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Viscosity Solution Methods in Risk Analysis

By

Khalilah Beal

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Craig C. Evans, Chair
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Abstract

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Professor Craig C. Evans, Chair

This dissertation concerns the problem of "collective ruin" in classical risk theory, which models the risk of an insurance business. Collective ruin is formulated via dynamical stochastic processes for a reserve, based on deterministic premia and random claims. The integro-differential equation (IDE) for u, the probability that the reserve remains nonnegative, is derived. To consider solutions for large initial reserves, a gauge parameter ϵ is introduced. Smooth solutions u_ϵ of the resulting singularly perturbed IDE are classified and characterized, and a complicated explicit representation formula is presented.

Motivated by Wentzel Kramers Brillouin (WKB) approximation methods, the IDE is expressed in terms of the changes of variable $u_\epsilon = 1 - e^{-w_\epsilon/\epsilon}$ and $u_\epsilon = e^{w_\epsilon/\epsilon}$. The resulting nonlinear integro-differential equations and their limits are studied within the framework of viscosity solutions. Using viscosity solutions allows a broad class of collective risk problems to be studied. Stability of viscosity solutions implies the limit as $\epsilon \to 0$ of w_ϵ satisfies a corresponding limiting equation. This equation has simpler structure and pointwise bounds for solutions are determined. Extension of the methods to the dynamic programming problems of optimally controlling the survivability on an infinite horizon via reinsurance and via new business are also discussed.

The results of this dissertation extend work of Knessl, Mangel, Peters, and others who treat cases with constant premiums and/or smooth distributions of claims.

For legal aliens, illegal aliens, and those who graciously help them.

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Finally, I would like to acknowledge the personal support and encouragement that my family and loved ones provided me during this journey.

Whatever affects one directly, affects all indirectly. I can never be what I ought to be until you are what you ought to be. This is the interrelated structure of reality.

Martin Luther King, Jr.

Chapter 1

Introduction

1.1 Overview

Here we give a nontechnical description of the problem, our model, and its variables.

Nontechnical motivation Risk theory is "a collection of related ideas for designing, managing, and regulating a risk enterprise", characterized by the fact that "the fundamental nature of the operation is such that expenditures may exceed receipts during some accounting periods in the normal course of operation" (Gerber [19]). Ruin theory, a topic within risk theory, has been researched as early as 1955 (Cramér [13]).

In this thesis, we address some problems involving probability of ruin, a subtopic of ruin theory. Informally, ruin occurs when a business becomes bankrupt. We use viscosity solution methods to provide asymptotic approximations to the probability of survival of an insurance business. In the final chapter, we extend our results about ruin probability to some optimization problems in ruin theory. We consider risk management via dynamic control of variables, such as writing new business and buying reinsurance.

Model and variables Our model is based upon the classic Cramér-Lundberg collective risk model, which models transactions of an insurance business (see, for example, Mikosch [30]). Essentially, claims arrive which must be paid out immediately, while revenue is collected from premiums. The practical goal is to determine premium rates so as to avoid bankruptcy. This translates to obtaining estimates for the corresponding probability of bankruptcy, depending on the premium rate.

Our model assumes

- the total number of claims occurring in a given period of time is randomly distributed;
- the claim sizes are independent and identically distributed nonnegative random variables;
- the claim sizes and claim arrival times are independent.

Our model has two variables: the distribution of claims' arrivals and sizes, and the premium rate. These variables separate and model the "two major reasons for big losses: frequent claims and large claims" (see Schmidli's book [42]). Realistically, premium rates depend on the market, as well as the risk of insuring. However, for this thesis, our scope is restricted to analyzing the impact of risk on premium rates. We use the probability of ruin to measure risk, where ruin is defined as the state of reserves being less than zero. Since big losses precipitate bankruptcy, our model is appropriate for use in determining ruin probabilities.

We maintain this model for the insurer's reserve when we consider stochastic control problems.

Technical motivation Much has been done towards premium rate calculation (see Gerber's monograph [19]). Canonical estimates, such as those in the texts of in Grandell's text [20] and Mikosch's text [30], are obtained using probabilistic or numerical analysis methods. The analyses rely on certain assumptions about the distribution of the claims. We use basic principles to prove lemmas typically approached via probability theory. The problem formulations involve linear nonlocal operators.

In this thesis, we consider nonlinear nonlocal operators arising from singular stochastic control problems.

Motivation for use of WKB method Historically, the theory of asymptotic series solutions only addressed linear ordinary differential equations with analytic coefficients. Hence, the Wentzel-Kramers-Brillouin (WKB) method was developed to obtain a one term approximation to the solution for equations of the type

$$w'' + \psi w = 0,$$

where $\psi = \psi(x; \lambda)$ for a possibly large parameter λ. A survey of the approaches used to obtain the WKB solutions and the resolution of turning point singularities may be found in Schlissel [40].

When applied to the probability of ruin equations studied in this thesis, the WKB method yields corresponding nonlinear integro-differential equations. Since the theory of viscosity solutions provides techniques well adapted to analyzing nonlinear problems (Crandall, Ishii, and Lions [14], Pham [35]), we use the framework of viscosity solutions to analyze the risk equation. The reward is this: with these generalized solutions, our results are valid for a broader class of claims distributions than those in Asmussen and Albrecher [4] and Peters and Mangel [34]. Moreover, we obtain results which are valid for a broader class of premium rate functions than those classically considered ([4], [19], and [20]).

As for the controlled processes, the infinitesimal generator is a nonlocal operator. Although much is known about second order partial differential operators, the problems we consider have fully nonlinear, nonlocal operators. We investigate the asymptotics of solutions of the optimal control equations. In some instances, our formal computations suggest that the limiting equation is deterministic.

1.2 Outline of this thesis

Throughout the thesis, we adopt the notation for estimates employed, for instance, by Evans [16] regarding both explicitly computable constants C. We use the Landau symbols O and o to express limiting behavior. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : [0, \infty) \times \Omega \to \mathbb{R}$ be a stochastic process. We use the standard notation of probability theory $X(t) := X(t, \cdot)$.

In Chapter 2 we introduce the risk equation and derive two integro-differential problems which are studied in the subsequent chapters. Chapter 3 is devoted to characterizing classical solutions of the risk equation, based on the regularity of the claims distribution F. In Chapter 4, we motivate the upcoming generalized solutions by analyzing three types of distributions. In Chapter 5, we use two nonlinear transformations to formulate the so-called viscosity solutions of the risk equation.

Because we are using viscosity solution framework, we do not need to assume solutions are smooth. We show that, under certain conditions, the limit of transformed equations has a solution. This, along with allowing the premium rate p to vary, is new.

1.2.1 Synopsis of chapters

Chapter 2 We begin by deriving the risk equation

$$u'(x)p(x) = \lambda u(x) - \lambda \textstyle\int_0^x u(x-y)\, F\{dy\}.$$

Here the unknown is $u\colon \mathbb{R}_+ \to \mathbb{R}$. The probability measure F, the positive smooth function $p\colon \mathbb{R}_+ \to \mathbb{R}$, and the positive constant $\lambda > 0$ are given. In the risk equation, u denotes the probability that the risk reserve remains nonnegative, and x represents the initial reserve amount. Thus, the boundary condition

$$u(\infty) = 1$$

indicates that the reserve will never go broke when starting with an infinite amount of money.

In order to analyze solutions of the risk equation for large x, we introduce a gauge parameter ϵ. Analyzing the resulting final-value problem

$$\begin{cases} \epsilon p_\epsilon(x) u_\epsilon'(x) - \lambda u_\epsilon(x) + \lambda \int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} = 0 & (x > 0) \\ u_\epsilon(\infty) = 1 \end{cases} \tag{1.2.1}$$

is the main focus of the first part of the thesis.

By inspection, any scalar multiple of a solution of (1.2.1) is also a solution. Hence uniqueness is not expected, and our first objective is to classify and characterize the solutions. Towards this end, we reformulate (1.2.1) as an initial value problem:

$$\begin{cases} \epsilon p_\epsilon(x) u_\epsilon'(x) - \lambda u_\epsilon(x) + \lambda \int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} = 0 & (x > 0) \\ u_\epsilon(0) = u_0 \end{cases} . \tag{1.2.2}$$

We show that bounded solutions of (1.2.2) are determined by $u'_\epsilon(0+)$.

Chapter 3 This chapter's goal is to determine properties of F under which classical solutions u_ϵ of (1.2.2) exist. Ultimately, the regularity of F determines that of a classical solution u_ϵ. We exhibit a representation formula for the solution and classify conditions under which solutions are bounded.

Chapter 4 We motivate developing generalized solutions by analyzing two types of distributions. First, we consider singular distributions. In this case, we express (1.2.2) as a system of equations. Although a smooth solution does not exist, we exhibit a recursively defined, piecewise continuous solution.

Then, we address the gamma distribution with shape parameter γ and scale parameter β. The case with standard gamma distribution and $\gamma = 1$ is the exponential distribution, and (1.2.2) is completely integrable. We explicitly solve the rescaled risk equation and prove that solutions of (1.2.2) are bounded exactly when $p_\epsilon(x)$ is greater than $\lambda\mathbb{E}[Y]$ for some $x > 0$. For general γ and constant premium rate, we use the "especially powerful technique" of Laplace transforms to study the resulting linear constant coefficient ordinary differential equation (Evans [16], p. 186).

Chapter 5 We consider viscosity solutions, thereby broadening the class of admissible distributions. Guided by the approaches in Knessl and Peters [26] and Peters and Mangel [34] of using the WKB method, we perform transformations on (1.2.2, 1.2.1). We analyze the resulting nonlinear first-order integro-differential equations

$$\lambda \int e^{w'(x)y} F\{dy\} - \lambda - p(x)w'(x) = 0 \qquad (x \geqslant 0) \tag{1.2.3}$$

$$\lambda \int e^{-w'(x)y} F\{dy\} - \lambda + p(x)w'(x) = 0 \qquad (x \geqslant 0). \tag{1.2.4}$$

We define viscosity solutions of (1.2.5, 1.2.6), and verify that (1.2.5, 1.2.6) are well-posed.

Let

$$p_\epsilon \to p \text{ uniformly on } \mathbb{R}_+ \text{ as } \epsilon \to 0,$$

and send $\epsilon \to 0$ in (1.2.5, 1.2.6) to obtain the limit equations

$$\lambda \int e^{w'(x)y} F\{dy\} - \lambda - p(x)w'(x) = 0 \qquad (x \geqslant 0) \tag{1.2.5}$$

$$\lambda \int e^{-w'(x)y} F\{dy\} - \lambda + p(x)w'(x) = 0 \qquad (x \geqslant 0). \tag{1.2.6}$$

These equations are em approximations to our risk equations (1.2.1) and (1.2.2).

We define an asymptotic limit of each transformed equation based on viscosity solution methods. There is a boundary layer at the origin. Depending on the magnitude of $\frac{p}{\lambda\mathbb{E}[Y]}$; there may be other boundary layers. There is a connection between the nonlinear eikonal equation resulting from WKB method and the limit equation (1.2.6); they are the same.

Chapter 6 We conclude with examples in stochastic control for which the methods developed in the previous chapters appear to be well suited. The equations are particularly interesting due to the nonlocal operators arising in the variational equations. Equations of the following types appear:

$$0 = \sup_{a\in A}\Big\{\epsilon(p-h(a))V_\epsilon'(x) - \lambda V_\epsilon(x) + \lambda\textstyle\int_0^\infty V_\epsilon\big(x-c(a,\epsilon y)\big)\,F\{dy\}\Big\},$$

$$0 = \sup_{0\leqslant b\leqslant 1}\Big\{p(b)w'(x) + \lambda - \lambda\textstyle\int_0^\infty e^{w'(x)by}\,F\{dy\}\Big\}.$$

1.2.2 Connections with previous research

Much has been written about asymptotic approximations to solutions of (1.2.1) with constant premium rate p and absolutely continuous distribution F. One of the contributions of this thesis is to provide an asymptotic analysis of (1.2.1) under general p and F.

Since $p = p(x)$ is allowed to vary, the claims process is nonhomogeneous in the space variable. Typically, Watson's Lemma and Laplace transforms are used to find asymptotic expansions of integrals when the smoothness conditions required for integration methods are not satisfied (see Bleistein and Handelsman [11]). Instead, in this thesis, we use viscosity solution theory to analyze the model with nonsmooth distributions (see Crandall, Ishii, and Lions [14]).

Chapter 2

The Risk Equation

We seek to model the reserve of an insurance company which has a given initial reserve, receives income from premiums, and pays out randomly occurring, randomly sized claims immediately upon their occurrences. We use a collective risk model, called the *Cramér-Lundberg model*, to model the reserve (see Ramasubramanian [36]). A survey of alternative risk models may be found in Hipp [22].

Once the model is established, we define the probability of ruin and derive the risk equations. These are the equations upon which this thesis focuses.

2.1 The model

Let $(\Omega, \mathcal{U}, \mathbb{P})$ be a probability space. We denote the *reserve process* with initial reserve $x \geqslant 0$ at time $t \geqslant 0$ by $X = X(t, x, \omega)$ for $\omega \in \Omega$, with $X(0, x, \cdot) = x$. We suppose the premium rate $p = p(x)$ is deterministic, and that the random process $C = C(t)$ records the total, or aggregate, claims up to time t.

We introduce next our assumptions concerning these expressions. Existence of each process below is standard. Proofs of existence and various properties can be found, for instance, in Çınlar [12]. After stating the assumptions of the parameters or variables, we will define the probability of ruin of reserve process.

a. The claim arrival process N For $t \geqslant 0$, $N(t)$ denotes the number of claims which have arrived during the time interval $[0, t]$. Then $N = \{N(t) : t \geqslant 0\}$ is the claim arrival process. We assume N is a time homogeneous Poisson process with intensity $\lambda > 0$. Consequently,

$$\mathbb{P}[N(t) = n] = \tfrac{(\lambda t)^n}{n!} e^{-\lambda t} \qquad (t > 0,\ n = 0, 1, 2, \ldots), \tag{2.1.1}$$

and $N(0) = 0$. For more information about N, see Appendix A.2.

b. The claim size process Y and distribution F For $n = 1, 2, \ldots$, let $Y(n)$ be the size of the nth claim. We assume that $Y = \{Y(n) : n \in \mathbb{N}\}$ is independent and identically distributed, with probability distribution F. It makes sense to assume Y is positive and, therefore, $\mathbb{E}[Y] > 0$.

We analyze the model via the distribution of Y, so care must be taken when specifying useful requirements on F. We now present notation and assumptions regarding F.

Following the notation in Feller [17], let $F\{\cdot\}$ denote the distribution measure and $F(\cdot)$ to denote the corresponding distribution function. Consequently,

$$F\{B\} := \mathbb{P}[Y \in B] \qquad (B \in \mathcal{B})$$

and

$$F(y) := \mathbb{P}[Y \leqslant y] \qquad (y \in \mathbb{R}),$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.

Hypothesis. We assume

F is proper. This means $F(x_0) = 0$ if and only if $x_0 = 0$.

Thus, our model assumes that there is no positive lower bound on the claim sizes. Equivalently, there is almost surely a claim of size ϵ, for every $\epsilon > 0$.

Notation. Throughout this thesis we use the following notation:

(i) We use Lebesgue-Stieltjes integration, so that the previous probabilities are denoted as follows:

$$\int \mathbb{1}_B(y)\, F\{dy\} = \int_B F\{dy\} = F\{B\},$$

where $\mathbb{1}_B$ is the indicator function of $B \in \mathcal{B}$. For $B = (a, b]$, we write

$$F\{B\} = \int_a^b F\{dy\},$$

and for a singleton $B = \{x_0\}$, we write $F\{B\} = F\{x_0\}$.

(ii) A point x is an *atom* of the distribution F if $F\{x\} > 0$. A distribution without atoms is called *continuous*.

(iii) The distribution F is *absolutely continuous with respect to Lebesgue measure* if it is of the form

$$F\{B\} = \int \mathbb{1}_B(y) f(y)\, dy \qquad (B \in \mathcal{B})$$

for some nonnegative Lebesgue integrable function f. In this case, f is called a *density of F with respect to Lebesgue measure*.

The distribution function $F(\cdot)$ is right continuous with left-hand limits. Therefore,

$$F(y) = F\{y\} + \lim_{\epsilon \to 0+} F(y - \epsilon) \qquad (y \geqslant 0);$$

and $F(\cdot)$ is continuous if and only if $F\{\cdot\}$ is continuous. Note that F is continuous at the origin. We call a continuous distribution F *differentiable at x* if $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ exists. These definitions allow us to unambiguously refer to the regularity of F. Lastly, we introduce a global assumption regarding the regularity of F.

Hypothesis. We assume

the atoms of F are isolated.

c. The aggregate claim amount process C In order to define the aggregate claim amount process, we first introduce the processes from which C is defined.

Consider the compound Poisson process

$$C(t) := \sum_{n=1}^{N(t)} Y(n) \qquad (t \geqslant 0), \tag{2.1.2}$$

which represents the total amount of claims arriving in the time interval $(0, t]$. By convention, the notation $\sum_{n=1}^{0}$ is the empty sum and $C(0) = 0$. We assume the claim arrival and size processes are independent.

We now derive the distribution function for C. We will see in fact that the computations used to derive the risk equation parallel those below. The argument is adapted from Karlin and Taylor [25].

Fix c, $t > 0$. The law of total probability implies

$$\mathbb{P}\left[C(t) \leqslant c\right] = \textstyle\sum_{k\geqslant 0} \mathbb{P}\left[C(t) \leqslant c \mid N(t) = k\right] \mathbb{P}\left[N(t) = k\right].$$

We use the probabilities in (2.1.1) to compute $\mathbb{P}\left[C(t) \leqslant c \mid N(t) = k\right]$, for $k = 1, 2, \ldots$. Since the claim arrival and size processes are independent, for each k

$$\mathbb{P}\left[C(t) \leqslant c \mid N(t) = k\right] \;=\; \mathbb{P}\left[\textstyle\sum_{n=1}^{k} Y(n) \leqslant c\right].$$

Since Y is independent, we employ the same conditioning argument to determine the latter probability. Fix $k \geqslant 1$. We condition on the size of $Y(k)$:

$$\begin{aligned}\mathbb{P}\left[\textstyle\sum_{n=1}^{k} Y(n) \leqslant c\right] &= \textstyle\sum_{y_k \geqslant 0} \mathbb{P}\left[\sum_{n=1}^{k} Y(n) \leqslant c \mid Y(k) = y_k\right] \\ &= \textstyle\int \mathbb{P}\left[\sum_{n=1}^{k-1} Y(n) < c - y_k\right] F\{dy_k\}.\end{aligned}$$

But Y is nonnegative, and so we find

$$\mathbb{P}\left[\textstyle\sum_{n=1}^{k} Y(n) \leqslant c\right] = \textstyle\int_0^c \mathbb{P}\left[\sum_{n=1}^{k-1} Y(n) < c - y_k\right] F\{dy_k\}.$$

Iterating this conditioning on the integrand yields

$$\begin{aligned}\mathbb{P}\left[\textstyle\sum_{n=1}^{k} Y(n) \leqslant c\right] = \\ &\textstyle\int_0^c \sum_{y_{k-1}\geqslant 0} \mathbb{P}\left[\sum_{n=1}^{k-2} Y(n) < c - y_k - y_{k-1} \mid Y(k-1) = y_{k-1}\right] F\{dy_k\} \\ &\vdots \\ &= \textstyle\int_0^c \int_0^{c-y_k} \cdots \int_0^{c-\sum_{i=1}^{k-1} y_i} F\left(c - \sum_{i=1}^{k} y_i\right) F\{dy_1\} \cdots F\{dy_k\}.\end{aligned}$$

Putting it all together, we get

$$\mathbb{P}\left[C(t) \leqslant c\right] = \sum_{k=0}^{\infty} \mathbb{P}\left[\sum_{n=1}^{k} Y(n) \leqslant c\right] \frac{e^{-\lambda t}(\lambda t)^k}{k!},$$

where for each $k \geqslant 1$,

$$\mathbb{P}\left[\sum_{n=1}^{k} Y(n) \leqslant c\right] = \int_0^c \int_0^{c-y_k} \cdots \int_0^{c-\sum_{i=1}^{k-1} y_i} F\left(c - \sum_{i=1}^{k} y_i\right) F\{dy_1\} \cdots F\{dy_k\}.$$

d. The premium rate p Concerning the premium rate, we assume that

$$p \text{ is a positive-valued, non-decreasing, and smooth function.} \tag{2.1.3}$$

It follows that $p(x) \geqslant p(0) > 0$ for all $x \geqslant 0$. Thus, our model assumes the insurer collects a premium amount based on the past reserves of the insurance company.

e. The survival probability u We combine the foregoing processes into the reserve process of our insurance company.

Definition. The *reserve process* $X = X(t, x)$ satisfies

$$X(t, x) = x + \int_0^t p(X(s, x))\, ds - C(t).$$

Occasionally, we express this in shorthand, stochastic differential form as

$$\begin{cases} dX = p(X)dt - dC \\ X(0, x) = x. \end{cases} \tag{2.1.4}$$

Finally, we are prepared to state the probabilistic expression upon which our measure of risk is founded.

Definition. We call

$$u(x) := \mathbb{P}\left[X(t, x) > 0 \text{ for all } t > 0\right] \tag{2.1.5}$$

the *probability of ultimate survival* u of the risk reserve X satisfying (2.1.4).

Remark. The construction of u in Peters and Mangel [34] relies on considering the limit as t_0 approaches ∞ of the probability of survival up to time $t_0 > 0$. In this thesis, u is not defined as the limit of finite time probabilities.

2.2 The risk equations

The ultimate goal of this section is to analyze the integro-differential equation for the probability u of ultimate survival. We begin by deriving the risk equations.

2.2.1 The classical risk equation

There are two common approaches to deriving an integro-differential equation satisfied by the probability of non-ruin. The first, introduced by Cramér [13] in 1955, uses an argument which assumes the ruin probability is differentiable. The second approach exploits the properties of renewal processes to derive the risk equation, and was proven by Feller [17] in 1971. Both assume p is constant. The former method is fine for our purposes, and we modify the argument in Grandell [20] to obtain the risk equation for the case where the premium rate is nonconstant.

During a short time interval $(0, dt]$ there are four possible cases: (1) no claim occurs, (2) one claim occurs and causes bankruptcy, (3) one claim occurs but does not cause bankruptcy, or (4) at least two claims occur. Call these events E_1, E_2, E_3, and E_4. Then by (2.1.5) and the law of total probability,

$$\begin{aligned} u(x) &= \mathbb{P}\left[X(t,x) > 0 \text{ for all } t > 0\right] \\ &= \sum_{i=1}^{4} \mathbb{P}\left[X(t,x) > 0 \text{ for all } t > 0 \mid E_i\right] \mathbb{P}[E_i]. \end{aligned} \tag{2.2.1}$$

We consider each summand of (2.2.1) in turn.

Regarding E_1:

$$\mathbb{P}\left[X(t,x) > 0 \text{ for all } t > 0 \mid E_1\right] = u\big(X(dt,x)\big).$$

Since the reserve process satisfies (2.1.4),

$$\begin{aligned} u\big(X(dt,x)\big) &= u\left(x + \textstyle\int_0^{dt} p\big(X(s,x)\big)\, ds - C(dt)\right) \\ &= u\big(x + p(X(0,x))dt + o(dt) - C(dt)\big) && \text{by (2.1.3)} \\ &= u\big(x + p(x)dt + o(dt) - C(dt)\big) && \text{by (2.1.4)} \\ &= u\left(x + p(x)dt + o(dt)\right) && \text{by (2.1.2)}. \end{aligned}$$

Assuming u is differentiable, we deduce that

$$u\big(X(dt,x)\big) = u\big(x + p(x)\, dt\big) + o(dt).$$

Thus

$$\mathbb{P}\left[X(t,x) > 0 \text{ for all } t > 0 \mid E_1\right] = u\big(x + p(x)\, dt\big) + o(dt).$$

Furthermore,

$$\mathbb{P}\left[E_1\right] \;=\; P[N(t) = dt] \;=\; e^{-\lambda dt} \;=\; 1 - \lambda dt + o(dt).$$

In summary,

$$\mathbb{P}\left[X(t,x) > 0 \text{ for all } t > 0 \mid E_1\right] \mathbb{P}[E_1] \;=\; (1 - \lambda dt) u(x + p(x)dt) \tag{2.2.2}$$

as $dt \to 0$.

Next we consider E_2. Since

$$\mathbb{P}\left[X(t,x) > 0 \text{ for all } t > 0 \mid E_2\right] = 0,$$

E_2 contributes nothing to the sum in (2.2.1).

Next is E_3:

$$\mathbb{P}\left[X(t,x) > 0, t > 0 \mid E_3\right] \mathbb{P}\left[E_3\right] \;=\; \lambda\, dt \int_0^{x+p(x)dt} u(x + p(x)dt - y)\, F\{dy\} + o(dt). \tag{2.2.3}$$

Finally, E_4: Recalling (2.1.1), we compute

$$\mathbb{P}\left[E_4\right] = \mathbb{P}\left[N(dt) \geqslant 2\right] \;=\; 1 - e^{-\lambda dt} - \lambda dt e^{-\lambda dt} \;=\; o(dt)$$

as $dt \to 0$. Therefore

$$\mathbb{P}[X(t,x) > 0, t > 0 \mid E_4]\, \mathbb{P}[E_4] \;=\; o(dt). \tag{2.2.4}$$

We combine (2.2.2)-(2.2.4) to get

$$u(x) = (1 - \lambda dt)u(x + p(x)dt) \;+\; \lambda dt \textstyle\int_0^{x+p(x)dt} u(x + p(x)dt - y)\, F\{dy\} \;+\; o(dt).$$

Expanding u about x up to order dt yields

$$\begin{aligned} u(x) \;&=\; (1 - \lambda dt)\big(u(x) + u'(x)p(x)dt + o(dt)\big) \\ &\qquad + \lambda dt \int_0^{x+p(x)dt} u\big(x + p(x)dt - y\big)\, F\{dy\} + o(dt) \\ &=\; u(x) + u'(x)p(x)dt - \lambda u(x)dt \\ &\quad + \lambda \int_0^{x+p(x)dt} u\big(x + p(x)dt - y\big)\, F\{dy\}\, dt + o(dt). \end{aligned}$$

Subtracting $u(x)$ and then formally dividing by dt yields

$$0 = p(x)u'(x) - \lambda u\big(x + p(x)dt\big) + \lambda \textstyle\int_0^{x+p(x)dt} u(x + p(x)dt - y)\, F\{dy\} + \frac{o(dt)}{dt}.$$

Finally, letting $dt \to 0$ gives

$$p(x)u'(x) = \lambda u(x) - \lambda \textstyle\int_0^x u(x - y)\, F\{dy\} \qquad (x \geqslant 0). \tag{2.2.5}$$

The identity (2.2.5) is called the ***classical risk equation***, and the boundary value problem

$$\begin{cases} p(x)u'(x) = \lambda u(x) - \lambda \int_0^x u(x - y)\, F\{dy\} & (x > 0) \\ u(\infty) = 1. \end{cases}$$

is the ***classical risk problem***.

2.2.2 The rescaled risk equation

For this thesis, we are interested in cases for which insurer transactions involve large sums of money. So we introduce a gauge parameter $\epsilon > 0$ into the reserve process and premium rate. Following Peters and Mangel [34], we rescale the reserve process according to

$$X_\epsilon(t, \epsilon x) := X(t, x) \quad \text{for small} \quad \epsilon > 0.$$

We denote the corresponding probability of ultimate survival

$$u_\epsilon(x) := \mathbb{P}[X_\epsilon(t, x) > 0 \text{ for all } t > 0]. \tag{2.2.6}$$

Hence

$$u_\epsilon(x) \;=\; \mathbb{P}[X_\epsilon(t, x) > 0 \text{ for all } t > 0] \;=\; \mathbb{P}[X\big(t, \tfrac{x}{\epsilon}\big) > 0 \text{ for all } t > 0] \;=\; u\big(\tfrac{x}{\epsilon}\big).$$

Consequently, (2.2.5) implies

$$\begin{aligned} u'_\epsilon(x) \;&=\; \tfrac{1}{\epsilon} u'\big(\tfrac{x}{\epsilon}\big) \;=\; \tfrac{\lambda}{\epsilon p\left(\frac{x}{\epsilon}\right)} \left(u\big(\tfrac{x}{\epsilon}\big) - \textstyle\int_0^{x/\epsilon} u\big(\tfrac{x}{\epsilon} - y\big)\, F\{dy\} \right) \\ &=\; \tfrac{\lambda}{\epsilon p\left(\frac{x}{\epsilon}\right)} \left(u_\epsilon(x) - \textstyle\int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} \right); \end{aligned}$$

equivalently,

$$\epsilon p\big(\tfrac{x}{\epsilon}\big) u_\epsilon(x) = u_\epsilon(x) - \textstyle\int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\}. \tag{2.2.7}$$

Ultimately, we are interested in solutions as $\epsilon \to 0$.

Instead of defining $p_\epsilon(x) := p\big(\frac{x}{\epsilon}\big)$ in (2.2.6), we generalize the rescaling of the premium rate function in [34].

Hypothesis. We assume

$$p_\epsilon = p_\epsilon(x) \text{ converges uniformly to } p = p(x) \text{ as } \epsilon \to 0^+ \text{ on } \mathbb{R}_+.$$

We are now prepared to formulate the equation.

Definition. We call

$$\epsilon p_\epsilon(x) u'_\epsilon(x) = \lambda u_\epsilon(x) - \lambda \int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} \qquad (x > 0). \tag{2.2.8}$$

the ***rescaled risk equation***.

The corresponding boundary value problems are

$$\begin{cases} \epsilon p_\epsilon(x) u'_\epsilon(x) = \lambda u_\epsilon(x) - \lambda \int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} & (x > 0) \\ u_\epsilon(\infty) = 1 \end{cases} \tag{2.2.9}$$

and

$$\begin{cases} \epsilon p_\epsilon(x) u'_\epsilon(x) = \lambda u_\epsilon(x) - \lambda \int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} & (x > 0) \\ u_\epsilon(0) = u_0 \end{cases} \tag{2.2.10}$$

for initial condition $u_0 \in \mathbb{R}$.

Analyzing the rescaled risk equation and the boundary value problems, especially in the asymptotic limit as $\epsilon \to 0$, are major goals of this thesis.

Chapter 3

Classical Solutions of the Risk Equation

We begin our treatment of the rescaled risk problems (2.2.9) and (2.2.10) by proving the existence of differentiable solutions $u \in L^1_{loc}(\mathbb{R}_+; F)$ of (2.2.8). We refer to such solutions as *classical*. Next, we discuss properties of classical solutions. In the last section, we investigate the asymptotic behavior of solutions as $\epsilon \to 0$ and derive strict error bounds for first approximations.

3.1 Solving the risk equation

In this section, we classify and characterize solutions of the rescaled risk equation (2.2.8), determine conditions on F and p_ϵ under which solutions of (2.2.8) are unique, and prove that a solution of (2.2.8) has one more derivative than F. The section concludes with pointwise bounds on solutions; these are used to provide error bounds in the asymptotic analysis.

Definition. For any $f \in L^\infty(\mathbb{R}_+)$ and $g \in L^1_{loc}(\mathbb{R}_+)$ we denote the (modified) *convolution of f with g* by $f * g$, where

$$\big(f * g\big)(x) := \textstyle\int_0^x f(x-s)g(s)\,ds \qquad (x \geqslant 0).$$

The *n-fold convolution of f*, denoted f^{n*}, is defined by

$$f^{n*} = \begin{cases} f & (n = 1) \\ f^{(n-1)*} * f & (n \geqslant 2) \end{cases}.$$

This agrees with the usual notion of convolution on $\mathbb{R}$ if f and g are extended to be 0 on the interval $(-\infty, 0]$.

3.1.1 Introducing H_ϵ

In previous works addressing the risk equation, (see Peters and Mangel [34]) the domain of integration in the integro-term is taken to be $\mathbb{R}$, with the convention that u is zero on the negative real axis. In other words, instead of $\int u(x-y)\mathcal{H}(x-y)\,F\{dy\}$ the term is $\int u(x-y)\,F\{dy\}$. One of the main contributions of this thesis is showing that this reduction is unnecessary. To glean more information about the rescaled risk equation (2.2.8), we first consider that (2.2.8) is equivalent to

$$p_\epsilon(x)u_\epsilon'(x) = \lambda\int_0^{x/\epsilon}\tfrac{u(x)-u_\epsilon(x-\epsilon y)}{\epsilon y}\,yF\{dy\} + \tfrac{\lambda u_\epsilon(x)}{\epsilon}\int_{x/\epsilon}^{\infty}F\{dy\}. \tag{3.1.1}$$

Sending $\epsilon\to 0$ motivates considering this scaled version of $1-F$, the tail distribution of Y.

Definition. For small $\epsilon>0$, let H_ϵ be the set function on $\mathcal{B}$ defined by

$$\epsilon H_\epsilon\{\epsilon B\} = \mathbb{P}\left[Y\notin B\right] \qquad (B\in\mathcal{B}). \tag{3.1.2}$$

As with the distribution F, we economize notation and use the letter H_ϵ to denote both the set function and the point function. This means

$$H_\epsilon(c) = \tfrac{1-F(\frac{c}{\epsilon})}{\epsilon} = \tfrac{1}{\epsilon}\int \mathbb{1}_{(c,\infty)}(\epsilon y)\,F\{dy\} \qquad (c\geqslant 0). \tag{3.1.3}$$

Let us verify that the expectation of Y is invariant with respect to the scaling introduced in (3.1.2). The case with $\epsilon=1$ is mentioned without proof in Grandell [20]. We record the calculation for general ϵ, below.

Lemma 3.1 (Mean preservation). *Let F be a probability distribution, $\epsilon>0$, and H_ϵ as in (3.1.2).*
Then

$$\int H_\epsilon(y)\,dy = \mathbb{E}[Y]. \tag{3.1.4}$$

Proof: Fix $\epsilon>0$. We compute

$$\begin{aligned}\textstyle\int H_\epsilon(y)\,dy &= \tfrac{1}{\epsilon}\int\left(\int\mathbb{1}_{(y,\infty)}(\epsilon y')F\{dy'\}\right)\mathbb{1}_{[0,\infty)}(y)dy\\ &= \tfrac{1}{\epsilon}\iint\mathbb{1}_{[0,\epsilon y')}(y)\mathbb{1}_{(0,\infty)}(y')\;F\{dy'\}dy\\ &= \tfrac{1}{\epsilon}\int\left(\int\mathbb{1}_{[0,\epsilon y')}(y)dy\right)\mathbb{1}_{(0,\infty)}(y')F\{dy'\}.\end{aligned}$$

Since the last expression is equal to $\int y'\,F\{dy'\}$, the proof is complete. □

Lemma 3.2 (Asymptotic behavior of H_ϵ). *For each $y>0$,*

$$H_\epsilon(y) = o(\epsilon^{-1}) \qquad \text{as} \quad \epsilon\to 0;$$

and for each $\epsilon > 0$,

$$H_\epsilon(y) = o(y^{-1}) \qquad \text{as} \quad y \to 0.$$

Proof: Fix $y > 0$. According to (3.1.2), $\frac{H_\epsilon(y)}{\epsilon^{-1}} = \int_{y/\epsilon}^\infty F\{dy'\}$. Thus

$$0 \leqslant \lim_{\epsilon\to 0} \tfrac{H_\epsilon(y)}{\epsilon^{-1}} \leqslant \lim_{\epsilon\to 0} \int_{y/\epsilon}^\infty F\{dy'\} = 0.$$

The last equality holds since $\int F\{dy'\}$ is finite. Therefore $H_\epsilon(y) = o(\epsilon^{-1})$. Similarly, since $\mathbb{E}[Y] = \int y' F\{y'\}$ is finite and

$$0 \leqslant yH_\epsilon(y) = \int_{y/\epsilon}^\infty \tfrac{y}{\epsilon} F\{dy'\} \leqslant \int_{y/\epsilon}^\infty y' F\{dy'\},$$

$yH_\epsilon(y) \to 0$ as $y \to \infty$. □

3.1.2 The convolution equation

We now present a formula for solutions of (2.2.8), under certain conditions. Computations derived in the proofs provide $L^\infty(\mathbb{R}_+)$ estimates for bounded solutions of the rescaled risk equation.

Transforming the risk equation

Motivation. Let us momentarily assume u_ϵ is a smooth solution of the rescaled risk equation (2.2.8). Fix $x > 0$. Then, integrating by parts and a substitution shows

$$\begin{aligned}\int_0^{x/\epsilon} u_\epsilon(x-\epsilon y)F\{dy\} &= u_\epsilon(0)F\left(\tfrac{x}{\epsilon}\right) + \int_0^x u'(x-s)F\left(\tfrac{s}{\epsilon}\right)\, ds \\ &= u_\epsilon(0)F\left(\tfrac{x}{\epsilon}\right) + \int_0^x u'_\epsilon(x-s)\Big(1-\epsilon H_\epsilon(s)\Big)\, ds \\ &= u(x) - \epsilon u(0)H_\epsilon(x) - (u' * \epsilon H_\epsilon)(x).\end{aligned}$$

Since u_ϵ solves (2.2.8),

$$\epsilon p_\epsilon(x)u'_\epsilon(x) = \lambda u_\epsilon(0)H_\epsilon(x) + \lambda\,(u' * \epsilon H_\epsilon)(x).$$

If $u_\epsilon(0) \neq 0$, we divide by $u_\epsilon(0)$ to see that

$$p_\epsilon(x)\tfrac{u'_\epsilon(x)}{u_\epsilon(0)} = \lambda H_\epsilon(x) + \Big(\tfrac{u'_\epsilon}{u_\epsilon(0)} * \lambda H_\epsilon\Big)(x) \qquad (x \geqslant 0).$$

Therefore, we seek a solution of the form

$$u_\epsilon(x) = u_\epsilon(0)\left(1 + \int_0^x v_\epsilon(s)\, ds\right),$$

where v_ϵ solves

$$p_\epsilon v_\epsilon = \lambda H_\epsilon + v_\epsilon * \lambda H_\epsilon. \tag{3.1.5}$$

These considerations motivate the following notation.

Definition. Fix $\epsilon > 0$. Let $T_\epsilon \colon L^1_{loc}(\mathbb{R}_+) \to L^1_{loc}(\mathbb{R}_+)$ be defined by

$$T_\epsilon f := \tfrac{f * \lambda H_\epsilon}{p_\epsilon}, \tag{3.1.6}$$

and let $S_\epsilon \colon L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ be defined by

$$S_\epsilon f = T_\epsilon f + \tfrac{\lambda H_\epsilon}{p_\epsilon}. \tag{3.1.7}$$

Lemma 3.3. *If*

$$\left\| \tfrac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1} < 1,$$

then T_ϵ is a strict contraction on $L^1(\mathbb{R}_+)$.

Proof: First, we show that

$$T_\epsilon \left(L^1_{loc}(\mathbb{R}_+) \right) \subset L^1_{loc}(\mathbb{R}_+). \tag{3.1.8}$$

Fix $f \in L^1_{loc}(\mathbb{R}_+)$ and $x > 0$. Then

$$\begin{aligned}
\int_0^x |T_\epsilon f(t)|\, dt = \int_0^x \tfrac{\lambda}{p_\epsilon(t)} \left| \int_0^t f(s) H_\epsilon(t-s)\, ds \right| dt &\leqslant \int_0^x \tfrac{\lambda}{p_\epsilon(t)} \int_0^t |f(s)|\, H_\epsilon(t-s)\, ds\, dt \\
&= \int_0^x |f(s)| \int_0^{x-s} \tfrac{\lambda H_\epsilon(t)}{p_\epsilon(t+s)}\, dt\, ds \\
&\leqslant \int_0^x |f(s)| \int_0^{x-s} \tfrac{\lambda H_\epsilon(t)}{p_\epsilon(t)}\, dt\, ds \\
&\leqslant \left\| \tfrac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1} \int_0^x |f(s)|\, dt\, ds.
\end{aligned}$$

Finally, assume $f \in L^1(\mathbb{R}_+)$. Then

$$\textstyle\int_0^\infty |T_\epsilon f(t)|\, dt \leqslant \left\| \tfrac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1} \int_0^\infty |f(s)|\, dt\, ds. \qquad \square$$

Existence for L^∞ solutions

Theorem 3.1. *Fix ϵ, $u_0 > 0$. There exists a (nontrivial) absolutely continuous solution $u_\epsilon \in L^\infty(\mathbb{R}_+)$ to the initial value problem (2.2.10), provided $\|\tfrac{\lambda H_\epsilon}{p_\epsilon}\|_{L^1} < 1$.*

Proof: First, we show the existence of a fixed point of S_ϵ. According to (3.2.1) and (3.1.7),

$$\|S_\epsilon v - S_\epsilon w\|_{L^1} = \|T_\epsilon (v - w)\|_{L^1} \qquad \left(v, w \in L^1(\mathbb{R}_+) \right).$$

Since $\|\frac{\lambda H_\epsilon}{p_\epsilon}\|_{L^1} < 1$, T_ϵ is a strict contraction on $L^1(\mathbb{R}_+)$. Thus, the Contraction Mapping Theorem (Martin [29]) implies the existence of a unique fixed point $v_\epsilon \in L^1(\mathbb{R}_+)$ of S_ϵ.

To complete the proof, let

$$u_\epsilon(x) := u_0\left(1 + \textstyle\int_0^x v_\epsilon(t)\,dt\right) \qquad (x \geqslant 0).$$

Then $u_\epsilon \in L^\infty(\mathbb{R}_+)$ is an absolutely continuous solution of (2.2.10). □

Corollary 3.2. *Assume $p_\epsilon(0) > \lambda\mathbb{E}[Y]$. Then there exists a (nontrivial) absolutely continuous solution $u_\epsilon \in L^\infty(\mathbb{R}_+)$ of (2.2.10).*

Proof: As p_ϵ is increasing,

$$\|\tfrac{\lambda H_\epsilon}{p_\epsilon}\|_{L^1} \leqslant \tfrac{\|\lambda H_\epsilon\|_{L^1}}{p_\epsilon(0)} = \tfrac{\lambda\mathbb{E}[Y]}{p_\epsilon(0)} < 1.$$ □

The following theorem justifies our restriction of initial values to positive constants in (2.2.10), and reduces the rescaled risk equation to a convolution equation.

Theorem 3.3. *Suppose $u \in C(\mathbb{R}_+)$ is differentiable.*

(i) *If $u_0 = 0$, then u_ϵ solves (2.2.8) if and only if $u_\epsilon \equiv 0$ on $\mathbb{R}_+$.*

(ii) *If $u_0 \neq 0$, then u_ϵ solves (2.2.8) if and only if there exists $v_\epsilon \in L^1_{loc}(\mathbb{R}_+)$ satisfying the **convolution equation***

$$p_\epsilon(x)v_\epsilon(x) = \lambda H_\epsilon(x) + \big(v_\epsilon * \lambda H_\epsilon\big)(x) \qquad (x \geqslant 0), \tag{3.1.9}$$

and

$$u_\epsilon(x) = u_\epsilon(0)\left(1 + \int_0^x v_\epsilon(s)\,ds\right) \qquad (x \geqslant 0). \tag{3.1.10}$$

Proof: 1. From Lemma B.7, u_ϵ solves (2.2.8) if and only if

$$\epsilon p_\epsilon(x)u_\epsilon'(x) = u_\epsilon(0)\lambda H_\epsilon(x) + \big(u_\epsilon' * \lambda H_\epsilon\big)(x) \qquad (x \geqslant 0). \tag{3.1.11}$$

Suppose $u_\epsilon(0) = 0$. Clearly, $u_\epsilon \equiv 0$ implies u_ϵ solves (2.2.8). Conversely, suppose u_ϵ solves (2.2.8). From (3.1.11), we see $p_\epsilon u_\epsilon' = u_\epsilon' * \lambda H_\epsilon$. Since p_ϵ is positive and nondecreasing and H_ϵ is nonnegative and nonincreasing, for any $x \geqslant 0$ we have

$$|u_\epsilon'(x)| \leqslant \tfrac{\lambda}{p_\epsilon(0)}\textstyle\int_0^x |u_\epsilon'(t)H_\epsilon(x-t)|\,dt \leqslant \tfrac{\lambda H_\epsilon(0)}{p_\epsilon(0)}\int_0^x |u_\epsilon'(t)|\,dt.$$

Thus, Lemma B.8 implies $|u_\epsilon'| \equiv 0$ on $[0,\infty)$. We deduce that $u_\epsilon \equiv 0$ on $[0,\infty)$.
2. Suppose $u_\epsilon(0) \neq 0$. In this case, (3.1.11) implies u_ϵ solves (2.2.8) if and only if

$$\epsilon p_\epsilon(x)\left(\tfrac{u_\epsilon}{u_\epsilon(0)}\right)'(x) = \lambda H_\epsilon(x) + \left(\tfrac{u_\epsilon}{u_\epsilon(0)}\right)' * \lambda H_\epsilon\,(x) \qquad (x \geqslant 0).$$

Clearly, $\frac{u_\epsilon}{u_\epsilon(0)} \in L^1_{loc}(\mathbb{R}_+)$. Thus, u_ϵ solves (2.2.8) if and only if (3.1.9) and (3.1.10) hold with $u_\epsilon(0)v_\epsilon = u_\epsilon$. □

Solving the convolution equation

In this section we solve the convolution equation (3.1.9). We conclude the section by proving a uniqueness result.

Motivation.(Constant premium rate) We formally derive a formula for the solution to (3.1.9) when the premium p_ϵ is constant. Momentarily, suppose $\epsilon = 1$. Let v, H, and p denote v_ϵ, H_ϵ, and p_ϵ, respectively. Assume v is a solution of (3.1.9). In this case $Tv = v * \frac{\lambda H}{p}$. This implies

$$\begin{aligned} v = Tv + \tfrac{\lambda H}{p} &= T\big(Tv + \tfrac{\lambda}{p}H\big) + \tfrac{\lambda}{p}H \\ &= T^2 v + \big(\tfrac{\lambda}{p}H\big)^{2*} + \tfrac{\lambda}{p}H \quad = T^2\big(Tv + \tfrac{\lambda}{p}H\big) + \big(\tfrac{\lambda}{p}H\big)^{2*} + \tfrac{\lambda}{p}H \\ &\qquad\qquad = T^3 v + \big(\tfrac{\lambda}{p}H\big)^{3*} + \big(\tfrac{\lambda}{p}H\big)^{2*} + \tfrac{\lambda}{p}H. \end{aligned}$$

By induction, we have

$$v = \Big(\tfrac{\lambda}{p}H\Big)^{n*} + T^n v \qquad (n \geqslant 1).$$

These computations suggest that *if* $\sum \big(\frac{\lambda}{p}H\big)^{j*}$ converges, *then* we have a representation formula for v. Let us now verify the validity of the above argument.

Notation. For $s \in \mathbb{R}$, let e_s denote the exponential function given by

$$\mathsf{e}_s(y) := e^{sy} \qquad (y \in \mathbb{R}).$$

Theorem 3.4 (Solving the convolution equation with constant premium). *Assume $p_\epsilon \equiv p$ is constant.*
Then

$$v_\epsilon(x) := \textstyle\sum_{n=1}^{\infty} \big(\frac{\lambda}{p}H_\epsilon\big)^{n*}(x) \qquad (x \geqslant 0) \tag{3.1.12}$$

converges pointwise on $\mathbb{R}_+$ and satisfies the convolution equation (3.1.9).

Proof: 1. For $\epsilon > 0$, choose $\theta = \theta(\epsilon) \geqslant 0$ satisfying

$$\textstyle\int \mathsf{e}_{-\epsilon\theta}(t) H_1(t)\, dt < \frac{p}{\lambda}.$$

According to Lemma B.4, such a θ exists. Also, using the definition of H_ϵ given by (3.1.2), we calculate

$$\textstyle\int \mathsf{e}_{-\epsilon\theta}(t) H_1(t)\, dt = \int \mathsf{e}_{-\theta}(t) H_\epsilon(t)\, dt. \tag{3.1.13}$$

Set

$$g_\epsilon := \tfrac{\lambda}{p}\mathsf{e}_{-\theta} H_\epsilon, \tag{3.1.14}$$

and note that

$$\|g_\epsilon\|_{L^1} = \tfrac{\lambda}{p}\textstyle\int \mathsf{e}_{-\theta}(t) H_\epsilon(t)\, dt < 1. \tag{3.1.15}$$

Fix $x \geqslant 0$ and let n be a non-negative integer. From Lemma B.6, we estimate

$$g_\epsilon^{n*} \leqslant g_\epsilon(0)\|g_\epsilon\|_{L^1}{}^{n-1}.$$

It follows from (3.1.15) and the Weierstrass M-test (see Remmert [37]) that

$$w_\epsilon(x) := \sum_{n=1}^{\infty} g_\epsilon^{n*}(x) \qquad (x \geqslant 0) \tag{3.1.16}$$

is uniformly convergent on $\mathbb{R}_+$ and

$$0 \leqslant w_\epsilon \leqslant \tfrac{g_\epsilon(0)}{1-\|g_\epsilon\|_{L^1}}. \tag{3.1.17}$$

2. Now we show

$$v_\epsilon := \mathsf{e}_\theta w_\epsilon$$

solves the convolution equation (3.1.9);

$$\mathsf{e}_\theta w_\epsilon = \tfrac{\lambda}{p}\Big(H_\epsilon + (\mathsf{e}_\theta w_\epsilon) * H_\epsilon\Big). \tag{3.1.18}$$

Equation (3.1.18) is equivalent to each of the following:

$$\begin{aligned}
\mathsf{e}_\theta \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*} &= \tfrac{\lambda}{p}H_\epsilon + \big(\mathsf{e}_\theta \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*}\big) * \tfrac{\lambda}{p}H_\epsilon && \text{by (3.1.16)}\\
\mathsf{e}_\theta \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*} &= \mathsf{e}_\theta g_\epsilon + \big(\mathsf{e}_\theta \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*}\big) * \mathsf{e}_\theta g_\epsilon && \text{by (3.1.14)}\\
\mathsf{e}_\theta \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*} &= \mathsf{e}_\theta g_\epsilon + \mathsf{e}_\theta \big(\textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*} * g_\epsilon\big) && \text{by Lemma B.5.}
\end{aligned}$$

Furthermore, according to the Monotone Convergence Theorem (see Royden [38]),

$$g_\epsilon * \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*} = \sum_{n=2}^{\infty} g_\epsilon^{n*}.$$

We deduce that (3.1.18) is equivalent to

$$\mathsf{e}_\theta \textstyle\sum_{n=1}^{\infty} g_\epsilon^{n*} \;=\; \mathsf{e}_\theta g_\epsilon + \mathsf{e}_\theta \sum_{n=2}^{\infty} g_\epsilon^{n*} \;=\; \mathsf{e}_\theta \sum_{n=1}^{\infty} g_\epsilon^{n*},$$

thereby completing the proof. □

As we will see, the reasoning used to prove this last theorem only needs to be tweaked to yield a proof for the case of nonconstant p_ϵ. We now solve the convolution equation in full generality.

Theorem 3.5 (Solving the convolution equation). *Define*

$$v_\epsilon(x) := \sum_{n=0}^{\infty} T_\epsilon^n\big(\tfrac{\lambda H_\epsilon}{p_\epsilon}\big)(x) \qquad (x \geqslant 0).$$

Then $v_\epsilon \in L^1_{loc}(\mathbb{R}_+)$ solves the convolution equation (3.1.9).

Proof: 1. Let g_ϵ denote the function

$$g_\epsilon(x) := \tfrac{\lambda H_\epsilon(x)}{p_\epsilon(x)} \qquad (x \geqslant 0).$$

We mention that g_ϵ satisfies the hypotheses of Lemma B.5. First, we claim

$$v_\epsilon \;=\; \textstyle\sum_{n=0}^{\infty} T_\epsilon^n g_\epsilon \;\in\; L^1_{loc}(\mathbb{R}_+). \tag{3.1.19}$$

To see this, suppose $n \geqslant 1$ and $\theta \in \mathbb{R}$. Then

$$0 \leqslant T_\epsilon^n g_\epsilon \leqslant g_\epsilon^{*(n+1)} = \mathsf{e}_\theta \left(\mathsf{e}_{-\theta} g_\epsilon\right)^{*(n+1)} \leqslant g_\epsilon(0) \|\mathsf{e}_{-\theta} g_\epsilon\|_{L^1}^n \mathsf{e}_\theta.$$

Choose θ such that $\|\mathsf{e}_{-\theta} g_\epsilon\|_{L^1} < 1$. Then

$$0 \leqslant \textstyle\sum_{n=1}^{\infty} T_\epsilon^n g_\epsilon \leqslant g_\epsilon(0) \dfrac{\|\mathsf{e}_{-\theta} g_\epsilon\|_{L^1}}{1 - \|\mathsf{e}_{-\theta} g_\epsilon\|_{L^1}} \mathsf{e}_\theta,$$

and

$$\textstyle\int_0^x \left(\sum_{n=1}^{\infty} T_\epsilon^n g_\epsilon\right)(s)\, ds \leqslant \dfrac{g_\epsilon(0)\|\mathsf{e}_{-\theta} g_\epsilon\|_{L^1}}{\theta\left(1 - \|\mathsf{e}_{-\theta} g_\epsilon\|_{L^1}\right)} \left(e^{\theta x} - 1\right).$$

This proves (3.1.19).
2. As $\{\sum_{k=1}^n T_\epsilon^k g_\epsilon\}_{n=1}^\infty \subset L^1_{loc}(\mathbb{R}_+)$ is uniformly convergent, we easily compute

$$T_\epsilon v_\epsilon = T_\epsilon \left(\textstyle\sum_{n=1}^{\infty} T_\epsilon^n g_\epsilon\right) = \sum_{n=1}^{\infty} \left(T_\epsilon T_\epsilon^n g_\epsilon\right) = \sum_{n=2}^{\infty} T_\epsilon^n g_\epsilon = v_\epsilon - g_\epsilon. \tag{3.1.20}$$

To conclude the proof, we substitute $\frac{\lambda H_\epsilon}{p_\epsilon} = g$ into (3.1.20). The result is (3.1.9). □

Essentially, the operator T_ϵ has only 0 as a fixed point because $\frac{H_\epsilon}{p_\epsilon}$ is nonnegative and nonincreasing. We use Gronwall's Inequality (Lemma B.8) to prove this.

Theorem 3.6 (Uniqueness). *There exists at most one absolutely continuous solution of the rescaled risk problem* (3.1.9).

Proof: According to Theorem 3.3, absolutely continuous solutions of (2.2.8) are determined by locally summable solutions of the convolution equation, up to initial value. So assume v_ϵ and w_ϵ solve (3.1.9), and set $z_\epsilon := v_\epsilon - w_\epsilon$. We claim

$$z_\epsilon \equiv 0.$$

Since v_ϵ and w_ϵ both solve (3.1.9), we have $p_\epsilon z_\epsilon = z_\epsilon * \lambda H_\epsilon$. Since H_ϵ and $\frac{1}{p_\epsilon}$ are non-negative and non-increasing,

$$|z_\epsilon(x)| \;\leqslant\; \tfrac{\lambda}{p_\epsilon(0)} \textstyle\int_0^x |z_\epsilon(t)| H_\epsilon(x-t)\, dt \;\leqslant\; \tfrac{\lambda H_\epsilon(0)}{p_\epsilon(0)} \int_0^x |z_\epsilon(t)|\, dt$$

for any $x \geqslant 0$. Thus Lemma B.8 implies $|z_\epsilon| \equiv 0$, and the claim is proved. □

3.2 Properties of classical solutions

In the last section, we derived a convolution equation and used it to prove the existence of unique classical solutions for (2.2.8). Presently, we show results on the

regularity and monotonicity of solutions of (2.2.8). We conclude this section by presenting pointwise estimates for solutions of (2.2.8).

For this section, we regard $\epsilon > 0$ as fixed.

3.2.1 Regularity

In order to streamline the presentation of results on regularity, we collect proofs of preliminary facts below.

Definition. The operator A_ϵ is given for $f \in (L^1_{loc}(\mathbb{R}_+); F)$ by

$$A_\epsilon f(x) := \int_0^{x/\epsilon} (f(x) - f(x - \epsilon y))\, F\{dy\} \qquad (x \geqslant 0)\,. \tag{3.2.1}$$

Lemma B.7 uses integration by parts to articulate the connection between A_ϵ and T_ϵ. We mention that the rescaled risk equation (2.2.8) can be expressed as

$$\epsilon p_\epsilon u_\epsilon' - \epsilon\lambda H_\epsilon u_\epsilon = \lambda A_\epsilon u_\epsilon.$$

Operator lemmas

The following lemma describes how the regularity of F determines the behavior of A_ϵ on $C(\mathbb{R}_+)$.

Lemma 3.4. *Suppose $u \in C(\mathbb{R}_+)$.*

(i) *If $u(x) = u(0)$, then $A_\epsilon u$ is continuous at x.*

(ii) *Otherwise, $A_\epsilon u$ is continuous at x exactly when F is continuous at $\frac{x}{\epsilon}$.*

Proof: 1. First we show that $A_\epsilon u$ is right continuous on $\mathbb{R}_+$. We claim that

$$\lim_{h \to 0^+} |A_\epsilon u(x+h) - A_\epsilon u(x)| = 0 \qquad (x \geqslant 0). \tag{3.2.2}$$

For $x \geqslant 0$ and small $h > 0$,

$$\begin{aligned} A_\epsilon u(x+h) - A_\epsilon u(x) = &\int \mathbb{1}_{(0,x]}(\epsilon y)(u(x+h) - u(x) - u(x+h-\epsilon y) + u(x-\epsilon y))\, F\{dy\} \\ &+ \int \mathbb{1}_{(x,x+h]}(\epsilon y)(u(x+h) - u(x+h-\epsilon y))\, F\{dy\}. \end{aligned}$$

Thus

$$\begin{aligned} |A_\epsilon u(x+h) - A_\epsilon u(x)| \leqslant\ & |u(x+h) - u(x)| F\left(\tfrac{x}{\epsilon}\right) \\ &+ \sup_{0 \leqslant z < x} |u(x+h) - u(z)| F\left(\tfrac{x}{\epsilon}\right) \\ &+ \sup_{0 \leqslant z < h} |u(z-\epsilon y) - u(z)| \left(F\left(\tfrac{x+h}{\epsilon}\right) - F\left(\tfrac{x}{\epsilon}\right)\right). \end{aligned} \tag{3.2.3}$$

Since u is locally uniformly continuous and F is bounded, the first two summands in (3.2.3) tend to 0 as h tends to 0. Furthermore, u is bounded near 0 and x, and F is right continuous at $\frac{x}{\epsilon}$. Thus, the last summand in (3.2.3) also tends to 0 as $h \to 0$. We conclude that (3.2.2) holds.
2. Suppose $u(x) \neq u(0)$. Since F is right continuous, F is continuous at $\frac{x}{\epsilon}$ exactly when F is left continuous at $\frac{x}{\epsilon}$:

$$\mathsf{D} := \left\{ x > 0 \mid F \text{ is not continuous at } \tfrac{x}{\epsilon} \right\}. \tag{3.2.4}$$

With this in mind, we characterize the conditions under which $A_\epsilon u$ is left continuous at $x > 0$. For small $h > 0$, consider

$$\begin{aligned} &A_\epsilon u(x-h) - A_\epsilon u(x) \\ &\quad = \textstyle\int \mathbb{1}_{(0,x-h]}(\epsilon y)(u(x-h) - u(x-h-\epsilon y) - u(x) + u(x-\epsilon y))\, F\{dy\} \\ &\quad - \textstyle\int \mathbb{1}_{(x-h,x]}(\epsilon y)(u(x) - u(x-\epsilon y))\, F\{dy\} \\ &\quad =: \mathsf{A}_1(h) - \mathsf{A}_2(h). \end{aligned}$$

We estimate

$$|\mathsf{A}_1(h)| \leqslant \sup_{h \leqslant z < x} |u(x-h) - u(z-h) - u(x) + u(z)| F\left(\tfrac{x-h}{\epsilon}\right).$$

Since u is locally uniformly continuous and F is bounded, $\lim_{h\to 0^+} \mathsf{A}_1(h) = 0$.

Now we analyze $\mathsf{A}_2(h)$. For small $h > 0$,

$$\mathbb{1}_{(x-h,x]}(\epsilon y)|u(x) - u(x-\epsilon y)| \leqslant \sup_{0 \leqslant z < h} |u(x) - u(z)|.$$

So the integrand of $\mathsf{A}_2(h)$ is uniformly bounded with respect to h. Therefore the Dominated Convergence Theorem (see Rudin [39]) implies

$$\begin{aligned} \lim_{h\to 0^+} \mathsf{A}_2(h) &= \textstyle\int \lim_{h\to 0^+} \mathbb{1}_{(x-h,x]}(\epsilon y)(u(x) - u(x-\epsilon y))\, F\{dy\} \\ &= \textstyle\int \mathbb{1}_{\{x\}}(\epsilon y)(u(x) - u(x-\epsilon y))\, F\{dy\} \\ &= (u(x) - u(0)) F\left\{\tfrac{x}{\epsilon}\right\}. \end{aligned}$$

We deduce that $\lim_{h\to 0^+} \mathsf{A}_2(h) = (u(x) - u(0))F\left\{\frac{x}{\epsilon}\right\}$. Consequently, $A_\epsilon u$ is left continuous at x exactly when F is left continuous at $\frac{x}{\epsilon}$. □

Remark. We see from the above proof that the *jump of* $A_\epsilon u$ *at* x is $(u(x)-u(0))F\left\{\frac{x}{\epsilon}\right\}$.

In the next lemma, we use the linearity of A_ϵ to classify conditions under which F is differentiable.

Lemma 3.5. *Assume F is continuous and suppose $u \in C^1(\mathbb{R}_+)$ is such that $u(x) \neq u(0)$ for all $x > 0$.*

(i) *$A_\epsilon u$ is differentiable at x exactly when F is differentiable at $\frac{x}{\epsilon}$.*

(ii) *If $A_\epsilon u$ is differentiable at x, then*

$$(A_\epsilon u)'(x) = \tfrac{u(x)-u(0)}{\epsilon} F'\big(\tfrac{x}{\epsilon}\big) + A_\epsilon u'(x). \tag{3.2.5}$$

Proof: 1. Fix $x > 0$ and small $h > 0$. Consider the difference quotient determining $(A_\epsilon u)'(x+)$:

$$\tfrac{A_\epsilon u(x+h)-A_\epsilon u(x)}{h} = \int_0^{x/\epsilon} \tfrac{u(x+h)-u(x+h-\epsilon y)-u(x)+u(x-\epsilon y)}{h}\, F\{dy\} + \int_{x/\epsilon}^{(x+h)/\epsilon} \tfrac{u(x+h)-u(x+h-\epsilon y)}{h}\, F\{dy\}$$

First we analyze

$$\mathsf{A}_1(h) := \int \tfrac{u(x+h)-u(x+h-\epsilon y)-u(x)+u(x-\epsilon y)}{h}\, \mathbb{1}_{(0,x]}(\epsilon y) F\{dy\}.$$

Since u' is locally bounded,

$$\sup_{0<\epsilon y\leqslant x} \tfrac{u(x+h)-u(x+h-\epsilon y)-u(x)+u(x-\epsilon y)}{h} = \tfrac{u(x+h)-u(x)}{h} + \sup_{0\leqslant z<x} \tfrac{u(z)-u(z+h)}{h} < \infty.$$

So, according to the Dominated Convergence Theorem, $\lim_{h\to 0} \mathsf{A}_1(h)$ exists and

$$\begin{aligned} \lim_{h\to 0} \mathsf{A}_1(h) &= \int \lim_{h\to 0} \tfrac{u(x+h)-u(x+h-\epsilon y)-u(x)+u(x-\epsilon y)}{h}\, \mathbb{1}_{(0,x]}(\epsilon y) F\{dy\}. \\ &= \int (u'(x) - u'(x-\epsilon y))\, \mathbb{1}_{(0,x]}(\epsilon y) F\{dy\} \\ &= A_\epsilon u'(x). \end{aligned} \tag{3.2.6}$$

Next we consider

$$\mathsf{A}_2(h) := \int \tfrac{u(x+h)-u(x)+u(0)-u(x+h-\epsilon y)}{h}\, \mathbb{1}_{(x,x+h]}(\epsilon y) F\{dy\}.$$

We claim that for all y,

$$\mathbb{1}_{(x,x+h]}(\epsilon y) \tfrac{|u(x+h-\epsilon y)-u(0)|}{h} \leqslant \tfrac{|u(x+h-\epsilon y)-u(0)|}{x+h-\epsilon y} \qquad (h > 0).$$

This is true because

$$\mathbb{1}_{(x,x+h]}(\epsilon y) \tfrac{|u(x+h-\epsilon y)-u(0)|}{h} = \mathbb{1}_{(x,x+h)}(\epsilon y) \tfrac{|u(x+h-\epsilon y)-u(0)|}{h}$$

and

$$\mathbb{1}_{(x,x+h)}(\epsilon y) \neq 0 \text{ implies } 0 < x + h - \epsilon y < h.$$

Therefore the integrand of $\mathsf{A}_2(h)$ is uniformly, with respect to the variable of integration, bounded by

$$\tfrac{|u(x+h)-u(x)|}{h} + \sup\{|u'(z)| \colon 0 < z < x\}.$$

Consequently,

$$|\mathsf{A}_2(h)| \leqslant \big(F\big(\tfrac{x+h}{\epsilon}\big) - F\big(\tfrac{x}{\epsilon}\big)\big) \left(\tfrac{|u(x+h)-u(x)|}{h} + \sup\{|u'(z)| \colon 0 < z < x\}\right).$$

Since F is left continuous and u' is bounded on $(0,x)$, we deduce that

$$\lim_{h\to 0} \mathsf{A}_2(h) = 0. \tag{3.2.7}$$

Lastly, let

$$\mathsf{A}_3(h) \;:=\; \int \tfrac{u(x)-u(0)}{h}\, \mathbb{1}_{(0,x]}(\epsilon y) F\{dy\} \;=\; \tfrac{u(x)-u(0)}{\epsilon} \tfrac{F\left(\frac{x+h}{\epsilon}\right)-F\left(\frac{x}{\epsilon}\right)}{\frac{h}{\epsilon}}.$$

So, if $u(x) \neq u(0)$, then $\mathsf{A}_3(h)$ converges as $h \to 0^+$ exactly when $F'\left(\frac{x}{\epsilon}+\right)$ exists. In that case,

$$\lim_{h\to 0} \mathsf{A}_3(h) = \tfrac{u(x)-u(0)}{\epsilon} F'\left(\tfrac{x}{\epsilon}+\right). \tag{3.2.8}$$

2. To conclude the proof, we consider the difference quotient determining $(A_\epsilon u)'(x-)$:

$$\tfrac{A_\epsilon u(x)-A_\epsilon u(x-h)}{h} = \int_0^{(x-h)/\epsilon} \tfrac{u(x)-u(x-\epsilon y)-u(x-h)+u(x-h-\epsilon y)}{h}\, F\{dy\} + \int_{(x-h)/\epsilon}^{x/\epsilon} \tfrac{u(x)-u(x-\epsilon y)}{h}\, F\{dy\}.$$

As before, we analyze each summand in turn.

Let

$$\mathsf{B}_1(h) := \int \tfrac{u(x)-u(x-\epsilon y)-u(x-h)+u(x-h-\epsilon y)}{h}\, \mathbb{1}_{(0,x-h]}(\epsilon y) F\{dy\}.$$

Then

$$|\mathsf{B}_1(h)| \leqslant \int \mathbb{1}_{(0,x)}(x-\epsilon y) \tfrac{|u(x)-u(x-\epsilon y)-u(x-h)+u(x-\epsilon y)|}{h}\, F\{dy\} \qquad (h>0).$$

Since u' is locally bounded on $\mathbb{R}_+$, the Dominated Convergence Theorem implies

$$\lim_{h\to 0^+} \mathsf{B}_1(h) \;=\; \int \mathbb{1}_{(0,x]}(\epsilon y)(u'(x)-u'(x-\epsilon y))\, F\{dy\} \;=\; A_\epsilon u'(x). \tag{3.2.9}$$

Next we estimate $\mathsf{B}_2(h) := \int \tfrac{u(0)-u(x-\epsilon y)}{h}\, \mathbb{1}_{(x-h,x]}(\epsilon y) F\{dy\}$. We have

$$\mathbb{1}_{(x-h,x]}(\epsilon y) \tfrac{|u(0)-u(x-\epsilon y)|}{h} \leqslant \tfrac{|u(0)-u(x-\epsilon y)|}{x-\epsilon y} \qquad (h>0, y \geqslant 0),$$

using reasoning similar to that used for A_2. Therefore the integrand of B_2 is uniformly, with respect to the variable of integration, bounded by $\sup\{|u'(y)|\colon 0<y<x\}$. Furthermore, $\mathsf{B}_2(h)$ is bounded by $\left(F\left(\frac{x}{\epsilon}\right) - F\left(\frac{x-h}{\epsilon}\right)\right) \sup\{|u'(z)|\colon 0<z<x\}$. Since F is continuous and u' is locally bounded,

$$\lim_{h\to 0^+} \mathsf{B}_2(h) = 0. \tag{3.2.10}$$

Finally, for $u(x) \neq u(0)$, we define $\mathsf{B}_3(h) := (u(x)-u(0)) \tfrac{F\left(\frac{x}{\epsilon}\right)-F\left(\frac{x-h}{\epsilon}\right)}{h}$. Then $\mathsf{B}_3(h)$ converges as $h \to 0^+$ exactly when $F'\left(\frac{x}{\epsilon}-\right)$ exists. In that case,

$$\lim_{h\to 0} \mathsf{B}_3(h) = \tfrac{u(x)-u(0)}{\epsilon} F'\left(\tfrac{x}{\epsilon}-\right). \tag{3.2.11}$$

3. From (3.2.6), (3.2.7), and (3.2.8), we have

$$\begin{cases} (A_\epsilon u)'(x+) \text{ exists if and only if } F'\left(\frac{x}{\epsilon}+\right) \text{ exists.} \\ \text{If } (A_\epsilon u)'(x+) \text{ exists, then } (A_\epsilon u)'(x+) = A_\epsilon u'(x) + \frac{u(x)-u(0)}{\epsilon} F'\left(\frac{x}{\epsilon}+\right). \end{cases} \tag{3.2.12}$$

From (3.2.9), (3.2.10), and (3.2.11), we have

$$\begin{cases} (A_\epsilon u)'(x-) \text{ exists if and only if } F'\left(\frac{x}{\epsilon}-\right) \text{ exists.} \\ \text{If } (A_\epsilon u)'(x-) \text{ exists, then } (A_\epsilon u)'(x-) = A_\epsilon u'(x) + \frac{u(x)-u(0)}{\epsilon} F'\left(\frac{x}{\epsilon}-\right). \end{cases} \tag{3.2.13}$$

Together, (3.2.13) and (3.2.12) imply (3.2.5). □

Main results

Equipped with the previous lemmas, we are prepared to present results on the regularity for solutions of (2.2.8).

Theorem 3.7. *If (2.2.8) has a classical solution, then F is continuous.*

Proof: Suppose u is a classical solution of (2.2.8), so that

$$u'(x) = \frac{\lambda}{\epsilon p_\epsilon(x)} \left(\epsilon H_\epsilon(x) + A_\epsilon u(x)\right).$$

According to Lemma 3.4, the jump of u' at x is $u(0)F\left\{\frac{x}{\epsilon}\right\} = u(0)\left(F\left(\frac{x}{\epsilon}\right) - F\left(\frac{x}{\epsilon}-\right)\right)$. Since derivatives cannot have jump discontinuities, F is continuous. □

Theorem 3.8 (Regularity). *Let k be a nonnegative integer, and assume $F \in C^k(\mathbb{R}_+)$. Suppose $u_\epsilon \in C(\mathbb{R}_+)$ is a classical solution of the rescaled risk equation. This means u_ϵ is differentiable on $(0,\infty)$ and satisfies (2.2.8).*
Then

$$u_\epsilon \in C^{k+1}\left(\mathbb{R}_+\right). \tag{3.2.14}$$

Proof: 1. Let u denote u_ϵ and D denote differentiation with respect to x. Define

$$\varphi(x) := \frac{x}{\epsilon}$$

and

$$\tilde{A}u(x) \;:=\; F * \varphi(x)u_\epsilon(x) - A_\epsilon u(x) \;=\; \int_0^{x/\epsilon} u(x-\epsilon y)\, F\{dy\}.$$

Then (2.2.8) is equivalent to

$$Du = \left(\varphi \circ \frac{\lambda}{p_\epsilon}\right)\left(u - \tilde{A}u\right),$$

and

$$D^n \tilde{A}u = \sum_{j=0}^{n-1} D^j u(0) D^{n-j}(F\circ\varphi) + \tilde{A}D^n u \tag{3.2.15}$$

for $n = 1, 2, \ldots$. Indeed,

$$D\tilde{A}u = u(0)D(F \circ \varphi) + \tilde{A}Du$$

from (3.2.5) of Lemma 3.5. Let us suppose (3.2.15) holds for $n = k$. Then

$$\begin{aligned} D^{k+1}\tilde{A}u \;=\; D[D^k\tilde{A}u] &= \sum_{j=0}^{k-1} D^j u(0) D^{k+1-j}(F \circ \varphi) + D\tilde{A}D^k u \\ &= \sum_{j=0}^{k-1} D^j u(0) D^{k+1-j}(F \circ \varphi) + D^k u(0) D(F \circ \varphi) + \tilde{A}DD^k u \\ &= \sum_{j=0}^{k} D^j u(0) D^{k+1-j}(F \circ \varphi) + \tilde{A}D^{k+1}u, \end{aligned}$$

so that (3.2.15) holds for $n = k + 1$. By induction, (3.2.15) holds for $n = 1, 2, \ldots$.
2. We will prove the Theorem by induction on k. The case $k = 0$ is Lemma 3.4. Fix a positive integer k and assume

$$u \in C^{k+1} \text{ whenever } F \in C^k \text{ and } u \text{ satisfies (2.2.8)}.$$

Now, suppose $F \in C^{k+1}$ and u satisfies (2.2.8). The induction hypothesis implies

$$u \in C^{k+1}; \tag{3.2.16}$$

therefore $D^{k+1}\tilde{A}u$ exists, with

$$D^{k+1}\tilde{A}u = \sum_{j=o}^{k+1} D^{k+1-j} u(0) D^j(F \circ \varphi) + \tilde{A}D^{k+1}u.$$

We claim $D^{k+1}\tilde{A}u$ is continuous. Since $D^{k+1}u$ and F are continuous, Lemma 3.4 implies $\tilde{A}D^{k+1}u$ is continuous. Also, $F \in C^{k+1}$ implies $D^{k+1}(F \circ \varphi)$ is continuous. Therefore the claim is true, i.e.

$$\tilde{A}u \in C^{k+1}. \tag{3.2.17}$$

Finally, since u satisfies the rescaled risk equation and $\frac{1}{p_\epsilon}$ is smooth, (3.2.16) and (3.2.17) imply $Du \in C^{k+1}$. □

3.2.2 Monotonicity

The next theorem implies solutions of (2.2.8) are monotonic. Since the derivation of the rescaled risk equation was motivated by a probability, it is reasonable to obtain nonnegative trajectories in the phase portrait of (2.2.8). Monotonicity coupled with a positive initial condition recovers the nonnegative trajectories.

Theorem 3.9. *Let $v_\epsilon \in L^1_{loc}(\mathbb{R}_+)$ be right-continuous. Suppose v_ϵ solves the convolution equation (3.1.9).*

Then

$$v_\epsilon > \tfrac{\lambda H_\epsilon}{p_\epsilon} \quad \textit{on} \quad \mathbb{R}_+. \tag{3.2.18}$$

Proof: 1. It suffices to show that

$$v_\epsilon(x) > 0 \qquad (x \geqslant 0).$$

Indeed, fix $\xi \geqslant 0$ and suppose $v_\epsilon > 0$ on $[0, \infty)$. Since H_ϵ cannot be identically zero,

$$\big(v_\epsilon * \lambda H_\epsilon\big)(\xi) = \lambda \textstyle\int_0^\xi v_\epsilon(t) H_\epsilon(\xi - t)\, dt \;>\; 0.$$

Since v_ϵ satisfies the convolution equation (3.1.9), we deduce that

$$v_\epsilon(\xi) = \tfrac{\lambda H_\epsilon(\xi) + (v_\epsilon * \lambda H_\epsilon)(\xi)}{p_\epsilon(\xi)} > \tfrac{\lambda H_\epsilon(\xi)}{p_\epsilon(\xi)}.$$

2. To prove that v_ϵ is positive on $\mathbb{R}_+$, it suffices to show that

$$\mathsf{O} := \{x \geqslant 0 \mid v_\epsilon > 0 \text{ on the interval } [0, x]\} \;=\; [0, \infty).$$

We argue that O is nonempty and clopen in $\mathbb{R}_+$. Evidently $\mathsf{O} \subset \mathbb{R}_+$, so the connectedness of $\mathbb{R}_+$ will yield the desired result.

First, we show that O is nonempty. Since v_ϵ satisfies (3.1.9), $v_\epsilon(0) = \frac{\lambda H_\epsilon}{p_\epsilon}(0)$. But $H_\epsilon(0) = \frac{1}{\epsilon}$ by definition (3.1.3). So $v_\epsilon(0) > 0$. This implies $0 \in \mathsf{O}$, thus confirming that O is nonempty.

3. Next, we show that O is open. Suppose $\xi \in \mathsf{O}$. Then $v_\epsilon(\xi) > 0$. Since v_ϵ is right continuous, $v_\epsilon > 0$ on $[\xi, \xi + \delta)$ for some positive δ. This, combined with the fact that $[0, \xi] \subset \mathsf{O}$ implies O is open.

Lastly, suppose $[0, \xi) \subset \mathsf{O}$. We claim that $\xi \in \mathsf{O}$. Since H_ϵ is right continuous and $H_\epsilon(0) > 0$,

$$H_\epsilon > 0 \quad \text{on} \quad [0, \delta) \quad \text{for some positive } \delta.$$

Moreover, for that δ, we also have $v_\epsilon > 0$ on $[\xi - \delta, \xi)$. Therefore,

$$\big(v_\epsilon * H_\epsilon\big)(\xi) \;=\; \textstyle\int_0^\xi v_\epsilon(\xi - t) H_\epsilon(t)\, dt \;\geqslant\; \int_0^\delta v_\epsilon(\xi - t) H_\epsilon(t)\, dt \;>\; 0.$$

Since p_ϵ is positive and v_ϵ satisfies (3.1.9), we deduce that

$$v_\epsilon(\xi) = \tfrac{\lambda H_\epsilon(\xi) + (v_\epsilon * \lambda H_\epsilon)(\xi)}{p_\epsilon(\xi)} \geqslant \tfrac{(v_\epsilon * \lambda H_\epsilon)(\xi)}{p_\epsilon(\xi)} > 0.$$

Since $\xi \in \mathsf{O}$ whenever $[0, \xi) \subset \mathsf{O}$, we conclude that O is closed, and the theorem is proved. □

Corollary 3.10 (Monotonicity). *Suppose $u_\epsilon \in C^1$ solves the rescaled risk equation (2.2.8) on $[0, \infty)$. Then,*

$$u_\epsilon' \geqslant 0 \qquad \textit{or} \qquad u_\epsilon' \leqslant 0. \tag{3.2.19}$$

Proof: According to (3.1.10), $u_\epsilon' = u_\epsilon(0) v_\epsilon$. Since $v_\epsilon > 0$, the sign of u_ϵ' is determined by that of $u_\epsilon(0)$. □

Corollary 3.10 verifies our intuition that increasing the initial reserve increases the probability of survival, and allows us to derive pointwise bounds on solutions to the rescaled risk equation.

3.2.3 Pointwise Estimates

The first theorem provides conditions on F and p_ϵ under which a bounded solution of (3.1.1) exists. The second theorem implies that solutions of (3.1.1) are unbounded unless $p_\epsilon > \lambda\mathbb{E}[Y]$.

Theorem 3.11. *Assume* $\left\| \frac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1} < 1$. *Then* $v_\epsilon \in L^1(\mathbb{R}_+)$ *with*

$$\|v_\epsilon\|_{L^1} \leqslant \frac{\left\| \frac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1}}{1 - \left\| \frac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1}}. \tag{3.2.20}$$

Moreover, $u \in L^\infty(\mathbb{R}_+)$ *with*

$$\|u_\epsilon\|_{L^\infty} \;=\; u_\epsilon(\infty) \;=\; \frac{u_\epsilon(0)}{1 - \left\| \frac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1}}. \tag{3.2.21}$$

Proof: To begin, we mention that (3.2.21) follows immediately from (3.2.20) and the convolution equation. So it suffices to prove the first claim.

According to the the convolution equation,

$$v_\epsilon \;=\; \frac{\lambda H_\epsilon}{p_\epsilon} + \frac{v_\epsilon * \lambda H_\epsilon}{p_\epsilon} \;\leqslant\; \frac{\lambda H_\epsilon}{p_\epsilon} + v_\epsilon * \frac{\lambda H_\epsilon}{p_\epsilon}.$$

Next, we integrate to obtain

$$\int_0^x v_\epsilon(t)\,dt \leqslant \int_0^x \frac{\lambda H_\epsilon}{p_\epsilon}(t)\,dt + \int_0^x v_\epsilon(t) \int_0^{x-t} \frac{\lambda H_\epsilon}{p_\epsilon}(s)\,ds\,dt.$$

Rearranging, we get

$$\int_0^x v_\epsilon(t) \left(1 - \int_0^{x-t} \frac{\lambda H_\epsilon}{p_\epsilon}(s)\,ds\right) dt \leqslant \int_0^x \frac{\lambda H_\epsilon}{p_\epsilon}(t)\,dt.$$

Finally, we use the hypothesis that $\left\| \frac{\lambda H_\epsilon}{p_\epsilon} \right\|_{L^1} < 1$ and the positivity of v_ϵ to conclude (3.2.20). □

Theorem 3.12 (Initial- and final-values). *Suppose* $p_\epsilon \in L^\infty(\mathbb{R}_+)$ *and* $v_\epsilon \in L^1(\mathbb{R}_+)$ *satisfies the convolution equation* (3.1.9).

(i) *Then*

$$p_\epsilon(\infty) > \lambda\mathbb{E}[Y]. \tag{3.2.22}$$

(ii) *If* $p_\epsilon \equiv p$ *is constant, then*

$$\|v_\epsilon\|_{L^1} = \frac{\lambda\mathbb{E}[Y]}{p - \lambda\mathbb{E}[Y]}. \tag{3.2.23}$$

Moreover, $u_\epsilon(\infty)$ exists and

$$u_\epsilon(\infty) = u_\epsilon(0)\left(1 - \tfrac{\lambda}{p}\mathbb{E}[Y]\right).$$

In particular, if $u_\epsilon(\infty) = 1$, then

$$u_\epsilon(0) = \left(1 - \tfrac{\lambda}{p}\mathbb{E}[Y]\right)^{-1}. \tag{3.2.24}$$

Proof: 1. First, we show that

$$\|v_\epsilon\|_{L^1}\left(p_\epsilon(\infty) - \lambda\mathbb{E}[Y]\right) \geqslant \lambda\mathbb{E}[Y], \tag{3.2.25}$$

which implies (3.2.22). Since $v_\epsilon \in L^1(\mathbb{R}_+)$ and v_ϵ satisfies (3.1.9),

$$p_\epsilon v_\epsilon \in L^1(\mathbb{R}_+),$$

with

$$\|p_\epsilon v_\epsilon\|_{L^1} = \|\lambda H_\epsilon\|_{L^1} + \|v_\epsilon * \lambda H_\epsilon\|_{L^1}.$$

Since p_ϵ is nondecreasing,

$$\|p_\epsilon v_\epsilon\|_{L^1} \leqslant p_\epsilon(\infty)\|v_\epsilon\|_{L^1}.$$

From Lemma B.3, we conclude

$$\|\lambda H_\epsilon\|_{L^1} + \|v_\epsilon\|_{L^1}\|\lambda H_\epsilon\|_{L^1} \leqslant p_\epsilon(\infty)\|v_\epsilon\|_{L^1}.$$

Equivalently,

$$\lambda\mathbb{E}[Y]\left(1 + \|v_\epsilon\|_{L^1}\right) \leqslant p_\epsilon(\infty)\|v_\epsilon\|_{L^1}.$$

This last inequality is equivalent to (3.2.25).

2. As $v_\epsilon \in L^1(\mathbb{R}_+)$ and v_ϵ satisfies (3.1.9),

$$\|v_\epsilon\|_{L^1} \leqslant \|\tfrac{\lambda H_\epsilon}{p}\|_{L^1} + \|v_\epsilon * \tfrac{\lambda H_\epsilon}{p}\|_{L^1}.$$

Also,

$$\begin{aligned} \|v_\epsilon * \tfrac{\lambda}{p}H_\epsilon\|_{L^1} &= \|v_\epsilon\|_{L^1}\|\tfrac{\lambda}{p}H_\epsilon\|_{L^1} && \text{by Lemma } B.3 \\ \|\tfrac{\lambda}{p}H_\epsilon\|_{L^1} &= \tfrac{\lambda}{p}\mathbb{E}[Y] && \text{by (3.1.4) of Lemma 3.1.} \end{aligned}$$

After the appropriate substitutions, we see that

$$\|v_\epsilon\|_{L^1} = \tfrac{\lambda}{p}\mathbb{E}[Y]\left(1 + \|v_\epsilon\|_{L^1}\right). \tag{3.2.26}$$

Therefore,

$$\tfrac{\|v_\epsilon\|_{L^1}}{1+\|v_\epsilon\|_{L^1}} = \tfrac{\lambda\mathbb{E}[Y]}{p} < 1.$$

Also, rearranging (3.2.26) leads to (3.2.23).

3. To prove the last claim, we mention that according to Theorem 3.3, $u_\epsilon(\infty)$ exists if and only if $v_\epsilon \in L^1(\mathbb{R}_+)$. In that case,

$$u_\epsilon(\infty) = u_\epsilon(0)\,(1 + \|v_\epsilon\|_{L^1}) = u_\epsilon(0)\left(1 + \left\|\tfrac{\lambda\mathbb{E}[Y]}{p-\lambda\mathbb{E}[Y]}\right\|_{L^1}\right),$$

with the last equality following from (3.2.23). Thus, the corollary is proved. □

Corollary 3.13. *Suppose $v_\epsilon \in L^1(\mathbb{R}_+)$ satisfies (3.1.10).*
Then $u_\epsilon \in L^\infty(\mathbb{R}_+)$ with

$$\|u_\epsilon\|_{L^\infty} = u_\epsilon(\infty) = \tfrac{u_\epsilon(0)}{1+\|v_\epsilon\|_{L^1}}, \tag{3.2.27}$$

and

$$\sup p_\epsilon > \lambda\mathbb{E}[Y]. \tag{3.2.28}$$

In other words, $p_\epsilon(\xi) > \lambda\mathbb{E}[Y]$ for some $\xi > 0$.

Proof: The conclusions $u_\epsilon \in L^\infty(\mathbb{R}_+)$ and (3.2.27) follow directly from (3.1.10). By hypothesis, $v_\epsilon \in L^1(\mathbb{R}_+)$ so $\lambda H_\epsilon + v_\epsilon * \lambda H_\epsilon \in L^1(\mathbb{R}_+)$ with

$$\|\lambda H_\epsilon + v_\epsilon * \lambda H_\epsilon\|_{L^1} = \lambda\mathbb{E}[Y](1 + \|v_\epsilon\|_{L^1}).$$

Therefore, according to the convolution equation, $p_\epsilon v_\epsilon \in L^1(\mathbb{R}_+)$ with

$$\|p_\epsilon v_\epsilon\|_{L^1} = \lambda\mathbb{E}[Y](1 + \|v_\epsilon\|_{L^1}) \;>\; \lambda\mathbb{E}[Y]\|v_\epsilon\|_{L^1}.$$

Notice that since p_ϵ is nondecreasing,

$$\lambda\mathbb{E}[Y]\|v_\epsilon\|_{L^1} \;<\; \|p_\epsilon v_\epsilon\|_{L^1} \;\leqslant\; \sup p_\epsilon\, \|v_\epsilon\|_{L^1}.$$

Therefore (3.2.28) holds. □

Corollary 3.14. *Assume $v_\epsilon \in L^1_{loc}(\mathbb{R}_+)$ solves (3.1.9) and $v_\epsilon(\infty) \neq 0$ exists.*
Then

$$p_\epsilon \in L^\infty(\mathbb{R}_+) \qquad \textit{with} \qquad p_\epsilon(\infty) = \lambda\mathbb{E}[Y].$$

Proof: From Lemma B.2, $\big(v_\epsilon * H_\epsilon\big)(\infty)$ exists and is $v_\epsilon(\infty)\mathbb{E}[Y]$. Also, since p_ϵ is non-decreasing and positive, $\frac{\lambda}{p_\epsilon}(\infty)$ exists. Since v_ϵ solves (3.1.9), we compute

$$v_\epsilon(\infty) = \left(\tfrac{\lambda H_\epsilon}{p_\epsilon} + \tfrac{v_\epsilon * \lambda H_\epsilon}{p_\epsilon}\right)(\infty) \;=\; \tfrac{\lambda}{p_\epsilon}(\infty)\, H_\epsilon(\infty) + \tfrac{\lambda}{p_\epsilon}(\infty)\, v_\epsilon(\infty)\mathbb{E}[Y].$$

Rearranging yields

$$v_\epsilon(\infty)\left(1 - \mathbb{E}[Y]\tfrac{\lambda}{p_\epsilon(\infty)}\right) = 0,$$

and the theorem is proved. □

For future reference, we collect estimates for v_ϵ which resulted from the proof of existence.

Theorem 3.15 (Pointwise estimates of v_ϵ). *Suppose $v_\epsilon \in L^1_{loc}(\mathbb{R}_+)$ is a solution of the convolution equation (3.1.9).*
Then

$$\frac{\lambda H_\epsilon(x)}{p_\epsilon(x)} \leqslant \frac{u_\epsilon'(x)}{u_\epsilon(x)} \leqslant \frac{\lambda}{\epsilon p_\epsilon(x)} \qquad (x > 0). \tag{3.2.29}$$

Proof: Fix $x > 0$. Since H_ϵ is non-increasing and v_ϵ is positive,

$$H_\epsilon(x) \int_0^x v_\epsilon(t)\,dt \leqslant \int_0^x v_\epsilon(x-t) H_\epsilon(t)\,dt \leqslant H_\epsilon(0) \int_0^x v_\epsilon(t)\,dt.$$

Since p_ϵ is positive,

$$\frac{\lambda H_\epsilon(x)}{p_\epsilon(x)} \left(1 + \int_0^x v_\epsilon(t)\,dt\right) \leqslant \frac{\lambda H_\epsilon(x) + v_\epsilon * \lambda H_\epsilon(x)}{p_\epsilon(x)} \leqslant \frac{\lambda H_\epsilon(0)}{p_\epsilon(x)} \left(1 + \int_0^x v_\epsilon(t)\,dt\right).$$

Lastly, we use the fact that $\epsilon H_\epsilon(0) = 1$ and Theorem 3.3 to complete the proof. □

As a straightforward consequence of this theorem and Lemma B.8, we have the following

Corollary 3.16 (Pointwise estimates of u_ϵ). *Suppose $u_\epsilon \in C^1(\mathbb{R}_+)$ is a nonnegative solution of (2.2.8). Then*

$$u_\epsilon(0) e^{\int_0^x \frac{\lambda H_\epsilon}{p_\epsilon}(t)\,dt} \leqslant u_\epsilon(x) \leqslant u_\epsilon(0) e^{\int_0^x \frac{\lambda}{\epsilon p_\epsilon}(t)\,dt}. \tag{3.2.30}$$

Chapter 4

Examples

To motivate later developments, we now present results for some specific distributions of Y. They are the discrete and gamma distributions.

Certain quantities involving p_ϵ and F are distinctive enough to warrant introducing the following

Notation. For $\epsilon > 0$, we define

$$q_\epsilon(x) := \tfrac{\lambda}{\epsilon} \textstyle\int_0^x \frac{ds}{p_\epsilon(s)} \qquad (x \geqslant 0).$$

If $p_\epsilon \equiv p$ is constant, we define

$$a := \tfrac{\lambda}{p} \quad \text{and} \quad b := \tfrac{1}{\mathbb{E}[Y]}.$$

We mention that whenever $p_\epsilon \equiv p$, we have $q_\epsilon(x) \equiv \frac{\lambda}{p}\frac{x}{\epsilon}$. This implies $a = \epsilon q_\epsilon'(0)$. In this sense, a captures the initial rate of growth of the (unscaled) premium function $p = p(x)$.

We assume for this chapter $u_0 > 0$.

4.1 A singular distribution

Theorem 4.1. *Assume $\{x_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ are sequences of positive numbers such that $\{x_n\}_{n=1}^\infty$ is increasing and $\sum_n \theta_n = 1$. Let F be given by*

$$F = \textstyle\sum_n \theta_n \delta_{x_n},$$

δ_{x_n} denoting the Dirac measure on $\mathbb{R}$ giving unit mass to the point x_n. Suppose u_ϵ is a classical solution of (2.2.8). Then $u_\epsilon \equiv 0$.

In this case, (2.2.8) is

$$\epsilon p_\epsilon(x) u_\epsilon'(x) - \lambda u_\epsilon(x) = -\lambda \sum_n \theta_n u_\epsilon(x - \epsilon x_n)\mathcal{H}(x - \epsilon x_n) \qquad (x > 0). \tag{4.1.1}$$

Proof: 1. First, we argue that $u_\epsilon(x) \equiv 0$ on $[0, \epsilon x_1]$. Consider the right hand side of (4.1.1) near the point ϵx_1:

$$-\lambda\theta_1 u_\epsilon(0) = \lim_{x\to\epsilon x_1^+} \lambda \sum_n \theta_n u_\epsilon(x - \epsilon x_n)\mathcal{H}(x - \epsilon x_n),$$

and

$$0 = \lim_{x\to\epsilon x_1^-} \lambda \sum_n \theta_n u_\epsilon(x - \epsilon x_n)\mathcal{H}(x - \epsilon x_n).$$

According to the intermediate value property (Rudin [39]) these limits must be equal. In other words, $u_\epsilon(0) = 0$. Since u_ϵ solves (4.1.1), $\epsilon p_\epsilon u_\epsilon' - \lambda u_\epsilon = 0$ on the interval $(0, \epsilon x_1)$. Therefore

$$u_\epsilon(x) \equiv 0 \qquad (0 \leqslant x \leqslant \epsilon x_1).$$

2. Now let

$$v(x) := u_\epsilon(x + \epsilon x_1) \qquad (x \geqslant 0).$$

It is straightforward to verify that v satisfies (4.1.1), with $F = \sum_{n=2} \theta_n \delta_{x_n}$. By the previous argument, we deduce that $v \equiv 0$ on $[0, \epsilon x_2]$. Equivalently, $u_\epsilon \equiv 0$ on $[0, \epsilon x_1 + \epsilon x_2]$. Iterating, we see that for any $n \geqslant 1$,

$$u_\epsilon \equiv 0 \qquad \text{on} \qquad [0, \epsilon x_1 + \epsilon x_2 + \cdots + \epsilon x_n].$$

To conclude, we notice that $\sum_n x_n = \infty$, by hypothesis. Therefore $u_\epsilon(x) = 0$ for any $x \geqslant 0$. □

All is not lost; the next theorem exhibits a continuous u_ϵ satisfying (4.1.1) at almost every point, in the case where F is atomic.

Theorem 4.2. *There exists $u_\epsilon \in C(\mathbb{R}_+)$ which is differentiable almost everywhere (with respect to Lebesgue measure) on $\mathbb{R}_+$ and satisfies*

$$\epsilon p_\epsilon(x) u_\epsilon'(x) - \lambda u_\epsilon(x) = -\lambda u_\epsilon\big(x - \tfrac{\epsilon}{b}\big)\mathcal{H}\big(x - \tfrac{\epsilon}{b}\big) \tag{4.1.2}$$

on the domain of u_ϵ'.

Proof: 1. We may assume $b = 1$ by, if needed, assuming $\epsilon_1 := b\epsilon$. To prove the theorem, it suffices to recursively define a continuous function u_ϵ such that u_ϵ' exists on $\mathbb{R}_+ \backslash \{0, \epsilon, 2\epsilon, \ldots\}$, and

$$\epsilon p_\epsilon(x) u_\epsilon'(x) - \lambda u_\epsilon(x) = -\lambda u_\epsilon(x - \epsilon)\mathcal{H}(x - \epsilon) \tag{4.1.3}$$

wherever u_ϵ' exists.
2. For $n = 0, 1, 2, \ldots$, define p_n and q_n by

$$\begin{cases} p_n(t) := p_\epsilon(t + n\epsilon) & (0 < t \leqslant \epsilon) \\ q_n(t) := \int_0^t \frac{\lambda ds}{\epsilon p_n(s)} & (0 < t \leqslant \epsilon) \end{cases}.$$

Let

$$u_0(t) := u_\epsilon(0) e^{q_\epsilon(t)},$$

and for $n = 1, 2, \ldots$, define u_n by

$$u_{n+1}(t) := e^{q_{n+1}(t)} u_n(\epsilon) - \int_0^t q'_{n+1}(s) e^{q_{n+1}(s)} u_n(s)\, ds \qquad (t \geqslant 0).$$

Then u_ϵ defined by

$$u_\epsilon(t + n\epsilon) := u_n(t) \qquad (0 \leqslant t < \epsilon, n = 0, 1, \ldots)$$

satisfies the desired properties. □

4.2 An absolutely continuous distribution

We assume Y has a gamma distribution with shape parameter γ and scale parameter β. According to Lemma B.10

$$b = \tfrac{1}{\beta\gamma},$$

and for fixed $\gamma > 0$,

$$F(y) = \tfrac{(\gamma b)^\gamma}{\Gamma(\gamma)} \int_0^y t^{\gamma-1} e^{-\gamma b t}\, dt \qquad (y \geqslant 0). \tag{4.2.1}$$

4.2.1 Exponential distribution, $\gamma = 1$

We start by considering the case with standard gamma distribution and $\gamma = 1$, the exponential distribution. Now (4.2.1) tells us $F(y) = 1 - e^{-by}$ and therefore,

$$H_\epsilon(y) = \tfrac{1}{\epsilon} e^{-\frac{by}{\epsilon}} \qquad (y \geqslant 0).$$

Thus, we may explicitly solve the rescaled risk equation by converting the convolution equation into an ordinary differential equation of first order.

Theorem 4.3. *Suppose $u_\epsilon \in C^1(\mathbb{R}_+)$ solves the exponentially distributed rescaled risk equation (2.2.8).*
Then

$$u_\epsilon(x) = u_\epsilon(0) \left[e^{-\frac{x}{\epsilon\mathbb{E}[Y]} + \int_0^x \frac{\lambda ds}{\epsilon p_\epsilon(s)}} + \int_0^{\epsilon\mathbb{E}[Y]x} \exp\left(\int_0^{\epsilon\mathbb{E}[Y]t} \frac{\lambda ds}{\epsilon p_\epsilon(s)} - t \right) dt \right] \qquad (x \geqslant 0).$$

For reference, we express the solution in terms of b and q_ϵ:

$$u_\epsilon(x) = u_\epsilon(0) \left(e^{q_\epsilon(x) - \frac{bx}{\epsilon}} + \tfrac{b}{\epsilon} \int_0^x e^{q_\epsilon(t) - \frac{bt}{\epsilon}}\, dt \right) \qquad (x \geqslant 0). \tag{4.2.2}$$

Proof: According to (3.1.9) of Theorem 3.3, for $x > 0$,

$$v_\epsilon(x) \;=\; \tfrac{\lambda}{p_\epsilon(x)} H_\epsilon(x) + \tfrac{\lambda}{p_\epsilon(x)} \big(v_\epsilon * H_\epsilon\big)(x) \;=\; q'_\epsilon(x) e^{-\frac{bx}{\epsilon}} + \int_0^x v(t) e^{-\frac{b(x-t)}{\epsilon}}\, dt.$$

Therefore,

$$v_\epsilon(x)e^{\frac{bx}{\epsilon}} - q'_\epsilon(x)\left(1 + \int_0^x v(t)e^{\frac{bt}{\epsilon}}\,dt\right) = 0.$$

Noting that the left-hand side of the equation is $\frac{d}{dx}\left[e^{-q_\epsilon(x)}\left(1 + \int_0^x v(t)e^{\frac{bt}{\epsilon}}\,dt\right)\right]$, we deduce that

$$1 + \int_0^x v_\epsilon(t)e^{\frac{bt}{\epsilon}}\,dt = e^{q_\epsilon(x)}.$$

Differentiating and rearranging, we calculate

$$v_\epsilon(x) = q'_\epsilon(x)e^{q_\epsilon(x)-\frac{b}{\epsilon}x}. \tag{4.2.3}$$

Combining (3.1.10) of Theorem 3.3 with (4.2.3) completes the proof:

$$u_\epsilon(x) = u_\epsilon(0)\left(1 + \int_0^x q'_\epsilon(t)e^{q_\epsilon(t)-\frac{bt}{\epsilon}}\,dt\right) = u_\epsilon(0)\left(e^{q_\epsilon(x)-\frac{bx}{\epsilon}} + \tfrac{b}{\epsilon}\int_0^x e^{q_\epsilon(t)-\frac{bt}{\epsilon}}\,dt\right). \quad \square$$

A solution u_ϵ of the exponentially distributed (2.2.8) is convex on $[x_0, \infty)$ whenever $p_\epsilon(x_0) > \lambda\mathbb{E}[Y]$. As a result, we can provide an alternative proof of Corollary 3.12 resulting in an estimate of L^∞-norm of u_ϵ.

Theorem 4.4 ($L^\infty(\mathbb{R}_+)$ solutions). *Suppose $u_\epsilon \in C^1(\mathbb{R}_+)$ solves the exponentially distributed rescaled risk equation (2.2.8).*
Then

$$u_\epsilon \in L^\infty(\mathbb{R}_+) \qquad \textit{exactly when} \qquad p_\epsilon(\bar{x}) > \lambda\mathbb{E}[Y] \ \textit{ for some } \ \bar{x} \geqslant 0. \tag{4.2.4}$$

Proof: 1. Suppose $p_\epsilon \leqslant \lambda\mathbb{E}[Y]$. Then for each $x \in \mathbb{R}_+$,

$$q_\epsilon(x) - \tfrac{x}{\epsilon\mathbb{E}[Y]} = \tfrac{1}{\epsilon\mathbb{E}[Y]}\int_0^x \left(\tfrac{\lambda\mathbb{E}[Y]}{p_\epsilon(t)} - 1\right) dt \quad \geqslant \quad 0,$$

so that

$$u_\epsilon(x) \;=\; u_\epsilon(0)\left(e^{q_\epsilon(x)-\frac{x}{\epsilon\mathbb{E}[Y]}} + \tfrac{1}{\epsilon\mathbb{E}[Y]}\int_0^x e^{q_\epsilon(t)-\frac{t}{\epsilon\mathbb{E}[Y]}}\,dt\right) \quad \geqslant \quad u_\epsilon(0)\left(1 + \tfrac{x}{\epsilon\mathbb{E}[Y]}\right).$$

Clearly $u_\epsilon \notin L^\infty(\mathbb{R}_+)$.
2. Suppose there exists $\bar{x} \in \mathbb{R}_+$ such that

$$\triangle := b - \tfrac{\lambda}{p_\epsilon(\bar{x})} > 0. \tag{4.2.5}$$

Then, since p_ϵ is increasing,

$$0 \leqslant q'_\epsilon(\bar{x} + t) \leqslant q'_\epsilon(\bar{x}) \qquad (t \geqslant 0). \tag{4.2.6}$$

Moreover,

$$q_\epsilon(\bar{x} + t) - \tfrac{b}{\epsilon}(\bar{x} + t) \;\leqslant\; q_\epsilon(\bar{x}) - \tfrac{b}{\epsilon}\bar{x} - \tfrac{\triangle}{\epsilon}x, \tag{4.2.7}$$

because

$$\begin{aligned} q_\epsilon(\bar{x}+t) &\leqslant q_\epsilon(\bar{x}) + q'_\epsilon(\bar{x})x && \text{since } q_\epsilon \text{ is convex,} \\ &= q_\epsilon(\bar{x}) + \tfrac{\lambda}{\epsilon p_\epsilon(\bar{x})}t \;=\; q_\epsilon(\bar{x}) + \left(\tfrac{b}{\epsilon} - \tfrac{\triangle}{\epsilon}\right)t && \text{by (4.2.5).} \end{aligned}$$

Thus

$$\begin{aligned} v_\epsilon(\bar{x}+t) &= q'_\epsilon(\bar{x}+t)e^{q_\epsilon(\bar{x}+t)-\frac{b}{\epsilon}(\bar{x}+t)} && \text{by (4.2.3),} \\ &\leqslant q'_\epsilon(\bar{x})e^{q_\epsilon(\bar{x}+t)-\frac{b}{\epsilon}(\bar{x}+t)} && \text{by (4.2.6),} \\ &\leqslant q'_\epsilon(\bar{x})e^{q_\epsilon(\bar{x})-\frac{b}{\epsilon}\bar{x}-\frac{\triangle}{\epsilon}t} \;=\; v_\epsilon(\bar{x})e^{-\frac{\triangle}{\epsilon}t} && \text{by (4.2.7).} \end{aligned}$$

So we deduce that for $x \geqslant 0$,

$$\begin{aligned} u_\epsilon(\bar{x}+x) - u_\epsilon(\bar{x}) \;=\; u_\epsilon(0)\textstyle\int_0^x v_\epsilon(\bar{x}+t)\,dt \;&\leqslant\; u_\epsilon(0)v_\epsilon(\bar{x})\textstyle\int_0^x e^{-\frac{\triangle}{\epsilon}t}\,dt \\ &= \tfrac{\epsilon}{\triangle}u'_\epsilon(\bar{x})\left(1-e^{-\frac{\triangle}{\epsilon}x}\right). \end{aligned}$$

Evidently, $u_\epsilon \in L^\infty(\mathbb{R}_+)$.

3. Furthermore,

$$\tfrac{\epsilon}{\triangle}u'_\epsilon(\bar{x}) \;=\; \tfrac{\epsilon p_\epsilon(\bar{x})u'_\epsilon(\bar{x})}{\triangle p_\epsilon(\bar{x})} \;<\; \tfrac{\lambda u_\epsilon(\bar{x})}{\triangle p_\epsilon(\bar{x})}.$$

As a result,

$$u_\epsilon(\bar{x}+x) \;\leqslant\; u_\epsilon(\bar{x})\left(1+\tfrac{\lambda}{\triangle p_\epsilon(\bar{x})}\left(-e^{-\frac{\triangle x}{\epsilon}}\right)\right) \;=\; u_\epsilon(\bar{x})\left(1+\tfrac{\lambda}{\triangle p_\epsilon(\bar{x})} - \tfrac{\lambda}{\triangle p_\epsilon(\bar{x})}e^{-\frac{\triangle x}{\epsilon}}\right).$$

Thus

$$\|u_\epsilon\|_{L^\infty} \leqslant u_\epsilon(\bar{x})\tfrac{p_\epsilon(\bar{x})}{p_\epsilon(\bar{x})-\lambda\mathbb{E}[Y]}.$$ □

Example 1 (Exponential distribution with constant premium)**.**
This example is addressed in [19] and [9].
Suppose $p_\epsilon(x) \equiv p$. Then $q_\epsilon(x) = \frac{ax}{\epsilon}$, so

$$u_\epsilon(x) = u_\epsilon(0)\left(e^{\frac{(a-b)x}{\epsilon}} + \tfrac{b}{\epsilon}\textstyle\int_0^x e^{\frac{(a-b)t}{\epsilon}}\,dt\right).$$

The solution depends on a and b:

$$u_\epsilon(x) = \begin{cases} u_\epsilon(0)\left(1+\frac{bx}{\epsilon}\right) & (a=b) \\ u_\epsilon(0)\frac{ae^{\frac{a-b}{\epsilon}x}-b}{a-b} & (a\neq b) \end{cases}.$$

Further, suppose $u_\epsilon(\infty) = 1$. According to Corollary 3.12, $b > a$ and $u_\epsilon(0) = 1 - \frac{a}{b}$. Therefore,

$$u_\epsilon(x) = 1 - \tfrac{a}{b}e^{\frac{(a-b)x}{\epsilon}} \qquad (x \geqslant 0).$$ □

4.2.2 Gamma distribution with constant premium

We apply the method of Laplace transforms to obtain a formula for u_ϵ. See Appendix B.3 for relevant notation and terminology.

Theorem 4.5. *Let $p_\epsilon \equiv p$ be constant and assume Y has a gamma distribution with shape parameter γ and scale parameter β. Suppose $u_\epsilon \in L^\infty(\mathbb{R}_+)$ solves the rescaled risk equation (2.2.8) and $u_\epsilon(\infty) = 1$.*
Then

$$\mathcal{L}u_\epsilon(s) = \epsilon\beta\gamma u_\epsilon(0)\frac{\tau^\gamma}{\tau^{\gamma+1} - \left(1 + \frac{\lambda\beta}{p}\right)\tau^\gamma + \frac{\lambda\beta}{p}}, \tag{4.2.8}$$

where

$$\tau := \epsilon\beta\gamma s + 1.$$

Thus, solving the rescaled risk equation (2.2.8) reduces to computing the inverse transform of (4.2.8).

Proof: 1. First, we record the relationships between λ, p, γ, b, and a. We have from Corollary 3.12 and Lemma B.10 that $1 = b\beta\gamma$ and $u_\epsilon(0) = 1 - \frac{a}{b}$.

Since u_ϵ is bounded, $\mathcal{L}u_\epsilon$ is well-defined on $(0,\infty)$ and $v_\epsilon \in L^1(\mathbb{R}_+)$. Therefore $\mathcal{L}v_\epsilon$ is well-defined on $\mathbb{R}_+$. To show that (4.2.8) holds, we use the convolution equation (3.1.9) satisfied by v_ϵ to determine $\mathcal{L}v_\epsilon$. Then, we exploit the relationship between u_ϵ and v_ϵ to compute $\mathcal{L}u_\epsilon$.

2. Taking the Laplace transform of (3.1.9) yields

$$\mathcal{L}v_\epsilon(s) = a\mathcal{L}H_\epsilon(s) + a\mathcal{L}H_\epsilon(s)\mathcal{L}v_\epsilon(s), \tag{4.2.9}$$

by B.15 and B.17. Next,

$$\mathcal{L}H_\epsilon(s) = \begin{cases} \frac{1}{\epsilon s} - \frac{1}{\epsilon s}\left(\frac{1}{\frac{\epsilon s}{b\gamma}+1}\right)^\gamma & (s > 0) \\ \frac{1}{b} & (s = 0) \end{cases} \tag{4.2.10}$$

by (B.16), Lemma B.11, and Lemma B.4. Upon substituting (4.2.10) into (4.2.9), we deduce that for $s > 0$,

$$\mathcal{L}v_\epsilon(s) \;=\; \frac{a\mathcal{L}H_\epsilon(s)}{1-a\mathcal{L}H_\epsilon(s)} \;=\; \frac{\frac{a}{\epsilon s} - \frac{a}{\epsilon s}\left(\frac{1}{\frac{\epsilon s}{b\gamma}+1}\right)^\gamma}{1 - \frac{a}{\epsilon s} + \frac{a}{\epsilon s}\left(\frac{1}{\frac{\epsilon s}{b\gamma}+1}\right)^\gamma}.$$

Thus, by (4.2.9),

$$\mathcal{L}u_\epsilon(s) \;=\; \frac{u_\epsilon(0)}{s}\left(1 - \frac{a}{\epsilon s} + \frac{a}{\epsilon s}\left(\frac{1}{\frac{\epsilon s}{b\gamma}+1}\right)^\gamma\right)^{-1}.$$

This is (4.2.8), and the theorem is proved. □

Example 2 (Gamma distribution with constant premium and $\gamma = 1$)**.**
In this case (4.2.8) is

$$\mathcal{L}u_\epsilon(s) = \frac{\epsilon}{b}\left(1-\frac{a}{b}\right)\frac{\tau}{\tau^2-\left(1+\frac{a}{b}\right)\tau+\frac{a}{b}}.$$

Simplifying yields

$$\mathcal{L}u_\epsilon(s) \;=\; \frac{\epsilon u_\epsilon(0)}{b-a}\left(\frac{1}{\tau-1}-\frac{\frac{a}{b}}{\tau-\frac{a}{b}}\right) \;=\; \frac{1}{s}-\frac{a}{b}\frac{1}{s-\frac{a-b}{\epsilon}}.$$

Therefore

$$u_\epsilon(x) = 1-\frac{a}{b}e^{-\frac{b-a}{\epsilon}x}. \tag{4.2.11}$$

We note that (4.2.11) agrees with Example 4.2.1.

Example 3 (Gamma distribution with constant premium and $\gamma = 2$)**.**
In this case (4.2.8) is

$$\mathcal{L}u_\epsilon(s) = \frac{\epsilon}{b}\left(1-\frac{a}{b}\right)\frac{\tau^2}{\tau^3-\left(1+\frac{a}{2b}\right)\tau^2+\frac{a}{2b}}.$$

Substituting $\tau = \frac{\epsilon s}{b}+1$ and simplifying yields

$$\mathcal{L}u_\epsilon(s) = \frac{1}{s}-\frac{s+\frac{b-a}{\epsilon}}{\left(s+\frac{b-a}{\epsilon}\right)^2-\frac{a^2}{\epsilon^2}-\frac{ab}{2\epsilon^2}}+\frac{b-2a}{\epsilon}\frac{1}{\left(s+\frac{b-a}{\epsilon}\right)^2-\frac{a^2}{\epsilon^2}-\frac{ab}{2\epsilon^2}}.$$

Therefore

$$u_\epsilon(x) = 1-e^{-\frac{b-a}{\epsilon}x}\left(\cos\left(\frac{a^2}{\epsilon^2}+\frac{ab}{2\epsilon^2}\right)x+\frac{\epsilon}{a}\left(\frac{b-2a}{b+2a}\right)\sinh\left(\frac{a^2}{\epsilon^2}+\frac{ab}{2\epsilon^2}\right)x\right). \tag{4.2.12}$$

Chapter 5

Viscosity Solutions

As shown by Theorem 4.2, when F is atomic, the only classical solution of (2.2.8) is trivial. We are thus motivated to define a generalized solution of the rescaled risk equation which accommodates a broader class of tractable distributions. The alternative is to consider only absolutely continuously distributed claim sizes. However, when we allow more diverse distributions, our model is more robust. Every law-abiding insurer periodically pays taxes and has a (deterministic) range for the amount of taxes to be paid. This can be incorporated into the model by adding a Dirac δ-type distribution to the claim size distribution F. Or, perhaps the insurer anticipates regular loan repayments to local gangsters for the seed money used to start the business. The insurer knows the amount owed, but not the specific collection time. Further, randomly assessed "late fees" can be incorporated into the model.

We assume for this chapter $u_\epsilon \in L^1_{loc}(\mathbb{R}_+)$ satisfies (2.2.8) with $u_\epsilon(0) > 0$. We also assume atoms of F are isolated.

5.1 Transformed equations

In this section we derive two transformations of (2.2.8). As in Peters and Mangel [34], the form of the explicit solutions obtained in Chapter 4 motivates an exponential transformation. We now present an alternative heuristic based on Theorem 3.16.

Motivation. Assume u_ϵ is sufficiently smooth and recall (from Theorem 3.16) the estimates

$$\tfrac{\lambda H_\epsilon}{p_\epsilon} \leqslant (\log u_\epsilon)' \leqslant \tfrac{\lambda}{\epsilon p_\epsilon}.$$

Setting

$$w_\epsilon := \log u_\epsilon$$

transforms these inequalities and (2.2.8) into

$$w_\epsilon(0) + \textstyle\int_0^x \frac{\lambda H_\epsilon(t)}{p_\epsilon(t)}\, dt \;\leqslant\; w_\epsilon(x) \;\leqslant\; w_\epsilon(0) + \int_0^x \frac{\lambda dt}{\epsilon p_\epsilon(t)}$$

and

$$\epsilon p_\epsilon(x) w_\epsilon'(x) - \lambda \epsilon H_\epsilon(x) = \lambda \textstyle\int_0^{x/\epsilon} 1 - e^{w_\epsilon(x-\epsilon y) - w_\epsilon(x)}\; F\{dy\},$$

respectively. Consequently, we investigate the rescaled risk equation under an exponential transformation.

5.1.1 An exponential transformation

For $\epsilon > 0$, define w_ϵ by

$$u_\epsilon(x) = u_\epsilon(0) e^{\frac{w_\epsilon(x)}{\epsilon}} \qquad (x \geqslant 0). \tag{5.1.1}$$

Notice that since $u_\epsilon(0) > 0$, such a w_ϵ is well-defined, and $w_\epsilon(0) = 0$. Then (2.2.8) implies

$$\begin{aligned} \epsilon p_\epsilon(x) u_\epsilon'(x) &= \lambda u_\epsilon(x) - \lambda \textstyle\int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} \\ &= \lambda u_\epsilon(0) e^{\frac{w_\epsilon(x)}{\epsilon}} - \lambda u_\epsilon(0) \textstyle\int_0^{x/\epsilon} e^{\frac{w_\epsilon(x-\epsilon y)}{\epsilon}}\, F\{dy\} \\ &= \lambda u_\epsilon(0) e^{\frac{w_\epsilon(x)}{\epsilon}} \left(1 - \textstyle\int_0^{x/\epsilon} e^{\frac{w_\epsilon(x-\epsilon y) - w_\epsilon(x)}{\epsilon}}\, F\{dy\}\right). \end{aligned}$$

But since $\epsilon u_\epsilon' = w_\epsilon' u_\epsilon$, we have the following fact.

Lemma 5.1. *Suppose u_ϵ solves the rescaled risk problem (2.2.10). Then w_ϵ, as given in (5.1.1), solves*

$$\begin{cases} \lambda \int_0^{x/\epsilon} e^{\frac{w_\epsilon(x-\epsilon y) - w_\epsilon(x)}{\epsilon}}\, F\{dy\} - \lambda + p_\epsilon(x) w_\epsilon'(x) = 0 & (x > 0) \\ w_\epsilon(0) = 0 \end{cases}. \tag{5.1.2}$$

5.1.2 Another exponential transformation

The following example and lemma justify considering a second exponential transformation.

Example 4 (Exponential distribution with constant p_ϵ)**.**
Suppose $p_\epsilon \equiv p$ is constant and the claim size process has an exponential distribution. Fix $x > 0$. Then as $\epsilon \to 0$, (see Appendix B.4)

$$w_\epsilon(x) = \begin{cases} (a-b)x + O(\epsilon) & (a > b) \\ O(\epsilon \log \epsilon) & (a = b) \\ O(\epsilon) & (a < b) \end{cases}.$$

□

But we know from Example 4.2.1 that the rescaled risk equation has a bounded (classical) solution if and only if $a < b$. This suggests the transformation given in (5.1.1) is not helpful in analyzing limits of bounded solutions of (2.2.8). The next lemma captures the sense in which the transformation (5.1.1) yields a trivial approximation for bounded solutions.

Lemma 5.2. *For $\epsilon > 0$, assume v_ϵ solves the convolution equation (3.1.9) and w_ϵ is given by the transformation in (5.1.1). Suppose $\{v_\epsilon\}_{\epsilon>0}$ is uniformly bounded in*

$L^1(\mathbb{R}_+)$ and $w_\epsilon \to w$ locally uniformly on $(0,\infty)$ as $\epsilon \to 0$.
Then to zero order,

$$w \asymp 0.$$

Proof: Let

$$\sup_{\epsilon>0} \|v_\epsilon\|_{L^1} =: C.$$

Then for any $x > 0$,

$$u_\epsilon(0) \leqslant u_\epsilon(x) \leqslant u_\epsilon(0)(1+C)$$

by monotonicity and (3.1.9). This implies

$$u_\epsilon(0) \leqslant u_\epsilon(0) e^{\frac{w_\epsilon(x)}{\epsilon}} \leqslant u_\epsilon(0)(1+C)$$

by (5.1.1). After factoring out $u_\epsilon(0) \geqslant 0$, we deduce that

$$1 \leqslant e^{\frac{w_\epsilon(x)}{\epsilon}} \leqslant 1 + C.$$

So $w_\epsilon = O(\epsilon)$ as $\epsilon \to 0$. Thus $w(x) \asymp 0$ is a zero order approximation of w on $(0,\infty)$. □

Our objective is to determine an exponential transformation appropriate for analyzing bounded solutions of the rescaled risk equation.

Hypothesis. We assume

$$u_\epsilon \in L^\infty(\mathbb{R}_+) \text{ and } u_\epsilon(\infty) = 1.$$

The value of this boundary condition serves to normalize our solutions. Indeed, according to Corollary 3.13, any nontrivial L^∞ (classical) solution of the rescaled risk equation tends to a positive multiple of its initial value. Also, requiring that $u_\epsilon(\infty) = 1$ hearkens back to the probabilistic origin of the rescaled risk equation.

In contrast to the transformation in the last section, we now define w_ϵ by

$$u_\epsilon = 1 - e^{-\frac{w_\epsilon}{\epsilon}}. \tag{5.1.3}$$

It is clear that for any $\epsilon > 0$, both w_ϵ and w_ϵ' are nonnegative.

We now present the transformed rescaled risk equation (proof on page 78).

Lemma 5.3. *Let w_ϵ be defined by the transformation (5.1.3). When u_ϵ solves the rescaled risk problem (2.2.9), w_ϵ solves*

$$\begin{cases} \lambda \int e^{\frac{w_\epsilon(x) - w_\epsilon(x-\epsilon y)\mathcal{H}(x-\epsilon y)}{\epsilon}} F\{dy\} - \lambda - p_\epsilon(x) w_\epsilon'(x) = 0 & (x > 0) \\ w_\epsilon(\infty) = \infty \end{cases}. \tag{5.1.4}$$

The equation in (5.1.4) may also be expressed as

$$\lambda \int_0^{x/\epsilon} e^{\frac{w_\epsilon(x) - w_\epsilon(x-\epsilon y)}{\epsilon}} F\{dy\} + \epsilon \lambda H_\epsilon(x) e^{w_\epsilon(x)} - \lambda - p_\epsilon(x) w_\epsilon'(x) = 0 \qquad (x \geqslant 0). \tag{5.1.5}$$

Definition. We refer to the initial value problem (5.1.2) and the final value problem (5.1.4) as the ***transformed rescaled risk problems***.

Now, we reconsider Example 5.1.2 with (5.1.3). The result is a nontrivial limit w.

Example 5 (Exponential distribution with constant p_ϵ)**.**
Recall that $b > a$, since we seek bounded solutions (see Example 4.2.1). Therefore

$$w_\epsilon(x) = (b-a)x + \epsilon \log \tfrac{b}{a};$$

and for $x > 0$,

$$w_\epsilon(x) = (b-a)x + O(\epsilon) \qquad (\epsilon \to 0).$$

Setting

$$w(x) := (b-a)x,$$

we have that w_ϵ converges uniformly to w on $\mathbb{R}_+$ as $\epsilon \to 0$. □

5.2 Viscosity solutions

In this section, we use the framework provided by the theory of viscosity solutions to analyze the transformed rescaled equations. Our definitions are motivated by the vanishing viscosity method, as used in Evans' [16] for first order Hamiltonians and Awatif [5] for first order integro-differential equations.

5.2.1 Definitions

Motivation for definition of viscosity solutions. Fix $\epsilon > 0$ and consider the following boundary value problem, for small $\delta > 0$:

$$\begin{cases} \lambda \int e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}\mathcal{H}(x-\epsilon y)} F\{dy\} - \lambda - p_\epsilon(x)w'(x) - \delta w''(x) = 0 \quad (x > 0) \\ w(0) = w_0 \end{cases}. \tag{5.2.1}$$

Suppose $w \in C^2(\mathbb{R}_+)$ is a solution of (5.2.1), $\varphi \in C^2(\mathbb{R}_+)$, and $w - \varphi$ attains a nonnegative global maximum at the point $\bar{x} > 0$. Then $w(\bar{x}) \geqslant \varphi(\bar{x})$, $w'(\bar{x}) = \varphi'(\bar{x})$, $w''(\bar{x}) \leqslant \varphi''(\bar{x})$, and

$$w(\bar{x}) - w(\bar{x} - \epsilon y)\mathcal{H}(\bar{x} - \epsilon y) \geqslant \varphi(\bar{x}) - \varphi(\bar{x} - \epsilon y)\,\mathcal{H}(\bar{x} - \epsilon y) \qquad (y \geqslant 0).$$

These, along with (5.2.1), imply

$$\lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)}{\epsilon}\mathcal{H}(\bar{x}-\epsilon y)} F\{dy\} - \lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) - \delta\varphi''(\bar{x}) \leqslant 0.$$

Sending $\delta \to 0$ suggests defining generalized solutions of (5.1.2) and (5.1.4) as below.

Limiting equations

First, we define viscosity solutions for (5.1.2) and (5.1.4) based on test functions φ. Then we verify that the definitions generalize those of classical solutions.

Consider the following conditions for $w \in L^1_{loc}(\mathbb{R}_+)$:

C1$_\epsilon$: For each $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ has a local maximum at $\bar{x} > 0$, then

$$\lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) - \lambda \int e^{\frac{\varphi(\bar{x}-\epsilon y)-\varphi(\bar{x})}{\epsilon}} \mathcal{H}(\bar{x}-\epsilon y) F\{dy\} \leqslant 0.$$

C2$_\epsilon$: For each $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ has a local minimum at $\bar{x} > 0$, then

$$\lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) - \lambda \int e^{\frac{\varphi(\bar{x}-\epsilon y)-\varphi(\bar{x})}{\epsilon}} \mathcal{H}(\bar{x}-\epsilon y) F\{dy\} \geqslant 0.$$

Definition. We say $w \in C(\mathbb{R}_+)$, with $w(0) \leqslant 0$, is a *viscosity subsolution* of (5.1.2) provided C1$_\epsilon$ holds and is a *supersolution* of (5.1.2) provided C2$_\epsilon$ holds. If w is both a subsolution and supersolution of (5.1.2), we say w is a *viscosity solution* of the same.

We continue with analogous conditions for (5.1.4).

C3$_\epsilon$: For $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ attains a nonnegative global maximum at $\bar{x} > 0$, then

$$\lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\,\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} - \lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) \leqslant 0.$$

C4$_\epsilon$: For $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ attains a nonpositive global minimum at $\bar{x} > 0$, then

$$\lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\,\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} - \lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) \geqslant 0.$$

Definition. We say $w \in C(\mathbb{R}_+)$, with $w(0) \geqslant 0$, is a *viscosity subsolution* of (5.1.4) provided C3$_\epsilon$ holds. We say w is a *supersolution* provided C4$_\epsilon$ holds. If w is both a subsolution and supersolution of (5.1.4), we say w is a *viscosity solution*.

Limit equations

Suppose for each $\epsilon > 0$ there exists $w_\epsilon \in C^1(\mathbb{R}_+)$ solving (5.1.4) and the set of solutions $\{w_\epsilon\}$ is uniformly Cauchy.

Then the Dominated Convergence Theorem implies

$$\lim_{\epsilon\to 0} \int_0^{x/\epsilon} e^{\frac{w_\epsilon(x)-w(x-\epsilon y)}{\epsilon}} F\{dy\} = \lambda \int e^{w'(x)y} F\{dy\}.$$

Also,

$$\lim_{\epsilon\to 0} \epsilon e^{\frac{w_\epsilon(x)}{\epsilon}} H_\epsilon(x) = 0,$$

since

$$\epsilon e^{\frac{w_\epsilon(x)}{\epsilon}} = \frac{\epsilon}{1-u_\epsilon(x)} = O(\epsilon) \quad \text{and} \quad H_\epsilon(x) = o(\epsilon)$$

as $\epsilon \to 0$.

Definition (Limit of the transformed rescaled risk problem). We call

$$\begin{cases} \lambda - p(x)w'(x) - \lambda \int e^{-w'(x)y} F\{dy\} = 0 \qquad (x \geqslant 0) \\ w(\infty) = \infty \end{cases} \tag{5.2.2}$$

the ***limit of the transformed risk problem*** (5.1.2), and

$$\begin{cases} \lambda \int e^{w'(x)y} F\{dy\} - \lambda - p(x)w'(x) = 0 \qquad (x \geqslant 0) \\ w(\infty) = \infty \end{cases} \tag{5.2.3}$$

the ***limit of the transformed risk problem*** (5.1.4).

We have analogous definitions for their viscosity solutions. Consider the following conditions for $w \in L^1_{loc}(\mathbb{R}_+)$:

C1: For each $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ has a local maximum at $\bar{x} > 0$, then

$$\lambda - p(\bar{x})\varphi'(\bar{x}) - \lambda \int e^{-\varphi'(\bar{x})y} F\{dy\} \leqslant 0.$$

C2: For $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ has a local minimum at $\bar{x} > 0$, then

$$\lambda - p(\bar{x})\varphi'(\bar{x}) - \lambda \int e^{-\varphi'(\bar{x})y} F\{dy\} \geqslant 0.$$

C3: For $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ attains a nonnegative global maximum at $\bar{x} > 0$, then

$$\lambda \int e^{\varphi'(\bar{x})y} F\{dy\} - \lambda - p(\bar{x})\varphi'(\bar{x}) \leqslant 0.$$

C4: For $\varphi \in C^2(\mathbb{R}_+)$, if $w - \varphi$ attains a nonpositive global minimum at $\bar{x} > 0$, then

$$\lambda \int e^{\varphi'(\bar{x})y} F\{dy\} - \lambda - p(\bar{x})\varphi'(\bar{x}) \geqslant 0.$$

Definition. Let $w \in C(\mathbb{R}_+)$ be increasing and suppose $w(0) \leqslant 0$. We say w is a *viscosity subsolution* of (5.2.2) provided C1 holds and is a *supersolution* of (5.2.2) provided C2 holds. If w is both a subsolution and supersolution of (5.2.2), we say w is a *viscosity solution* of the same.

Let $w \in C(\mathbb{R}_+)$ be increasing and suppose $w(0) \geqslant 0$. We say w is a *viscosity subsolution* of (5.2.3) provided C3 holds. We say w is a *supersolution* provided C4 holds. If w is both a subsolution and supersolution of (5.2.3), we call w a *viscosity solution*.

5.2.2 Consistency

We now verify that the definition above is a generalization of the classical definition of a solution. First we address consistency with respect to the classical definition. Then we address a case in which w is a classical solution of (5.1.4) almost everywhere, so that our definitions generalize the classical definitions.

Limiting equations

Lemma 5.4 (Consistency for C^1 solutions). *Every $C^1(\mathbb{R}_+)$ solution of (5.1.4) is a viscosity solution of (5.1.4).*

Proof: 1. Let $w \in C^1(\mathbb{R}_+)$ solve (5.1.4). We will show that C3$_\epsilon$ and C4$_\epsilon$ hold. Suppose $\bar{x}$, $\delta > 0$ and $\varphi \in C^2(\mathbb{R}_+)$ satisfy

$$(w-\varphi)(\bar{x}) \geqslant (w-\varphi)(y)\,\mathcal{H}(y) \quad \text{for } 0 \leqslant y \leqslant \bar{x}+\delta. \tag{5.2.4}$$

By hypothesis, w and φ are differentiable at $\bar{x}$ and $w-\varphi$ has a local maximum at $\bar{x}$. It follows that $(w-\varphi)'(\bar{x}) = 0$. We use this equality to prove that

$$\lambda + p_\epsilon(\bar{x})\varphi'(\bar{x}) \geqslant \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\,\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\}.$$

Notice that

$$\begin{aligned}
\lambda + p_\epsilon(\bar{x})\varphi'(\bar{x}) &= \lambda + p_\epsilon(\bar{x})w'(\bar{x}) && \text{as } \varphi'(\bar{x}) = w'(\bar{x}) \\
&= \lambda \int e^{\frac{w(\bar{x})-w(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} && \text{as } w \text{ solves (5.1.4) at } \bar{x} \\
&\geqslant \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} && \text{by (5.2.4).}
\end{aligned}$$

Consequently, w is a subsolution of (5.1.4).
2. Now, suppose

$$(w-\varphi)(\bar{x}) \leqslant (w-\varphi)(y)\,\mathcal{H}(y)$$

whenever $y \in [0, \bar{x}+\delta]$. As before, $w'(\bar{x}) = \varphi'(\bar{x})$. Thus,

$$\begin{aligned}
\lambda + p_\epsilon(\bar{x})\varphi'(\bar{x}) \;=\; \lambda + p_\epsilon(\bar{x})w'(\bar{x}) \;&=\; \lambda \int e^{\frac{w(\bar{x})-w(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} \\
&\leqslant\; \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\}.
\end{aligned}$$

So w is a subsolution. Therefore, w is a viscosity solution of (5.1.4). □

Recall that the example in Section 4.1 is a case where the only classical solution of (2.2.8) is the trivial solution. However, we exhibited a function which satisfied (2.2.8) almost everywhere. Within the framework of viscosity solutions, we are able to obtain nontrivial solutions of equations which have no classical solution. Now, we present a theorem which shows that the viscosity solutions do not merely characterize classical solutions, but generalize them.

We now show consistency for our definition with respect to continuous piecewise $C^1(\mathbb{R}_+)$ functions. As a consequence, (5.1.4) will have solutions for a broader class of distributions since the regularity requirement for a solution is relaxed.

Theorem 5.1 (Consistency for piecewise C^1 functions). *Fix $\epsilon > 0$ and assume $w \in C(\mathbb{R}_+)$ is differentiable almost everywhere. Suppose $w'(x+)$ and $w'(x-)$ exist for all $x > 0$. If w solves (5.1.4) wherever w is differentiable, then w is a viscosity solution of (5.1.4).*

Proof: 1. We verify that w satisfies C3$_\epsilon$ and C4$_\epsilon$. Let $\varphi \in C^2(\mathbb{R}_+)$, and suppose δ, $\bar{x} > 0$ satisfy

$$(w-\varphi)(\bar{x}) \geqslant (w-\varphi)(y)\,\mathcal{H}(y) \quad \text{for} \quad 0 \leqslant y \leqslant \bar{x}+\delta. \tag{5.2.5}$$

Either w is differentiable at $\bar{x}$, or else, w is continuously differentiable on a deleted neighborhood of $\bar{x}$. Lemma 5.4 addresses the former case, so we may assume the latter.
2. Our goal is to prove

$$\lambda + p_\epsilon(\bar{x})\varphi'(\bar{x}) \geqslant \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\,\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\};$$

we use the right-continuity of the mapping

$$x \mapsto \int e^{\frac{w(x)-w(x-\epsilon y)\mathcal{H}(x-\epsilon y)}{\epsilon}} F\{dy\} \qquad (x \geqslant 0) \tag{5.2.6}$$

to do so (see Lemma B.1). In fact,

$$\begin{aligned}
\lambda + p_\epsilon(\bar{x})\varphi'(\bar{x}) &\geqslant \lambda + p_\epsilon(\bar{x})w'(\bar{x}+) && \text{since } (w-\varphi)'(\bar{x}+) \leqslant 0 \\
&= \lambda \int e^{\frac{w(\bar{x})-w(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} && \text{by right-continuity of (5.2.6)} \\
&\geqslant \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} && \text{by (5.2.5).}
\end{aligned}$$

Thus w satisfies C3$_\epsilon$.
3. On the other hand, if

$$(w-\varphi)(\bar{x}) \leqslant (w-\varphi)(y)\,\mathcal{H}(y) \quad \text{for} \quad 0 \leqslant y \leqslant \bar{x}+\delta, \tag{5.2.7}$$

then

$$\begin{aligned}
\lambda + p_\epsilon(\bar{x})\varphi'(\bar{x}) &\leqslant \lambda + p_\epsilon(\bar{x})w'(\bar{x}+) && \text{since } (w-\varphi)'(\bar{x}+) \geqslant 0 \\
&= \lambda \int e^{\frac{w(\bar{x})-w(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} && \text{by right-continuity of (5.2.6)} \\
&\leqslant \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} && \text{by (5.2.7).}
\end{aligned}$$

Therefore w is a viscosity solution of (5.1.4). □

5.2.3 Stability

The definition of a viscosity solution of (5.1.4) considers behavior at a local maximum, whereas the definition of a viscosity solution of (5.2.3) considers behavior at a global maximum. We present a calculus fact which describes the sense in which a local maximum may be assumed to be a strict global maximum. With this fact, the properties satisfied by the solutions of (5.1.4) propagate to the solution of the limit equation (5.2.3).

Lemma 5.5. *Assume $w \in C(\mathbb{R}_+)$ and $\varphi \in C^2(\mathbb{R}_+)$. Fix $\epsilon > 0$. Suppose $w-\varphi$ attains*

a local maximum (minimum) at $\bar{x} > 0$. Then there exists $\rho \in C^2(\mathbb{R}_+)$ and $\delta > 0$ such that

(i) *$w - \rho$ attains a strict global maximum (minimum) on the interval $[0, \bar{x} + \delta]$ at $\bar{x}$,*

and

(ii) $\rho'(\bar{x}) = \varphi'(\bar{x})$.

Proof: 1. Let $\delta > 0$ be such that $\delta < \bar{x}$ and

$$w - \varphi \text{ attains a global maximum on } [\bar{x} - \delta, \bar{x} + \delta] \text{ at } \bar{x}. \tag{5.2.8}$$

We may assume

$$w(\bar{x}) \geqslant \varphi(\bar{x}); \tag{5.2.9}$$

if needed, replace φ with $\varphi + (w - \varphi)(\bar{x})$. Since $w \in C(\mathbb{R}_+)$, the quantity

$$C := \sup\{w(x) - \varphi(x) \mid 0 \leqslant x \leqslant \bar{x} - \delta\} \geqslant 0$$

is finite. Define $\rho \in C^2(\mathbb{R}_+)$ by

$$\rho(x) := \varphi(x) + (x - \bar{x})^4 \mathcal{H}(x - \bar{x} + \delta) + C \left(\tfrac{x-\bar{x}}{\delta}\right)^4 \mathcal{H}(\bar{x} - \delta - x).$$

Evidently, ρ satisfies (ii) and $w(\bar{x}) - \rho(\bar{x}) \geqslant 0$. To verify that ρ satisfies (i), we need only consider that

$$(w - \rho)(x) = \begin{cases} (w - \varphi)(x) - C \left(\tfrac{x-\bar{x}}{\delta}\right)^4 & (0 \leqslant x < \bar{x} - \delta) \\ (w - \varphi)(x) - (x - \bar{x})^4 & (\bar{x} - \delta \leqslant x \leqslant \bar{x} + \delta) \end{cases}.$$

If $x < \bar{x} - \delta$, then

$$(w - \rho)(x) < (w - \varphi)(x) - M \leqslant 0 \leqslant w(\bar{x}) - \rho(\bar{x}).$$

On the other hand, if $0 < |x - \bar{x}| \leqslant \delta$, then

$$(w - \rho)(x) = (w - \varphi)(\bar{x}) - (x - \bar{x})^4 < (w - \rho)(\bar{x}).$$

The inequalities follow from (5.2.9) and (5.2.8).
2. The case where $w - \varphi$ attains a local minimum at $\bar{x}$ follows immediately from the above result. Indeed, in this case $-w - (-\varphi)$ attains a local maximum at $\bar{x}$. □

Theorem 5.2 (Stability). *Suppose for each $\epsilon > 0$, w_ϵ is a viscosity solution of (5.1.4) and $w_\epsilon \to w$ locally uniformly on $\mathbb{R}_+$ as $\epsilon \to 0$.*
Then w is a viscosity solution of (5.2.3), with $w_0 = \lim_{\epsilon \to 0} w_\epsilon(0)$.

Proof: 1. The locally uniform convergence of w_ϵ to w implies $w(0) = w_0$. To prove the theorem, we must verify that w is a supersolution and subsolution. To that end, let φ be a test function and suppose the point $\bar{x} > 0$ is a local maximum of $w - \varphi$. According to Lemma 5.5, we may assume there exists $\delta > 0$ such that $\bar{x}$ is a strict global maximum of $w - \varphi$ on $[0, \bar{x} + \delta]$. We claim that

$$\lambda + p(0)\varphi'(\bar{x}) \geqslant \lambda \int e^{\varphi'(\bar{x})y} F\{dy\}. \tag{5.2.10}$$

2. For each $\epsilon > 0$, $w_\epsilon - \varphi$ is continuous on $[0, \bar{x} + \delta]$; thus, there exists a point $x_\epsilon \in [0, \bar{x} + \delta]$ such that

$$(w_\epsilon - \varphi)(x) < (w_\epsilon - \varphi)(x_\epsilon) \qquad (0 \leqslant x \leqslant \bar{x} + \delta). \tag{5.2.11}$$

Choose a sequence $\{\epsilon(j)\}$ of positive numbers and $x_* \in [0, \bar{x} + \delta]$ satisfying

$$x_{\epsilon(j)} \to x_* \qquad \text{as} \qquad \epsilon(j) \to 0.$$

First, we assert that $x_* = \bar{x}$. To clarify the following argument, we introduce some temporary notation:

$$\varphi_j := w_{\epsilon(j)} - \varphi, \qquad \varphi_0 := w - \varphi, \qquad x_j := x_{\epsilon(j)}.$$

Since φ_j attains a strict global maximum at x_j,

$$\varphi_0(\bar{x}) - \varphi_0(x_j) \ \leqslant \ (\varphi_j - \varphi_0)(x_j) \ - \ (\varphi_j - \varphi_0)(\bar{x}).$$

Now

$$|\varphi_j - \varphi_0| \ \leqslant \ \sup_{\epsilon < \epsilon(j)} |w - w_\epsilon|,$$

and

$$x_* - x_j \ , \ \sup_{\epsilon < \epsilon(j)} |w - w_\epsilon| \ \to \ 0 \qquad \text{as} \qquad \epsilon(j) \to 0.$$

Therefore $\varphi_0(\bar{x}) \leqslant \varphi_0(x_*)$. Recalling that a strict global maximum of φ_0 on $[0, \bar{x} + \delta]$ occurs at $\bar{x}$, we deduce that $x_* = \bar{x}$.

Now, we apply the definition of viscosity solution to see that

$$\lambda + p_{\epsilon(j)}\varphi'(x_j) \geqslant \lambda \int e^{\frac{\varphi(x_j) - \varphi(x_j - \epsilon(j)y)\mathcal{H}(x_j - \epsilon(j)y)}{\epsilon(j)}} F\{dy\}. \tag{5.2.12}$$

Since φ is smooth and $x_j \to \bar{x}$ as $\epsilon(j) \to 0$,

$$\lambda + p(x_*)\varphi'(\bar{x}) \geqslant \lambda \int e^{\varphi'(\bar{x})y} F\{dy\}. \tag{5.2.13}$$

Thus (5.2.10) holds.

3. To show that w is a subsolution, assume $w - \varphi$ attains a local minimum at $\bar{x}$. Then, repeat the above arguments with the inequalities in (5.2.11), (5.2.12), and (5.2.13) reversed. □

5.2.4 Properties of solutions

We begin with a comparison theorem.

Theorem 5.3. *Assume*

(i) F *is eventually* 1. *This means there exists* $x_0 > 0$ *such that* $F(x_0) = 1$.

(ii) $u \in C(\mathbb{R}_+)$ *is a subsolution of* (5.1.4) *and* $v \in C(\mathbb{R}_+)$ *is a supersolution of* (5.1.4),

(iii) $u(0) \leqslant v(0)$,

(iv) $p_\epsilon \equiv p$.

Then $u \leqslant v$ *on* $\mathbb{R}_+$.

Proof: If not, there exists $N > 0$ and $x_\sigma > 0$ such that

$$\max_{[0,N]} u - v \;=:\; u(x_\sigma) - v(x_\sigma) \;>\; 0.$$

So define

$$\Phi(x, y) := u(x) - v(y) - \mu(x + y).$$

We have $\bar{x} \geqslant 0$ and $\bar{y} \geqslant 0$ satisfying

$$\max_{[0,N]^2} \Phi = \Phi(\bar{x}, \bar{y}) \geqslant 0 \quad \text{and} \quad \max_{[0,N]^2} \Phi \geqslant \Phi(x_\sigma, x_\sigma) = \sigma - 2x_\sigma.$$

Now define φ and ρ by

$$\varphi(x) := v(\bar{y}) + \mu\bar{y} + \mu x \quad \text{and} \quad \rho(y) := u(\bar{x}) - \mu\bar{x} - \mu y.$$

Then $u - \varphi$ attains a maximum at $\bar{x}$ and $v - \rho$ attains a minimum at $\bar{y}$. Therefore,

$$\lambda \int e^{\frac{\varphi(\bar{x}) - \varphi(\bar{x} - \epsilon y)}{\epsilon} \mathcal{H}(\bar{x} - \epsilon y)} F\{dy\} - \lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) \leqslant 0. \tag{5.2.14}$$

Without loss of generality, assume $\bar{x} \geqslant \epsilon x_0$, so that

$$\int e^{\frac{\varphi(\bar{x}) - \varphi(\bar{x} - \epsilon y)}{\epsilon} \mathcal{H}(\bar{x} - \epsilon y)} F\{dy\} = \int e^{\frac{\varphi(\bar{x}) - \varphi(\bar{x} - \epsilon y)}{\epsilon}} F\{dy\}.$$

Then (5.2.14) is

$$\lambda \int e^{\mu y} F\{dy\} - \lambda - p\mu \leqslant 0,$$

or equivalently,

$$\mu \geqslant \tfrac{\lambda}{p} \left(\int e^{\mu y} F\{dy\} - 1 \right). \tag{5.2.15}$$

Similarly,

$$\lambda \int e^{-\mu y} F\{dy\} - \lambda + p\mu \geqslant 0,$$

so that

$$\mu \geqslant \tfrac{\lambda}{p}\left(1 - \int e^{-\mu y}\, F\{dy\}\right). \tag{5.2.16}$$

Adding (5.2.15) to (5.2.16) yields

$$\mu \geqslant \tfrac{\lambda}{p} \int \sinh(\mu y)\, F\{dy\}.$$

But Jensen's Inequality implies

$$\int \sinh(\mu y)\, F\{dy\} \;\geqslant\; \sinh(\mu \mathbb{E}[Y]),$$

so that we have

$$\tfrac{p}{\lambda \mathbb{E}[Y]} \geqslant \tfrac{\sinh(\mu \mathbb{E}[Y])}{\mu \mathbb{E}[Y]} > 1.$$

□

We continue by exhibiting supersolutions and subsolutions of (5.1.2) and (5.1.4). Let

$$q_\epsilon(x) := \textstyle\int_0^x \frac{\lambda}{p_\epsilon(t)}\, dt \qquad (x \geqslant 0).$$

Theorem 5.4. *q_ϵ is a viscosity subsolution of (5.1.2). $-q_\epsilon$ is a viscosity supersolution of (5.1.4). 0 is a viscosity subsolution of (5.1.4). 0 is a viscosity supersolution of (5.1.2).*

Proof: 1. Clearly, $q_\epsilon \in C(\mathbb{R}_+)$ since $p_\epsilon \in C^\infty(\mathbb{R}_+)$, and $q_\epsilon(0) = 0$. To prove q_ϵ is a viscosity subsolution of (5.1.2), we only need to verify that q_ϵ satisfies C1$_\epsilon$. To that end, let $\varphi \in C^2(\mathbb{R}_+)$ and $\bar{x} > 0$ be such that $q_\epsilon - \varphi$ has a global maximum at $\bar{x} > 0$. Then

$$\tfrac{\lambda}{p_\epsilon(\bar{x})} \;=\; q_\epsilon'(\bar{x}) \;=\; \varphi'(\bar{x});$$

where the first equality follows from the definition of q_ϵ. Thus

$$0 \;=\; 1 - \tfrac{p_\epsilon(\bar{x})}{\lambda}\varphi'(\bar{x}) \;\geqslant\; 1 - \tfrac{p_\epsilon(\bar{x})}{\lambda}\varphi'(\bar{x}) - \textstyle\int_0^{\bar{x}/\epsilon} e^{\frac{\varphi(\bar{x}-\epsilon y)-\varphi(\bar{x})}{\epsilon}} F\{dy\},$$

since the integral is nonnegative. Therefore C1$_\epsilon$ holds.
2. To prove that $-q_\epsilon$ is a viscosity supersolution of (5.1.4), we only need to verify that $-q_\epsilon$ satisfies C4$_\epsilon$. Let $\varphi \in C^2(\mathbb{R}_+)$ and $\bar{x} > 0$ be such that $-q_\epsilon - \varphi$ has a nonpositive global minimum at $\bar{x} > 0$. Then

$$\lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} - \lambda - p_\epsilon(\bar{x})\varphi'(\bar{x}) \;=\; \lambda \int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}} F\{dy\} \;\geqslant\; 0.$$

Therefore C4$_\epsilon$ holds.
3. To prove that 0 is a subsolution of (5.1.4), we only need to verify that 0 satisfies C3$_\epsilon$. Let $\varphi \in C^2(\mathbb{R}_+)$ and $\bar{x} > 0$ be such that $-\varphi$ has a nonpositive global maximum at $\bar{x} > 0$. Then $\varphi'(\bar{x}) = 0$ and

$$\varphi(\bar{x}) - \varphi(\bar{x} - \epsilon y)\mathcal{H}(\bar{x} - \epsilon y) \leqslant 0 \qquad (y \geqslant 0).$$

Therefore

$$\lambda\int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}}F\{dy\}-\lambda-p_\epsilon(\bar{x})\varphi'(\bar{x}) \;=\; \lambda\int e^{\frac{\varphi(\bar{x})-\varphi(\bar{x}-\epsilon y)\mathcal{H}(\bar{x}-\epsilon y)}{\epsilon}}F\{dy\}-\lambda \;\leqslant\; 0.$$

Therefore C3$_\epsilon$ holds.
4. To prove that 0 is a supersolution of (5.1.2), we only need to verify that 0 satisfies C2$_\epsilon$. Let $\varphi\in C^2(\mathbb{R}_+)$ and $\bar{x}>0$ be such that $-\varphi$ has a global minimum at $\bar{x}>0$. Then $\varphi'(\bar{x})=0$ and $\int_0^{\bar{x}/\epsilon}e^{\frac{\varphi(\bar{x}-\epsilon y)-\varphi(\bar{x})}{\epsilon}}F\{dy\}\leqslant F\left(\frac{\bar{x}}{\epsilon}\right)$. Therefore C2$_\epsilon$ holds. □

Lemma 5.6 (Increasing supersolutions of (5.2.2)). *Assume* $\varphi'>0$ *and*

$$\lambda-p\varphi'-\lambda\int e^{-\varphi' y}F\{dy\}\geqslant 0.$$

Then $\frac{p}{\lambda\mathbb{E}[Y]}<1$.

Proof: Since

$$\lambda-p\varphi'-\lambda\int e^{-\varphi' y}F\{dy\}=\lambda\mathbb{E}[Y](-\varphi')\left(\frac{p}{\lambda\mathbb{E}[Y]}-\int\frac{e^{-\varphi' y}-1}{-\varphi' y}\frac{yF\{dy\}}{\mathbb{E}[Y]}\right),$$

we conclude that

$$\frac{p}{\lambda\mathbb{E}[Y]}\leqslant\int\frac{e^{-\varphi' y}-1}{-\varphi' y}\frac{yF\{dy\}}{\mathbb{E}[Y]}.$$

But

$$\frac{e^{-\varphi' y}-1}{-\varphi' y}\leqslant 1\qquad(y\geqslant 0),$$

so the conclusion holds. □

Lemma 5.7 (Increasing subsolutions of (5.2.2)). *Assume* $\mathbb{E}[Y^2]<\infty$. *Suppose* $\varphi'>0$ *and*

$$\lambda-p\varphi'-\lambda\int e^{-\varphi' y}F\{dy\}\leqslant 0.$$

Then $\varphi'>\frac{\mathbb{E}[Y]}{\mathbb{E}[Y^2]}\log\frac{\lambda\mathbb{E}[Y]}{p}$.

Proof: Since

$$\lambda-p\varphi'-\lambda\int e^{-\varphi' y}F\{dy\}=\lambda\mathbb{E}[Y](-\varphi')\left(\frac{p}{\lambda\mathbb{E}[Y]}-\int\frac{e^{-\varphi' y}-1}{-\varphi' y}\frac{yF\{dy\}}{\mathbb{E}[Y]}\right),$$

we conclude that

$$\int\frac{e^{-\varphi' y}-1}{-\varphi' y}\frac{yF\{dy\}}{\mathbb{E}[Y]}\leqslant\frac{p}{\lambda\mathbb{E}[Y]}.$$

But Jensen's Inequality implies

$$\frac{e^{-\varphi'\xi}-1}{-\varphi'\xi}\leqslant\int\frac{e^{-\varphi' y}-1}{-\varphi' y}\frac{yF\{dy\}}{\mathbb{E}[Y]},$$

where $\xi=\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}$. Moreover, since $\varphi'>0$,

$$e^{-\varphi'\xi}\leqslant\frac{e^{-\varphi'\xi}-1}{-\varphi'\xi}.$$

Finally, since $\frac{p}{\lambda\mathbb{E}[Y]} < 1$, we have the following positive lower bound for φ':

$$\frac{\mathbb{E}[Y]}{\mathbb{E}[Y^2]} \log \frac{\lambda\mathbb{E}[Y]}{p} < \varphi'. \qquad \square$$

Lemma 5.8 (Bounds for subsolutions of (5.2.3)). *Assume* $\mathbb{E}[Y^2] < \infty$. *Suppose* $p > \lambda\mathbb{E}[Y]$ *and*

$$\lambda \int e^{zy}\, F\{dy\} - \lambda - pz \leqslant 0. \tag{5.2.17}$$

Then

$$0 \;\leqslant\; z \;\leqslant\; \frac{\sqrt{9+24\left(\frac{p}{\lambda\mathbb{E}[Y]}-1\right)}-3}{2\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}.$$

Proof: 1. First we argue by contradiction that $z \geqslant 0$. Rewrite (5.2.18) as

$$\lambda\mathbb{E}[Y]z\left(\int \tfrac{e^{zy}-1}{zy}\, \tfrac{yF\{dy\}}{\mathbb{E}[Y]} - \tfrac{p}{\lambda\mathbb{E}[Y]}\right) \leqslant 0. \tag{5.2.18}$$

If $z < 0$, then (5.2.18) implies

$$\int \tfrac{e^{zy}-1}{zy}\, \tfrac{yF\{dy\}}{\mathbb{E}[Y]} \;\geqslant\; \tfrac{p}{\lambda\mathbb{E}[Y]} \;>\; 1.$$

Since $z < 0$,

$$\tfrac{e^{zy}-1}{zy} < 1 \qquad (y > 0).$$

Thus $\int \frac{e^{zy}-1}{zy}\, \frac{yF\{dy\}}{\mathbb{E}[Y]} < 1$. This contradiction implies $z \geqslant 0$.
2. Now, (5.2.18) implies

$$\int \tfrac{e^{zy}-1}{zy}\, \tfrac{yF\{dy\}}{\mathbb{E}[Y]} \leqslant \tfrac{p}{\lambda\mathbb{E}[Y]};$$

Jensen's Inequality implies

$$\frac{e^{z\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}-1}{z\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}} \leqslant \int \tfrac{e^{zy}-1}{zy}\, \tfrac{yF\{dy\}}{\mathbb{E}[Y]},$$

since $\int y^2\, \frac{F\{dy\}}{\mathbb{E}[Y]} = \frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}$. Since $z \geqslant 0$,

$$1 + \frac{z\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}{2} + \frac{\left(\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}z\right)^2}{3!} \leqslant \frac{e^{z\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}-1}{z\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}.$$

Thus

$$1 + \frac{z\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}{2} + \frac{\left(\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}z\right)^2}{3!} \leqslant \frac{p}{\lambda\mathbb{E}[Y]},$$

so that

$$0 \;\leqslant\; z \;\leqslant\; \frac{\sqrt{9+24\left(\frac{p}{\lambda\mathbb{E}[Y]}-1\right)}-3}{2\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]}}. \qquad \square$$

Chapter 6

Stochastic Control and Ruin Probabilities

6.1 Introduction

We discuss in this final chapter some extensions of the ideas developed in previous chapters. We now consider problems in which the reserve process of an insurer may be influenced by a random control process. In each of the control problems below, the dynamics provided by the Cramér-Lundberg model serve as the case with no control. In this sense, we are exhibiting applications of the results of the previous chapters (see Schmidli [42] and Hipp [21]).

Let $(\Omega, \mathcal{U}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. We assume the filtration $\mathcal{F}(\cdot)$ is right continuous.

Definition. We say that C is a compound Poisson process with the usual parameters N, λ, T, Y, and F if

(i) $N = \{N(t) : t \geqslant 0\}$ is a time homogeneous Poisson process with intensity λ,

(ii) $T = \{T(n) : n \in \mathbb{N}\}$ is the arrival process associated with N (see Appendix A.2),

(iii) $Y = \{Y(n) : n \in \mathbb{N}\}$ is positive, independent and identically distributed, with distribution F,

(iv) N and Y are independent, and

(v) $C(t) = \sum_{n=1}^{N(t)} Y(n)$.

In this case, $N(t)$ represents the number of claims which have arrived during the time interval $[0, t]$, $T(n)$ is the time at which the nth claim arrives, and $Y(n)$ is the size of the nth claim.

6.2 New business

We consider the following model for an insurance business. The insurer wishes to minimize infinite time ruin probability via optimal choice of new business. We follow the problem formulation in Hipp and Taksar's [22].

For this section, we make the following assumption.

Hypothesis.

$$L := \inf\{y \geqslant 0 \mid F(y) = 1\} < \infty. \tag{6.2.1}$$

Since F is right-continuous and $F(0) = 0$, we have $L > 0$ and $F(L) = 0$. This hypothesis is interpreted as assuming there is a bound on the size of claims which may be presented to the insurer.

Problem formulation

We focus on a fixed insurance portfolio. We suppose p is the constant premium intensity of the portfolio. The total claims up to time $t \geqslant 0$ are modeled by a compound Poisson process with the usual parameters N, λ, T, Y, and F: $C(t) = \sum_{n=1}^{N(t)} Y(n)$. We assume the second portfolio is similarly modeled, with premium intensity $\tilde{p}$ and total claims up to time t modeled by a compound Poisson process $\tilde{C}$ with the usual parameters $\tilde{N}$, $\tilde{\lambda}$, $\tilde{T}$, $\tilde{Y}$, and $\tilde{F}$. Additionally, we assume $\tilde{F} \neq F$.

Let $X^\alpha = X^\alpha(t, x)$ denote the reserve process of the insurer starting from an initial capital $x \geqslant 0$ and governed by a given strategy $\alpha = \alpha(t)$. Admissible controls are those that are α predictable and right continuous.

For new business, the controlled reserve process evolves according to

$$X^\alpha(t, x) = x + pt + \tilde{p}\int_0^t \alpha(s)\, ds - C(t) - \sum_{n=1}^{\tilde{N}(t)} \alpha(\tilde{T}(n))\tilde{Y}(n).$$

Dynamic programming equation

The dynamic programming equation for our problem (see Hipp and Taksar [22]) is

$$0 = \sup_{0 \leqslant a \leqslant 1} \left\{ \begin{array}{r} \lambda \int_0^x u(x-y)\, F\{dy\} + a\tilde{\lambda} \int_0^x u(x-y)\, \tilde{F}\{dy\} \\ -(\lambda + a\tilde{\lambda})u(x) + (p + a\tilde{p})u'(x) \end{array} \right\}. \tag{6.2.2}$$

Consider now for $\epsilon > 0$ the perturbation:

$$0 = \sup_{0 \leqslant a \leqslant 1} \left\{ \begin{array}{r} \lambda \int_0^{x/\epsilon} u_\epsilon(x-\epsilon y)\, F\{dy\} + a\tilde{\lambda} \int_0^{x/\epsilon} u_\epsilon(x-\epsilon y)\, \tilde{F}\{dy\} \\ -(\lambda + a\tilde{\lambda})u_\epsilon(x) + \epsilon(p + a\tilde{p})u_\epsilon'(x) \end{array} \right\}. \tag{6.2.3}$$

We are interested in determining if and how u_ϵ approximates u. First, we first perform

the WKB motivated transformation $u_\epsilon = 1 - e^{\frac{w_\epsilon}{\epsilon}}$. The resulting equation is

$$0 = pw'(x) + \lambda - \lambda \int_0^{x/\epsilon} e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}} F\{dy\} - \lambda \epsilon H_\epsilon(x) e^{\frac{w(x)}{\epsilon}} + \sup_{0 \leqslant a \leqslant 1} \left\{ a \left(\tilde{p} w'(x) + \tilde{\lambda} - \tilde{\lambda} \int_0^{x/\epsilon} e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}} \tilde{F}\{dy\} - \tilde{\lambda} \epsilon \tilde{H}_\epsilon(x) e^{\frac{w(x)}{\epsilon}} \right) \right\}. \quad (6.2.4)$$

Supposing that as $\epsilon \to 0$, the functions w_ϵ converge in some sense to a limit w, we invoke the hypothesis that $F(L) = 0$ to compute the limiting equation:

$$0 = pw'(x) + \lambda - \lambda \int e^{w'(x)y} F\{dy\} + \sup_{0 \leqslant a \leqslant 1} \left\{ a \left(\tilde{p} w'(x) + \tilde{\lambda} - \tilde{\lambda} \int e^{w'(x)y} \tilde{F}\{dy\} \right) \right\}. \quad (6.2.5)$$

Therefore, we expect solutions u_ϵ of (6.2.3) to satisfy

$$u_\epsilon \;=\; 1 - e^{\frac{w}{\epsilon}} \qquad \text{with} \qquad w \text{ solving (6.2.5)}.$$

6.3 Reinsurance

In this section, we perturb fully nonlinear nonlocal operators. The limiting equations are *nonlinear approximations* which can be considered as reduced in order.

We consider the following model for an insurance business. The insurer wishes to minimize infinite time ruin probability via controlling the treaty variable. See Waters [45] and Hipp [21].

6.3.1 General case

Problem formulation

Let X^α denote the reserve process of the insurer governed by control α. The dynamics of the reserve are described by

$$X^\alpha(t,x) = x + pt - \int_0^t h(\alpha(s))\, ds - \sum_{n=1}^{N(t)} c\left(\alpha(T(n)), Y(n)\right). \quad (6.3.1)$$

Here, $p \geqslant 0$ is a constant, $t \geqslant 0$, and $x \geqslant 0$. $C(t) = \sum_{n=1}^{N(t)} Y(n)$ is a compound Poisson process with the usual parameters N, λ, T, Y, and F.

The control process $\alpha = \{\alpha(t)\}_{t \geqslant 0}$ is progressively measurable with respect to $\mathcal{F}$, and takes values in closed $A \subset \mathbb{R}$. The set of admissible controls $\mathcal{A}$ consists of all processes predictable with respect to the filtration generated by C.

We assume $h = h(\cdot)$ is continuous on A and takes values in $\mathbb{R}_+$. Regarding $c : A \times \mathbb{R}_+ \to \mathbb{R}_+$, we assume

- $c(\cdot, y)$ is continuous for each $y \in \mathbb{R}_+$

- $c(a,\cdot)$ is nondecreasing for each $a \in A$
- $0 \leqslant c(a,y) \leqslant y$ for all $(a,y) \in A \times \mathbb{R}_+$
- there exists $\bar{a} \in A$ such that $c(\bar{a},y) = y$ for all $y \in \mathbb{R}_+$.

Physical interpretation

The function h represents the reinsurance premium paid by the insurer to the reinsurer. The rule $c = c(a,y)$ describes how each of the insurer's claims is divided between the insurer and reinsurer. Given a claim of size Y, the insurer pays $c(a,Y)$ while the reinsurer pays $Y - c(a,Y)$.

The control $\alpha = \alpha(t)$ quantifies the insurer's decision regarding a treaty variable made at time t. Ultimately, the goal of the insurer is to maximize the probability of survival by controlling the treaty variable(s).

Dynamic programming equations

For clarity, we introduce the following notation used for domains of integration. Let

$$D(a,x) := \{y \geqslant 0 \mid x \geqslant c(a,y)\} \qquad (a \in A)$$

and

$$D(a,\epsilon,x) = \{y \geqslant 0 \mid x \geqslant c(a,\epsilon y)\} \qquad (a \in A, \epsilon > 0),$$

for each $x \in \mathbb{R}_+$.

Given rule c, reinsurance premium h, and control state space A, we consider

$$0 = \sup_{a\in A}\{(p - h(a))u'(x) - \lambda u(x) + \lambda \textstyle\int_{D(a,y)} u(x - c(a,y))\, F\{dy\}\},$$

with natural boundary condition $u(\infty) = 1$. We mention that the choice of A depends on the rule c.

We now rescale the dynamic programming equation to get

$$0 = \sup_{a\in A}\left\{\epsilon(p - h_\epsilon(a))u'(x) - \lambda u(x) + \lambda \textstyle\int_{D(a,\epsilon,x)} u(x - c(a,\epsilon y))\, F\{dy\}\right\}, \qquad (6.3.2)$$

with boundary condition $u(\infty) = 1$. Since the reinsurance premium typically depends on the aggregate claims process $C = C(t)$, we denote the dependence of the rescaled h on ϵ by h_ϵ.

Now we transform the dynamic programming equation. After making the substitution $u = 1 - e^{-\frac{w}{\epsilon}}$,(6.3.2) becomes

$$0 = \sup_{a\in A}\left\{(p - h_\epsilon(a))w'(x) + \lambda - \lambda \textstyle\int_{D(a,\epsilon,x)} e^{\frac{w(x)-w(x-c(a,\epsilon y))}{\epsilon}}\, F\{dy\}\right\}. \qquad (6.3.3)$$

Our next objective is to formulate the limiting equation. First, we rearrange

(6.3.3) to get

$$0 = pw'(x) + \lambda + \sup_{a\in A}\left\{-h_\epsilon(a)w'(x) - \lambda\int_{D(a,\epsilon,x)} e^{\frac{w(x)-w(x-c(a,\epsilon y))}{\epsilon}}\, F\{dy\}\right\}. \tag{6.3.4}$$

Since $0 \leqslant c(a,\epsilon y) \leqslant \epsilon y$ for all $(a,y) \in A \times \mathbb{R}_+$, we must have $c(a,0) = 0$. Assuming c is smooth, we formally send $\epsilon \to 0$ to get the limiting equation

$$0 = \sup_{a\in A}\left\{(p - h_0(a))w'(x) + \lambda - \lambda\int_{D_0(a,x)} e^{yw'(x)\partial_y c(a,0)}\, F\{dy\}\right\}. \tag{6.3.5}$$

Here we use h_0 and D_0 to denote the uniform (with respect to A) limits of $h_\epsilon(a)$ and $D(a,\epsilon,x)$, respectively, as $\epsilon \to 0$. Similarly, regarding D as the set valued function $\mathbb{1}_{D(a,\epsilon,x)}(y)$, we let $D_0(a,x)$ denote the uniform limit of $D(a,\epsilon,x)$, over A, as $\epsilon \to 0$.

6.3.2 Examples

Below we exhibit two types of reinsurance. We list the dynamic programming equations without derivation. See, for instance, Hipp [21], Waters [45], or Schmidli [42] for the derivation of the dynamic programming equation from the dynamic programming principle.

A. Proportional reinsurance

Formulating the problem The rule c and reinsurance premium h are defined as follows:

$$\begin{aligned} c(a,y) &:= ay \qquad (0 \leqslant a \leqslant 1, y \geqslant 0);\\ h(a) &:= a\rho\lambda\mathbb{E}[Y] \qquad (0 \leqslant a \leqslant 1), \end{aligned} \tag{6.3.6}$$

for some fixed $\rho > 1$ satisfying $p < \rho\lambda\mathbb{E}[Y]$.

When the proportionality factor $a \in [0,1]$ is chosen, an insurer pays aY of a claim of size Y, while the reinsurer pays $(1-a)Y$.

We consider the problem of optimizing the probability of survival by controlling the proportionality factor.

Dynamic programming equation We have:

$$0 = \sup_{0\leqslant a\leqslant 1}\left\{(p - h(a))u'(x) - \lambda u(x) + \lambda\int_{D(a,x)} u(x-ay)\, F\{dy\}\right\}. \tag{6.3.7}$$

Here, $D(a,x) = \left[0, \frac{x}{a}\right]$ for $0 < a \leqslant 1$, $D(0,x) = \mathbb{R}_+$, and the boundary condition is $u(\infty) = 1$.

The rescaled equation is

$$0 = \sup_{a\in[0,1]}\left\{(p - a\rho\lambda\mathbb{E}[Y])w'(x) + \lambda - \lambda\int_{D(a,\epsilon,x)} e^{\frac{w(x)-w(x-\epsilon a y)}{\epsilon}}\, F\{dy\}\right\}, \tag{6.3.8}$$

where $D(a,\epsilon,x) = \left[0, \frac{x}{\epsilon a}\right]$ for $a \neq 0$ and $D(0,\epsilon,x) = \mathbb{R}_+$.

The limiting equation is

$$0 = \sup_{a\in[0,1]} \left\{(p - a\rho\lambda\mathbb{E}[Y])w'(x) + \lambda - \lambda \int e^{aw'(x)y}\, F\{dy\}\right\}. \tag{6.3.9}$$

B. Excess of loss reinsurance

In the following example we consider optimal nonproportional reinsurance control. We will need the following

Notation. $u^+ := \max(u, 0)$ and $u^- := -\min(u, 0)$
See Hipp and Taksar [23].

Unlimited excess of loss reinsurance We begin by formulating the problem. The rule c and reinsurance premium h are defined as follows:

$$\begin{aligned} c(a, y) &:= \min(a, y) \qquad (0 \leqslant a \leqslant \infty, y \geqslant 0); \\ h(a) &:= \rho\lambda \int (y - a)^+ F\{dy\} \qquad (0 \leqslant a \leqslant \infty), \end{aligned} \tag{6.3.10}$$

for some fixed $\rho > \max\left(1, \frac{p}{\lambda\mathbb{E}[Y]}\right)$.

Physical interpretation In this case, the insurer pays the smaller of the *priority* a and the claim of size Y. If $Y < a$, the reinsurer pays nothing, but if $Y > a$, the reinsurer pays the remaining $Y - a$.

Requiring $\rho > \frac{p}{\lambda\mathbb{E}[Y]}$ ensures reinsurance is expensive, in the sense that choosing priority $a = 0$ costs more than the insurer's premiums.

The goal is to maximize the probability of survival by controlling the priority a.

Dynamic programming equation We consider the equation

$$0 = \sup_{0\leqslant a\leqslant\infty} \left\{(p - h(a))u'(x) - \lambda u(x) + \lambda \int_0^a u(x - y)\, F\{dy\} + \lambda u(x - a)(1 - F(a))\right\},$$

with boundary condition $u(\infty) = 1$.

Now, the rescaled equation is

$$0 = \sup_{0\leqslant a\leqslant\infty} \left\{\epsilon(p - h_\epsilon(a))u'(x) - \lambda u(x) + \lambda \int_{D(a,\epsilon,x)} u(x - \min(a, \epsilon y))\, F\{dy\}\right\}, \tag{6.3.11}$$

where $h_\epsilon(a) := \rho\lambda \int (\epsilon y - a)^+ F\{dy\}$ and $D(a, \epsilon, x) = \begin{cases} [0, x/\epsilon] & (x \leqslant a) \\ \mathbb{R}_+ & (a \leqslant x) \end{cases}$.

We substitute $u = 1 - e^{-\frac{w}{\epsilon}}$ into (6.3.11) to get the transformed equation

$$0 = \sup_{0 \leqslant a \leqslant \infty} \left\{ (p - h_\epsilon(a)) w'(x) e^{-\frac{w(x)}{\epsilon}} - \lambda \left(1 - e^{-\frac{w(x)}{\epsilon}}\right) + \lambda \int_{D(a,\epsilon,x)} \left(1 - e^{-\frac{w(x - \min(a, \epsilon y))}{\epsilon}}\right) F\{dy\} \right\}.$$

This is equivalent to

$$0 = \sup_{0 \leqslant a \leqslant \infty} \left\{ \lambda + (p - h_\epsilon(a)) w'(x) - \lambda \int_{D(a,\epsilon,x)} e^{\frac{w(x) - w(x - \min(a, \epsilon y))}{\epsilon}} F\{dy\} - \lambda e^{\frac{w(x)}{\epsilon}} \epsilon H_\epsilon(x) \mathcal{H}(a - x) \right\}. \tag{6.3.12}$$

For each $a \in [0, \infty]$,

$$\begin{aligned} \int_{D(a,\epsilon,x)} e^{\frac{w(x) - w(x - \min(a, \epsilon y))}{\epsilon}} F\{dy\} &= \epsilon H_\epsilon(a) e^{\frac{w(x) - w(x-a)}{\epsilon}} \mathcal{H}(x - a) \\ &\quad + \mathcal{H}(x - a) \int_0^{a/\epsilon} e^{\frac{w(x) - w(x - \epsilon y)}{\epsilon}} F\{dy\} \\ &\quad + \mathcal{H}(a - x) \int_0^{x/\epsilon} e^{\frac{w(x) - w(x - \epsilon y)}{\epsilon}} F\{dy\} \end{aligned}$$

Thus, we express (6.3.12) as

$$\begin{aligned} 0 = \lambda + p w'(x) \\ - \min \Bigg(\inf_{0 \leqslant a \leqslant x} \left\{ h_\epsilon(a) w'(x) + \lambda \textstyle\int_0^{a/\epsilon} e^{\frac{w(x) - w(x - \epsilon y)}{\epsilon}} F\{dy\} + \lambda \epsilon H_\epsilon(a) e^{\frac{w(x) - w(x-a)}{\epsilon}} \right\}, \\ \inf_{x \leqslant a \leqslant \infty} \left\{ h_\epsilon(a) w'(x) + \lambda \textstyle\int_0^{x/\epsilon} e^{\frac{w(x) - w(x - \epsilon y)}{\epsilon}} F\{dy\} + \lambda \epsilon H_\epsilon(x) e^{\frac{w(x)}{\epsilon}} \right\} \Bigg). \end{aligned} \tag{6.3.13}$$

To compute the limiting equation, we first note that $h_\epsilon \to 0$ as $\epsilon \to 0$ uniformly on A. Indeed, for $a \in \mathbb{R}_+$,

$$\begin{aligned} h_\epsilon(a) = \rho\lambda \int_{a/\epsilon}^{\infty} (\epsilon y - a) F\{dy\} &= \rho\lambda \int_{a/\epsilon}^{\infty} \int_a^{\epsilon y} dt \, F\{dy\} \\ &= \rho\lambda \int_a^{\infty} \int_{t/\epsilon}^{\infty} F\{dy\} \, dt \\ &= \epsilon\rho\lambda \int_a^{\infty} H_\epsilon(t) \, dt \;\leqslant\; \epsilon\rho\lambda \int_0^{\infty} H_\epsilon(t) \, dt \;=\; \epsilon\rho\lambda\mathbb{E}. \end{aligned}$$

Next, we analyze the first supremum in the maximum of (6.3.13):

$$\sup_{0\leqslant a\leqslant x}\left\{\lambda+(p-h_\epsilon(a))w'(x)-\lambda\textstyle\int_0^{a/\epsilon}e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}}\,F\{dy\}-\lambda\epsilon H_\epsilon(a)e^{\frac{w(x)-w(x-a)}{\epsilon}}\right\}$$
$$=\lambda+pw'(x)-\inf_{0\leqslant a\leqslant x}\left\{h_\epsilon(a)w'(x)+\lambda\textstyle\int_0^{a/\epsilon}e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}}\,F\{dy\}+\lambda\epsilon H_\epsilon(a)e^{\frac{w(x)-w(x-a)}{\epsilon}}\right\}$$

Now we analyze the second supremum in the maximum of (6.3.13):

$$\sup_{x\leqslant a\leqslant\infty}\left\{\lambda+(p-h_\epsilon(a))w'(x)-\lambda\textstyle\int_0^{x/\epsilon}e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}}\,F\{dy\}-\lambda\epsilon H_\epsilon(x)e^{\frac{w(x)}{\epsilon}}\right\}$$
$$=\lambda+pw'(x)-\lambda\textstyle\int_0^{x/\epsilon}e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}}\,F\{dy\}-\lambda\epsilon H_\epsilon(x)e^{\frac{w(x)}{\epsilon}}-\inf\{h_\epsilon(a)w'(x)\mid a\in[x,\infty]\}$$
$$=\lambda+pw'(x)-\lambda\textstyle\int e^{w'(x)y}\,F\{dy\}.$$

The last equality follows from an assumption that $\epsilon H_\epsilon(x)e^{\frac{w(x)}{\epsilon}}\to 0$ as $\epsilon\to 0$. Putting it all together, we see that the limit equation is

$$0=\max\left(\begin{array}{l}\lambda+pw'(x)-\lambda\displaystyle\int e^{w'(x)y}\,F\{dy\},\\ \lambda+pw'(x)-\displaystyle\inf_{0\leqslant a\leqslant x}\left\{\lambda\textstyle\int_0^{a/\epsilon}e^{\frac{w(x)-w(x-\epsilon y)}{\epsilon}}\,F\{dy\}+\lambda\epsilon H_\epsilon(a)e^{\frac{w(x)-w(x-a)}{\epsilon}}\right\}\end{array}\right).\tag{6.3.14}$$

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Appendix A

Notation

A.1 Notation for functions.

One-sided limits.

Let $f \in L^1_{loc}(\mathbb{R}_+)$ and $c \in \mathbb{R}_+$. When either of the following one-sided limits exists, we denote it as follows:

$$f(c+) := \lim_{h \to 0^+} f(c+h) \qquad \text{(limit of } f \text{ at } c \text{ from above)},$$
$$f(c-) := \lim_{h \to 0^+} f(c-h) \qquad \text{(limit of } f \text{ at } c \text{ from below)}.$$

If $\lim_{x \to \infty} f(x)$ exists, we write

$$f(\infty) := \lim_{x \to \infty} f(x).$$

Special functions.

Recall that the indicator function $\mathbb{1}_B$ equals 1 in B and vanishes outside B, for $B \in \mathcal{B}$. If $B = \mathbb{R}_+$, the indicator function is denoted by $\mathcal{H}$. In particular,

$$\mathcal{H}(y) = \mathbb{1}_{[0,\infty)}(y) \qquad (y \in \mathbb{R}),$$

and

$$F(c) = \int \mathcal{H}(c-y)\, F\{dy\} \qquad (c \in \mathbb{R}).$$

We mention that $\mathcal{H}$ is the Heaviside function (see Schwartz [43]).

Function spaces.

Let μ be a Borel measure.

(i) $L^1(\mathbb{R}_+;\mu) := \{u\colon \mathbb{R}_+ \to \mathbb{R} \mid \int_0^\infty |u(y)|\, \mu\{dy\} \text{ exists finitely}\}$
$L^1_{loc}(\mathbb{R}_+;\mu) := \{u\colon \mathbb{R}_+ \to \mathbb{R} \mid \int_0^x |u(y)|\, \mu\{dy\} \text{ is finite for every } x \geqslant 0\}$
$L^\infty_{loc}(\mathbb{R}_+;\mu) := \{u\colon \mathbb{R}_+ \to \mathbb{R} \mid u \text{ is locally bounded with respect to } \mu\}$.
When μ is Lebesgue measure, we abbreviate the aforementioned spaces by

$L^1(\mathbb{R}_+)$, $L^1_{loc}(\mathbb{R}_+)$, and $L^\infty_{loc}(\mathbb{R}_+)$. We denote integration with respect to μ by dx.

(ii) $C(\mathbb{R}_+) := \{u\colon \mathbb{R}_+ \to \mathbb{R} \mid u \text{ is continuous}\}$

For $k = 1, 2, \ldots,$

$C^k(\mathbb{R}_+) := \{u\colon \mathbb{R}_+ \to \mathbb{R} \mid u \text{ is } k\text{-times continuously differentiable}\}$.

(iii) $\mathcal{CA} = \{u\colon \mathbb{R}_+ \mid u \text{ is right continuous and } u(c-) \text{ exists for each } c > 0.\}$.

A.2 Stochastic notation.

Notation for stochastic processes.

The following definitions can be found, for instance, in Evans [15] and Ramasubramanian [36].

Poisson process. A nonnegative integer-valued stochastic process $N(\cdot)$ is called a Poisson process with rate $\lambda > 0$ if

(i) $N(0) = 0$,

(ii) $N(s) \leqslant N(t)$ whenever $0 \leqslant s \leqslant t$,

(iii) for all $s, t \geqslant 0$ and $n = 0, 1, \ldots,$

$$P[N(t+s) - N(s) = n] \;=\; \tfrac{(\lambda t)^n}{n!} e^{-\lambda t}$$

(iv) for all times $0 \leqslant t_1 < t_2 < \ldots < t_n$, the corresponding $n-1$ increments $N(t_2) - N(t_1), \cdots, N(t_n) - N(t_{n-1})$ are independent.

Arrival time process. Let $N = \{N(t) : t \geqslant 0\}$ be a Poisson process with rate λ, and suppose $N(t)$ is the number of claims occurring by time t, for $t \geqslant 0$. It will be helpful to characterize N in terms of arrival times. Consider the process T satisfying $T(0) = 0$ and

$$T(n) = \inf\{t \geqslant 0 \mid N(t) = n\} \qquad (n \in \mathbb{N}).$$

Then $T(n)$ denotes the time of arrival of the nth claim and

$$N(t) = \max\{n \geqslant 0 \mid T(n) \leqslant t\}.$$

Let

$W_1 :=$ the waiting time until the first claim arrives

$W_i :=$ the interarrival time between claim $Y(i-1)$ and $Y(i)$ $\qquad (i \geqslant 2)$.

Consequently,

$$N(t) = \max\{k \mid W(1) + \cdots + W(k) \leqslant t\},$$

$N(t)$	number of claims arriving during the time interval $[0, t]$
$Y(i)$	size of the ith claim
$F(x)$	probability that a claim is less than size x
λ	expected rate of number of claims arriving per unit time
μ	expected size of a claim
$p = p(\cdot)$	rate of income per unit time
$X(t)$	surplus at time t
$u(x)$	probability of ultimate survival given initial surplus x
$u(x, t)$	probability of ultimate survival during the time interval $[0, t]$ given initial surplus x

Table A.1: Probabilistic Interpretation.

and the $Y(n)$th claim occurs at time $T(n) = W(1) + \cdots + W(n)$ for $n \geqslant 1$.

Wiener process. A real-valued stochastic process $W(\cdot)$ is called a Wiener process if

(i) $W(0) = 0$ almost surely,

(ii) For all $0 \leqslant s \leqslant t$, the density of $W(t) - W(s)$ is

$$f(x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}} \qquad (x \in \mathbb{R}),$$

(iii) for all times $0 < t_1 < t_2 < \cdots < t_n$, the corresponding n increments $W(t_1)$, $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent.

Physical interpretations of stochastic quantities

Now, we introduce the notation of relevant variables and constants and their stochastic interpretations.

$\tau(x)$	time to ruin for $X(t)$, given initial surplus x
$b = b(\cdot)$	control process
$X^b(t)$	surplus at time t governed by control b
$u^b(x)$	probability of ultimate survival for $\{X^b(s)\}$ occurring given initial surplus x
$u^b(x,t)$	probability of ultimate survival for $\{X^b(s)\}$ occurring during the time interval $[0,t]$ given initial surplus x
$\tau^b(x)$	time to ruin for $\{X^b(s)\}$ given initial surplus x

Table A.2: Probabilistic Interpretation.

Appendix B

Some computations

B.1 Calculus

Consider the operators B_ϵ and S_ϵ defined by

$$\begin{aligned} B_\epsilon u(x) &:= \int_0^{x/\epsilon} u(y)\, F\{dy\} \qquad \left(u \in \left(L^1_{loc}(\mathbb{R}_+); F\right), x \geqslant 0\right) \\ S_\epsilon u(x) &:= \int e^{-\frac{u(x-\epsilon y)\mathcal{H}(x-\epsilon y)}{\epsilon}}\, F\{dy\} \qquad \left(u \in \left(L^1_{loc}(\mathbb{R}_+); F\right), x \geqslant 0\right). \end{aligned} \tag{B.1}$$

Lemma B.1 (Jumps). *Fix $\epsilon > 0$. Suppose $u \in C(\mathbb{R}_+)$.*
Then

(i) *$B_\epsilon u$ and $S_\epsilon u$ are right continuous.*

(ii) *For each $x > 0$, $B_\epsilon u(x-)$ exists with*

$$B_\epsilon u(x) - B_\epsilon u(x-) = u\left(\tfrac{x}{\epsilon}\right) F\left\{\tfrac{x}{\epsilon}\right\}. \tag{B.2}$$

(iii) *For each $x > 0$, $S_\epsilon u(x-)$ exists with*

$$S_\epsilon u(x) - S_\epsilon u(x-) = \left(e^{\frac{u(0)}{\epsilon}} - 1\right) F\left\{\tfrac{x}{\epsilon}\right\}. \tag{B.3}$$

Proof: Fix $x > 0$, and suppose $h > 0$ is small. First we consider the limit of $B_\epsilon u$ at x from below. Since u is continuous,

$$\begin{aligned} B_\epsilon u(x) - B_\epsilon u(x-h) &= \textstyle\int u(y) \mathbb{1}_{(x-h,x]}(\epsilon y) F\{dy\} \\ &\leqslant \textstyle\int \max_{[0,x]} |u| \mathbb{1}_{(x-h,x]}(\epsilon y) F\{dy\}. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{h\to 0^+} \left(B_\epsilon u(x) - B_\epsilon u(x-h)\right) &= \textstyle\int \lim_{h\to 0^+} u(y) \mathbb{1}_{(x-h,x]}(\epsilon y) F\{dy\} \\ &= \textstyle\int u(y) \mathbb{1}_{\{x\}}(\epsilon y) F\{dy\} \\ &= u\left(\tfrac{x}{\epsilon}\right) F\left\{\tfrac{x}{\epsilon}\right\}. \end{aligned}$$

Next, we consider the limit of $S_\epsilon u$ at x from above. Denote the integrand of $S_\epsilon u(x+h)$ by

$$g(y,h) := e^{-\frac{u(x+h-\epsilon y)\mathcal{H}(x+h-\epsilon y)}{\epsilon}}.$$

Then

$$S_\epsilon u(x+h) = \int g(y;h)\,\mathbb{1}_{(0,x)}(\epsilon y)F\{dy\} + g\big(\tfrac{x}{\epsilon};h\big)F\big\{\tfrac{x}{\epsilon}\big\} + \int \mathbb{1}_{(x,\infty)}(\epsilon y)g(y;h)\,F\{dy\}.$$

By Lebesgue's dominated convergence theorem (Applebaum [3]),

$$\begin{aligned}\lim_{h\to 0^-}\int g(y;h)\,F\{dy\} &= \int \mathbb{1}_{(0,x)}(\epsilon y)g(y;0)\,F\{dy\} + F\big\{\tfrac{x}{\epsilon}\big\}\\ &\quad + \mathbb{1}_{(x,\infty)}(\epsilon y)g(y;0)\,F\{dy\}\\ &= \int \mathbb{1}_{(0,x)}(\epsilon y)g(y;0)\,F\{dy\} + F\big\{\tfrac{x}{\epsilon}\big\}\\ &\quad + F\big(\tfrac{x}{\epsilon},\infty\big).\end{aligned}$$

Therefore

$$\lim_{h\to 0^-}\int g(y;h)\,F\{dy\} = \int \mathbb{1}_{(0,x)}(\epsilon y)e^{-\frac{u(x-\epsilon y)}{\epsilon}}\,F\{dy\} + F\big[\tfrac{x}{\epsilon},\infty\big). \tag{B.4}$$

As g is right-continuous, for any $y \geqslant 0$,

$$\lim_{h\to 0^+} g(y;h) = g(y;0).$$

So, by Lebesgue's dominated convergence theorem,

$$\lim_{h\to 0^+}\int g(y;h)\,F\{dy\} \;=\; \int g(y;0)\,F\{dy\} \;=\; \int e^{-\frac{u\mathcal{H}(x-\epsilon y)}{\epsilon}}\,F\{dy\}. \tag{B.5}$$

Combining (B.4) and (B.5), we see that (B.2) and (B.3) hold. □

Remark. This last lemma means the jump at $\bar{x}$ of $S_\epsilon u$ is $\left|1-e^{-\frac{u(0)}{\epsilon}}\right| F\big\{\frac{\bar{x}}{\epsilon}\big\}$.

B.2 Convolution lemmas

Lemma B.2. *Assume $f \in L^1_{loc}(\mathbb{R}_+)$ is nonnegative and $f(\infty)$ exists. Suppose $g \in L^1(\mathbb{R}_+)$ is nonnegative and non-increasing.*
Then

$$(f*g)(\infty) \text{ exists,}$$

with

$$(f*g)(\infty) = f(\infty)\|g\|_{L^1}. \tag{B.6}$$

Proof: Since $f(\infty)$ exists, for every $\epsilon > 0$ there exists N such that

$$|f(\infty) - f(x)| \leqslant \epsilon \qquad (x \geqslant N).$$

For $x > N$, we estimate $(f * g)(x)$ by partitioning the domain of integration. Since $x > N$, we have

$$(f * g)(x) = \int_0^N f(s)g(x-s)\,ds + \int_N^x f(s)g(x-s)\,ds,$$

with

$$g(x)\int_0^N f(s)\,ds \quad \leqslant \quad \int_0^N f(s)g(x-s)\,ds \quad \leqslant \quad g(x-N)\int_0^N f(s)\,ds$$

and

$$(f(\infty)-\epsilon)\int_0^{x-N} g(s)\,ds \quad \leqslant \quad \int_N^x f(s)g(x-s)\,ds \quad \leqslant \quad (f(\infty)+\epsilon)\int_0^{x-N} g(s)\,ds.$$

Therefore,

$$(f * g)(x) \geqslant g(x)\int_0^N f(s)\,ds + \big(f(\infty)-\epsilon\big)\int_0^{x-N} g(s)\,ds,$$

and

$$(f * g)(x) \leqslant g(x-N)\int_0^N f(s)\,ds + \big(f(\infty)+\epsilon\big)\int_0^{x-N} g(s)\,ds.$$

Sending $x \to \infty$ and $\epsilon \to 0$ completes the proof, since $g(\infty) = 0$. □

Lemma B.3 (Convolution and function spaces). *Assume f, $g \in L^1_{loc}(\mathbb{R}_+)$.*

(i) *If $f \in L^\infty(\mathbb{R}_+)$, then*

$$f * g \in L^1_{loc}(\mathbb{R}_+),$$

with the pointwise estimate

$$\int_0^x |f * g(s)|\,ds \leqslant \|f\|_{L^\infty}\int_0^x (x-s)\,|g(s)|\,ds \qquad (x \geqslant 0).$$

(ii) *If $f \in L^\infty(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+)$, then*

$$f * g \in L^\infty(\mathbb{R}_+),$$

with the estimate

$$\|f * g\|_{L^\infty} \leqslant \|f\|_{L^\infty}\|g\|_{L^1}.$$

(iii) *If $f, g \in L^1(\mathbb{R}_+)$, then*

$$f * g \in L^1(\mathbb{R}_+),$$

with the estimate

$$\|f * g\|_{L^1} \leqslant \|f\|_{L^1}\|g\|_{L^1}.$$

Moreover, if f and g are nonnegative, then

$$\|f * g\|_{L^1} = \|f\|_{L^1}\|g\|_{L^1}. \tag{B.7}$$

Proof: 1. We prove (i) and (ii) by first noting that

$$|f * g|(x) \;\leqslant\; \textstyle\int_0^x |f(x-s)g(s)|\,ds \;\leqslant\; \|f\|_{L^\infty} \int_0^x |g(s)|\,ds, \tag{B.8}$$

for each $x \in \mathbb{R}_+$. Therefore,

$$\textstyle\int_0^x |f * g(s)|\,ds \;\leqslant\; \|f\|_{L^\infty} \int_0^x \int_0^s |g(t)|\,dt\,ds \;=\; \|f\|_{L^\infty} \int_0^x (x-t)|g(t)|\,dt.$$

Further, if $g \in L^1(\mathbb{R}_+)$, then according to (B.8),

$$|f * g|(x) \;\leqslant\; \|f\|_{L^\infty} \textstyle\int_0^x |g(t)|\,dt \;\leqslant\; \|f\|_{L^\infty} \|g\|_{L^1}.$$

2. To prove (iii), we estimate

$$\begin{aligned} \|f * g\|_{L^1} &\leqslant \textstyle\int_0^\infty \int_0^x |f(x-s)g(s)|\,ds\,dx \\ &= \textstyle\int_0^\infty \int_s^\infty |f(x-s)g(s)|\,dx\,ds \;=\; \int_0^\infty \int_0^\infty |f(x)||g(s)|\,dx\,ds \\ &= \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

Since f and g are nonnegative,

$$\|f * g\|_{L^1} \;=\; \textstyle\int_0^\infty \int_0^x f(x-s)g(s)\,ds\,dx \;=\; \int_0^\infty \int_s^\infty f(x-s)g(s)\,dx\,ds.$$

Finally, a change of variables $t = x - s$ applied to the last expression yields

$$\textstyle\int_0^\infty \int_0^\infty f(t)g(s)\,dt\,ds = \|f\|_{L^1} \|g\|_{L^1}.$$ □

Recall that for $s \in \mathbb{R}$, the function e_s denotes

$$\mathsf{e}_s(y) := e^{sy} \qquad (y \in \mathbb{R}).$$

Lemma B.4. *Assume $f \in L^1(\mathbb{R}_+)$ is nonnegative, right continuous, and nonincreasing.*
Then there exists $\theta \geqslant 0$ such that

$$\int_0^\infty \mathsf{e}_{-\theta}(x) f(x)\,dx < 1.$$

Proof: Note that for all $s > 0$,

$$0 \;\leqslant\; s \textstyle\int_0^\infty f(t)\mathsf{e}_{-s}(t)\,dt \;\leqslant\; \|f\|_{L^1}.$$ □

Lemma B.5 (Convolution and $\mathsf{e}_{(\cdot)}$**).** *Assume $f \in L^\infty(\mathbb{R}_+)$ and $g \in L^1_{loc}(\mathbb{R}_+)$. Suppose $s \geqslant 0$, and let n be a positive integer.*
Then

$$(\mathsf{e}_s f) * (\mathsf{e}_s g) \in L^1_{loc}(\mathbb{R}_+),$$

and

$$\mathsf{e}_s(f * g) = (\mathsf{e}_s f) * (\mathsf{e}_s g). \tag{B.9}$$

Consequently,

$$\mathsf{e}_s\big(f^{n*}\big) = \big(\mathsf{e}_s f\big)^{n*}. \tag{B.10}$$

Proof: Fix $x \in \mathbb{R}_+$. We compute

$$\begin{aligned}(\mathsf{e}_s f) * (\mathsf{e}_s g)(x) &= \textstyle\int_0^x (\mathsf{e}_s f)(x-t)(\mathsf{e}_s g)(t)\,dt \\ &= \textstyle\int_0^x \mathsf{e}_s(x-t)\mathsf{e}_s(t) f(x-t) g(t)\,dt \\ &= \mathsf{e}_s(x)(f*g)(x),\end{aligned}$$

so (B.9) holds. This implies

$$\textstyle\int_0^x |(\mathsf{e}_s f * \mathsf{e}_s g)(t)|\,dt \;=\; \int_0^x \mathsf{e}_s(t)|(f*g)(t)|\,dt \;\leqslant\; \mathsf{e}_s(x)\int_0^x |f*g(s)|\,ds.$$

By Lemma B.5, $f*g \in L^1_{loc}(\mathbb{R}_+)$. We conclude that $\mathsf{e}_s f * \mathsf{e}_s g \in L^1_{loc}(\mathbb{R}_+)$. Moreover,

$$\mathsf{e}_s\left(f^{n*}\right) \;=\; \mathsf{e}_s\left(f * f^{(n-1)*}\right) \;=\; \mathsf{e}_s f * \left(\mathsf{e}_s f^{(n-1)*}\right) \;=\; \cdots \;=\; (\mathsf{e}_s f)^{n*}.$$

Thus (B.10) also holds. □

Lemma B.6. *Assume $f \in L^1(\mathbb{R}_+)$ is non-negative, right-continuous, and non-increasing. For $n = 1, 2, \ldots$,*

$$f^{n*}(y) \leqslant f(0)\left(\int_0^y f(t)\,dt\right)^{n-1} \qquad (y \geqslant 0), \tag{B.11}$$

with equality exactly when f is constant on the interval $[0, y)$.
Therefore,

$$f^{n*} \leqslant f(0)\|f\|_{L^1}^{n-1}.$$

Proof: We prove (B.11) by induction. Fix $y > 0$. As f is non-increasing and right-continuous,

$$f^{*1}(y) = f(y) \;\leqslant\; f(0),$$

with equality if and only if f is constant on $[0, y)$. Assume $k \geqslant 1$ and the inequality in (B.11) is valid for $n = k$. Since f is nonnegative, the induction hypothesis implies

$$f^{*(k+1)}(y) = \textstyle\int_0^y f^{*k}(t) f(y-t)\,dt \leqslant f(0)\left(\int_0^y f(t)\,dt\right)^{k-1}\int_0^y f(y-t)\,dt,$$

with equality if and only if f is constant on $[0, t)$ for every $t \in [0, y]$. Thus, after a change of variables, we obtain

$$f^{*(k+1)}(y) \leqslant f(0)\left(\textstyle\int_0^y f(t)\,dt\right)^k,$$

with equality if and only if f is constant on $[0, y)$. Since f is non-negative,

$$\textstyle\int_0^y f(t)\,dt \leqslant \|f\|_{L^1},$$

and the lemma is proved. □

Lemma B.7 (Integration by parts). *Suppose $u \in C(\mathbb{R}_+)$ is differentiable. Then*

$$\tfrac{\lambda A_\epsilon u}{\epsilon p_\epsilon(x)} = Tu(x) - \tfrac{\lambda H_\epsilon}{p_\epsilon(x)}(u(x) - u(0)). \tag{B.12}$$

Proof: We compute:

$$\begin{aligned}
A_\epsilon u(x) &= \textstyle\int_0^{x/\epsilon} (u(x) - u(x - \epsilon y))\, F\{dy\} \\
&= \textstyle\int_0^{x/\epsilon} \int_0^{\epsilon y} u'(x-t)\, dt F\{dy\} \\
&= \textstyle\int_0^{x} \int_{t/\epsilon}^{x/\epsilon} F\{dy\} u'(x-t)\, dt \\
&= \textstyle\int_0^{x} \left(F\left(\frac{x}{\epsilon}\right) - F\left(\frac{t}{\epsilon}\right)\right) u'(x-t)\, dt \\
&= \left(F\left(\tfrac{x}{\epsilon}\right) - 1\right)\right) (u(x) - u(0)) + \textstyle\int_0^x \left(1 - F\left(\frac{t}{\epsilon}\right)\right) u'(x-t)\, dt \\
&= (u(0) - u(x))\epsilon H_\epsilon(x) + \textstyle\int_0^x \epsilon H_\epsilon(t) u'(x-t)\, dt \\
&= (u(0) - u(x))\epsilon H_\epsilon(x) + u' * \epsilon H_\epsilon(x). \qquad \square
\end{aligned}$$

Lemma B.8 (Gronwall's Inequality). *Let $f(t)$ be a nonnegative, summable function on the interval $[0, T]$. Suppose f satisfies for almost every $t \in [0, T]$ the integral inequality*

$$f(t) \leqslant C_1 \int_0^t f(s)\, ds + C_2 \tag{B.13}$$

for constants $C_1, C_2 \geqslant 0$.
Then

$$f(t) \leqslant C_2 \left(1 + C_1 t e^{C_1 t}\right) \tag{B.14}$$

for a.e. $t \in [0, T]$.

A proof can be found in Evans [16].

B.3 Laplace Transforms

The following properties of the Laplace transform can be found in Feller [17].

Definition. The *Laplace transform* of $f(x)$ is defined by

$$\mathcal{L}f(s) = \textstyle\int_0^\infty e^{-st} f(t)\, dt,$$

whenever this integral exists. Clearly, if $f \in L^\infty(\mathbb{R}_+)$, then $\mathcal{L}f(s)$ exists for all $s \in (0, \infty)$. If $f \in L^1(\mathbb{R}_+)$, then $\mathcal{L}f(s)$ exists for all $s \in [0, \infty)$ and $\mathcal{L}f \in L^\infty(\mathbb{R}_+)$.

Lemma B.9 (Properties of Laplace transforms). *Suppose $\theta \in \mathbb{R}$ and consider the functions $f = f(x)$, $g = g(x)$. The following statements hold whenever their limits exist (finitely).*

(i) *If* $h(x) := \theta f(x) + g(x)$, *then*

$$\mathcal{L}h(s) = \theta \mathcal{L}f(s) + \mathcal{L}g(s). \tag{B.15}$$

(ii) *If* $h(x) := f(\theta x)$ *for* $\theta \neq 0$, *then*

$$\theta \mathcal{L}h(\theta s) = \mathcal{L}f(s). \tag{B.16}$$

(iii) *If* $h(x) := (f * g)(x)$, *then*

$$\mathcal{L}h(s) = \mathcal{L}f(s)\mathcal{L}g(s). \tag{B.17}$$

(iv) *If* $h'(x) := f(x)$, *then*

$$\mathcal{L}f(s) = h(0) + s\mathcal{L}h(s). \tag{B.18}$$

Proof of (iii). We compute:

$$\begin{aligned}
\mathcal{L}h(s) \;=\; \textstyle\int_0^\infty e^{-st}(f * g)(t)\,dt &= \textstyle\int_0^\infty e^{-st}\int_0^t f(t-y)g(y)\,dy\,dt \\
&= \textstyle\int_0^\infty \int_0^t e^{-s(t-y)} f(t-y)e^{-sy}g(y)\,dy\,dt \\
&= \textstyle\int_0^\infty \int_y^\infty e^{-s(t-y)} f(t-y)e^{-sy}g(y)\,dt\,dy \\
&= \textstyle\int_0^\infty e^{-sy}g(y)\int_y^\infty e^{-s(t-y)} f(t-y)\,dt\,dy \\
&= \textstyle\int_0^\infty e^{-sy}g(y)\int_0^\infty e^{-s(t)} f(t)\,dt\,dy \\
&= \mathcal{L}g(s)\mathcal{L}f(s). \qquad \square
\end{aligned}$$

Gamma distribution.

Definition. The *gamma function* Γ is defined by

$$\Gamma(\gamma) = \textstyle\int_0^\infty t^{\gamma-1}e^{-t}\,dt \qquad (\gamma \neq -1, -2, \ldots).$$

The *gamma distribution with shape parameter* $\gamma > 0$, *scale parameter* $\beta > 0$, *and location parameter* μ, is defined by

$$F(y) = \tfrac{\frac{1}{\beta}}{\Gamma(\gamma)} \textstyle\int_\mu^y \left(\frac{t-\mu}{\beta}\right)^{\gamma-1} e^{-\frac{t-\mu}{\beta}}\,dt \qquad (y \geqslant \mu).$$

The case where $\mu = 0$ and $\beta = 1$ is called the *standard gamma distribution*. We assume hereafter that $\mu = 0$.

Lemma B.10. *Suppose* Y *has a gamma distribution with shape parameter* γ *and scale parameter* β. *Then*

$$\mathbb{E}[Y] = \beta\gamma. \tag{B.19}$$

Proof: We calculate:

$$\mathbb{E}[Y] \;=\; \tfrac{1}{\Gamma(\gamma)} \textstyle\int_0^\infty \left(\frac{t}{\beta}\right)^{\gamma} e^{-\frac{t}{\beta}}\,dt \;=\; \tfrac{\beta}{\Gamma(\gamma)} \textstyle\int_0^\infty t^{\gamma}e^{-t}\,dt \;=\; \tfrac{\beta\gamma}{\Gamma(\gamma)} \textstyle\int_0^\infty t^{\gamma-1}e^{-t}\,dt \;=\; \beta\gamma. \quad \square$$

Lemma B.11 (Laplace transform of the gamma distribution). *Consider the standard gamma distribution*

$$f(y) := \tfrac{\frac{1}{\beta}}{\Gamma(\gamma)} \int_0^y \left(\tfrac{t}{\beta}\right)^{\gamma-1} e^{-\frac{t}{\beta}}\, dt \qquad (y \geqslant 0).$$

Then

$$\mathcal{L}f(s) = \tfrac{\frac{1}{\beta}^{\gamma}}{s\left(\frac{1}{\beta}+s\right)^{\gamma}} \qquad (s > 0). \tag{B.20}$$

Proof: We calculate $f'(y) = \frac{\frac{1}{\beta}}{\Gamma(\gamma)}\left(\frac{y}{\beta}\right)^{\gamma-1} e^{-\frac{y}{\beta}}$ and

$$\begin{aligned}
\mathcal{L}(f')(s) &= \tfrac{1}{\beta^{\gamma}\Gamma(\gamma)} \int_0^{\infty} e^{-\left(s+\frac{1}{\beta}\right)t}\, t^{\gamma-1}\, dt \\
&= \tfrac{1}{\beta^{\gamma}\left(s+\frac{1}{\beta}\right)^{\gamma}\Gamma(\gamma)} \int_0^{\infty} e^{-t} t^{\gamma-1}\, dt = \tfrac{1}{\beta^{\gamma}\left(s+\frac{1}{\beta}\right)^{\gamma}\Gamma(\gamma)}\Gamma(\gamma) \\
&= \left(\tfrac{\frac{1}{\beta}}{s+\frac{1}{\beta}}\right)^{\gamma}.
\end{aligned}$$

Therefore, (B.18) implies

$$\left(\tfrac{\frac{1}{\beta}}{s+\frac{1}{\beta}}\right)^{\gamma} \;=\; s\mathcal{L}f(s) - f(0) \;=\; s\mathcal{L}f(s).$$

Thus

$$\mathcal{L}f(s) = \tfrac{1}{s}\left(\tfrac{\frac{1}{\beta}}{s+\frac{1}{\beta}}\right)^{\gamma}.$$

□

B.4 Examples

Lemma B.12 (Example 5.1.2). *Suppose $p_\epsilon \equiv p$ is constant, the claim size process has an exponential distribution, and w_ϵ is defined by (5.1.1). Then, as $\epsilon \to 0$,*

$$w_\epsilon(x) = \begin{cases} (a-b)x + O(\epsilon) & (a > b) \\ O(\epsilon \log \epsilon) & (a = b) \\ O(\epsilon) & (a < b). \end{cases}$$

Proof: Upon substituting (5.1.1) into the formula obtained in Example 4.2.1, we compute

$$w_\epsilon(x) = \begin{cases} \epsilon \log \left| \frac{a}{a-b} e^{\frac{a-b}{\epsilon}x} - \frac{b}{a-b} \right| & (a \neq b) \\ \epsilon \log\left(1 + \frac{bx}{\epsilon}\right) & (a = b). \end{cases}$$

Equivalently,

$$w_\epsilon(x) = \begin{cases} (a-b)x + \epsilon \log\left(\frac{a - be^{-\frac{a-b}{\epsilon}x}}{a-b}\right) & (a > b) \\ \epsilon \log(\epsilon + bx) - \epsilon \log \epsilon & (a = b) \\ \epsilon \log\left(\frac{b - ae^{-\frac{b-a}{\epsilon}x}}{b-a}\right) & (a < b). \end{cases}$$

Computing the relevant limits as ϵ tends to 0 yields the result. □

Proof of Lemma 5.3. For $x > 0$,

$$\begin{aligned} p_\epsilon(x) w_\epsilon'(x) e^{-\frac{w_\epsilon(x)}{\epsilon}} &= \epsilon p_\epsilon(x) u_\epsilon'(x) && \text{by (5.1.3)} \\ &= \lambda u_\epsilon(x) - \lambda \textstyle\int_0^{x/\epsilon} u_\epsilon(x - \epsilon y)\, F\{dy\} && \text{by (2.2.8)}, \end{aligned}$$

so that

$$(p_\epsilon(x) w_\epsilon'(x) + \lambda) e^{-\frac{w_\epsilon(x)}{\epsilon}} = \lambda \left(\textstyle\int_{x/\epsilon}^{\infty} F\{dy\} + \int_0^{x/\epsilon} e^{-\frac{w_\epsilon(x-\epsilon y)}{\epsilon}} F\{dy\} \right). \tag{B.21}$$

Multiplying (B.21) by $e^{\frac{w_\epsilon(x)}{\epsilon}}$ results in (5.1.5), while expressing the right-hand side of (B.21) as one integral results in (5.1.4). Indeed: for $y \geqslant 0$,

$$e^{-\frac{w_\epsilon(x-\epsilon y)}{\epsilon}} \mathbb{1}_{(0,x]}(\epsilon y) + \mathbb{1}_{(x,\infty)}(\epsilon y) \;=\; \begin{cases} e^{-\frac{w_\epsilon(x-\epsilon y)}{\epsilon}} & (\epsilon y \geqslant x) \\ 1 & (x < \epsilon y) \end{cases} \;=\; e^{-\frac{w_\epsilon(x-\epsilon y)\mathcal{H}(x-\epsilon y)}{\epsilon}}.$$

With this, we see that (B.21) and (5.1.4) are equivalent. □

Lemma B.13 (Chapter 4, Example 5). *Suppose $p_\epsilon \equiv p$ is constant, and F is the gamma distribution with $\gamma = 1$. If $u_\epsilon \in L^\infty(\mathbb{R}_+)$ solves (2.2.8) and $u_\epsilon(\infty) = 1$, then*

$$u_\epsilon(x) = 1 - e^{-\frac{b-a}{\epsilon}x} \left(\cos\left(\frac{a^2}{\epsilon^2} + \frac{ab}{2\epsilon^2}\right) x + \frac{\epsilon}{a}\left(\frac{b-2a}{b+2a}\right) \sinh\left(\frac{a^2}{\epsilon^2} + \frac{ab}{2\epsilon^2}\right) x \right). \tag{B.22}$$

Proof: In this case (4.2.8) is

$$\begin{aligned} \mathcal{L}u_\epsilon(s) &= \tfrac{\epsilon}{b}\left(1 - \tfrac{a}{b}\right) \tfrac{\tau^2}{\tau^3 - \left(1 + \frac{a}{2b}\right)\tau^2 + \frac{a}{2b}} \\ &= 2\epsilon\left(1 - \tfrac{a}{b}\right) \tfrac{\tau^2}{(\tau-1)(2b\tau^2 - a\tau - a)}. \end{aligned}$$

Decomposing the rational function yields:

$$\begin{aligned} \tfrac{\tau^2}{(\tau-1)(2b\tau^2 - a\tau - a)} &= \tfrac{\frac{1}{2(b-a)}}{\tau - 1} + \tfrac{1}{2b} \tfrac{\frac{a}{a-b}\left(\tau - \frac{a}{b}\right)}{\left(\tau - \frac{a}{b}\right)^2 - \frac{a^2}{b^2} - \frac{a}{2b}} + \tfrac{1}{2b} \tfrac{\frac{a}{a-b}\left(\frac{a}{b} - \frac{1}{2}\right)}{\left(\tau - \frac{a}{b}\right)^2 - \frac{a^2}{b^2} - \frac{a}{2b}} \\ &= \tfrac{1}{2(b-a)} \left(\tfrac{1}{\tau - 1} - \tfrac{\left(\tau - \frac{a}{b}\right) + \left(\frac{a}{b} - \frac{1}{2}\right)}{\left(\tau - \frac{a}{b}\right)^2 - \frac{a^2}{b^2} - \frac{a}{2b}} \right). \end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}u_\epsilon(s) &= 2\epsilon\left(1-\tfrac{a}{b}\right)\tfrac{\tau^2}{(\tau-1)(2b\tau^2-a\tau-a)}\\ &= \tfrac{\epsilon}{b}\left(\tfrac{1}{\tau-1}-\tfrac{\left(\tau-\frac{a}{b}\right)+\left(\frac{a}{b}-\frac{1}{2}\right)}{\left(\tau-\frac{a}{b}\right)^2-\frac{a^2}{b^2}-\frac{a}{2b}}\right).\end{aligned}$$

Since $\tau = \frac{\epsilon s}{b}+1$,

$$\tfrac{\epsilon}{b}\tfrac{1}{\tau-1} = \tfrac{1}{s} \text{ and } \tau-\tfrac{a}{b} = \tfrac{\epsilon}{b}\left(s+\tfrac{b-a}{\epsilon}\right).$$

Therefore

$$\begin{aligned}\mathcal{L}u_\epsilon(s) &= \tfrac{1}{s}-\tfrac{\epsilon^2}{b^2}\tfrac{s+\frac{b-a}{\epsilon}}{\left(\frac{\epsilon}{b}\left(s+\frac{b-a}{\epsilon}\right)\right)^2-\frac{a^2}{b^2}-\frac{a}{2b}}-\tfrac{\epsilon}{b}\left(\tfrac{a}{b}-\tfrac{1}{2}\right)\tfrac{1}{\left(\frac{\epsilon}{b}\left(s+\frac{b-a}{\epsilon}\right)\right)^2-\frac{a^2}{b^2}-\frac{a}{2b}}\\ &= \tfrac{1}{s}-\tfrac{s+\frac{b-a}{\epsilon}}{\left(s+\frac{b-a}{\epsilon}\right)^2-\frac{a^2}{\epsilon^2}-\frac{ab}{2\epsilon^2}}-\left(\tfrac{a}{\epsilon}-\tfrac{b}{2\epsilon}\right)\tfrac{1}{\left(s+\frac{b-a}{\epsilon}\right)^2-\frac{a^2}{\epsilon^2}-\frac{ab}{2\epsilon^2}}.\end{aligned}$$

Since (see Abramowitz and Stegun [2])

$$\begin{aligned}\mathcal{L}^{-1}\left\{\tfrac{1}{s}\right\} &= x\mapsto 1\\ \mathcal{L}^{-1}\left\{\tfrac{s-B}{(s-B)^2-C}\right\} &= x\mapsto e^{Bx}\cos(Cx)\\ \mathcal{L}^{-1}\left\{\tfrac{1}{(s-B)^2-C}\right\} &= x\mapsto \tfrac{e^{Bx}}{C}\sinh(Cx),\end{aligned}$$

we deduce

$$u_\epsilon(x) = 1-e^{-\frac{b-a}{\epsilon}x}\left(\cos\left(\tfrac{a^2}{\epsilon^2}+\tfrac{ab}{2\epsilon^2}\right)x-\tfrac{\frac{a}{\epsilon}-\frac{b}{2\epsilon}}{\frac{a^2}{\epsilon^2}+\frac{ab}{2\epsilon^2}}\sinh\left(\tfrac{a^2}{\epsilon^2}+\tfrac{ab}{2\epsilon^2}\right)x\right).$$

Simplifying yields (B.22). □