denotes the length).

Back to the proof of the theorem. We first fix $C \hookrightarrow W$ for W some étale open neighbourhood of x and find N_C such that for any other test curve $C' \hookrightarrow W$, satisfying $C \equiv C'$ mod $\mathfrak{m}_x^{N_C}$, we have $sw(C) = sw(C')$. Then we show N_C is bounded uniformly as C varies.

Let $\overline{W} \to W$ be the base change of $\overline{X} \to X$ via $W \to X$. Consider the following diagram:

$$\begin{tikzcd}
\tilde{C} + E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{W} \arrow[d] \\
\overline{C} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \overline{W} \arrow[d] \\
C \arrow[r, hook] & W
\end{tikzcd}$$

where $\tilde{W}$ is obtained from $\overline{W}$ by successive blowups at closed points until $\overline{C}$ is resolved, $\tilde{C}$ is the strict transform of $\overline{C}$, E is the collection of exceptional divisors (with multiplicities). Note $\tilde{C}$ is smooth and equals to the normalisation of $\overline{C}$.[15] We require that we blowup each time

[14] i.e., C, C' arise as zero loci of ttfun's on the étale open.

[15] e.g. by Zariski's Main Theorem.

simultaneously at all points above x in the strict transforms of $\overline{C}$, so that the G-action always extends. Let M_1 = maximum of the multiplicities in E. Let $M_2 = \max_{\sigma \neq id \in G}\{ep(I_{\tilde{\sigma},\tilde{W}})\}$, where $\tilde{\sigma}$ is the extension (by the universal property of normalisations) of σ to $\tilde{W}$.

Claim: $N_C := M_1 + M_2.(D.C)_x.|G|$ satisfies our purpose. Here $(D.C)_x$ is the intersection number of D and C at x. A simple computation shows that in the nonexceptional situation, if ξ is not conormal to D, then $(D.C)_x = 1$; if ξ is conormal to D, then $(D.C)_x = 2$.

Proof of the claim: if $C' \hookrightarrow W$ is another test curve, let $\overline{\overline{C'}}$ be its normalisation. By the above recollection on Swan conductors, to show $sw(C) = sw(C')$, it suffices to show there exists a bijection of points $\{\tilde{x}\} \leftrightarrow \{\overline{\overline{x'}}\}$ of points of $\tilde{C}, \overline{\overline{C'}}$ above x and for corresponding points the quantities $\lambda_{\mathcal{O}_{\tilde{C},\tilde{x}}}(\mathcal{O}_{\tilde{C},\tilde{x}}/I_{\tilde{\sigma},\tilde{C}}.\mathcal{O}_{\tilde{C},\tilde{x}})$, $\lambda_{\mathcal{O}_{\overline{\overline{C'}},\overline{\overline{x'}}}}(\mathcal{O}_{\overline{\overline{C'}},\overline{\overline{x'}}}/I_{\overline{\overline{\sigma'}},\overline{\overline{C'}}}.\mathcal{O}_{\overline{\overline{C'}},\overline{\overline{x'}}})$ equal for each $\sigma \neq id$ in G. But if $C \equiv C' \mod \mathfrak{m}_x^{N_C}$, then $\tilde{C} \equiv \tilde{C}' \mod I_{E_{red}}^{N_C - M_1}$, which implies: a) $\{\tilde{x}\} := \tilde{C} \cap E_{red} = \tilde{C}' \cap E_{red}$; b) a fortiori $\tilde{C} \equiv \tilde{C}' \mod \mathfrak{m}_{\tilde{x}}^{N_C - M_1}$ so $\tilde{C}'$ is also smooth (hence equal to $\overline{\overline{C'}}$). From now on we abbreviate $\mathcal{O}_{\tilde{C},\tilde{x}}$ (resp. $\mathcal{O}_{\tilde{C}',\tilde{x}}$) by $\mathcal{O}$ (resp. $\mathcal{O}'$) and drop the subscripts on "λ". Consider $\lambda_{\mathcal{O}_{\tilde{C},\tilde{x}}}(\mathcal{O}_{\tilde{C},\tilde{x}}/I_{\tilde{\sigma},\tilde{C}}.\mathcal{O}_{\tilde{C},\tilde{x}})$. We have the following estimations:

$$\lambda(\mathcal{O}/I_{\tilde{\sigma},\tilde{C}}.\mathcal{O}) = \lambda(\mathcal{O}/I_{\tilde{\sigma},\tilde{W}}.\mathcal{O}) \leq M_2.\lambda(\mathcal{O}/\sqrt{I_{\tilde{\sigma},\tilde{W}}}.\mathcal{O})$$

$$\leq M_2.\lambda(\mathcal{O}/\sqrt{I_{\tilde{D}}}.\mathcal{O}) \leq M_2.\lambda(\mathcal{O}/I_{\tilde{D}}.\mathcal{O}) = M_2.(\tilde{D}.\tilde{C})_{\tilde{x}} \leq M_2.(\tilde{D}.\tilde{C}) = M_2.(D.C)_x.|G|$$

where $I_{\tilde{D}}$ is the ideal corresponding to $\tilde{D}$ in $\tilde{W}$. For the last equality, we used the projection formula from intersection theory in the form of [Liu02, 9.2.13]. Since $\tilde{C} \equiv \tilde{C}'$ mod $\mathfrak{m}_{\tilde{x}}^{M_2.(D.C)_x.|G|}$, $\lambda(\mathcal{O}/I_{\tilde{\sigma},\tilde{C}}.\mathcal{O})$ and $\lambda(\mathcal{O}'/I_{\tilde{\sigma},\tilde{C}'}.\mathcal{O}')$ must equal because $\tilde{C}$ and $\tilde{C}'$ are equal in the $(M_2.(D.C)_x.|G|)$-th infinitesimal neighbourhood of $\tilde{x} \in \tilde{W}$. This proves the claim.

It remains to show $N_C = M_1 + M_2.(D.C)_x.|G|$ is bounded with respect to C. By Lemma 2.3.14 below, $M_1 \leq 2^{M_C - 1}.\max_{\overline{x}}\{\mathrm{mult}_{\overline{x}}(\overline{C})\}$, where M_C is the smallest number of blowup stages needed to resolve $\overline{C}$. By Lemma 2.3.15, $M_2 \leq (2p+1)^{M_C}.\max_{\sigma \neq id \in G}\{ep(I_{\sigma,\overline{X}})\}$. By Proposition 2.3.18, M_C is bounded independent of C. Finally, $\mathrm{mult}_{\overline{x}}(\overline{C}) \leq |G|.(C.D)_x$. In fact, denote $\pi : \overline{X} \to X$, then $\mathrm{mult}_{\overline{x}}(\overline{C}) \leq \pi_*(\overline{C}.\pi^* D) = |G|.(C.D)_x$, where the first inequality follows from the fact that at an intersction point of two curves with no common components, the intersection number is greater than or equal to the product of their multiplicities at that point, and the second equality follows from the projection formula in *loc. cit.* □

Lemma 2.3.14. *Let C be a curve on a smooth surface X. Suppose C is singular at a closed point x. Let $\mathrm{mult}_x(C)$ be the multiplicity of C at x. Then, after M stages of blowups at closed points (Terminology 2.3.11), the largest multiplicity of the* inverse image *of C at its singularities is $\leq 2^{M-1}.\mathrm{mult}_x(C)$.*

Here the "inverse image of C" means $\overline{X} \times_X C$ ($\overline{X}$ being the result after blowups), which includes the exceptional divisors together with their multiplicities.

Proof. After the first blowup, $C_1 :=$ the strict transform of C has $\mathrm{mult}_{\overline{x}}(C_1) \le \mathrm{mult}_x(C)$ at each point $\overline{x}$ above x, and the exceptional divisor E_1 has multiplicity $= \mathrm{mult}_x(C)$ ([Kol07, 1.38, 1.40]). In the second blowup (at a closed point in E_1), C_2 still has multiplicities $\le \mathrm{mult}_x(C)$, and $\mathrm{mult}(E_2) \le \max_{\{\overline{x}\}}\{\mathrm{mult}_{\overline{x}}(C_1)\} + \mathrm{mult}(E_1) \le 2.\mathrm{mult}_x(C)$. Iterate. □

Lemma 2.3.15. *Let X be a smooth surface, $\sigma \neq id$ be an automorphism of X, X^σ_{red} be the fixed locus, $x_0 \in X^\sigma_{red}$ be a closed point. Assume X^σ_{red} is a prime divisor smooth at x_0. Then, after M stages of blowups at closed points which are fixed by (extensions of) σ, we have $ep(I_{\tilde{\sigma}}) \le (2p+1)^M.ep(I_\sigma)$, where $\tilde{\sigma}$ denotes the last extension of σ.*

In the statement and proof of this lemma, we suppress the space in the notation for the ideal of fixed points. For example, $I_{\tilde{\sigma},\tilde{W}}$ as in the proof above would have been abbreviated by $I_{\tilde{\sigma}}$. We will implicitly use Lemma 2.3.13 in the proof.

Proof. Clearly, it suffices to consider M successive blowups, each time at a closed point in the new exceptional divisors. σ induces an automorphism of the Zariski localisation of X at x_0. So we may assume $X = \mathrm{Spec}(A)$ is local. We need to analyse all possible configurations of fixed points in the successive blowups.

The configuration we start with is $X^\sigma_{red} =$ a smooth curve. The blowup replaces x_0 by an exceptional divisor isomorphic to $\mathbb{P}^1$ (after reduction). Denote the extension of σ by $\overline{\sigma}$. Then $\overline{\sigma}$ acts linearly on $\mathbb{P}^1$ (which is just the derivative of σ at x_0). There are three possibilities: a) $\overline{\sigma}|_{\mathbb{P}^1}$ fixes $\mathbb{P}^1$; b) $\overline{\sigma}|_{\mathbb{P}^1}$ fixes two points, each with multiplicity 1; c) $\overline{\sigma}|_{\mathbb{P}^1}$ fixes a single point with multiplicity 2. Then perform the second blowup, and so on. In Figure 2.2, we draw all possible local configuration changes under blowups. Solid lines and points represent fixed points. The dotted arrows represent the point of blowup. Each crossing is a simple normal crossing. One can analyse the change of $ep(I_\sigma)$ in all cases and find the desired estimate. We illustrate with two cases, the others are similar.

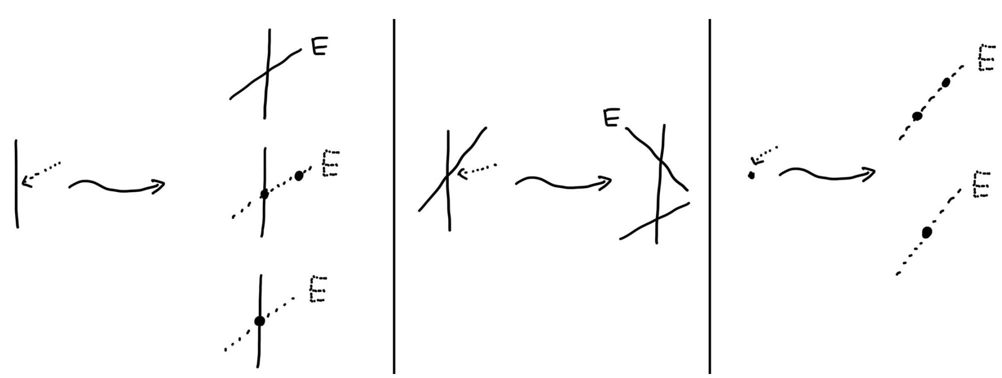

Figure 2.2: Possible configuration changes of fixed points

To separate notations, in the following we will use W, w, σ, etc. to denote the starting space, point, action, etc. and $\overline{W}, \overline{w}, \overline{\sigma}$, etc. to denote those after one blowup. We abbreviate $ep(I_\sigma), ep(I_{\overline{\sigma}})$ as $ep, \overline{ep}$. Let π be the projection $\overline{W} \to W$.

General setup and observations: choose local coordinates (x, y) at w, use coordinates $((x, y), [u : v])$ on $\overline{W}$. If $W = \mathrm{Spec}(A)$ then $\overline{W}$ is $\mathrm{Spec}(A[\frac{x}{y}])$ in the (y, u)-chart and is $\mathrm{Spec}(A[\frac{y}{x}])$ in the (x, v)-chart. Denote σ^* the corresponding action on A. Then, essentially by definition, $I_\sigma = (\sigma^*(a) - a)$ where a ranges through elements in A. Let $\sigma^*(x) = x + f, \sigma^*(y) = y + f$, so $f, g \in I_\sigma$. Note $\pi^*(I_\sigma) \subseteq I_{\overline{\sigma}}$.[16]

Case i). Line to cross (left column top): we may assume $W^\sigma_{red} = \{x = 0\}$. We have $\sqrt{I_\sigma} = (x)$ and $x^{ep} \in I_\sigma$. We look at (y, u)-chart, the other chart is similar. Denote by $I_{\overline{\sigma},(y,u)}$ the restriction of $I_{\overline{\sigma}}$ to the (y, u)-chart. Then $\sqrt{I_{\overline{\sigma},(y,u)}} = (yu)$. Since $\pi^*(I_\sigma) \subseteq I_{\overline{\sigma}}$, x^{ep} lies in $I_{\overline{\sigma},(y,u)}$. But in the (y, u)-chart $x = yu$, so $(yu)^{ep} \in I_{\overline{\sigma}}$, hence $\overline{ep}_{(y,u)} \leq ep$.

Case ii). Point to point (right column bottom): we have $\sqrt{I_\sigma} = (x, y)$ and $(x, y)^{ep} \subseteq I_\sigma$. Without loss of generality, assume that the new fixed point $\overline{w}$ lies in the (y, u)-chart, with coordinate $(0, u_1)$. Consider the completion of $\overline{W}$ at $\overline{w}$. Denote by $I_{\overline{\sigma},\hat{w}}$ the completion of $I_{\overline{\sigma}}$ at $\overline{w}$. Then $\sqrt{I_{\overline{\sigma},\hat{w}}} = (y, u - u_1)$.

We now exhibit some elements in $I_{\overline{\sigma},\hat{w}}$, which will be sufficient for estimating $\overline{ep}$. Since $\overline{\sigma}^*(u) = \overline{\sigma}^*(x/y) = \frac{x+f}{y+g} = \frac{u+f/y}{1+g/y}$, we have $\overline{\sigma}^*(u) - u = \frac{f/y - ug/y}{1+g/y} \in I_{\overline{\sigma},(y,u)}$. Upon localising to $\overline{w}$, $1 + g/y$ becomes a unit, so $f/y - ug/y \in I_{\overline{\sigma},\hat{w}}$. On the other hand, we know $(x, y)^{ep} \subseteq I_\sigma$ and $\pi^*(I_\sigma) \subseteq I_{\overline{\sigma}}$. This implies $y^{ep} \in I_{\overline{\sigma},\hat{w}}$.

Expand f, g in power series in y and $u' = u - u_1$, the terms in $f/y - ug/y$ which do not involve y are precisely the terms in the quadratic equation on $\mathbb{P}^1$ for the fixed point $\overline{w}$. So $f/y - ug/y \in k[[y, u']]$ is of the form au'^2 or $au'^2 + y(...)$ with $(...) \neq 0$, for some $a \neq 0$ in k. It is an exercise to see that in the former case $\overline{ep} \leq ep + 2$, and in the latter case $\overline{ep} \leq (2p + 1).ep$.[17] $\square$

Before stating Proposition 2.3.18 which bounds the number of blowup stages, we fix the following setup. Let X be a smooth surface over an algebraically closed field k, D a prime divisor, $U := X - D$, $U' \to U$ be a Galois cover with Galois group G, $\pi : X' \to X$ be the normalisation of X in U', $D' := X' \times_X D$, $x \in D$ be a closed point. Note π is finite. The following lemma is standard, we include a proof for completeness.

Lemma 2.3.16. *With the above setup, let $X_{\hat{x}}$ be the completion of X at x, $X'_{\hat{x}} := X' \times_X X_{\hat{x}}$. Then $X'_{\hat{x}} \cong \bigsqcup_{x' \in \pi^{-1}(x)} X'_{\hat{x}'}$, where $X'_{\hat{x}'}$ is the completion of X' at x'. Moreover, for each x',*

[16]This is one precise sense how the blowup "improves the situation".

[17]Proof in the latter case: $\forall p^m \in \mathbb{N}$, $(au'^2 + y(...))^{p^m} = a^{p^m} u'^{2p^m} + y^{p^m}(...)^{p^m} \in I_{\overline{\sigma},\hat{w}}$. If p^m is the smallest integer $\geq ep$ (which is clearly $< p.ep$), then, using $y^{ep} \in I_{\overline{\sigma},\hat{w}}$, we get $u'^{2p^m} \in I_{\overline{\sigma},\hat{w}}$, so $\overline{ep} \leq ep + 2.p^m$.

$X'_{\hat{x}'} \to X_{\hat{x}}$ is the normalisation of $X_{\hat{x}}$ in the function field $K(X'_{\hat{x}'})$, which is a finite separable extension of $K(X_{\hat{x}})$.

Proof. Let $X_x = \mathrm{Spec}(A), X_{\hat{x}} = \mathrm{Spec}(\hat{A})$. Then $X' \times_X X_x = \mathrm{Spec}(A'_L)$, where A'_L is the normalisation of A in the finite separable extension $L = K(X')$ of $K(X) = Q(A)$ (Q denotes the total ring of fractions). As A is Japanese, A'_L is a finite A-module, so $A'_L \otimes_A \hat{A} \cong \hat{A}'_L$ by [Mat80, 23.L, Theorem 55], where $\hat{A}'_L$ is the completion of A'_L as an A-algebra at $\mathfrak{m}_A$. $\hat{A}'_L$, being finite over a local ring, is semi-local, and its maximal ideals are in bijection with $\pi^{-1}(x)$. Apply (24.C) in *loc. cit.*, we get $\hat{A}'_L \cong \Pi_{x' \in \pi^{-1}(x)} (\hat{A}'_L)_{\hat{x}'}$ (the product of completions of $\hat{A}'_L$ at x').[18] This proves the first statement.

For the second statement, note we have the inclusions $\hat{A} \subseteq \hat{A}'_L \subseteq Q(\hat{A}'_L)$. It suffices to show $\hat{A}'_L$ is normal. In fact, $\hat{A}'_L$ is finite hence integral over $\hat{A}$, if it is normal, then it is the integral closure of $\hat{A}$ in $Q(\hat{A}'_L)$. Further, $Q(\hat{A}'_L) \cong \Pi_{x' \in \pi^{-1}(x)} Q((\hat{A}'_L)_{\hat{x}'})$, so for each x', $(\hat{A}'_L)_{\hat{x}'}$ is the integral closure of $\hat{A}$ in $Q((\hat{A}'_L)_{\hat{x}'})$, which is a finite extension of $Q(\hat{A})$. The separability follows from the separability of U' over U and base change. To see $\hat{A}'_L$ is normal, consider

$$\begin{tikzcd} A'_L \arrow[r] & \hat{A}'_L \\ A \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \hat{A} \arrow[u] \end{tikzcd}$$

The formal fibres of A'_L are base changes of formal fibres of A under finite field extensions. Since A is excellent, its formal fibres are geometrically regular, so the formal fibres of A'_L are also geometrically regular. So the normality of A'_L implies the normality of $\hat{A}'_L$ by (21.E) in *loc. cit.* □

Now consider $X'_{\hat{x}'} \to X_{\hat{x}}$ for a fixed x'. Suppose further that X' is smooth at x'. Denote the function rings of $X_{\hat{x}}$ and $X'_{\hat{x}'}$ by A and R, respectively (note the notations are different from those in the above proof). Let $C = V(h), h \in \mathfrak{m}_A$ be a smooth curve in $X_{\hat{x}}$, with no common components with $D_{\hat{x}}$, where $D_{\hat{x}} := D \times_X X_{\hat{x}} = V(g)$, for some $g \in \mathfrak{m}_A$. Denote $A' = A/hA, R' = R/hR$. Let $\mathfrak{C}(R/A)$ be the Dedekind codifferent. We refer to [Stacks, 0BW0 and related tags] for generalities on Dedekind codifferents. We summarise what we need in the following:

Facts 2.3.17 (about Dedekind codifferents)**.** *Notations here are separate from the above. Let $A \to B$ be a map between Noetherian rings. Assume the map is finite, any nonzerodivisor in A maps to a nonzerodivisor in B, and $K \to L$ is étale, where $K := Q(A), L := Q(A) \otimes_A B$. Under these assumptions, $\mathfrak{C}(B/A)$ is defined to be $\{x \in L | \mathrm{Tr}_{L/K}(xb) \in A, \forall b \in B\}$. Assume further $A \to B$ is flat. We have:*
a) $\mathfrak{C}(B/A)$ is a finite A-module (being the inverse of an ideal of a Noetherian ring ([0BW1])).

[18] Note $V(\mathfrak{m}_{\hat{A}'_L}) = V(\mathfrak{m}_{\hat{A}} \hat{A}'_L)$, so $\hat{A}'_L$ is already complete with respect to $\mathfrak{m}_{\hat{A}'_L}$.

b) If $A \to B$ is a local complete intersection map, then $\mathfrak{C}(B/A)$ equals the inverse of the Kähler different ([0BWG, 0BW5]). Note, with our assumptions, the Kähler different is the same as the Jacobian ideal (see [SH06, 4.4] for generalities on Jacobian ideals). Both equal the ideal generated by (the image in B of) $\Delta := \det(\partial f_i/\partial x_j)$, where $B \cong A[x_1, ..., x_n]/(f_1, ..., f_n)$ is any presentation of B as an A-algebra.

Proposition 2.3.18. *With the above setup, let M_C be the number of blowup stages needed to resolve $C' := X'_{\hat{x}'} \times_{X_{\hat{x}}} C \hookrightarrow X'_{\hat{x}'}$. Then $M_C \leq r.s.(C.D_{\hat{x}})$, where r is the smallest integer such that $g^r A \subseteq \mathrm{Ann}_A(\mathfrak{C}(R/A)/R)$, and s is the smallest number of generators of $\mathfrak{C}(R/A)/R$ as an A-module.*

The proof is due to Bernd Ulrich[19].

Proof. First note $M_C \leq \dim_k(\overline{R}'/R')$, where $\overline{R}'$ is the normalisation of R', i.e., the integral closure of R' in $Q(R')$. This follows from a simple application of [Nor57, Theorem 4, Theorem 5] and the well-known fact that finitely many blowups resolve R'. Now consider the inclusions in $Q(R')$: $R' \subseteq \overline{R}' \subseteq \mathfrak{C}(\overline{R}'/A') \subseteq \mathfrak{C}(R'/A')$. We get $M_C \leq \dim_k(\overline{R}'/R') = \lambda_A(\overline{R}'/R') \leq \lambda_A(\mathfrak{C}(R'/A')/R')$. Claim: $\mathfrak{C}(R'/A') = \mathfrak{C}(R/A)'(:= \mathfrak{C}(R/A) \otimes_A A')$. Consequently $\mathfrak{C}(R'/A')/R' = \mathfrak{C}(R/A)'/R' = (\mathfrak{C}(R/A)/R)'(:= (\mathfrak{C}(R/A)/R) \otimes_A A')$. Assume this for now, we prove the proposition.

It remains to bound $\lambda_A(\mathfrak{C}(R/A)/R)')$. Denote $N = \mathfrak{C}(R/A)/R$. N is a finite A-module, supported in $D_{\hat{x}}$ (because the $A \to R$ is étale elsewhere). Let r and s be as in the statement of the proposition. Then N is naturally an $A/g^r A$-module, $N' = N \otimes_{A/g^r A} A/(g^r, h)$. We have $(A/g^r A)^s \twoheadrightarrow N$, hence $(A/(g^r, h)^s \twoheadrightarrow N'$ after tensoring with $A/(g^r, h)$. So $\lambda_A(N') \leq \lambda_A((A/(g^r, h))^s) = r.s.(C.D_{\hat{x}})$.

Finally, we prove the claim. By assumption, A and R are regular, so R is a local complete intersection over A. Choose any presentation $R \cong A[x_1, ..., x_n]/(f_1, ..., f_n)$, where $(f_1, ..., f_n)$ is a regular sequence in $A[x_1, ..., x_n]$. Then, by Fact b) above, $\mathfrak{C}(R/A) = R\Delta^{-1}$, where Δ denotes the image of $\det(\partial f_i/\partial x_j)$ in R. Base change to over A', we get $R' \cong A'[x_1, ..., x_n]/(\tilde{f}_1, ..., \tilde{f}_n)$, where $\tilde{f}_i$ denotes the image of f_i in $A'[x_1, ..., x_n]$. Since $\mathrm{Spec}(R')(= C')$ is of codimension 1 in $\mathrm{Spec}(R)(= X'_{\hat{x}'})$, $(\tilde{f}_1, ..., \tilde{f}_n)$ is still a regular sequence in $A'[x_1, ..., x_n]$. So, by Fact b) again, $\mathfrak{C}(R'/A') = R'\tilde{\Delta}^{-1}$, where $\tilde{\Delta}$ denotes the image of $\det(\partial \tilde{f}_i/\partial x_j)$ in R'. We conclude that $\mathfrak{C}(R'/A') = (\mathfrak{C}(R/A))'$. □

[19] Ulrich's original argument in fact also applies to the case without the assumption of $X'_{\hat{x}'}$ being smooth. We present this special case for the sake of simplicity. In that generality, the last paragraph needs to be modified: instead of Fact b), one applies [KW88, 3.5] to compute $\mathfrak{C}(R/A)$ (resp. $\mathfrak{C}(R'/A')$), and get $\Delta^{-1}((f_1, ..., f_n) : I)$ (resp. $\tilde{\Delta}^{-1}((\tilde{f}_1, ..., \tilde{f}_n) : \tilde{I})$), where $R \cong A[x_1, ..., x_n]/I$ is a presentation and $(f_1, ..., f_n)$ is a regular sequence in I (which exists because R is Cohen-Macaulay, being normal of dimension 2).

This completes the proof of Theorem 2.3.10. The version of our main result as stated in the Introduction now follows easily from it (plus three lemmas to be proved in the next section).

Theorem 2.3.19 (Generic finite depth). *Let X be a smooth surface over an algebraically closed field k of characteristic $p > 2$, $\mathcal{F} \in D(X)$. Then, there exists a Zariski open dense $V = X - \{$finitely many closed points$\}$ and a Zariski open dense $S \subseteq SS(\mathcal{F}|_V)$ such that for any $(x, \xi) \in S$, there exists an integer $N \geq 2$ such that the depth of $\mathcal{F}$ at (x, ξ) is $\leq N$. Moreover, we have an upper bound: if $\mathcal{F}$ is locally constant in some punctured neighbourhood of x, then $N = 2$; if x lies in a ramification divisor D of $\mathcal{F}$ (still assuming $(x, \xi) \in S$), then $N \leq 2^{M-1}.i_x.|G| + (2p+1)^M . \max_{\sigma \neq id \in G}\{ep(I_{\sigma,\overline{X}})\}.i_x.|G|$, where $i_x = 1$ for (x, ξ) not conormal to D, $i_x = 2$ if the component of $SS\mathcal{F}$ (x, ξ) lies in is the conormal of D, $M = r.s.i_x$, and r, s are as in Proposition 2.3.18 (applied to $\overline{X}_{\hat{x}'} \to X_{\hat{x}}$ for any x' above x).*

Note r and s do not depend on the choice of x', as the Galois action acts transitively on $\{x'\}$.

Proof. It is clear that away from finitely many closed points and by standard dévissage (recollement and induction on amplitudes) (the dévissage works because of Lemma 2.4.16), we are reduced to three cases: 1) $\mathcal{F}$ is locally constant on a punctured neighbourhood of x; 2) $\mathcal{F} = i_*\mathcal{L}$ where i is the closed immersion of a smooth curve D on X, and L is a local system on D; 3) the situation of Theorem 2.3.10. Case 1) follows from Lemma 2.4.17 and Lemma 2.4.16, case 2) follows from Proposition 2.4.6 and Proposition 2.4.10, case 3) follows from Theorem 2.3.10. $\square$

Example 2.3.20. *Consider the sheaf in Example 2.2.6. $(x, \xi) = ((0,0), dy)$. One checks that the normalisation of $k[x, y]$ in $k(x, y)[t]/(t^p - t - y/x^p)$ is $k[x, y, xt]/((xt)^p - x^{p-1}(xt) - y)$. Let $\tau = xt$, then $\overline{X} = \mathrm{Spec}(k[x, \tau])$. There is a single point $\overline{x} = \{x = \tau = 0\}$ above x. One checks that $i_x = 1$; $G = \mathbb{Z}/p$; $\sigma \in \mathbb{Z}/p$ acts on $\overline{X}$ by $(x, \tau) \mapsto (x, \tau + \sigma x)$, so $\forall \sigma \neq 0, I_{\sigma,\overline{X}} = (x)$ hence $ep(I_{\sigma,\overline{X}}) = 1$; finally, using the notations of Proposition 2.3.18, $\Delta = -x^{p-1}$, $\mathfrak{C}(R/A)/R = R\frac{1}{x^{p-1}}/R$, $r = p-1$, $s = p$. So our bound gives $\mathrm{depth}(\mathcal{F})_{((0,0),dy)} \leq 2^{p(p-1)-1}.p + (2p+1)^{p(p-1)}.p$.*

We comment that, in the above example, by directly computing the Swan of test curves using explicit equations, one can show $\mathrm{depth}(\mathcal{F}) = p$ at every point $((0, y), \xi) \in SS\mathcal{F}$ with $\xi \neq 0$ (see Example 2.4.19 for an illustration in a simpler case). So our estimate is by no means sharp.

We now discuss Saito's result and its relation with ours.

Theorem 2.3.21. *[Sai15, 2.14] Let X be a smooth surface over a field k which is algebraically closed of characteristic $p > 0$, Λ be a finite field of characteristic $\ell \neq p$, $\mathcal{F} \in D(X, \Lambda)$ be of the form $\mathcal{F} = j_!\mathcal{F}_U$, where U is an open dense subscheme of X and $\mathcal{F}_U$ is a local system* *on it concentrated in degree 0. Let $Z = X - U$. Let $(x,\xi) \in SS\mathcal{F}$ be a smooth point, x closed. Let $f : X \to \mathbb{A}^1$ be a morphism such that (x,ξ) is an isolated characteristic point of f with respect to $\mathcal{F}$. Assume f is flat and its restriction to $Z - x$ is étale. Then there exists a positive integer N such that for any $g : X \to \mathbb{A}^1$ satisfying $f \equiv g \mod \mathfrak{m}_x^N$, there exists an isomorphism $\phi_f(\mathcal{F})_x \xrightarrow{\sim} \phi_g(\mathcal{F})_x$ as objects in $D_c^b(\Lambda[G_\eta])$.*

i) Saito's result fixes a test function f which has an isolated characteristic point (not necessarily transverse), while our result is a *uniform* bound for *transverse* test functions.

ii) The isomorphism of vanishing cycles in Saito's result is an isomorphism of G_η- representations. In our result, although $\phi_f(\mathcal{F})$ being a local system certainly implies $\phi_{f_s}(\mathcal{F})$'s are isomorphic as (complexes of) vector spaces for all closed points s in T, it is not clear what representation-theoretic data is contained in our notion of stability. On the other hand, our loss in representation-theoretic data gained us more functoriality. For example, one has a version of the 2-out-of-3 property for the depth, see Lemma 2.4.16.

We end this section with two conjectures.

Conjecture 2.3.22. *Let X be a smooth variety over an algebraically closed field k of characteristic $p \neq 2$.*[20] *Then $\mathcal{F} \in D(X)$ has finite depth at all smooth points of $SS\mathcal{F}$.*

Conjecture 2.3.23. *Let X be a smooth variety over an algebraically closed field k of characteristic $p \neq 2$, $\mathcal{F} \in D(X)$, (x,ξ) be a smooth point of $SS\mathcal{F}$. Then there exists a positive integer N (depending on (x,ξ)) such that for any étale neighbourhood U of x and $f, g : U \to \mathbb{A}^1$ satisfying a) f and g are ttfun's at (x,ξ); b) $f \equiv g \mod \mathfrak{m}_x^N$, there exists an isomorphism $\phi_f(\mathcal{F})_x \xrightarrow{\sim} \phi_g(\mathcal{F})_x$ as objects in $D_c^b(\mathbb{Z}/\ell^n[G_\eta])$.*

2.4 μc sheaves

We maintain the same setup in §2.3. As mentioned in the introduction, our motivation is to build a microlocal sheaf theory in this setting. A microlocal sheaf theory "lives" on the cotangent bundle, but as discussed in §2.2, due to the complexity of π_1 (or wild ramification), microlocal data is huge, reflected in the fact that vanishing cycles depend on higher jets of the ttfun. This suggests at least two directions to go: i) work on a space larger than T^*X (e.g., higher jet bundles), ii) restrict the class of sheaves. The previous section is a step in i): we showed that on a surface, generically, the vanishing cycles "live" on some finite jet bundle. In this section, we explore the second route.

An immediate thought is to restrict to tame sheaves. However, this is not satisfactory, as tameness is not even preserved under the Radon transform (see Example 2.4.21), while

[20] We need $p \neq 2$ so that there are enough ttfun's and the depth makes sense.

as mentioned in §2.2, a fundamental feature of microlocal sheaf theory is contact invariance, of which the Radon transform is the prototypical case. Inspired by the situation in complex analytic context, we instead consider the class of sheaves with the strongest stability.

Definition 2.4.1 (μc, μc^s sheaves)**.** *$\mathcal{F} \in D(X)$ is μc at a smooth point $(x, \xi) \in SS\mathcal{F}$ if for all ttfam's of $\mathcal{F}$ at (x, ξ), $\phi_f(\mathcal{F})$ is a local system. $\mathcal{F}$ is μc if it is μc at all smooth points of $SS\mathcal{F}$.*

$\mathcal{F} \in D(X)$ is μc^s at a smooth point $(x, \xi) \in SS\mathcal{F}$ if for all smooth morphism $p : Y \to X$ and $(y, \eta) \in T^*Y$ with $y \mapsto x, \eta = dp(\xi)$, and all ttfam's of $p^*\mathcal{F}$ at (y, η), $\phi_f(\mathcal{F})$ is a local system. $\mathcal{F}$ is μc^s if it is μc^s at all smooth points of $SS\mathcal{F}$.

We record a question we do not know how to answer yet, it is a special case of Question 2.3.9.

Question 2.4.2. *Is μc equivalent to μc^s?*

A μc sheaf is just a sheaf of depth 2 at all smooth points of its SS. We give them a special name as they are closest to the complex analytic case and are good candidates for microlocal constructions. Actually, we have the analogues of Propositions 2.2.1, 2.2.4. (Note we have no control on the representation structure, see item ii) after Theorem 2.3.21.)

Lemma 2.4.3. *i) Let $\mathcal{F} \in D(X)$ be μc and (x, ξ) be a smooth point in $SS\mathcal{F}$. Then for any two ttfun's f, g at (x, ξ), there exists a (noncanonical) isomorphism $\phi_f(\mathcal{F})_x \cong \phi_g(\mathcal{F})_x$ as objects in $D^b_c(\mathbb{Z}/\ell^n)$. We call this the microstalk of $\mathcal{F}$ at (x, ξ).*

ii) For μc^s sheaves, the microstalks are invariant under the Radon transform: let $\mathcal{F} \in D(\mathbb{P})$ be μc^s, and (x, ξ) be a smooth point of $SS\mathcal{F}$ with $\xi \neq 0$. Let (a, α) be a point in $SSR\mathcal{F}$ corresponding to (x, ξ). Let f, g be ttfun's for $\mathcal{F}, R\mathcal{F}$ at $(x, \xi), (a, \alpha)$ respectively. Then there exists an isomorphism $\phi_g(\mathcal{F})_a \cong \phi_f(\mathcal{F})_x$ as objects in $D^b_c(\mathbb{Z}/\ell^n)$.

Proof. i) $\mathcal{F}$ being μc implies that in a ttfam (T, U, V, g), the stalks of $\phi_g(\mathcal{F})$ are all isomorphic. So it suffices to show that any two ttfun's can be connected by a ttfam. Fix an étale coordinate $\{x_1, ..., x_n\}$ at x. Let f be a ttfun on some étale neighbourhood U of x. The restriction of f to the strict localisation $X_{(x)} \cong \mathrm{Spec}(k\{x, y\})$ is of the form $f|_{X_{(x)}} = \sum \xi_i x_i + \sum a_{ij} x_i x_j + H$, where ξ_i are components of ξ and H means higher order terms. Consider the $\mathbb{A}^1 = \mathrm{Spec}(k[s])$-family: $f_s := f + (s-1)(f - (\sum \xi_i x_i + \sum a_{ij} x_i x_j))$, then $f_s|_{X_{(x)}} = \sum \xi_i x_i + \sum a_{ij} x_i x_j + sH$. Note these are also defined on U, and since we have not changed $\leq$ second order terms, ν is still a transverse intersection point of Γ_{df_s} and $SS\mathcal{F}$. f_s is a ttfun on some Zariski open neighbourhood V_s of $x \in U$. Put them together, we get a ttfam $(\mathbb{A}^1, U, V, f)$, connecting $f_1 = f$ to $f_0 = \sum \xi_i x_i + \sum a_{ij} x_i x_j$. Now consider $Q =$ the space of all quadratic forms $\{b_{ij}\}$ such that $\sum \xi_i x_i + \sum b_{ij} x_i x_j$ is a ttfun on some Zariski open neighbourhood of $x \in X$. It is an open dense subspace of an affine space. Let $f_{\{b_{ij}\}} = \sum \xi_i x_i + \sum b_{ij} x_i x_j$. This defines a ttfam $(Q, U', V', f_{\{b_{ij}\}})$ for some Zariski open U'

of X, connecting all ttfun's parametrised by Q.

ii) By Corollary 2.4.14, $R\mathcal{F}$ is also μc^s, so its microstalks are well-defined. The same computation as in the proof of Proposition 2.3.2 then gives the result. □

Remark 2.4.4. *Here is a direct proof of Proposition 2.3.1: by the proof of the above Lemma i), all ttfun's can be connected via ttfam's. By [Sai17b, 1.16], dimtot is constant in a ttfam (note that by the definition of the ttfam, the nonacyclicity locus is mapped isomorphically to the base $\mathbb{A}^1_T$, so being flat implies being locally constant in the terminology of [Sai17b, 1.16]).*

The rest of this section is devoted to:

i) showing some basic sheaves are μc (μc^s). In particular, tame simple normal crossing sheaves are μc^s;
ii) showing some functorialities of the μc condition. In particular, μc^s sheaves are preserved under the Radon transform;
iii) examples.

Basic objects

Recall definitions and remarks in §2.3.

Lemma 2.4.5. *If X is a smooth curve, then any $\mathcal{F} \in D(X)$ is μc.*

Proof. Let (x, ξ) be a smooth point of $SS\mathcal{F}$. Notice that for any ttfam at (x, ξ), in Diagram 2.3, f_T is an isomorphism and $\mathcal{F}_T$ is a constant sheaf, so $(f_T, \mathcal{F}_T)$ is ULA. □

Proposition 2.4.6. *Local systems are μc^s.*

Proof. It suffices to show they are μc because pullback of local systems are local systems. The problem being étale local, we may assume the sheaf is constant. Let $\mathcal{F} \in D(X)$ be a constant sheaf, $x \in X$. Let (T, U, V, f) be a ttfam for $\mathcal{F}$ at $(x, \xi = 0)$. On each slice $V_s \xrightarrow{f_s} \mathbb{A}^1_s$, f_s being a ttfun implies it has a nondegenerate quadratic singularity at x over $0 \in \mathbb{A}^1_s$ (in the sense of [SGA7, Exp. XV, 1.2.1]). We want to show Diagram 2.3 is ULA. Consider the following diagram:

$$\begin{tikzcd}
\overline{V_T} \arrow[r, hook] \arrow[d] & \overline{V} \arrow[d, "\pi"'] & \overline{V} - \overline{V_T} \arrow[l, hook', "\bar{j}"'] \arrow[d, "\cong"] \\
V_T \arrow[r, hook, "i"] \arrow[d, "f_T"'] & V \arrow[dl, "h"] & V - V_T \arrow[l, hook', "j"'] \\
T & &
\end{tikzcd}$$

where h is the composition of f and the projection $\mathbb{A}^1_T \to T$, $\pi : \overline{V} \to V$ is the blowup of V along x_T. Note $\overline{V_T} \hookrightarrow \overline{V}$ is a simple normal crossing divisor over T. By the distinguished triangle $j_! j^* \mathcal{F} \to \mathcal{F} \to i_* \mathcal{F}_T \to$ and the fact that $(h, \mathcal{F})$ is ULA, to show $(f_T, \mathcal{F}_T)$ is ULA, it suffices to show $(h, j_! j^* \mathcal{F})$ is ULA. But $j_! j^* \mathcal{F} \cong \pi_* \overline{j}_! \mathcal{G}$, where $\mathcal{G}$ is the pullback of $j^* \mathcal{F}$ to $\overline{V} - \overline{V_T}$. By [Sai17b, 4.11], $SS(\overline{j}_! \mathcal{G}) = T^*_{\overline{V}} \overline{V} \cup T^*_{\overline{V_T}} \overline{V}$,[21] so $\overline{V} \to T$ is $SS(\overline{j}_! \mathcal{G})$-transversal, so $(h\pi, \overline{j}_! \mathcal{G})$ is ULA. By the compatibility of vanishing cycles and proper pushforwards, $\Phi_h(\pi_* \overline{j}_! \mathcal{G}) \cong \overleftarrow{\pi}_* \Phi_{h\pi}(\overline{j}_! \mathcal{G}) = 0$. □

Proposition 2.4.7. *Let $D \subseteq X$ be a simple normal crossing divisor (sncd), $j : U \to X$ be its complement. If $\mathcal{F} \in D(X)$ is of the form $\mathcal{F} = j_! \mathcal{F}_U$ for $\mathcal{F}_U$ a local system tame along D, then $\mathcal{F}$ is μc (hence μc^s because its smooth pullbacks are of the same form).*

Recall that in this situation $SS\mathcal{F} = T^*_X X \cup T^*_D X$ by [Sai17b, 4.11] (see Footnote [21] for the notation).

Proof. The question being étale local, we may assume that we are in the situation $D = \cup_{i=1}^r D_i \hookrightarrow X = \mathbb{A}^n_k$, where $0 < r \le n$, $D_i = \{x_i = 0\}$, with $\{x_1, ..., x_n\}$ the standard coordinates on $\mathbb{A}^n_k$. The locally constant locus has been dealt with in the previous proposition. It suffices to show $\mathcal{F}$ is μc at (x, ξ) for $x =$ origin, $\xi = dx_1 + ... + dx_r$.

Let (T, U, V, f) be a ttfam for $\mathcal{F}$ at (x, ξ). We want to show $\Phi_{f_T}(\mathcal{F}_T) = 0$ in Diagram 2.3. For this, we need to understand the geometry of $D_T := (D \times T) \cap V_T \hookrightarrow V_T$ near x_T. First look at each slice $D = D \times \{s\} \hookrightarrow V_s$.

Claim: the embedded singularity $D \cap H_s \hookrightarrow H_s$ can be resolved in two steps: first blow up at x, then blow up along the intersection of the exceptional divisor with the strict transform of $D_1 \cap ... \cap D_{r-1} \cap H_s$. (For $r = 1$, there is only one blowup.)[22]

The claim is shown in the next two lemmas. We first assume this and finish the proof. It follows from the claim that the embedded singularity $D_T \hookrightarrow V_T$ can be resolved by first blowing up along x_T, then blowing up along the intersection of the exceptional divisor with the strict transform of $D_{1,T} \cap ... \cap D_{r-1,T}$, where $D_{i,T} := (D_i \times T) \cap V_T$. We get the following diagram:

$$\begin{tikzcd} V_T \arrow[d, "f_T"'] & \overline{V_T} \arrow[l, "\pi"'] \arrow[dl, "g_T"] \\ T & \end{tikzcd}$$

where π is proper and induces an isomorphism over $V_T - D_T$, and $\pi^{-1}(D_T) \hookrightarrow \overline{V_T}$ is a sncd *relative to T*. Note $\mathcal{F}_T = \pi_* \pi^* \mathcal{F}_T$ (because $\mathcal{F}_T$ is a !-extension from the open), and $\pi^* \mathcal{F}_T$ is

[21] We use the following notation: for $D = \cup_{i=1}^r D_i \hookrightarrow X$ a sncd, $T^*_D X := \cup_I T^*_{D_I} X$, where I ranges through nonempty subsets of $\{1, 2, ..., r\}$, and $D_I := \cap_{i \in I} D_i$.

[22] Of course the choice $\{1, 2, ..., r-1\}$ is unimportant: one can choose any $r-1$ elements in $\{1, 2, ..., r\}$.

still a sncd tame sheaf (by [KS10, 4.4]). So $\Phi_{f_T}(\mathcal{F}_T) = \overleftarrow{\pi}_*\Phi_{g_T}(\pi^*\mathcal{F}_T) = 0$, where the last equality comes from the fact that SS of a sncd tame sheaf is conormal. □

Lemma 2.4.8. *Let $D = \cup_{i=1}^r D_i \hookrightarrow X = \mathbb{A}_k^n$, where $0 < r \leq n$, $D_i = \{x_i = 0\}$, with $\{x_1, ..., x_n\}$ the standard coordinates on $\mathbb{A}_k^n$. Denote $\underline{D} = D_1 \cap ... \cap D_r$. Let f be a ttfun of $T^*_{\underline{D}}X$ at (x, ξ) where $x =$ origin, $\xi = dx_1 + ... + dx_r$.*[23] *Denote $H = f^{-1}(0)$. Then $D_1 \cap H, ..., D_{r-1} \cap H$ form a sncd on H and $x_r|_H$ is a ttfun of $T^*_{D_1\cap...\cap D_{r-1}\cap H}H$ at $(x, -(dx_1 + ... + dx_{r-1})|_H)$. (For $r = 1$, $T^*_{D_1\cap...\cap D_{r-1}\cap H}H := T^*_H H$.)*

Proof. It follows easily from f being a ttfun that:
i) $D_1, ..., D_{r-1}$ indeed form a sncd on H;
ii) $\Gamma_{dx_r|_H}$ intersects $T^*_{D_1\cap...\cap D_{r-1}\cap H}H$ precisely at $(x, -(dx_1 + ... + dx_{r-1})|_H)$.

We want to show the intersection is transverse. By the "⇒" part of the proof of Proposition 2.4.10, it suffices to show this in the ambient space X, i.e., that $\Gamma_{dx_r}.\langle dx_1, ..., dx_{r-1}, df\rangle = 1.(x, dx_r)$, where $\langle dx_1, ..., dx_{r-1}, df\rangle$ denotes the pushforward of $T^*_{D_1\cap...\cap D_{r-1}\cap H}H$ into X. The $r = n$ case is easy. We assume $r \leq n - 1$ in the following. Note Γ_{dx_r} and $\langle dx_1, ..., dx_{r-1}, df\rangle$ intersect precisely at (x, dx_r), so it suffices to show that their tangents at (x, dx_r) are linearly independent. The computation is straightforward, here are the results:

Use coordinates $\{x_1, ..., x_n; p_1, ..., p_n\}$ on T^*X.
Γ_{dx_r}: tangent space at (x, dx_r) is spanned by $\{\partial_{x_1}, ..., \partial_{x_n}\}$.
$\langle dx_1, ..., dx_{r-1}, df\rangle$: tangent space at (x, dx_r) is spanned by $\{\partial_{p_1}, ..., \partial_{p_{r-1}}, \sum_{i=1}^n \partial_{p_i}$, $(\partial_{x_{r+1}} + \sum_{i=1}^n f_{i,r+1}\partial_{p_i}), ..., (\partial_{x_n} + \sum_{i=1}^n f_{i,n}\partial_{p_i})\}$, where $f_{i,j}$ denotes the derivative of f in x_i followed by in x_j.

These are linearly independent if and only if the matrix $\{f_{i,j}\}_{i,j\in\{r+1,...,n\}}$ is nondegenerate. But this follows exactly from the assumption that f is a ttfun for $T^*_{\underline{D}}X$. □

Lemma 2.4.9. *Same set up as in the previous lemma. Then the embedded singularity $D \cap H \hookrightarrow H$ can be resolved in two steps: first blow up at x, then blow up along the intersection of the exceptional divisor with the strict transform of $D_1 \cap ... \cap D_{r-1} \cap H$. (For $r = 1$, there is only one blowup.).*

Proof. As everything happens on H, for convenience, we make the following notation changes in this proof (new ⇠ old): X ⇠ H, D_i ⇠ $D_i \cap H$ (for $i = 1, 2, ..., r-1$), D ⇠ $\cup_{i=1}^{r-1} D_i \cap H$, $\underline{D}$ ⇠ $\cap_{i=1}^{r-1} D_i \cap H$, H ⇠ $D_r \cap H$, f ⇠ $x_r|_H$. We also rename n and r such that our new X is of dimension n, and new D has r components. In this new notation, the statement becomes: the embedded singularity $D \cup H \hookrightarrow X$ can be resolved by first blowing up at x, then blowing up along the intersection of the exceptional divisor with the strict transform

[23] For C a conic closed subset of T^*X and (x, ξ) a smooth point of C, we say f is a ttfun of C at (x, ξ) if f satisfies the same conditions as in Definition 2.1.1, with "$SS\mathcal{F}$" replaced by "C".

of $\underline{D}$.

The problem being étale local, we may assume $X = \mathbb{A}^n_k$, with the standard coordinates $\{x_1, ..., x_n\}$, and x is the origin. The $r = n$ case is simple. We check the $r \leq n-1$ cases. The condition on f implies that f is of the form $x_1 + ... + x_r + Q + Q' + P + (...)$ where Q is a nondegenerate quadratic form in $\{x_{r+1}, ..., x_n\}$, Q' is a quadratic form in $\{x_1, ..., x_r\}$, P is a linear combination of monomials of the form $x_a x_\alpha$ for $a \in \{1, ..., r\}$, $\alpha \in \{r+1, ..., n\}$, and $(...)$ means higher degree terms. In the rest of the proof, $a, b...$ always mean an index in $\{1, ..., r\}$; $\alpha, \beta...$ always mean an index in $\{r+1, ..., n\}$; $i, j...$ always mean an index in $\{1, ..., n\}$; $\sum$ means over all allowed indices unless specified. By a linear change of coordinates in $\{x_\alpha\}$, we may assume $Q = \sum x_\alpha^2$.

Blow up at x. Use new coordinates $((x_1, ..., x_n), [p_1 : ... : p_n])$ with relations $\{x_i p_j = x_j p_i\}_{\text{all } i,j}$. We look at $p_n = 1$ piece, the others can be checked by the same method. On this piece, we may use coordinates $\{x_n, p_1, ..., p_{n-1}\}$, and we have $x_i = x_n p_i$ for $i = 1, ..., n-1$. The exceptional divisor is $E = \{x_n = 0\}$. We list the strict transforms of relevant things:
D_a: $D'_a = \{p_a = 0\}$;
$\underline{D}$: $\underline{D}' = \{p_1 = ... = p_r = 0\}$;
H: $H' = \{f^{(1)} = \sum p_a + (x_n + \sum_{\alpha=r+1}^{n-1} x_n p_\alpha^2) + Q'/x_n + P/x_n + (...) = 0\}$. Here Q'/x_n consists of (linear combinations of) terms of the form $x_n p_a p_b$, and P/x_n of the forms $x_n p_a p_\alpha, x_n p_a$.

The conormals of $D'_a, \underline{D}'$ are spanned by $dp_a, \{dp_1, ..., dp_r\}$ respectively. We compute: $df^{(1)}|_E = \sum dp_a + (1 + \sum_{\alpha=r+1}^{n-1} p_\alpha^2) dx_n + A$, where A consists terms of the form $p_a p_b dx_n$, $p_a p_\alpha dx_n$, $p_a dx_n$. It is an exercise to deduce from these that:
i) $\{D'_1, ..., D'_r, H', E\}$ form a sncd except along $\underline{D}' \cap H' \cap E = \underline{D}' \cap E$;
ii) along $\underline{D}' \cap E$: outside the conic $C := \{x_n = p_1 = ... = p_r = 0, 1 + \sum_{\alpha=r+1}^{n-1} p_\alpha^2 = 0\}$, any $r-1$ members of $\{D'_1, ..., D'_r, H', E\}$ form a sncd; on C, $\{D'_1, ..., D'_r, E\}$ form a sncd, and $df^{(1)} = \sum dp_a$.

It remains to resolve the singularity along $\underline{D}' \cap E$. Blow up along $\underline{D}' \cap E$. It is another exercise to see, using ii), that the singularity outside C is resolved. We check that the singularity is also resolved over C.

Use new coordinates $((x_n, p_1, ..., p_{n-1}), [q_n : q_1 : ... : q_r])$ with relations $\{x_n q_i = q_n p_i, p_i q_j = q_i p_j\}_{\text{all } i,j}$. We look at $q_n = 1$ piece, the others can be checked by the same method. On this piece, we may use coordinates $\{x_n, q_1, ..., q_r, p_{r+1}, ..., p_{n-1}\}$, and we have $p_a = x_n q_a$ for $a = 1, ..., r$. The exceptional divisor is $F = \{x_n = 0\}$. We list the strict transform of relevant things:
E: E' lies at infinity and is irrelevant on this piece;
D'_a: $D''_a = \{q_a = 0\}$;
$\underline{D}'$: $\underline{D}'' = \{q_1 = ... = q_r = 0\}$;
H': $H'' = \{f^{(2)} = \sum q_a + (1 + \sum_{\alpha=r+1}^{n-1} p_\alpha^2) + Q'/x_n^2 + P/x_n^2 + (...) = 0\}$. Here Q'/x_n^2 consists

of terms of the form $x_n^2 q_a q_b$, and P/x_n^2 of the form $x_n q_a p_\alpha, x_n q_a$.

We compute: $df^{(2)}|_{x_n=0} = \sum dq_a + (\sum_{\alpha=r+1}^{n-1} 2p_\alpha dp_\alpha) + A$, where A consists of terms of the form $q_a p_\alpha dx_n, q_a dx_n$. Recall that we want to show $\{D_1'', ..., D_r'', H'', F, E'\}$ form a sncd. E' is irrelevant here, and it suffices to check over C, i.e., on the locus $\{x_n = 0, 1 + \sum_{\alpha=r+1}^{n-1} p_\alpha^2 = 0\}$. But along this locus, $\{p_{r+1}, ..., p_{n-1}\}$ are not all 0, so $df^{(2)}|_{x_n=0}$ contains some dp_α component and is thus not contained in the span of $\{dx_n, dq_1, ..., dq_r\}$, consequently $\{D_1'', ..., D_r'', H'', F, E'\}$ form a sncd. □

Properties

Proposition 2.4.10 (closed immersion)**.** *Let $i : Z \hookrightarrow X$ be a closed immersion of smooth varieties, $\mathcal{F} \in D(Z)$. Then*
i) $\mathcal{F}$ is μc if and only if $i_*\mathcal{F}$ is μc;
ii) Same for μc^s;
i') More generally, if (z, ζ) is a smooth point in $SS\mathcal{F}$ and (x, ξ) is in $SS(i_*\mathcal{F})$ such that $x = i(z), di(\xi) = \zeta$, then $\mathrm{depth}(\mathcal{F})_{(z,\zeta)} = \mathrm{depth}(i_*\mathcal{F})_{(x,\xi)}$.

Proof. i) Let (z, ζ) be a point in $SS\mathcal{F}$, and (x, ξ) be a point in $SS(i_*\mathcal{F}) = i_\circ SS\mathcal{F}$ such that $x = i(z), di(\xi) = \zeta$. First note that (x, ξ) is a smooth point of $SS(i_*\mathcal{F})$ if and only if (z, ζ) is a smooth point of $SS\mathcal{F}$. This follows from the observation that in the following correspondence u is smooth and v is a closed immersion.

$$\begin{tikzcd} & Z \times_X T^*X \arrow[dl, "u"'] \arrow[dr, "v"] & \\ T^*Z & & T^*X \end{tikzcd}$$

<u>$\mathcal{F}$ $\mu c \Rightarrow i_*\mathcal{F}$ μc</u>: Let (T, U, V, f) be a ttfam at (x, ξ) for $i_*\mathcal{F}$, we want to show $\phi_f(i_*\mathcal{F})$ is locally constant. Consider the restriction of (T, U, V, f) to Z:

$$\begin{tikzcd} Z \times T \arrow[d, hook, "i"] & U_Z \times T \arrow[l] \arrow[d, hook] \arrow[dl, phantom, "\llcorner"] & V_Z \arrow[l, hook'] \arrow[d, hook, "i'"] \arrow[dl, phantom, "\llcorner"] & z_T \arrow[l, hook'] \arrow[d, "\sim"] \\ X \times T & U \times T \arrow[l] & V \arrow[l, hook'] \arrow[d, "f"] & x_T \arrow[l, hook'] \\ & & \mathbb{A}^1_T & \end{tikzcd}$$

By the compatibility of vanishing cycles (over general bases) with proper pushforwards, $\Phi_f((i_*\mathcal{F})_V) \cong \overleftarrow{i'}_* \Phi_{fi'}(\mathcal{F}_{V_Z})$. But $\Phi_f((i_*\mathcal{F})_V)$ is supported on $z_T \overleftarrow{\times}_{\mathbb{A}^1_T} \mathbb{A}^1_T$ and $\overleftarrow{i'}$ restricted to $z_T \overleftarrow{\times}_{\mathbb{A}^1_T} \mathbb{A}^1_T$ is an isomorphism, so $\phi_f(i_*\mathcal{F}) = \Phi_f(i_*\mathcal{F}_V)|_{\overleftarrow{T}T} \cong \Phi_{fi'}(\mathcal{F}_{V_Z})|_{\overleftarrow{T}T} = \phi_{fi'}(\mathcal{F})$. So, $\mathcal{F}$ being μc, it suffices to show the restriction of (T, U, V, f) to Z is a ttfam for $\mathcal{F}$ at (z, ζ). In the definition of the ttfam i), ii) are clear. We check iii):

The computation being local, we may assume $V_s = X$. Consider the correspondence above. Abbreviate $SS\mathcal{F}$ as C. We want to compute $C.\Gamma_{d(f_s|_Z)} = C.uv^{-1}\Gamma_{df_s}$. First note $(vu^{-1}C).\Gamma_{df_s}(= (i_\circ SS\mathcal{F}).\Gamma_{df_s} = 1.(x,\xi))$, $(u^{-1}C).v^{-1}\Gamma_{df_s}$, $C.uv^{-1}\Gamma_{df_s}$ are all supported at a single point because f_s is a ttfun, u is smooth and v is a closed immersion. Then compute: $(vu^{-1}C).\Gamma_{df_s} = (v_*u^{-1}C).\Gamma_{df_s} = (u^{-1}C).v^*\Gamma_{df_s}$, where the second equality comes from v being a closed immersion, third equality comes from the projection formula in intersection theory. A simple computation in a local coordinate shows that the intersection of Γ_{df_s} and $Z \times_X T^*X$ is transverse. So $v^{-1}\Gamma_{df_s}$ is also smooth and $(u^{-1}C).v^*\Gamma_{df_s} = (u^{-1}C).v^{-1}\Gamma_{df_s}$, i.e. $u^{-1}C$ and $v^{-1}\Gamma_{df_s}$ intersect transversely at a single point. So C and $uv^{-1}\Gamma_{df_s}$ also intersect transversely at a single point.

$\underline{\mathcal{F}\ \mu c \Leftarrow i_*\mathcal{F}\ \mu c}$: Let (T, Z', V, f) be a ttfam at (z,ζ) for $\mathcal{F}$. If $\zeta = 0$, then $\mathcal{F}$ is locally constant near z and the assertion is clear. Assume $\zeta \neq 0$. By [EGAIV, 18.1.2], we can extend Z' to an étale neighbourhood X' of x in X. By the following Lemma 2.4.11, after possibly shrinking Z' and X', there exists an étale neighbourhood $\tilde{X}' \xrightarrow{\beta} X'$ of x in X and maps α, r satisfying the following diagram:

$$\begin{tikzcd} Z' \arrow[r, hook, "\alpha"'] & \tilde{X}' \arrow[l, bend right, "r"'] \arrow[r, "\beta"'] & X' \end{tikzcd}$$

where α is a closed immersion, β is étale, r is a retraction, and $\beta\alpha$ coincides with the closed immersion $Z' \hookrightarrow X'$. Consider the pullback of (T, Z', V, f) via r:

$$\begin{tikzcd} \tilde{X}' \times T \arrow[d, "r\times id"'] & \tilde{V} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dd, bend right, "\tilde{f}"'] & z_T \arrow[l, hook'] \arrow[d, equal] \\ Z' \times T & V \arrow[l, hook'] \arrow[d, "f"] & z_T \arrow[l, hook'] \\ & \mathbb{A}^1_T & \end{tikzcd}$$

On each slice, $\tilde{f}_s = f_s r$ is an extension of f_s. A similar intersection theoretic computation as above shows that $(T, \tilde{X}', \tilde{V}, \tilde{f})$ is a ttfam at $(z, (d\tilde{f}_s)_z)$ for $i_*\mathcal{F}$. Then again by the compatibility of vanishing cycles with proper pushforwards, $\phi_f(\mathcal{F}) \cong \phi_{\tilde{f}}(i_*\mathcal{F})$, and the latter is a local system by assumption.

ii) "$\Rightarrow$" is clear. For "$\Leftarrow$": The question being étale local, we may reduce to showing that the pullback of μc a sheaf $\mathcal{F}$ along $\mathbb{A}^m \times Z \to Z$ is μc. But it equals the restriction to $\mathbb{A}^m \times Z$ of the pullback of $\mathcal{F}$ along $\mathbb{A}^m \times X \to X$, which is μc by i).

i′) This follows from the same method as for i) plus Lemma 2.4.12. $\square$

The following lemma is well-known. We learned it from Owen Barrett.

Lemma 2.4.11. *Let $i : Z \hookrightarrow X$ be a closed immersion of smooth schemes over a field k, $z \in Z$ be a point. Then in some Zariski open neighbourhood X' of z in X, there exists an étale neighbourhood $\tilde{X}' \xrightarrow{\beta} X'$ of z in X' and maps α, r satisfying the following diagram:*

$$\begin{tikzcd} Z' \arrow[r, hook, "\alpha"'] & \tilde{X}' \arrow[l, bend right, "r"'] \arrow[r, "\beta"'] & X' \end{tikzcd}$$

where $Z' = Z \times_X X'$, α is a closed immersion, β is étale, r is a retraction, and $\beta\alpha = i$. Moreover, the retraction r is smooth along Z'.

Proof. Z being smooth, there exists an étale map $Z' \to \mathbb{A}^m_k$ for some Zariski open neighbourhood Z' of z in Z. Extend Z' to a Zariski open X' in X. By further shrinking, we may assume Z', X' are affine. Consider the following pushout, and choose a retraction r':

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d, "f"] \\ \mathbb{A}^m_k \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & X'' \arrow[l, bend left, "r'"] \end{tikzcd}$$

(If $X'' = \mathrm{Spec}A$, $\mathbb{A}^m_k = \mathrm{Spec}(k[x_1, ..., x_m])$, choosing an r' amounts to choosing a lift for each x_i of $A \twoheadrightarrow k[x_1, ..., x_m]$.) We construct $\tilde{X}'$ etc. using the following two pullback diagrams:

$$\begin{tikzcd} Z' \arrow[d] & Z' \times_{\mathbb{A}^m_k} X' \arrow[l, "pr_1"'] \arrow[d, "pr_2"] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{A}^m_k & X' \arrow[l, "r'f"] \end{tikzcd} \qquad \begin{tikzcd} Z' \times_{\mathbb{A}^m_k} Z' \arrow[r, hook, "i'"] \arrow[d, "pr_2"] \arrow[dr, phantom, "\lrcorner", very near start] & Z' \times_{\mathbb{A}^m_k} X' \arrow[d, "pr_2"] \\ Z' \arrow[r, hook, "i"'] \arrow[u, bend left, "\Delta"] & X' \end{tikzcd}$$

In the right diagram, note $Z' \times_{\mathbb{A}^m_k} Z'$ is a disjoint union of several copies of Z' because pr_2 is étale. Δ is an isomorphism to the diagonal copy. Let $\tilde{X}' = Z' \times_{\mathbb{A}^m_k} X' - (Z' \times_{\mathbb{A}^m_k} Z' - \Delta(Z'))$, $\alpha = i'\Delta$, $\beta = pr_2$, $r = pr_1$. It is an exercise to see that these satisfy the requirement. Note the smoothness of r along Z' follows from the smoothness of $Z', \tilde{X}'$ and the injectivity of dr on cotangent spaces (see, e.g., [Liu02, 6.2.10]). $\square$

Lemma 2.4.12. *Let $f : X \to Y$ be a morphism of schemes, $y \in Y$. If $g, h \in \mathcal{O}_{y,Y}$ are such that $f \equiv g \mod \mathfrak{m}_{y,Y}^N$ for some $N \in \mathbb{N}$, then $g \circ f \equiv h \circ f \mod \mathfrak{m}_{x,X}^N$ for any $x \in f^{-1}(y)$.*

Proof. Let $\varphi : \mathcal{O}_{x,X} \leftarrow \mathcal{O}_{y,Y}$ be the induced local ring map. $g \equiv h \mod \mathfrak{m}_{y,Y}^N \Rightarrow g \circ f \equiv h \circ f \mod \varphi(\mathfrak{m}_{y,Y}^N) = \varphi(\mathfrak{m}_{y,Y})^N$, a fortiori $g \circ f \equiv h \circ f \mod \mathfrak{m}_{x,X}^N$. $\square$

Like tame sheaves, μc sheaves are not stable under general proper pushforwards, however they are stable under pushforwards which resemble (the pushforward part of) an integral transform.

Proposition 2.4.13 (special pushforward)**.** *Let $f : Y \to X$ be a morphism of smooth varieties, $\mathcal{F}$ be a μc sheaf on Y.*
i) If f is special with respect to $\mathcal{F}$, then $f_*\mathcal{F}$ is μc;
ii) Same for μc^s;
i′) More generally, if f is special with respect to $\mathcal{F}$, then for any pair $(x, \xi), (y, \eta)$ (see notations below), we have $\mathrm{depth}(f_*\mathcal{F})_{(x,\xi)} \leq \mathrm{depth}(\mathcal{F})_{(y,\eta)}$.

Here we say $f : Y \to X$ is special with respect to $\mathcal{F}$ if
a) it is smooth and proper;
b) for any smooth point $(x, \xi) \in SS(f_*\mathcal{F})$ with $\xi \neq 0$, there exists a unique point $(y, \eta) \in (SS\mathcal{F})|_{f^{-1}(x)}$ such that $df(\xi) = \eta$. Furthermore, (y, η) is a smooth point of $SS\mathcal{F}$; c) $f_+SS\mathcal{F} = f_\circ SS^+\mathcal{F}$. Here f_+ is the map from cycles *on T^*Y to* cycles *on T^*X defined as follows: take the intersection theoretic pull and push under the correspondence $T^*Y \leftarrow Y \times_X T^*X \to T^*X$, then set the coefficient of the zero section to be 1. We will use a similar notation for pullbacks.*

Note, being special implies that the pull back of any ttfun for $f_*\mathcal{F}$ at (x, ξ) with $\xi \neq 0$ is a ttfun for $\mathcal{F}$ at (y, η) (c.f. the proof of Lemma 2.3.3).

Proof. i) Let (x, ξ) be a smooth point of $SS(f_*\mathcal{F})$ with $\xi \neq 0$, (y, η) be the point in $SS\mathcal{F}$ corresponding to it. Let (T, U, V, g) be a ttfam for $f_*\mathcal{F}$ at (x, ξ). Consider the pullback of this ttfam along f:

$$\begin{tikzcd}
Y \times T \arrow[d, "f \times id"] & \tilde{U} \times T \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] & \tilde{V} \arrow[l, hook'] \arrow[d] \arrow[dd, bend right, "h"'] \arrow[dl, phantom, "\llcorner"] & y_T \arrow[l, hook'] \arrow[d, "\sim"] \\
X \times T & U \times T \arrow[l] & V \arrow[l, hook'] \arrow[d, "g"] & x_T \arrow[l, hook'] \\
 & & \mathbb{A}^1_T &
\end{tikzcd}$$

f being special with respect to $\mathcal{F}$ implies that, for any closed point $s \in T$, the slice $\tilde{V}_s \xrightarrow{h_s} \mathbb{A}^1_s$, satisfies condition iii) in the definition of a ttfam for $\mathcal{F}$ at (y, η). Conditions i), ii) are clearly satisfied, so $(T, \tilde{U}, \tilde{V}, h)$ is a ttfam for $\mathcal{F}$ at (y, η). Since $\mathcal{F}$ is μc, $\phi_h(\mathcal{F})$ is a local system. By the compatibility of vanishing cycles (over general bases) with proper pushforwards, we conclude that $\phi_g(f_*\mathcal{F})$ is also a local system.

ii) For the same statement for μc^s, it suffices to check that being special with respect to a sheaf is preserved under smooth pullback.

Let $g : W \to X$ be a smooth map. We have the following diagram:

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y_W \arrow[l, "g'"'] \arrow[d, "f'"] \arrow[dl, phantom, "\llcorner"] \\ X & W \arrow[l, "g"] \end{tikzcd}$$

We want to show f' is special with respect to $g'^*\mathcal{F}$.

a): Clear;

b): We need to know the intersection (away from the zero section) of $f'^\circ SS(f'_* g'^*\mathcal{F}) = f'^\circ SS(g^* f_*\mathcal{F}) = f'^\circ g^\circ SS(f_*\mathcal{F}) = g'^\circ f^\circ SS(f_*\mathcal{F})$ and $SS(g'^*\mathcal{F}) = g'^\circ SS\mathcal{F}$. Clearly, on the fibre $f'^{-1}(x')$, for any $x' \in W$, the intersection is none empty if and only if, on $f^{-1}(g(x'))$, the intersection of $f^\circ SS(f_*\mathcal{F})$ and $SS\mathcal{F}$ is nonempty, and if so the intersection is a single smooth point of $SS(g'^*\mathcal{F})$;

c): $f'_+ SS(g'^*\mathcal{F}) = f'_+ g'^+ SS\mathcal{F} = g^+ f_+ SS\mathcal{F} = g^\circ f_\circ SS^+\mathcal{F} = f'_\circ g'^\circ SS^+\mathcal{F} = f'_\circ SS^+(g'^*\mathcal{F})$, where the second equality comes from the base change formula in intersection theory.

i′) This follows from the same method as for i) plus Lemma 2.4.12. □

Recall the notations in Radon setup 2.2.3 for the next corollary and remark.

Corollary 2.4.14 (Radon transform)**.** *The Radon transform preserves μc^s sheaves: if $\mathcal{F} \in D(\mathbb{P})$ is μc^s, then $R\mathcal{F} \in D(\mathbb{P}^\vee)$ is μc^s.*

Proof. It suffices to observe that, by the proof of Lemma 2.3.3, q is special with respect to $p^*\mathcal{F}$ for any $\mathcal{F} \in D(\mathbb{P})$. □

Remark 2.4.15. *Note that we actually proved a pointwise statement: if $\mathcal{F} \in D(\mathbb{P})$ is μc^s at (x, ξ), $\xi \neq 0$, then $R\mathcal{F} \in D(\mathbb{P}^\vee)$ is μc^s at (a, α), where (a, α) is any point corresponding to (x, ξ).*

Lemma 2.4.16 (distinguished triangle)**.** *Being μc is compatible with distinguished triangles: let $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to$ be a distinguished triangle, assume $SS\mathcal{G} = SS\mathcal{F} \cup SS\mathcal{H}$, then*
i) $\mathcal{F}$ and $\mathcal{H}$ μc implies $\mathcal{G}$ μc;
ii) Same for μc^s;
i′) More generally, for any smooth point $\nu \in SS\mathcal{G}$, $\mathrm{depth}(\mathcal{G})_\nu \leq \max\{\mathrm{depth}(\mathcal{F})_\nu, \mathrm{depth}(\mathcal{H})_\nu\}$.

Proof. These all follow from applying Remark 2.3.6 v), and using the distinguished triangle

$$\Phi_{f_T}\mathcal{F}_T \longrightarrow \Phi_{f_T}\mathcal{G}_T \longrightarrow \Phi_{f_T}\mathcal{H}_T \longrightarrow$$

Note that we need the assumption $SS\mathcal{G} = SS\mathcal{F} \cup SS\mathcal{H}$ because otherwise $SS\mathcal{F}$ and $SS\mathcal{H}$ may cancel each other and a smooth point of $SS\mathcal{G}$ may be a nonsmooth point of $SS\mathcal{F}$ or $SS\mathcal{H}$ (see, e.g., Example 2.4.20). □

Item i) in the following lemma is well-known, we include a proof for completeness.

Lemma 2.4.17 (purity)**.** *Let Z be a smooth closed subvariety of X of* codim ≥ 2*, $j : U \hookrightarrow X$ be the complement. Let $\mathcal{F} = j_!\mathcal{F}_U$ with $\mathcal{F}_U$ a local system. Then*
i) $SS\mathcal{F} = T^*_Z X \cup T^*_X X$;
ii) $\mathcal{F}$ is μc^s.

Proof. i) By induction on amplitudes and the compatibility of μc with distinguished triangles, it suffices to deal with the case where $\mathcal{F}_U$ is concentrated in a single degree. By Theorem of Purity (e.g. [SGA1, Exp. X]), $\mathcal{F}_U$ extends to some local system (concentrated in a single degree) $\overline{\mathcal{F}}$ on X. Consider the exact sequence $0 \to \mathcal{F} \to \overline{\mathcal{F}} \to i_*\mathcal{F}_Z \to 0$. The second and third terms have $SS = T^*_Z X \cup T^*_X X$, so the first term has $SS \subseteq T^*_Z X \cup T^*_X X$. But $\mathcal{F}$ is not locally constant on Z so $SS\mathcal{F} \neq T^*_X X$, so the inclusion is an equality by dimensional reasons.

ii) By induction on amplitudes, we reduce to the case $\mathcal{F}_U$ is concentrated in a single degree. Then it follows from the same exact sequence above and Lemma 2.4.16, proposition 2.4.10, and Lemma 2.4.6. □

Corollary 2.4.18. *Let Z be a smooth closed subvariety of X of* codim ≥ 2*, $j : U \hookrightarrow X$ be the complement. Let $\mathcal{F} \in D(X)$, assume it is* not *a local system. Then*
i) $SS\mathcal{F} = T^*_Z X \cup T^*_X X$ if and only if $\mathcal{F}$ is a local system on U and Z;
ii) If so, $\mathcal{F}$ is μc^s.

In real and complex analytic contexts, statement i) is true without assumptions on the codimension ([KS90, 8.4.1]). In the positive characteristic algebraic context, it is false for codim $= 1$, see Example 2.4.20.

Proof. By the previous lemma, ii) and "$\Leftarrow$" in i) are clear. For "$\Rightarrow$" in i): suppose $SS\mathcal{F} = T^*_Z X \cup T^*_X X$. Consider the distinguished triangle $j_!\mathcal{F}_U \to \mathcal{F} \to i_*\mathcal{F}_Z \to$. The first and second terms both have $SS = T^*_Z X \cup T^*_X X$, so the third has $SS \subseteq T^*_Z X$. But $SS(i_*\mathcal{F}_Z) = i_\circ SS\mathcal{F}_Z$, these force $SS(i_*\mathcal{F}_Z) = T^*_Z X$ and $SS\mathcal{F}_Z = T^*_Z Z$, hence $\mathcal{F}_Z$ is a local system. □

Examples

Example 2.4.19. *The sheaves in Examples 2.2.6, 2.2.7 are not μc (for $p > 3$) by the computations there and Lemma 2.4.3. Example 2.2.7 shows that SS being Lagrangian does not* *imply being μc. We do not know if being μc implies SS being Lagrangian. Furthermore, similar computations at other points show that Example 2.2.6 is not μc anywhere. On the other hand, as we show now, Example 2.2.7 is μc everywhere (along the smooth locus of $SS\mathcal{F}$) except above the origin.*

Consider $((0,1), dx) \in SS\mathcal{F}$, the other points are similar. Change coordinates, we may assume the sheaf is given by $t^p - t = (y+1)/x^{p-1}$ and the point in question is $(x_0, \xi) = ((0,0), dx)$. By the same reasoning as in the first paragraph of the proof of Theorem 2.3.10, it suffices to show: for any smooth curves on an étale open neighbourhood of x_0 passing through x_0 with conormal at x_0 proportional to dx, $sw(C)$ is independent of the curve. By Implicit Function Theorem, any such curve is of the form $\{x = c_2y^2 + c_3y^3 + c_4y^4 + ...\}, c_2 \neq 0$ in the formal neighbourhood of x_0. The restriction of the sheaf is given by Artin-Schreier equation

$$t^p - t = \frac{y+1}{(c_2y^2 + c_3y^3 + c_4y^4 + ...)^{p-1}} = \frac{y+1}{y^{2p-2}(c_2 + c_3y + c_4y^2 + ...)^{p-1}}$$

which has Swan conductor $2p-1$, independent of C.

In the following, we will use coordinates $[x : y : z]$ on $\mathbb{P}^2$ and $[a : b : c]$ on its dual.

Example 2.4.20. *($p > 2$) Consider the Artin-Schreier sheaf on $\mathbb{P}^2$ determined by the equation $t^p - t = yz^{p-2}/x^{p-1}$, !-extended along $\{x = 0\}$. Note on the affine $\{[x : y : 1]\}$, this is just Example 2.2.7. One can compute: $SS\mathcal{F} = T^*_{\mathbb{P}}\mathbb{P} \cup T^*_{\{x=0\}}\mathbb{P} \cup T^*_{[0:0:1]}\mathbb{P} \cup T^*_{[0:1:0]}\mathbb{P}$, $SSR\mathcal{F} = T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee \cup T^*_{\{b=0\}}\mathbb{P}^\vee \cup T^*_{\{c=0\}}\mathbb{P}^\vee \cup T^*_{[1:0:0]}\mathbb{P}^\vee$. Focus on a neighbourhood of the point $[0 : 1 : 0] \in \mathbb{P}^\vee$. Claim: although $SSR\mathcal{F}$ is the zero section union the conormal to a smooth divisor near this point, $R\mathcal{F}$ is* not *locally constant on the divisor near this point. Indeed, as a varies, the points $[a : 1 : 0]$ correspond to the lines $\{aX + Y = 0\}$ on $\mathbb{P}$ and the stalks $(R\mathcal{F})_{[a:1:0]} \cong R\Gamma(\{aX + Y = 0\}, \mathcal{F})$ has a jump at $a = 0$.*[24]

This shows that the same statement as in Corollary i) 2.4.18 for $\operatorname{codim} = 1$ *is false. This is in steep contrast with the real and complex analytic cases.*

Example 2.4.21. *($p > 2$) Let $Z \hookrightarrow \mathbb{P}^2$ be the closed subscheme with equation $z^{p-1}y = x^p$. Let $\mathcal{F}$ be the constant sheaf on Z, $*$-extended to $\mathbb{P}$. One can compute: $SS\mathcal{F} = T^*_Z\mathbb{P}$, $SSR\mathcal{F} = T^*_{\{b=0\}}\mathbb{P}^\vee \cup \Lambda$, where Λ is described as follows: on the affine $\{[x : y : 1]\}$, $\Lambda|_{\{c=1\}} = \{((0,b), \langle \frac{1}{b^{1/p}}da + \frac{1}{b}db \rangle)\}$ (for $b = 0$ this means $((0,0), \langle db \rangle)$), Λ is the closure of $\Lambda|_{\{c=1\}}$ in $T^*\mathbb{P}^\vee$. By Remark 2.4.6, Proposition 2.4.10 and Remark 2.4.15, $R\mathcal{F}$ is μc^s except possibly along $\{b = 0\}$. Its SS shows that it has wild ramification along $\{a = 0\}$.*

This tells us: a) being tame is not stable under proper pushforwards; b) μc, μc^s sheaves can have wildly ramifications.

[24] This can be seen, e.g., by computing the Euler-Poincaré characteristics using Grothendieck-Ogg-Shafarevich.

2.5 Appendix to Chapter 2: analogies and contrasts among sheaf theories

We list some analogies and contrasts among the following contexts from the microlocal perspective (well-known to experts):

i) Ét.: bounded constructible complexes of étale $\mathbb{Z}/\ell^n$-sheaves on smooth algebraic varieties over algebraically closed fields of positive characteristic $p \neq l$;
ii) Dist.: complex valued tempered distributions on $\mathbb{R}^n$;
iii) D-mod$_\mathrm{h}$: bounded holonomic complexes of algebraic D-modules on smooth complex algebraic varieties;
iv) $\mathbb{C}$-ana.: bounded $\mathbb{C}$-constructible complexes of $\mathbb{C}$-sheaves on complex analytic manifolds. By Riemann-Hilbert this is equivalent to bounded regular holonomic complexes of analytic D-modules.

<u>6-functor formalisms</u> All except Dist. have 6-functor formalisms. Special features:
Ét.: the subclass of tame sheaves is not preserved under (proper) pushforward;
Dist.: having polynomial growth is not preserved under integrations;
D-mod$_\mathrm{h}$: subclass of regular holonomic D-modules is stable under 6-functors.

<u>Singular supports (SS) and characteristic cycles (CC):</u> SS and CC are defined for Ét., D-mod$_\mathrm{h}$ and $\mathbb{C}$-ana.. CC's satisfy index formulas. SS is also defined for Dist. (which are called wavefronts instead). SS's are closed conic subsets in T^*X. Special features:
Ét.: SS's are half-dimensional;
Dist.: no special feature;
D-mod$_\mathrm{h}$: SS's are Lagrangian; for general coherent (not necessarily holonomic) D-modules SS's are coisotropic;
$\mathbb{C}$-ana.: SS's are Lagrangian.

<u>Fourier transforms:</u> All of them have Fourier transforms (on $X = \mathbb{A}^n$). Special features:
Ét., Dist., D-mod$_\mathrm{h}$: equivalence on the whole category;
$\mathbb{C}$-ana.: not an equivalence on the whole category but becomes an equivalence after restriction to conic sheaves.

<u>Microlocal data:</u>
Ét.: large data contained in wild ramifications (in dimension one: representation of local Galois groups);
Dist.: large data contained in (essential) singularities[25];
D-mod$_\mathrm{h}$: large data contained in irregular singularities (in dimension one: Stokes data); for

[25] e.g., Great Picard's Theorem: at an essential singularity x of a complex analytic function f, in any punctured neighbourhood of x, f takes all complex values infinitely many times, with at most one exception.

general analytic D-modules, microlocalisation can be carried out and is the content of the theory of algebraic analysis (microfunctions, microdifferential operators...);
$\mathbb{C}$-ana.: relatively small data, microlocalisation can be carried out.

Extension properties:
Ét.: fix $\mathbb{A}^1_{k,(0)} - \{0\} \to \mathbb{G}_{m,k} \hookrightarrow \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. Given a local system on $\mathbb{A}^1_{k,(0)} - \{0\}$, there are in general many ways to extend it to a local system on $\mathbb{G}_{m,k}$. However, there exists a unique extension (up to isomophism) which is special ([Kat86]);
Dist.: given a smooth function on a small punctured disk at the origin of $\mathbb{R}$, there are many ways to extend to a smooth function on $\mathbb{R} - \{0\}$;
D-mod$_\mathrm{h}$: fix $D^\circ \to \mathbb{G}_{m,\mathbb{C}} \hookrightarrow \mathbb{A}^1_\mathbb{C} \hookrightarrow \mathbb{P}^1_\mathbb{C}$, where D° is the punctured formal disk at the origin. Given a vector bundle with a flat connection on D°, there are in general many ways to extend it to a vector bundle with a flat connection on $\mathbb{G}_{m,\mathbb{C}}$. However, there exists a unique extension (up to isomophism) which is special ([Kat87, II.2.4]);
$\mathbb{C}$-ana.: there is a unique way to extend a local system on a punctured small disk at the origin of $\mathbb{C}$ to a local system on $\mathbb{C}^\times$.

Chapter 3

The Fourier transform and characteristic cycles of monodromic ℓ-adic sheaves

3.1 Introduction

Let $V = \mathrm{Spec}(\mathbf{C}[x_1, x_2, ..., x_d])$ be a finite dimensional vector space over $\mathbf{C}$. Denote by $D(V)$ the triangulated category of bounded coherent algebraic D-modules on V. $\mathcal{M} \in D(V)$ is called monodromic if the Euler vector field $\mathrm{eu} = \Sigma_i x_i \frac{\partial}{\partial x_i}$ acts locally finitely on each $\mathcal{H}^i(\mathcal{M})$ (i.e., for any local section s, $\{\mathrm{eu}^n(s)\}_{n\in\mathbf{N}}$ span a finite dimensional $\mathbf{C}$-vector space). Denote by $D_{mon}(V)$ the full subcategory of coherent monodromic D-modules. Let F denote the Fourier transform of D-modules (c.f. [KL85, 7.1]). It is easy to see that being monodromic is preserved under the Fourier transform. We have:

Theorem 3.1.1 (Brylinski-Malgrange, [Bry86, 7.25])**.** *1) If $\mathcal{M} \in D_{mon}(V)$, then $CC(\mathcal{M}) = CC(F\mathcal{M})$. 2) Further assume $\mathcal{M}$ is regular holonomic, then so is $F\mathcal{M}$.*

Here V' denotes the dual of V, and T^*V is implicitly canonically identified with T^*V' via $T^*V = V \times V' \cong V' \times V \cong T^*V'$. Note that statement 1) in Theorem 3.1.1 as we stated is more general than Brylinski-Malgrange's original version, but in fact their proof works in this generality.

The main theorem of this paper is the analogue of statement 1) for ℓ-adic sheaves[1]. Let V be a finite dimensional vector space over an algebraically closed field of characteristic $p > 0$. Let Λ be either a finite extension of $\mathbf{F}_\ell$ (the finite case), or a finite extension of $\mathbf{Q}_\ell$, or $\overline{\mathbf{Q}}_\ell$ (the rational case), for ℓ a prime different from p. Denote by $D(V)$ the triangulated category of bounded constructible Λ-étale sheaves. We will prove:

[1]Note that the analogue of statement 2) is false, i.e., being monodromic tame is not preserved under the Fourier transform. Example: let k be an algebraically closed field of characteristic $p > 0$, $V = \mathrm{Spec}(k[x,y,z]), Z = \{z^{p-1}y = x^p\} \hookrightarrow V$. Consider $\mathcal{F} := \underline{\Lambda}_Z$, which is evidently monodromic and tame. But, combining [Bry86, 9.13] and the computation in Example 2.4.21, one sees that $F\mathcal{F}$ is not tame.

Theorem 3.1.2 (Corollary 3.4.2)**.** *If $\mathcal{F} \in D(V)$ is monodromic, then $CC(\mathcal{F}) = CC(F\mathcal{F})$ and $SS(\mathcal{F}) = SS(F\mathcal{F})$.*

Here F denotes the ℓ-adic Fourier transform or its finite coefficient analogue (c.f. [Lau87]). $\mathcal{F} \in D(V)$ is called monodromic if all $\mathcal{H}^i(\mathcal{F})$ are *tame* local systems on all $\mathbf{G}_m$-orbits. This is preserved under the Fourier transform (Proposition 3.2.5.4).

In fact, we will prove the following theorem, which implies Theorem 3.1.2 by the additivity of characteristic cycles and singular supports with respect to irreducible constituents. We first introduce a terminology.

Definition 3.1.3 (F-good)**.** *$\mathcal{F} \in D(V)$ is F-good if for each irreducible constituent $\mathcal{P}$,*[2] *$CC(\mathcal{P}) = CC(F\mathcal{P})$.*

Theorem 3.1.4 (Theorem 3.4.1)**.** *Monodromic sheaves are F-good.*

Our proof of Theorem 3.1.4 consists of a precise realisation of the following intuition: a monodromic sheaf "decomposes" into a "projective component" and a "radial component" (the twist). The case where the twist is trivial can be proved utilising the relation between the Radon transform and the Fourier transform (c.f. [Bry86, 9.13]) and the fact that characteristic cycles behave well under the Radon transform ([Sai17b, 7.5]). The general case then follows, because the radial component is tame by monodromicity, and thus does not affect the characteristic cycle. Our original way of making the last sentence precise uses the notion of having the same wild ramification (see [Kat18; Kat21] and references therein). Beilinson pointed out that the general case in fact follows formally from the trivial twist case by untwisting the sheaf after pulling back to $V \times \mathbf{A}^1$. This leads to a much simpler proof. Both proofs are presented.

In §3.2, we make a preliminary study on monodromic sheaves and F-good sheaves. In §3.3, we prove Theorem 3.1.4 in the trivial twist case, and give a formula for the coefficient of T_0^*V in $CC(\mathcal{F})$. In §3.4, we prove Theorem 3.1.4. The Appendix reviews basic facts about characteristic cycles of sheaves with rational coefficients and the notion of having the same wild ramification.

In [Zho24], we will apply our results to give a microlocal characterisation of admissible (or character) sheaves on reductive Lie algebras in positive characteristic.

Conventions for Chapter 3

We fix an algebraically closed field k of characteristic $p > 0$ and a prime $\ell \neq p$. A variety means a finite type reduced separated scheme over k. For a variety X, $D(X)$ denotes $D_c^b(X, \Lambda)$ ([Del80, 1.1]). Λ is either a finite extension of $\mathbf{F}_\ell$, or a finite extension of $\mathbf{Q}_\ell$,

[2] This means $\mathcal{P}$ is an irreducible subquotient of some ${}^p\mathcal{H}^i(\mathcal{F})$.

or $\overline{\mathbf{Q}}_\ell$. We refer to the former as the finite coefficients case, and the latter as the rational coefficients case. In all statements below, Λ is understood to be either finite or rational unless otherwise specified. For $\mathcal{F} \in D(X)$, by an irreducible constituent of $\mathcal{F}$ we mean an irreducible subquotient of some ${}^p\mathcal{H}^i(\mathcal{F})$.

All derived categories are in the triangulated sense. All sheaf-theoretic functors are derived. A "sheaf" means an object of $D(X)$. A "local system" means an object of $D(X)$ whose cohomology sheaves are locally constant (if Λ is finite) or lisse (if Λ is rational) with finite type stalks.

V denotes a finite dimensional vector spaces over k, V' denotes its dual, $\mathring{V}$ denotes $V - \{0\}$, $\mathbf{P}(V)$ denotes the projectivisation of $\mathring{V}$, q denotes the projection $\mathring{V} \to \mathbf{P}(V)$.

$\mathbf{G}_m$ acts on V by scaling. For $n \geq 1$ in $\mathbf{N}$, we call $\theta(n) : \mathbf{G}_m \times V \to V, (\lambda, v) \mapsto \lambda^n v$ the n-twisted scaling action. We fix a non-trivial character $\psi : \mathbf{Z}/p \to \Lambda^\times$. Fourier transforms are denoted by F and are with respect to this character unless otherwise specified. As we work over an algebraically closed field, we may ignore Tate twists.

By a Kummer sheaf $\mathcal{K}$, we mean the !-extension to $\mathbf{A}^1$ of a rank 1 local system in degree -1 on $\mathbf{G}_m$ corresponding to a non-trivial continuous character from the tame fundamental group $\pi_1^t(\mathbf{G}_m, 1)$ to $\Lambda^\times$. We sometimes abuse notations and denote its restriction to $\mathbf{G}_m$ also by $\mathcal{K}$. We denote by $\mathcal{K}^{-1}$ the Kummer sheaf corresponding to the inverse character of that of $\mathcal{K}$.

We refer to [Bei16; Sai17b] for the theory of singular support and characteristic cycle for sheaves with finite coefficients, and to [UYZ20; Bar23] for the case of rational coefficients.

3.2 Preliminaries on monodromic sheaves and F-good sheaves

The setup is as in the Conventions. Λ can be either finite or rational, unless otherwise specified. For completeness, we have included more materials in this section than are actually needed in the sequel. Recall:

Definition 3.2.1 (monodromic sheaves, [Ver83])**.** *A sheaf $\mathcal{F}$ on V is* <u>monodromic</u> *if the restriction of all $\mathcal{H}^i(\mathcal{F})$ to all $\mathbf{G}_m$-orbits are tame local systems.*

When Λ is finite, we have the following crucial equivalent characterisation of monodromic sheaves.

Proposition 3.2.2 ([Ver83, 5.1])**.** *Let Λ be finite. Then, $\mathcal{F} \in D(V)$ is monodromic if and only if $\exists\ n > 0$ in $\mathbf{N}$ prime to p such that there exists an isomorphism $\theta(n)^*\mathcal{F} \tilde{\to} pr^*\mathcal{F}$. Here*

$\theta(n) : \mathbf{G}_m \times V \to V, (\lambda, v) \mapsto \lambda^n v$ is the n-twisted scaling action and $pr : \mathbf{G}_m \times V \to V$ is the projection.

Proof. The "only if" direction is proved in *loc. cit.* In *loc. cit.*, it is not stated that n can be chosen to be prime to p, but the proof in fact shows this.

For the "if" direction, just observe that $\theta(n)^*\mathcal{F}\tilde{\to}pr^*\mathcal{F}$ implies $\theta(n)^*\mathcal{H}^i(\mathcal{F})\tilde{\to}pr^*\mathcal{H}^i(\mathcal{F})$. So for each $x \in V$, $(\theta(1)^*\mathcal{H}^i(\mathcal{F}))|_{\mathbf{G}_m\times\{x\}}$ is a sheaf concentrated in degree 0 and trivialised by the cover $\mathbf{G}_m \times \{x\} \to \mathbf{G}_m \times \{x\}, \lambda \mapsto \lambda^n, p \nmid n$. So $(\theta(1)^*\mathcal{H}^i(\mathcal{F}))|_{\mathbf{G}_m\times\{x\}}$, hence $\mathcal{H}^i(\mathcal{F})|_{\mathbf{G}_m.x}$, is necessarily a tame local system. □

The "if" direction is false for Λ rational:

Example 3.2.3. *Let $\Lambda = \mathbf{Q}_\ell$.*

1) Let $\mathcal{K}$ be a Kummer sheaf whose corresponding representation of the tame fundamental group $\pi_1^t(\mathbf{G}_m, 1)$ does not factor through a finite quotient, then $\mathcal{K}$ cannot be trivialised by any finite cover (it has "infinite monodromy"), hence an n as in the proposition does not exist.

2) Consider the !-extension to $\mathbf{A}^1$ of the local system $\mathcal{L}$ of rank 2 concentrated in degree -1 on $\mathbf{G}_m$ corresponding to the representation $\rho : \pi_1^t(\mathbf{G}_m, 1) \to \mathbf{GL}_2(\Lambda), t \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, where t is a topological generator of $\pi_1^t(\mathbf{G}_m, 1)$. This local system also has "infinite monodromy", and an n as in the proposition does not exist.

Let Λ be finite or rational, and $\mathcal{F}$ be a monodromic sheaf. If there exists an n as in Proposition 3.2.2, we say $\mathcal{F}$ is <u>finite monodromic</u> and refer to the (multiplicatively) smallest n as the <u>twist</u> of $\mathcal{F}$, if furthermore the twist is 1, we say $\mathcal{F}$ has <u>trivial twist</u>.

Lemma 3.2.4. *1) Being finite monodromic is preserved under taking irreducible constituents, $\otimes$, and Verdier dual $\mathbb{D}$.*
2) Being monodromic is preserved under taking cones, irreducible constituents, $\otimes$, and Verdier dual $\mathbb{D}$. In particular, a sheaf is monodromic if and only if its irreducible constituents are.

For Λ rational, having finite monodromic irreducible constituents does not imply the sheaf itself is finite monodromic, as Example 3.2.3.2 shows.

Proof. 1) Let $\mathcal{F}, \mathcal{G}$ be finite monodromic sheaves, and $n > 0$ in $\mathbf{N}$ prime to p such that there exist isomorphisms $\theta(n)^*\mathcal{F}\tilde{\to}pr^*\mathcal{F}$, $\theta(n)^*\mathcal{G}\tilde{\to}pr^*\mathcal{G}$.

$\theta(n)$ and pr are smooth maps with connected geometric fibres, so $\theta(n)^*, pr^*$ are perverse t-exact and embeds $Perv(V)$ into $Perv(\mathbf{G}_m \times V)$ as a full subcategory closed under taking subquotients ([BBDG, 4.2.5]). We may thus take irreducible constituents on both sides of $\theta(n)^*\mathcal{F}\tilde{\to}pr^*\mathcal{F}$ and get the analogous isomorphisms for the irreducible constituents of $\mathcal{F}$. So the irreducible constituents are also finite monodromic.

The preservation under $\otimes$ and $\mathbb{D}$ is easily verified: $\theta(n)^*(\mathcal{F}\otimes\mathcal{G})\tilde{\to}pr^*(\mathcal{F}\otimes\mathcal{G})$, so $\mathcal{F}\otimes\mathcal{G}$ is finite monodromic. $\theta(n)^*\mathbb{D}\mathcal{F}\cong\mathbb{D}\theta(n)^!\mathcal{F}\tilde{\to}\mathbb{D}pr^!\mathcal{F}\cong pr^*\mathbb{D}\mathcal{F}$, so $\mathbb{D}\mathcal{F}$ is finite monodromic.

2) We first show the preservation under taking cones. Let $\mathcal{F}\to\mathcal{G}\to\mathcal{H}\to$ be a distinguished triangle, with $\mathcal{F},\mathcal{G}$ monodromic. The long exact sequence associated to $\mathcal{H}^i$ easily implies that $\mathcal{H}^i(\mathcal{H})$ sit inside exact sequences of the form $0\to coker_i\to\mathcal{H}^i(\mathcal{H})\to ker_i\to 0$. Restrict to any $\mathbf{G}_m$-orbit $\mathcal{O}$, $coker_i$ and ker_i become cokernels and kernels of map between tame local systems, so are themselves tame local systems. $\mathcal{H}^i(\mathcal{H})|_{\mathcal{O}}$ are thus also tame local systems.

To show the preservation under taking irreducible constituents, because of the preservation under taking cones, we may do induction on the amplitude to reduce to the case of a monodromic sheaf $\mathcal{F}$ concentrated in degree 0. Recall that for a sheaf $\mathcal{G}$ concentrated in degree 0, being monodromic is equivalent to $\theta_\lambda^*\mathcal{G}\cong\mathcal{G}$, $\forall\lambda\in k^\times$, where $\theta_\lambda: V\to V$ is the map of multiplication by λ ([Ver83, 3.2]). So $\theta_\lambda^*\mathcal{F}\cong\mathcal{F}$. Since θ_λ^* restricts to an equivalence $Perv(V)\to Perv(V)$, we may take irreducible constituents on both sides and get $\theta_\lambda^*\mathcal{P}\cong\mathcal{P}$, for each irreducible constituent $\mathcal{P}$ of $\mathcal{F}$. Further take $\mathcal{H}^i$, we get $\theta_\lambda^*\mathcal{H}^i(\mathcal{P})\cong\mathcal{H}^i(\mathcal{P})$. By [Ver83, 3.2] again, $\mathcal{H}^i(\mathcal{P})$ is monodromic.

We now show the preservation under $\otimes$. Let $\mathcal{F},\mathcal{G}$ be monodromic sheaves. Because of the preservation under taking cones, we may do induction on the amplitude to reduce to the case where $\mathcal{F},\mathcal{G}$ are concentrated in degree 0. Then, for any $\mathbf{G}_m$ orbit $\mathcal{O}$, $\mathcal{F}|_{\mathcal{O}}$ and $\mathcal{G}|_{\mathcal{O}}$ are tame local systems in degree 0. It follows that $(\mathcal{F}|_{\mathcal{O}})\otimes(\mathcal{G}|_{\mathcal{O}})$ is a tame local system. So $\mathcal{H}^i(\mathcal{F}\otimes\mathcal{G})|_{\mathcal{O}}=\mathcal{H}^i((\mathcal{F}|_{\mathcal{O}})\otimes(\mathcal{G}|_{\mathcal{O}}))$ is a tame local system, $\mathcal{F}\otimes\mathcal{G}$ is monodromic.

To show the preservation under $\mathbb{D}$, because of the preservation under taking cones, we may reduce to the case of perverse irreducible monodromic sheaves. The finite coefficient case is dealt with in 1). For Λ rational, we may further assume $\Lambda=\overline{\mathbf{Q}}_\ell$ because of the easily verified fact that, for Λ rational, $\mathcal{F}$ is monodromic if and only if $\mathcal{F}\otimes_\Lambda\overline{\mathbf{Q}}_\ell$ is. In this case, the statement follows from Proposition 3.2.5 items 1) and 3), and the compatibility of the Fourier transform and linear actions of algebraic groups ([Lau87, 1.2.3.4]). □

Proposition 3.2.5. *1) If $\mathcal{F}\in D(V)$ is perverse, then $\mathcal{F}$ is (finite monodromic) with trivial twist if and only if $\mathcal{F}$ is $\mathbf{G}_m$-equivariant. This is preserved under the Fourier transform.*
2) Let $\mathcal{F}\in D(V)$ be perverse irreducible finite monodromic with twist $n>1$. Assume Λ contains a primitive n-th root of unity[3]. Then there exists a Kummer sheaf $\mathcal{K}$ on $\mathbf{G}_m$ (unique up to isomorphism), trivialised by the power n cover of $\mathbf{G}_m$, such that $\theta(1)^*\mathcal{F}\cong\mathcal{K}\boxtimes\mathcal{F}[-1]$. Furthermore, $\theta(1)^*F\mathcal{F}\cong\mathcal{K}^{-1}\boxtimes F\mathcal{F}[-1]$.

[3]This can always be achieved by adjoining a primitive n-th root of unity, see Remark 3.2.7.

3) Let $\Lambda = \overline{\mathbf{Q}}_\ell$. Let $\mathcal{F} \in D(V)$ be perverse irreducible monodromic with twist $n > 1$. Then there exists a Kummer sheaf $\mathcal{K}$ on $\mathbf{G}_m$ (unique up to isomorphism), trivialised by the power n cover of $\mathbf{G}_m$, such that $\theta(1)^*\mathcal{F} \cong \mathcal{K} \boxtimes \mathcal{F}[-1]$. Furthermore, $\theta(1)^*F\mathcal{F} \cong \mathcal{K}^{-1} \boxtimes F\mathcal{F}[-1]$.
4) $\mathcal{F} \in D(V)$ is monodromic (resp. finite monodromic) if and only if $F\mathcal{F}$ is monodromic (resp. finite monodromic). In the finite monodromic case, they have the same twist.

Note that in situations 2) and 3), the restriction of $\mathcal{F}$ to any $\mathbf{G}_m$-orbit not equal to $\{0\}$ is of the form $\mathcal{C} \otimes \mathcal{K}$, for some *constant* sheaf $\mathcal{C}$ (depending on the orbit). We also say that $\mathcal{K}$ is the twist of $\mathcal{F}$.

Proof. 1) The first statement follows from the characterisation of perverse sheaves being equivariant under actions of connected algebraic groups (c.f. [Ach21, 6.2.17]). The second statement follows from the compatibility of the Fourier transform and linear actions of algebraic groups.

2) Being perverse irreducible, $\mathcal{F}$ is of the form $j_{!*}\mathcal{L}$ for some irreducible local system $\mathcal{L}$ on some smooth irreducible locally closed conic subvariety $S \hookrightarrow \mathring{V}$.[4] The restriction of $\theta(n)^*\mathcal{F} \tilde{\rightarrow} pr^*\mathcal{F}$ to $\mathbf{G}_m \times S$ gives $\theta(n)|^*_{\mathbf{G}_m \times S}\mathcal{L} \tilde{\rightarrow} pr|^*_{\mathbf{G}_m \times S}\mathcal{L}$.

Claim: $\theta(1)|^*_{\mathbf{G}_m \times S}\mathcal{L}$ is isomorphic to $\mathcal{K} \boxtimes \mathcal{L}[-1]$ for some Kummer sheaf $\mathcal{K}$ on $\mathbf{G}_m$ (necessarily unique up to isomorphism), trivialised by the power n cover of $\mathbf{G}_m$.

Accepting this claim, the conclusions follow: using the well-known characterisation of $j_{!*}$ (c.f. [Ach21, 3.3.4]), it is easily seen that $\theta(1)^*\mathcal{F} \cong \mathcal{K} \boxtimes pr^*\mathcal{F}[-1]$. In fact, $\theta(1)^*\mathcal{F} \cong \theta(1)^*(j_{!*}\mathcal{L}) \cong j_{!*}(\theta(1)^*\mathcal{L}) \cong j_{!*}(\mathcal{K} \boxtimes pr^*\mathcal{L}[-1])$. Apply 3.3.4 in *loc. cit.*, we get $\mathcal{K} \boxtimes j_{!*}pr^*\mathcal{L}[-1]$ is the middle extension of $\mathcal{K} \boxtimes pr^*\mathcal{L}[-1]$, hence isomorphic to $\theta(1)^*\mathcal{F}$. The last statement follows from the compatibility of the Fourier transform and linear actions of algebraic groups.

It remains to prove the claim. Denote $e(n) : \mathbf{G}_m \to \mathbf{G}_m, \lambda \mapsto \lambda^n$. Consider $\mathcal{L}' := (e(n) \times id)_*(e(n) \times id)^*\theta(1)|^*_{\mathbf{G}_m \times S}\mathcal{L}$. Since $e(n) \times id$ is finite étale, $\mathcal{L}'$ is a local system concentrated in a single degree. Denote its corresponding $\pi_1(\mathbf{G}_m \times S)$ (we omit the base points in the notation from here on) representation by $\rho' : \pi_1(\mathbf{G}_m \times S) \to \mathrm{Aut}_\Lambda(L')$. ρ' factors through $\pi_1(\mathbf{G}_m \times S) \to \pi_1^t(\mathbf{G}_m) \times \pi_1(S) \to \mathbf{Z}/n \times \pi_1(S)$. The adjunction $id \to (e(n) \times id)_*(e(n) \times id)^*$ realises $\theta(1)|^*_{\mathbf{G}_m \times S}\mathcal{L}$ as a sub-local-system of $\mathcal{L}'$. Its corresponding representation $\rho : \pi_1(\mathbf{G}_m \times S) \to \mathrm{Aut}_\Lambda(L_1)$ thus also factors through $\mathbf{Z}/n \times \pi_1(S)$. Since $\theta(1)|^*_{\mathbf{G}_m \times S}\mathcal{L}$ is irreducible, ρ is irreducible. By our assumption on Λ, we can apply Lemma 3.2.6 case 1) and get $L_1 \cong K \boxtimes L$ as $\mathbf{Z}/n \times \pi_1(S)$ representations, for some 1-dimensional representation K of $\mathbf{Z}/n$. Consequently $\theta(1)|^*_{\mathbf{G}_m \times S}\mathcal{L} \cong \mathcal{K} \boxtimes \mathcal{G}$, for some Kummer sheaf $\mathcal{K}$

[4] Proof that S can be chosen to be conic (note the proof only requires $\mathcal{F}$ being perverse irreducible monodromic): assume $\mathcal{F}$ is not a local system, let D be its ramification divisor. We show D is conic. Let x be any closed point of D. Then, for some i, $\mathcal{H}^i(\mathcal{F})$ is a local system near x. But $\mathcal{H}^i(\mathcal{F})$ is monodromic, hence isomorphic to itself under the pullback by the λ-scaling, $\forall \lambda \in k^\times$ ([Ver83, 3.2]), so $\mathcal{H}^i(\mathcal{F})$ is not a local system near $\lambda.x, \forall \lambda \in k^\times$. This forces D to be conic, as D is a divisor.

and some sheaf $\mathcal{G}$. Looking at the restriction of $\theta(1)|^*_{\mathbf{G}_m\times S}\mathcal{L}$ to $1\times S\hookrightarrow \mathbf{G}_m\times S$, we see $\mathcal{G}$ is necessarily isomorphic to $\mathcal{L}[-1]$. $\mathcal{K}$ is clearly trivialised by the power n cover of $\mathbf{G}_m$.

3) The argument is similar to 2), we indicate the differences. Consider $\theta(1)|^*_{\mathbf{G}_m\times S}\mathcal{L}$ as above. Fix a torsion free integral model for $\theta(1)|^*_{\mathbf{G}_m\times S}\mathcal{L}$. For each of its reductions mod ℓ^r, the corresponding $\pi_1(\mathbf{G}_m\times S)$-representation (over $\mathbf{Z}/\ell^r$) factors through $\mathbf{Z}/m\times\pi_1(S)$ for varying m. Take the limit over r, we get that the (continuous) $\pi_1(\mathbf{G}_m\times S)$-representation over $\overline{\mathbf{Q}}_\ell$ corresponding to $\theta(1)|^*_{\mathbf{G}_m\times S}\mathcal{L}$ factors through $\pi_1^t(\mathbf{G}_m)\times\pi_1(S)$. It is necessarily irreducible. Apply Lemma 3.2.6 case 2) (easily modified to take continuity into account), we get an external product decomposition of $\theta(1)|^*_{\mathbf{G}_m\times S}\mathcal{L}$. The rest is similar.

4) The statements concerning the finite monodromic case follow from the compatibility of the Fourier transform and linear actions of algebraic groups, and the fact that being finite monodromic implies $a^*\mathcal{F}\cong\mathcal{F}$, where a is the antipodal map. The statement concerning the monodromic (Λ rational) case follows from 3) above, Lemma 3.2.4.1, and the easily verified fact that, for Λ rational, $\mathcal{F}$ is monodromic if and only if $\mathcal{F}\otimes_\Lambda\overline{\mathbf{Q}}_\ell$ is. □

Lemma 3.2.6. *Let H be an abelian group, G be any group, Λ be a field. Assume either 1) $H=\mathbf{Z}/n$ for $n>1$ in $\mathbf{N}$, and Λ contains a primitive n-th root of unity, or 2) Λ is algebraically closed. Then, for any finite dimensional irreducible Λ-representation M of $H\times G$, there exist irreducible Λ-representations K of H and L of G and an isomorphism $K\boxtimes L\cong M$ as representations of $H\times G$. Note, necessarily,* $\dim_\Lambda(K)=1$, $\dim_\Lambda(L)=\dim_\Lambda(M)$.

Here $\boxtimes$ denotes the external tensor product of group representations. It is also denoted by $\otimes$ in the literature.

Proof. In case 1), all finite dimensional representations of $\mathbf{Z}/n$ are semisimple (note that n is necessarily invertible in Λ), and irreducible ones are 1-dimensional. View M as a representation of $\mathbf{Z}/n=\mathbf{Z}/n\times\{1\}$, it decomposes as $M=\oplus_i K_i^{\oplus r_i}$, for some 1-dimensional representations K_i of $\mathbf{Z}/n$, and $r_i>1$. Since the $G=\{1\}\times G$ action commutes with the $\mathbf{Z}/n$ action, each $K_i^{\oplus r_i}$ is a sub-representation of $\mathbf{Z}/n\times G$. By the irreducibility of M, there is only one of them. Denote it by $M=K^r$. View M as a representation of G, and denote it by L, then clearly $M\cong K\boxtimes L$ as representations of $\mathbf{Z}/n\times G$.

In case 2), since $H\times\{1\}$ and $\{1\}\times G$ commute, and H is abelian, the homomorphism $H\times\{1\}\to End_\Lambda M$ lands in $End_{H\times G-rep}M$. By Schur's Lemma, $End_{H\times G-rep}M\cong\Lambda$. We see that each element of $H\times\{1\}$ must act through scaling. Denote the corresponding 1-dimensional representation of H by K, and M regarded as a G-representation (clearly irreducible) by L. Then $M\cong K\boxtimes L$ as representations of $H\times G$. □

Remark 3.2.7. *We own the following observation to Beilinson: for Λ finite, and $\mathcal{F}\in D(V)$ a perverse irreducible monodromic sheaf, the twist n is always prime to ℓ. Proof: using the* *same notations as in the fourth paragraph of the proof of Proposition 3.2.5.2, we claim the representation $\rho : \mathbf{Z}/n \times \pi_1(S) \to \mathrm{Aut}_\Lambda L_1$ must factor through $((\mathbf{Z}/n)/\{\ell-torsion\}) \times \pi_1(S)$. Since it follows easily from the definition of the twist that n is the smallest positive integer for which a factorisation $\pi_1(\mathbf{G}_m \times S) \to \mathbf{Z}/n \times \pi_1(S) \to \mathrm{Aut}_\Lambda L_1$ exists, n must be prime to ℓ. To see the claim, note that $\mathbf{Z}/n \times \{1\}$ is in the centre of $\mathbf{Z}/n \times \pi_1(S)$, so it lands in $End_{\mathbf{Z}/n\times\pi_1(S)-rep} L_1$. As ρ is irreducible, the latter is a division algebra over Λ by Schur's Lemma. If $m \in \mathbf{Z}/n \times \{1\}$ is ℓ-torsion, say $\ell^r m = 0$, then $\rho(\ell^r m) = \rho(m)^{\ell^r} = id$, so $\rho(m)^{\ell^r} - id = (\rho(m) - id)^{\ell^r} = 0$, so $\rho(m) = id$.*

Proposition 3.2.8. *Let $\mathcal{F} \in D(V)$ be perverse irreducible monodromic, with non-trivial twist $\mathcal{K}$. Assume either 1) $\mathcal{F}$ is finite monodromic with twist n and Λ contains a primitive n-th root of unity, or 2) $\Lambda = \overline{\mathbf{Q}}_\ell$. Let (W, σ) be the data of an open conic subvariety W of $\mathring{V} := V - \{0\}$ together with a section σ of the projection $q : \mathring{V} \to \mathbf{P}(V)$ (restricted to W). Then, $\mathcal{F}|_W \cong \underline{\mathcal{F}_\sigma} \boxtimes \mathcal{K}$ for some perverse irreducible sheaf $\underline{\mathcal{F}_\sigma}$ on $\mathbf{P}(V)$ (unique up to isomorphism).*

Here, σ determines an isomorphism $W \cong_\sigma \underline{W} \times \mathbf{G}_m$, and the $\boxtimes$ is with respect to this isomorphism. We emphasise that $\underline{\mathcal{F}_\sigma}$ depends on σ.

Proof. $\mathcal{F}$ is of the form $j_{!*}\mathcal{L}$ for some irreducible local system $\mathcal{L}$ on some smooth irreducible locally closed conic subvariety $S \hookrightarrow \mathring{V}$, $\mathcal{F}|_W \cong j_{!*}(\mathcal{L}|_{W\cap S})$. Consider the sheaf $(\mathcal{L}|_{W\cap S}) \otimes pr_2^*\mathcal{K}^{-1}$, where $pr_2 : W \cong_\sigma \underline{W} \times \mathbf{G}_m \to \mathbf{G}_m$ is the second projection. By the comment after the statement of Proposition 3.2.5, $(\mathcal{L}|_{W\cap S}) \otimes pr_2^*\mathcal{K}^{-1}$ is a local system concentrated in a single degree and constant on each closed fibre of the projection $pr_1 : \underline{W \cap S} \times \mathbf{G}_m \to \underline{W \cap S}$. Apply Lemma 3.2.9, we get $(\mathcal{L}|_{W\cap S}) \otimes pr_2^*\mathcal{K}^{-1} \cong pr_1^*\mathcal{L}'[2]$ for some (perverse) local system $\mathcal{L}'$ on $\underline{W \cap S}$. (In fact $\mathcal{L}'$ must be isomorphic to $((\mathcal{L}|_{W\cap S}) \otimes pr_2^*\mathcal{K}^{-1})|_{\underline{W\cap S}\times\{1\}}[-2]$.) So $\mathcal{L}|_{W\cap S} \cong \mathcal{L}' \boxtimes \mathcal{K}$. Reasoning as in the third paragraph in proof of Proposition 3.2.5.2, the well-known characterisation of $j_{!*}$ implies $\mathcal{F}|_W \cong j_{!*}(\mathcal{L}|_{W\cap S}) \cong j_{!*}(\mathcal{L}' \boxtimes \mathcal{K}) \cong (j_{!*}\mathcal{L}') \boxtimes \mathcal{K} =: \underline{\mathcal{F}_\sigma} \boxtimes \mathcal{K}$. $\square$

Lemma 3.2.9. *Let $f : X \to Y$ be a smooth morphism between varieties of relative dimension d, with geometrically connected fibres. If $\mathcal{F}$ is a sheaf on X concentrated in degree 0, such that for each closed point $y \in Y$, there exists an isomorphism $\mathcal{F}|_{X_y} \cong \underline{\Lambda}^r$, for some $r_i \in \mathbf{N}$ independent of y. Then the canonical map $\mathcal{F} \to f^*\mathcal{H}^{2d}(f_!\mathcal{F})$ is an isomorphism. If $\mathcal{F}$ is perverse, then $\mathcal{H}^{2d}(f_!\mathcal{F})$ is the unique (up to isomorphism) sheaf on Y (necessarily perverse and concentrated in degree 0) whose pullback is isomorphic to $\mathcal{F}$.*

Here the map $\mathcal{F} \to f^*\mathcal{H}^{2d}(f_!\mathcal{F})$ is obtained by taking $\mathcal{H}^0$ of the adjunction map $\mathcal{F} \to f^! f_!\mathcal{F}$.

Proof. [5] It suffices to show $\mathcal{F} \to f^*\mathcal{H}^{2d}(f_!\mathcal{F})$ is an isomorphism for each closed point $x \in X$. This, in turn, is implied by $\mathcal{F}|_{X_y} \tilde{\to} (f^*\mathcal{H}^{2d}(f_!\mathcal{F}))|_{X_y} \cong f^*\mathcal{H}^{2d}(f_!(\mathcal{F}|_{X_y}))$ for each closed point $y \in Y$, where the last isomorphism is from proper base change. Using $\mathcal{F}|_{X_y} \cong \underline{\Lambda}^r$, the question reduces to showing $\underline{\Lambda}_{X_y} \tilde{\to} p^*\mathcal{H}^{2d}(p_!\underline{\Lambda}_{X_y})$, which is clear (here we use the connectedness

[5]This proof follows the suggestion of Will Sawin in https://mathoverflow.net/questions/225468

of X_y). The assertion when $\mathcal{F}$ is perverse is a direct consequence of the fact that f^* induces a fully faithful embedding of $Perv(Y)$ into $Perv(X)$ ([BBDG, 4.2.5], here we use again the geometrically-connectedness of fibres). □

Remark 3.2.10. *Proposition 3.2.8 can fail without the assumptions on* Λ. *In fact, if* $\mathcal{L}|_{W\cap S}$ *corresponds to an irreducible representation of* $\pi_1(\underline{W \cap S}) \times \mathbf{Z}/n$ *which cannot be written as an external tensor product (which can exist without the assumptions on* Λ*), then* $\mathcal{L}|_{W\cap S}$ *cannot be written as an external tensor product.*

We now turn to F-good sheaves. Recall:

Definition 3.2.11 (F-good sheaves)**.** $\mathcal{F} \in D(V)$ *is* F-good *if for each irreducible constituent* $\mathcal{P}$, $CC(\mathcal{P}) = CC(F\mathcal{P})$.

Remark 3.2.12. *F-good sheaves are not necessarily monodromic. For example, let* $\mathcal{L}$ *be a local system concentrated in degree 0 on* $\mathbf{G}_m$, $\mathcal{F}$ *be the !-extension of* $\mathcal{L}[1]$ *to* $\mathbf{A}^1$. *If* $\mathcal{L}$ *is purely of slope* < 1 *at* ∞, *then* $\mathcal{F}$ *is F-good, as one can verify using Laumon's local Fourier transforms (c.f. [Lau87, 2.3.1, 2.4.3]). But such an* $\mathcal{F}$ *is not monodromic if* $\mathcal{L}$ *is not tame.*

Lemma 3.2.13. *1) Being F-good is preserved under taking cones, taking irreducible constituents, and Verdier dual* $\mathbb{D}$.
2) Let $f : W \to V$ *be a linear injection (resp. surjection) between finite dimensional vector spaces,* $\mathcal{F}$ *(resp.* $\mathcal{G}$*) be an F-good sheaf on* W *(resp.* V*). Then* $f_*\mathcal{F}$ *(resp.* $f^*\mathcal{G}$*) is F-good.*

Proof. 1) That being F-good is preserved under taking irreducible constituents is clear. If $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to$ is a distinguished triangle, the long exact sequence associated to ${}^p\mathcal{H}^i$ easily implies that irreducible constituents of $\mathcal{G}$ is a subset of the union of irreducible constituents of $\mathcal{F}$ and $\mathcal{H}$. So the F-goodness of $\mathcal{F}$ and $\mathcal{H}$ implies the F-goodness of $\mathcal{G}$. Finally, let $\mathcal{F}$ be F-good, we show $\mathbb{D}\mathcal{F}$ is F-good: as $\mathbb{D}$ is an anti-equivalence preserving $Perv(V)$, it suffices to prove $CC(\mathbb{D}\mathcal{F}) = CC(F\mathbb{D}\mathcal{F})$ for $\mathcal{F}$ perverse irreducible. Apply the formula $F\mathbb{D} \cong a^*\mathbb{D}F$ to $\mathcal{F}$ (where a is the "multiplication by -1" on V), using the monodromicity of $\mathbb{D}F\mathcal{F}$, we get $F\mathbb{D}\mathcal{F} \cong \mathbb{D}F\mathcal{F}$. So $CC(F\mathbb{D}\mathcal{F}) = CC(\mathbb{D}F\mathcal{F})$. By $CC\mathbb{D} = CC$ (see [Sai17b, 5.13.4] for Λ finite, and Proposition 3.5.3 for Λ rational), and use the assumption that $\mathcal{F}$ is F-good, we get $CC(\mathbb{D}\mathcal{F}) = CC(\mathcal{F}) = CC(F\mathcal{F}) = CC(\mathbb{D}F\mathcal{F}) = CC(F\mathbb{D}\mathcal{F})$.

2) This follows directly from the compatibility of the Fourier transform with linear maps ([Lau87, 1.2.2.4] and its dual version), and the behaviour of CC under closed immersions and smooth pullbacks (see [Sai17b, 5.13.2, 5.17] for Λ finite, the rational case follows easily from the finite case and the definition of CC, reviewed in the Appendix). □

3.3 The case of the trivial twist

As above, Λ can be either finite or rational. We prove the special case of Theorem 3.1.4 where the sheaf has trivial twist, which will be the basis for the proof of the general case. In the terminology of the intuition mentioned in the Introduction, we deal with the "projective components" in this section.

Proposition 3.3.1. *If $\mathcal{F} \in D(V)$ is perverse irreducible with trivial twist, then $\mathcal{F}$ is F-good.*

Recall that, for $\mathcal{F}$ perverse, having trivial twist is equivalent to being $\mathbf{G}_m$-equivariant, and this is preserved under the Fourier transform (Proposition 3.2.5.1). We fix some notations. Let $\pi : \tilde{V} \to V$ be the blowup of V at 0, $\tilde{q} : \tilde{V} \to \mathbf{P}(V)$ be the natural projection, j be the inclusion $\mathring{V} \subseteq V$. We use the same letters with " $'$ " to denote the corresponding maps on the dual side.

Proof. We first prove $CC(\mathcal{F}) = CC(F\mathcal{F})$ away from the 0-sections and 0-fibres (i.e., with the components supported on $V \times 0$ and $0 \times V'$ removed). Consider the following diagram, where each sequence is a distinguished triangle:

$$\begin{tikzcd}
 & & & \mathcal{F}_! \arrow[dl] \arrow[dr] & & & \\
 & & \mathcal{F} \arrow[dl] & & \tilde{\mathcal{F}} \arrow[dr] & & \\
 & \mathcal{F}_0 \arrow[dl] & & & & \tilde{\mathcal{F}}_0 \arrow[dr] & \\
 {} & & & & & & {}
\end{tikzcd}$$

Here $\mathcal{F}_! := j_!(\mathcal{F}|_{\mathring{V}})$, $\tilde{\mathcal{F}} := \pi_* \tilde{q}^* \mathcal{F}$, and $\mathcal{F}_0$ (resp. $\tilde{\mathcal{F}}_0$) is the stalk of $\mathcal{F}$ (resp. $\tilde{\mathcal{F}}$) at 0, viewed as skyscraper sheaves. As $\mathcal{F}$ is $\mathbf{G}_m$-equivariant, $\mathcal{F}|_{\mathring{V}}$ descends to some $\mathcal{G}$ on $\mathbf{P}(V)$. By the compatibility of the Radon transform and the Fourier transform ([Bry86, 9.13]), $(F\tilde{\mathcal{F}})|_{\mathring{V}'} \cong q'^* R\mathcal{G}$, where R is the Radon transform on $\mathbf{P}(V)$. By the smooth pullback formula for CC and the compatibility of CC with the Radon transform (see [Sai17b, 7.5] for Λ finite, and Proposition 3.5.3.3 for Λ rational) this easily implies $CC(F\tilde{\mathcal{F}}) = CC(\mathcal{F})$ away from the 0-sections and 0-fibres. By the above diagram and its Fourier dual, $CC(\mathcal{F}) = CC(\tilde{\mathcal{F}}) - CC(\tilde{\mathcal{F}}_0) + CC(\mathcal{F}_!)$, $CC(F\mathcal{F}) = CC(F\tilde{\mathcal{F}}) - CC(F\tilde{\mathcal{F}}_0) + CC(F\mathcal{F}_!)$. Since the last two terms in each equality are supported on the 0-sections and 0-fibres, we conclude that $CC(\mathcal{F}) = CC(F\mathcal{F})$ away from the 0-sections and 0-fibres.

To prove the full equality $CC(\mathcal{F}) = CC(F\mathcal{F})$, consider $i : V \hookrightarrow V \times \mathbf{A}^1, v \mapsto (v, 0)$ and its dual $p' : (V \times \mathbf{A}^1)' \to V'$. $i_*\mathcal{F}$ is still perverse irreducible with trivial twist, so, by the above paragraph, $CC(i_*\mathcal{F}) = CC(Fi_*\mathcal{F}) = CC(p'^* F\mathcal{F}[1])$ away from the 0-sections and 0-fibres (of $V \times \mathbf{A}^1$). The 0-section in $CC(\mathcal{F})$ corresponds to the component $\{(v, adt), v \in V, a \in k\}$ in $CC(i_*\mathcal{F})$ (t is the linear coordinate on $\mathbf{A}^1$), and the 0-fibre in $CC(F\mathcal{F})$ corresponds to the component $\{(0, \xi), \xi \in V \cong T^*_{0'}V'\}$ in $CC(p'^* F\mathcal{F}[1])$. They are away from the 0-sections and 0-fibres (of $V \times \mathbf{A}^1$), hence equal. This proves the 0-section in $CC(\mathcal{F})$ equals the 0-fibre in $CC(F\mathcal{F})$. Apply the Fourier inversion, we get the 0-section in $CC(F\mathcal{F})$ equals the 0-fibre in $CC(\mathcal{F})$. Hence the full equality $CC(\mathcal{F}) = CC(F\mathcal{F})$. □

Corollary 3.3.2. *If $\mathcal{F} \in D(V)$ is such that all its irreducible constituents have trivial twists (equivalently, $\mathbf{G}_m$-equivariant), then $\mathcal{F}$ is F-good.*

Corollary 3.3.3. *If $\mathcal{F} \in D(V)$ is such that $\mathcal{F}|_{\mathring{V}} \cong q^*\underline{\mathcal{F}}$ for some $\underline{\mathcal{F}} \in D(\mathbf{P}(V))$, then $\mathcal{F}$ is F-good.*

Proof. By the corollary above, it suffices to show all irreducible constituents of $\mathcal{F}$ are $\mathbf{G}_m$-equivariant. Let $\mathcal{P}$ be an irreducible constituent. If $\mathcal{P}$ is supported at $\{0\}$, this is clear. If not, then $\mathcal{P}$ is of the form $j_{!*}\mathcal{L}$ for $\mathcal{L}$ some perverse irreducible local system on some smooth irreducible subvariety in $\mathring{V}$. So $\mathcal{P}|_{\mathring{V}}$ is still perverse irreducible. Since the restriction to $\mathring{V}$ is perverse t-exact, $\mathcal{P}|_{\mathring{V}}$ is an irreducible constituent of $\mathcal{F}|_{\mathring{V}} \cong q^*\underline{\mathcal{F}}$. Since q^* is also perverse t-exact and induces a fully faithful embedding of $Perv(\mathbf{P}(V))$ into $Perv(\mathring{V})$ closed under taking subquotients ([BBDG, 4.2.5]), it is easily seen that irreducible constituents of $q^*\underline{\mathcal{F}}$ are exactly q^* of irreducible constituents of $\underline{\mathcal{F}}$. So $\mathcal{P}|_{\mathring{V}} \cong q^*\underline{\mathcal{P}}$ for some perverse irreducible $\underline{\mathcal{P}}$ on $\mathbf{P}(V)$. So $\mathcal{P}|_{\mathring{V}}$, hence $\mathcal{P}$, is $\mathbf{G}_m$-equivariant. □

Remark 3.3.4. *It follows from the proof that the irreducible constituents of such an $\mathcal{F}$ have trivial twists, so $\mathcal{F}$ is necessarily monodromic (Lemma 3.2.4). Note that, for Λ finite, $\mathcal{F}$ needs not have trivial twist; for Λ rational, $\mathcal{F}$ needs not be of finite monodromy. The Fourier transform of $j_{!*}\mathcal{L}$ for $\mathcal{L}$ as in Example 3.2.3.2 and $j : \mathbf{G}_m \to \mathbf{A}^1$ gives such an example ($j_{!*}\mathcal{L}$ is in fact the maximal extension of $\underline{\Lambda}_{\mathbf{G}_m}[1]$).*

For a monodromic sheaf $\mathcal{F} \in V$, its singular support is $\mathbf{G}_m$-stable[6]. The 0-fibre $SS(\mathcal{F}) \cap T_0^*V$ is either T_0^*V or $\overline{SS(\mathcal{F})|_{\mathring{V}}} \cap T_0^*V$ (closure taken in T^*V). As an application of the above, we record a formula, applicable to perverse sheaves with trivial twists, which allows us to tell which case happens. This will not be used in the sequel.

Proposition 3.3.5. *Let $\mathcal{F} \in D(V)$ be such that $\mathcal{F}|_{\mathring{V}} \cong q^*\underline{\mathcal{F}}[1]$ for some sheaf $\underline{\mathcal{F}}$ on $\mathbf{P}(V)$. Then, the coefficient of T_0^*V in $CC(\mathcal{F})$ equals $rk_0\mathcal{F} - \chi(\mathbf{P}(V), \underline{\mathcal{F}}[1]) + \chi(H, \underline{\mathcal{F}}[1])$, where χ denotes the Euler characteristic, and H is a general hyperplane on $\mathbf{P}(V)$.[7] If $\mathcal{F}$ is also perverse, then $SS(\mathcal{F}) \cap T_0^*V = T_0^*V$ if and only if $rk_0\mathcal{F} - \chi(\mathbf{P}(V), \underline{\mathcal{F}}[1]) + \chi(H, \underline{\mathcal{F}}[1]) \neq 0$.*

Proof. It suffices to prove the first statement, as the second statement follows from the first and the effectivity of characteristic cycles of perverse sheaves. Denote $\dim V$ by d. It follows from Corollary 3.3.3 that the coefficient of T_0^*V in $CC(\mathcal{F})$ equals $(-1)^d.rk(F\mathcal{F})$, where rk denotes the generic rank. Using the same diagram and notations at the beginning of the proof of Proposition 3.3.1, we get $rk(F\mathcal{F}) = rk(F\mathcal{F}_0) + rk(F\tilde{\mathcal{F}}) - rk(F\tilde{\mathcal{F}}_0)$. Compute:
$rk(F\mathcal{F}_0) = (-1)^d.rk_0(\mathcal{F})$;

[6]This can be seen, e.g., using Proposition 3.2.8.

[7]More precisely: there exists an open dense $U \subseteq \mathbf{P}(V')$, such that $\chi(\tilde{H}, \underline{\mathcal{F}}[1])$, as a function of hyperplanes $\tilde{H}$ (parametrised by closed points of $\mathbf{P}(V')$), is constant. $\chi(H, \underline{\mathcal{F}}[1])$ is defined to be this constant.

$rk(F\tilde{\mathcal{F}}_0) = (-1)^d.rk(\tilde{\mathcal{F}}_0) = (-1)^d.\chi(\mathbf{P}(V), \underline{\mathcal{F}}[1])$;
$rk(F\tilde{\mathcal{F}}) = rk(q^*R\underline{\mathcal{F}}[1]) = rk(R\underline{\mathcal{F}}[1]) = (-1)^{d-2}.\chi(H, \underline{\mathcal{F}}[1])$,
where in the last line we have used the compatibility of the Radon transform and the Fourier transform. The statement easily follows. □

3.4 Proof of the main theorem

As above, Λ can be either finite or rational.

Theorem 3.4.1. *Monodromic sheaves are F-good.*

We present two proofs. The first one follows Beilinson's key idea of untwisting the sheaf after pulling back to $V \times \mathbf{A}^1$, reducing the general case to the trivial twist case.

Proof. Since irreducible constituents of a monodromic sheaf are monodromic (Lemma 3.2.4), it suffices to prove the claim for perverse irreducible monodromic sheaves. Further, since being monodromic is clearly preserved under a finite extension of the coefficient field, and characteristic cycles do not change under this extension (c.f. the discussion after Definition 3.5.1), we may assume that Λ contains a primitive n-th root of unity[8], where n is the twist of the sheaf in consideration. In the rational case, we may further assume $\Lambda = \overline{\mathbf{Q}}_\ell$.

Let $\mathcal{F} \in D(V)$ be perverse irreducible monodromic. We want to show $CC(\mathcal{F}) = CC(F\mathcal{F})$. If $\mathcal{F}$ has trivial twist, then it is F-good by Proposition 3.3.1. Assume $\mathcal{F}$ has non-trivial twist $\mathcal{K}$. Consider $\mathcal{F} \boxtimes \mathcal{K}$ on $V \times \mathbf{A}^1$. We claim that $\mathcal{F} \boxtimes \mathcal{K}^{-1}$ satisfies $(\mathcal{F} \boxtimes \mathcal{K}^{-1})|_{(V\times\mathbf{A}^1)-\{0\}} \cong q^*(\underline{\mathcal{F} \boxtimes \mathcal{K}^{-1}})$ for some $\underline{\mathcal{F} \boxtimes \mathcal{K}^{-1}} \in D(\mathbf{P}(V \times \mathbf{A}^1))$, where $q : (V \times \mathbf{A}^1) - \{0\} \to \mathbf{P}(V \times \mathbf{A}^1)$ is the projection. Accepting this claim, Corollary 3.3.3 implies that $CC(\mathcal{F} \boxtimes \mathcal{K}^{-1}) = CC(F(\mathcal{F} \boxtimes \mathcal{K}^{-1}))$. As, in general, $CC(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = CC(\mathcal{F}_1) \boxtimes CC(\mathcal{F}_2)$ (for Λ finite, see [Sai17a, 2.2]; for Λ rational, this is verified in Proposition 3.5.3).4 and F commutes with $\boxtimes$ (c.f. [Lau87, 1.2.2.7]), it easily follows that $CC(\mathcal{F}) = CC(F\mathcal{F})$.

It remains to show the claim. $\mathcal{F}$ is of the form $j_{!*}\mathcal{L}$ for some irreducible local system $\mathcal{L}$ on some smooth irreducible locally closed conic subvariety $S \hookrightarrow \mathring{V}$. So $\mathcal{F} \boxtimes \mathcal{K}^{-1} \cong (j_{!*}\mathcal{L}) \boxtimes \mathcal{K}^{-1} \cong j_{!*}(\mathcal{L} \boxtimes \mathcal{K}^{-1})$. By our construction, $\mathcal{L} \boxtimes \mathcal{K}^{-1}$ is a local system concentrated in a single degree, which is constant when restricted to each $\mathbf{G}_m$ orbit in $S \times \mathbf{A}^1$ (note that $\mathcal{K}^{-1}$ is 0 at $\{0\} \in \mathbf{A}^1$). By Lemma 3.2.9, $\mathcal{L} \boxtimes \mathcal{K}^{-1} \cong q^*(\underline{\mathcal{L} \boxtimes \mathcal{K}^{-1}})$, for some $\underline{\mathcal{L} \boxtimes \mathcal{K}^{-1}} \in D(\mathbf{P}(S \times \mathbf{A}^1))$. Then $\mathcal{F} \boxtimes \mathcal{K}^{-1} \cong j_{!*}q^*(\underline{\mathcal{L} \boxtimes \mathcal{K}^{-1}})$. Its restriction to $(V \times \mathbf{A}^1) - \{0\}$ is isomorphic to $q^*j_{!*}(\underline{\mathcal{L} \boxtimes \mathcal{K}^{-1}})$. □

We now present our original proof, which uses the local decomposition (Proposition 3.2.8) and the notion of having the same wild ramification to reduce to the trivial twist case.

[8] In the extended coefficient field Λ', we will use the character $\psi' : \mathbf{Z}/p \to \Lambda \hookrightarrow \Lambda'$ to define the Fourier transform, where the first arrow is the character ψ for Λ. This ensures $(F_\psi\mathcal{F}) \otimes_\Lambda \Lambda' \cong F_{\psi'}(\mathcal{F} \otimes_\Lambda \Lambda')$.

Proof. As explained at the beginning of the previous proof, it suffices to consider perverse irreducible monodromic sheaves, and we may assume, in the finite case, that Λ contains a primitive n-th root of unity where n is the twist of the sheaf in consideration, or, in the rational case, that $\Lambda = \overline{\mathbf{Q}}_\ell$.

We do induction on $d = \dim V$. For $d = 1$, there are three types of perverse irreducible monodromic sheaves: i) the rank 1 skyscraper at $\{0\}$, ii) the rank 1 constant sheaf in degree -1 on V, iii) (!-extension of) Kummer sheaves $\{\mathcal{K}\}$. Their Fourier transforms are easy to compute: i') the rank 1 constant sheaf in degree -1 on V', ii') the rank 1 skyscraper at $\{0'\}$, iii') (!-extension of) Kummer sheaves $\{\mathcal{K}^{-1}\}$ ([Lau87, 1.4.3.2]). In each case, F-goodness can be directly verified.

Now consider the case $d > 1$. Let $\mathcal{F} \in D(V)$ be a perverse irreducible monodromic sheaf. If $\mathcal{F}$ has trivial twist, then it is F-good by Proposition 3.3.1. Assume $\mathcal{F}$ has non-trivial twist $\mathcal{K}$ (recall, by our convention, $\mathcal{K}$ is in degree -1). Fix linear coordinates $(x_1, x_2, ..., x_d)$ (i.e. an isomorphism $V \cong \mathbf{A}^d = \mathrm{Spec}(k[x_1, x_2, ..., x_d])$), this induces coordinates $[x_1 : x_2 : ... : x_d]$ on $\mathbf{P}(V)$. Let $D_1 = \{x_1 = 0\}$, U_1 its complement in V. The projection $q : \mathring{V} \to \mathbf{P}(V)$ maps U_1 to $\underline{U_1} = \{x_1 \neq 0\} \subseteq \mathbf{P}(V)$. Fix the section $\sigma_1 : \underline{U_1} \to U_1, [x_1 : x_2, ..., x_d] \mapsto (1, \frac{x_2}{x_1}, ..., \frac{x_d}{x_1})$.

Apply Proposition 3.2.8 to (U_1, σ_1), we get a decomposition $\mathcal{F}|_{U_1} \cong \underline{\mathcal{F}}_{\sigma_1} \boxtimes \mathcal{K}$. We denote the !-extension of $\underline{\mathcal{F}}_{\sigma_1}$ to $\mathbf{P}(V)$ by $\underline{\mathcal{F}_1}$. Then $j_!(\mathcal{F}|_{U_1}) \cong j_!(\underline{\mathcal{F}}_{\sigma_1} \boxtimes \mathcal{K}) \cong (j_! q^* \underline{\mathcal{F}_1}) \otimes pr_1^* \mathcal{K}$, where j denotes the inclusions into V (we use the same j for the inclusion from U_1 as well as $\mathring{V}$), and $pr : V \to \mathbf{A}^1$ denotes the projection to the first coordinate. The last isomorphism follows from the observation that the map $U_1 \cong_{\sigma_1} \underline{U_1} \times \mathbf{G}_m$ followed by the projection to $\mathbf{G}_m$ coincides with the map pr_1 (restricted to U_1).

We have the distinguished triangle: $j_!(\mathcal{F}|_{U_1}) \to \mathcal{F} \to i_{1*}(\mathcal{F}|_{D_1}) \to$, where i_{1*} is the inclusion of D_1 to V. As $\mathcal{F}|_{D_1}$ is clearly monodromic, it is F-good by the induction hypothesis. Using the compatibility of the Fourier transform with linear maps between vector spaces, it is easily seen that $i_{1*}(\mathcal{F}|_{D_1})$ is F-good. As being F-good is stable under taking cones (Lemma 3.2.13), it suffices to show $j_!(\mathcal{F}|_{U_1})$ is F-good, i.e., to show $CC(Fj_!(\mathcal{F}|_{U_1})) = CC(j_!(\mathcal{F}|_{U_1}))$. In the following, we assume $j_!(\mathcal{F}|_{U_1})$ is nonzero (with non-trivial twist $\mathcal{K}$).

We compute: $Fj_!(\mathcal{F}|_{U_1}) = F((j_! q^* \underline{\mathcal{F}_1}) \otimes pr_1^* \mathcal{K}) = (Fj_! q^* \underline{\mathcal{F}_1}) * F(pr_1^* \mathcal{K})[d]$, where $- * -$ denotes the convolution: let s be the sum map: $V' \times V' \to V', (v_1, v_2) \mapsto v_1 + v_2$, then $- * - : D(V') \times D(V') \to D(V'), (\mathcal{G}_1, \mathcal{G}_2) \mapsto s_!(\mathcal{G}_1 \boxtimes \mathcal{G}_2)$. Further compute: $F(pr_1^* \mathcal{K}) = i'_{1!} F\mathcal{K}[1-d] = i'_{1!} \mathcal{K}^{-1}[1-d]$, where i'_1 is the inclusion of the x'_1-axis into V' (we use the dual coordinates on V').

<u>Claim</u>: $(Fj_! q^* \underline{\mathcal{F}_1}) \boxtimes (i'_{1!} \mathcal{K}^{-1})$ has the same wild ramification (swr) as $(Fj_! q^* \underline{\mathcal{F}_1}) \boxtimes (i'_{1!} \underline{\Lambda}_{\mathbf{G}_m}[1])$.

Accepting the claim, then by Theorem 3.5.5 and Theorem 3.5.7, $(Fj_! q^* \underline{\mathcal{F}_1}) * F(pr_1^* \mathcal{K})[d]$

has the swr as $(Fj_!q^*\underline{\mathcal{F}_1}) * (i'_{1!}\underline{\Lambda}_{\mathbf{G}_m}[1])[1]$, and they have the same characteristic cycle. Now, $i'_{1!}\underline{\Lambda}_{\mathbf{G}_m}[1] = FFi'_{1!}\underline{\Lambda}_{\mathbf{G}_m}[1] = Fpr_1^*\mathcal{H}[d-1]$, where $\mathcal{H} := Fi'_{1!}\underline{\Lambda}_{\mathbf{G}_m}[1]$ is a sheaf on $\mathbf{A}^1$ whose restriction to $\mathbf{G}_m$ is constant and concentrated in degree -1. So $(Fj_!q^*\underline{\mathcal{F}_1}) * (i'_{1!}\underline{\Lambda}_{\mathbf{G}_m}[1]) = F((j_!q^*\underline{\mathcal{F}_1}) \otimes pr_1^*\mathcal{H})[-1]$. Note that $(j_!q^*\underline{\mathcal{F}_1}) \otimes pr_1^*\mathcal{H}$ is isomorphic to $j_!q^*\underline{\mathcal{F}_1}[1]$, which is F-good by Corollary 3.3.3. Put these together, we get the equalities $CC(Fj_!(\mathcal{F}|_{U_1})) = CC(F((j_!q^*\underline{\mathcal{F}_1}) \otimes pr_1^*\mathcal{H})[1-d]) = CC(j_!q^*\underline{\mathcal{F}_1}[1])$. Recall that we want to show this equals $CC(j_!(\mathcal{F}|_{U_1}))$.

Claim: $j_!(\mathcal{F}|_{U_1}) = (j_!q^*\underline{\mathcal{F}_1}) \otimes pr_1^*\mathcal{K}$ has the swr as $(j_!q^*\underline{\mathcal{F}_1}) \otimes pr_1^*\underline{\Lambda}_{\mathbf{G}_m,!}[1] = j_!q^*\underline{\mathcal{F}_1}[1]$, where $\underline{\Lambda}_{\mathbf{G}_m,!}$ denotes the !-extended to $\mathbf{A}^1$ of the constant sheaf on $\mathbf{G}_m$.

Accepting this claim and combining it with the previous equalities, we get the desired equality: $CC(Fj_!(\mathcal{F}|_{U_1})) = CC(j_!q^*\underline{\mathcal{F}_1}[1]) = CC(j_!(\mathcal{F}|_{U_1}))$.

It remains to prove the two claims. We prove the second claim, the first is completely analogous. We first consider the Λ finite case. Denote $j_!q^*\underline{\mathcal{F}_1}$ by $\mathcal{A}$, $\mathcal{K}$ by $\mathcal{B}$. Let $\overline{C}$ be a smooth proper curve, $g : C \subseteq \overline{C}$ an open dense, $f : C \to X$ a map, $s \in \overline{C}$ a closed point. We want to show $a_s(g_!f^*\mathcal{A}[1]) = a_s(g_!f^*(\mathcal{A}\otimes pr_1^*\mathcal{B}))$. Recall $a_s = rk_{\overline{\eta}_s} + sw_{\overline{\eta}_s} - rk_s$. Clearly, the ranks are the same for $g_!f^*\mathcal{A}[1]$ and $g_!f^*(\mathcal{A} \otimes pr_1^*\mathcal{B})$ (note $\mathcal{A}$ is 0 along D_1). For the Swan conductors, observe $(\mathcal{A} \otimes pr_1^*\mathcal{B})_{\overline{\eta}_s} = \mathcal{A}_{\overline{\eta}_s} \otimes (pr_1^*\mathcal{B})_{\overline{\eta}_s}$, $(pr_1^*\mathcal{B})_{\overline{\eta}_s}$ is 0 (if η_s is mapped to 0 by $pr_1 \circ f$) or tame of rank 1 concentrated in degree -1 (otherwise). In both cases, $sw_{\overline{\eta}_s}(\mathcal{A}\otimes pr_1^*\mathcal{B}) = sw_{\overline{\eta}_s}(\mathcal{A}\otimes pr_1^*\underline{\Lambda}_{\mathbf{G}_m,!}[1]) = sw_{\overline{\eta}_s}(\mathcal{A}[1])$. This completes the proof for the Λ finite case. For the $\Lambda = \overline{\mathbf{Q}}_\ell$ case, it suffices to make the following changes to this paragraph: $\mathcal{A}$ denotes the reduction of any integral model of $j_!q^*\underline{\mathcal{F}_1}$, and $\mathcal{B}$ denotes the reduction of any torsion free integral model of $\mathcal{K}$. Note, $\mathcal{B}$ is then a rank 1 local system concentrated in degree -1 trivialised by a power n cover of $\mathbf{G}_m$, $p \nmid n$, hence a Kummer sheaf. □

Corollary 3.4.2. *If $\mathcal{F} \in D(V)$ is monodromic, then $CC(\mathcal{F}) = CC(F\mathcal{F})$ and $SS(\mathcal{F}) = SS(F\mathcal{F})$.*

Proof. The characteristic cycle (resp. singular support) of a sheaf is the sum (resp. union) of the characteristic cycles (resp. singular supports) of its irreducible constituents. The Fourier transform preserves the irreducible constituents. So it suffices to prove $CC(\mathcal{F}) = CC(F\mathcal{F})$ and $SS(\mathcal{F}) = SS(F\mathcal{F})$ for $\mathcal{F}$ perverse irreducible monodromic. The first equality follows from the theorem above. The support of the characteristic cycle of a perverse sheaf equals its singular support (for Λ finite, this is [Sai17b, 5.17]; for Λ rational, this is verified in Proposition 3.5.3.2), the second equality follows. □

3.5 Appendix to Chapter 3: review of the characteristic cycle and the notion of having the same wild ramification

The characteristic cycle of a sheaf with rational coefficient

We refer to [UYZ20, §5] and [Zhe15] for details. Let X be a variety over k. Denote the Grothendieck group of constructible $\overline{\mathbf{F}}_\ell$ (resp. $\overline{\mathbf{Z}}_\ell$, resp. $\overline{\mathbf{Q}}_\ell$)-sheaves on X by $K(X,\overline{\mathbf{F}}_\ell)$ (resp. $K(X,\overline{\mathbf{Z}}_\ell)$, resp. $K(X,\overline{\mathbf{Q}}_\ell)$). There are natural group homomorphisms:

$$K(X,\overline{\mathbf{F}}_\ell) \underset{i_*}{\overset{i^*}{\leftrightarrows}} K(X,\overline{\mathbf{Z}}_\ell) \xrightarrow{j^*} K(X,\overline{\mathbf{Q}}_\ell)$$

where i^*, i_*, and j^* are induced by the reduction, restricting scalars, and tensoring to $\overline{\mathbf{Q}}_\ell$, respectively. It is known that $i_*=0$, i^* is surjective, and j^* is an isomorphism. Define the <u>decomposition homomorphism</u> $d: K(X,\overline{\mathbf{Q}}_\ell)\to K(X,\overline{\mathbf{F}}_\ell)$ as $i^*\circ (j^*)^{-1}$.

Definition 3.5.1 (CC for rational coefficients, [UYZ20, 5.3.2])**.** *Let Λ be rational. For $\mathcal{F}\in D(X)$, $CC(\mathcal{F}) := CC(d[\mathcal{F}\otimes_\Lambda \overline{\mathbf{Q}}_\ell])$. Here "[ ]" denotes the class in $K(X,\overline{\mathbf{Q}}_\ell)$.*

We will drop "[ ]" and "$-\otimes_\Lambda \overline{\mathbf{Q}}_\ell$" from the notation if there is no risk of confusion. Here by $CC(d[\mathcal{F}\otimes_\Lambda \overline{\mathbf{Q}}_\ell])$ we mean the characteristic cycle of any representative for $\mathcal{F}\otimes_\Lambda \overline{\mathbf{Q}}_\ell$ which is defined over some finite extension of $\mathbf{F}_\ell$. This is well-defined because CC is additive and does not change under coefficient field extensions (which can be seen, for example, using the Milnor formula and the fact that Swan conductors do not change under coefficient field extensions).

Concretely, $CC(\mathcal{F})$ can be computed as follows: let Q be a large enough finite extension of $\mathbf{Q}_\ell$ on which $\mathcal{F}$ is defined. Denote by Z its ring of integers, and by F the residue field. Choose any integral model $\mathcal{F}_0$ for $\mathcal{F}$ (i.e. a Z-sheaf $\mathcal{F}_0$ such that $\mathcal{F}_0\otimes_Z Q\cong \mathcal{F}$). Let $\overline{\mathcal{F}}_0=\mathcal{F}_0\otimes_Z F$ be the reduction. Then $[\overline{\mathcal{F}}_0]=d[\mathcal{F}\otimes_E \overline{\mathbf{Q}}_\ell]$, and $CC(\mathcal{F})=CC(\overline{\mathcal{F}}_0)$.

As the operations $f^*, f_*, f^!, f_!, \otimes$ and $\mathcal{RH}om$ are exact functors between triangulated categories, they induce the corresponding operations on the Grothendieck groups, denoted by the same letters. The decomposition homomorphism commutes with all these operations. For $f^*, f_*, f^!, f_!$, this is stated in [UYZ20, 5.2.7], for $\otimes$ and $\mathcal{RH}om$, this is verified in the following.

Lemma 3.5.2. *Let Λ be rational. Let $\mathcal{F},\mathcal{G}$ be sheaves on a variety. Then $d(\mathcal{F}\otimes\mathcal{G}) = (d\mathcal{F})\otimes(d\mathcal{G}), d(\mathcal{RH}om(\mathcal{F},\mathcal{G})) = \mathcal{RH}om(d\mathcal{F},d\mathcal{G})$.*

Proof. Suppose $\mathcal{F},\mathcal{G}$ are defined over a finite extension Q of $\mathbf{Q}_\ell$, denote by Z (resp. F) the ring of integers (resp. residue field) of Q. Let $\mathcal{F}_0,\mathcal{G}_0$ be any integral models for $\mathcal{F},\mathcal{G}$, denote their reductions by $\overline{\mathcal{F}}_0, \overline{\mathcal{G}}_0$.

Essentially by the definition of $- \otimes_Q -$ ([Zhe15, 6.1]), $\mathcal{F}_0 \otimes_Z \mathcal{G}_0$ is an integral model for $\mathcal{F} \otimes_Q \mathcal{G}$. So $d(\mathcal{F} \otimes \mathcal{G}) = (\mathcal{F}_0 \otimes_Z \mathcal{G}_0) \otimes_Z F = \overline{\mathcal{F}}_0 \otimes_F \overline{\mathcal{G}}_0 = (d\mathcal{F}) \otimes (d\mathcal{G})$, where in the second equality we used [Zhe15, 5.3].

Essentially by the definition of $\mathcal{RH}om_Q(-,-)$, $\mathcal{RH}om_Z(\mathcal{F}_0, \mathcal{G}_0)$ is an integral model for $\mathcal{RH}om_Q(\mathcal{F}, \mathcal{G})$. So, $d(\mathcal{RH}om(\mathcal{F}, \mathcal{G})) = \mathcal{RH}om_Z(\mathcal{F}_0, \mathcal{G}_0) \otimes_Z F = \mathcal{RH}om_F(\mathcal{F} \otimes_Z F, \mathcal{G} \otimes_Z F) = \mathcal{RH}om(d\mathcal{F}, d\mathcal{G})$, where in the second equality we used [Zhe15, 5.7]. □

One can thus transport results of characteristic cycles proved in the finite coefficient case to the rational coefficient case. Here are a few that we need but not explicitly stated in [UYZ20].

Proposition 3.5.3. *Let Λ be rational.*
1) If $\mathcal{F}$ is a sheaf on a smooth variety, then $CC\mathbb{D}(\mathcal{F}) = CC(\mathcal{F})$.
2) Let $\mathcal{F}$ be a sheaf on a smooth variety. Then $CC(\mathcal{F})$ is supported on $SS(\mathcal{F})$. If $\mathcal{F}$ is perverse, nonzero, then the coefficients in $CC(\mathcal{F})$ is positive on each irreducible component of $SS(\mathcal{F})$. In particular, the support of $CC(\mathcal{F})$ equals $SS(\mathcal{F})$.
3) If $\mathcal{G}$ is a sheaf on a projective space $\mathbf{P}$, then $CC(R\mathcal{G}) = LCC(\mathcal{G})$, where R is the Radon transform and L is the Legendre transform (as defined above 7.5 in [Sai17b]).
4) Let X, Y be smooth varieties, $\mathcal{F}_1, \mathcal{F}_2$ be sheaves on X, Y, respectively. Then $CC(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = CC(\mathcal{F}_1) \boxtimes CC(\mathcal{F}_2)$ (see [Sai17a, §2] for the meaning of the notation).

Proof. 1) We want to show $CC(\mathbb{D}\mathcal{F}) = CC(\mathcal{F})$. By definition, $CC(\mathbb{D}\mathcal{F}) = CC(d\mathbb{D}\mathcal{F}), CC(\mathcal{F}) = CC(d\mathcal{F})$. By the corresponding result for finite coefficients ([Sai17b, 5.13.4]), it suffices to show $d \circ \mathbb{D} = \mathbb{D} \circ d$, which is immediate from the commutativity of d with $\mathcal{RH}om$.

2) Let $\mathcal{F}_0$ be an integral model for $\mathcal{F}$, and $\overline{\mathcal{F}}_0$ its reduction mod ℓ, such that $SS(\mathcal{F}) = SS(\overline{\mathcal{F}}_0)$ (which exists, by [Bar23, 1.5 (v)]). By definition, $CC(\mathcal{F}) = CC(\overline{\mathcal{F}}_0)$. The first claim follows. The second claim follows from the Milnor formula ([UYZ20, 5.3.3]) and the well-known fact that the vanishing cycle shifted by -1 is perverse t-exact (c.f. [Ill94, 4.6]).

3) and 4) follow from the commutativity of d with f^*, f_*, and $\otimes$, and the corresponding results for finite coefficients ([Sai17b, 7.12; Sai17a, 2.2]). □

The notion of having the same wild ramification

In situations relevant to us, this notion is equivalent to having universally the same conductors ([Kat21, 6.11]). We will only review (and use) the latter, as it is easier to state and verify (in our situation). We refer to [Kat18; Kat21] and references therein for details.

Definition 3.5.4 (universally the same conductors for finite coefficients, [Kat18, 2.5])**.** *Let Λ be finite. Let X be a variety over k. We say $\mathcal{F}, \mathcal{F}' \in D(X)$ have universally the same* *conductors (usc), if for all smooth proper curve $\overline{C}$, all open dense $j : C \subseteq \overline{C}$, all map $f : C \to X$, all closed point $s \in \overline{C}$, we have $a_s(j_! f^* \mathcal{F}) = a_s(j_! f^* \mathcal{F}')$, where $a_s := rk_{\overline{\eta}_s} + sw_{\overline{\eta}_s} - rk_s$ is the Artin conductor at s.*

Theorem 3.5.5 ([Kat18, 4.6.ii, 4.7])**.** *Let Λ be finite. Let $f : X \to Y$ be a map between varieties.*
1) If $\mathcal{F}, \mathcal{F}' \in D(X)$ have usc, then $f_! \mathcal{F}, f_! \mathcal{F}' \in D(Y)$ have usc.
2) Assume X is smooth. If $\mathcal{F}, \mathcal{F}' \in D(X)$ have usc, then $CC(\mathcal{F}) = CC(\mathcal{F}')$.

Note that as a_s, $j_!$ and f^* are additive, having usc descends to the Grothendieck group. This suggests that we can transport this notion to rational coefficients and get the analogue of the above theorem.

Definition 3.5.6 (same wild ramification for rational coefficients)**.** *Let Λ be rational. Let X be a variety over k. We say $\mathcal{F}, \mathcal{F}' \in D(X)$ have the same wild ramification (swr), or have universally the same conductors (usc), if $d(\mathcal{F}), d(\mathcal{F}')$ do.*

Theorem 3.5.7. *Let Λ be rational, $f : X \to Y$ be a map between varieties.*
1) If $\mathcal{F}, \mathcal{F}' \in D(X)$ have the swr, then $f_! \mathcal{F}, f_! \mathcal{F}' \in D(Y)$ have the swr.
2) Assume X is smooth. If $\mathcal{F}, \mathcal{F}' \in D(X)$ have the swr, then $CC(\mathcal{F}) = CC(\mathcal{F}')$.

Proof. 1) This follows from the corresponding statement for Λ finite and the fact that the decomposition homomorphism commutes with $f_!$.

2) This follows from the corresponding statement for Λ finite and the definition of CC for Λ rational. ∎

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