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Euler Equations on 2D Singular Domains

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The dissertation of Zonglin Han is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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DEDICATION

I dedicate my dissertation to my family.

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ABSTRACT OF THE DISSERTATION

Euler Equations on 2D Singular Domains

by

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Professor Andrej Zlatoš, Chair

The Euler equations are a fundamental yet celebrated set of mathematical equations that describe the motions of inviscid, incompressible fluid on planar domains. They play a critical role in various fields of study including fluid dynamics, aerodynamics, hydrodynamics and so on. Though it was first written out in 1755, there are still many open questions regarding to this rich system, including some fundamental questions of Euler equations on singular domains. Unlike existence of weak solutions that were proven on considerably general domains, uniqueness of such solutions are still quite open on singular domains, even on convex domains. In this thesis, we will show uniqueness of weak solutions on singular domains given two different assumptions of initial vorticity $\omega_0 \in L^\infty$:

1. ω_0 is constant near the boundary.

2. ω_0 is constant near the boundary and has a sign (non-positive or non-negative).

Under the first assumption, the previous best uniqueness results can only be applied to $C^{1,1}$ domains except at finitely many corners with interior angles less than π. Here, we will extend the result to fairly general singular domains which are only slightly more restrictive than the exclusion of corners with angles larger than π, thus including all convex domains. We derive this by showing that the Euler particle trajectories cannot reach the boundary in finite time and hence the vorticity cannot be created by the boundary. We will also show that if the given geometric condition is not satisfied, then we can construct a domain and a bounded initial vorticity such that some particle could reach the boundary in finite time.

Under the second assumption with the sign condition, the previous best uniqueness result can only be applied to $C^{1,1}$ domains with finitely many corners with interior angels larger than $\frac{\pi}{2}$. Here, we will extend the result to a class of general singular open bounded simply connected domains, which can be possibly nowhere C^1 and there are no restrictions on the size of each angle.

Chapter 1

Introduction

Euler equations model the flow of an inviscid, incompressible fluid with constant density. Mathematically, for any open domain $\Omega \subseteq \mathbb{R}^d$ as well as positive time $t > 0$, these motions are modeled by

$$\partial_t u + (u \cdot \nabla)\, u = -\nabla p, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

with u the fluid velocity and p its pressure. We also endow the domains with impermeable boundaries, as well as the *no-flow* (or *slip*) boundary condition, which is

$$u \cdot n = 0 \tag{1.3}$$

on $\mathbb{R}^+ \times \partial\Omega$, with n the unit outer normal to Ω. In this thesis, we will focus on planar spatial domains ($d = 2$), on which the Euler equations can be reformulated as the active scalar equation (vorticity form):

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1.4}$$

on $\mathbb{R}^+ \times \Omega \subseteq \mathbb{R}^+ \times \mathbb{R}^2$, with

$$\omega := \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1$$

be the *vorticity* of the flow. This conveniently removes the pressure from the system, and one can now also find the (divergence-free) velocity from the vorticity via the Biot-Savart law

$$u = \nabla^\perp \Delta^{-1} \omega, \tag{1.5}$$

with Δ the Dirichlet Laplacian on Ω and $\nabla^\perp \psi := (-\partial_{x_2}\psi, \partial_{x_1}\psi)$.

This system was widely studied since 1755 when Euler first formulated it. Well-posedness is also known on smooth domains but is still open on more singular ones. The focus of this thesis is to show uniqueness of weak solutions on fairly large class of singular domains. In this chapter, we first briefly review previous existence and uniqueness results on domains with different kinds of singular boundaries. Then, we will introduce some definitions and tools from complex analysis. Lastly we will show existence and uniqueness of solutions on smooth domains, as well as existence of weak solutions on open bounded simply connected domains.

1.1 Previous Existence and Uniqueness Results

Well-posedness of the equations (1.1)-(1.2) has been the focus of many works for many decades. In 1933, Wolibner [32] and Hölder [12] showed global well-posedness for strong solutions on bounded domains with smooth boundaries. After that, due to the property of L^∞ norm of ω_0 in the equation (1.4), Yudovich proved global well-posedness for weak solutions with $\omega_0 \in L^\infty(\Omega)$, and we say such solutions are in the *Yudovich class* (see also [1, 21, 23, 28]). Existence of global weak solutions can also be proved for $\omega_0 \in L^p(\Omega)$ (see [7]). However, all these results rely on the fact that the boundary of the domain Ω belongs to $C^{1,1}$ or better.

For more singular domains, global existence of weak solutions has been shown to hold on convex domains by Taylor for $\omega_0 \in L^p(\Omega)$ [27]. Then Gérard-Varet and Lacave improved the result to a class of very general domains including all open bounded simply connected domains [9, 10]. We will prove existence of weak solutions in the Yudovich class on open bounded simply connected domains later in this section by following the idea from [9]. On the other hand, there are fewer results for uniqueness of solutions on domains with less regular boundaries. Most uniqueness results require the velocity to be close to Lipschitz, and sufficient smoothness of $\partial\Omega$ is typically needed to obtain a priori estimates on the Riesz transform $\nabla\nabla^{\perp}\Delta^{-1}\omega = \nabla u$. For example, Yudovich used Calderón-Zygmund inequalities

$$\|\nabla u(t,\cdot)\|_{L^p} \leq Cp\|\omega(t,\cdot)\|_{L^p},$$

with some uniform C for all $p \in [2,\infty)$, and the *log-Lipschitz* estimate

$$\sup_{x,y\in\Omega} \frac{|u(t,x)-u(t,y)|}{|x-y|\max\{1,-\ln|x-y|\}} \leq C\|\omega(t,\cdot)\|_{L^\infty} \tag{1.6}$$

(see, e.g., [23]). We will also prove the *log-Lipschitz* estimate for velocity on smooth domains later in this chapter.

Nevertheless, on less regular domains these estimates no longer hold. Jerison and Kenig showed that ∇u may not even be integrable on some C^1 domains [13]. Uniqueness results on non-$C^{1,1}$ domains are only proved on piecewise $C^{1,1}$ domains with possibly finitely many acute corners. Bardos, Di Plinio, and Temam [2] first showed it on rectangles, and then Lacave, Miot, and Wang [18] showed the uniqueness result on domains which are $C^{2,\gamma}$, with $\gamma > 0$, except at finitely many acute corners. Later Di Plinio and Temam [6] further improved the results to domains that are $C^{1,1}$ except at finitely many acute corners.

On general piecewise $C^{1,1}$ domains, the velocity is not close to Lipschitz near the corners with angles greater than $\frac{\pi}{2}$ even for initially smooth ω. Nevertheless Lacave and

Zlatǒs ([16],[19]) recently showed the uniqueness of weak solutions on piecewise $C^{1,1}$ domains with finitely many corners with possibly non-acute angles by adding further assumptions on initial vorticity. The key idea here is to establish uniqueness for solutions remaining forever constant near $\partial\Omega$ where the velocity is not close to Lipschitz. For smooth domains, any particle trajectory starting from inside of the domain Ω cannot reach $\partial\Omega$ in finite time since the particle cannot reach the boundary faster than double-exponentially, which also implies that the solution will remain constant near the boundary in time if it is initially. It turns out that this extends to piecewise $C^{1,1}$ domains with finite obtuse or reflex angles, which Lacave ([16]) showed when also ω_0 is non-negative (or non-positive) besides being constant near $\partial\Omega$. In this setting, the weak solution is unique and will remain constant near the boundary. Later on, Lacave and Zlatǒs ([19]) showed the same result for $\partial\Omega$ being $C^{1,1}$ except at finitely many corners with angles less than π and without the sign condition on ω_0. For both papers, the Euler particle trajectories cannot approach the boundary faster than double-exponentially and will remain inside Ω for all $t > 0$. Moreover, for the latter one, the authors also constructed a counter example showing that the particle trajectories can reach the boundary in finite time even if $\partial\Omega$ is smooth except at a single corner with an arbitrary angle from $(\pi, 2\pi)$. However, it still remains open whether non-Lagrangian solutions exist or not on these or other singular domains. Based on the results of [16] and [19], it is natural to ask the following questions:

1. Without the sign condition on ω_0, can we prove uniqueness of weak solutions on convex domains, or on even more general domains with no angles larger than π?

2. With the sign condition on ω_0, can we prove uniqueness of weak solutions on domains with corners with arbitrary angles from $(0, 2\pi)$?

In this thesis, we will provide positive answers for both questions.

1.2 Trajectory Approach and Well-posedness on Smooth Domains

We consider solutions to the Euler equations on Ω from the *Yudovich class*

$$\left\{(\omega, u) \in L^\infty\left((0,\infty); L^\infty(\Omega) \times L^2(\Omega)\right) \,\middle|\, \omega = \nabla \times u, \text{ and (1.2)–(1.3) hold weakly}\right\},$$

where the weak form of (1.2)–(1.3) is

$$\int_\Omega u(t,\cdot) \cdot \nabla h \, dx = 0 \qquad \forall h \in H^1_{\text{loc}}(\Omega) \text{ with } \nabla h \in L^2(\Omega), \tag{1.7}$$

for almost all $t > 0$ (see [9, 10]). ω and u are equivalently related by the Biot-Savart law (1.5), which can be expressed as

$$u(t,x) = \int_\Omega K_\Omega(x,y)\omega(t,y)dy \tag{1.8}$$

where $K_\Omega = \nabla_x^\perp G_\Omega$ and $G_\Omega(x,y)$ is the Green's function for the Dirichlet Laplacian on domain Ω. Since u is uniquely determined by ω, we will simply say that ω is from the Yudovich class. We say that ω from the Yudovich class is a *weak solution* to the Euler equations on Ω, on time interval $(0,T)$ and with initial condition $\omega_0 \in L^\infty(\Omega)$, if

$$\int_0^T \int_\Omega \omega\left(\partial_t\varphi + u \cdot \nabla\varphi\right) dxdt = -\int_\Omega \omega_0 \varphi(0,\cdot)\, dx \qquad \forall\, \varphi \in C_0^\infty\left([0,T) \times \Omega\right). \tag{1.9}$$

This is obviously the definition of weak solutions to the transport equation (1.4), but it is also equivalent to the relevant weak velocity formulation of the Euler equations on Ω (see [10, Remark 1.2]). When $T = \infty$, we call such solutions *global*.

Another way to characterize the system is to consider the particle trajectories X_t^x.

For any divergence free velocity field $u(t,x)$, we define the particle trajectory X_t^x as

$$\frac{d}{dt}X_t^x = u(t, X_t^x) \qquad \text{and} \qquad X_0^x = x, \tag{1.10}$$

on an interval $(0, t_x)$ such that

$$t_x := \sup\{t > 0 \,|\, X_s^x \in \Omega \text{ for all } s \in (0,t)\}. \tag{1.11}$$

Here if the particle starting at $x \in \Omega$ reaches the boundary in finite time, then t_x is the first such time. Notice that since (1.4) is a transport equation, it is also natural to consider whether the general weak solution is transported by u. In other words, we want to see if

$$\omega(t, X_t^x) = \omega_0(x), \tag{1.12}$$

for a.e. $t \in (0,\infty)$ and a.e. $x \in \Omega$ such that $t_x > t$. This will be shown to be true for general open domains in Chapter 2 (Lemma 2.3.1), but that does not a priori exclude the possibility of vorticity creation and depletion on $\partial\Omega$ unless $t_x = \infty$ for a.e. $x \in \Omega$ (then $\nabla \cdot u \equiv 0$ shows that $|\Omega \setminus \{X_t^x \,|\, x \in \Omega \text{ and } t_x > t\}| = 0$). If both these properties hold, so that $\omega(t,\cdot)$ is the push-forward of ω_0 via X_t^x for each $t \in (0,\infty)$, we call such ω a *Lagrangian solution.*

For the classical approach, in order to make the trajectories well-defined, we require the velocity to be at least log-Lipschitz in space. In fact, assuming $\omega_0 \in L^\infty$ will ensure this level of regularity of u on smooth domains. Furthermore, we can derive at most double-exponential approach rate for the trajectories, uniqueness of weak solutions, as well as that the vorticity is transported by the velocity. This approach uses the estimates on the Dirichlet Green's function and its derivatives. We will present the outline of the proof for log-Lipschitz property of velocity on smooth domains from Chapter 2 in [23], and we will prove it using this idea and complex analysis in later sections. First we need to state some estimates on the Dirichlet Green's function on smooth domains:

Proposition 1.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary, then the Dirichlet Green's function $G_\Omega(x,y)$, with $x \neq y$, can be written as*

$$G_\Omega(x,y) = \frac{1}{2\pi} \log|x-y| + h(x,y), \tag{1.13}$$

where $h(x,y)$ is a harmonic function such that for each $y \in \Omega$, we have

$$\Delta_x h = 0 \text{ on } \Omega, \quad h|_{x \in \partial\Omega} = -\frac{1}{2\pi} \log|x-y|. \tag{1.14}$$

Moreover we have that $G_\Omega(x,y)$ is symmetric $x, y \in \Omega$ and equals zero if either x or y is on $\partial\Omega$. Lastly there exists a constant C depending only on Ω such that the following estimates on G_Ω and its derivatives hold:

$$|G_\Omega(x,y)| \leq C\left(|\log|x-y|| + 1\right) \tag{1.15}$$

$$|\nabla G_\Omega(x,y)| \leq C\frac{1}{|x-y|} \tag{1.16}$$

$$|\nabla^2 G_\Omega(x,y)| \leq C\frac{1}{|x-y|^2} \tag{1.17}$$

We will skip the proof for these classical estimates, and now we are ready to show log-Lipschitz continuity of u when $\omega_0 \in L^\infty$. Define

$$\phi(r) = \begin{cases} r(1-\log r) & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r \geq 1 \end{cases} \tag{1.18}$$

Theorem 1.2.2. *If $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary, $\omega \in L^\infty([0,\infty) \times \Omega)$ and u is generated by ω via (1.8), then there exists a $C > 0$ which depends only on Ω such that for any $x, x' \in \Omega$ and $t > 0$, we have*

$$|u(t,x) - u(t,x')| \leq C||\omega||_{L^\infty}\phi(|x-x'|) \tag{1.19}$$

The idea from [23] is that by using the property of $\omega_0 \in L^\infty$ and (1.16), it is enough to show that

$$\int_\Omega |K_\Omega(x,y) - K_\Omega(x',y)|dy \leq C\phi(|x-x'|). \tag{1.20}$$

where $r = |x-x'| < 1$. The integration is divided into two parts by defining $A = \{y \in \Omega \mid |x-y| < 2r\}$, $A_1 = A \cap \Omega$ and $A_2 = A^c \cap \Omega$. The integration over A_1 is bounded by $C_\Omega r$ using the triangle inequality and (1.16). The integration over A_2 uses the Mean Value Theorem and observing that for any point x'' on the segment connecting x and x', we have $|x''-y| \geq \frac{1}{2}|x-y|$ for all $y \in \Omega$. Then using (1.17) and a direct computation yield the result. However, here we assumed that the line segment connecting x and x' is totally contained in Ω. This can be fixed by using a curve, whose length is less than $C'|x-x'|$ and C' doesn't depend on the points because of the smoothness of the boundary. Instead of showing this with more detail, we will prove it in the next section using the Riemann Mapping.

Now we want to show existence and uniqueness of solutions on smooth domains. In fact we will actually show existence and uniqueness of solutions to the following system, which is Lagrangian formulation of the Euler equations:

$$\frac{dX_t^x}{dt} = u(t, X_t^x), \quad X_x^0 = x \tag{1.21}$$

$$\omega(x,t) = \omega_0((X_t^x)^{-1}) \tag{1.22}$$

$$u(t,x) = \int_\Omega K_\Omega(x,y)\omega(t,y)dy \tag{1.23}$$

Theorem 1.2.3. *If Ω is smooth and $\omega_0 \in L^\infty(\Omega)$, then there exists a unique triple (ω, u, X_t^x) satisfying* (1.21),(1.22) *and* (1.23).

Before proving this theorem, we first need to state an ODE result that guarantees global existence and uniqueness of the Cauchy problem for log-Lipschitz functions:

Lemma 1.2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Assume that the velocity field $b(t,x)$ satisfies*

$$b \in L^\infty([0,\infty) \times \bar{\Omega}), \ \ |b(t,x) - b(t,y)| \leq C\phi(|x-y|), \ b(t,x) \cdot n|_{\partial\Omega} = 0, \ \ \forall t > 0, \ \ \forall x, y \in \Omega \tag{1.24}$$

where ϕ is defined in (1.18) *and n is the unit normal to $\partial\Omega$ at point x. Then the problem*

$$\frac{dx}{dt} = b(t,x), \ \ x(0) = x_0 \tag{1.25}$$

has a unique global solution $\forall x_0 \in \Omega$. Moreover, if $x_0 \in \Omega$, then $x(t) \in \Omega$ for all $t \geq 0$.

The proof of this ODE result will be skipped here but could be find in [21]. Now we would like to show Hölder continuity on the trajectories generated by log-Lipschitz velocities.

Lemma 1.2.5. *Suppose $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, and assume that X_t^x is generated by a log-Lipschitz vector field $b(x,t)$ satisfying the assumptions of Lemma 1.2.4. Then for every $x, y \in \Omega$ and $t \geq 0$ with $|x-y|, |X_t^x - X_t^y| < \frac{1}{2}$, we have*

$$|x-y|^{e^{Ct}} \leq |X_t^x - X_t^y| \leq |x-y|^{e^{-Ct}}, \tag{1.26}$$

where C only depends on the constant C from the (1.19).

Proof. For any fixed $x, y \in \Omega$, let $F(t) = |X_t^x - X_t^y|$. Then we have

$$\begin{aligned} \left| \frac{d}{dt} F(t)^2 \right| &= 2|(X_t^x - X_t^y) \cdot (b(t, X_t^x) - b(t, X_t^y))| \\ &\leq 2CF(t)\phi(F(t)), \end{aligned}$$

which yields (notice that $F(t) \leq \frac{1}{2}$)

$$|F'(t)| \leq CF(t) \log \frac{1}{F(t)},$$

and this leads to

$$e^{Ct}\log F(0) \le \log F(t) \le e^{-Ct}\log F(0).$$

Taking the exponential and using the definition of $F(t)$ finishes the proof. $\square$

Note that the same inequality could be applied to $(X_t^x)^{-1}$. Lastly before proving Theorem 1.2.3, we need to prove the measure preserving property of trajectories generated by log-Lipschitz velocities. If the velocity u is smooth, let $X(t,x) := X_t^x$ be the trajectories generated by u. The Jacobian of $x \mapsto X(t,x)$ is $J(t,x) = \det(\frac{\partial X_i(t,x)}{\partial x_j})_{i,j=1,2}$, where $X_i(t,x)$ is the i-th coordinate of $X(t,x)$. The measure preserving property is equivalent to $\frac{dJ}{dt} = 0$ since $J(0,x) = 1$. Define the matrix H as $H_{ij}(t,x) = \frac{\partial X_i(t,x)}{\partial x_j}$, and we have

$$\frac{dH_{ij}}{dt} = \sum_{k=1}^{2} \frac{\partial u_i}{\partial x_k}\frac{\partial X_k(t,x)}{\partial x_j} \tag{1.27}$$

Let A be any 2×2 matrix, and the minors $\{M_{ij}\}$ of A are just $\{A_{3-i,3-j}\}$. Hence we have

$$\det A = \sum_{j=1}^{2}(-1)^{i+j}A_{3-i,3-j}A_{ij},$$

and

$$\frac{\partial}{\partial A_{ij}}\det A = (-1)^{i+j}A_{3-i,3-j}.$$

Combining these we have

$$\frac{d}{dt}(\det A) = \sum_{i,j=1}^{2}(-1)^{i+j}A_{3-i,3-j}\frac{dA_{ij}}{dt}. \tag{1.28}$$

Moreover, recall that we can write the elements of A^{-1} as

$$(A^{-1})_{ij} = \frac{1}{\det A}(-1)^{i+j}A_{3-j,3-i},$$

which gives us

$$\sum_{j=1}^{2}(-1)^{i+j}A_{3-i,3-j}A_{kj} = \delta_{ik}\det A \tag{1.29}$$

where $\delta_{ik} = 1$ if $i = k$ and 0 otherwise. Apply (1.27), (1.28) and (1.29) to $A = H$ gives us

$$\frac{dJ}{dt} = \sum_{i,j=1}^{2}(-1)^{i+j}A_{3-i,3-j}\frac{d}{dt}(H_{ij}) = \sum_{i,j=1}\frac{\partial u_i}{\partial x_j}J\delta_{ij} = J(\nabla\cdot u) = 0.$$

This shows the measure preserving property for trajectory generated by smooth velocity. Now we will show this property for trajectories generated by log-Lipschitz velocities.

Lemma 1.2.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and $b(t,x)$ satisfy 1.2.4 and be divergence free in distributional sense. Then the flow map $X(t,x)$ generated by b is measure preserving on Ω.*

Proof. From Lemma 1.2.4 and Lemma 1.2.5, we see that $X(t,x)$ is bijective and Hölder continuous, hence it suffices to show X is measure preserving on any open ball inside Ω. Let $R \subset \Omega$ be an open ball and let $T > 0$. By Lemma 1.2.4, there exists a $d > 0$ such that $\operatorname{dist}(\partial\Omega, X(t,R)) \geq d$ for all $0 \leq t \leq T$. Now for any $\delta < \frac{d}{2}$, define

$$R_\delta := \{x \in \Omega \mid \operatorname{dist}(x,R) < \delta\} \subset \Omega.$$

Let $\eta(x) \in C_0^\infty(\mathbb{R}^2)$ be non-negative with $\eta(x) = 0$ for $x \notin \mathbb{D}$ and $\int_{\mathbb{R}^2}\eta(x) = 1$, as well as we let $\eta_\epsilon(x) = \frac{1}{\epsilon^2}\eta(\frac{x}{\epsilon})$. Then for any $\epsilon < \frac{d}{2}$, we define the smooth divergence free function $b_\epsilon = b * \eta_\epsilon$ such that the flow map $X_\epsilon(t,x)$ generated by b_ϵ. Since b is log-Lipschitz and $|b(t,x) - b_\epsilon(t,x)| \leq \int_{\mathbb{R}^2}|b(t,x) - b(t,y)|\eta_\epsilon(\frac{x-y}{\epsilon})dy \leq \phi(\epsilon)$, we have

$$\begin{aligned}|X(t,x) - X_\epsilon(t,x)| &\leq \int_0^t |b(s,X(s,x)) - b(s,X_\epsilon(s,x))| + |b(s,X_\epsilon(s,x)) - b_\epsilon(s,X_\epsilon(s,x))|ds \\ &\leq C\left(\int_0^t \phi\big(|X(s,x) - X_\epsilon(s,x)|\big)ds + \phi(\epsilon)t\right).\end{aligned}$$

Now for $0 \le t \le T$, let $g(t)$ be a function such that $g(0) = 0$ and satisfying

$$g'(t) = C\big(\phi(g(t)) + \phi(\epsilon)\big),$$

and let $h(t)$ be such that $h(0) = C\phi(\epsilon)T$ and solving

$$h'(t) = C\phi(h(t)).$$

Then we get $|X(t,x) - X_\epsilon(t,x)| \le g(t) \le h(t)$, and if we choose ϵ small enough to make $h(t) < 1$ for all $t \in [0,T]$, we see that

$$h(t) = h(0)^{\exp(-Ct)} \exp(1 - \exp(-Ct)).$$

This and $\phi(\epsilon) < \epsilon^{0.5}$ for $\epsilon < 1$ implies that if we let $\beta_T := \frac{\exp(-CT)}{2} > 0$, then we have (for ϵ small enough such that $CT\epsilon^{0.5} < 1$)

$$|X(t,x) - X_\epsilon(t,x)| \le \Big(CT\phi(\epsilon)\Big)^{\exp(-Ct)} \exp(1 - \exp(-Ct)) \le C_T \epsilon^{\beta_T}, \tag{1.30}$$

which also ensures that when we choose ϵ small enough with $C_T\epsilon^{\beta_T} < \frac{d}{2}$ so that $X_\epsilon(t,x)$ will not exit Ω up to time T for all $x \in R_\delta$, hence the above arguments make sense. Now let $f \in C_0^\infty(R_\delta)$ such that $||\nabla f||_{L^\infty} \le C'\frac{1}{\delta}$ and $f(x) = 1$ for $x \in R$. By the definition of f and the measure preserving property of X_ϵ (since it is generated by a smooth velocity), we have for all $0 \le t \le T$,

$$|X^{-1}(t,R)| = \int_\Omega \mathbb{1}_R(X(t,x))dx \le \int_\Omega f(X(t,x))dx \le \int_\Omega \mathbb{1}_{R_\delta}(X(t,x))dx = |X^{-1}(t,R_\delta)|, \tag{1.31}$$

$$|R| = |X_\epsilon^{-1}(t,R)| \le \int_\Omega f(X_\epsilon(t,x))dx \le |X_\epsilon^{-1}(t,R_\delta)| = |R_\delta| \tag{1.32}$$

and

$$|\int_{\Omega} f(X(t,x)) - f(X_\epsilon(t,x))dx| \le ||\nabla f||_{L^\infty}|\Omega| \sup_{x\in R_\delta, 0\le t\le T} |X(t,x) - X_\epsilon(t,x)| \le C_{T,\Omega}\frac{1}{\delta}\epsilon^{\beta_T}. \tag{1.33}$$

Hence by taking $\epsilon < \delta^{\frac{2}{\beta_T}}$, letting δ to 0, and using the last three inequalities will give us $|X^{-1}(t,R)| \le |R|$. A similar argument for $X^{-1}(t,R)$ gives us $|X(t,R)| \le |R|$. Combining these two inequalities and the fact that $X(t,\cdot)$ is injective on R, we have $|X^{-1}(t,R)| = |R| = |X(t,R)|$ for all open balls $R \subset \Omega$, hence it is also true for all open sets $R \subset \Omega$, which implies that the flow map is measure preserving. □

Now we are ready to prove existence and uniqueness of solutions on smooth domains.

Proof of Theorem 1.2.3. Given a bounded initial vorticity ω_0, we define an iterative sequence of approximations as

$$u^n(t,x) = \int_\Omega K_\Omega(x,y)\omega^{n-1}(t,y)dy, \tag{1.34}$$

$$\frac{d}{dt}X^n(t,x) = u^n(t, X^n(t,x)), \tag{1.35}$$

$$\omega^n(t,x) = \omega_0((X^n)^{-1}(t,x)) \tag{1.36}$$

Notice that these are well-defined because of Lemma 1.2.4 and $||\omega^n(t)||_{L^\infty} = ||\omega_0||_{L^\infty}$. Moreover Lemma 1.2.5 implies that all $X^n(t,x)$ are also Hölder continuous and the constant C in Lemma 1.2.5 is uniform for all n (but depends on Ω and $||\omega_0||_{L^\infty}$). This implies that for every $T > 0$, we have

$$||X^n(t,\cdot)||_{C^{\alpha_T}([0,T]\times\bar{\Omega})} \le C_T \tag{1.37}$$

for some $\alpha_T, C_T > 0$ and independent of n. The Arzela-Ascoli Theorem implies (by taking a subsequence, but we will keep using the same index for simplicity of the notations) that $X^n(t,x)$ convergences uniformly to a function $X(t,x) \in C^{\alpha_T}([0,T]\times\bar{\Omega})$, which also implies $X(t,x) \in \Omega$ for all x, t. By the Dominated Convergence Theorem, uniform convergence and measure preserving lemma 1.2.6, $X(t,\cdot)$ is then also measure preserving for all $t > 0$.

To show it is surjective, fix $0 \leq t \leq T$, $y \in \Omega$ and $d = \frac{\text{dist}(y,\partial\Omega)}{2}$. Choose $y_1 \in B(y,d)$ such that $X(t,x_1) = y_1$ with $x_1 \in \Omega$, which is possible by the measure preserving property. Now let $d' = \frac{\text{dist}(x_1,\partial\Omega)}{2}$. For each $n > 1$, we choose y_n such that $|y - y_n| < (\frac{d'}{\pi^2 n^2})^{\exp(-Ct)}$ with C from lemma 1.2.5, $x_n \in \Omega$ and $X(t,x_n) = y_n$. This is possible again by the measure preserving property. We see that there exists a subsequence of x_n such that it converges to $x \in \Omega$ (by the first inequality in Lemma 1.2.5) and the continuity of $X(t,\cdot)$ implies $X(t,x) = y$. Hence $X(t,x)$ is a bijection on Ω and its inverse $X^{-1}(t,x)$ is well defined, measure preserving and satisfying Lemma 1.2.5. Thus we can define

$$\omega(t,x) = \omega_0(X^{-1}(t,x)), \tag{1.38}$$

$$u(t,x) = \int_\Omega K_\Omega(x,y)\omega(t,y)dy. \tag{1.39}$$

If we can prove uniform convergence of $u_n \to u$ on $[0,T] \times \Omega$, we will conclude the existence claim. Define $y_t^n(z) = X^n\left(t, X^{-1}(t,z)\right)$ for $z \in \Omega$, which is measure preserving. Given $\epsilon > 0$, choose N such that if $n > N$, we have $|X^n(t,x) - X(t,x)| < \delta$ for all $(t,x) \in [0,T] \times \Omega$ for some $\delta < 1$ chosen later. Hence we have for all $z \in \Omega$, $|y_t^n(z) - z| < \delta$ for such n for all $0 \leq t \leq T$. By the measure preserving property, we have for all $x \in \Omega$,

$$\begin{aligned} |u(t,x) - u^n(t,x)| &= \left| \int_\Omega \Big(K_\Omega(x, X(t,z)) - K_\Omega(x, X^n(t,z)) \Big) \omega_0(z) dz \right| \\ &\leq ||\omega_0||_{L^\infty} \int_\Omega \Big| K_\Omega(x,z) - K_\Omega(x, y_t^n(z)) \Big| dz \end{aligned}$$

Now since Ω is smooth, then there exists a $C_\Omega > 0$ such that every $x, y \in \Omega$ can be joined by a smooth curve with the length of curve $l_{x,y}$ satisfying

$$l_{x,y} \leq C_\Omega |x - y|, \tag{1.40}$$

if $|x-y| \le \frac{1}{C_\Omega}$. Then by (1.16) and triangle inequality

$$\int_{B(x,(C_\Omega+1)\delta)\cap\Omega} \Big|K_\Omega(x,z) - K_\Omega(x,y_t^n(z))\Big| dz \le 2C \int_{B(x,2(C_\Omega+1)\delta)} \frac{1}{|x-z|} dz = 8\pi C(C_\Omega+1)\delta.$$

For the integration over rest of Ω, by (1.17), (1.40) and Mean Value Theorem we have

$$\begin{aligned}\int_{B(x,(C_\Omega+1)\delta)^c\cap\Omega} \Big|K_\Omega(x,z) - K_\Omega(x,y_t^n(z))\Big| dz &\le C_\Omega\delta \int_{B^c(x,(C_\Omega+1)\delta)\cap\Omega} |\nabla K_\Omega(x,p(z))| dz \\ &\le C'_\Omega\delta \int_{B^c(x,(C_\Omega+1)\delta)\cap\Omega} \frac{1}{|x-p(z)|^2} dz \\ &\le C'_\Omega\delta \int_{B^c(x,\delta)\cap\Omega} \frac{1}{|x-z|^2} dz \\ &\le C''_\Omega\delta\log(\frac{1}{\delta})\end{aligned}$$

where $p(z)$ is some point on the curve connecting z and $y_t^n(z)$ satisfying (1.40). Lastly, choose $\delta < \frac{1}{C_\Omega}$ small enough such that $|u(t,x) - u^n(t,x)|$ doesn't exceed ϵ.

The argument above shows uniform convergence of u^n on $[0,T]\times\Omega$, and by taking $n\to\infty$ in

$$X^n(t,x) = x + \int_0^t u^n(s, X^n(s,x))ds,$$

we have

$$X(t,x) = x + \int_0^t u(s, X(s,x))ds,$$

which concludes that the triple (ω, u, X) satisfies (1.21)-(1.23).

Now if we get two such triples (ω, u, X) and $(\bar\omega, \bar u, Y)$, define

$$D(t) = \frac{1}{|\Omega|}\int_\Omega |X(t,x) - Y(t,x)| dx.$$

By Theorem 1.2.2 and the measure preserving property of X, Y, Y^{-1} in space, we have

$$
\begin{aligned}
|X(t,x) - Y(t,x)| &\leq \int_0^t \Big| u(s, X(s,x) - u(s, Y(s,x) \Big| ds + \int_0^t \Big| u(s, Y(s,x) - \bar{u}(s, Y(s,x) \Big| ds \\
&\leq C ||\omega_0||_{L^\infty} \int_0^t \phi(|X(s,x) - Y(s,x)|) ds \\
&+ ||\omega_0||_{L^\infty} \int_0^t \int_\Omega \Big| K_\Omega(Y(s,x), X(s,y)) - K_\Omega(Y(s,x), Y(s,y)) \Big| dyds \\
&= C ||\omega_0||_{L^\infty} \int_0^t \phi(|X(s,x) - Y(s,x)|) ds \\
&+ ||\omega_0||_{L^\infty} \int_0^t \int_\Omega \Big| K_\Omega(x, X(s,y)) - K_\Omega(x, Y(s,y)) \Big| dyds.
\end{aligned}
$$

Applying the inequalities above and inequality (1.20) to $D(t)$, we have

$$
\begin{aligned}
D(t) &\leq \frac{C}{|\Omega|} ||\omega_0||_{L^\infty} \int_0^t \int_\Omega \phi(|X(s,x) - Y(s,x)|) dsdx \\
&+ \frac{1}{|\Omega|} ||\omega_0||_{L^\infty} \int_0^t \int_\Omega \int_\Omega \Big| K_\Omega(x, X(s,y)) - K_\Omega(x, Y(s,y)) \Big| dxdyds \\
&\leq \frac{2C ||\omega_0||_{L^\infty}}{|\Omega|} \int_0^t \int_\Omega \phi(|X(s,x) - Y(s,x)|) dsdx \\
&\leq C(\Omega, ||\omega_0||_{L^\infty}) \int_0^t \phi(D(s)) ds.
\end{aligned}
$$

Since $D(0) = 0$ and the inequalities above imply that $D \equiv 0$ and hence the solution is unique. □

1.3 Riemann Mapping and Boundary Behavior

In this section we will present some complex analysis definitions and results which will be useful later. A complete introduction and rigorous proofs of this material can be found in [25] and we will skip the details here.

It is well known that for any open bounded simply connected domain $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$, there is a conformal bijection $\mathcal{T} : \Omega \to \mathbb{D}$. We let $\mathcal{S}$ to be $\mathcal{T}^{-1}$, and by the Kelllogg-

Warschawski Theorem, both can be extended to the boundary continuously. In fact, if the domain is smooth, then all derivatives of $\mathcal{T}$ and $\mathcal{S}$ can also be extended to the boundary continuously.

After using the Riemann mapping and the explicit form of Green's function on the disc, the singular part of the Green's function on original domain 'transfers' to the derivatives of the Riemann mapping, for which we have a good representation formula stated later. Hence by using this trick we may get more accurate estimates near the singular part of the boundary. In particular, using the Biot-Savart Law and the Dirichlet Green's function $G_{\mathbb{D}}(\xi, z) = \frac{1}{2\pi} \ln \frac{|\xi - z|}{|\xi - z^*||z|}$ for $\mathbb{D}$ (with $z^* := z|z|^{-2}$ and $(a,b)^\perp := (-b, a)$), we have

$$u(t,x) = \frac{1}{2\pi} D\mathcal{T}(x)^T \int_\Omega \left(\frac{\mathcal{T}(x) - \mathcal{T}(y)}{|\mathcal{T}(x) - \mathcal{T}(y)|^2} - \frac{\mathcal{T}(x) - \mathcal{T}(y)^*}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2} \right)^\perp \omega(t,y)\, dy. \tag{1.41}$$

Before stating the representation formula, we need to introduce several definitions. Let $\Omega \subset \mathbb{R}^2$ be any open bounded simply connected Lipschitz domain. For any $\theta \in \mathbb{R}$, the *unit forward tangent vector* to Ω at $\mathcal{S}(e^{i\theta}) \in \partial\Omega$ is the unit vector

$$\bar{\nu}_{\mathcal{T}}(\theta) := \lim_{\phi \to \theta+} \frac{\mathcal{S}(e^{i\phi}) - \mathcal{S}(e^{i\theta})}{|\mathcal{S}(e^{i\phi}) - \mathcal{S}(e^{i\theta})|}, \tag{1.42}$$

provided this limit exists. If it does for each $\theta \in \mathbb{R}$, and the limits $\lim_{\phi \to \theta\pm} \bar{\nu}_{\mathcal{T}}(\phi)$ both exist at each $\theta \in \mathbb{R}$, then the domain Ω is said to be *regulated*. In this case obviously $\lim_{\phi \to \theta+} \bar{\nu}_{\mathcal{T}}(\phi) = \bar{\nu}_{\mathcal{T}}(\theta)$, while the argument of the complex number $\bar{\nu}_{\mathcal{T}}(\theta) \left[\lim_{\phi \to \theta-} \bar{\nu}_{\mathcal{T}}(\phi)\right]^{-1}$ equals π minus the interior angle of Ω at $\mathcal{S}(e^{i\theta})$. We then let

$$\bar{\beta}_{\mathcal{T}}(\theta) := \bar{\arg}\, \bar{\nu}_{\mathcal{T}}(\theta), \tag{1.43}$$

where $\bar{\arg}$ is the argument of a complex number plus some integer multiple of 2π. This multiple is chosen so that $\bar{\beta}_{\mathcal{T}}(0) \in [0, 2\pi)$ and $\bar{\beta}_{\mathcal{T}}(\theta) - \lim_{\phi \to \theta-} \bar{\beta}_{\mathcal{T}}(\phi) \in [-\pi, \pi]$ for each

$\theta \in \mathbb{R}$, and if Ω has cusps, we do it so that this difference is π at exterior cusps (with interior angle 0) and $-\pi$ at interior cusps (with interior angle 2π). Of course, then this difference is again π minus the interior angle of Ω at $\mathcal{S}(e^{i\theta})$. Since we only consider Lipschitz domains here (i.e., without cusps), we will always have $\bar{\beta}_{\mathcal{T}}(\theta) - \lim_{\phi \to \theta-} \bar{\beta}_{\mathcal{T}}(\phi) \in (-\pi, \pi)$.

The above defines the right-continuous function $\bar{\beta}_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}$ uniquely, and it satisfies $\bar{\beta}_{\mathcal{T}}(\theta + 2\pi) = \bar{\beta}_{\mathcal{T}}(\theta) + 2\pi$ for all $\theta \in \mathbb{R}$. We will see that whether Euler particle trajectories for general bounded solutions ω can reach $\partial\Omega$ in finite time depends on how quickly is $\bar{\beta}_{\mathcal{T}}$ allowed to decrease locally (which happens when $\bar{\nu}_{\mathcal{T}}$ turns clockwise), with no restrictions on its increase. This will be quantified in terms of a modulus of continuity for one of two components of $\bar{\beta}_{\mathcal{T}}$, with the other component being an arbitrary non-decreasing function. We will split $\bar{\beta}_{\mathcal{T}}$ in Chapter 2.

We call a function

$$m : [0, 2\pi] \to [0, \infty) \tag{1.44}$$

with $m(0) = 0$ a *modulus* if it is continuous, non-decreasing, and satisfies $m(a+b) \le m(a) + m(b)$ for any $a, b \in [0, 2\pi]$ with $a + b \le 2\pi$. If some $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(\theta) - f(\phi)| \le m(r)$ for all $r \in [0, 2\pi]$ and all $\theta, \phi \in \mathbb{R}$ with $|\theta - \phi| \le r$, we say that f has *modulus of continuity* m.

Now we are ready to state a key theorem (Theorem 3.15 in [25]) that gives a representation of the derivative of the Riemann mapping in terms of $\bar{\beta}_{\mathcal{T}}$ above:

Theorem 1.3.1. *Let f map $\mathbb{D}$ conformally onto a regulated domain. Then for $z \in \mathbb{D}$, we have*

$$\log f'(z) = \log |f'(0)| + \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \left(\bar{\beta}_{\mathcal{T}}(t) - t - \frac{\pi}{2} \right) dt \tag{1.45}$$

where $\bar{\beta}_{\mathcal{T}}(t)$ is from (1.43)*.*

Though not obvious, this is actually the generalized form of the Schwarz-Christoffel formula. Lastly, in this section we will prove the log-Lipschitzness of the velocity u on smooth domains, using the Riemann Mapping.

Proof of Theorem 1.2.2. For any smooth domain Ω, let $\mathcal{T}$ be a Riemann mapping from Ω to $\mathbb{D}$ and let $\mathcal{S}$ be its inverse. For $x, x' \in \Omega$, let $\xi = \mathcal{T}(x)$ and $\xi' = \mathcal{T}(x')$. Because $\mathcal{T}$, $\mathcal{S}$ are uniform continuous (by Kellogg-Warschawski) implying that there exists $\tilde{C}, \bar{C} > 0$ such that $\tilde{C}|\xi - \xi'| \geq |x - x'| \geq \bar{C}|\xi - \xi'|$ for all $x, x' \in \Omega$. It is enough to consider $x, x' \in \Omega$ such that $|x - x'| < \max\{1, \bar{C}\}$, hence $r = |\xi - \xi'| < 1$. By using the expression (1.41), change variables by letting $z = \mathcal{T}(y)$, $\omega_0 \in L^\infty$ and Kellogg-Warschawski theorem for smooth domains (which implies both $D\mathcal{T}$, and $D\mathcal{S}$ are uniformly bounded on $\Omega, \mathbb{D}$ respectively, see [25] Chapter 3), we obtain

$$\begin{aligned} |u(t,x) - u(t,x')| &\leq \frac{1}{2\pi}|D\mathcal{T}(x)| \int_{\mathbb{D}} |K_{\mathbb{D}}(\xi, z) - K_{\mathbb{D}}(\xi', z)||\omega(t, \mathcal{S}(z))|| \det D\mathcal{S}(z)|dz \\ &+ \frac{1}{2\pi}|D\mathcal{T}(x) - D\mathcal{T}(x')| \int_{\mathbb{D}} |K_{\mathbb{D}}(\xi', z)||\omega(t, \mathcal{S}(z))|| \det D\mathcal{S}(z)|dz \\ &\leq C\bigg(\int_{\mathbb{D}} |K_{\mathbb{D}}(\xi, z) - K_{\mathbb{D}}(\xi', z)|dz + |D\mathcal{T}(x) - D\mathcal{T}(x')| \int_{\mathbb{D}} |K_{\mathbb{D}}(\xi', z)|dz \bigg), \end{aligned}$$

where C depends on Ω and $||\omega_0||_{L^\infty}$. We will estimate the two components separately.

For the second one, Kellogg-Warschawski theorem implies the uniform continuity of $D\mathcal{T}$, and (1.16) give us

$$|D\mathcal{T}(x) - D\mathcal{T}(x')| \int_{\mathbb{D}} |K_{\mathbb{D}}(\xi', z)|dz \leq C'r,$$

where C' depends on Ω. The estimation of the first integration follows a standard approach. Let $A = \{z \in \mathbb{D} \mid |z - \xi| < 2r\}$, $A_1 = \mathbb{D} \cap A$ and $A_2 = \mathbb{D} \cap A^c$. By triangle inequality and

(1.16), we have

$$\begin{aligned}\int_{A_1} |K_{\mathbb{D}}(\xi, z) - K_{\mathbb{D}}(\xi', z)| dz &\leq \int_{A_1} |K_{\mathbb{D}}(\xi, z)| + |K_{\mathbb{D}}(\xi', z)| dz \\ &\leq C'' \int_{A_1} \frac{1}{|x-z|} + \frac{1}{|x'-z|} dz \\ &\leq 16\pi C'' r\end{aligned}$$

For the integral over A_2, by the Mean Value Theorem and by (1.17), there exists $\xi'' \in \mathbb{D}$, lying on the line segment connecting ξ and ξ', such that

$$\begin{aligned}\int_{A_2} |K_{\mathbb{D}}(\xi, z) - K_{\mathbb{D}}(\xi', z)| dz &\leq r \int_{A_2} \frac{1}{|\xi'' - z|^2} dz \\ &\leq 2r \int_{\mathbb{D} \setminus B(\xi, r)} \frac{1}{|\xi - z|^2} dz \\ &\leq 2\pi r (1 - \log(r))\end{aligned}$$

where the second inequality is because $|\xi'' - z| > \frac{1}{2}|\xi - z|$ for all ξ'' lying on the line segment connecting ξ and ξ'. Combining all the above, and $\tilde{C} r \geq |x - x'| \geq \bar{C} r$ for all $x, x' \in \Omega$ with $r = |\mathcal{T}(x) - \mathcal{T}(x')|$, we get the result. □

1.4 Existence of Weak Solutions on Open Bounded Simply Connected Domains

In this section, we will prove existence of weak solutions satisfying (1.7) and (1.9) on any general open bounded simply connected domain Ω. The idea is to first construct smooth approximations Ω_n of Ω and smooth approximations $\omega_n(0, \cdot)$ of the initial data $\omega(0, \cdot)$. By Theorem 1.2.3 there are unique weak solutions (u_n, ω_n) solving the (1.4)-(1.5) on Ω_n for each n. We would like to find some uniform bounds on u_n to show convergence of the sequence, as well as to show that the limit will solve the Euler equations weakly on Ω. Notice that in this section we will write the initial vorticity ω_0 as $\omega(0,\cdot)$.

Theorem 1.4.1. *If $\Omega \subset \mathbb{R}^2$ is an open bounded simply connected domain with $\omega(0,\cdot) \in L^\infty(\Omega)$, then there exists*

$$u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \quad \textit{with} \;\; \omega = \nabla \times u \in L^\infty(\mathbb{R}^+; L^\infty(\Omega)) \tag{1.46}$$

satisfying (1.7) *and* (1.9).

Proof. Let $\mathcal{T}$ be a Riemann mapping from Ω to $\mathbb{D}$ and $\mathcal{S}$ be its inverse such that $\mathcal{S}'(0) > 0$. Let $\Omega_n := \mathcal{S}(B(0, 1-\frac{1}{n})$. Then Ω_n are smooth Jordan domains with $\Omega_n \subset \Omega$, and the Kellogg-Warschaski theorem shows that Ω_n converges to Ω in the Hausdorff metric. Further, by convolutions with the molliffier functions and truncations, we can find a sequence $\omega_n(0,\cdot) \in C_c^\infty(\Omega)$ which converges strongly to $\omega(0,\cdot)$ in $L^p(\Omega)$ for all $1 \leq p < \infty$ and

$$||\omega_n(0,\cdot)||_{L^p} \leq ||\omega(0,\cdot)||_{L^p} \;\; \forall p \in [1,\infty]. \tag{1.47}$$

Since for any $K \subset \Omega$ compact, $K \subset \Omega_n$ for all $n \geq N_K$ for some N_K, we can construct ω_n so that $\mathrm{supp}(\omega_n(0,\cdot)) \subset \Omega_n$ for all $n \in \mathbb{N}$.

Because each Ω_n has a smooth boundary, Theorem 1.2.3 shows there exists unique (u_n, ω_n) solving equations (1.4)-(1.5) weakly on Ω_n with initial vorticity $\omega_n(0,\cdot)$. Hence we have

$$u_n(t,x) = \nabla^\perp \Psi_n(t,x) \tag{1.48}$$

where Ψ_n solves

$$\Delta \Psi_n = \omega_n = \mathrm{curl}\, u_n \;\text{ in }\; \Omega_n, \;\; \Psi_n|_{\partial\Omega_n} = 0 \tag{1.49}$$

Now we want to prove convergence of ω_n, Ψ_n and u_n. Let D be a fixed ball such that $\Omega_n \subset D$ for all n. For each ω_n and Ψ_n, we extend $\omega_n(\cdot,x) = \Psi_n(\cdot,x) = 0$ for $x \in D \backslash \Omega_n$. Since ω_n is a bounded sequence, by Alaoglu Theorem there exists a subsequence of $\{\omega_n\}$ (for simplicity of notation, we will still use ω_n denoting this subsequence) such that

$$\omega_n \to \omega \text{ weakly }^* \text{ in } L^\infty(\mathbb{R}^+ \times D). \tag{1.50}$$

For Ψ_n, first by the Dirichlet energy estimate for the Poisson equation we have:

$$\frac{1}{2}||\nabla \Psi_n(t,\cdot)||^2_{L^2(\Omega_n)} - ||\Psi_n \omega_n(t,\cdot)||_{L^1(\Omega_n)} \leq 0$$
$$||\nabla \Psi_n(t,\cdot)||^2_{L^2(\Omega_n)} \leq 2||\Psi_n \omega_n(t,\cdot)||_{L^1(\Omega_n)} \leq 2||\Psi_n(t,\cdot)||_{L^2(\Omega_n)}||\omega_n(t,\cdot)||_{L^2(\Omega_n)}$$

for all t, n. Then by the Poincaré inequality on D, there is a $C_D > 0$ such that

$$||\Psi_n||_{H^1_0(\Omega_n)} \leq C_D||\nabla \Psi_n||_{L^2(\Omega_n)} \leq C_D\sqrt{||\Psi_n||_{L^2(\Omega_n)}||\omega_n||_{L^2(\Omega_n)}}. \tag{1.51}$$

This and ω_n being uniformly bounded (because ω is) gives us

$$||\Psi_n||_{H^1_0(\Omega_n)} \leq C'_D. \tag{1.52}$$

Since the total energy $\frac{1}{2}||u_n(t)||_{L^2(\Omega_n)}$ is conserved in time (because of the no-flow boundary condition), we can conclude that C'_D is a uniform $L^2(\Omega_n)$ bound for u_n for all n, t.

In order to derive weak star convergence of Ψ_n, we also need uniform estimate on the H^1_0 norm of $\partial_t \Psi_n$ on D. Note that this time derivative satisfies the Poisson Equation

$$\Delta \partial_t \Psi_n = \partial_t \omega_n = -\text{div}(u_n \omega_n) \text{ in } \Omega_n, \ \ \partial_t \Psi_n|_{\partial \Omega_n} = 0. \tag{1.53}$$

Using integration by parts, equation (1.53) and the divergence theorem, we have:

$$\begin{aligned}\int_D |\nabla \partial_t \Psi_n|^2 dx &= \int_{\Omega_n} |\nabla \partial_t \Psi_n|^2 dx \\ &= \int_{\Omega_n} \partial_t \Psi_n \Delta \partial_t \Psi_n dx \\ &= \int_{\Omega_n} u_n \omega_n \cdot \nabla(\partial_t \Psi_n) dx \\ &\leq ||\omega_n||_{L^\infty(\Omega_n)} ||u_n||_{L^2(\Omega_n)} ||\nabla \partial_t \Psi_n||_{L^2(\Omega_n)}\end{aligned}$$

Using the uniform bound for ω_n and u_n derived above, as well as the Poincaré inequality on D, we thus have

$$||\partial_t \Psi_n(t, \cdot)||_{H_0^1(\Omega_n)} \leq \tilde{C}_D \tag{1.54}$$

for all t, n. By the Banach-Alaoglu Theorem and the Aubin-Lions-Simon Lemma (see [26]), there exists $\Psi \in W^{1,\infty}(\mathbb{R}^+, H_0^1(D))$ and a subsequence of Ψ_n (we will use index n for simplicity of notation) such that

$$\Psi_n \to \Psi \;\text{ weakly}^* \;\text{ in }\; W^{1,\infty}(\mathbb{R}^+, H_0^1(D)), \tag{1.55}$$

and

$$\Psi_n \to \Psi \;\text{ strongly }\;\text{ in }\; C((0,T), H_0^1(D)) \;\; \forall\, T < \infty. \tag{1.56}$$

Note that although the extensions of Ψ_n to 0 on $D \backslash \Omega_n$ make $\Delta \Psi_n$ be the delta function on $\partial \Omega_n$, since Ω_n converges to Ω in the Hausdorff metric, hence for every compact set K, $K \subset \Omega_n$ for $n > N$ for some $N \in \mathbb{N}$. This means that for almost all $t > 0$,

$$\Delta \Psi(t, \cdot) = \omega(t, \cdot) \;\text{ in }\; \mathcal{D}'(\Omega). \tag{1.57}$$

Now by Theorems 3.2.3 or 3.2.7 in [11], we can conclude that for each fixed t, there is a subsequence of $\{\Psi_n(t,\cdot)\}$ (for simplicity of notations, we will use the same notation denoting this subsequence) that converges weakly in $H_0^1(D)$ to a limit in $H_0^1(\Omega)$. This means that $\Psi(t,\cdot) \in H_0^1(\Omega)$ for all t. By this we have

$$\int_0^T \int_D |\nabla\Psi|^2 = \int_0^T \int_\Omega |\nabla\Psi|^2 = -\int_0^T \int_\Omega \omega\Psi = -\int_0^T \int_D \omega\Psi \tag{1.58}$$

and similarly we have

$$\int_0^T \int_D |\nabla\Psi_n|^2 = \int_0^T \int_{\Omega_n} |\nabla\Psi_n|^2 = -\int_0^T \int_{\Omega_n} \omega_n\Psi_n = -\int_0^T \int_D \omega_n\Psi_n \to -\int_0^T \int_D \omega\Psi \tag{1.59}$$

where the last convergence is by (1.52), (1.56), and by the L^2 norm for $\omega(t,\cdot)$ being conserved and bounded uniformly in time. This shows strong convergence of Ψ_n to Ψ in $L^2(0,T;H_0^1(D))$, which also yields strong L^2 convergence of u_n to u where $u = \nabla^\perp\Psi \in L^\infty(\mathbb{R}^+;L^2(\Omega))$ and $\omega = \text{curl } u \in L^\infty(\mathbb{R}^+ \times \Omega)$. By construction, the divergence free and boundary tangent conditions are satisfied. We also have $u_n(0,\cdot) \to u(0,\cdot)$ strongly in L^2. This is by the same argument as above we can derive that $u_n(0,\cdot)$ has a convergent subsequence, whose limit can be written as $\tilde{u}(0,\cdot) = \nabla^\perp\tilde{\Psi}(0,\cdot)$ with $\tilde{\Psi}(0,\cdot) \in H_0^1(\Omega)$ and $\Delta\Psi(0,\cdot) = \omega_0$, which also satisfies the divergence free and boundary tangent conditions. Hence so does the difference $v = u(0,\cdot) - \tilde{u}(0,\cdot)$, which is also curl free. This implies that v is smooth on Ω by considering the Cauchy-Riemann equation for the function

$$z \mapsto v_1(z) + iv_2(z),$$

where $v = (v_1, v_2)$. Moreover by the Green's Theorem and v_0 is curl free, for any smooth Jordan curve Γ in Ω, we have

$$\oint_\Gamma v \cdot d\boldsymbol{s} = 0, \tag{1.60}$$

which shows $v = \nabla p$ for some smooth p inside Ω. Define

$$H(\Omega) := \text{completion of}\{u \in \mathcal{D}(\Omega)\ :\ \text{div}\ u = 0\} \text{ in the norm of } L^2 \tag{1.61}$$

and

$$G(\Omega) := \{u \in L^2(\Omega)\ :\ u = \nabla p \text{ for some } p \in H^1_{loc}(\Omega)\} \tag{1.62}$$

where (1.7) can be re-written in terms of (1.62) (replacing ∇h with u in $G(\Omega)$). Then Lemma III 2.1 in [8] states that if v satisfies (1.7), then it's in $H(\Omega)$. Moreover the previous argument shows it is also in $G(\Omega)$, and the intersection of $H(\Omega)$ and $G(\Omega)$ is $\{0\}$. Hence $v \equiv 0$.

Using above strong L^2 convergences, and that for each $\psi \in C_0^\infty([0,\infty) \times \Omega)$, when n is large enough such that supp $\psi \subset \Omega_n$, we have

$$\int_0^\infty \int_\Omega \omega_n(\partial_t \psi + u_n \cdot \nabla \psi) = -\int_\Omega \omega_n(0,\cdot)\psi(0,\cdot). \tag{1.63}$$

We get (1.9) by taking limits on both sides. This shows existence of solutions. □

1.5 A General Uniqueness Lemma

In this section, we would like to state and reproduce a general uniqueness lemma (lemma 1.4 from [19]) in the case that initial vorticity is constant near the singular parts of the boundary. The idea of the proof is that when the particles are away from the singular parts of the boundary, the standard estimates on the Green's function still hold, which implies that the log-Lipschitz property of the velocity still holds on any fixed subset of Ω that is away from the singular part of the boundary (while the relevant constant depends on the subset). Then we will use the log-Lipschitz estimate, transportation of the vorticity and the measure perserving property of the flow to show that the mean absolute difference between any two solutions, with the same $\omega(0,\cdot)$, will be zero. Let $\Omega \subseteq \mathbb{R}^2$ be any open bounded simply connected domain and $\alpha > 0$. We define

$$\Gamma_\alpha := \{x \in \partial\Omega \ : \ \partial\Omega \cap B(x, \epsilon) \notin C^{2,\alpha} \text{ for all } \epsilon > 0\} \tag{1.64}$$

In the following proof, to make the notation simpler we will let $X(t,x) = X_t^x$ for $x \in \Omega$.

Theorem 1.5.1. *Let Ω be an open bounded simply connected domain and let u be a global weak solution to the Euler equations on Ω from the Yudovich class such that $\omega(t,x) = \omega(0, X^{-1}(t,x))$ for all $t > 0$. If there is a constant $a \in \mathbb{R}$ such that supp$(\omega(0,\cdot) - a) \cap \Gamma_\alpha = \emptyset$ for some $a > 0$, then u is the unique such solution with initial value ω_0 until the first time t such that supp$(\omega(t,\cdot) - a) \cap \Gamma_\alpha \neq \emptyset$*

Proof. Without loss of generality, we assume $||\omega_0||_{L^\infty} \leq 1$. Let $T > 0$ be any time such that $\cup_{0 \leq t \leq T}\text{supp}(\omega(t,\cdot) - a) \cap \Gamma_\alpha = \emptyset$. Let $\Omega_0 \subset \Omega$ be an open set such that $\omega_0(x) = a$ if $x \in \Omega \backslash \Omega_0$ with the measure of $\partial\Omega_0 = 0$. Moreover, by definition of T, we have $\overline{\cup_{t\in[0,T)} X(t,\Omega_0)} \cap \Gamma_\alpha = \emptyset$. Let G_Ω be the Dirichlet Green's function on Ω, with $G_\Omega(x,y) = G_{\mathbb{D}}(\mathcal{T}(x), \mathcal{T}(y))$ where $\mathcal{T}$ is a Riemann mapping from Ω to $\mathbb{D}$. By Kellogg-Warschawski, $\mathcal{T}(x)$ is at least C^2 for x is away from Γ_α. Hence we have standard Green's function estimates away from Γ_α. Specifically, there exists a domain Ω_T with $\Omega \cap \overline{\cup_{t\in[0,T)} X(t,\Omega_0)} \subseteq \Omega_T \subset \Omega$ and a constant C_{Ω_T} such that if $x, y \in \Omega_T$ with $|x - y| < \frac{1}{C_{\Omega_T}}$, then x, y can be joined by a smooth curve whose length is no more than $C_{\Omega_T}|x - y|$, and the estimates (1.16) and (1.17) still hold for any $x \in \Omega_T$ and $y \in \Omega$.

By contradiction, if there's another solution $\tilde{\omega}$ with its associated flow map Y and $\omega(0, Y^{-1}(t,0)) = \tilde{\omega}(t,\cdot)$, we will show $X = Y$ almost surely by showing that the following function equals zero:

$$D(t) := \frac{1}{|\Omega_0|} \int_{\Omega_0} |X(t,x) - Y(t,x)| dx. \tag{1.65}$$

The Biot-Savart Laws for $\omega, \tilde{\omega}$ can be written as:

$$u(t,x) := \int_\Omega K_\Omega(x,y)\omega(t,y)dy \tag{1.66}$$

and

$$v(t,x) := \int_{\Omega} K_{\Omega}(x,y)\tilde{\omega}(t,y)dy. \tag{1.67}$$

Recall the definition (1.18) of $\phi(r)$. Applying the argument used to show u is log-Lipschitz on smooth domains yields (since we have the same estimates for the Green's function here)

$$\max\left\{\int_{\Omega}|K_{\Omega}(x,y)-K_{\Omega}(x',y)|dy, \int_{\Omega}|K_{\Omega}(y,x)-K_{\Omega}(y,x')|dy\right\} \le C\phi(|x-x'|) \tag{1.68}$$

for all $x, x' \in \Omega_T$ with C depends on Ω, Ω_T and C_T. Now let $T' \le T$ be the largest T' such that $\Omega \cap \overline{\cup_{t\in[0,T')} Y(t,\Omega_0)} \subseteq \Omega_T$. By $||\omega_0||_{L^\infty} \le 1$, we also have

$$|u(t,x) - u(t,x')| \le C\phi(|x'-x|) \tag{1.69}$$

for all $t < T'$ and $x, x' \in \Omega_T$. Moreover, by the definition of Ω_0, we have for any $z \in \Omega$

$$\int_{\Omega\setminus\Omega_0} K_{\Omega}(z, X(s,y))\omega_0(y)dy = a\left(\int_{\Omega} K_{\Omega}(z, X(s,y))dy - \int_{\Omega_0} K_{\Omega}(z, X(s,y))dy\right) \tag{1.70}$$

and the equation is also true if we replace X with Y. These inequalities and (1.70) plus the measure-preserving property of X and Y in space gives us

$$\begin{aligned}
&\frac{1}{|\Omega_0|}\int_{\Omega_0}|u(s,Y(s,x))-v(s,Y(s,x)|dsdx\\
&=\frac{1}{|\Omega_0|}\int_0^t\int_{\Omega_0}\left|\int_\Omega K_\Omega(Y(s,x),y)\omega(s,y)-K_\Omega(Y(s,x),y)\tilde{\omega}(s,y)dy\right|dxds\\
&=\frac{1}{|\Omega_0|}\int_0^t\int_{\Omega_0}\left|\int_\Omega\Big(K_\Omega(Y(s,x),X(s,y))-K_\Omega(Y(s,x),Y(s,y))\Big)\omega_0(y)dy\right|dxds\\
&\leq\frac{1+a}{|\Omega_0|}\int_0^t\int_{\Omega_0}\int_{\Omega_0}|K_\Omega(Y(s,x),X(s,y))-K_\Omega(Y(s,x),Y(s,y))|dydxds\\
&\leq\frac{2}{|\Omega_0|}\int_0^t\int_{\Omega_0}\int_{\Omega_0}|K_\Omega(Y(s,x),X(s,y))-K_\Omega(Y(s,x),Y(s,y))|dydxds\\
&\leq\frac{2}{|\Omega_0|}\int_0^t\int_{\Omega}\int_{\Omega_0}|K_\Omega(x,X(s,y))-K_\Omega(x,Y(s,y))|dydxds\\
&\leq\frac{2C}{|\Omega_0|}\int_0^t\int_{\Omega_0}\phi(|X(s,y)-Y(s,y)|)dyds\\
&\leq 2C\int_0^t\phi(D(s))ds
\end{aligned}$$

for all $t<T'$, where the last inequality is by Jensen's inequality. By triangle inequality, the log-Lipschitz estimate and the inequality above, we have

$$\begin{aligned}
D(t)&\leq\frac{1}{|\Omega_0|}\int_{\Omega_0}\int_0^t|u(s,X(s,x))-v(s,Y(s,x))|dsdx\\
&\leq\frac{1}{|\Omega_0|}\int_{\Omega_0}\int_0^t|u(s,X(s,x))-u(s,Y(s,x))|dsdx\\
&+\frac{1}{|\Omega_0|}\int_{\Omega_0}\int_0^t|u(s,Y(s,x))-v(s,Y(s,x))|dsdx\\
&\leq 3C\int_0^t\phi(D(s))dx
\end{aligned}$$

for all $t<T'$ with $D(0)=0$. This implies $D(t)\equiv 0$ for all $t\in[0,T')$, which means $X=Y$ and $\omega=\tilde{\omega}$ on $[0,T')\times\Omega$. Thus we have $T'=T$, and so $Y\equiv X$. □

1.6 Outline

This thesis focuses on proving uniqueness of weak solutions from Yudovich class on singular domains. In Chapter 2, we will show a general sufficient condition on the geometry of the domain's boundary that guarantees global uniqueness of the weak solution when ω_0 is constant near the whole boundary $\partial\Omega$. Moreover, we will show that this condition is sharp in the sense by constructing domains that come arbitrarily close to satisfying it, and on which Euler particles for bounded ω may reach the boundary in finite time. In Chapter 3, we will show uniqueness of the weak solutions on more general domains, including domains that contains corners with angles larger than π which are excluded in the result of Chapter 2, by further assuming that ω has a sign.

Chapter 2

Planar Domains without Reflex Corners

2.1 Main Results

In this chapter, we will consider general *regulated* (i.e. (1.42) exists for all θ) bounded Lipschitz domains. We will obtain a general condition guaranteeing that Euler particle trajectories for bounded weak solutions on these domains never reach $\partial\Omega$, and also prove uniqueness of global weak solutions for all vorticities initially constant near $\partial\Omega$. This condition is only slightly more restrictive than exclusion of corners with angles greater than π, which was shown to be necessary in [19], and it places no restrictions on those segments of $\partial\Omega$ where the argument of the forward tangent vector (1.43) is non-decreasing. This shows that our results will include all convex Ω. Specifically, our condition is satisfied precisely when the argument of the forward tangent vector to $\partial\Omega$, composed with the Riemann mapping for Ω, can be written as a sum of an arbitrary increasing function and a second function that has a modulus of continuity m (recall (1.44)) from a precisely defined class of moduli (which includes, e.g., m with $m(r) = \frac{\pi}{2|\log r|}$ for all small enough $r > 0$). Moreover, for any concave modulus m from this class, we find the exact (up to a constant factor in time) asymptotic rate of the fastest possible approach of Euler particle trajectories to $\partial\Omega$ among all domains as above. We also show that no vorticity can be created by the boundary of these possibly singular domains, a result that even extends in a weaker form to general bounded domains (see Corollary 2.1.4). We also note that a priori the ODE (1.10) only holds for almost all $t \in (0, t_x)$ (with X_t^x, the particle trajectory, continuous in time, for t_x see (1.11)), but we will show that u is continuous and therefore (1.10) holds for all $t \in [0, t_x)$ (see Corollary 2.1.4 below).

Finally, we show that our condition is essentially sharp. Specifically, for each concave modulus not in the above class of moduli (e.g., m with $m(r) = \frac{a}{2|\log r|}$ for all small enough $r > 0$, with any fixed $a > \pi$), we construct a domain as above in which particle trajectories can reach the boundary in finite time. Hence this work pushes right up to the limits of the philosophy from [17, 16, 19, 22], within the class of regulated domains at least, under the assumption that the vorticity initially constant near the singular part of the boundary where the velocity may be far from Lipschitz. Our Theorem 2.1.1(ii) and Corollary 2.1.4 below represent a first step in this effort.

To state the results more rigorously, first recall the definitions (1.42), (1.43),(1.44). Based on these, we let

$$q_m(s) := s \exp\left(\frac{2}{\pi}\int_s^1 \frac{m(r)}{r} dr\right),$$

and if $\int_0^1 \frac{ds}{q_m(s)} = \infty$, we let $\rho_m : \mathbb{R} \to (0,1)$ be the inverse function to $y \mapsto \ln \int_y^1 \frac{ds}{q_m(s)}$, so

$$\rho_m\left(\ln \int_y^1 \frac{ds}{q_m(s)}\right) = y.$$

Then ρ_m is decreasing with $\lim_{t\to-\infty} \rho_m(t) = 1$ and $\lim_{t\to\infty} \rho_m(t) = 0$, and we shall see that it is the maximal asymptotic approach rate of Euler particle trajectories to $\partial\Omega$ (up to a constant factor in time) among all domains for which $\bar{\beta}_{\mathcal{T}}$ above has modulus of continuity m. In fact, our main results show that this statement extends to domains with $\bar{\beta}_{\mathcal{T}}$ being a sum of a function with modulus m and any non-decreasing function (see hypothesis **(H)** below). Note also that $\int_0^1 \frac{ds}{q_m(s)} = \infty$ holds whenever $\int_0^1 \frac{m(r)}{r} dr < \infty$, and functions with such moduli m are called *Dini continuous*.

In our main results, we will assume the following hypothesis.

(H) Let $\Omega \subseteq \mathbb{R}^2$ be a regulated open bounded Lipschitz domain with $\partial\Omega$ a Jordan curve. Let $\mathcal{T} : \Omega \to \mathbb{D}$ be a Riemann mapping and let $\beta_{\mathcal{T}}, \tilde{\beta}_{\mathcal{T}}$ be functions on $\mathbb{R}$ with 2π-periodic (distributional) derivatives such that $\beta_{\mathcal{T}}$ is non-decreasing, $\tilde{\beta}_{\mathcal{T}}$ has some modulus of continuity m with q_m and ρ_m defined above, and the argument of the (counterclockwise) forward tangent vector to $\partial\Omega$ is $\bar{\beta}_{\mathcal{T}} = \beta_{\mathcal{T}} + \tilde{\beta}_{\mathcal{T}}$.

Note that if $\beta_{\mathcal{T}}, \tilde{\beta}_{\mathcal{T}}$ are as above and their sum is the argument of the forward tangent vector to a Jordan curve $\partial\Omega$, then the bounded domain Ω must automatically be regulated.

As mentioned above, neither **(H)** nor our results place any continuity restrictions on $\beta_{\mathcal{T}}$. In particular, the following main result of the present paper holds for any convex domain Ω, since then one can let $\beta_{\mathcal{T}} := \bar{\beta}_{\mathcal{T}}$ and $\tilde{\beta}_{\mathcal{T}} \equiv 0$ (and therefore $m \equiv 0$).

Theorem 2.1.1. *Assume* **(H)** *and that* $\int_0^1 \frac{ds}{q_m(s)} = \infty$. *Let* $\omega_0 \in L^\infty(\Omega)$ *and let* ω *from the Yudovich class be any global weak solution to the Euler equations on* Ω *with initial condition* ω_0 *(such solutions are known to exist by [9]).*

(i) We have $t_x = \infty$ *for all* $x \in \Omega$, *and for any* $R < 1$ *and all large enough* $t > 0$,

$$\sup_{|\mathcal{T}(x)| \le R} |\mathcal{T}(X_t^x)| \le 1 - \rho_m(300\|\omega\|_{L^\infty} t) \tag{2.1}$$

(except when $\omega \equiv 0$, *but then* $X_t^x \equiv x$*). And if* $\tilde{\beta}_{\mathcal{T}}$ *is Dini continuous, then the right-hand side of* (2.1) *can be replaced by the* m*-independent bound* $1 - \exp(-e^{300\|\omega\|_{L^\infty} t})$.

(ii) We have $\{X_t^x \,|\, x \in \Omega\} = \Omega$ *for all* $t > 0$, *and for a.e.* $(t, x) \in \mathbb{R}^+ \times \Omega$ *we have* $\omega(t, X_t^x) = \omega_0(x)$. *Moreover,* u *is continuous on* $[0, \infty) \times \Omega$ *and* (1.10) *holds for all* $(t, x) \in [0, \infty) \times \Omega$.

(iii) If $\operatorname{supp}(\omega_0 - a) \cap \partial\Omega = \emptyset$ *for some* $a \in \mathbb{R}$, *then the solution* ω *is unique.*

Remarks. 1. This naturally extends to solutions on time intervals $(0,T)$ for $T \in (0,\infty)$.

2. Part (i) also shows that $\inf_{|\mathcal{T}(x)|\le R} d(X_t^x, \partial\Omega) \ge \rho_m(300\|\omega\|_{L^\infty} t)$ for any $R<1$, due to $\mathcal{T}$ being Hölder continuous for Lipschitz Ω (see, e.g., [20, Theorem 2]). This is because our proof shows that (i) also holds with 299 in place of 300, and one can easily show that $\rho_m(300ct) \le \frac{1}{N}\rho_m(299ct)^N$ for any fixed $c, N > 0$ and all large enough $t > 0$.

3. A "borderline" case for the condition $\int_0^1 \frac{ds}{q_m(s)} = \infty$ is $m(r) = \frac{a}{|\log r|}$ for all small $r > 0$ (with $a \ge 0$). Here $\int_0^1 \frac{ds}{q_m(s)} = \infty$ holds precisely when $a \le \frac{\pi}{2}$, while $\int_0^1 \frac{m(r)}{r} dr = \infty$ for all $a > 0$. In this case ρ_m is still a double exponential when $a < \frac{\pi}{2}$, as for Dini continuous $\tilde{\beta}_{\mathcal{T}}$, but a triple exponential when $a = \frac{\pi}{2}$. The double-exponential rate is known to be the maximal possible boundary approach rate for *smooth* domains, due to (1.6) holding there, but (1.6) fails even for general convex domains. See also the remark after Theorem 2.1.2 below.

Our second main result, which applies to concave moduli m, shows that Theorem 2.1.1(i) is essentially sharp, even for stationary solutions. This involves analysis of Euler particle trajectories on some special domains, which are more sophisticated versions of domains with concave corners considered in [15, 19].

Theorem 2.1.2. *For any concave modulus m, there is a domain Ω satisfying* **(H)** *and a stationary weak solution ω from the Yudovich class to the Euler equations on Ω such that the following hold.*

(i) If $\int_0^1 \frac{ds}{q_m(s)} < \infty$, then $X_t^x \in \partial\Omega$ for some $x \in \Omega$ and $t > 0$.

(ii) If $\int_0^1 \frac{ds}{q_m(s)} = \infty$, then $|\mathcal{T}(X_t^x)| \ge 1 - \rho_m(ct)$ for some $x \in \Omega$, $c > 0$, and all $t \ge 0$.

Remark. Note that if $m(r) = a(L_1(\frac{1}{r}) \dots L_{k-1}(\frac{1}{r}))^{-1} + \frac{\pi}{2}\sum_{j=1}^{k-2}(L_1(\frac{1}{r}) \dots L_j(\frac{1}{r}))^{-1}$ for all small enough $r > 0$, with $k \ge 2$, $a \in [0, \frac{\pi}{2})$, and $L_j(r)$ being $\ln r$ composed j times, then ρ_m is essentially a k-tuple exponential. Therefore all such boundary approach rates do occur on some domains Ω to which Theorem 2.1.1(i) applies.

We also note that Theorem 2.1.1 has a natural analog when the forward tangent vector is defined via arc-length parametrization of $\partial\Omega$, rather than via $\mathcal{S}$. If $\sigma : [0, 2\pi] \to \partial\Omega$ is the (counter-clockwise) constant speed parametrization of $\partial\Omega$ (extended to be 2π-periodic on $\mathbb{R}$, and obviously unique up to translation), then Lemma 1 in [31] shows that $\mathcal{T} \circ \sigma$ and its inverse (modulo 2π) are Hölder continuous. If we therefore use

$$\bar{\nu}_\Omega(\theta) := \lim_{\phi \to \theta+} \frac{\sigma(\phi) - \sigma(\theta)}{|\sigma(\phi) - \sigma(\theta)|}, \tag{2.2}$$

instead of (1.42), and the corresponding $\tilde{\beta}_\Omega$ (with $\bar{\beta}_\Omega, \beta_\Omega, \tilde{\beta}_\Omega$ chosen analogously to $\bar{\beta}_\mathcal{T}, \beta_\mathcal{T}, \tilde{\beta}_\mathcal{T}$) has some modulus of continuity m, then $\tilde{\beta}_\mathcal{T}$ has modulus of continuity $\tilde{m}(r) := m(Cr^\gamma)$ for some $C, \gamma > 0$. But since a simple change of variables shows that $\int_0^1 \frac{m(r)}{r} dr < \infty$ is equivalent to $\int_0^1 \frac{m(Cr^\gamma)}{r} dr < \infty$, we obtain the following result.

Corollary 2.1.3. *Theorem 2.1.1 continues to hold when* (1.42) *and* $\bar{\beta}_\mathcal{T}, \beta_\mathcal{T}, \tilde{\beta}_\mathcal{T}$ *in* **(H)** *are replaced by* (2.2) *and* $\bar{\beta}_\Omega, \beta_\Omega, \tilde{\beta}_\Omega$*, respectively, and if* $\tilde{\beta}_\Omega$ *is also Dini continuous.*

Remarks. 1. Of course, while $\bar{\beta}_\mathcal{T}, \beta_\mathcal{T}, \tilde{\beta}_\mathcal{T}$ depend on $\mathcal{T}$, they can also be made to only depend on Ω because we are free to choose $\mathcal{T}$.

2. Note that if an open bounded simply connected Lipschitz domain Ω can be touched from the outside by a disc of uniform radius at each point of $\partial\Omega$ (i.e., Ω satisfies the *uniform exterior sphere condition*), and we replace (1.42) by (2.2), then these hypotheses are satisfied with $m(r) = Cr$ for some constant C. Hence Corollary 2.1.3 holds for all such domains.

Finally, we provide here a version of Theorem 2.1.1(ii) for general open bounded domains, which follows from its proof and is also of independent interest. To the best of our knowledge, such results previously required $\partial\Omega$ to be at least piecewise $C^{1,1}$ (see, e.g., [16, 18, 19]).

Corollary 2.1.4. *Let* ω *from the Yudovich class be a weak solution to the Euler equations on an open bounded domain* $\Omega \subseteq \mathbb{R}^2$*, on time interval* $(0, T)$ *for some* $T \in (0, \infty]$ *and with initial condition* $\omega_0 \in L^\infty(\Omega)$*. Then* $\omega(t, X_t^x) = \omega_0(x)$ *for a.e.* $t \in (0, T)$ *and a.e.* $x \in \Omega$ *with*

$t_x > t$, the velocity u is continuous on $[0,T)\times\Omega$ (as well as on $[0,T]\times\Omega$ if $T<\infty$), and (1.10) holds for all $x\in\Omega$ and $t\in[0,t_x)$.

Remark. So even when $\partial\Omega$ is very irregular, vorticity might be created (at $\partial\Omega$) only if enough particle trajectories "depart" from the boundary into Ω, so that $\Omega\setminus\{X_t^x \,|\, x\in\Omega\}$ has positive measure for some $t\in(0,T)$.

2.2 Proof of Theorem 2.1.1(i)

Take any $x\in\Omega$ and let

$$d(t) := 1 - |\mathcal{T}(X_t^x)|$$

be the distance of $\mathcal{T}(X_t^x)$ from $\partial\mathbb{D}$. Then we have

$$d'(t) = -\frac{\mathcal{T}(X_t^x)}{|\mathcal{T}(X_t^x)|}\cdot D\mathcal{T}(X_t^x)\frac{d}{dt}X_t^x$$

as long as $|\mathcal{T}(X_t^x)|\in(0,1)$. Since $D\mathcal{T}$ is of the form $\begin{pmatrix} a & b \\ -b & a\end{pmatrix}$ because $\mathcal{T}$ is analytic, we have $D\mathcal{T}D\mathcal{T}^T = (\det D\mathcal{T})I_2$. The Biot-Savart law (1.41) for $\frac{d}{dt}X_t^x$ now shows that

$$\begin{aligned} d'(t) =& -\frac{\det D\mathcal{T}(X_t^x)}{2\pi|\mathcal{T}(X_t^x)|}\int_\Omega\left(\frac{-\mathcal{T}(X_t^x)\cdot\mathcal{T}(y)^\perp}{|\mathcal{T}(X_t^x)-\mathcal{T}(y)|^2}+\frac{\mathcal{T}(X_t^x)\cdot\mathcal{T}(y)^{*\perp}}{|\mathcal{T}(X_t^x)-\mathcal{T}(y)^*|^2}\right)\omega(t,y)\,dy \\ =&\frac{\det D\mathcal{T}(X_t^x)(1-|\mathcal{T}(X_t^x)|^2)}{2\pi|\mathcal{T}(X_t^x)|}\int_\Omega\frac{|\mathcal{T}(y)|^2(1-|\mathcal{T}(y)|^2)\mathcal{T}(X_t^x)\cdot\mathcal{T}(y)^\perp}{|\mathcal{T}(X_t^x)-\mathcal{T}(y)|^2\,||\mathcal{T}(y)|^2\mathcal{T}(X_t^x)-\mathcal{T}(y)|^2}\omega(t,y)\,dy. \end{aligned}$$

where $z^* := z|z|^{-2}$ and $(a,b)^\perp := (-b,a)$. After the change of variables $z=\mathcal{T}(y)$, we obtain

$$|d'(t)| \le d(t)\frac{2\|\omega\|_{L^\infty}}{\pi|\mathcal{T}(X_t^x)|}\det D\mathcal{T}(X_t^x)\int_\mathbb{D}\frac{(1-|z|)|\mathcal{T}(X_t^x)\cdot z^\perp|}{|\mathcal{T}(X_t^x)-z|^2\,||z|^2\mathcal{T}(X_t^x)-z|^2}\det D\mathcal{T}^{-1}(z)\,dz.$$

This estimate already appeared in [19], but we will use the following crucial result to tightly bound its right-hand side for much more general domains.

Lemma 2.2.1. *Assume* **(H)** *and that* $\int_0^1 \frac{ds}{q_m(s)} = \infty$*. There is* $C < 147\pi$ *and a (*$\mathcal{T}$*-dependent) constant* $C_{\mathcal{T}} > 0$ *such that if* $|\xi| \in [\frac{1}{2}, 1)$*, then*

$$\det D\mathcal{T}(\mathcal{T}^{-1}(\xi)) \int_{\mathbb{D}} \frac{(1-|z|)|\xi \cdot z^{\perp}|}{|\xi - z|^2 \, ||z|^2\xi - z|^2} \det D\mathcal{T}^{-1}(z)\, dz \\ \le C\, Q_m(1-|\xi|) \left(\int_{1-|\xi|}^1 \frac{ds}{sQ_m(s)} + C_{\mathcal{T}} \right), \qquad (2.3)$$

with $Q_m(s) := s^{-1} q_m(s) = \exp\left(\frac{2}{\pi} \int_s^1 \frac{m(r)}{r} dr \right)$*.*

Remark. Note that Q_m is non-increasing, and $\lim_{s\to 0} s^\alpha Q_m(s) = 0$ for all $\alpha > 0$ because $s^\alpha = \exp(-\alpha \int_s^1 \frac{dr}{r})$.

Lemma 2.2.1 with $\xi := \mathcal{T}(X_t^x)$ now yields

$$d'(t) \ge -C\|\omega\|_{L^\infty} q_m(d(t)) \left(\int_{d(t)}^1 \frac{ds}{q_m(s)} + C_{\mathcal{T}} \right)$$

with some $C < 300$ and $C_{\mathcal{T}} > 0$ when $d(t) \in (0, \frac{1}{50}]$. Hence

$$\frac{d}{dt} \ln \left(\int_{d(t)}^1 \frac{ds}{q_m(s)} + C_{\mathcal{T}} \right) \le C\|\omega\|_{L^\infty},$$

and so

$$\ln \int_{d(t)}^1 \frac{ds}{q_m(s)} \le C\|\omega\|_{L^\infty} t + \ln \left(\int_{\min\{d(0),1/2\}}^1 \frac{ds}{q_m(s)} + C_{\mathcal{T}} \right)$$

for all $t \ge 0$. Therefore

$$d(t) \ge \rho_m \left(C\|\omega\|_{L^\infty} t + \ln \left(\int_{\min\{d(0),1/2\}}^1 \frac{ds}{q_m(s)} + C_{\mathcal{T}} \right) \right). \qquad (2.4)$$

This is no less than $\rho_m\left(300\|\omega\|_{L^\infty} t\right)$ for all large $t \ge 0$, uniformly in all x with $|\mathcal{T}(x)| \le R$ (for any $R < 1$, except when $\omega \equiv 0$). And if $M := \int_0^1 \frac{m(r)}{r} dr < \infty$, then $\rho_m(z) \ge \exp(-e^{z+2M/\pi})$ (because $\rho_m(z)$ equals y such that $e^z = \int_y^1 \frac{ds}{q_m(s)} \ge e^{-2M/\pi} \int_y^1 \frac{ds}{s}$), so this is no less than $\exp(-e^{300\|\omega\|_{L^\infty} t})$ for all large $t \ge 0$, uniformly in all x with $|\mathcal{T}(x)| \le R$.

Hence, to conclude Theorem 2.1.1(i), it only remains to prove Lemma 2.2.1. Its proof, which relies on the crucial representation formula (2.7) for $D\mathcal{T}$, is somewhat involved. We postpone it to Section 2.4 and first show how to obtain Theorem 2.1.1(ii,iii) from Theorem 2.1.1(i).

2.3 Proofs of Theorem 2.1.1(ii,iii) and Corollary 2.1.4

Theorem 2.1.1(iii) follows immediately from Theorem 2.1.1(ii) and Proposition 3.2 in [19], which shows that solutions from Theorem 2.1.1(ii) are unique as long as they remain constant near $\partial\Omega$ (constancy near the non-$C^{2,\gamma}$ portion of $\partial\Omega$ for some $\gamma > 0$, where u may be far from Lipschitz, is in fact sufficient). It therefore suffices to prove Theorem 2.1.1(ii).

The first claim follows from the fact that the estimate (2.4) equally applies to the solutions of the time-reversed ODE $\frac{d}{ds}Y(s) = -u(t-s, Y(s))$ with $Y(0) \in \Omega$ (which of course satisfy $Y(s) = X_{t-s}^{Y(t)}$). The proof of the second claim was obtained in [16, 18, 19] for some sufficiently regular domains by looking at (1.4) as a (passive) transport equation with given u and ω_0, and proving uniqueness of its solutions (using also that $t_x = \infty$ for all $x \in \Omega$). This is because $\tilde{\omega}(t, X_t^x) := \omega_0(x)$ can be shown to be its weak solution in the sense of (1.9). The uniqueness proofs used the DiPerna-Lions theory, which required relevant extensions of u and ω to $\mathbb{R}^2 \setminus \Omega$ (the latter by 0). This necessitated $\partial\Omega$ to be piecewise $C^{1,1}$, in addition to having $t_x = \infty$ for all $x \in \Omega$, so that the extension of u is sufficiently regular for the DiPerna-Lions theory to be applicable.

We avoid this extension argument, and hence also extra regularity hypotheses on Ω, thanks to the following result concerning weak solutions to the transport equation (1.4).

Lemma 2.3.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open and $T > 0$. Let $u \in L^\infty_{\text{loc}}([0,T] \times \Omega)$ satisfy*

$$\sup_{t\in[0,T]} \sup_{x,y\in K} \frac{|u(t,x) - u(t,y)|}{|x-y| \max\{1, -\ln|x-y|\}} < \infty \tag{2.5}$$

for any compact $K \subseteq \Omega$, as well as (1.2) on $(0,T) \times \Omega$. If $\omega \in L^\infty_{\text{loc}}([0,T] \times \Omega)$ is a weak *solution to the linear PDE* (1.4) *with initial condition* $\omega_0 \in L^\infty_{\rm loc}(\Omega)$ *and* X^x_t *is from* (1.10)*, then we have* $\omega(t, X^x_t) = \omega_0(x)$ *for a.e.* $t \in (0,T)$ *and a.e.* $x \in \Omega$ *with* $t_x > t$*.*

Proof. Let $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ be smooth open bounded sets in $\mathbb{R}^2$ with $\bar\Omega_n \subseteq \Omega = \bigcup_{n\ge1}\Omega_n$. Since ω is also a weak solution to (1.4) on Ω_n and exit times $t_{x,n}$ of X^x_t from Ω_n then satisfy $\lim_{n\to\infty} t_{x,n} = t_x$ for each $x \in \Omega$, it obviously suffices to prove that $\omega(t, X^x_t) = \omega_0(x)$ for a.e. $t \in (0,T)$ and a.e. $x \in \Omega_n$ such that $t_{x,n} > t$. We can therefore assume that Ω is smooth and bounded, (2.5) holds with K replaced by Ω, and u, ω, ω_0 are all bounded. We can also assume without loss that $\omega \ge 0$ and $\omega_0 \ge 0$, by adding a large constant to them.

Extend the particle trajectories from (1.10) by $X^x_t := \lim_{s\uparrow t_x} X^x_s \in \partial\Omega$ for all $t \ge t_x$, and let $\Omega_t := \{X^x_t \,|\, x \in \Omega \;\&\; t_x > t\}$ for all $t \in [0,T)$ (these sets are open due to (2.5)). Then the lemma essentially follows from Theorem 2 in [3] but in order to apply it, we need to show that ω weakly satisfies some boundary conditions on $(0,T) \times \partial\Omega$ (even though these do not affect the result). To this end we employ Theorem 3.1 and Remark 3.1 in [4], which show that there is indeed some $\kappa \in L^\infty((0,T)\times\partial\Omega)$ such that

$$\int_0^T \int_\Omega \omega\left(\partial_t\varphi + u\cdot\nabla\varphi\right)\,dxdt = -\int_\Omega \omega_0\varphi(0,\cdot)\,dx + \int_0^T\int_{\partial\Omega}(u\cdot n)\varphi\kappa\,d\sigma dt$$

holds for all $\varphi \in C_0^\infty\big([0,T)\times\bar\Omega\big)$.

Theorem 2 in [3] now shows that there is a positive measure η on Ω such that

$$\int_{\Omega_t} \psi(y)\omega(t,y)\,dy = \int_\Omega \psi(X^x_t)\,d\eta(x) \tag{2.6}$$

for almost all $t \in (0,T)$ and all $\psi \in C_0^\infty(\Omega_t)$. (In fact, the measure in [3] is supported on the set of all maximal solutions to the ODE $\frac{d}{dt}Y(t) = u(t, Y(t))$ on $(0,T)$, and the relevant formula holds for all $\psi \in C_0^\infty(\mathbb{R}^d)$. But this becomes (2.6) when restricted to the ψ above, with η the restriction of the measure from [3] to the set of solutions $\{\{X^x_t\}_{t\in(0,T)} \,|\, x\in\Omega\}$. This is because uniqueness of solutions for the ODE shows that the other solutions have

$Y(t) \notin \Omega_t$ for any $t \in (0,T)$.) By taking $t \to 0$ in (2.6), we obtain

$$\int_\Omega \psi(y)\omega_0(y)\,dy = \int_\Omega \psi(x)\,d\eta(x)$$

for any $\psi \in C_0^\infty(\Omega)$, so $d\eta(x) = \omega_0(x)dx$. Letting ψ in (2.6) be approximate delta functions near all $y \in \Omega_t$ then shows that for almost all $t \in (0,T)$ we have $\omega(t, X_t^x) = \omega_0(x)$ whenever x and X_t^x are Lebesgue points of ω_0 and $\omega(t,\cdot)$, respectively. This finishes the proof. □

Since $t_x = \infty$ for all $x \in \Omega$, Lemma 2.3.1 with $T \to \infty$ now proves the second claim in Theorem 2.1.1(ii). As in [19], uniform boundedness of u on any compact subset of Ω then yields $\omega \in C([0,\infty); L^1(\Omega))$, and continuity of u on $[0,\infty) \times \Omega$ follows from this and the Biot-Savart law. Then also (1.10) holds pointwise, finishing the proof of Theorem 2.1.1(ii).

This argument actually applies on general open bounded $\Omega \subseteq \mathbb{R}^2$, without needing $t_x = \infty$ for all $x \in \Omega$. This is because boundedness of ω implies $u \in L^\infty((0,T) \times K)$ for any compact $K \subseteq \Omega$ as well as (2.5) (for solutions on a time interval $(0,T)$ with $T < \infty$), and these three facts then again yield $\omega \in C([0,T]; L^1(\Omega))$ (with $\omega(0,\cdot) := \omega_0$ and $\omega(T,\cdot)$ defined by continuity). This yields Corollary 2.1.4.

2.4 Proof of Lemma 2.2.1

We can assume that $\tilde{\beta}_{\mathcal{T}}(0) = 0$, which is achieved by subtracting $\tilde{\beta}_{\mathcal{T}}(0)$ from $\tilde{\beta}_{\mathcal{T}}$ and adding it to $\beta_{\mathcal{T}}$. Since $\mathcal{T}$ is analytic, we have $\det D\mathcal{T}(z) = |\mathcal{T}'(z)|^2$, where $\mathcal{T}'$ is the complex derivative when $\mathcal{T}$ is considered as a function on $\mathbb{C}$. The same is true for its inverse $\mathcal{S}$, and we also have $\mathcal{S}'(z) = \mathcal{T}'(\mathcal{S}(z))^{-1}$.

Since Ω is regulated, Theorem 3.15 in [25] shows that

$$\mathcal{S}'(z) = |\mathcal{S}'(0)| \exp\left(\frac{i}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\bar{\beta}_{\mathcal{T}}(\theta) - \theta - \frac{\pi}{2}\right) d\theta\right) \tag{2.7}$$

for all $z \in \mathbb{D}$, and from $\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta = 2\pi \in \mathbb{R}$ and $\mathrm{Im}\, \frac{e^{i\theta}+z}{e^{i\theta}-z} = 2\mathrm{Im}\, \frac{z}{e^{i\theta}-z}$ we get

$$\det D\mathcal{S}(z) = \det D\mathcal{S}(0) \exp\left(-\frac{2}{\pi} \int_0^{2\pi} \mathrm{Im}\, \frac{z}{e^{i\theta}-z} \left(\bar\beta_{\mathcal{T}}(\theta) - \theta\right) d\theta \right) \tag{2.8}$$

(with $\bar\beta_{\mathcal{T}}(\theta) - \theta$ being 2π-periodic).

We note that if $\bar\beta_{\mathcal{T}}$ is itself Dini continuous (so we can have $\tilde\beta_{\mathcal{T}} = \bar\beta_{\mathcal{T}}$ and $\int_0^1 \frac{m(r)}{r}\, dr < \infty$), then the integral in (2.8) is uniformly bounded by some m-dependent constant. Indeed, letting $\theta_z := \arg z$, this follows from $\mathrm{Im}\, \frac{z}{e^{i(\theta_z-\theta')}-z} = -\mathrm{Im}\, \frac{z}{e^{i(\theta_z+\theta')}-z}$ and $|\frac{z}{e^{i(\theta_z+\theta')}-z}| \le \frac{\pi}{2|\theta'|}$ for all θ' (which show that $\int_0^{2\pi} \mathrm{Im}\, \frac{z}{e^{i\theta}-z}(\bar\beta_{\mathcal{T}}(\theta_z) - \theta) d\theta$ is uniformly bounded), and from the latter bound also implying

$$\left| \frac{z}{e^{i\theta}-z} \left(\bar\beta_{\mathcal{T}}(\theta) - \bar\beta_{\mathcal{T}}(\theta_z)\right) \right| \le \frac{\pi}{2} \frac{m(|\theta-\theta_z|)}{|\theta-\theta_z|}.$$

One can also easily show that $\int_{\mathbb{D}} \frac{(1-|z|)|\xi\cdot z^\perp|}{|\xi-z|^2\, ||z|^2\xi - z|^2} dz \le C|\ln(1-|\xi|)|$ for some $C > 0$ when $|\xi| \in [\frac{1}{2}, 1)$, using (2.20) below and the argument following it, with the exponential terms removed. So (2.3) with the right-hand side $C_m |\ln(1-|\xi|)|$ follows immediately in this case. The rest of this section (and Section 2.6) proves (2.3) in the general case.

We will now split the exponential in (2.8) into the parts corresponding to $\beta_{\mathcal{T}}$ and $\tilde\beta_{\mathcal{T}}$. Let $\kappa := \frac{1}{2\pi}(\tilde\beta_{\mathcal{T}}(2\pi) - \tilde\beta_{\mathcal{T}}(0))$, so that $\tilde\beta_{\mathcal{T}}(\theta) - \kappa\theta$ and $\beta_{\mathcal{T}}(\theta) - (1-\kappa)\theta$ are both 2π-periodic (note that we also have $\kappa \in [-\frac{m(2\pi)}{2\pi}, \min\{1, \frac{m(2\pi)}{2\pi}\}]$ because $\beta_{\mathcal{T}}$ is non-decreasing). Integration by parts then shows that

$$\int_0^{2\pi} \frac{z}{e^{i\theta}-z} \left(\beta_{\mathcal{T}}(\theta) - (1-\kappa)\theta\right) d\theta = i \int_0^{2\pi} \ln(1 - ze^{-i\theta})\, d\left(\beta_{\mathcal{T}}(\theta) - (1-\kappa)\theta\right),$$

so from $\int_0^{2\pi} \ln(1-ze^{-i\theta}) d\theta = \ln 1 = 0$ we obtain

$$\int_0^{2\pi} \mathrm{Im}\, \frac{z}{e^{i\theta}-z} \left(\beta_{\mathcal{T}}(\theta) - (1-\kappa)\theta\right) d\theta = \int_0^{2\pi} \ln|e^{i\theta} - z|\, d\beta_{\mathcal{T}}(\theta). \tag{2.9}$$

In order to simplify notation, let β be the positive measure with distribution function $\beta_{\mathcal{T}}$, and define the function $\tilde{\beta}(\theta) := \tilde{\beta}_{\mathcal{T}}(\theta) - \kappa\theta$. Then $\tilde{\beta}$ has modulus of continuity $\tilde{m}(r) := m(r) + |\kappa| r$, and we have $\tilde{m}(r) \le m(r) + \frac{m(2\pi)}{2\pi} r \le 3m(r)$ for $r \in [0, 2\pi]$. This is because any modulus satisfies $m(2^{-n}a) \ge 2^{-n} m(a)$ for any $a \in [0, 2\pi]$ and $n \in \mathbb{N}$ (by induction), and thus $m(b) \ge \frac{b}{2a} m(a)$ whenever $0 \le b \le a \le 2\pi$ since m is non-decreasing. We also let

$$|\beta| := \beta((0, 2\pi]) = \beta_{\mathcal{T}}(2\pi) - \beta_{\mathcal{T}}(0) = 2\pi(1-\kappa) \in [0, 2\pi + m(2\pi)].$$

Next, for any $z \in \mathbb{D}$, bounded measurable $A \subset \mathbb{R}$, and $\theta^* \in \mathbb{R}$, let

$$\mathcal{I}(z, A) := \frac{2}{\pi} \int_A \ln |e^{i\theta} - z| \, d\beta(\theta),$$
$$\mathcal{J}(z, A, \theta^*) := \frac{2}{\pi} \int_A \operatorname{Im} \frac{z}{e^{i\theta} - z} \, (\tilde{\beta}(\theta) - \tilde{\beta}(\theta^*)) \, d\theta,$$

as well as

$$\mathcal{I}(z) := \mathcal{I}(z, (0, 2\pi]),$$
$$\mathcal{J}(z) := \mathcal{J}(z, (0, 2\pi], \theta^*)$$

(with the latter independent of θ^* due to $\int_0^{2\pi} \operatorname{Im} \frac{z}{e^{i\theta}-z} d\theta = 0$). Then (2.8) and (2.9) yield

$$\det D\mathcal{S}(z) = \det D\mathcal{S}(0) \, e^{-\mathcal{I}(z) - \mathcal{J}(z)}$$

and

$$\det D\mathcal{T}(\mathcal{S}(z)) = \det D\mathcal{S}(0)^{-1} \, e^{\mathcal{I}(z) + \mathcal{J}(z)} \tag{2.10}$$

(recall that $\tilde{\beta}(0) = 0$). In view of this, (2.3) becomes

$$\int_{\mathbb{D}} \frac{(1-|z|)|\xi \cdot z^{\perp}|}{|\xi - z|^2 \, ||z|^2 \xi - z|^2} e^{\mathcal{I}(\xi) - \mathcal{I}(z)} e^{\mathcal{J}(\xi) - \mathcal{J}(z)} \, dz \le C \, Q_m(1 - |\xi|) \left(\int_{1-|\xi|}^1 \frac{ds}{s Q_m(s)} + C_{\mathcal{T}} \right). \tag{2.11}$$

To prove this, we need the following lemma, whose proof we postpone to Section 2.6.

Lemma 2.4.1. *Let β be a (positive) measure on $\mathbb{R}$ and let $I := [\theta^* - 2\delta, \theta^* + 2\delta]$ for some $\theta^* \in \mathbb{R}$ and $\delta \in (0, \frac{\pi}{2}]$. Let $H \subset \mathbb{D}$ be an open region such that if $re^{i(\theta^*+\phi)} \in H$ for some $r \in (0,1)$ and $|\phi| \le \pi$, then $re^{i(\theta^*+\phi')} \in H$ whenever $|\phi'| \le |\phi|$ (i.e., H is symmetric and angularly convex with respect to the line connecting 0 and $e^{i\theta^*}$). If $\alpha \ge 1$, then*

$$\int_H f(z) \left[g(z) + \frac{1}{\beta(I)} \int_I h(|e^{i\theta} - z|) d\beta(\theta) \right]^\alpha dz \le \int_H f(z) \left[g(z) + h(|e^{i\theta^*} - z|) \right]^\alpha dz \quad (2.12)$$

holds for any non-increasing $h : (0, \infty) \to [0, \infty)$ and non-negative $f, g \in L^1(H)$ such that $f(re^{i(\theta^*+\phi')}) \ge f(re^{i(\theta^*+\phi)})$ and $g(re^{i(\theta^*+\phi')}) \ge g(re^{i(\theta^*+\phi)})$ whenever $r \in (0,1)$ and $|\phi'| \le |\phi|$.

Remark. The right-hand side of (2.12) is just the left-hand side for the Dirac measure at θ^* with mass $\beta(I)$. That is, concentrating all the mass of β on I into θ^* cannot decrease the value of the integral in (2.12).

Next, we claim that there is $\delta > 0$ such that $\beta([\theta - 2\delta, \theta + 2\delta]) \le \frac{4}{3}\pi$ for all $\theta \in \mathbb{R}$ (any number from $(\pi, \frac{3}{2}\pi)$ would work in place of $\frac{4}{3}\pi$ here). Let $\delta' > 0$ be such that any interval of length $4\delta'$ contains at most one θ with $\beta(\{\theta\}) \ge \frac{\pi}{9}$ (there are only finitely many such θ in $(0, 2\pi]$). Then for each $\theta \in [0, 2\pi]$, find $\delta_\theta \in (0, \delta']$ such that $\beta([\theta - 2\delta_\theta, \theta + 2\delta_\theta]) \le \beta(\{\theta\}) + \frac{\pi}{9}$. Since $\{(\theta - 2\delta_\theta, \theta + 2\delta_\theta) \,|\, \theta \in [-\pi, 3\pi]\}$ is an open cover of $[-\pi, 3\pi]$, there is a finite sub-cover $\{(\theta_k - 2\delta_{\theta_k}, \theta_k + 2\delta_{\theta_k}) \,|\, k = 1, \dots, N\}$. If we let $\delta := \min\{\delta_{\theta_k} \,|\, k = 1, \dots, N\} > 0$, then indeed $\beta([\theta - 2\delta, \theta + 2\delta]) \le (\pi + \frac{\pi}{9}) + (\frac{\pi}{9} + \frac{\pi}{9}) = \frac{4}{3}\pi$ for all $\theta \in [0, 2\pi]$ (and so for all $\theta \in \mathbb{R}$). This is because $[\theta - 2\delta, \theta + 2\delta] \subseteq [\theta_k - 2\delta_{\theta_k}, \theta_k + 2\delta_{\theta_k}] \cup [\theta_j - 2\delta_{\theta_j}, \theta_j + 2\delta_{\theta_j}]$ for some k, j such that $|\theta_k - \theta_j| \le 4\delta'$, and hence at most one of $\beta(\{\theta_k\})$ and $\beta(\{\theta_j\})$ is greater than $\frac{\pi}{9}$ (unless $k = j$), while obviously each is at most π.

Moreover, let us decrease this constant so that $\delta \in (0, \frac{\ln 2}{10^3(1+m(2\pi))}]$ and $m(2\delta) \le \frac{\ln 2}{300}$. With this ($\mathcal{T}$-dependent) δ, we can now prove the following estimates (recall (2.11)).

Lemma 2.4.2. *Let $\beta, \tilde{\beta}, m$ and δ be as above. There are $C_{|\beta|,\delta}$ and C_m (depending only on*

$|\beta|, \delta$ and on m, respectively, so only on $\mathcal{T}$) such that for any $\xi \in \mathbb{D}$ we have

$$\int_{\mathbb{D}} z^{-1}(1-|z|)^{5/6} e^{\mathcal{I}(\xi)-\mathcal{I}(z)} dz \leq C_{|\beta|,\delta}, \tag{2.13}$$

and for all $z, \xi \in \mathbb{D}$ also

$$e^{\mathcal{J}(\xi)-\mathcal{J}(z)} \leq C_m \frac{Q_m(\min\{1-|\xi|, |\xi-z|\})}{Q_m(|\xi-z|)} \frac{Q_m(\min\{1-|z|, |\xi-z|\})}{Q_m(|\xi-z|)}. \tag{2.14}$$

Moreover, if $|\xi - z| \leq 4\delta$, then for $\theta_\xi := \arg \xi$ and $I := [\theta_\xi - 2\delta, \theta_\xi + 2\delta]$ we have

$$e^{\mathcal{J}(\xi,I,\theta_\xi)-\mathcal{J}(z,I,\theta_\xi)} \leq 2\, \frac{Q_m(\min\{1-|\xi|, |\xi-z|\})}{Q_m(|\xi-z|)} \frac{Q_m(\min\{1-|z|, |\xi-z|\})}{Q_m(|\xi-z|)}. \tag{2.15}$$

Proof. Let us start with (2.14). Let $\theta_\xi := \arg \xi$ and $\theta_z := \arg z$, as well as

$$A := \left\{ \theta \in (0, 2\pi] \,\middle|\, \min\{d(\theta, \theta_\xi), d(\theta, \theta_z)\} \geq \frac{1}{2}|\xi - z| \right\},$$

where d is the distance in $[0, 2\pi]$ with 0 and 2π identified. Then from

$$\left| \frac{\xi}{e^{i\theta} - \xi} - \frac{z}{e^{i\theta} - z} \right| = \left| \frac{e^{i\theta}(\xi - z)}{(e^{i\theta} - \xi)(e^{i\theta} - z)} \right| \leq \pi^2 \frac{|\xi - z|}{d(\theta, \theta_\xi)\, d(\theta, \theta_z)}$$

we obtain

$$|\mathcal{J}(\xi, A, \pi) - \mathcal{J}(z, A, \pi)| \leq 4\pi |\xi - z| \tilde{m}(\pi) \left(\int_{|\xi-z|/2}^{\pi} \frac{dr}{r(r+2a)} + \int_b^a \frac{dr}{r(2a-r)} \right),$$

where $a := \frac{1}{2} d(\theta_\xi, \theta_z)$ and $b := \min\{\frac{1}{2}|\xi - z|, a\} \leq a$, and we separately integrated over the 2 or 4 regions obtained by cutting A at the two midpoints between θ_ξ and θ_z. That is,

$$|\mathcal{J}(\xi, A, \pi) - \mathcal{J}(z, A, \pi)| \leq 4\pi \tilde{m}(\pi) \left(2 + \frac{b}{a} \ln \frac{a}{b} \right) \leq 10\pi \tilde{m}(\pi) \leq C_m.$$

On the complement $A^c := (0, 2\pi] \setminus A$ we can estimate the two $\mathcal{J}$ terms individually.

To conclude (2.14), it now suffices to show

$$|\mathcal{J}(z, A^c, \pi)| \le C_m + \ln \frac{Q_m(\min\{1-|z|, |\xi - z|\})}{Q_m(|\xi - z|)} \tag{2.16}$$

because an analogous estimate then follows for $\mathcal{J}(\xi, A^c, \pi)$ as well. First note that if we let $A' := \{\theta \in A^c \,|\, d(\theta, \theta_z) > \frac{1}{2}|\xi - z|\}$, then

$$|\mathcal{J}(z, A', \pi)| \le 2\tilde{m}(\pi) \int_{d(A',\theta_z)}^{3d(A',\theta_z)} \frac{dr}{r} \le C_m.$$

With $A'' := \{\theta \in A^c \,|\, d(\theta, \theta_z) \le \frac{1}{2}\min\{1-|z|, |\xi - z|\}\}$ we also have

$$|\mathcal{J}(z, A'', \pi)| \le \frac{2}{\pi}\tilde{m}(\pi) \le C_m$$

due to $|e^{i\theta} - z| \ge 1 - |z|$. This proves (2.16) when $|\xi - z| \le 1 - |z|$. If instead $|\xi - z| > 1 - |z|$, then we also use $\operatorname{Im} \frac{z}{e^{i(\theta_z - r)} - z} = -\operatorname{Im} \frac{z}{e^{i(\theta_z + r)} - z}$ (note that the region $A^c \setminus (A' \cup A'')$ is symmetric across θ_z, so $\int_{A^c \setminus (A' \cup A'')} \operatorname{Im} \frac{z}{e^{i(\theta_z + r)} - z} d\theta = 0$) and $|e^{i\theta} - z| \ge \sin|\theta - \theta_z|$ to estimate

$$|\mathcal{J}(z, A^c \setminus (A' \cup A''), \pi)| \le \frac{2}{\pi} \int_{(1-|z|)/2}^{|\xi - z|/2} \frac{\tilde{m}(2r)}{\sin r} dr \le C_m + \ln \frac{Q_m(1-|z|)}{Q_m(|\xi - z|)},$$

with the last inequality due to

$$\int_{a/2}^{b/2} \frac{\tilde{m}(2r)}{\sin r} dr \le \int_a^b \frac{\tilde{m}(s)}{\sin s} ds \le \int_a^b \frac{m(s)}{s} ds + \int_a^b (10\tilde{m}(s) + |\kappa|)\, ds \le \int_a^b \frac{m(s)}{s} ds + C_m$$

for $0 \le a \le b \le 2$ (because $\sup_{s \in [0,2]}(\frac{1}{\sin s} - \frac{1}{s}) \le 10$). Hence (2.16) follows, proving (2.14).

To obtain (2.15), we repeat this argument with some minor adjustments. For

$$A := \left\{ \theta \in I \,\middle|\, \min\{d(\theta, \theta_\xi), d(\theta, \theta_z)\} \ge \frac{1}{2}|\xi - z| \right\},$$

we obtain the bound

$$|\mathcal{J}(\xi,A,\theta_\xi)-\mathcal{J}(z,A,\theta_\xi)|\le 4\pi\tilde m(2\delta)\left(2+\frac{b}{a}\ln\frac{a}{b}\right)\le 10\pi\tilde m(2\delta)\le 30\pi m(2\delta)\le\frac{\ln 2}{3}$$

(recall that $\tilde m(s)\le 3m(s)$). Hence it suffices to show (2.16) with $A^c:=I\setminus A$, and with θ_ξ and $\frac{\ln 2}{3}$ in place of π and C_m. As above, we now obtain

$$|\mathcal{J}(z,A',\theta_\xi)|\le 2\tilde m(2\delta)\int_{d(A',\theta_z)}^{3d(A',\theta_z)}\frac{dr}{r}\le 4\ln 3\,m(2\delta)\le\frac{\ln 2}{9}$$

and

$$|\mathcal{J}(z,A'',\theta_\xi)|\le\frac{2}{\pi}\tilde m(2\delta)\le\frac{4}{\pi}m(2\delta)\le\frac{\ln 2}{9}.$$

Finally, if $|\xi-z|>1-|z|$, then we also get

$$|\mathcal{J}(z,A^c\setminus(A'\cup A''),\theta_\xi)|\le\frac{2}{\pi}\int_{(1-|z|)/2}^{|\xi-z|/2}\frac{\tilde m(2r)}{\sin r}dr\le\frac{\ln 2}{9}+\ln\frac{Q_m(1-|z|)}{Q_m(|\xi-z|)}$$

because $\int_a^b(10\tilde m(s)+|\kappa|)\,ds\le 4\delta(21m(2\pi))\le\frac{\ln 2}{9}$ when $0\le a\le b\le 4\delta$.

Now we prove (2.13). We obviously have

$$\max\{\mathcal{I}(\xi),|\mathcal{I}(z)|\}\le\frac{2\ln 2}{\pi}|\beta| \tag{2.17}$$

for all $\xi\in\mathbb{D}$ and all $z\in B(0,\frac{1}{2})$, so it suffices to prove

$$\int_{\mathbb{D}}(1-|z|)^{5/6}e^{-\mathcal{I}(z)}dz\le C_{|\beta|,\delta}. \tag{2.18}$$

The integrand is clearly bounded above by $(\frac{\delta}{2})^{-2|\beta|/\pi}$ on $B(0,1-\frac{\delta}{2})$. Since $\mathbb{D}\setminus B(0,1-\frac{\delta}{2})$ can be covered by $O(\frac{1}{\delta})$ disks with centers on $\partial\mathbb{D}$ and radii δ, it suffices to prove (2.18) with $H:=B(e^{i\theta^*},\delta)\cap\mathbb{D}$ in place of $\mathbb{D}$, for any $\theta^*\in\mathbb{R}$.

Let $I:=[\theta^*-2\delta,\theta^*+2\delta]$ and $\alpha:=\frac{2\beta(I)}{\pi}\in[0,\frac{8}{3}]$. Since $\mathcal{I}(z,(0,2\pi]\setminus\bigcup_{k\in\mathbb{Z}}(I+2k\pi))$

is bounded below by $\frac{2|\beta|}{\pi} \ln \frac{\delta}{2}$ for all $z \in H$, it in fact suffices to prove

$$\int_H (1-|z|)^{5/6} e^{-\mathcal{I}(z,I)} dz \leq C. \tag{2.19}$$

If $\alpha \in [0,1]$, then from $1-|z| \leq |e^{i\theta}-z|$ for all $(z,\theta) \in \mathbb{D} \times \mathbb{R}$ we indeed have

$$\begin{aligned}\int_H (1-|z|)^{5/6} e^{-\mathcal{I}(z,I)} dz &= \int_H (1-|z|)^{-\alpha+5/6} \exp\left(\frac{2}{\pi}\int_I \ln \frac{1-|z|}{|e^{i\theta}-z|} d\beta(\theta)\right) dz \\ &\leq \int_{\mathbb{D}} (1-|z|)^{-1/6} dz,\end{aligned}$$

as needed.

If $\alpha \in [1, \frac{8}{3}]$, then we instead use Jensen's inequality and Lemma 2.4.1 with $f(z) = (1-|z|)^{5/6}$, $g(z) = 0$, and $h(s) = \frac{1}{s}$ to obtain

$$\begin{aligned}\int_H (1-|z|)^{5/6} e^{-\mathcal{I}(z,I)} dz &\leq \int_H (1-|z|)^{5/6} \exp\left[\alpha \ln\left(\frac{1}{\beta(I)}\int_I \frac{1}{|e^{i\theta}-z|} d\beta(\theta)\right)\right] dz \\ &= \int_H (1-|z|)^{5/6} \left(\frac{1}{\beta(I)}\int_I \frac{1}{|e^{i\theta}-z|} d\beta(\theta)\right)^{\alpha} dz \\ &\leq \int_H (1-|z|)^{5/6} |e^{i\theta^*}-z|^{-\alpha} dz \\ &\leq \int_H |e^{i\theta^*}-z|^{-\alpha+5/6} dz \\ &\leq 12\pi.\end{aligned}$$

This proves (2.19) and hence also (2.13). □

Now we are ready to prove Lemma 2.2.1

Proof of Lemma 2.2.1. For the sake of simplicity, we first prove the result with $C < 10^5$, and at the end indicate the changes required to obtain $C < 147\pi$. Consider the ($\mathcal{T}$-dependent) δ from above. Recall that we only need to prove (2.11), and note that $\xi \cdot z^\perp = (\xi - z) \cdot z^\perp$ implies

$$\frac{|\xi \cdot z^\perp|}{|\xi - z|^2 \, ||z|^2\xi - z|^2} \leq \frac{1}{|\xi - z| \, |z| \, ||z|\xi - \frac{z}{|z|}|^2} = \frac{1}{|\xi - z| \, |z|^3 \, |\xi - \frac{z}{|z|^2}|^2}. \tag{2.20}$$

Together with (2.17) and (2.14) this yields C_m such that for any $\xi \in \mathbb{D} \setminus B(0, \frac{1}{2})$ we have

$$\int_{B(0,\frac{1}{4})} \frac{(1-|z|)|\xi \cdot z^\perp|}{|\xi - z|^2 ||z|^2\xi - z|^2} e^{\mathcal{I}(\xi)-\mathcal{I}(z)} e^{\mathcal{J}(\xi)-\mathcal{J}(z)} dz \le C_m Q_m(1-|\xi|)$$

because $|z|\,|\xi - \frac{z}{|z|^2}| = ||z|\xi - \frac{z}{|z|}| \ge 1 - |z|$ and the last fraction in (2.14) is bounded above by $\exp\left(\frac{2}{\pi}\int_{3/4}^{5/4} \frac{m(r)}{r} dr\right)$ when $z \in B(0, \frac{1}{4})$ (note that the dependence of the constant on $|\beta|$ need not be indicated here because $0 \le |\beta| \le 2\pi + m(2\pi)$).

If now $|\xi| \in [\frac{1}{2}, 1)$ and $z \in B(\xi, \frac{1-|\xi|}{2})$, then $\mathcal{I}(\xi) - \mathcal{I}(z) \le \frac{2|\beta|}{\pi}$ due to $|e^{i\theta} - \xi||e^{i\theta} - z|^{-1} \le 2$ for all $\theta \in \mathbb{R}$. Hence using $|\xi - \frac{z}{|z|^2}| \ge 1 - |\xi| \ge \frac{1-|z|}{2}$ in (2.20) (because $\frac{z}{|z|^2} \notin \mathbb{D}$) and $|\xi - z| \le \min\{1-|\xi|, 1-|z|\}$ in (2.14) yields

$$\int_{B(\xi,\frac{1-|\xi|}{2})} \frac{(1-|z|)|\xi \cdot z^\perp|}{|\xi - z|^2 ||z|^2\xi - z|^2} e^{\mathcal{I}(\xi)-\mathcal{I}(z)} e^{\mathcal{J}(\xi)-\mathcal{J}(z)} dz \le C_m \int_{B(\xi,\frac{1-|\xi|}{2})} \frac{1}{|\xi - z|(1-|\xi|)} dz \le C_m \pi.$$

For all other $z \in \mathbb{D} \setminus B(0, \frac{1}{4})$, we can bound the right-hand side of (2.20) above by $\frac{64}{|\xi - z|^3}$, using that $|\frac{z}{|z|^2}| - 1 \ge 1 - |z|$ implies $|\xi - \frac{z}{|z|^2}| \ge |\xi - z|$. This, (2.14), (2.13), and Q_m being non-increasing and satisfying the bounds $Q_m(1-|\xi|) \ge 1$ and $Q_m(1-|z|) \le C_m(1-|z|)^{-1/6}$ (see the remark after Lemma 2.2.1) now yield

$$\int_{\mathbb{D}\setminus(B(\xi,\delta^3)\cup B(0,\frac{1}{4}))} \frac{(1-|z|)|\xi \cdot z^\perp|}{|\xi - z|^2 ||z|^2\xi - z|^2} e^{\mathcal{I}(\xi)-\mathcal{I}(z)} e^{\mathcal{J}(\xi)-\mathcal{J}(z)} dz \le C_{m,\delta}\, Q_m(1-|\xi|)$$

(note that the constant now also depends on δ). To obtain (2.11), it therefore suffices to prove

$$\int_{H_\xi} \frac{1-|z|}{|\xi - z|^3} e^{\mathcal{I}(\xi)-\mathcal{I}(z)} e^{\mathcal{J}(\xi)-\mathcal{J}(z)} dz \le C\, Q_m(1-|\xi|) \left(\int_{1-|\xi|}^1 \frac{ds}{sQ_m(s)} + 1\right) \tag{2.21}$$

when $|\xi| \in [1-2\delta^3, 1)$, with $H_\xi := [B(\xi, \delta^3) \setminus B(\xi, \frac{1-|\xi|}{2})] \cap \mathbb{D}$ and a universal $C < 10^5(1-3\delta^3)^3$. Since $(1-3\delta^3)^3 \ge (1 - \frac{3}{10^9})^3 > 1 - \frac{1}{10^8}$, it suffices to obtain $C \le 10^5 - 1$ here

Let $\theta_\xi := \arg \xi$, and again let $I := [\theta_\xi - 2\delta, \theta_\xi + 2\delta]$ as well as $\alpha := \frac{2\beta(I)}{\pi} \in [0, \frac{8}{3}]$.

Then $|e^{i\theta}-\xi|\ge\delta$ for all $\theta\notin\bigcup_{k\in\mathbb{Z}}(I+2k\pi)$, hence for all such θ and all $z\in B(\xi,\delta^3)$ we have $\frac{|e^{i\theta}-\xi|}{|e^{i\theta}-z|}\le\frac{1}{1-\delta^2}\le 1+\frac{\pi}{2|\beta|}$ (the last inequality follows from $\delta^2\le\frac{\pi}{\pi+2|\beta|}$, which is due to $\frac{\pi}{\pi+2|\beta|}\ge\frac{\pi}{5\pi+2m(2\pi)}\ge\frac{\ln 2}{10^3(1+m(2\pi))}\ge\delta$). This yields for all $z\in B(\xi,\delta^3)$,

$$\mathcal{I}(\xi)-\mathcal{I}(z)=\frac{2}{\pi}\int_{(0,2\pi]}\ln\frac{|e^{i\theta}-\xi|}{|e^{i\theta}-z|}d\beta(\theta)\le 1+\frac{2}{\pi}\int_I\ln\frac{|e^{i\theta}-\xi|}{|e^{i\theta}-z|}d\beta(\theta). \tag{2.22}$$

Similarly, for the same z and θ we have $|\frac{\xi}{e^{i\theta}-\xi}-\frac{z}{e^{i\theta}-z}|=\frac{|\xi-z|}{|e^{i\theta}-\xi||e^{i\theta}-z|}\le\frac{\delta}{1-\delta^2}\le\frac{1}{4\tilde{m}(2\pi)}$, so

$$\mathcal{J}(\xi)-\mathcal{J}(z)=\mathcal{J}(\xi,(0,2\pi],\theta_\xi)-\mathcal{J}(z,(0,2\pi],\theta_\xi)\le 1+\mathcal{J}(\xi,I,\theta_\xi)-\mathcal{J}(z,I,\theta_\xi). \tag{2.23}$$

Using (2.15), combined with $Q_m(\frac{1}{2}(1-|\xi|))Q_m(1-|\xi|)^{-1}\le e^{2m(2\delta^3)/\pi}\le e^{1/100\pi}$ (recall that $|\xi-z|\ge\frac{1-|\xi|}{2}$) and $Q_m(a)Q_m(b)^{-1}\le\exp(\frac{1}{6}\int_a^b\frac{1}{r}dr)=b^{1/6}a^{-1/6}$ for $0<a\le b\le\delta^3$ (because $m(\delta^3)\le m(2\delta)\le\frac{\pi}{12}$), we thus obtain

$$e^{\mathcal{J}(\xi)-\mathcal{J}(z)}\le 2\cdot 3^{1/6}e^{1+1/100\pi}\,\frac{Q_m(1-|\xi|)}{Q_m(|\xi-z|)}\,\frac{|\xi-z|^{1/6}}{(1-|z|)^{1/6}}, \tag{2.24}$$

where we also used that $1-|z|\le 3|\xi-z|$ for all $z\in\mathbb{D}\setminus B(\xi,\frac{1-|\xi|}{2})$. Estimates (2.22) and (2.24), together with $2\cdot 3^{1/6}e^{1+1/100\pi}\le 3e$ and

$$\int_{\frac{1}{2}(1-|\xi|)}^{1-|\xi|}\frac{ds}{sQ_m(s)}\le\frac{\ln 2}{Q_m(1-|\xi|)}\le\int_{1-|\xi|}^{2(1-|\xi|)}\frac{ds}{sQ_m(s)}\le\int_{1-|\xi|}^{1}\frac{ds}{sQ_m(s)}, \tag{2.25}$$

now show that (2.21) will follow from

$$\int_{H_\xi}\frac{(1-|z|)^{5/6}}{|\xi-z|^{17/6}}\exp\left(\frac{2}{\pi}\int_I\ln\frac{|e^{i\theta}-\xi|}{|e^{i\theta}-z|}d\beta(\theta)\right)\frac{dz}{Q_m(|\xi-z|)}\le C\left(\int_{\frac{1}{2}(1-|\xi|)}^{1}\frac{ds}{sQ_m(s)}+1\right) \tag{2.26}$$

whenever $|\xi|\in[1-2\delta^3,1)$, with some universal $C\le\frac{10^5-1}{6e^2}$.

Consider now the case $\alpha\in[0,1]$. We have $1-|z|\le|e^{i\theta}-z|$ for all $(z,\theta)\in\mathbb{D}\times\mathbb{R}$, and $1-|z|\le 3|\xi-z|$ for all $z\in H_\xi$. This and the triangle inequality yield

$$\frac{|e^{i\theta}-\xi|}{|e^{i\theta}-z|}\le \frac{|\xi-z|}{|e^{i\theta}-z|}+1\le 4\frac{|\xi-z|}{1-|z|} \tag{2.27}$$

for all $(z,\theta)\in H_\xi\times I$. Therefore the left-hand side of (2.26) is bounded above by

$$\begin{aligned}4^\alpha\int_{H_\xi}\frac{(1-|z|)^{-\alpha+5/6}}{|\xi-z|^{-\alpha+17/6}}\frac{dz}{Q_m(|\xi-z|)} &\le 4^\alpha 3^{1-\alpha}\int_{H_\xi}\frac{(1-|z|)^{-1/6}}{|\xi-z|^{11/6}}\frac{dz}{Q_m(|\xi-z|)}\\ &\le 4\int_{\frac12(1-|\xi|)}^{1}\left(\int_{A_s}\frac{s^{1/6}}{(1-|\xi+se^{i\phi}|)^{1/6}}d\phi\right)\frac{ds}{sQ_m(s)}\\ &= 4\int_{\frac12(1-|\xi|)}^{1}\left(\int_{A_s}(s^{-1}-|s^{-1}\xi+e^{i\phi}|)^{-1/6}d\phi\right)\frac{ds}{sQ_m(s)},\end{aligned}$$

with

$$A_s:=\{\phi\in(0,2\pi]\,\big|\,|\xi+se^{i\phi}|<1\}=\{\phi\in(0,2\pi]\,\big|\,|s^{-1}\xi+e^{i\phi}|<s^{-1}\}.$$

It is not difficult to see that the inside integral is maximized when $s=1-|\xi|$ (i.e., $(0,2\pi]\setminus A_s$ is a single point) for any $|\xi|\in[1-2\delta^3,1)$, in which case the integrand is bounded above by $[\frac12(1-\cos(\phi-\theta_\xi))]^{-1/6}=[\sin\frac12(\phi-\theta_\xi)]^{-1/3}$ because $\delta\le\frac{1}{10^3}$. But then the inside integral is bounded above by $2\int_0^\pi\left(\frac{\phi}{\pi}\right)^{-1/3}d\phi=3\pi$. Hence (2.26) holds with $C=12\pi$.

Next consider the case $\alpha\in[1,\frac83]$, and define the functions $g(z):=\min\{\frac{1}{|\xi-z|},\frac{2}{1-|\xi|}\}$ and

$$f(z):=\min\left\{\frac{(1-|z|)^{5/6}}{|\xi-z|^{-\alpha+17/6}Q_m(|\xi-z|)},\frac{2^{-\alpha+17/6}(1-|z|)^{5/6}}{(1-|\xi|)^{-\alpha+17/6}Q_m(\frac12(1-|\xi|))}\right\},$$

as well as $H'_\xi:=B(\xi,\delta^3)\cap\mathbb{D}\supseteq H_\xi$. We can now use Jensen's inequality, (2.27), and

Lemma 2.4.1 to bound the left-hand side of (2.26) above by

$$\begin{aligned}
&\int_{H_\xi} \frac{(1-|z|)^{5/6}}{|\xi-z|^{17/6}} \left(\frac{1}{\beta(I)}\int_I \frac{|e^{i\theta}-\xi|}{|e^{i\theta}-z|} d\beta(\theta)\right)^\alpha \frac{dz}{Q_m(|\xi-z|)} \\
&\quad \le \int_{H_\xi} \frac{(1-|z|)^{5/6}}{|\xi-z|^{17/6}} \left(1+\frac{1}{\beta(I)}\int_I \frac{|\xi-z|}{|e^{i\theta}-z|} d\beta(\theta)\right)^\alpha \frac{dz}{Q_m(|\xi-z|)} \\
&\quad = \int_{H_\xi} \frac{(1-|z|)^{5/6}}{|\xi-z|^{-\alpha+17/6}} \left(\frac{1}{|\xi-z|}+\frac{1}{\beta(I)}\int_I \frac{1}{|e^{i\theta}-z|} d\beta(\theta)\right)^\alpha \frac{dz}{Q_m(|\xi-z|)} \\
&\quad \le \int_{H'_\xi} f(z)\left(g(z)+\frac{1}{\beta(I)}\int_I \frac{1}{|e^{i\theta}-z|} d\beta(\theta)\right)^\alpha dz \\
&\quad \le \int_{H'_\xi} f(z)\left(g(z)+\frac{1}{|e^{i\theta_\xi}-z|}\right)^\alpha dz \\
&\quad \le \frac{3^{5/6}2^\alpha \pi}{Q_m(\frac{1}{2}(1-|\xi|))} + \int_{H_\xi} \frac{(1-|z|)^{5/6}}{|\xi-z|^{-\alpha+17/6}} \left(\frac{1}{|\xi-z|}+\frac{1}{|e^{i\theta_\xi}-z|}\right)^\alpha \frac{dz}{Q_m(|\xi-z|)} \\
&\quad \le 24\pi + 4\int_{H_\xi} \frac{(1-|z|)^{5/6}}{|\xi-z|^{-\alpha+17/6}} \left(\frac{1}{|\xi-z|^\alpha}+\frac{1}{|e^{i\theta_\xi}-z|^\alpha}\right) \frac{dz}{Q_m(|\xi-z|)}.
\end{aligned}$$

Notice that Lemma 2.4.1 applies because $\frac{2}{\pi}m(\delta^3) \le \frac{2}{\pi}m(2\delta) \le \frac{1}{6} \le \frac{17}{6} - \alpha$ shows that $s^{-\alpha+17/6}Q_m(s)$ is increasing on $(0, \delta^3]$. Using again $1-|z| \le 3|\xi - z|$ for $z \in H_\xi$ yields

$$\int_{H_\xi} \frac{(1-|z|)^{5/6}}{|\xi-z|^{17/6}} \frac{dz}{Q_m(|\xi-z|)} \le 3^{5/6} \int_{H_\xi} \frac{1}{|\xi-z|^2} \frac{dz}{Q_m(|\xi-z|)} \le 6\pi \int_{\frac{1}{2}(1-|\xi|)}^1 \frac{ds}{sQ_m(s)},$$

and then we also have with $H^* := B(e^{i\theta_\xi}, \frac{1-|\xi|}{2}) \cap \mathbb{D}$,

$$\begin{aligned}
\int_{H_\xi \setminus H^*} \frac{(1-|z|)^{5/6}}{|\xi-z|^{-\alpha+17/6}|e^{i\theta_\xi}-z|^\alpha} \frac{dz}{Q_m(|\xi-z|)} &\le 3^\alpha \int_{H_\xi\setminus H^*} \frac{(1-|z|)^{5/6}}{|\xi-z|^{17/6}} \frac{dz}{Q_m(|\xi-z|)} \qquad (2.28)\\
&\le 162\pi \int_{\frac{1}{2}(1-|\xi|)}^1 \frac{ds}{sQ_m(s)}.
\end{aligned}$$

Finally, from $1-|z| \le |e^{i\theta_\xi} - z|$, $\alpha \le \frac{8}{3}$, and $Q_m \ge 1$ on $[0,1]$ we obtain

$$\int_{H^*} \frac{(1-|z|)^{5/6}}{|\xi-z|^{-\alpha+17/6}|e^{i\theta_\xi}-z|^\alpha} \frac{dz}{Q_m(|\xi-z|)} \le \left(\frac{1-|\xi|}{2}\right)^{\alpha-17/6} \int_{H^*} |e^{i\theta_\xi}-z|^{-\alpha+5/6} dz \le 12\pi.$$

This proves (2.26) with $C = 672\pi \le \frac{10^5-1}{6e^2}$.

Finally, to obtain $C < 147\pi$, we perform the following adjustments to the above argument. We choose $\delta > 0$ so that $\beta([\theta - 2\delta, \theta + 2\delta]) \le 1.01\pi$ for all $\theta \in \mathbb{R}$, so we always have $\alpha \in [0, 2.02]$. The 1 in (2.22) and (2.23) can be replaced by an arbitrary positive constant by lowering δ further. Similarly the 2 in (2.15) can be replaced by an arbitrary constant greater than 1, and the power $\frac{1}{6}$ in (2.24) by an arbitrarily small positive power (which allows us to turn the $3^{1/6}$ in (2.24) into an arbitrary constant greater than 1; this power then also propagates through the rest of the proof). This means that the constant in (2.24) with the new power can be made arbitrarily close to 1. The right-hand side of (2.25) can be multiplied by an arbitrarily small positive constant if we replace the upper bound in the second integral by a large multiple of $1 - |\xi|$ instead of $2(1 - |\xi|)$ (which is again possible when $\delta > 0$ is small enough), so it follows that it suffices to prove (2.26) with some $C < 147\pi$. Since in (2.28) we can actually replace 3^α by $(\sqrt{5})^\alpha \le 5^{1.01} < 5.1$, we indeed obtain (2.26) with $C = 4(6\pi + 30.6\pi) < 147\pi$. While further lowering of C is possible, we do not do so here. □

2.5 Proof of Theorem 2.1.2

Let $\Omega \subseteq \mathbb{R}^2$ be a regulated open bounded Lipschitz domain with $\partial\Omega$ a Jordan curve. Also assume that Ω is symmetric with respect to the real axis, $0 \in \partial\Omega$, and $(1-\varepsilon, 1) \times \{0\} \subseteq \Omega$ for some $\varepsilon > 0$. Let $\Omega^\pm := \Omega \cap (\mathbb{R} \times \mathbb{R}^\pm)$ and $\Omega^0 := \Omega \cap (\mathbb{R} \times \{0\})$ (these are obviously all simply connected). Then there is a Riemann mapping $\mathcal{T} : \Omega \to \mathbb{D}$ with $\mathcal{T}(\Omega^0) = (-1, 1)$ and $\mathcal{T}(0) = 1$, and therefore also $\mathcal{T}(\Omega^\pm) = \mathbb{D}^\pm := \mathbb{D} \cap (\mathbb{R} \times \mathbb{R}^\pm)$. Assume also that there are $\beta_\mathcal{T}, \tilde\beta_\mathcal{T}$ as in **(H)**, and $\tilde\beta_\mathcal{T}$ has bounded variation. Then $\mathcal{I}(z), \mathcal{J}(z)$ from the last section are the integrals

$$\mathcal{I}(z) = \frac{2}{\pi} \int_{(-\pi,\pi]} \ln |e^{i\theta} - z| \, d\beta_{\mathcal{T}}(\theta),$$
$$\mathcal{J}(z) = \frac{2}{\pi} \int_{(-\pi,\pi]} \ln |e^{i\theta} - z| \, d\tilde{\beta}_{\mathcal{T}}(\theta), \tag{2.29}$$

where we replaced integration over $(0, 2\pi]$ by $(-\pi, \pi]$ for convenience, and the second formula follows similarly to (2.9).

Given any concave modulus m and $r_0 \in (0, \frac{1}{2}]$ with $m(2r_0) \leq \frac{\pi}{6}$, assume that there are Ω and $\mathcal{T}$ as above with $\beta_{\mathcal{T}} \equiv 0$ on $(-1, 1)$ and $\tilde{\beta}_{\mathcal{T}}(\theta) = \frac{\pi}{2} - \frac{\operatorname{sgn}(\theta)}{2} m(2\min\{|\theta|, r_0\})$ for $\theta \in (-\pi, \pi]$. Concavity of m then guarantees that $\tilde{\beta}_{\mathcal{T}}$ indeed has modulus of continuity m. Notice also that $d\tilde{\beta}_{\mathcal{T}}(\theta) = -\chi_{(-r_0,r_0)} m'(2|\theta|) d\theta$ on $(-\pi, \pi]$, as well as $|\beta_{\mathcal{T}}| = 2\pi + m(2r_0) \leq 7$. We show at the end of this section that such Ω and $\mathcal{T}$ do exist for any m and $r_0 \in (0, \frac{1}{2}]$ with $m(2r_0) \leq \frac{\pi}{6}$.

We will first show that if $\int_0^1 \frac{ds}{q_m(s)} < \infty$ and $x \in \Omega^0$, then the trajectory X_t^x for the stationary weak solution $\omega := \chi_{\Omega^+} - \chi_{\Omega^-}$ to the Euler equations on Ω will reach $0 \in \partial\Omega$ in finite time. This will prove Theorem 2.1.2(i).

Due to symmetry, the particle trajectories X_t^x for this solution coincide with those for the stationary solution $\omega \equiv 1$ on Ω^+. We will therefore now employ the Biot-Savart law on Ω^+. Let $\mathcal{R} : \mathbb{D}^+ \to \mathbb{D}$ be a Riemann mapping with $\mathcal{R}(1) = 1$, so that $\mathcal{T}^+ := \mathcal{R}\mathcal{T} : \Omega^+ \to \mathbb{D}$ is a Riemann mapping with $\mathcal{T}^+(0) = 1$. The (time-independent) Biot-Savart law for $\omega \equiv 1$ on Ω^+ can therefore be written as

$$u(x) = D\mathcal{T}^+(x)^T \int_{\Omega^+} \nabla_\xi^\perp G_{\mathbb{D}}(\mathcal{T}^+(x), \mathcal{T}^+(y)) \, dy, \tag{2.30}$$

with $G_{\mathbb{D}}(\xi, z) = \frac{1}{2\pi} \ln \frac{|\xi - z|}{|\xi - z^*||z|}$ the Dirichlet Green's function for $\mathbb{D}$. If $x \in \Omega^0 \subseteq \partial\Omega^+$, we have $\mathcal{T}^+(x) \in \partial\mathbb{D}$, where $G_{\mathbb{D}}(\cdot, z)$ vanishes for any fixed $z \in \mathbb{D}$ (and $G_{\mathbb{D}}(\cdot, z) < 0$ on $\mathbb{D}$), so

$$\nabla_\xi^\perp G_{\mathbb{D}}(\mathcal{T}^+(x), \mathcal{T}^+(y)) = |\nabla_\xi G_D(\mathcal{T}^+(x), \mathcal{T}^+(y))| \mathcal{T}^+(x)^\perp.$$

This suggests one to evaluate

$$D\mathcal{T}^+(x)^T\mathcal{T}^+(x)^\perp = D\mathcal{T}^+(x)^T(\det D\mathcal{T}^+(x))^{-1/2}D\mathcal{T}^+(x)(1,0),$$

where $(1,0)$ is the counterclockwise unit tangent to Ω^+ at $x\in\Omega^0$, and we used that the action of the matrix $D\mathcal{T}^+(x)$ is just multiplication by a complex number with magnitude $\sqrt{\det D\mathcal{T}^+(x)}$. Since $D\mathcal{T}^+$ is of the form $\begin{pmatrix} a & b \\ -b & a\end{pmatrix}$, we have

$$D\mathcal{T}^+(x)^T D\mathcal{T}^+(x) = (\det D\mathcal{T}^+(x))I_2,$$

so (2.30) for $x\in\Omega^0$ becomes

$$u_1(x)=\sqrt{\det D\mathcal{T}^+(x)}\int_{\Omega^+}|\nabla_\xi G_{\mathbb{D}}(\mathcal{T}^+(x),\mathcal{T}^+(y))|\,dy \qquad \text{and} \qquad u_2(x)=0.$$

Since Ω^0 is a smooth segment of $\partial\Omega^+$, standard estimates show that $D\mathcal{T}^+(x)$ is continuous and non-vanishing on Ω^0. Since $\frac{d}{dt}X_t^x=u(X_t^x)$, it follows that for each $x\in\Omega^0$, the trajectory X_t^x either reaches 0 in finite time or converges to 0 as $t\to\infty$. It therefore suffices to analyze $u_1(x)$ for $x\in\Omega^0$ close to 0.

If $x\in\Omega^+\cup\Omega^0$ is not close to the left end of Ω^0, then $\mathcal{T}(x)\in\overline{\mathbb{D}^+}$ is not close to -1, so standard estimates yield $\sqrt{\det D\mathcal{R}(\mathcal{T}(x))}\in[c|\mathcal{T}(x)-1|,c^{-1}|\mathcal{T}(x)-1|]$ for some $c=c_{\mathcal{T}}\in(0,1]$ (because $D\mathcal{R}(z)\sim z-1$ for z near 1, and $D\mathcal{R}$ only vanishes at ±1). So for all $x\in\Omega^+\cup\Omega^0$ not close to the left end of Ω^0 we have

$$\sqrt{\det D\mathcal{T}^+(x)}\left(|\mathcal{T}(x)-1|\sqrt{\det D\mathcal{T}(x)}\right)^{-1}\in[c,c^{-1}]. \tag{2.31}$$

From (2.10) we have

$$\det D\mathcal{T}(x)=\det D\mathcal{T}(\mathcal{T}^{-1}(0))e^{\mathcal{I}(\mathcal{T}(x))+\mathcal{J}(\mathcal{T}(x))}. \tag{2.32}$$

Since $\beta_{\mathcal{T}}$ is supported away from $\theta = 0$, the term $e^{\mathcal{I}(\mathcal{T}(x))}$ is bounded above and below by positive numbers, uniformly in all x that are either close to 0 or not close to $\partial\mathbb{D}$. Moreover, (2.29) and the specific form of $\tilde{\beta}_{\mathcal{T}}$ give us for $z \in \mathbb{D}$,

$$\mathcal{J}(z) \geq -\frac{4}{\pi}\int_0^{r_0} \ln(|z-1|+\theta)m'(2\theta)d\theta = -\frac{2}{\pi}m(2r_0)\ln(|z-1|+r_0) + \frac{2}{\pi}\int_0^{r_0} \frac{m(2\theta)}{|z-1|+\theta}d\theta.$$

We can now estimate (with a constant C_{m,r_0} changing from one inequality to another)

$$\begin{aligned}
\left|\int_0^{r_0} \frac{m(2\theta)}{|z-1|+\theta}d\theta - \int_{|z-1|}^1 \frac{m(r)}{r}dr\right| &\leq \left|\int_{|z-1|/2}^{1/2} \frac{m(2\theta)}{|z-1|+\theta}d\theta - \int_{|z-1|/2}^{1/2} \frac{m(2\theta)}{\theta}d\theta\right| + C_{m,r_0} \\
&\leq \left|\int_{|z-1|/2}^{1/2} \frac{|z-1|m(2\theta)}{\theta(|z-1|+\theta)}d\theta\right| + C_{m,r_0} \\
&\leq \|m\|_{L^\infty}\left|\int_{|z-1|/2}^{1/2} \frac{|z-1|}{\theta^2}d\theta\right| + C_{m,r_0} \\
&\leq C_{m,r_0}.
\end{aligned}$$

For $z \in \mathbb{D}^0 := \mathbb{D} \cap (\mathbb{R} \times \{0\})$, we now obtain

$$\left|\mathcal{J}(z) - \frac{2}{\pi}\int_{|z-1|}^1 \frac{m(r)}{r}dr\right| \leq C_{m,r_0} \tag{2.33}$$

from this and from an opposite estimate via $\mathcal{J}(z) \leq -\frac{4}{\pi}\int_0^{r_0} \ln(\frac{1}{2}(|z-1|+\theta))m'(2\theta)d\theta$. Hence, for a new $c = c_{\mathcal{T},r_0,m} > 0$ and all $x \in \Omega^0$ not close to the left end of Ω^0 we obtain

$$\det D\mathcal{T}(x)\, Q_m(|\mathcal{T}(x)-1|)^{-1} \in [c, c^{-1}].$$

This and (2.31) show that there is $c = c_{\mathcal{T},r_0,m} > 0$ such that for all $x \in \Omega^0$ close to 0 we have

$$u_1(x) \geq c|\mathcal{T}(x)-1|\sqrt{Q_m(|\mathcal{T}(x)-1|)}\int_{\Omega^+} |\nabla_\xi G_{\mathbb{D}}(\mathcal{T}^+(x), \mathcal{T}^+(y))|\, dy \qquad \text{and} \qquad u_2(x) = 0.$$

If now $X_t^x \in \Omega^0$ is close to 0 and we let $d(t) := 1 - |\mathcal{T}(X_t^x)| = |\mathcal{T}(X_t^x) - 1|$, then

$$d'(t) = -\left| D\mathcal{T}(X_t^x) \frac{d}{dt} X_t^x \right| = -\sqrt{\det D\mathcal{T}(X_t^x)}\, u_1(X_t^x)$$

because $D\mathcal{T}$ is a multiple of I_2 on Ω^0. Therefore we have (with a new $c > 0$)

$$d'(t) \le -c d(t) Q_m(d(t)) \int_{\Omega^+} |\nabla_\xi G_{\mathbb{D}}(\mathcal{T}^+(X_t^x), \mathcal{T}^+(y))| \, dy. \tag{2.34}$$

Since $|\nabla_\xi G_{\mathbb{D}}(\xi, z)|$ is uniformly bounded away from 0 in $(\xi, z) \in \partial\mathbb{D} \times \kappa\mathbb{D}$ for any fixed $\kappa \in (0,1)$, the integral is bounded below by a positive constant. But then $d'(t) \le -c q_m(d(t))$, which implies

$$\int_{d(t)}^1 \frac{ds}{q_m(s)} \ge ct + \int_{d(0)}^1 \frac{ds}{q_m(s)}$$

for some $c = c_{\mathcal{T},m,r_0} \in (0,1]$. Since the left-hand side is bounded in t if $\int_0^1 \frac{ds}{q_m(s)} < \infty$, we must have $d(t) = 0$ for some $t < \infty$. This proves that X_t^x reaches $0 \in \partial\Omega$ in finite time, and hence Theorem 2.1.2(i).

This construction also allows us to prove Theorem 2.1.2(ii). When $\int_0^1 \frac{ds}{q_m(s)} = \infty$, we can estimate the integral in (2.34) better after first rewriting it via a change of variables as

$$\int_{\mathbb{D}} |\nabla_\xi G_{\mathbb{D}}(\mathcal{T}^+(X_t^x), z)| \left[\det D\mathcal{T}^+((\mathcal{T}^+)^{-1}(z))\right]^{-1} dz. \tag{2.35}$$

Now with $\xi := \mathcal{T}^+(X_t^x)$ (and still assuming $X_t^x \in \Omega^0$) we have

$$|\nabla_\xi G_{\mathbb{D}}(\xi, z)| = \left| \frac{\xi - z}{|\xi - z|^2} - \frac{\xi - z^*}{|\xi - z^*|^2} \right| \ge \frac{10c}{|\xi - z|} \ge \frac{c}{|z - 1|}$$

for some $c > 0$ (which will below change from one inequality to another and may also depend on $\mathcal{T}, m, r_0$) and all $z \in \mathbb{D} \cap (B(1,1) \setminus B(1, |\xi - 1|))$ that also lie in the sector with vertex 1, angle $\frac{\pi}{2}$, and axis being the real axis (call this set $\mathcal{C}_\xi$ and note that $\mathcal{C}_\xi \subseteq \mathcal{C}_1$).

If $z \in \mathcal{C}_1$, then for $y := (\mathcal{T}^+)^{-1}(z)$ (so $\mathcal{T}(y) = \mathcal{R}^{-1}(z)$) we have as above

$$\det D\mathcal{T}^+(y) \le c|\mathcal{T}(y) - 1|^2 Q_m(|\mathcal{T}(y) - 1|) = c|\mathcal{T}(y) - 1| q_m(|\mathcal{T}(y) - 1|).$$

Indeed, this follows from (2.31), (2.32), and also (2.33) for $\mathcal{T}(y)$ in place of z. The latter extends here even though $\mathcal{T}(y) \in \mathcal{R}^{-1}(\mathcal{C}_1) \subseteq \mathbb{D}^+$ and so $\mathcal{T}(y) \notin \mathbb{D}^0$ because for some y-independent $C > 0$ we have $\mathcal{J}(\mathcal{T}(y)) \le -\frac{4}{\pi}\int_0^{r_0} \ln(\frac{1}{C}(|\mathcal{T}(y) - 1| + \theta)) m'(2\theta) d\theta$ (recall (2.29)). This in turn is due to the distance of any $v \in \mathcal{R}^{-1}(\mathcal{C}_1)$ to $\partial\mathbb{D}$ being comparable to $|v - 1|$, since $\mathcal{C}_1$ has the same property.

So for $z \in \mathcal{C}_\xi$, the integrand in (2.35) can be bounded below by a multiple of

$$\frac{1}{|z - 1||\mathcal{R}^{-1}(z) - 1| q_m(|\mathcal{R}^{-1}(z) - 1|)} \ge \frac{c^3}{|z - 1|^{3/2} q_m(c|z - 1|^{1/2})},$$

with the inequality due to $|\mathcal{R}(v) - 1| \in [c|v - 1|^2, c^{-1}|v - 1|^2]$ for all $v \in \overline{\mathbb{D}^+}$ as well as $q_m(a^{-1}b) = a^{-1}bQ_m(a^{-1}b) \le a^{-1}bQ_m(b) = a^{-1}q_m(b)$ for $a \in (0, 1]$. The integral is therefore bounded below by a multiple of

$$\int_{|\xi-1|}^1 \frac{dr}{\sqrt{r}\, q_m(c\sqrt{r})} = \frac{2}{c}\int_{c\sqrt{|\xi-1|}}^c \frac{ds}{q_m(s)}.$$

Finally, since $|\xi - 1| = |\mathcal{R}(\mathcal{T}(X_t^x)) - 1| \le c^{-1}|\mathcal{T}(X_t^x) - 1|^2 = c^{-1}d(t)^2$, from (2.34) and $cq_m(c^{-1}d) \le q_m(d)$ for $c \in (0, 1]$ and $d \in (0, c]$ we obtain

$$d'(t) \le -cq_m(d(t)) \left(\int_{c^{-1}d(t)}^1 \frac{ds}{q_m(s)} - C \right) \le -c^2 q_m(c^{-1}d(t)) \left(\int_{c^{-1}d(t)}^1 \frac{ds}{q_m(s)} - C \right)$$

whenever $X_t^x \in \Omega^0$ is close enough to 0, with some $c = c_{\mathcal{T},m,r_0} \in (0, 1]$ and $C = C_{\mathcal{T},m,r_0} \ge 0$. But dividing this by the right-hand side and integrating yields (with a new C)

$$\ln \int_{d(t)}^1 \frac{ds}{q_m(s)} \ge \ln \left(\int_{c^{-1}d(t)}^1 \frac{ds}{q_m(s)} - C \right) \ge ct + \ln \left(\int_{c^{-1}d(0)}^1 \frac{ds}{q_m(s)} - C \right) \ge ct$$

for all $t > 0$, as long as $x \in \Omega^0$ is close enough to 0 (so the last parenthesis is ≥ 1). This now yields Theorem 2.1.2(ii).

Construction of a Domain Corresponding to a Given Modulus

We will now show that a domain as above does exist. We will do this by taking the desired $\bar{\beta}_{\mathcal{T}} = \beta_{\mathcal{T}} + \tilde{\beta}_{\mathcal{T}}$ and obtaining the domain $\Omega := \mathcal{S}(\mathbb{D})$ via the corresponding mapping $\mathcal{S}$ from (2.7). Since $\bar{\beta}_{\mathcal{T}}$ has bounded variation, we can now use the equivalent formula

$$\mathcal{S}'(z) = \mathcal{S}'(0) \exp\left(-\frac{1}{\pi}\int_{(-\pi,\pi]} \ln(1 - ze^{-i\theta})\, d\bar{\beta}_{\mathcal{T}}(\theta)\right) \tag{2.36}$$

(see [25, Corollary 3.16]). Our Ω will in fact be a perturbed isosceles triangle, with one vertex and the center of the opposite "side" on the real axis, and the modulus m will be "attained" at the center of that side (where Ω will therefore be concave).

Given any concave modulus m and $r_0 \in (0, \frac{1}{2}]$ with $m(2r_0) \leq \frac{\pi}{6}$, let us define $\tilde{\beta}(\theta) := \frac{\pi}{2} - \frac{\operatorname{sgn}(\theta)}{2}\, m(2\min\{|\theta|, r_0\})$ on $(-\pi, \pi]$ (and let its derivative be 2π-periodic). Then let β be such that $\beta(0) = 0$ and

$$d\beta|_{(-\pi,\pi]} := \left(\frac{2\pi}{3} + \pi m_0\right)\delta_\pi + \frac{2\pi}{3}\delta_{\pi/3} + \frac{2\pi}{3}\delta_{-\pi/3},$$

where $m_0 := \frac{1}{\pi}m(2r_0)$ and δ_{θ_0} is the Dirac measure at $\theta = \theta_0$. Clearly $\bar{\beta} := \beta + \tilde{\beta}$ satisfies $\bar{\beta}(\pi) = \bar{\beta}(-\pi) + 2\pi$, and $\bar{\beta} - \frac{\pi}{2}$ is odd on $\mathbb{R}$ (which is needed for symmetry of Ω with respect to the real axis).

We now use (2.36) with the choice $\mathcal{S}'(0) := 1$ to define

$$\mathcal{V}(z) := \exp\left(-\frac{1}{\pi}\int_{(-\pi,\pi]} \ln(1 - ze^{-i\theta})\, d\bar{\beta}(\theta)\right) = (1 + z^3)^{-2/3} v(z),$$

where we consider the branch of the logarithm with $\ln : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ (since

$\mathrm{Re}(1 - ze^{-i\theta}) > 0$), use that $\Pi_{k=0}^{2}(1 - ze^{-i(2k-1)\pi/3}) = 1 + z^3$, and also define

$$v(z) := (1+z)^{-m_0} \exp\left(\frac{2}{\pi} \int_0^{r_0} \ln(1 - ze^{-i\theta}) m'(2\theta)\, d\theta \right).$$

Since $\mathrm{Im} \ln(1 - ze^{-i\theta}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $(z, \theta) \in \mathbb{D} \times \mathbb{R}$, the imaginary part of the above exponent belongs to $(-\frac{\pi}{2} m_0, \frac{\pi}{2} m_0)$. This and $\mathrm{Re}(1 + z) > 0$ now yield

$$|\arg v(z)| < \pi m_0 = m(2r_0) \leq \frac{\pi}{6}$$

for all $z \in \mathbb{D}$. Since also $|\arg(1 + z^3)| < \frac{\pi}{2}$, it follows that $\mathrm{Re}\, \mathcal{V}(z) > 0$ for all $z \in \mathbb{D}$. But then the mapping $\mathcal{S} : \mathbb{D} \to \mathbb{C}$ given by

$$\mathcal{S}(z) := \int_1^z \mathcal{V}(\xi)\, d\xi$$

is injective, and $\mathcal{T} := \mathcal{S}^{-1}$ is a Riemann mapping for $\Omega := \mathcal{S}(\mathbb{D})$ with $\partial\Omega$ is a Jordan curve. Note that Ω is bounded because $\mathcal{V}(z) = O(\sum_{k=0}^{2} |e^{i(2k-1)\pi/3} - z|^{-5/6})$. Since $\mathcal{V}((-1, 1)) \subseteq \mathbb{R}^+$, we have $\mathcal{S}((-1, 1)) \subseteq \mathbb{R}$, and then $\mathcal{S}((-1, 1)) = \Omega^0$, with $\mathcal{S}(1) = 0 \in \partial\Omega$ its right endpoint.

Observe that $\arg(\mathcal{V}(e^{i\phi}))$ is uniformly continuous on $(e^{i(2k-1)\pi/3}, e^{i(2k+1)\pi/3})$ for $k = 0, 1, 2$. This is because the same is true for the argument of $(1 + e^{3i\phi})^{-2/3}(1 + e^{i\phi})^{-m_0}$, while

$$\arg\left(\mathcal{V}(e^{i\phi})(1 + e^{3i\phi})^{2/3}(1 + e^{i\phi})^{m_0}\right) = \frac{2}{\pi} \int_0^{r_0} \arg(1 - e^{i(\phi-\theta)}) m'(2\theta)\, d\theta,$$

which is continuous in ϕ because m is continuous. We therefore have that for each $\varepsilon > 0$ there are points $0 = \phi_0 < \cdots < \phi_N = 2\pi$ (with $e^{i(2k-1)\pi/3}$ being among them) and $a_1, \ldots, a_N \in \mathbb{R}$ such that $|\arg(\mathcal{S}(e^{i\phi'}) - \mathcal{S}(e^{i\phi})) - a_n| < \varepsilon$ whenever $\phi_{n-1} < \phi < \phi' < \phi_n$. Then Ω is a regulated domain by Theorem 3.14 in [25]. So it has a unit forward tangent vector from

(1.42) for each $\theta \in \mathbb{R}$, and (2.7) shows that with its argument $\bar{\beta}_{\mathcal{T}}$ from (1.43) we have

$$\mathcal{V}(z) = \mathcal{S}'(z) = \exp\left(\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\bar{\beta}_{\mathcal{T}}(\theta)-\theta-\frac{\pi}{2}\right)d\theta\right) \tag{2.37}$$

because $\mathcal{S}'(0) = \mathcal{V}(0) = 1$. In the definition of $\mathcal{V}$, we can replace $\bar{\beta}(\theta)$ by the 2π-periodic function $\bar{\beta}(\theta) - \theta - \frac{\pi}{2}$ because $\int_0^{2\pi} \ln(1 - ze^{-i\theta})d\theta = \ln 1 = 0$.

Integration by parts then yields

$$\begin{aligned}\mathcal{V}(z) &= \exp\left(\frac{i}{\pi}\int_{-\pi}^{\pi}\frac{z}{e^{i\theta}-z}\left(\bar{\beta}(\theta)-\theta-\frac{\pi}{2}\right)d\theta\right)\\ &= \exp\left(\frac{i}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}-1\right)\left(\bar{\beta}(\theta)-\theta-\frac{\pi}{2}\right)d\theta\right).\end{aligned}$$

From this and (2.37) we find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\bar{\beta}_{\mathcal{T}}(\theta)-\bar{\beta}(\theta)\right)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\bar{\beta}(\theta)-\theta-\frac{\pi}{2}\right)d\theta + 2k\pi = 2k\pi$$

for some $k \in \mathbb{Z}$ and all $z \in \mathbb{D}$ (because $\bar{\beta}(\theta) - \frac{\pi}{2}$ and θ are odd). Hence $\bar{\beta}_{\mathcal{T}} - \bar{\beta} \equiv 2k\pi$, so Ω and $\mathcal{T}$ are indeed the domain and Riemann mapping we wanted to construct.

2.6 Proof of Lemma 2.4.1

Monotone Convergence Theorem shows that it suffices to consider bounded f, g, h. We will prove this via a series of "foldings" of $\beta|_I$ onto smaller and smaller intervals that shrink toward θ^*. We will show that at each step the relevant integral cannot decrease.

Define $\beta^0 := \beta|_I$ and let β^1 be the measure for which

$$\beta^1(A) = \begin{cases}\beta^0(A) & \text{if } A \subseteq (-\infty, \theta^* - 2\delta) \cup (\theta^*, \infty),\\ 0 & \text{if } A \subseteq [\theta^* - 2\delta, \theta^* - \delta),\\ \beta^0(A \cup (2(\theta^* - \delta) - A)) & \text{if } A \subseteq [\theta^* - \delta, \theta^*]\end{cases}$$

for any measurable $A \subseteq \mathbb{R}$.

That is, we obtain β^1 from β^0 by reflecting $\beta^0|_{[\theta^*-2\delta,\theta^*-\delta)}$ across $\theta^*-\delta$ onto $(\theta^*-\delta,\theta^*]$. In particular, β^1 is supported on $[\theta^*-\delta,\theta^*+2\delta]$ and both measures have total mass $\beta(I)$.

We now let

$$G^j(z) := g(z) + \frac{1}{\beta(I)}\int_I h(|e^{i\theta}-z|)\,d\beta^j(\theta),$$

and want to show that

$$\int_H f(z)G^0(z)^\alpha dz \leq \int_H f(z)G^1(z)^\alpha dz. \tag{2.38}$$

Let $\tilde{H} := \{re^{i\phi} \in H \,|\, \phi \in [\theta^*-\delta-\pi, \theta^*-\delta]\}$ and let $H' := \{re^{i(2(\theta^*-\delta)-\phi)} \,|\, re^{i\phi} \in \tilde{H}\}$ be its reflection across the line connecting 0 and $e^{i(\theta^*-\delta)}$. The properties of H ensure that $H' \subseteq H$. If now $z \in H \setminus \tilde{H}$, then $|e^{i\theta}-z| \geq |e^{i(2(\theta^*-\delta)-\theta)}-z|$

for any $\theta \in [\theta^*-2\delta,\theta^*-\delta)$. This and h being non-increasing show that $G^0(z) \leq G^1(z)$ for all $z \in H \setminus \tilde{H}$, and in particular for all $z \in H \setminus (\tilde{H} \cup H')$. To conclude (2.38), it hence suffices to show that

$$f(z)G^0(z)^\alpha + f(z')G^0(z')^\alpha \leq f(z)G^1(z)^\alpha + f(z')G^1(z')^\alpha \tag{2.39}$$

holds for any $z = re^{i\phi} \in \tilde{H}$, with $z' := re^{i(2(\theta^*-\delta)-\phi)} \in H'$ its reflection across the line connecting 0 and $e^{i(\theta^*-\delta)}$.

Note that the properties of f and g show that $f(z') \geq f(z)$ and $g(z') \geq g(z)$. Let

$$\begin{aligned}
b_+ &:= g(z) + \frac{1}{\beta(I)}\int_{[\theta^*-\delta,\theta^*+2\delta]} h(|e^{i\theta}-z|)\,d\beta^0(\theta) \qquad (\geq 0),\\
b_- &:= \frac{1}{\beta(I)}\int_{[\theta^*-2\delta,\theta^*-\delta)} h(|e^{i\theta}-z|)\,d\beta^0(\theta) \qquad (\geq 0),\\
b'_+ &:= g(z') + \frac{1}{\beta(I)}\int_{[\theta^*-\delta,\theta^*+2\delta]} h(|e^{i\theta}-z'|)\,d\beta^0(\theta) \qquad (\geq 0),\\
b'_- &:= \frac{1}{\beta(I)}\int_{[\theta^*-2\delta,\theta^*-\delta)} h(|e^{i\theta}-z'|)\,d\beta^0(\theta) \qquad (\geq 0).
\end{aligned}$$

Then $G^0(z) = b_+ + b_-$, $G^0(z') = b'_+ + b'_-$, $G^1(z) = b_+ + b'_-$, and $G^1(z') = b'_+ + b_-$, so

$$G^0(z) + G^0(z') = G^1(z) + G^1(z').$$

We also have $b'_+ \geq b_+$ and $b'_- \leq b_-$ due to $g(z') \geq g(z)$, h being non-increasing, and the definition of z'. This implies

$$0 \leq G^1(z) \leq \min\{G^0(z), G^0(z')\} \leq \max\{G^0(z), G^0(z')\} \leq G^1(z').$$

The last two relations, together with convexity of the function x^α on $[0, \infty)$, now yield

$$G^0(z)^\alpha + G^0(z')^\alpha \leq G^1(z)^\alpha + G^1(z')^\alpha.$$

From this and $(f(z') - f(z))(G^1(z')^\alpha - G^0(z')^\alpha) \geq 0$ we obtain (2.39), and therefore (2.38).

An identical (modulo reflection) argument shows that if β^2 is obtained from β^1 by reflecting $\beta^1|_{(\theta^*+\delta, \theta^*+2\delta]}$ across $\theta^* + \delta$ onto $[\theta^*, \theta^* + \delta)$, then we have

$$\int_H f(z) G^1(z)^\alpha dz \leq \int_H f(z) G^2(z)^\alpha dz.$$

We can then repeat this with $\frac{\delta}{2}$ in place of δ because β^2 is supported on $[\theta^* - \delta, \theta^* + \delta]$ and has total mass $\beta(I)$. Continuing in this way, we obtain a sequence of measures $\beta^0, \beta^2, \beta^4, \ldots$, each β^{2j} having total mass $\beta(I)$ and supported on $[\theta^* - 2^{1-j}\delta, \theta^* + 2^{1-j}\delta]$, such that

$$\int_H f(z) G^{2j}(z)^\alpha dz \leq \int_H f(z) G^{2(j+1)}(z)^\alpha dz$$

for $j = 0, 1, \ldots$. Since the integrands are uniformly bounded and converge pointwise to $f(z)(g(z) + h(|e^{i\theta^*} - z|))^\alpha$ as $j \to \infty$, Dominated Convergence Theorem finishes the proof.

2.7 Acknowledgment

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Chapter 3

General Planar Domains

As we saw in Chapter 2, its main result is sharp and we cannot prove uniqueness of weak solutions via this approach for general bounded vorticities on more general domains. However, the example we constructed in Section 2.5 highly depends on oddness of the vorticity across a line of a symmetry of the domain, so this approach might still work for some more general domains (including domains containing corners with any interior angles) if we add a sign condition on initial vorticity as in [16]. The main result of this chapter is to show that the weak solutions are indeed Lagrangian on domains with possibly infinitely many corners without any restrictions on their angles and less boundary smoothness in between the corners. We will show that boudnary approach rate of particle trajectories is still no faster than double-exponential in this case, and then uniqueness of weak solutions (again when ω_0 is constant near the boundary) will follow from Theorem 1.5.1. The idea of the proof is similar to the one in last chapter but it uses a different Lyapunov functional.

3.1 Main Results

In this chapter, instead of using (1.42), we use another natural analog way to define the unit forward tangent. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open Lipschitz domain with $\partial\Omega$ a Jordan curve, let $L := |\partial\Omega|$ be the arc-length of $\partial\Omega$, and let $\sigma : [0, L] \to \mathbb{C}$ be a counter-clockwise arc-length parametrization of $\partial\Omega$ (so $\sigma(L) = \sigma(0)$). For any $\theta \in [0, L)$, the *unit forward tangent vector* to Ω at $\sigma(\theta)$ is the unit vector

$$\bar{\tau}(\theta) := \lim_{\phi \to \theta+} \frac{\sigma(\phi) - \sigma(\theta)}{|\sigma(\phi) - \sigma(\theta)|}, \tag{3.1}$$

provided the limit exists (we also let $\bar{\tau}(L) := \bar{\tau}(0)$). If it does for each $\theta \in [0, L)$, and $\bar{\tau}$ has one-sided limits everywhere on $[0, L]$, then Ω is said to be *regulated*. In that case $\bar{\tau}$ is right-continuous, and if we identify $\mathbb{R}^2$ with $\mathbb{C}$ and let $\arg(z) \in (-\pi, \pi]$ for $z \neq 0$, then

$$\bar{\alpha}(\theta) := \arg\left(\frac{\bar{\tau}(\theta)}{\lim_{\phi \to \theta-} \bar{\tau}(\phi)}\right) \in (-\pi, \pi) \tag{3.2}$$

for $\theta \in (0, L]$ is such that $\pi - \bar{\alpha}(\theta)$ is the *interior angle* of Ω at $\sigma(\theta)$. Note that $\bar{\alpha}(0)$ is not defined, and $\bar{\alpha}(\theta) \in (-\pi, \pi)$ for $\theta \in (0, L]$ because Ω is Lipschitz. So corners of Ω are precisely the points $\sigma(\theta)$ with $\theta \in (0, L]$ and $\bar{\alpha}(\theta) \neq 0$, and regulated domains clearly have countably many of them. If also $\sum_{\theta \in (0,L]} |\bar{\alpha}(\theta)| < \infty$, then

$$\bar{\beta}_c(\theta) := \bar{\arg}\left(\bar{\tau}(\theta)\right) - \sum_{\theta' \leq \theta} \bar{\alpha}(\theta') \tag{3.3}$$

is a continuous function on $[0, L]$ provided we let $\bar{\arg}(\bar{\tau}(\theta))$ be the argument of $\bar{\tau}(\theta)$ plus an appropriate θ-dependent integer multiple of 2π. We will also assume that $\bar{\beta}_c$ is *Dini continuous* on $[0, L]$. That is, it has a modulus of continuity $m : [0, L] \to [0, \infty)$ (1.44) with $\int_0^L \frac{m(r)}{r} dr < \infty$ holds for all $\theta, \theta' \in [0, L]$). We recall that any Hölder modulus of continuity is also a Dini modulus. The following theorem is the main result for this chapter.

Theorem 3.1.1. *Assume that a bounded open Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ with $\partial\Omega$ a Jordan curve is regulated. Let $\bar{\tau}$ be the forward tangent vector to Ω from* (3.1)*, let $\bar{\alpha}$ be from* (3.2)*, and assume that $\sum_{\theta \in (0,L]} |\bar{\alpha}(\theta)| < \infty$ and $\bar{\beta}_c$ from* (3.3) *is Dini continuous. Consider any $0 \leq \omega_0 \in L^\infty(\Omega)$ and let $\omega \geq 0$ from the Yudovich class be any global weak solution to the Euler equations on Ω with initial value ω_0 (such ω is known to exist by [9]).*

(i) We have $t_x = \infty$ *for all* $x \in \Omega$ *and* $\{X_t^x \,|\, x \in \Omega\} = \Omega$ *for all* $t > 0$*, and there is a constant* $C_\omega < \infty$ *such that for any* $\varepsilon > 0$ *and all large enough* $t > 0$*,*

$$\sup_{\mathrm{dist}(x,\partial\Omega)\geq\varepsilon} \mathrm{dist}(X_t^x, \partial\Omega) \geq \exp(-e^{C_\omega t}) \tag{3.4}$$

(except when $\omega \equiv 0$*, but then* $X_t^x \equiv x$*). Moreover,* $\omega(t, X_t^x) = \omega_0(x)$ *for almost every* $(t,x) \in (0,\infty) \times \Omega$ *(i.e.,* ω *is Lagrangian), and* u *is continuous on* $[0,\infty) \times \Omega$ *and* (1.10) *holds pointwise.*

(ii) If $\mathrm{supp}\,(\omega_0 - a) \cap \partial\Omega = \emptyset$ *for some* $a \geq 0$*, then* ω *is the unique non-negative weak solution with initial value* ω_0*.*

Remarks. 1. Hence the well-known double-exponential bound on the rate of approach of particle trajectories to the boundaries of smooth domains (going back to [12, 32]) still holds on the domains considered here, even though u can be far from log-Lipschitz near $\partial\Omega$ and even unbounded at corners with angles $> \pi$. A partial explanation is that $\omega \geq 0$ forces u to "circulate" around $\partial\Omega$ counter-clockwise, thus keeping any particle trajectory near any corner for only a short time during each passage through its neighborhood. However, our domains can even have everywhere singular boundaries (e.g., a dense set of corners), so all of $\partial\Omega$ could be the set of potential trouble spots rather than just a few individual corners.

2. Part (i) of this result suggests a natural open question: is there any planar domain Ω and a weak solution $\omega \geq 0$ to the Euler equation on it that has a particle trajectory starting inside Ω and reaching $\partial\Omega$ in finite time? Of course, a second one is whether such solutions, if they exist, can fail to be Lagrangian (this is currently open even for unsigned ω).

Let us briefly discuss our approach and its relation to [16, 19] and the result in Chapter 2. In all four papers, the central ingredient is a non-negative Lyapunov functional on $(0,\infty) \times \overline{\Omega}$ that vanishes only on $(0,\infty) \times \partial\Omega$ and its change on Euler particle trajectories can be controlled sufficiently well to show that it can never become 0 unless it is 0 initially.

Lacave first chose this functional to be the stream function $\Psi := -\Delta^{-1}\omega$ of the fluid velocity u [16] because its rate of change in the flow direction u is 0. When ω does not have a sign, then Ψ can vanish inside Ω, and [19] as well as last chapter therefore used instead the time-independent function $1 - |\mathcal{T}(x)|$, with $\mathcal{T} : \Omega \to \mathbb{D}$ a Riemann mapping. In the present paper we consider again solutions $\omega \geq 0$, and so revisit the idea of using the stream function. However, in Lemmas 3.2.2–3.2.5 we obtain sharper and more general estimates on Ψ and $\partial_t \Psi$ than [16], which allows us to include much more general domains, with arbitrary corners as well as considerably less regular boundaries overall.

In the next section we state these estimates and use them to prove Theorem 3.1.1, leaving the proofs of the estimates and of a formula for $\partial_t \Psi$ for the last two sections.

3.2 Proof of Theorem 3.1.1

We complete the proof in three steps. We always assume that Ω satisfies the hypotheses from Theorem 3.1.1, and (ω, u) is a weak solution to (1.1)–(1.3) on $(0, \infty) \times \Omega$, as defined next.

3.2.1 Weak Solutions and Space-time Differentiability of the Stream Function

We again consider here weak solutions to (1.1)–(1.3) from the *Yudovich class*

$$\left\{(\omega, u) \in L^\infty\left((0,\infty); L^\infty(\Omega) \times L^2(\Omega)\right) \,\middle|\, \omega = \nabla \times u \text{ and (1.2)–(1.3) all hold weakly}\right\},$$

where the weak form of (1.2)–(1.3) is

$$\int_\Omega u(t,x) \cdot \nabla h(x)\, dx = 0 \qquad \forall h \in H^1_{\text{loc}}(\Omega) \text{ with } \nabla h \in L^2(\Omega)$$

for a.e. $t \in (0, \infty)$ (see [9, 10]). It is well-known that $\nabla \times u \in L^\infty((0,\infty) \times \Omega)$ implies that u is bounded and log-Lipschitz on any compact $K \subseteq \Omega$ at a.e. time $t \in (0, \infty)$ (and uniformly in these times), after possibly redefining it on a measure zero spatial set for each such t. If we also redefine u at the exceptional measure-zero set of times (and also at $t = 0$), then for any compact $K \subseteq \Omega$ we will have

$$\sup_{t \geq 0} \sup_{x,y \in K} \left(|u(t,x)| + \frac{|u(t,x) - u(t,y)|}{|x-y| \max\{1, -\ln|x-y|\}} \right) < \infty \tag{3.5}$$

(this is also shown in the proof of Lemma 3.2.1 below). Let now $X_t^{s,x}$ for $(s,x) \in [0,\infty) \times \Omega$ be the unique continuous function satisfying

$$\frac{d}{dt} X_t^{s,x} = u(t, X_t^{s,x}) \qquad \text{and} \qquad X_s^{s,x} = x \tag{3.6}$$

a.e. on the maximal interval $I^{s,x} \subseteq [0,\infty)$ (containing s) such that $X_t^{s,x} \in \Omega$ for all $t \in I^{s,x} \setminus \partial I^{s,x}$. That is, $I^{s,x}$ is the (backward and forward) life-span of the particle trajectory $X_t^{s,x}$. Of course, $X_t^{0,x} = X_t^x$ and $I^{0,x} = [0, t_x]$ (or $[0,\infty)$ if $t_x = \infty$) for all $x \in \Omega$.

We say that (ω, u) from the Yudovich class is a *weak solution* to (1.1)–(1.3) on $(0,T) \times \Omega$ (for some $T \in (0,\infty]$) with some initial condition $\omega_0 \in L^\infty(\Omega)$, if

$$\int_0^T \int_\Omega \omega \left(\partial_t \varphi + u \cdot \nabla \varphi \right) \, dx dt = - \int_\Omega \omega_0(x) \varphi(0,x) \, dx \qquad \forall \, \varphi \in C_c^\infty \left([0,T) \times \Omega \right). \tag{3.7}$$

This is in fact the definition of a weak solution ω to the transport equation (1.4) when u is some given vector field, but it is also equivalent to the relevant weak velocity formulation of the Euler equations on Ω (see [10, Remark 1.2]). When $T = \infty$, we call such solutions *global*. Existence of a global weak solution is guaranteed by [9] for any $\omega_0 \in L^\infty(\Omega)$ on very general domains (while uniqueness is still open on most singular domains), and so for the sake of notational simplicity we will always assume that $T = \infty$.

Lemma 2.3.1 now shows that for a.e. $t \in (0,\infty)$, a weak solution (ω, u) satisfies

$\omega(t, X_t^x) = \omega_0(x)$ for a.e. $x \in \Omega$ such that $t_x < t$. We can therefore redefine ω on a set of measure 0 so that $\omega(t, X_t^x) = \omega_0(x)$ holds for all $x \in \Omega$ and all $t \in (0, t_x)$. Let now $s_1 \in (0, \infty)$ be any Lebesgue point of ω as a function from $(0, \infty)$ to $L^1(\Omega)$. Replacing φ in (3.7) by $\varphi\psi_\varepsilon$, where $\psi_\varepsilon \in C^\infty([0, \infty))$ satisfies $\chi_{[s_1,\infty)} \leq \psi_\varepsilon \leq \chi_{[s_1-\varepsilon,\infty)}$, and taking $\varepsilon \to 0$ shows that (ω, u) is also a weak solution to (1.1)–(1.3) on $(s_1, \infty) \times \Omega$ with initial condition $\omega(s_1, \cdot)$ (i.e., (3.7) holds with $(0, \omega_0)$ replaced by $(s_1, \omega(s_1, \cdot))$. Doing the same with any $\varphi \in C_c^\infty((0, \infty) \times \Omega)$ and $\chi_{[0,s_1]} \leq \psi_\varepsilon \leq \chi_{[0,s_1+\varepsilon]}$ shows that (ω, u) is also a weak solution to (1.1)–(1.3) on $(0, s_1) \times \Omega$ with terminal condition $\omega(s_1, \cdot)$ (which becomes an initial condition if we reverse the direction of time and replace u by $-u$). This and Lemma 2.3.1 show that we can redefine ω on a set of measure 0 so that $\omega(t, X_t^{s_1,x}) = \omega(s_1, x)$ holds for all $x \in \Omega$ and all $t \in I^{s_1,x} \setminus \partial I^{s_1,x}$ (clearly the values on the curve $(t, X_t^{s_1,x})$ will not change for any x such that $0 \in I^{s_1,x}$). We can continue this way, with $s_2, s_3, \dots$ consecutively in place of s_1, where $\{s_j\}_{j\geq 1}$ is dense in $(0, \infty)$. This allows us to change ω on a measure zero set so that for all $s \in [0, \infty)$ (and with $\omega(0, \cdot) := \omega_0$) we will from now have

$$\omega\left(t, X_t^{s,x}\right) = \omega(s, x) \qquad \forall x \in \Omega \text{ and } t \in I^{s,x} \setminus \partial I^{s,x}. \tag{3.8}$$

It is well known that since Ω is simply connected, ω from any weak solution (ω, u) uniquely defines the velocity u via its *stream function*

$$\Psi(t, \cdot) := -\Delta^{-1}\omega(t, \cdot)$$

for all $t \geq 0$ (the negative sign is chosen so that $\Psi \geq 0$ when $\omega \geq 0$). Namely, after redefinition of u on a measure zero set we have $u = -\nabla^\perp\Psi$, where $(v_1, v_2)^\perp := (-v_2, v_1)$ and $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. We can now use (3.8) to show that Ψ is space-time differentiable (we postpone the proof of this to the last section).

Lemma 3.2.1. *We have $\Psi \in C^1([0,\infty) \times K)$ for each compact $K \subseteq \Omega$, and $\nabla \Psi = u^\perp$ and*

$$\partial_t \Psi(t,x) = -\frac{1}{2\pi} \int_\Omega \left(\frac{\mathcal{T}(y) - \mathcal{T}(x)}{|\mathcal{T}(y) - \mathcal{T}(x)|^2} - \frac{\mathcal{T}(y) - \mathcal{T}(x)^*}{|\mathcal{T}(y) - \mathcal{T}(x)^*|^2} \right)^T D\mathcal{T}(y)\, u(t,y)\, \omega(t,y) dy \quad (3.9)$$

for each $(t,x) \in [0,\infty) \times \Omega$, where $\mathcal{T} : \Omega \to \mathbb{D}$ is any Riemann mapping.

Note that since this shows that now $u = -\nabla^\perp \Psi$ is continuous on $[0,\infty) \times \Omega$, this version of u still satisfies (3.5). Since u is uniquely determined by ω, from now on we will refer to ω as a weak solution to (1.4) (with $u := \nabla^\perp \Delta^{-1} \omega$), instead of to (ω, u) as a weak solution to (1.1)–(1.3).

3.2.2 Formulation on the Unit Disc via Riemann Mapping

Let next $\mathcal{T} : \Omega \to \mathbb{D}$ be a Riemann mapping as in Lemma 3.2.1, extended continuously to $\partial\Omega$, and let $\mathcal{S} := \mathcal{T}^{-1}$. We will now use $\mathcal{T}$ to rewrite u and $\partial_t \Psi$ in terms of integrals over $\mathbb{D}$. We have

$$\Psi(t,x) = -\left[\Delta^{-1}\omega(t,\cdot)\right](x) = -\frac{1}{2\pi} \int_\Omega \ln \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|}\, \omega(t,y) dy, \quad (3.10)$$

and then

$$u(t,x) = -\nabla^\perp \Psi(t,x) = \frac{1}{2\pi} D\mathcal{T}(x)^T R(t, \mathcal{T}(x)) \quad (3.11)$$

for any $(t,x) \in [0,\infty) \times \Omega$, where

$$R(t,\xi) := \int_{\mathbb{D}} \left(\frac{\xi - z}{|\xi - z|^2} - \frac{\xi - z^*}{|\xi - z^*|^2} \right)^\perp \det D\mathcal{S}(z)\, \omega(t, \mathcal{S}(z)) dz \quad (3.12)$$

for $(t,\xi) \in [0,\infty) \times \mathbb{D}$. We note that the second equality in (3.11) holds because $\mathcal{T} = (\mathcal{T}^1, \mathcal{T}^2)$

is analytic, which means that

$$D\mathcal{T} = \begin{pmatrix} \partial_{x_1}\mathcal{T}^1 & \partial_{x_2}\mathcal{T}^1 \\ \partial_{x_1}\mathcal{T}^2 & \partial_{x_2}\mathcal{T}^2 \end{pmatrix} = \begin{pmatrix} \partial_{x_1}\mathcal{T}^1 & \partial_{x_2}\mathcal{T}^1 \\ -\partial_{x_2}\mathcal{T}^1 & \partial_{x_1}\mathcal{T}^1 \end{pmatrix} \tag{3.13}$$

and so for any $v \in \mathbb{R}^2$ we have

$$\left(\begin{pmatrix} \partial_{x_1}\mathcal{T}^1 & \partial_{x_1}\mathcal{T}^2 \\ \partial_{x_2}\mathcal{T}^1 & \partial_{x_2}\mathcal{T}^2 \end{pmatrix} v \right)^{\perp} = \begin{pmatrix} \partial_{x_2}\mathcal{T}^2 & -\partial_{x_2}\mathcal{T}^1 \\ -\partial_{x_1}\mathcal{T}^2 & \partial_{x_1}\mathcal{T}^1 \end{pmatrix} v^{\perp},$$

Lemma 3.2.1 and $u \cdot \nabla\Psi \equiv 0$ now yield for any $x \in \Omega$ and $t \in [0, t_x)$,

$$\frac{d}{dt}\Psi(t, X_t^x) = -\frac{1}{2\pi}\int_{\Omega} \left(\frac{\mathcal{T}(y) - \mathcal{T}(X_t^x)}{|\mathcal{T}(y) - \mathcal{T}(X_t^x)|^2} - \frac{\mathcal{T}(y) - \mathcal{T}(X_t^x)^*}{|\mathcal{T}(y) - \mathcal{T}(X_t^x)^*|^2} \right)^T D\mathcal{T}(y)\, u(t,y)\, \omega(t,y) dy$$

(the parenthesis is replaced by $\frac{\mathcal{T}(y)}{|\mathcal{T}(y)|^2}$ when $\mathcal{T}(X_t^x) = 0$). If we substitute (3.11) here and use

$$D\mathcal{T}(y) D\mathcal{T}(y)^T = \det D\mathcal{T}(y)\, I_2 \tag{3.14}$$

(note that $\det D\mathcal{T} = (\partial_{x_1}\mathcal{T}^1)^2 + (\partial_{x_2}\mathcal{T}^1)^2 > 0$),

after a change of variables we obtain

$$\frac{d}{dt}\Psi(t, X_t^x) = -\frac{1}{4\pi^2}\int_{\mathbb{D}} \left(\frac{z - \mathcal{T}(X_t^x)}{|z - \mathcal{T}(X_t^x)|^2} - \frac{z - \mathcal{T}(X_t^x)^*}{|z - \mathcal{T}(X_t^x)^*|^2} \right) \cdot R(t,z)\, \omega(t, \mathcal{S}(z)) dz.$$

Finally, from this and the identity

$$\left| \frac{z}{|z|^2} - \frac{w}{|w|^2} \right| = \frac{|z-w|}{|z|\,|w|} \tag{3.15}$$

for all $z, w \in \mathbb{C} \setminus \{0\}$ we see that (with the fraction below replaced by $\frac{1}{|z|}$ when $\mathcal{T}(X_t^x) = 0$)

$$\left|\frac{d}{dt}\Psi(t, X_t^x)\right| \leq \frac{1}{4\pi^2} \int_{\mathbb{D}} \frac{|\mathcal{T}(X_t^x) - \mathcal{T}(X_t^x)^*|}{|z - \mathcal{T}(X_t^x)|\,|z - \mathcal{T}(X_t^x)^*|}\,|R(t,z)|\,|\omega(t, \mathcal{S}(z))|dz. \tag{3.16}$$

It will also be convenient to re-parametrize the forward tangent vector $\bar{\tau}$ to Ω to 1.42, which is

$$\bar{\nu}(\theta) := \lim_{\phi \to \theta+} \frac{\mathcal{S}(e^{i\phi}) - \mathcal{S}(e^{i\theta})}{|\mathcal{S}(e^{i\phi}) - \mathcal{S}(e^{i\theta})|},$$

with $\theta \in \mathbb{R}$. Then of course $\bar{\nu}(\theta) = \bar{\tau}(\Gamma(e^{i\theta}))$ for all $\theta \in \mathbb{R}$, where $\Gamma := \left(\sigma|_{(0,L]}\right)^{-1} \circ \mathcal{S}$. We now let $\{\bar{\theta}_j\}_{j\geq 1} \subseteq (0, L]$ be the set of all points such that Ω has a corner at $\sigma(\bar{\theta}_j)$, and define

$$\theta_j := \pi + \arg\left(-\Gamma^{-1}(\bar{\theta}_j)\right) \in (0, 2\pi] \qquad \text{and} \qquad \alpha_j := \frac{\bar{\alpha}(\bar{\theta}_j)}{\pi} \in (-1, 0) \cup (0, 1)$$

for $j \geq 1$. That is, Ω has corners at $\{\mathcal{S}(e^{i\theta_j})\}_{j\geq 1}$ with angles $\{\pi - \pi\alpha_j\}_{j\geq 1}$. Then we define

$$\beta_c(\theta) := \bar{\beta}_c(\Gamma(e^{i\theta})) \qquad \text{and} \qquad \beta_d(\theta) := \pi \sum_{\theta_j \leq \theta} \alpha_j$$

for $\theta \in (0, 2\pi]$ and extend these two functions to $\mathbb{R}$ so that for all $\theta \in \mathbb{R}$ we have

$$\beta_c(\theta + 2\pi) = \beta_c(\theta) + 2\pi\kappa \qquad \text{and} \qquad \beta_d(\theta + 2\pi) = \beta_d(\theta) + 2\pi(1 - \kappa),$$

where $\kappa := \frac{\bar{\beta}_c(L) - \bar{\beta}_c(0)}{2\pi}$ (which means that $\sum_{\theta \in (0,L]} \bar{\alpha}(\theta) = 2\pi(1 - \kappa)$). Then of course β_c is continuous, β_d is piecewise constant, and $\beta := \beta_c + \beta_d$ is the argument of τ in the sense that $e^{i\beta(\theta)} = \tau(\theta)$ for all $\theta \in \mathbb{R}$ (we also have $\beta(\theta + 2\pi) = \beta(\theta) + 2\pi$).

Lemma 1 in [31] shows that Γ and Γ^{-1} are both Hölder continuous, which means that β_c is Dini continuous because $\bar{\beta}_c$ is. Indeed, if $\bar{m}$ is a modulus of continuity for $\bar{\beta}_c$, then β_c has modulus of continuity $m(r) := \bar{m}(Cr^\gamma)$ for some $C, \gamma > 0$, and a simple change of variables shows that $\int_0^1 \frac{\bar{m}(Cr^\gamma)}{r} dr < \infty$ if and only if $\int_0^1 \frac{\bar{m}(r)}{r} dr < \infty$.

We next state the following important formula for $\det D\mathcal{S}$.

Lemma 3.2.2. *We have*

$$\det D\mathcal{S}(z) = \det D\mathcal{S}(0)\, \Pi_{j\geq 1}|z-e^{i\theta_j}|^{-2\alpha_j} \exp\left(-\frac{2}{\pi}\int_0^{2\pi} \operatorname{Im}\frac{z}{e^{i\theta}-z}\,(\beta_c(\theta)-\kappa\theta)\,d\theta\right)$$

for each $z \in \mathbb{D}$ (this holds even without β_c being Dini continuous), as well as

$$\sup_{z\in\mathbb{D}}\left|\int_0^{2\pi} \operatorname{Im}\frac{z}{e^{i\theta}-z}\,(\beta_c(\theta)-\kappa\theta)\,d\theta\right| < \infty.$$

Proof. Since $\mathcal{S}$ is analytic, $\det D\mathcal{S}(z) = |\mathcal{S}'(z)|^2$, where $\mathcal{S}'$ is the complex derivative when $\mathcal{S}$ is considered as a function on $\mathbb{C}$.

Since Ω is regulated, Theorem 3.15 in [25] shows that

$$\mathcal{S}'(z) = |\mathcal{S}'(0)|\,\exp\left(\frac{i}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\beta(\theta)-\theta-\frac{\pi}{2}\right)d\theta\right)$$

for all $z\in\mathbb{D}$, and from $\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta = 2\pi\in\mathbb{R}$ and $\operatorname{Im}\frac{e^{i\theta}+z}{e^{i\theta}-z} = 2\operatorname{Im}\frac{z}{e^{i\theta}-z}$ we get

$$\det D\mathcal{S}(z) = \det D\mathcal{S}(0)\,\exp\left(-\frac{2}{\pi}\operatorname{Im}\int_0^{2\pi}\frac{z}{e^{i\theta}-z}\,(\beta(\theta)-\theta)\,d\theta\right) \tag{3.17}$$

(note that $\beta(\theta)-\theta$ is 2π-periodic). We split the integral into two parts, one of which is

$$\int_0^{2\pi}\frac{z}{e^{i\theta}-z}\,(\beta_d(\theta)-(1-\kappa)\theta)\,d\theta = i\int_0^{2\pi}\ln(1-ze^{-i\theta})\,d\left(\beta_d(\theta)-(1-\kappa)\theta\right),$$

where we used integration by parts. Since $\int_0^{2\pi}\ln(1-ze^{-i\theta})d\theta = \ln 1 = 0$, it follows that

$$\begin{aligned}\exp\left(-\frac{2}{\pi}\operatorname{Im}\int_0^{2\pi}\frac{z}{e^{i\theta}-z}\big(\beta_d(\theta)-(1-\kappa)\theta\big)d\theta\right) &= \exp\left(-\frac{2}{\pi}\int_0^{2\pi}\ln|e^{i\theta}-z|\,d\beta_d(\theta)\right)\\ &= \Pi_{j\geq1}|z-e^{i\theta_j}|^{-2\alpha_j}.\end{aligned}$$

This and (3.17) prove the first claim.

Let $\tilde{\beta}(\theta) = \beta_c(\theta)-\kappa\theta$, which is also 2π-periodic. If β_c has a Dini modulus of continuity

m, then $\tilde{\beta}$ has Dini modulus $\tilde{m}(r) := m(r) + |\kappa| r$. So for any $z \in \mathbb{D}$ and $\theta_z := \arg(z)$ we obtain using $\operatorname{Im} \frac{z}{e^{i(-\theta+\theta_z)}-z} = -\operatorname{Im} \frac{z}{e^{i(\theta+\theta_z)}-z}$ and $\left|\frac{z}{e^{i(\theta+\theta_z)}-z}\right| \le \frac{\pi}{2|\theta|}$ for any $\theta \in \mathbb{R}$ the estimate

$$\left| \int_0^{2\pi} \operatorname{Im} \frac{z}{e^{i\theta} - z} \tilde{\beta}(\theta) d\theta \right| = \left| \int_{-\pi}^{\pi} \operatorname{Im} \frac{z}{e^{i(\theta+\theta_z)} - z} \left(\tilde{\beta}(\theta + \theta_z) - \tilde{\beta}(\theta_z) \right) d\theta \right| \le \left| \frac{\pi}{2} \int_{-\pi}^{\pi} \frac{\tilde{m}(|\theta|)}{|\theta|} d\theta \right|.$$

Since this is finite, the second claim follows.

□

In view of (3.16), (3.12), and this lemma, of particular concern to us will be corners corresponding to $\alpha_j > 0$ (i.e., those with angles less than π; note that the velocity u on Ω in fact vanishes at these, while it may be infinite at the other corners).

We therefore let $\alpha_j^+ := \max\{\alpha_j, 0\}$ and define $\beta_d^+(\theta) := \pi \sum_{\theta_j \le \theta} \alpha_j^+$ for all $\theta \in (0, 2\pi]$. We then extend β_d^+ to $\mathbb{R}$ so that $\beta_d^+(\theta + 2\pi) = \beta_d^+(\theta) + \pi \sum_{j \ge 1} \alpha_j^+$, and choose $\delta \in (0, \frac{1}{8}]$ such that

$$\frac{\beta_d^+(\theta + 3\delta) - \beta_d^+(\theta - 3\delta)}{\pi} \le \alpha_* := \frac{1 + \max_{j\ge1} \alpha_j^+}{2} \tag{3.18}$$

for each $\theta \in \mathbb{R}$. Note that $\alpha_* \in [\frac{1}{2}, 1)$ because $\max_{j \ge 1} \alpha_j^+ < 1$ by $\sum_{j \ge 1} |\alpha_j| < \infty$.

3.2.3 Estimates on the Stream Function and Conclusion of the Proof

We now state the following three crucial estimates, whose proofs we postpone to the next section. In them, constants C_Ω and C'_Ω only depend on Ω.

Lemma 3.2.3. *There is $C_\Omega > 0$ such that for each $(t, \xi) \in [0, \infty) \times \mathbb{D}$ we have*

$$|\Psi(t, \mathcal{S}(\xi))| \le C_\Omega \|\omega(t, \cdot)\|_{L^\infty} (1 - |\xi|)^{2 \min\{1 - \alpha_*, 1/4\}}.$$

Lemma 3.2.4. *If $\omega \geq 0$, then for each $(t,\xi) \in [0,\infty) \times \mathbb{D}$ we have*

$$\Psi(t,\mathcal{S}(\xi)) \geq \frac{1-|\xi|}{100\pi} \int_{\mathbb{D}} \frac{(1-|z|)\det D\mathcal{S}(z)}{\max\{|z-\xi|, 1-|\xi|\}^2}\, \omega(t,\mathcal{S}(z))dz.$$

Lemma 3.2.5. *There is $C'_\Omega > 0$ such that for each $(t,\xi) \in [0,\infty) \times (\mathbb{D} \setminus B(0,\frac{3}{4}))$ we have*

$$\int_{\mathbb{D}} \frac{|R(t,z)|}{|z-\xi||z-\xi^*|} dz \leq$$
$$C'_\Omega |\ln(1-|\xi|)| \left(\int_{\mathbb{D}} \frac{(1-|z|)\det D\mathcal{S}(z)}{\max\{|z-\xi|, 1-|\xi|\}^2} |\omega(t,\mathcal{S}(z))| dz + \|\omega(t,\cdot)\|_{L^\infty} \right).$$

Remarks. 1. Lemmas 3.2.3 and 3.2.4 are sharper and more general versions of Lemmas 3.1 and 3.2 in [16]. Our use of Lemma 3.2.5 to estimate $\partial_t \Psi$ is analogous to the use of Proposition 2.4 and Lemma 3.5 in [16], but instead of bounding $|R|$ above by essentially $\|\omega\|_{L^\infty}$ and leaving ω as a function, we bound ω by $\|\omega\|_{L^\infty}$ and leave $|R|$ in (3.16). This is because for the domains Ω considered here, R can blow up at $\partial\mathbb{D}$ (see (3.30) below). In particular, this happens at corners with angles $\leq \frac{\pi}{2}$, which is why such corners had to be excluded in [16].

2. Lemma 3.2.5 easily extends to $\xi \in B(0,\frac{3}{4})$ but we will not need this.

From now assume also that $\omega \geq 0$. Since $(1-|z|)\det D\mathcal{S}(z)$ is bounded below by a positive constant on $B(0,r)$ for any $r < 1$ due to Lemma 3.2.2, for any $a > 0$ there is $c_a > 0$ such that the second integral in Lemma 3.2.5 is bounded below by $c_a \|\omega(t,\cdot)\|_{L^\infty}$ whenever

$$\|\omega(t,\cdot)\|_{L^1} \geq a\|\omega(t,\cdot)\|_{L^\infty}.$$

From this, the above lemmas, and (3.16) it follows that when $|\mathcal{T}(X_t^x)| \geq \frac{3}{4}$ (in which case also $|\mathcal{T}(X_t^x) - \mathcal{T}(X_t^x)^*| \le 3(1 - |\mathcal{T}(X_t^x)|))$, then we have

$$\begin{aligned} \left| \frac{d}{dt}\Psi(t, X_t^x) \right| &\le \frac{75C'_\Omega}{\pi} \frac{1 + c_a}{c_a} \|\omega(t, \cdot)\|_{L^\infty} |\ln(1 - |\mathcal{T}(X_t^x)|)| \Psi(t, X_t^x) \\ &\le C_{a,\Omega} \|\omega(t, \cdot)\|_{L^\infty} \Psi(t, X_t^x) \left| \ln \frac{\Psi(t, X_t^x)}{C_\Omega \|\omega(t, \cdot)\|_{L^\infty}} \right|, \end{aligned} \tag{3.19}$$

where $C_{a,\Omega} > 0$ is some constant that only depends on (a, Ω).

For each $\varepsilon > 0$ let $\Omega_\varepsilon := \Omega \setminus \bigcup_{x \in \partial\Omega} B(x, \varepsilon)$. For each $\varepsilon > 0$ such that $\Omega_{2\varepsilon} \neq \emptyset$, let

$$T_\varepsilon := \mathrm{dist}(\Omega_{2\varepsilon}, \Omega \setminus \Omega_\varepsilon) \|u\|^{-1}_{L^\infty((0,\infty) \times \Omega_\varepsilon)} > 0.$$

Then $X_t^x \in \Omega_\varepsilon$ for all $(t, x) \in [0, T_\varepsilon] \times \Omega_{2\varepsilon}$, and therefore (3.8) yields $\omega(t, X_t^x) = \omega_0(x)$ for all $(t, x) \in [0, T_\varepsilon] \times \Omega_{2\varepsilon}$. Taking $\varepsilon \to 0$ we obtain

$$\|\omega_0\|_{L^\infty} \le \liminf_{t \to 0} \|\omega(t, \cdot)\|_{L^\infty} \le \|\omega\|_{L^\infty},$$

and then from $\nabla \cdot u \equiv 0$ also

$$\|\omega(t, \cdot)\|_{L^1} \ge \|\omega_0\|_{L^1(\Omega_{2\varepsilon})} \ge \|\omega_0\|_{L^1} - |\Omega \setminus \Omega_{2\varepsilon}| \, \|\omega_0\|_{L^\infty} \ge \|\omega_0\|_{L^1} - |\Omega \setminus \Omega_{2\varepsilon}| \, \|\omega\|_{L^\infty} \tag{3.20}$$

for each $\varepsilon > 0$ and all $t \in [0, T_\varepsilon]$.

If now $\omega_0 \not\equiv 0$, let $a := \frac{1}{2}\|\omega_0\|_{L^1} \|\omega\|^{-1}_{L^\infty} > 0$ and let $\varepsilon > 0$ be such that $|\Omega \setminus \Omega_{2\varepsilon}| \le a$. From (3.20) we obtain

$$\|\omega(t, \cdot)\|_{L^1} \ge a\|\omega\|_{L^\infty} \ge a\|\omega(t, \cdot)\|_{L^\infty}$$

for all $t \in [0, T_\varepsilon]$. Thus (3.19) yields

$$\left| \frac{d}{dt}\Psi(t, X_t^x) \right| \le C_{a,\Omega} \|\omega\|_{L^\infty} \Psi(t, X_t^x) \left| \ln \frac{\Psi(t, X_t^x)}{C_\Omega \|\omega\|_{L^\infty}} \right| \tag{3.21}$$

for all $(t, x) \in [0, T_\varepsilon] \times \Omega$ such that $|\mathcal{T}(X_t^x)| \ge \frac{3}{4}$. This and Gronwall's inequality show that

$X_t^x \in \Omega$ for all $(t,x) \in [0,T_\varepsilon] \times \Omega$. Therefore $\omega(t,X_t^x) = \omega_0(x)$ for all $(t,x) \in [0,T_\varepsilon] \times \Omega$, and in particular, $\|\omega(T_\varepsilon,\cdot)\|_{L^1} = \|\omega_0\|_{L^1}$. We can therefore repeat this argument with the same a and ε on the time interval $[T_\varepsilon, 2T_\varepsilon]$, then on $[2T_\varepsilon, 3T_\varepsilon]$, etc.

It follows that ω is a Lagrangian solution to (1.4) on $(0,\infty) \times \Omega$ and $\|\omega(t,\cdot)\|_{L^p} = \|\omega_0\|_{L^p}$ for all $(t,p) \in [0,\infty) \times [1,\infty]$. Integrating (3.21) shows that there is a constant C_ω (depending on $\|\omega_0\|_{L^\infty}, \|\omega_0\|_{L^1}, \Omega$) such that for each $\varepsilon > 0$ and all large enough $t > 0$ we have $\Psi(t,X_t^x) \geq \exp(-e^{C_\omega t})$ whenever $\Psi(0,x) \geq \varepsilon$. Since Lemma 3.2.3 yields $C''_\omega > 0$ such that $\Psi(t,\mathcal{S}(\xi)) \leq C''_\omega(1-|\xi|)^{2\min\{1-\alpha_*,1/4\}}$ for all $(t,\xi) \in [0,\infty) \times \mathbb{D}$, and $\mathcal{T}$ is Hölder continuous on $\overline{\Omega}$ (see [31, Lemma 1]), this shows (3.4). Using also that (1.10) can clearly be solved backwards in time with the same estimate on the boundary approach rate, we find that $\{X_t^x \,|\, x \in \Omega\} = \Omega$, thus finishing the proof of Theorem 3.1.1(i) for $\omega_0 \not\equiv 0$.

If $\omega_0 \equiv 0$, then $\omega \equiv 0$ is clearly a Lagrangian solution to (1.4) on $(0,\infty) \times \Omega$ with $X_t^x = x$, which satisfies Theorem 3.1.1(i) except for (3.4).

If $\omega \geq 0$ is a different global weak solution, then the above arguments with time 0 replaced by any $T' > 0$ such that $\omega(T',\cdot) \not\equiv 0$ show that for all $t \in (T',\infty)$ we have $\|\omega(t,\cdot)\|_{L^1} = \|\omega(T',\cdot)\|_{L^1}$. But then $\|\omega(t,\cdot)\|_{L^1}$ must be constant on the time interval (T'',∞), where $T'' \in [0,\infty)$ is the infimum of times t with $\omega(t,\cdot) \not\equiv 0$ (and that constant is then positive). This contradicts continuity of ω as an $L^1(\Omega)$-valued function of time because $\omega(0,\cdot) = \omega_0 \equiv 0$.

Theorem 3.1.1(ii) follows immediately from Theorem 3.1.1(i) and Proposition 3.2 in [19]. We note that the latter result shows that Lagrangian solutions are unique as long as they remain constant near $\partial\Omega$ (more specifically, near the non-$C^{2,\gamma}$ portion of $\partial\Omega$ for some $\gamma > 0$).

3.3 Proofs of Lemmas 3.2.3–3.2.5

Let us first state an auxiliary technical result.

Lemma 3.3.1. *Let β be a (positive) measure on $\mathbb{R}$ and let $I := (\theta^* - 2\delta, \theta^* + 2\delta)$ for some $\theta^* \in \mathbb{R}$ and $\delta \in (0, \frac{\pi}{2}]$. Let $H \subset \mathbb{D}$ be an open region such that if $re^{i(\theta^*+\phi)} \in H$ for some $r \in (0,1)$ and $|\phi| \le \pi$, then $re^{i(\theta^*+\phi')} \in H$ whenever $|\phi'| \le |\phi|$ (i.e., H is symmetric and angularly convex with respect to the line connecting 0 and $e^{i\theta^*}$). If $F : [0,\infty) \to [0,\infty)$ is non-decreasing and convex, then*

$$\int_H f(z) F\left(g(z) + \frac{1}{\beta(I)} \int_I h(|z - e^{i\theta}|) d\beta(\theta)\right) dz \le \int_H f(z) F\left(g(z) + h(|z - e^{i\theta^*}|)\right) dz$$

holds for any non-increasing $h : (0,\infty) \to [0,\infty)$ and non-negative $f, g \in L^1(H)$ such that $f(re^{i(\theta^*+\phi')}) \ge f(re^{i(\theta^*+\phi)})$ and $g(re^{i(\theta^*+\phi')}) \ge g(re^{i(\theta^*+\phi)})$ whenever $r \in (0,1)$ and $|\phi'| \le |\phi|$.

The proof of this result is identical to that of Lemma 2.4.1 in last chapter, which was stated with $F(s) = s^\alpha$ for some $\alpha \ge 1$, because the only properties of F used in it were that it is non-decreasing and convex. We will be using it here with $F(s) := e^s$, $g \equiv 0$, and $h(s) := 2\beta(I) \ln_+ \frac{2}{s}$, so that for any β, I, H, f as above we have

$$\int_H f(z) \exp\left(-2 \int_I \ln|z - e^{i\theta}| d\beta(\theta)\right) dz \le \int_H f(z) |z - e^{i\theta^*}|^{-2\beta(I)} dz. \tag{3.22}$$

Since Lemmas 3.2.3–3.2.5 are all stated at a single time t, we will drop t from our notation in the proofs below. Hence we will have $\omega(x), \Psi(x)$, and $R(z)$. For $z \in \mathbb{D}$ we will also denote

$$\Lambda(z) := \det D\mathcal{S}(z) \, |\omega(\mathcal{S}(z))| \ge 0.$$

We note that

$$\int_{\mathbb{D}} \Lambda(z) dz \le \|\omega\|_{L^\infty} \int_{\mathbb{D}} \det D\mathcal{S}(z) dz = |\Omega| \, \|\omega\|_{L^\infty}, \tag{3.23}$$

and that constants $C_1, C_2, \dots$ below will always be allowed to depend (only) on Ω.

3.3.1 Proof of Lemma 3.2.4

We have

$$\begin{aligned}\frac{|\xi - z|^2}{|\xi - z^*|^2|z|^2} &= 1 - \frac{|\xi z - z^* z|^2 - |\xi - z|^2}{|\xi - z^*|^2|z|^2} \\ &= 1 - \frac{(|\xi|^2|z|^2 - 2\mathrm{Re}\,(\xi\bar{z}) + 1) - (|\xi|^2 - 2\mathrm{Re}\,(\xi\bar{z}) + |z|^2)}{|\xi - z^*|^2|z|^2} \\ &= 1 - \frac{(1 - |\xi|^2)(1 - |z|^2)}{|\xi - z^*|^2|z|^2} \end{aligned} \tag{3.24}$$

for $\xi, z \in \mathbb{D}$ with $z \neq 0$, which also means that $\frac{|\xi - z|^2}{|\xi - z^*|^2|z|^2} \in (0, 1)$ when $z \neq 0, \xi$. Hence

$$-\ln\frac{|\xi - z|}{|\xi - z^*||z|} = -\frac{1}{2}\ln\left(1 - \frac{(1 - |\xi|^2)(1 - |z|^2)}{|\xi - z^*|^2|z|^2}\right) \geq \frac{1}{2}\frac{(1 - |\xi|^2)(1 - |z|^2)}{|\xi - z^*|^2|z|^2},$$

and so for each $\xi \in \mathbb{D}$ we have

$$\Psi(\mathcal{S}(\xi)) \geq \frac{1}{4\pi}\int_{\mathbb{D}}\frac{(1 - |\xi|^2)(1 - |z|^2)}{|\xi - z^*|^2|z|^2}\Lambda(z)dz \geq \frac{1 - |\xi|}{4\pi}\int_{\mathbb{D}}\frac{(1 - |z|)}{|\xi - z^*|^2|z|^2}\Lambda(z)dz.$$

Given any $z, \xi \in \mathbb{D}$, let $M := \max\{|z - \xi|, 1 - |\xi|\}$. Then $1 - |z| \leq 1 - |\xi| + |z - \xi| \leq 2M$, so

$$|\xi - z^*|\,|z| \leq |\xi - z| + |z - z^*|\,|z| = |z - \xi| + 1 - |z|^2 \leq |z - \xi| + 2(1 - |z|) \leq 5M$$

when $z \neq 0$, and the result follows.

3.3.2 Proof of Lemma 3.2.3

Identity (3.24) and $-\ln(1 - r) \leq (\frac{r}{1-r})^{\frac{1}{2}}$ for $r \in [0, 1)$ (equality holds for $r = 0$ and the right-hand side has a larger derivative on $(0, 1)$) show that

$$-\ln\frac{|\xi - z|}{|\xi - z^*||z|} \leq \frac{1}{2}\left(\frac{\frac{(1-|\xi|^2)(1-|z|^2)}{|\xi - z^*|^2|z|^2}}{\frac{|\xi - z|^2}{|\xi - z^*|^2|z|^2}}\right)^{\frac{1}{2}} \leq \frac{(1 - |\xi|)^{\frac{1}{2}}(1 - |z|)^{\frac{1}{2}}}{|\xi - z|}.$$

Hence it suffices to show that there is $C_1 > 0$ such that

$$\int_{\mathbb{D}} \frac{(1-|z|)^{\frac{1}{2}}}{|z-\xi|}\Lambda(z)dz \le C_1\|\omega\|_{L^\infty}(1-|\xi|)^{2\hat{\alpha}-\frac{1}{2}}, \tag{3.25}$$

where $\hat{\alpha} := \min\{1-\alpha_*, \frac{1}{4}\}$ (note that $2\hat{\alpha} - \frac{1}{2} \le 0$). From (3.23) we see that it in fact suffices to replace $\mathbb{D}$ by $A_1 := B(\xi,\delta) \cap \mathbb{D}$ in (3.25).

Let us decompose A_1 into $A_2 := B(\xi,\varepsilon) \cap A_1$ with $\varepsilon := \frac{1-|\xi|}{2}$, $A_3 := B(\tilde{\xi},\varepsilon) \cap A_1$ with $\tilde{\xi} := \frac{\xi}{|\xi|}$, and $A_4 := A_1 \backslash (A_2 \cup A_3)$. Now Lemma 3.2.2 and (3.22) with $H := A_1$, $I := (\arg(\xi) - 2\delta, \arg(\xi) + 2\delta)$, $f(z) := \frac{(1-|z|)^{1/2}}{|z-\xi|}$, and $\beta := \sum_{\theta_j \in I} \alpha_j^+ \delta_{\theta_j}$, where δ_{θ_j} is the Dirac mass at θ_j, yield

$$\begin{aligned}
\int_{A_1} \frac{(1-|z|)^{\frac{1}{2}}}{|z-\xi|}\Lambda(z)dz \le C_2||\omega||_{L^\infty} \int_{A_1} \frac{(1-|z|)^{\frac{1}{2}}}{|z-\xi|}\Pi_{\theta_j\in I}|z-e^{i\theta_j}|^{-2\alpha_j^+}dz \\
\le & C_2||\omega||_{L^\infty} \int_{A_1} \frac{(1-|z|)^{\frac{1}{2}}}{|z-\xi|}|z-\tilde{\xi}|^{-2\alpha_*}dz \\
\le & C_2||\omega||_{L^\infty} \left(\int_{A_2} \frac{\varepsilon^{\frac{1}{2}-2\alpha_*}}{|z-\xi|}dz + \int_{A_3} \frac{|z-\tilde{\xi}|^{\frac{1}{2}-2\alpha_*}}{\varepsilon}dz + \int_{A_4} 3^3|z-\xi|^{-\frac{1}{2}-2\alpha_*}dz \right) \\
\le & C_3||\omega||_{L^\infty}\varepsilon^{\frac{3}{2}-2\alpha_*} \le 2C_3||\omega||_{L^\infty}(1-|\xi|)^{2\hat{\alpha}-\frac{1}{2}}
\end{aligned}$$

because (3.18) shows that $\sum_{\theta_j\in I}\alpha_j^+ \le \alpha_* < 1$. This therefore finishes the proof of (3.25).

3.3.3 Proof of Lemma 3.2.5

First integrate over $A_0 := \mathbb{D}\backslash B(\xi,\delta)$. Then (3.15), (3.23), and

$$|z-\tilde{z}^*| \ge |\tilde{z}^*| - 1 \ge \frac{|\tilde{z}-\tilde{z}^*|}{2} \ge 1 - |\tilde{z}| \tag{3.26}$$

for any $z, \tilde{z} \in \mathbb{D}$ yield

$$\begin{aligned}
\int_{A_0} \frac{|R(z)|}{|z-\xi||z-\xi^*|} dz &\le \frac{1}{\delta^2} \int_{A_0} \int_{\mathbb{D}} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \\
&\le \frac{2}{\delta^2} \int_{\mathbb{D}} \int_{A_0} \frac{dz}{|z-\tilde{z}|} \Lambda(\tilde{z}) d\tilde{z} \\
&\le \frac{4\pi}{\delta^2} \int_{\mathbb{D}} \Lambda(\tilde{z}) d\tilde{z} \\
&= \frac{4\pi|\Omega|}{\delta^2} ||\omega||_{L^\infty}.
\end{aligned}$$

So it remains to integrate over $A_1 := B(\xi, \delta) \cap \mathbb{D}$. From (3.26), $|\xi| - \delta \ge \frac{5}{8}$, and (3.23) we have

$$\int_{A_1} \frac{1}{|z-\xi||z-\xi^*|} \int_{B(0,1/2)} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \le C_1 |\ln(1-|\xi|)| \, ||\omega||_{L^\infty}, \tag{3.27}$$

where we also used that with $B_\xi := B(\xi, \frac{|\xi-\xi'|}{2}) \cap \mathbb{D}$ and $B_{\xi'} := B(\xi', \frac{|\xi-\xi'|}{2}) \cap \mathbb{D}$ we have

$$\int_{\mathbb{D}} \frac{dz}{|z-\xi||z-\xi'|} \le 3 \int_{\mathbb{D} \setminus (B_\xi \cup B_{\xi'})} \frac{dz}{|z-\xi|^2} + \frac{4}{|\xi-\xi'|} \int_{B_\xi} \frac{dz}{|z-\xi|} \le 6\pi \ln_+ \frac{1}{|\xi-\xi'|} + 50 \tag{3.28}$$

for any $\xi, \xi' \in \mathbb{C}$.

We now let $\varepsilon := 1 - |\xi|$ and split A_1 into $A_2 := B(\xi, \frac{\varepsilon}{4})$ and $A_3 := A_1 \setminus A_2$. We start with A_2, and let $E_1 := B(\xi, \frac{\varepsilon}{2})$ and $E_2 := \mathbb{D} \setminus (B(0, \frac{1}{2}) \cup B(\xi, \frac{\varepsilon}{2}))$. We also denote $M(\xi, z) := \max\{|z-\xi|, 1-|\xi|\}$. When $(z, \tilde{z}) \in A_2 \times E_1$, then (3.26), $|z-\xi^*| \ge \varepsilon$, and (3.28) show that

$$\begin{aligned}
\int_{A_2} \frac{1}{|z-\xi||z-\xi^*|} \int_{E_1} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz &\le \frac{2}{\varepsilon} \int_{E_1} \Lambda(\tilde{z}) \int_{A_2} \frac{dz}{|z-\xi||z-\tilde{z}|} d\tilde{z} \\
&\le \frac{C_2}{\varepsilon} \int_{E_1} \Lambda(\tilde{z}) \, |\ln|\tilde{z}-\xi|| \, d\tilde{z}.
\end{aligned}$$

From Lemma 3.2.2 and (3.18) we see that $\det D\mathcal{S}(\tilde{z}) \le C_3(1-|\tilde{z}|)^{-2}$ for some C_3 and all $\tilde{z} \in \mathbb{D}$, hence

$$\int_{B(\xi,\varepsilon^2)} \Lambda(\tilde{z})\, |\ln|\tilde{z}-\xi||\, d\tilde{z} \le 4C_3\varepsilon^{-2}||\omega||_{L^\infty} \int_{B(\xi,\varepsilon^2)} |\ln|\tilde{z}-\xi||\, d\tilde{z} \le C_4||\omega||_{L^\infty}\varepsilon^2|\ln\varepsilon|.$$

From the last two estimates and $M(\xi,\tilde{z}) = \varepsilon \le 2(1-|\tilde{z}|)$ for $\tilde{z} \in E_1$ it now follows that

$$\begin{aligned}\int_{A_2} \frac{1}{|z-\xi||z-\xi^*|} \int_{E_1} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|}\Lambda(\tilde{z})d\tilde{z}dz &\le C_2C_4||\omega||_{L^\infty}\varepsilon|\ln\varepsilon| + \frac{C_2|\ln\varepsilon|}{\varepsilon}\int_{E_1}\Lambda(\tilde{z})d\tilde{z} \\ &\le C_5|\ln\varepsilon| \left(\int_{E_1} \frac{1-|\tilde{z}|}{M(\xi,\tilde{z})^2}\Lambda(\tilde{z})d\tilde{z} + ||\omega||_{L^\infty} \right).\end{aligned}$$

Moreover, for all $(z,\tilde{z}) \in A_2 \times E_2$ we have $|z-\tilde{z}^*| \ge |z-\tilde{z}| \ge \frac{|\tilde{z}-\xi|}{2} \ge \frac{1-|\xi|}{4}$ and $|\tilde{z}-\tilde{z}^*| \le 3(1-|\tilde{z}|)$, therefore

$$\begin{aligned}\int_{A_2} \frac{1}{|z-\xi||z-\xi^*|} \int_{E_2} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|}\Lambda(\tilde{z})d\tilde{z}dz &\le 48\int_{E_2} \frac{1-|\tilde{z}|}{M(\xi,\tilde{z})^2}\Lambda(\tilde{z})d\tilde{z} \int_{A_2} \frac{dz}{|z-\xi||z-\xi^*|} \\ &\le C_6|\ln(1-|\xi|)| \int_{E_2} \frac{1-|\tilde{z}|}{M(\xi,\tilde{z})^2}\Lambda(\tilde{z})d\tilde{z},\end{aligned}$$

where we also used (3.28). The last two estimates and (3.27) show that

$$\int_{A_2} \frac{|R(z)|}{|z-\xi||z-\xi^*|}dz \le (C_1+C_5+C_6)|\ln(1-|\xi|)| \left(\int_{\mathbb{D}} \frac{1-|z|}{M(\xi,z)^2}\Lambda(z)dz + ||\omega||_{L^\infty} \right),$$

so it remains to integrate over A_3.

Let $F_1 := B(\xi, \frac{\varepsilon}{8})$, $F_2 := \mathbb{D} \setminus (B(0,\frac{1}{2}) \cup B(\xi, 2\delta))$, and $F_3 := (B(\xi,2\delta)\cap\mathbb{D}) \setminus B(\xi,\frac{\varepsilon}{8})$. Then for all $(z,\tilde{z}) \in A_3 \times F_1$ we have $|z-\tilde{z}^*| \ge |z-\tilde{z}| \ge \frac{1-|\xi|}{8} \ge |\tilde{z}-\xi|$ and $|\tilde{z}-\tilde{z}^*| \le 3(1-|\tilde{z}|)$, which together with (3.28) yields

$$\begin{aligned}\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_1} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|}\Lambda(\tilde{z})d\tilde{z}dz &\le 192\int_{F_1} \frac{1-|\tilde{z}|}{M(\xi,\tilde{z})^2}\Lambda(\tilde{z})d\tilde{z} \int_{A_3} \frac{dz}{|z-\xi^*||z-\xi|} \\ &\le C_7|\ln(1-|\xi|)| \int_{F_1} \frac{1-|\tilde{z}|}{M(\xi,\tilde{z})^2}\Lambda(\tilde{z})d\tilde{z}.\end{aligned}$$

And from (3.26), (3.23), and (3.28) we obtain

$$\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_2} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \leq \frac{2}{\delta} \int_{F_2} \Lambda(\tilde{z}) d\tilde{z} \int_{A_3} \frac{dz}{|z-\xi||z-\xi^*|}$$
$$\leq C_8 |\ln(1-|\xi|)| \, ||\omega||_{L^\infty}.$$

For the integral involving $(z, \tilde{z}) \in A_3 \times F_3$, let $F_4 := F_3 \cap B(0, 1 - \varepsilon^{\frac{1}{1-\alpha_*}})$ and for $\tilde{z} \in F_3$ let $A_{\tilde{z}} := B(\tilde{z}, \frac{|\tilde{z}-\xi|}{2}) \cap A_3$. From $|\xi|, |\tilde{z}| \geq \frac{1}{2}$ and (3.15) we get

$$|\tilde{z} - \xi^*| \leq |\tilde{z} - \tilde{z}^*| + 4|\tilde{z} - \xi| \leq |\tilde{z} - \tilde{z}^*| + 8|\tilde{z} - z| \leq 10|z - \tilde{z}^*|$$

when also $z \notin A_{\tilde{z}}$. This, (3.28), $|\tilde{z} - \tilde{z}^*| \leq 3(1 - |\tilde{z}|)$, and $|\tilde{z} - \xi^*| \geq |\tilde{z} - \xi| \geq \frac{1-|\xi|}{8}$ for $\tilde{z} \in F_3$, and $|\tilde{z} - \xi^*| \geq \frac{|\tilde{z}-\xi|}{2}$ show that

$$\begin{aligned}
&\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_4} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \\
&\qquad \leq 4 \int_{F_4} \int_{A_{\tilde{z}}} \frac{1}{|\tilde{z}-\xi||\tilde{z}-\xi^*|} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) dz d\tilde{z} \\
&\qquad\quad + 20 \int_{F_4} \int_{A_3 \setminus A_{\tilde{z}}} \frac{1}{|z-\xi||z-\xi^*|} \frac{|\tilde{z}-\tilde{z}^*|}{|\tilde{z}-\xi||\tilde{z}-\xi^*|} \Lambda(\tilde{z}) dz d\tilde{z} \\
&\qquad \leq C_9 \int_{F_4} \frac{|\tilde{z}-\tilde{z}^*|}{|\tilde{z}-\xi||\tilde{z}-\xi^*|} \Lambda(\tilde{z}) \left(|\ln(1-|\tilde{z}|)| + |\ln(1-|\xi|)| \right) d\tilde{z} \\
&\qquad \leq C_{10} |\ln(1-|\xi|)| \int_{F_4} \frac{1-|\tilde{z}|}{M(\xi,\tilde{z})^2} \Lambda(\tilde{z}) d\tilde{z}.
\end{aligned}$$

Finally, let $F_5 := F_3 \setminus F_4$. From (3.26), (3.28), Lemma 3.2.2, and (3.22) with $H := F_5$,

$I := (\arg(\xi) - 3\delta, \arg(\xi) + 3\delta)$, $f \equiv 1$, and $\beta := \sum_{\theta_j \in I} \alpha_j^+ \delta_{\theta_j}$ we obtain

$$\begin{aligned}
\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_5} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \\
&\leq 2 \int_{F_5} \Lambda(\tilde{z}) \int_{A_3} \frac{dz}{|z-\xi||z-\xi^*||z-\tilde{z}|} d\tilde{z} \\
&\leq 8 \int_{F_5} \Lambda(\tilde{z}) \left(\int_{A_{\tilde{z}}} \frac{dz}{|\tilde{z}-\xi|^2 |\tilde{z}-z|} + \int_{A_3 \setminus A_{\tilde{z}}} \frac{dz}{|z-\xi||z-\xi^*||\tilde{z}-\xi|} \right) d\tilde{z} \\
&\leq C_{11} \int_{F_5} \Lambda(\tilde{z}) \left(\frac{1}{|\tilde{z}-\xi|} + \frac{|\ln(1-|\xi|)|}{|\tilde{z}-\xi|} \right) d\tilde{z} \\
&\leq C_{12} \frac{|\ln(1-|\xi|)|}{\varepsilon} ||\omega||_{L^\infty} \int_{F_5} \Pi_{\theta_j \in I} |\tilde{z} - e^{i\theta_j}|^{-2\alpha_j^+} d\tilde{z} \\
&\leq C_{13} \frac{|\ln(1-|\xi|)|}{\varepsilon} ||\omega||_{L^\infty} \int_{F_5} \left| \tilde{z} - \frac{\xi}{|\xi|} \right|^{-2\alpha_*} d\tilde{z} \\
&\leq C_{14} |\ln(1-|\xi|)| \, ||\omega||_{L^\infty},
\end{aligned}$$

where in the last inequality we used that $|F_5| \leq \varepsilon^{\frac{1}{1-\alpha_*}}$, which is less than the area of a disc with radius $\varepsilon^{\frac{1}{2-2\alpha_*}}$. Combining the above estimates and (3.27) yields

$$\int_{A_3} \frac{|R(z)|}{|z-\xi||z-\xi^*|} dz \leq (C_1 + C_7 + C_8 + C_{10} + C_{14}) |\ln(1-|\xi|)| \left(\int_{\mathbb{D}} \frac{1-|z|}{M(\xi,z)^2} \Lambda(z) dz + ||\omega||_{L^\infty} \right),$$

and the result follows.

3.4 Proof of Lemma 3.2.1

We see from (3.11), a change of variables in the integral from (3.9), and (3.14) that we need to show boundedness and continuity of R and

$$Q(t,\xi) := \int_{\mathbb{D}} \left(\frac{z-\xi}{|z-\xi|^2} - \frac{z-\xi^*}{|z-\xi^*|^2} \right) \cdot R(t,z)\, \omega(t, \mathcal{S}(z)) dz$$

on $[0,\infty) \times K$ for any compact $K \subseteq \mathbb{D}$, as well as that $\partial_t \Psi(t,x) = -\frac{1}{2\pi} Q(t, \mathcal{T}(x))$ holds for each $(t,x) \in [0,\infty) \times \Omega$.

So fix any such K and let $d := \operatorname{dist}(K, \partial\mathbb{D}) > 0$, then fix any $(t, \xi) \in [0, \infty) \times K$ and let $B := B(\xi, \frac{d}{2})$ and $B' := \overline{B(\xi, \frac{d}{4})}$. With $C_d := \sup_{|z| \le 1-d/2} \det D\mathcal{S}(z)$, and using (3.15), $|w - z^*| \ge |w - z|$ for all $z, w \in \mathbb{D}$, (3.28), and (3.23), we obtain for any $(t', \xi') \in [0, \infty) \times B'$,

$$\begin{aligned} |R(t,\xi) - R(t,\xi')| \le & ||\omega||_{L^\infty} \left(\int_B + \int_{\mathbb{D}\setminus B} \right) \left(\frac{|\xi - \xi'|}{|\xi - z||\xi' - z|} + \frac{|\xi - \xi'|}{|\xi - z^*||\xi' - z^*|} \right) \det D\mathcal{S}(z) dz \\ \le & 2||\omega||_{L^\infty} |\xi - \xi'| \left(6\pi C_d \ln_+ \frac{1}{|\xi - \xi'|} + 50 C_d + \frac{8|\Omega|}{d^2} \right) \end{aligned}$$

and (using also $|z - z^*| \le 2|\xi' - z^*|$ and Hölder's inequality)

$$\begin{aligned} |R(t,\xi') - R(t',\xi')| \le & \int_{\mathbb{D}} \frac{|z - z^*|}{|\xi' - z||\xi' - z^*|} \det D\mathcal{S}(z) |\omega(t, \mathcal{S}(z)) - \omega(t', \mathcal{S}(z))| dz \\ \le & 2 \left(\int_{\mathbb{D}} |\xi' - z|^{-\frac{3}{2}} \det D\mathcal{S}(z) dz \right)^{\frac{2}{3}} ||\omega(t, \cdot) - \omega(t', \cdot)||_{L^3(\Omega)}. \end{aligned} \tag{3.29}$$

(Note also that the first of these estimates and (3.30) below prove (3.5).) Since the last integral is bounded in $\xi' \in B'$ by Lemma 3.2.2 and (3.23), and ω is continuous as an $L^p(\Omega)$-valued function of $t \in [0, \infty)$ for any $p \in [1, \infty)$ due to boundedness of ω, local boundedness of u, and (3.8), these two estimates show that R is continuous at (t, ξ). Boundedness of R on $[0, \infty) \times K$ follows from the estimate

$$|R(t,\xi)| \le C_\Omega ||\omega||_{L^\infty} (1 - |\xi|)^{1 - 2\alpha_*} \tag{3.30}$$

for all $(t, \xi) \in [0, \infty) \times \mathbb{D}$, with α_* from (3.18) and some Ω-dependent constant C_Ω. To obtain it, first note that $|z - z^*| \le 2|\xi - z^*|$ and (3.23) yield (with δ from (3.18))

$$\int_{\Omega\setminus B(\xi,\delta)} \frac{|z - z^*|}{|\xi - z||\xi - z^*|} \det D\mathcal{S}(z) dz \le \frac{2}{\delta} \int_{\Omega\setminus B(\xi,\delta)} \det D\mathcal{S}(z) dz \le \frac{2|\Omega|}{\delta}.$$

Then use Lemma 3.2.2, and (3.22) with $H := B(\xi, \delta)$, $I := (\arg(\xi) - 2\delta, \arg(\xi) + 2\delta)$,

$f(z) := \frac{1}{|\xi - z|}$, and $\beta := \sum_{\theta_j \in I} \alpha_j^+ \delta_{\theta_j}$ to get (with $\varepsilon := \frac{1-|\xi|}{2}$ and $\tilde{\xi} = \frac{\xi}{|\xi|}$)

$$\int_{B(\xi,\delta)} \frac{|z - z^*|}{|\xi - z||\xi - z^*|} \det D\mathcal{S}(z) dz \le C' \int_{B(\xi,\delta)} \frac{|\tilde{\xi} - z|^{-2\alpha_*}}{|\xi - z|} dz$$
$$\le C' \left(\int_{B(\xi,\varepsilon)} \frac{\varepsilon^{-2\alpha_*}}{|\xi - z|} dz + \int_{B(\tilde{\xi},\varepsilon)} \frac{|\tilde{\xi} - z|^{-2\alpha_*}}{\epsilon} dz + 9 \int_{B(\xi,\delta)\setminus(B(\xi,\varepsilon)\cup B(\tilde{\xi},\varepsilon))} |\xi - z|^{-1-2\alpha_*} dz \right)$$
$$\le C''(1 - |\xi|)^{1-2\alpha_*}$$

with some Ω-dependent constant C', C'' because $\sum_{\theta_j \in I} \alpha_j^+ \le \alpha_* < 1$ by (3.18). The last two estimates now imply (3.30).

Let us now turn to Q. Fix any K as above, then fix any $(t, \xi) \in [0, \infty) \times K$ and let d, B, B' be as above (without loss assume that $d \le \frac{1}{4}$). Then for any $(t', \xi') \in [0, \infty) \times B'$ we have from (3.15),

$$|Q(t,\xi) - Q(t,\xi')| \le ||\omega||_{L^\infty} \int_{\mathbb{D}} \left(\frac{|\xi - \xi'|}{|\xi - z||\xi' - z|} + \frac{|\xi^* - \xi'^*|}{|\xi^* - z||\xi'^* - z|} \right) |R(t,z)| dz,$$

where the second fraction is just $\frac{1}{|\xi^* - z|}$ when $\xi' = 0$ and $\frac{1}{|\xi'^* - z|}$ when $\xi = 0$. Using (3.15), splitting the integration to $z \in B$ and $z \in \mathbb{D}\setminus B$, and applying (3.30) and (3.28) yields

$$|Q(t,\xi) - Q(t,\xi')| \le C'||\omega||_{L^\infty} |\xi' - \xi| \left(d^{1-2\alpha_*} \left(1 + \ln_+ \frac{1}{|\xi - \xi'|} \right) + d^{-2} \right)$$

for some Ω-dependent constant C'. Next, we have

$$|Q(t,\xi') - Q(t',\xi')| \le ||\omega||_{L^\infty} \int_{\mathbb{D}} \frac{|\xi' - \xi'^*|}{|\xi' - z|\,|\xi'^* - z|} |R(t,z) - R(t',z)| dz$$
$$+ \int_{\mathbb{D}} \frac{|\xi' - \xi'^*|}{|\xi' - z|\,|\xi'^* - z|} |R(t',z)|\,|\omega(t,\mathcal{S}(z)) - \omega(t',\mathcal{S}(z))| dz.$$

Splitting the first integration into $z \in B'$ and $z \in \mathbb{D}\setminus B'$, and then using $|\xi' - \xi'^*| \le 2|\xi'^* - z|$, (3.29), and (3.30) shows that the first integral is bounded above by

$$C_d||\omega(t,\cdot)-\omega(t',\cdot)||_{L^3(\Omega)}+\frac{4}{d}\int_{\mathbb{D}}|R(t,z)-R(t',z)|\,dz$$

for some (Ω, d)-dependent constant C_d. This converges to 0 as $t' \to t$ by continuity of $\omega : [0,\infty) \to L^3(\Omega)$, together with (3.29) and integrability of the right-hand side of (3.30). Using $|\xi' - \xi'^*| \le 2|\xi'^* - z|$, (3.30), and Lemma 3.2.2, the second integral is bounded by

$$C'\left[\int_{\mathbb{D}}\left(\frac{(1-|z|)^{1-2\alpha_*}}{|\xi'-z|\det D\mathcal{S}(z)^{\frac{1}{p}}}\right)^q dz\right]^{\frac{1}{q}}\left(\int_{\mathbb{D}}\det D\mathcal{S}(z)|\omega(t,\mathcal{S}(z))-\omega(t',\mathcal{S}(z))|^p dz\right)^{\frac{1}{p}}$$
$$\le C_d||\omega(t,\cdot)-\omega(t',\cdot)||_{L^p(\Omega)}$$

for some Ω-dependent C' and (d,Ω)-dependent C_d, provided $p \in (2,\infty)$ is large enough so that with $q := \frac{p}{p-1}$ we have $(1-2\alpha_* - \frac{1}{p}\sum_j \alpha_j^+)q > -1$. The above estimates thus together show that Q is continuous at (t,ξ). We can also use (3.15), $|\xi - \xi^*| \le 2|\xi^* - z|$, and (3.30) to get

$$|Q(t,\xi)| \le 2C_\Omega||\omega||^2_{L^\infty}\int_{\mathbb{D}}\frac{(1-|z|)^{1-2\alpha_*}}{|\xi-z|}dz \tag{3.31}$$

for all $(t,\xi) \in [0,\infty)\times\mathbb{D}$, showing boundedness of Q on $[0,\infty)\times K$ for each compact $K \subseteq \mathbb{D}$. Hence it remains to show $\partial_t\Psi(t,x) = -\frac{1}{2\pi}Q(t,\mathcal{T}(x))$ pointwise, which will follow from

$$-\frac{1}{2\pi}\int_{t_0}^{t_1}Q(t,\mathcal{T}(x_0))dt = \Psi(t_1,x_0)-\Psi(t_0,x_0) \tag{3.32}$$

for all $0 \le t_0 < t_1$ and $x_0 \in \Omega$ because Q is continuous. So fix any such (t_0,t_1,x_0). Let

$$\phi(x) := -\frac{1}{2\pi}\ln\frac{|\mathcal{T}(x_0)-\mathcal{T}(x)|}{|\mathcal{T}(x_0)-\mathcal{T}(x)^*||\mathcal{T}(x)|} = -\frac{1}{2\pi}\ln\frac{|\mathcal{T}(x)-\mathcal{T}(x_0)|}{|\mathcal{T}(x)-\mathcal{T}(x_0)^*||\mathcal{T}(x_0)|}$$

(so $\Psi(t_j, x_0) = \int_\Omega \phi(x)\omega(t_j, x)dx$ for $j = 0, 1$) and

$$\psi(x) := \nabla\phi(x) = -\frac{1}{2\pi} D\mathcal{T}(x)^T \left(\frac{\mathcal{T}(x) - \mathcal{T}(x_0)}{|\mathcal{T}(x) - \mathcal{T}(x_0)|^2} - \frac{\mathcal{T}(x) - \mathcal{T}(x_0)^*}{|\mathcal{T}(x) - \mathcal{T}(x_0)^*|^2} \right)$$

for each $x \in \Omega$ (recall (3.13)). Also, for each $r \in (0, \frac{t_1 - t_0}{2})$ let $g_r \in C_c^\infty([0, \infty))$ be such that

$$\chi_{[t_0+r, t_1-r]} \le g_r \le \chi_{(t_0, t_1)}$$

and g_r is non-increasing on $[0, t_1]$ and non-decreasing on $[t_1, \infty)$; and for each $h \in (0, 1]$ let $f_h \in C^\infty([0, \infty))$ be such that

1. $f_h(x) = 0$ for $x \in [0, \frac{h}{3}]$,
2. $f_h(x) = x$ for $x \in [h, \frac{1}{h}]$,
3. $f_h(x) = \frac{1}{h} + h$ for $x \in [\frac{1}{h} + h, \infty)$,
4. $0 \le f_h'(x) \le 2$ for $x \in [0, \infty)$.

Now for any $h, r \in (0, \min\{1, \frac{t_1 - t_0}{2}\})$ and $(t, x) \in [0, \infty) \times \Omega$ let

$$\varphi_{r,h}(t, x) := g_r(t) f_h(\phi(x)).$$

Then clearly $\varphi_{r,h} \in C_c^\infty([0, \infty) \times \Omega)$ and $\varphi_{r,h}(0, \cdot) \equiv 0$, so plugging it into (3.7) yields

$$\int_0^\infty \int_\Omega \omega(t, x) g_r(t) f_h'(\phi(x))\, u(t, x) \cdot \psi(x) dx dt + \int_0^\infty \int_\Omega \omega(t, x) g_r'(t) f_h(\phi(x)) dx dt = 0.$$

Since $\omega(t, x) g_r(t) f_h'(\phi(x)) \psi(x)$ is a bounded function and $u \in L^\infty((0, \infty); L^2(\Omega))$, we can use the Dominated Convergence Theorem to pass to the limit $r \to 0$ and obtain

$$\int_{t_0}^{t_1} \int_\Omega \omega(t, x) f_h'(\phi(x))\, u(t, x) \cdot \psi(x) dx dt + \int_\Omega \omega(t_0, x) f_h(\phi(x)) dx - \int_\Omega \omega(t_1, x) f_h(\phi(x)) dx = 0,$$

where in the second integral above we used that ω is continuous as an $L^1(\Omega)$-valued function of $t \in [0,\infty)$. If we can show that $u \cdot \psi \in L^\infty((0,\infty); L^1(\Omega))$, then taking $h \to 0$ will yield

$$\int_{t_0}^{t_1} \int_\Omega \psi(x)^T u(t,x)\, \omega(t,x) dx dt = \int_\Omega \phi(x)\omega(t_1,x)dx - \int_\Omega \phi(x)\omega(t_0,x)dx$$

via the Dominated Convergence Theorem. But this is precisely (3.32) due to (3.11) and (3.14). If $B := B(x_0, \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega))$, then $u \cdot \psi \in L^\infty((0,\infty); L^1(B))$ because u is bounded on $[0,\infty) \times B$ by (3.30). From (3.13) we see that there is C_{x_0} such that

$$|\psi(x)| \le C_{x_0} \|D\mathcal{T}(x)\| \le 2C_{x_0} |\det D\mathcal{T}(x)|^{\frac{1}{2}}$$

for all $x \in \Omega \setminus B$, so $\psi \in L^2(\Omega)$ by $\int_\Omega \det D\mathcal{T}(x)dx = |\mathbb{D}|$. So $u \cdot \psi \in L^\infty((0,\infty); L^1(\Omega \setminus B))$, which indeed yields $u \cdot \psi \in L^\infty((0,\infty); L^1(\Omega))$ and thus finishes the proof.

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