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A Weak Equivalence from the Subdivision of the Orbit Category to the Link Orbit Category for Finite Abelian Groups

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A Weak Equivalence from the Subdivision of the Orbit Category to the Link Orbit Category for Finite Abelian Groups

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Katherine Lara Muckle

June 2026

Dissertation Committee:
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To Jamie, who kept me company via contact naps during my cleverer moments,

and to my little sister, the first Dr. Muckle

ABSTRACT OF THE DISSERTATION

A Weak Equivalence from the Subdivision of the Orbit Category to the Link Orbit Category for Finite Abelian Groups

by

Katherine Lara Muckle

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2026
Prof. Sarah Yeakel, Chairperson

Given a finite group G, an *equivariant map* between spaces with left G-actions is a map that preserves the action. We can keep track of the equivariant maps between the quotient sets of G using the *orbit category.* An *isovariant map* is an equivariant map that preserves stabilizer groups. Yeakel defines the *link orbit category*, which roughly can be interpreted as recording the isovariant maps between the quotient sets of G [10]. The *subdivision of the orbit category* [6] is a stratified version of the orbit category that explicitly encodes paths through the orbit category. The *classifying space functor*, B, assigns to each category a topological space and interprets functors as continuous maps, and a functor is a *weak equivalence* if its image under B is a homotopy equivalence. In this thesis, for finite abelian groups, I construct a *weak equivalence* from the subdivision of the orbit category to the link orbit category using Quillen's Theorem A [8].

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Chapter 1

Introduction

1.1 Setting Up the Question and Conventions

Let G be a finite group, and let X and Y be compactly generated spaces with continuous left G-actions. An *equivariant map* $\alpha : X \to Y$ is a function that preserves the G-action so that $\alpha(g \cdot x) = g \cdot \alpha(x)$ for all $g \in G$ and $x \in X$, i.e., we obtain the same result whether we perform the group action before or after applying α.

Example 1.1.1. Let $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ and $X = (\mathbb{Z}/2\mathbb{Z})/\langle 0\rangle = \{0 + \langle 0\rangle, 1 + \langle 0\rangle\}$ where the G-action is defined by $g \cdot (x + H) = g + x + H$. Then there is an equivariant map

$$\alpha_1 : (\mathbb{Z}/2\mathbb{Z})/\langle 0\rangle \to (\mathbb{Z}/2\mathbb{Z})/\langle 0\rangle$$

that switches the elements: $\alpha_1(0 + \langle 0\rangle) = 1 + \langle 0\rangle$ and $\alpha_1(1 + \langle 0\rangle) = 0 + \langle 0\rangle$. In this situation when the group and set are so small, we can check equivariance directly for each case; for example, if $g = 1$ and $x = 1 + \langle 0 \rangle$, then the calculation

$$1 + \alpha_1(1 + \langle 0 \rangle) = 1 + 0 + \langle 0 \rangle = 1 + \langle 0 \rangle = \alpha_1(0 + \langle 0 \rangle) = \alpha_1(1 + 1 + \langle 0 \rangle)$$

shows that we may either have $g = 1$ act on the group element before or after mapping via α_1. $\triangle$

The following lemma gives us a way to label equivariant maps between the quotient sets of G based on the group elements.

Lemma 1.1.2. Let G be a group and let H be a subgroup of G. Define the action of G on G/H as $g \cdot xH := gxH$. Let $K \leq G$. A map $\alpha : G/H \to G/K$ is equivariant if and only if there exists $\gamma \in G$ such that $\alpha(xH) = x\gamma K$ and $\gamma^{-1}H\gamma \subseteq K$. *(See Lemmas 2.1.6 and 2.1.7 in this document.)*

Different elements in G may induce the same equivariant map $G/H \to G/K$. Therefore each equivariant map is given by an equivalence class of elements in G; in fact, γ and γ' induce the same map if and only if they are in the same K-coset in G (Lemma 2.1.10).

The *orbit category*, denoted $\mathcal{O}_G$, has as objects the quotient sets of G and as morphisms the equivariant maps between them. We denote an object G/H simply using the subgroup H, and we denote a morphism from H to K by $\gamma_i^{\mathcal{O}}$ where $i \in G$ induces the equivariant map $xH \mapsto xiK$. If $H \xrightarrow{\gamma_i^{\mathcal{O}}} K \xrightarrow{\gamma_j^{\mathcal{O}}} L$, then $\gamma_j^{\mathcal{O}} \circ \gamma_i^{\mathcal{O}} = \gamma_{ij}^{\mathcal{O}}$. As a little example, the orbit category $\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$ is given in Figure 1.1.

$$\begin{tikzcd}
\langle 0 \rangle \arrow[loop above, "\gamma_1^{\mathcal{O}}"] \arrow[loop below, "\gamma_0^{\mathcal{O}}"] \arrow[rr, bend left=10, "\gamma_0^{\mathcal{O}}"] & & \langle 1 \rangle \arrow[loop below, "\gamma_0^{\mathcal{O}}"]
\end{tikzcd}$$

Figure 1.1: The orbit category for $\mathbb{Z}/2\mathbb{Z}$.

In general, if α is an equivariant map, then the stabilizer group of $x \in X$, denoted G_x, is a subset of the stabilizer group $G_{\alpha(x)}$. An *isovariant map* is an equivariant map that preserves all stabilizer groups, i.e., $G_x = G_{\alpha(x)}$ for all $x \in X$. Note that the map given in Example 1.1.1 is isovariant, but the equivariant map from $(\mathbb{Z}/2\mathbb{Z})/\langle 0 \rangle$ to $(\mathbb{Z}/2\mathbb{Z})/\langle 1 \rangle$ is not isovariant since at least one stabilizer group increases in size: $G_{0+\{0\}} = \{0\}$ but $G_{\alpha(0+\{0\})} = \{0,1\}$.

The orbit category allows for a special adjunction called a Quillen equivalence between $G\mathcal{T}op$ and $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}op)$, where $G\mathcal{T}op$ is the category of spaces with a left G-action and the equivariant maps between them. Yeakel defines the *link orbit category*, $\mathcal{L}_G$, to keep track of isovariant maps between the quotient sets of G such that there is an analogous adjunction between isvt$-G\mathcal{T}op$ and $\mathrm{Fun}(\mathcal{L}_G^{op}, \mathcal{T}op)$ [10]. (See Figure 1.2.)

$$\begin{tikzcd}
G\mathcal{T}op \arrow[d, shift left] & \mathrm{isvt}G\mathcal{T}op \arrow[d, shift left] \\
\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}) \arrow[u, shift left] & \mathrm{Fun}(\mathcal{L}_G^{op}, \mathcal{T}op) \arrow[u, shift left]
\end{tikzcd}$$

Figure 1.2: Quillen equivalences and $\mathcal{L}_G$.

The link orbit category has as objects strictly increasing chains of subgroups of G, denoted $\mathbb{H} = H_0 < \ldots < H_n$. The morphisms track the isovariant maps between G/H_i and G/K_j if H_i and K_j are elements in the chains, and composition is again obtained by composing group elements (see Definitions 2.6.1 and 2.6.2 for the formal definition). In fact, if G is abelian, a morphism $\mathbb{H} \to \mathbb{K}$ exists if and only if $\mathbb{H}$ is a subchain of $\mathbb{K}$, and the collection of morphisms between these chains is induced by the same group elements that induce the equivariant self-maps of G/H_0 (Remark 2.6.8).

Example 1.1.3. The link orbit category $\mathcal{L}_{\mathbb{Z}/2\mathbb{Z}}$ is given below in Figure 1.3.

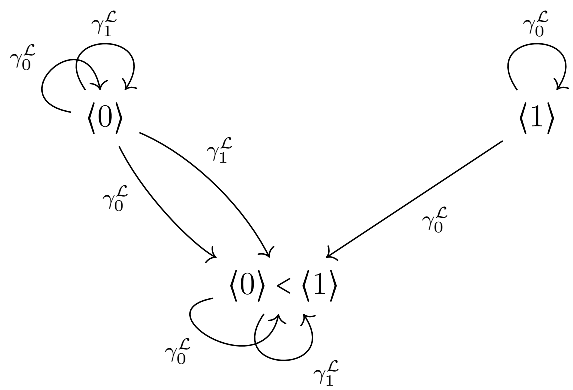

Figure 1.3: Link orbit category for $\mathbb{Z}/2\mathbb{Z}$.

To give some further intuition, note that in $\mathcal{L}_{\mathbb{Z}/2\mathbb{Z}}$, there is no map from $\langle 0 \rangle$ to $\langle 1 \rangle$ because while we do have an equivariant map from $(\mathbb{Z}/2\mathbb{Z})/\langle 0 \rangle$ to $(\mathbb{Z}/2\mathbb{Z})/\langle 1 \rangle$, this map is not isovariant. The two maps from $\langle 0 \rangle < \langle 1 \rangle$ to itself represent the two isovariant self-maps of $(\mathbb{Z}/2\mathbb{Z})/\langle 0 \rangle$ (the "switch" map and the identity), and these maps also represent the identity map on $(\mathbb{Z}/2\mathbb{Z})/\langle 1 \rangle$; this identity map is induced by both $\gamma = 0$ and $\gamma = 1$, but $\gamma_0^{\mathcal{L}}$ and $\gamma_1^{\mathcal{L}}$ on $\langle 0 \rangle < \langle 1 \rangle$ are not identified in the link orbit category because the two self-maps induced by $\gamma = 0$ and $\gamma = 1$ on $(\mathbb{Z}/2\mathbb{Z})/\langle 0\rangle$ are distinct. The isovariant self-maps on $(\mathbb{Z}/2\mathbb{Z})/\langle 0\rangle$ are also tracked by the two maps from $\langle 0\rangle$ to $\langle 0\rangle < \langle 1\rangle$. △

Isovariant maps are stratified equivariant maps in the sense that isovariant maps send fixed points to fixed points and free points to free points. This relationship suggests that perhaps a "stratified" orbit category might be in some way equivalent to the link orbit category. We can in fact perform a stratification by way of the *subdivision of the orbit category*, $\mathrm{Sd}\mathcal{O}_G$. The subdivision of the orbit category (using the definition of subdivision given by May in [5]) has as objects chains of non-identity morphisms from $\mathcal{O}_G$:

$$H_0 \xrightarrow{\varphi_1} H_1 \xrightarrow{\varphi_2} H_2 \xrightarrow{\varphi_3} \ldots \xrightarrow{\varphi_n} H_n$$

Let $[n]$ be the category with objects $0, \ldots, n$, and there exists a morphism $k \to l$ if $k \leq l$. Then we can think of the n-chains in $\mathrm{Sd}\mathcal{O}_G$ as functors from $[n] \to \mathcal{O}_G$, as represented below:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 1 & \longrightarrow & \ldots & \longrightarrow & n \\ \downarrow & & \downarrow & & & & \downarrow \\ H_0 & \xrightarrow{\varphi_1} & H_1 & \xrightarrow{\varphi_2} & \ldots & \xrightarrow{\varphi_n} & H_n \end{array}$$

Note that the morphism $H_0 \to H_n$ is the composition $\varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$. I use the more concise notations Φ^n or ${}^0\varphi_1{}^1\varphi_2{}^2\ldots{}^{n-1}\varphi_n{}^n$ to denote an n-dimensional chain. When the dimension is not necessary information, I will simply call the chain Φ. One nice way to think of one of these chains is as a path of morphisms through the orbit category.

Two of these paths have a morphism between them if one path can be inserted into the other allowing for composition, i.e., for two objects Φ^n and Ψ^m, there exists a functor $d : [n] \to [m]$ that is a strictly increasing injection when viewed on the objects such that $\Phi = \Psi \circ d$. If such a d exists, we call the morphism $d_* : \Phi \to \Psi$. Certain insertions are considered the same if they differ by a string of consecutive maps in the chain that compose to the identity, and so the morphisms in the subdivision of the orbit category are technically equivalence classes of path insertions (see Definitions 2.7.5 and 2.8.1.) The intuition for the equivalence relation comes from May [6], but I have formalized his perspective in the Scoot Lemma 2.8.2.

Example 1.1.4. In this example, I calculate $\mathrm{Sd}\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$. For this and other examples later in this document, I use an α-notation for clarity to denote the morphisms of the orbit category (see Figure 1.4).

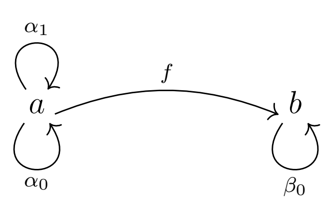

Figure 1.4: The orbit category for the group with 2 elements in α-notation.

In this notation, the non-identity morphisms in $\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$ are α_1 and f, so the nondegenerate n-chains look like

$$\overset{0}{a},\ \overset{0}{b},\ {}^0f^1,\ {}^0\alpha_1{}^1\alpha_1{}^2\dots{}^{n-1}\alpha_1{}^n,\ \text{and}\ {}^0\alpha_1{}^1\alpha_1{}^2\dots{}^{n-2}\alpha_1{}^{n-1}f^n$$

There is a map $d_* : {}^0\alpha_1{}^1\alpha_1{}^2 \to {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4f^5$ defined by the injection $0 \mapsto 0$, $1 \mapsto 3$, and $2 \mapsto 4$. In Figure 1.5, I have calculated the entire category $\mathrm{Sd}\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$. To demonstrate the equivalence classes for the subdivision, the map d_* in this example is equivalent to the morphism δ_* defined by $0 \mapsto 2$, $1 \mapsto 3$, and $2 \mapsto 4$. Because $\alpha_1 \circ \alpha_1$ composes to an identity, there are at most two maps between any of the objects once we account for the equivalence relation on the morphisms. △

$$
\begin{tikzcd}
\overset{0}{a} \arrow[d, bend right] \arrow[d, bend left] \arrow[rrd] & & \overset{0}{b} \arrow[d] \\
{}^0\alpha_1{}^1 \arrow[d, bend right] \arrow[d, bend left] \arrow[rrd] & & {}^0f^1 \arrow[d, bend right] \arrow[d, bend left] \\
{}^0\alpha_1{}^1\alpha_1{}^2 \arrow[d, bend right] \arrow[d, bend left] \arrow[rrd] & & {}^0\alpha_1{}^1f^2 \arrow[d, bend right] \arrow[d, bend left] \\
{}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3 \arrow[d, bend right] \arrow[d, bend left] \arrow[rrd] & & {}^0\alpha_1{}^1\alpha_1{}^2f^3 \arrow[d, bend right] \arrow[d, bend left] \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

Figure 1.5: The subdivision of the orbit category for $\mathbb{Z}/2\mathbb{Z}$.

The sort of categorical equivalence we will be pursuing is *weak equivalence*, which is based on homotopy equivalence of topological spaces. The *classifying space functor* $B : \mathcal{C}at \Rightarrow \mathcal{T}op$ associates to each (small) category $\mathcal{C}$ a topological space via the geometric realization of the nerve [2]. A functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ in $\mathcal{C}at$ is said to be a *weak equivalence* if $BF : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence in $\mathcal{T}op$. A category is said to be *contractible* if the geometric realization of its nerve is contractible in $\mathcal{T}op$.

We can now state the main theorem of this thesis:

Theorem 1.1.5. If G is a finite abelian group, then there exists a weak equivalence from the subdivision of the orbit category of G to the link orbit category of G. *(See Theorem 3.3.11 in this document.)*

Integral to the proof of this theorem are two results of Quillen:

Theorem 1.1.6. (Quillen's Theorem A) [8] Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. The functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ is a weak equivalence if the left fiber F/d is contractible for every object d of $\mathcal{D}$.

Theorem 1.1.7. Any filtered category is contractible [8].

There are a few terms to define in order to understand these statements. The *left fiber* of a functor with respect to an object d in $\mathcal{D}$ is a category that captures the notion of a more general preimage. It includes both the objects that are sent directly to d and the objects that are sent to other objects with morphisms to d (see Definition 2.4.7). A nonempty category $\mathcal{C}$ is said to be a *filtered category* if (1) for every pair of objects, there exists a third object such that both objects map to that

$$\begin{tikzcd}
c_1 \arrow[dr] & & c_2 \arrow[dl] \\
& c_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[rr, bend left, "h\circ f = h\circ g"] & c_2 \arrow[r, "h"'] & c_3
\end{tikzcd}$$

First condition Second condition

Figure 1.6: Filtered category conditions.

object, and (2) for any pair of parallel morphisms, there exists a third morphism that collapses the pair (see Figure 1.6 and Definition 2.2.5).

Therefore, the goal of the thesis is to construct a functor from the subdivision of the orbit category to the link orbit category such that the left fibers of the functor are filtered categories if the underlying group is finite and abelian.

1.2 Defining the Functor

Originally, my proof for the group $\mathbb{Z}/p\mathbb{Z}$ was based on the sup functor, which is the functor del Hoyo uses in his proof that there is a weak equivalence from Sd$\mathcal{C}$ to $\mathcal{C}$ [2]. In essence, if you insert a subpath into a path, sup picks out the tail of maps that are not included in the insertion. However, this functor did not extend well to $\mathbb{Z}/p^2\mathbb{Z}$, as it loses too much information about the paths in $\mathcal{O}_G$. In order to prove the theorem for all abelian groups, I define the contravariant inf functor, which picks out the "beginning difference" of the insertion (Definition 3.1.1) and retains more information in our situation. From the inf functor, I construct a functor $[\ell\,\mathrm{inf}]$ from the subdivision of the orbit category to the link orbit category.

Viewing the objects of $\mathrm{Sd}\mathcal{O}_G$ as paths in $\mathcal{O}_G$, the functor $[\ell\inf]$ maps each path to the path of objects in the orbit category that it passes through, and the morphisms are assigned to the beginning difference between the entire path and the subpath.

To aid in the following Example, I have labeled the orbit category $\mathcal{O}_{\mathbb{Z}/p^2\mathbb{Z}}$ in Figure 1.7 using the $\gamma_i^{\mathcal{O}}$-notation and using an α-notation for objects in the subdivision of the orbit category.

$$\begin{tikzcd}
\langle 0\rangle \arrow[loop above, "\gamma_0^{\mathcal{O}},\dots,\gamma_{p^2-1}^{\mathcal{O}}"] \arrow[rr, "\gamma_0^{\mathcal{O}}"] \arrow[dr, "\gamma_0^{\mathcal{O}},\dots,\gamma_{p-1}^{\mathcal{O}}"'] & & \langle 1\rangle \arrow[loop above, "\gamma_0^{\mathcal{O}}"] \\
& \langle p\rangle \arrow[ur, "\gamma_0^{\mathcal{O}}"'] \arrow[loop below, "\gamma_0^{\mathcal{O}},\dots,\gamma_{p-1}^{\mathcal{O}}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle 0\rangle \arrow[loop above, "\alpha_0,\dots,\alpha_{p^2-1}"] \arrow[rr, "f_a"] \arrow[dr, "\omega_0,\dots,\omega_{p-1}"'] & & \langle 1\rangle \arrow[loop above, "\beta_0"] \\
& \langle p\rangle \arrow[ur, "f_c"'] \arrow[loop below, "\theta_0,\dots,\theta_{p-1}"] &
\end{tikzcd}$$

Figure 1.7: The orbit category for $\mathbb{Z}/p^2\mathbb{Z}$

Example 1.2.1. The link orbit category $\mathcal{L}_{\mathbb{Z}/4\mathbb{Z}}$ is given below in Figure 1.8 (where $p = 2$) with the object image of $[\ell\inf]$ in blue boxes. I have labeled the images by their general forms, e.g., $\alpha\omega\theta$ represents any path that includes at least one α_i, one ω_j, and one θ_k. To see how $[\ell\inf]$ relates $\mathrm{Sd}\mathcal{O}_{\mathbb{Z}/4\mathbb{Z}}$ and $\mathcal{L}_{\mathbb{Z}/4\mathbb{Z}}$, observe that $[\ell\inf]$ maps the objects ${}^0\omega_1{}^1$ and ${}^0\alpha_1{}^1\alpha_2{}^2\alpha_1{}^3\omega_0{}^4$ to $\langle 0\rangle < \langle p\rangle$ because these paths pass through exactly the objects $\langle 0\rangle$ and $\langle p\rangle$ in the orbit category. If we define $d : [1] \to [4]$ as $0 \mapsto 2$ and $1 \mapsto 4$, then we have a morphism ${}^0\omega_1{}^1 \xrightarrow{d_*} {}^0\alpha_1{}^1\alpha_2{}^2\alpha_1{}^3\omega_0{}^4$ in the subdivision of the orbit category, and $[\ell\,\mathrm{inf}]d_* = \alpha_3^{\mathcal{L}}$ to show that when we insert ${}^0\omega_1{}^1$, the beginning difference in the two paths is $\alpha_2 \circ \alpha_1 = \alpha_3$.

For another more general intuition into the morphisms, note that the morphisms

$$\omega_0^{\mathcal{L}}, \ldots, \omega_{p-1}^{\mathcal{L}} : \langle p \rangle \to \langle 0 \rangle < \langle p \rangle$$

show what the possible beginning differences are when we insert paths of the forms c or θ into paths of the form ω, $\alpha\omega$, $\omega\theta$, or $\alpha\omega\theta$. (As mentioned previously, these maps can also be interpreted as the self-equivariant maps of G/H_0; in this case $H_0 = \langle p \rangle$.) $\triangle$

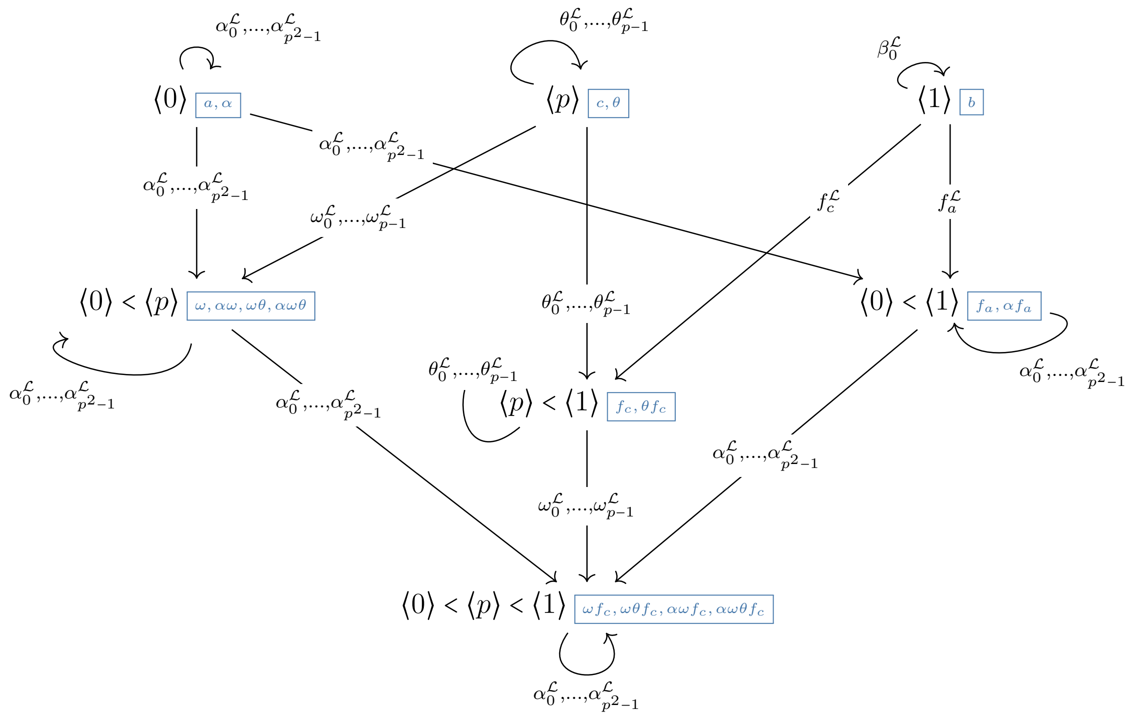

Figure 1.8: The link orbit category $\mathcal{L}_{\mathbb{Z}/p^2\mathbb{Z}}$ and the object image of $\ell\,\mathrm{inf}$.

1.3 Proving the Weak Equivalence

By Quillen's results stated earlier, the functor $[\ell\,\mathrm{inf}] : \mathrm{Sd}\mathcal{O}_G \to \mathcal{L}_G$ is a weak equivalence if for every object T in $\mathcal{L}_G$, the left fibers $[\ell\,\mathrm{inf}]/T$ are filtered categories. In order for each left fiber to satisfy the second condition of being filtered (Definition 2.2.5), we actually have the stronger property given by the following Lemma:

Lemma 1.3.1. There is at most one morphism between any two objects in $[\ell\,\mathrm{inf}]/T$ for all objects T in $\mathcal{L}_G$. *(See Lemma 3.3.4 in this document.)*

We prove this by showing that $[\ell\,\mathrm{inf}]$ is faithful (Lemma 3.2.9).

It remains to show that for each left fiber, the first condition of being a filtered category (Definition 2.2.5) is satisfied, i.e., for any two objects $(\Phi, \gamma_i^{\mathcal{L}})$ and $(\Psi, \gamma_j^{\mathcal{L}})$ in $[\ell\,\mathrm{inf}]/T$, we have an object $(\Omega, \gamma_k^{\mathcal{L}})$ and morphisms $d_\star$ and $\delta_\star$ such that

$$(\Phi, \gamma_i^{\mathcal{L}}) \xrightarrow{d_\star} (\Omega, \gamma_k^{\mathcal{L}}) \text{ and } (\Psi, \gamma_j^{\mathcal{L}}) \xrightarrow{d_\star} (\Omega, \gamma_k^{\mathcal{L}})$$

To prove this, I first construct intermediary objects $(\Gamma_\Phi, \gamma_0^{\mathcal{L}})$ and $(\Gamma_\Psi, \gamma_0^{\mathcal{L}})$ as in Figure 1.9. Defining these objects requires ensuring that Φ and Ψ can be properly inserted as subpaths into Γ_Φ and Γ_Ψ with the correct beginning difference. (See Corollary 3.3.10 and the lemmas preceding it, Lemmas 3.3.7 and 3.3.8.) This shows that the left fibers are contractible, and thus $[\ell\,\mathrm{inf}]$ is a weak equivalence.

$$\begin{tikzcd}
(\Phi, \gamma_i^{\mathcal{O}}) \arrow[d] & & (\Psi, \gamma_j^{\mathcal{O}}) \arrow[d] \\
(\Gamma_\Phi, \gamma_0^{\mathcal{L}}) \arrow[dr] & & (\Gamma_\Psi, \gamma_0^{\mathcal{L}}) \arrow[dl] \\
& (\Omega, \gamma_0^{\mathcal{L}}) &
\end{tikzcd}$$

Figure 1.9: Proving that the left fiber satisfies the first filtered condition.

1.4 Chapter Summaries

In Chapter 2, I present some necessary preliminary material. While most of the results in Chapter 2 are not new, they offer an alternative perspective that forms a firm foundation for my original work in Chapter 3. First, in Section 2.1, I discuss equivariant and isovariant maps with a focus on such maps between the quotient sets of a finite group, G. I emphasize the simplifications we can make when G is abelian, which help us in later sections to gain a nice intuition for the link orbit category in this setting.

Next, in Section 2.2, I include some simple categorical definitions and facts. Lemma 2.2.4 is a rewriting of one of del Hoyo's statements [2] that I found particularly helpful in handling the equivalence relation needed to define the subdivision.

I address the category Δ in Section 2.3, which is a technical piece in the definition of the subdivision. (Friedman's paper is a wonderful reference for this category [3].) In Section 2.4, I recall the definition of weak equivalence [2] and state Quillen's Theorem

A together with his result concerning the contractibility of filtered categories [8], which provide the motivation for my framework in Chapter 3.

Following this, I define the orbit and link orbit categories in Sections 2.5 and 2.6, respectively, and illustrate with examples. The abelianness of G provides us with a simpler description of the link orbit category (Remark 2.6.8).

My treatment of the pre-subdivision category in Section 2.7 expands on May's work [6]. I utilize his definition but introduce some new notation. I have included a detailed example and calculate the subdivision for $G = \mathbb{Z}/2\mathbb{Z}$, which, to my knowledge, has not previously appeared in the literature.

In order to move from the pre-subdivision to the subdivision in Section 2.8, we must put an equivalence relation on the morphisms. I formally use May's definition [6], but del Hoyo's alternate viewpoint defining the subdivision from the left fiber $\Delta/\mathcal{C}$ [2] aided my understanding immensely. May briefly mentions that his definition allows us to "account for degeneracies", i.e., consecutive groups of maps that compose to an identity, when deciding whether two morphisms should be equivalent; I formalize this statement in the Scoot Lemma 2.8.2 and give a detailed proof and examples. (The Scoot Lemma is very general and does not rely on the abelianness of the group.)

I begin Chapter 3 by defining the inf functor in Section 3.1, a contravariant functor from the pre-subdivision to the orbit category that I created when I found that del Hoyo's sup functor [2] suppresses too much information about the objects in the subdivision of the orbit category. In Section 3.2, I use this functor to define the functor $\ell\,\mathrm{inf}$ from the pre-subdivision of the orbit category to the link orbit category. I prove that this functor is compatible with the equivalence relation required to pass to the subdivision of the orbit category, which allows me to define the induced functor $[\ell\,\mathrm{inf}]$. I prove that each left fiber of this functor is filtered in Section 3.3, and a result from Quillen tells us that filtered categories are contractible [8]. Consequently, $[\ell\,\mathrm{inf}]$ is a weak equivalence by Quillen's Theorem A.

1.5 Beyond Abelian Groups

Based upon certain proof steps, I suspect that expanding this result to non-abelian groups would take some dramatic changes to this framework. One early important instance of this is in the proof that the functor $[\ell\,\mathrm{inf}]$ is in fact a functor (Lemma 3.2.4). I also depend heavily on the abelianness of G when proving that the left fibers of $[\ell\,\mathrm{inf}]$ satisfy the first filtered condition (Lemma 3.3.7). Perhaps even more crucially than these specific lemmas, the link orbit category becomes more complicated to understand for non-abelian groups and a broader intuition becomes necessary.

Chapter 2

Preliminaries

In this chapter, I present some necessary preliminary material. While most of the results in Chapter 2 are not new, they offer an alternative perspective that forms a firm foundation for my original work in Chapter 3. First, in Section 2.1, I discuss equivariant and isovariant maps with a focus on such maps between the quotient sets of a finite group, G. I emphasize the simplifications we can make when G is abelian, which help us in later sections to gain a nice intuition for the link orbit category in this setting.

Next, in Section 2.2, I include some simple categorical definitions and facts. Lemma 2.2.4 is a rewriting of one of del Hoyo's statements [2] that I found particularly helpful in handling the equivalence relation needed to define the subdivision.

I address the category Δ in Section 2.3, which is a technical piece in the definition of the subdivision. (Friedman's paper is a wonderful reference for this category [3].) In

Section 2.4, I recall the definition of weak equivalence [2] and state Quillen's Theorem A together with his result concerning the contractibility of filtered categories [8], which provide the motivation for my framework in Chapter 3.

Following this, I define the orbit and link orbit categories in Sections 2.5 and 2.6, respectively, and illustrate with examples. The abelianness of G provides us with a simpler description of the link orbit category (Remark 2.6.8).

My treatment of the pre-subdivision category in Section 2.7 expands on May's work [6]. I utilize his definition but introduce some new notation. I have included a detailed example and calculate the subdivision for $G = \mathbb{Z}/2\mathbb{Z}$, that, to my knowledge, has not previously appeared in the literature.

In order to move from the pre-subdivision to the subdivision in Section 2.8, we must put an equivalence relation on the morphisms. I formally use May's definition [6], but del Hoyo's alternate viewpoint defining the subdivision from the left fiber $\Delta/\mathcal{C}$ [2] aided my understanding immensely. May briefly mentions that his definition allows us to "account for degeneracies", i.e., consecutive groups of maps that compose to an identity, when deciding whether two morphisms should be equivalent; I formalize this statement in the Scoot Lemma 2.8.2 and give a detailed proof and examples. (The Scoot Lemma is very general and does not rely on the abelianness of the group.)

2.1 Equivariant and Isovariant Maps

Definition 2.1.1. Let G be a finite group, and let X and Y be compactly generated spaces with continuous left G-actions. An *equivariant map* between X and Y is a function $\alpha : X \to Y$ that preserves the G-action so that $\alpha(g \cdot x) = g \cdot \alpha(x)$ for all $g \in G$ and $x \in X$, i.e., we obtain the same result whether we perform the group action before or after applying α.

Definition 2.1.2. Let G be a group and X be a G-set. Let $x \in X$. Then the *stabilizer group of* x, denoted G_x, is the subgroup of all $g \in G$ such that $g \cdot x = x$.

Lemma 2.1.3. If $\alpha : X \to Y$ is an equivariant map between G-sets, then $G_x \subseteq G_{\alpha(x)}$.

Proof. Suppose $g \cdot x = x$. Then $g \cdot \alpha(x) = \alpha(g \cdot x) = \alpha(x)$. □

Example 2.1.4. Let D^2 be the closed unit disk in $\mathbb{R}^2$, and let $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. For each $(x, y) \in D^2$, let $0 \cdot (x, y) = (x, y)$ and $1 \cdot (x, y) = (x, -y)$. Let $\alpha : D \to \{(0, 0)\}$ be defined by $\alpha(x, y) = (0, 0)$. Then α is an equivariant map. Further, $G_{(1,0)} = \{0\}$ but $G_{\alpha(1,0)} = \{0, 1\}$, so $G_{(1,0)} \subset G_{\alpha(1,0)}$. △

Remark 2.1.5. Example 2.1.4 shows that the set inclusion in Lemma 2.1.3 might be a strict inclusion. An equivariant map may increase the size of the stabilizer group as there may be elements in $G_{\alpha(x)}$ that are not in G_x.

Lemma 2.1.6. Let $\gamma \in G$ be fixed. The map

$$\alpha : G/H \to G/K \text{ defined by } \alpha(xH) \coloneqq x\gamma K$$

is well-defined if and only if $\gamma^{-1}H\gamma \subseteq K$.

Proof.

($\Rightarrow$) Note that

$$\alpha(H) = \alpha(eH) = e\gamma K = \gamma K$$

and for any $h \in H$,

$$\alpha(H) = \alpha(hH) = h\gamma K$$

and so $h\gamma K = \gamma K$. This implies that $\gamma^{-1}h\gamma K = K$, so $\gamma^{-1}h\gamma \in K$.

($\Leftarrow$) If $x_1H = x_2H$, then $x_2 = x_1h$ for some $h \in H$. Then for some $k \in K$,

$$\alpha(x_2H) = x_2\gamma K = x_1h\gamma K = x_1\gamma kK = x_1\gamma K = \alpha(x_1H)$$ □

Lemma 2.1.7. Let G act on the quotient sets of G where the action is defined by $g \cdot xH \coloneqq gxH$. (If not otherwise stated, assume we are using this G-action on the quotient sets of G for the remainder of this document.) A map $\alpha : G/H \to G/K$ is equivariant if and only if there exists $\gamma \in G$ such that $\alpha(xH) = x\gamma K$ for all $x \in G$.

Proof.

($\Leftarrow$) If $g \in G$, then $\alpha(gxH) = gx\gamma K$ and $g \cdot \alpha(xH) = gx\gamma K$.

($\Rightarrow$) There exists $\gamma \in G$ such that $\alpha(eH) = \gamma K$, and so $\alpha(xH) = x\alpha(eH) = x\gamma K$ for all $x \in G$. □

Corollary 2.1.8. Suppose G is abelian. Then the map $\alpha : G/H \to G/K$ defined by $\alpha(xH) = x\gamma K$ is equivariant if and only if $H \leq K$.

Proof. See Lemmas 2.1.6 and 2.1.7. □

Remark 2.1.9. Note that different values of $\gamma \in G$ may induce the same equivariant map $G/H \to G/K$. The next assertion, Lemma 2.1.10, tells us precisely when this occurs.

Lemma 2.1.10. The group elements γ and γ' in G induce the same equivariant map $G/H \to G/K$ if and only if they are in the same K-coset, gK for some $g \in G$. As a result, for finite G, there are $|G|/|K|$ distinct equivariant maps from $G/H \to G/K$.

Proof.

($\Rightarrow$) Suppose $\alpha : G/H \to G/K$ is equivariant defined as $\alpha(xH) = x\gamma K = x\gamma' K$. Then $\gamma \in \gamma K = \gamma' K \ni \gamma'$, and so γ and γ' both belong to the coset γK.

($\Leftarrow$) Suppose $\gamma, \gamma' \in gK$. Then $\gamma = gk_1$ and $\gamma' = gk_2$ for some $k_1, k_2 \in K$, and so

$$x\gamma K = xgk_1 K = xgK = xgk_2 K = x\gamma' K$$

□

Example 2.1.11. Let $G = \mathbb{Z}/N\mathbb{Z} = \{0, ..., N-1\}$. Let H and K be subgroups of G such that $H \leq K$, $H = \langle h \rangle$, and $K = \langle k \rangle$ (when $k = 0$, instead let $k = N$). We would like to describe all distinct equivariant maps from G/H to G/K. The cosets of K are $0 + \langle k \rangle, 1 + \langle k \rangle, ..., k-1 + \langle k \rangle$, so by Lemma 2.1.10, there are k equivariant maps given by $\gamma = 0, 1, 2, ..., k-1$.

How do we compose these equivariant maps? Suppose $L = \langle l \rangle \leq G$ such that $K \leq L$. Further suppose that $G/H \xrightarrow{\alpha} G/K \xrightarrow{\beta} G/L$ where $\alpha(xH) = x + a + K$ and $\beta(xK) = x + b + L$. Then

$$(\beta \circ \alpha)(xH) = x + a + b + L$$

and $\beta \circ \alpha$ is induced by $\gamma = (a + b) \mod l$. △

Lemma 2.1.12. Suppose that we have the equivariant maps $\alpha : G/H \to G/K$ and $\beta : G/H \to G/L$. Further, suppose there exists an equivariant map $\epsilon : G/K \to G/L$ such that $\beta = \epsilon \circ \alpha$ (see Figure 2.1). Because the maps are equivariant, there must exist γ_a, γ_b, and γ_c in G such that $\alpha(gH) = g\gamma_a K$, $\beta(gH) = g\gamma_b L$, and $\epsilon(gK) = g\gamma_c L$. Then ϵ is unique and given by the map $\epsilon(gK) = g\gamma_a^{-1}\gamma_b L$. (Note that G need not be abelian in this Lemma.)

$$\begin{tikzcd}
G/H \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & G/L \\
& G/K \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Figure 2.1: Equivariant map uniqueness commutative diagram.

Proof. For any $g \in G$, the commutative diagram implies that $g\gamma_a\gamma_c L = g\gamma_b L$, and so

$$\gamma_a\gamma_c L = \gamma_b L \Rightarrow \gamma_c L = \gamma_a^{-1}\gamma_b L \Rightarrow \gamma_c = \gamma_a^{-1}\gamma_b l$$

for some $l \in L$ (and in fact there may be many such l's). However, regardless of our choice of l, we obtain the same map ϵ since $\epsilon(gK) = g\gamma_a^{-1}\gamma_b l L = g\gamma_a^{-1}\gamma_b L$ □

Definition 2.1.13. An *isovariant map* between G-sets X and Y is an equivariant map $\alpha : X \to Y$ that preserves all stabilizer groups, i.e., $G_x = G_{\alpha(x)}$ for all $x \in X$.

Example 2.1.14. Using the same group and G-sets as Example 2.1.4, let

$$\alpha : \{(0,0)\} \to D^2$$

$$\alpha(0,0) := (1,0)$$

Note that $1 \cdot (1,0) = (1,0)$. Then $G_{(0,0)} = \{0,1\}$ and $G_{\alpha(0,0)} = \{0,1\}$, and so α is an isovariant map. △

Lemma 2.1.15. Let $\alpha : G/H \to G/K$ be equivariant where $\alpha(xH) = x\gamma K$. Then α is isovariant if and only if $\gamma^{-1}H\gamma = K$. (If G is abelian, then α is isovariant if and only if $H = K$.)

Proof.

($\Rightarrow$) Since α is isovariant, $G_H = G_{\alpha(H)}$. I will show that $G_H = H$ and $G_{\alpha(H)} = \gamma K \gamma^{-1}$.

- First, $G_H = H$ since $G_H = \{g \in G : gH = H\}$, and for some $h_1, h_2 \in H$,

$$gH = H \Rightarrow gh_1 = h_2 \Rightarrow g = h_2 h_1^{-1} \Rightarrow g \in H,$$

and if $h \in H$, then $hH = h$.

- Secondly, suppose $g \in G_{\alpha(H)}$. Then $g\alpha(H) = \alpha(H)$, and so for some $k_1, k_2 \in K$,

$$g\gamma K = \gamma K \Rightarrow g\gamma k_1 = \gamma k_2 \Rightarrow g = \gamma k_2 k_1^{-1} \gamma^{-1} \Rightarrow g \in \gamma K \gamma^{-1}$$

On the other hand, $\gamma k \gamma^{-1} \alpha(H) = \gamma k \gamma^{-1} \gamma K = \gamma K = \alpha(H)$. Thus $G_{\alpha(H)} = \gamma K \gamma^{-1}$.

($\Leftarrow$) Let $g \in G_{\alpha(xH)}$. Then $g \cdot \alpha(xH) = \alpha(xH)$, and so $gx\gamma K = x\gamma K$, and $gx\gamma(\gamma^{-1}H\gamma) = x\gamma(\gamma^{-1}H\gamma)$. Simplifying, this shows that $gxH = xH$, and so $g \in G_{xH}$. □

2.2 Categorical Concepts

This section contains some general categorical results and definitions that we will need for this document.

Definition 2.2.1. A *small* category has a set's worth of morphisms (and thus a set's worth of objects due to the bijection between objects and identity morphisms).

Remark 2.2.2. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories, and $F : \mathcal{C} \Rightarrow \mathcal{D}$ is a functor. Further suppose that $\mathcal{D}$ has at most one morphism between any two objects. Then if $f : x \to y$ is a morphism in $\mathcal{C}$, Ff is completely determined by Fx and Fy, i.e., we can write $F(x \to y) = Fx \to Fy$ unambiguously. This is because for any morphism $f : x \to y$ in $\mathcal{C}$, F must take f to some morphism of the form $Fx \to Fy$. If there is only one such morphism, there is only one such choice.

Definition 2.2.3. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories, and $F : \mathcal{C} \Rightarrow \mathcal{D}$ is a functor. The functor F is *faithful* if for all objects x, y in $\mathcal{C}$, the map $\mathrm{Hom}(x, y) \longrightarrow \mathrm{Hom}(Fx, Fy)$ is injective, i.e., if $Fx \to Fy$ is a $\mathcal{D}$-morphism, then there is at most one morphism between x and y that F maps to $Fx \to Fy$.

The proof of the following Lemma is based on a part of del Hoyo's argument in [2] Lemma 18 .

Lemma 2.2.4. Suppose $F : \mathcal{C} \Rightarrow \mathcal{D}$ is a (covariant or contravariant) functor between categories, $\sim$ is the smallest equivalence relation generated by the relation $\approx$, and for all morphisms f_1, f_2 in $\mathcal{C}$, if $f_1 \approx f_2$ then $F(f_1) = F(f_2)$. Then if f and g are morphisms in $\mathcal{C}$ and $f \sim g$, then $F(f) = F(g)$.

Proof. Let $R_{\approx} = \{(f_1, f_2) : f_1 \approx f_2\}$ and $R_{\sim} = \{(f_1, f_2) : f_1 \sim f_2\}$. Define the equivalence relation

$$f_1 \sim_F f_2 \iff F(f_1) = F(f_2)$$

and let $R_{\sim_F} = \{(f_1, f_2) : f_1 \sim_F f_2\}$. Then $R_{\approx} \subseteq R_{\sim_F}$, and since $R_{\sim}$ is the smallest equivalence relation containing $R_{\approx}$, we must have $R_{\approx} \subseteq R_{\sim} \subseteq R_{\sim_F}$. Thus if $f \sim g$, then $F(f) = F(g)$. $\square$

Definition 2.2.5. A small, nonempty category $\mathcal{C}$ is said to be a *filtered category* if both of the following hold:

1. For any two objects $c_1, c_2 \in \mathcal{C}$, there exists an object $c_3 \in \mathcal{C}$ and morphisms $c_1 \to c_3$ and $c_2 \to c_3$.

2. For any two parallel morphisms $f, g : c_1 \to c_2$ in $\mathcal{C}$, there exists a morphism $h : c_2 \to c_3$ such that $h \circ f = h \circ g$.

$$\begin{tikzcd} c_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[rr, bend left, "h \circ f = h \circ g"] & c_2 \arrow[r, "h"'] & c_3 \end{tikzcd}$$

Figure 2.2: Second filtered category condition.

Remark 2.2.6. In Definition 2.2.5, note that if all parallel morphisms in a category are equal, i.e., there is at most one morphism between any two objects in the category, then (2) is satisfied with $c_3 = c_2$ and $h = id_{c_2}$.

2.3 The Category Δ

My understanding of this category comes primarily from Friedman's paper [3].

Definition 2.3.1. Define $[n] := \{0, 1, ..., n\}$. Let Δ be the category with data:

`objects:` $[n]$ where $n \in \mathbb{Z}_{\geq 0}$

`morphisms:` the (not strictly) order-preserving maps between $[n]$ and $[m]$

Remark 2.3.2. All of the order-preserving maps are generated as compositions of the injections $d_i : [n] \to [n+1]$ and the surjections $s_i : [n+1] \to [n]$ defined as

$$d_i(j) = \begin{cases} j & j \leq i - 1 \\ j + 1 & j \geq i \end{cases}$$

and

$$s_i(j) = \begin{cases} j & j \leq i \\ j - 1 & j \geq i + 1 \end{cases}$$

Lemma 2.3.3. (Simplicial Identities) In the category Δ, we have the following simplicial identities:

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ if } i < j$$

$$d_i \circ s_j = s_{j-1} \circ d_i \text{ if } i < j$$

$$d_j \circ s_j = d_{j+1} \circ s_j = id$$

$$d_i \circ s_j = s_j \circ d_{i-1} \text{ if } i > j+1$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j$$

Example 2.3.4. We can visualize $d_2 : [4] \to [5]$ as an injection in which 2 is deleted from the codomain, and $s_2 : [4] \to [3]$ as a surjection in which 2 is repeated in the codomain, as visualized in Figure 2.3. $\triangle$

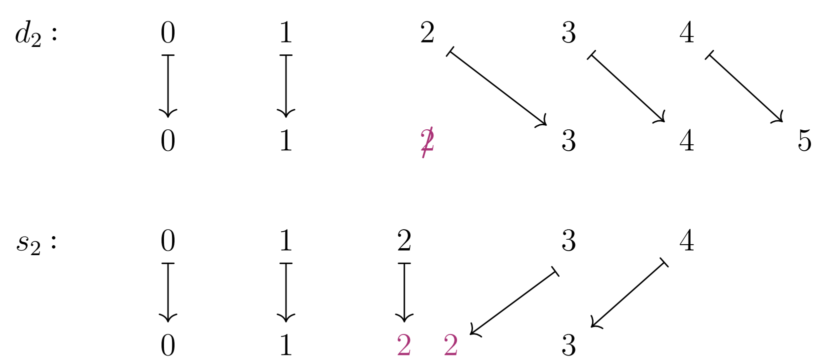

Figure 2.3: Simplicial maps example.

Lemma 2.3.5. Suppose $d : [n] \to [m]$ is an injection in Δ where $k_1, ..., k_{m-n} \in [m] \backslash d([n])$ and $k_1 < ... < k_{m-n}$. Then $d = d_{k_{m-n}} d_{k_{m-n-1}} ... d_{k_1}$. In particular, if written in strictly decreasing order, the indices of the d_i-maps indicate which elements are excluded from the range.

Definition 2.3.6. We can further consider *the category* $[n]$ with objects $0, 1, ..., n$ and morphisms $k \to l$ if $k \leq l$. This implies that if $k \leq l$, then there is exactly one map from $k \to l$ in $[n]$. If $k = l$, this map is the identity map on k. From this perspective, the morphisms in Δ are functors $[n] \to [m]$.

Remark 2.3.7. Because there is only one morphism between any two objects in $[n]$, any functors into the category $[n]$ are completely determined by their definition on objects. So if $j_1 \to j_2$ is a morphism in $[n]$ and $d : [n] \to [m]$ is a morphism in Δ, then

$$d(j_1 \to j_2) = d(j_1) \to d(j_2)$$

without any ambiguity (see Remark 2.2.2)

2.4 Weak Equivalence

Definition 2.4.1. Let $\mathcal{C}$ be a small category, and let $\mathrm{Fun}([n], \mathcal{C})$ denote the set of functors from $[n]$ to $\mathcal{C}$. The *nerve of* $\mathcal{C}$ is a functor $N\mathcal{C} : \Delta^{op} \to \mathrm{Set}$ such that:

`on objects:` $N\mathcal{C}([n]) = \mathrm{Fun}([n], \mathcal{C})$ (denoted $N\mathcal{C}_n$)

`on morphisms:` $N\mathcal{C}(d_i^{op} : [m] \to [n])(F) := F \circ d_i : [n] \to \mathcal{C}$, and

$N\mathcal{C}(s_i^{op} : [n] \to [m])(F) := F \circ s_i : [m] \to \mathcal{C}$ and expand functorially.

$$\begin{tikzcd}
{[m]} \arrow[d, "d_i^{op}"] & \overset{N\mathcal{C}}{\Longrightarrow} & \mathrm{Fun}([m], \mathcal{C}) \arrow[d, "N\mathcal{C}(d_i^{op})(F) := F \circ d_i"] \\
{[n]} & & \mathrm{Fun}([n], \mathcal{C})
\end{tikzcd}$$

Figure 2.4: The nerve functor.

Remark 2.4.2. We define $N\mathcal{C}_k$ as the set of all functors $[k] \Rightarrow \mathcal{C}$. In particular, $N\mathcal{C}_0$ can be viewed as all of the objects of $\mathcal{C}$, $N\mathcal{C}_1$ as the set of all morphisms in $\mathcal{C}$, and $N\mathcal{C}_2$ as the set of all pairs of two composable $\mathcal{C}$-morphisms.

Definition 2.4.3. Quillen defines the *classifying space functor* $B : \mathcal{C}at \Rightarrow \mathcal{T}op$ such that B associates to each (small) category $\mathcal{C}$ a topological space via the geometric realization of $N\mathcal{C}$ [8]. A functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ in $\mathcal{C}at$ is said to be a *weak equivalence* if $BF : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence in $\mathcal{T}op$, i.e., there exists $g : B\mathcal{D} \to B\mathcal{C}$ in $\mathcal{T}op$ such that $BF \circ g$ is homotopic to $id_{B\mathcal{D}}$ and $g \circ BF$ is homotopic to $id_{B\mathcal{C}}$.

Remark 2.4.4. Based upon Definition 2.4.3, it is not necessarily the case that g in Definition 2.4.3 is in the image of B. Additionally, B is not full [8], so $B\mathcal{C}$ and $B\mathcal{D}$ may be homotopically equivalent in $\mathcal{T}op$, but there may be no weak equivalence between $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.4.5. A (small) category $\mathcal{C}$ is *contractible* if the geometric realization of its nerve, $B\mathcal{C}$ is contractible in $\mathcal{T}op$.

Theorem 2.4.6. Any filtered category is contractible [8].

Definition 2.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. Let d be an object in $\mathcal{D}$. The *left fiber* F/d is the category with the following data:

`objects:` the pairs (c, δ) with c an object of $\mathcal{C}$ and $\delta : Fc \to d$ a morphism of $\mathcal{D}$

`morphisms:` There exists an arrow $(c, \delta) \xrightarrow{\kappa_*} (c', \delta')$ if there exists a map $\kappa : c \to c'$ in $\mathcal{C}$ such that $\delta' \circ F\kappa = \delta$.

Theorem 2.4.8. (Quillen's Theorem A) Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. The functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ is a weak equivalence if F/d is contractible for every object d of $\mathcal{D}$ [8].

2.5 The Orbit Category

The orbit category of a group G keeps track of equivariant maps between the quotient sets of G.

Definition 2.5.1. The *orbit category of* G, denoted $\mathcal{O}_G$, has the following data:

`objects:` G-sets G/H where $H \leq G$, which we will denote by the subgroup H when the group is given.

`morphisms:` equivariant maps of G-sets where the group action is defined by $\gamma \cdot gH := \gamma gH$. I will use the notation $\gamma_i^{\mathcal{O}} : G/H \to G/K$ to denote the map induced by $i \in G$. Then composition in the orbit category is defined by $\gamma_g^{\mathcal{O}} \circ \gamma_h^{\mathcal{O}} = \gamma_{hg}^{\mathcal{O}}$.

Example 2.5.2. Two orbit categories are given in Figures 2.5 and 2.6. $\triangle$

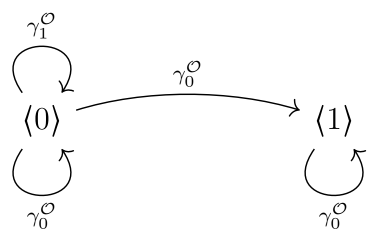

Figure 2.5: The orbit category for $\mathbb{Z}/2\mathbb{Z}$

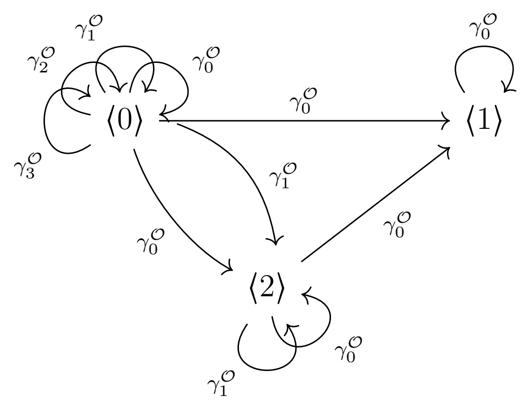

Figure 2.6: The orbit category for $\mathbb{Z}/4\mathbb{Z}$

2.6 The Link Orbit Category

The orbit category allows for a special adjunction called a Quillen equivalence [8] between $G\mathcal{T}op$ and $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}op)$. Yeakel defines the *link orbit category*, $\mathcal{L}_G$, to keep track of isovariant maps between the quotient sets of G such that there is an analogous adjunction between isvt–$G\mathcal{T}op$ and $\mathrm{Fun}(\mathcal{L}_G^{op}, \mathcal{T}op)$ [10]. (See Figure 2.7.)

$$\begin{array}{ccc} G\mathcal{T}op & & \text{isvt}G\mathcal{T}op \\ \uparrow\ \downarrow & & \uparrow\ \downarrow \\ \text{Fun}(\mathcal{O}_G^{op}, \mathcal{T}) & & \text{Fun}(\mathcal{L}_G^{op}, \mathcal{T}op) \end{array}$$

Figure 2.7: Quillen equivalences and $\mathcal{L}_G$.

Definition 2.6.1. Given a group G, define the small category $\widetilde{\mathcal{L}_G}$ as follows:

objects: strictly increasing chains of subgroups of G, $\mathbb{H} = H_0 < ... < H_n$

morphisms: Let $\iota : [n] \to [m]$ be an order-preserving inclusion. Then a morphism of $\widetilde{\mathcal{L}_G}$, $H_0 < ... < H_n \to K_0 < ... < K_m$, is a pair (ι, γ), where $\gamma \in G$ is an element such that $H_j = \gamma K_{\iota(j)} \gamma^{-1}$ for all $j \in [n]$. That is, γ induces a collection of $n+1$ maps,

$$R_\gamma^j : G/H_j \to G/K_{\iota(j)}$$

$$xH_j \mapsto x\gamma K_{\iota(j)}$$

for each $j \in [n]$. (The conjugacy relationship forces these maps to be isovariant by Lemma 2.1.15.) Composition in $\widetilde{\mathcal{L}_G}$ is defined by $(\iota, \gamma) \circ (\iota', \gamma') = (\iota \circ \iota', \gamma' \gamma)$.

Definition 2.6.2. Define the *link orbit category*, $\mathcal{L}_G$, as follows:

objects: strictly increasing chains of subgroups of G, $\mathbb{H} = H_0 < ... < H_n$

morphisms: equivalence classes of $\widetilde{\mathcal{L}_G}$-morphisms in which two parallel morphisms $(\iota, \gamma), (\iota', \gamma') : H \to K$ are identified if $\iota = \iota'$, and γ and γ' induce the same equivariant maps, i.e.,

$$R_\gamma^j = R_{\gamma'}^j : G/H_j \to G/K_{\iota(j)} = G/K_{\iota'(j)}$$

for all j. Composition in $\mathcal{L}_G$ is defined by $[(\iota, \gamma)]_{\mathcal{L}} \circ [(\iota', \gamma')]_{\mathcal{L}} = [(\iota \circ \iota', \gamma' \gamma)]_{\mathcal{L}}$.

Lemma 2.6.3. Suppose G is a finite, abelian group, and $\mathbb{H}$ and $\mathbb{K}$ are objects in $\widetilde{\mathcal{L}_G}$ such that H_0, ..., H_n appear in $\mathbb{K}$. Then we have $|G|$ maps $\mathbb{H} \to \mathbb{K}$ in $\widetilde{\mathcal{L}_G}$ where each map is induced by an element of G.

Proof. First, recall from Lemma 2.1.15 that a map $G/H \to G/K$ defined by $xH \mapsto x\gamma K$ is isovariant if and only if $\gamma^{-1}H\gamma = K$. Since G is abelian, the second part of this statement simply says that $H = K$. So, using the definition of $\widetilde{\mathcal{L}_G}$, there is a map $\mathbb{H} \to \mathbb{K}$ in $\widetilde{\mathcal{L}_G}$ corresponding to $\gamma \in G$ if and only if H_0, ..., H_n appear in $\mathbb{K}$. So as long as H_0, ..., H_n appear in $\mathbb{K}$, we have $|G|$ maps $\mathbb{H} \to \mathbb{K}$ in $\widetilde{\mathcal{L}_G}$ where each map is induced by an element of G. □

Remark 2.6.4. Lemma 2.6.3 also implies that for $(G, +)$ finite and abelian, there is only one possible ι that includes the subgroups in the chain $\mathbb{H}$ into $\mathbb{K}$, and so in this situation, we do not need to keep track of ι when notating the morphisms of $\widetilde{\mathcal{L}_G}$. Because of this, I will use the notation $\gamma_g^{\mathcal{L}}$ for $[(\iota, g)]_{\mathcal{L}}$ where $g \in G$ induces the isovariant map. Then if we have the maps $\gamma_i^{\mathcal{L}} : \mathbb{H} \to \mathbb{K}$ and $\gamma_j^{\mathcal{L}} : \mathbb{K} \to \mathbb{J}$, then composition in $\mathcal{L}_G$ is defined as $\gamma_j^{\mathcal{L}} \circ \gamma_i^{\mathcal{L}} = \gamma_{i+j}^{\mathcal{L}}$.

Remark 2.6.5. We need to understand which maps get identified in $\mathcal{L}_G$ for G finite and abelian. By Lemma 2.6.3, we can start by looking at the elements of G that induce isovariant self-maps of G/H_0. This gives us an initial list of maps in $\mathcal{L}_G$ that could potentially be expanded by looking at isovariant self-maps of G/H_j for some

$j \geq 1$. However, Lemma 2.6.6 and Corollary 2.6.7 will assert that the list built by only considering H_0 is in fact the complete list of maps $\mathbb{H} \to \mathbb{K}$.

Lemma 2.6.6. Let G be finite and abelian. If $R^0_\gamma = R^0_{\gamma'} : G/H_0 \to G/H_0$ is an isovariant self-map defined by $gH_0 \mapsto g\gamma H_0$ (or $gH_0 \mapsto g\gamma' H_0$), then $R^j_\gamma = R^j_{\gamma'} : G/H_j \to G/H_j$ defined by $gH_j \mapsto g\gamma H_j$ (or $gH_j \mapsto g\gamma' H_j$) for all $j \geq 1$.

Proof. Note that since $R^0_\gamma = R^0_{\gamma'}$, we know that $g\gamma H_0 = g\gamma' H_0$, and so there exists $h_0 \in H_0$ such that $g\gamma = g\gamma' h_0$. Now, suppose $g\gamma h \in g\gamma H_j$. Then $g\gamma h = g\gamma' h_0 h = g\gamma' h'$ for some $h' \in H_j$. Thus $g\gamma h \in g\gamma' H_j$, and $g\gamma H_j \subset g\gamma H_j$. By a symmetrical argument, we get the reverse set inclusion, and so $g\gamma H_j = g\gamma' H_j$, and so $R^j_\gamma = R^j_{\gamma'}$. $\square$

Corollary 2.6.7. Let G be a finite abelian group, and let $\gamma_i^{\mathcal{L}}, \gamma_j^{\mathcal{L}} : \mathbb{H} \to \mathbb{K}$ be maps in $\mathcal{L}_G$ such that we only have $R^0_j = R^0_i$. Then $\gamma_i^{\mathcal{L}} = \gamma_j^{\mathcal{L}}$ in $\mathcal{L}_G$.

Proof. See Remark 2.6.5 and Lemma 2.6.6. $\square$

Remark 2.6.8. In the case that G is finite and abelian, $\mathcal{L}_G$ becomes somewhat simpler to describe based on the previous Lemmas.

`objects:` strictly increasing chains of subgroups of G, denoted $\mathbb{H} = H_0 < \ldots < H_n$

`morphisms:` We have a morphism $\mathbb{H} \to \mathbb{K}$ in $\mathcal{L}_G$ if and only if H_i appears in $\mathbb{K}$ for every H_i in $\mathbb{H}$, i.e., $\mathbb{H}$ is a subchain of $\mathbb{K}$. Distinct morphisms are induced by the same elements of G that induce the self-equivariant maps of G/H_0 (Remark 2.6.5) and are in fact then isovariant by Lemma 2.1.15. So when a map exists from $\mathbb{H} \to \mathbb{K}$, there is a bijection between the H_0-cosets of G and morphisms $\mathbb{H} \to \mathbb{K}$ given by mapping $g + H_0 \mapsto \gamma_g^{\mathcal{L}}$ (Corollary 2.1.10).

Remark: We can interpret the composition of maps in $\mathcal{L}_G$ using this bijection. Suppose we have $\mathbb{H} \xrightarrow{\gamma_i^{\mathcal{L}}} \mathbb{K} \xrightarrow{\gamma_j^{\mathcal{L}}} \mathbb{L}$. Then we define $(j + K_0) \circ (i + H_0) := (i + j) + H_0$. This is well-defined since $K_0 \leq H_0$.

Example 2.6.9. Here are some examples of link orbit categories in Figure 2.8 and Figure 2.9 for the groups $\mathbb{Z}/p\mathbb{Z}$ with p prime and $\mathbb{Z}/4\mathbb{Z}$. △

$$\begin{tikzcd}
\langle 0 \rangle \arrow[loop above, "\gamma_0^{\mathcal{L}},\dots,\gamma_{p-1}^{\mathcal{L}}"] \arrow[dr, "\gamma_0^{\mathcal{L}},\dots,\gamma_{p-1}^{\mathcal{L}}"'] & & \langle 1 \rangle \arrow[loop above, "\gamma_0^{\mathcal{L}}"] \arrow[dl, "\gamma_0^{\mathcal{L}}"'] \\
& \langle 0 \rangle < \langle 1 \rangle \arrow[loop below, "\gamma_0^{\mathcal{L}},\dots,\gamma_{p-1}^{\mathcal{L}}"] &
\end{tikzcd}$$

Figure 2.8: The link orbit category for the group of order p.

Lemma 2.6.10. Let G be finite and abelian. Suppose we have the maps $\gamma_a^{\mathcal{L}} : \mathbb{K} \to \mathbb{L}$ and $\gamma_b^{\mathcal{L}} : \mathbb{H} \to \mathbb{L}$ in $\mathcal{L}_G$. Further, suppose there exists a map $\gamma_c^{\mathcal{L}} : \mathbb{H} \to \mathbb{K}$ in $\mathcal{L}_G$ such that $\gamma_b^{\mathcal{L}} = \gamma_a^{\mathcal{L}} \circ \gamma_c^{\mathcal{L}}$ (see Figure 2.10). Then $\gamma_c^{\mathcal{L}}$ is given by the map $\gamma_c^{\mathcal{L}} = (\gamma_b \gamma_a^{-1})^{\mathcal{L}}$ (and so is uniquely determined by $\gamma_a^{\mathcal{L}}$ and $\gamma_b^{\mathcal{L}}$).

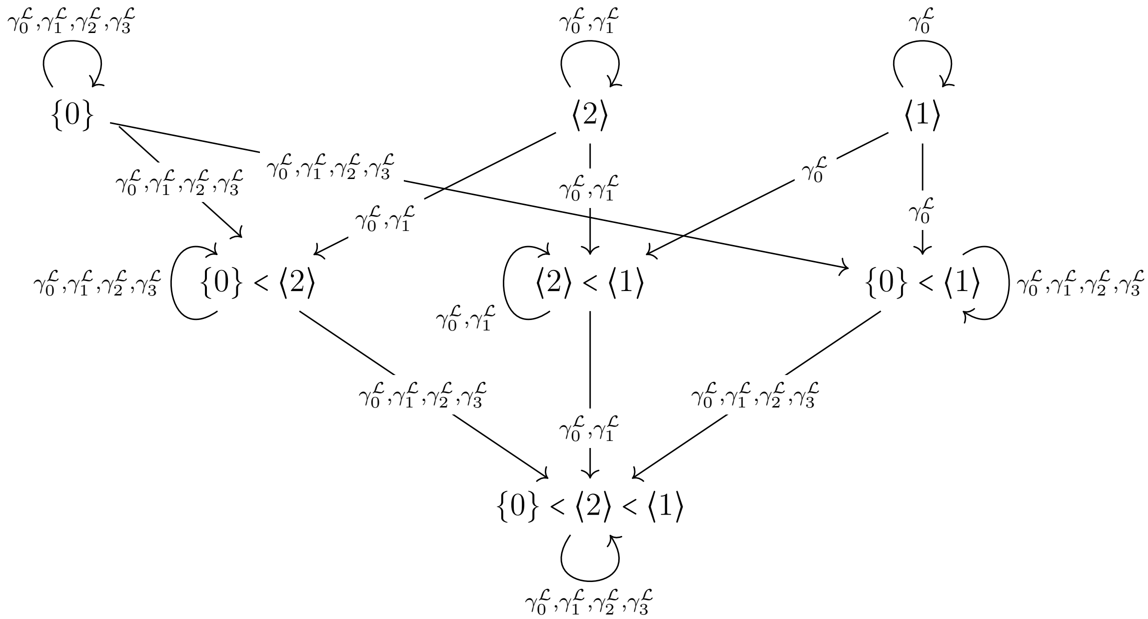

Figure 2.9: The link orbit category for the cyclic group with 4 elements.

$$
\begin{tikzcd}
\mathbb{H} \arrow[rr, "\gamma_b^{\mathcal{L}}"] \arrow[dr, "\gamma_c^{\mathcal{L}}"'] & & \mathbb{L} \\
& \mathbb{K} \arrow[ur, "\gamma_a^{\mathcal{L}}"'] &
\end{tikzcd}
$$

Figure 2.10: Link orbit category uniqueness commutative diagram.

Proof. Let $\mathbb{H} = H_0 < ... < H_n$. Since $\gamma_b^{\mathcal{L}}$ is a map in $\mathcal{L}_G$, there exists a collection of maps $R^j_{\gamma_b} : G/H_j \to G/H_j$ defined by $gH_j \mapsto g\gamma_b H_j$ for all $j \in [n]$. On the other hand, the map $\gamma_a^{\mathcal{L}} \circ \gamma_c^{\mathcal{L}} = (\gamma_c\gamma_a)^{\mathcal{L}}$ implies that there exists the collection $R^j_{\gamma_c\gamma_a} : G/H_j \to G/H_j$ defined by $gH_j \mapsto g\gamma_c\gamma_a H_j$ for all $j \in [n]$. The diagram commutes, and so for each j,

$$g\gamma_b H_j = g\gamma_c\gamma_a H_j$$

which implies that: $\gamma_b H_j = \gamma_c\gamma_a H_j \Rightarrow H_j = \gamma_b^{-1}\gamma_c\gamma_a H_j \Rightarrow \gamma_b^{-1}\gamma_c\gamma_a = h_j$ for some $h_j \in H_j$, and so $\gamma_c = \gamma_b h_j \gamma_a^{-1} \Rightarrow \gamma_c = \gamma_b\gamma_a^{-1} h_j$ since G is abelian.

Now, $\gamma_c^{\mathcal{L}}$ is a map in $\mathcal{L}$, so we have the collection $R^j_{\gamma_c} : G/H_j \to G/H_j$ defined by $gH_j \mapsto g\gamma_c H_j$. But $g\gamma_c H_j = g\gamma_b\gamma_a^{-1}h_jH_j = g\gamma_b\gamma_a^{-1}H_j$, and so $\gamma_c^{\mathcal{L}}$. □

2.7 The Pre-subdivision Category

I will use primarily May's formulation to first describe the pre-subdivision category $\mathfrak{C}\mathcal{C}$ and then describe Sd$\mathcal{C}$ as $\mathfrak{C}\mathcal{C}$ under a morphism-gluing equivalence relation [6].

Definition 2.7.1. Let $\mathcal{C}$ be a small category and let

$$\varphi_1 : c_0 \to c_1, \ \ \ldots, \ \ \varphi_n : c_{n-1} \to c_n$$

be $\mathcal{C}$-morphisms such that the composition $\varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$ is well-defined. Then

$$(c_0 \xrightarrow{\varphi_1} c_1 \xrightarrow{\varphi_2} c_2 \xrightarrow{\varphi_3} \ldots \xrightarrow{\varphi_n} c_n)$$

is a *chain* or *path* in $\mathcal{C}$. If $\varphi_i \neq \mathrm{id}_{c_{i-1}}$ for all $i \in [n]$, then the chain is a *nondegenerate chain*. Otherwise, the chain is called a *degenerate chain*. The integer n is the *dimension* or *length* of the chain.

Remark 2.7.2. While a $\mathcal{C}$-morphism in a nondegenerate chain can not be an identity, it is possible that compositions of these $\mathcal{C}$-morphisms in a nondegenerate chain compose to an identity (and we will see many instances of this).

Remark 2.7.3. We can notate these n-chains of $\mathcal{C}$-morphisms in the following manner:

$${}^0\varphi_1{}^1 \ldots {}^{k-1}\varphi_k{}^k \ldots {}^{n-1}\varphi_n{}^n := (c_0 \xrightarrow{\varphi_1} c_1 \xrightarrow{\varphi_2} c_2 \xrightarrow{\varphi_3} \ldots \xrightarrow{\varphi_n} c_n)$$

For an even shorter notation, we can use Φ^n where $\Phi^n(k-1 \to k) = \varphi_k$, and expanding functorially, $\Phi^n(j \to k) = \varphi_k \circ \ldots \circ \varphi_{j+1}$. Note that there are $k-j$ maps in this composition. When we don't wish to specify the length of the chain, we can simply use Φ.

Remark 2.7.4. We will often think of n-chains as functors from $[n] \to \mathcal{C}$ where $\Phi^n(j) = c_j$, $\Phi^n(j \to j) = id_{c_j}$, and

$$(c_0 \xrightarrow{\varphi_1} c_1 \xrightarrow{\varphi_2} c_2 \xrightarrow{\varphi_3} \ldots \xrightarrow{\varphi_n} c_n)(j_1 \to j_2) = \varphi_{j_2} \circ \ldots \circ \varphi_{j_1+1} : c_{j_1} \to c_{j_2} \text{ if } j_1 \neq j_2$$

Definition 2.7.5. The *pre-subdivision category* $\mathcal{CC}$ has the following data:

`objects:` the functors $[n] \Rightarrow \mathcal{C}$, which can be expressed as nondegenerate chains of $\mathcal{C}$-morphisms, i.e., if $c_0, \ldots, c_n$ are objects in $\mathcal{C}$ with non-identity $\mathcal{C}$-morphisms

$$\varphi_1 : c_0 \to c_1, \quad \ldots, \quad \varphi_n : c_{n-1} \to c_n$$

then we may represent an object of $\mathcal{CC}$ as:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 1 & \longrightarrow & 2 & \ldots \longrightarrow & n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ c_0 & \xrightarrow{\varphi_1} & c_1 & \xrightarrow{\varphi_2} & c_2 & \ldots \xrightarrow{\varphi_n} & c_n \end{array}$$

or using any of the shorter notations in Remark 2.7.3. Additionally, we include all functors of the form $[0] \to \mathcal{C}$.

Remark: The functors of the form $[0] \to \mathcal{C}$ recover the objects of $\mathcal{C}$. So if c is an object of $\mathcal{C}$, we can notate the functor from $[0] \Rightarrow \mathcal{C}$ that maps $0 \mapsto c$ as $\overset{0}{c}$.

$$\begin{tikzcd} {[n]} \arrow[r, "\Phi^n"] \arrow[d, "d"'] & \mathcal{C} \\ {[m]} \arrow[ur, "\Psi^m"'] & \end{tikzcd}$$

Figure 2.11: Defining the morphisms of the pre-subdivision category.

`morphisms:` the maps $d_* : \Phi^n \longrightarrow \Psi^m$ such that $d : [n] \to [m]$ is an order-preserving injection (in particular, d is a functor from $[n]$ to $[m]$), and $\Psi^m \circ d = \Phi^n$

Remark: Given this requirement and the fact that Φ^n is nondegenerate, d must be an injection, for if d were not an injection and $d(k) = d(l)$ for some $k < l$, then

$$(\Psi^m \circ d)(k \to l) = \Psi^m(d(k) \to d(l)) = \Psi^m(id) = id = \Phi^n(k \to l)$$

If follows that we must have $n \leq m$, for if $n > m$, then d would not be an injection.

Remark 2.7.6. Intuitively, the objects in the pre-subdivision are paths through the orbit category. The morphisms in $\mathfrak{C}\mathcal{C}$ consist of all allowable insertions of Φ into Ψ, so we have a morphism from Φ to Ψ if Φ is a subpath of Ψ, and the different morphisms signify different subpath insertions. I will illustrate this in Example 2.7.8.

Lemma 2.7.7. If $d_* : \Phi^n \to \Psi^m$ and $\delta : \Psi^m \to \Omega^q$, then $(\delta \circ d)_* = \delta_* \circ d_*$

Example 2.7.8. Let's do some calculations to gain a better understanding of $\mathfrak{C}\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$. With some relabeling, recall that $\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$ is the category:

$$\begin{tikzcd} a \arrow[loop above, "\alpha_1"] \arrow[loop below, "\alpha_0"] \arrow[rr, bend left, "f"] & & b) \arrow[loop below, "\beta_0"] \end{tikzcd}$$

Figure 2.12: The orbit category for the group with 2 elements in α-notation.

where $\alpha_1 \circ \alpha_1 = \alpha_0 = id_a$ and $f \circ \alpha_1 = f$. In this category, the non-identity $\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$-morphisms φ_k may be α_1 or f, so the nondegenerate n-chains look like

$$\overset{0}{a},\ \ \overset{0}{b},\ \ {}^0f^1,\ \ {}^0\alpha_1{}^1\alpha_1{}^2...{}^{n-1}\alpha_1{}^n, \quad \text{or} \quad {}^0\alpha_1{}^1\alpha_1{}^2...\alpha_1{}^{n-1}f^n$$

- Let $\Phi = {}^0\alpha_1{}^1\alpha_1{}^2$ and $\Psi = {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$. The maps between ${}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$ are $(d_0)_*$ and $(d_3)_*$. To see why this is true for $(d_0)_*$, view the injection d_0 as the functor $d_0 : [2] \to [3]$. Then we have

$$d_0(0 \to 1 \to 2) = 1 \to 2 \to 3$$

This indicates that we can insert ${}^0\alpha_1{}^1\alpha_1{}^2$ into the last two maps in ${}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$. Similarly, $(d_3)_*$ indicates that we can insert ${}^0\alpha_1{}^1\alpha_1{}^2$ into the first two maps in ${}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$. However,

$$d_1(0 \to 1 \to 2) = 0 \to 2 \to 3$$

does not represent an allowable insertion. We can see this by way of Definition 2.7.5 since

$$(\Psi \circ d_1)(0 \to 1) = ({}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3)(d_1(0 \to 1)) = ({}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3)(0 \to 2) = \alpha_1 \circ \alpha_1 = \ id_a$$

but

$$\Phi(0 \to 1) = ({}^0\alpha_1{}^1\alpha_1{}^2)(0 \to 1) = \alpha_1$$

Similarly, d_2 does not represent an allowable insertion, and d_0, d_1, d_2, and d_3 are the nondegenerate morphisms $[2] \to [3]$ in Δ.

• Now, I claim that the map

$$(d_2 \circ d_1)_* : {}^0\alpha_1{}^1 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$$

is in $\mathcal{CO}_{\mathbb{Z}/2\mathbb{Z}}$ since $d_2 \circ d_1(0 \to 1) = 0 \to 3$, and we can insert ${}^0\alpha_1{}^1$ as ${}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$.

The map $(d_2 \circ d_1)_*$ illustrates another important point about $\mathcal{CO}_{\mathbb{Z}/2\mathbb{Z}}$; the category $\mathcal{CO}_{\mathbb{Z}/2\mathbb{Z}}$ is *not* generated by the maps between objects of consecutive dimensions, i.e., $\mathcal{C}$ is not generated by maps of the form $\Phi^n \longrightarrow \Psi^{n+1}$. In this particular instance, the map $(d_1)_*$ is not a map between ${}^0\alpha_1{}^1 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2$, and $(d_2)_*$ is not a map between ${}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3$. (Even if we use simplicial identities and consider $d_2 \circ d_1 = d_1 \circ d_1$, this does not fix the issue.)

• As a final note in this example, there are no morphisms from an object with larger dimension to an object with a smaller dimension. (There are no allowable insertions between such objects.)

In this case, the category $\mathcal{CO}_{\mathbb{Z}/2\mathbb{Z}}$ roughly looks like the category in Figure 2.13. There are many more morphisms between the objects specified than the ones I have drawn; the map $(d_1 \circ d_1)_* = (d_2 \circ d_1)_*$ serves as a reminder of the maps that are not generated by compositions of the drawn morphisms. $\triangle$

$$
\begin{tikzcd}
{}^{0}a \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_1)_*"] \arrow[drr, "(d_1)_*" description] & & {}^{0}b \arrow[d, "(d_0)_*"] \\
{}^{0}\alpha_1{}^{1} \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_2)_*"] \arrow[drr, "(d_2)_*" description] \arrow[dd, bend right=60, "(d_2 \circ d_1)_*"'] & & {}^{0}f^{1} \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_1)_*"] \\
{}^{0}\alpha_1{}^{1}\alpha_1{}^{2} \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_3)_*"] \arrow[drr, "(d_3)_*" description] & & {}^{0}\alpha_1{}^{1}f^{2} \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_2)_*"] \\
{}^{0}\alpha_1{}^{1}\alpha_1{}^{2}\alpha_1{}^{3} \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_4)_*"] \arrow[drr, "(d_4)_*" description] & & {}^{0}\alpha_1{}^{1}\alpha_1{}^{2}f^{3} \arrow[d, shift right=1.5ex, bend right, "(d_0)_*"'] \arrow[d, shift left=1.5ex, bend left, "(d_3)_*"] \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

Figure 2.13: The pre-subdivision of the orbit category for $\mathbb{Z}/2\mathbb{Z}$.

2.8 The Subdivision

Definition 2.8.1. (This definition is due to May in [6].) Let $\mathcal{C}$ be a small category. Then the *subdivision of* $\mathcal{C}$, denoted Sd$\mathcal{C}$, is the quotient category of $\mathfrak{C}\mathcal{C}$ with equivalence classes of morphisms under the equivalence relation $\sim$, which is generated by the relation $\approx$, i.e., $\sim$ is the smallest equivalence relation containing $\approx$. To define $\approx$, let $d_*, \delta_* : \Phi^n \to \Psi^m$ be in $\mathfrak{C}\mathcal{C}$ such that there exist maps ν, σ, α, and β in Δ such that σ is a surjection with right inverses α and β (so we have $\sigma \circ \alpha = \sigma \circ \beta = id$), and $d = \nu \circ \alpha$ and $\delta = \nu \circ \beta$. Then we define $\approx$ by setting $d_* \approx \delta_*$ if

$$\Phi^n \circ \sigma = \Psi^m \circ \nu.$$

Note that $\Phi^n \circ \sigma$ may be a degenerate chain – and so not in $\mathcal{CC}$– and ν might *not* be an injection. If $d_* \approx \delta_*$, we say that d_* and δ_* are *elementary equivalent*.

$$\begin{tikzcd}
\Phi^n \arrow[rr, shift left, "(\nu\circ\alpha)_*=d_*"] \arrow[rr, shift right, "(\nu\circ\beta)_*=\delta_*"'] \arrow[dr, shift right=3, "\alpha_*"'] \arrow[dr, shift left=3, "\beta_*"] & & \Psi^m \\
& \Phi^n \circ \sigma = \Psi^m \circ \nu \arrow[ul, "\sigma_*"] \arrow[ur, "\nu_*"'] &
\end{tikzcd}$$

Figure 2.14: Defining elementary equivalence.

Intuitively, this equivalence relation relates morphisms in $\mathcal{CC}$ that differ by an identity in $\mathcal{C}$, i.e., we can "scoot past an identity" and find an equivalent morphism in the subdivision. We can see which identity is being considered based on the placement of identities in the degenerate chain, $\Phi^n \circ \sigma$. (I will expand on this intuition in Lemma 2.8.2 and Examples 2.8.3 and 2.8.4.)

While a morphism in Sd$\mathcal{C}$ should formally be the equivalence class $[d_*]_\sim$, I will often abuse notation and write d_* for both a morphism in the pre-subdivision $\mathcal{CC}$ and for a morphism in Sd$\mathcal{C}$. It will occassionally be useful to also use the same notation for a morphism in which the domain or codomain is a degenerate chain as in the previous diagram. This viewpoint is consistent with del Hoyo's definition of the subdivision as arrising from the left fiber $\Delta/\mathcal{C}$, and so such a morphism can be thought of as a morphism in $\Delta/\mathcal{C}$ or in $[\Delta/\mathcal{C}]$. [2]

Lemma 2.8.2. (Scoot Lemma) Let $\mathcal{C}$ be a small category, let Φ^n and Ψ^m be objects in $\mathcal{CC}$, and let $d, \delta : [n] \to [m]$ be Δ-maps. Suppose that there exists $k \in [n]$ for which

$$\Psi^m(d(k) \to \delta(k)) = \psi_{\delta(k)} \circ ... \circ \psi_{d(k)+1} = id_{\Psi^m(d(k))}$$

and $\delta(j) = d(j)$ for all $j \in [n] \backslash \{k\}$. Then:

(a) d_* is a morphism in $\mathcal{CC}$ if and only if δ_* is a morphism in $\mathcal{CC}$, and

(b) if d_* and δ_* are morphisms in $\mathcal{CC}$, then $d_* \sim \delta_*$.

Remark: The intuition gained as a result of the Scoot Lemma is stated in May [6] in the commentart following Definition 13.2.2.

Remark: Because Ψ^m is nondegenerate, none of the ψ_i's can be the identity. This implies that the composition $\psi_{\delta(k)} \circ ... \circ \psi_{d(k)+1}$ must contain at least two distinct morphisms, namely $\psi_{\delta(k)}$ and $\psi_{d(k)+1}$. Thus $\delta(k) > d(k) + 1$, and so $\delta(k) \geq d(k) + 2$ and $\delta(k) - d(k) \geq 2$.

Proof. **(a)** If d_* is a morphism in $\mathcal{CC}$, we know that $\Psi^m \circ d = \Phi^n$, and if δ_* is a morphism in $\mathcal{CC}$, we know that $\Psi^m \circ \delta = \Phi^n$. We will show that $\Psi^m \circ \delta = \Psi^m \circ d$, and so regardless of which morphism belongs to $\mathcal{CC}$, we will have the result. To this end, let $j_1 \to j_2$ be a morphism in the category $[n]$.

- If $j_2 \leq k - 1$, or $j_1 \geq k + 1$, or $j_1 \leq k - 1$ and $j_2 \geq k + 1$, or $j_1 = j_2 \neq k$, then

$$\delta(j_1 \to j_2) = \delta(j_1) \to \delta(j_2) = d(j_1) \to d(j_2) = d(j_1 \to j_2)$$

$$(\Psi^m \circ \delta)(j_1 \to j_2) = (\Psi^m \circ d)(j_1 \to j_2)$$

- If $j_2 = k$ and $j_1 \leq k-1$, then

$$\begin{aligned}(\Psi^m \circ \delta)(j_1 \to k) &= \psi_{\delta(k)} \circ \psi_{\delta(k)-1} \circ \ldots \circ \psi_{\delta(j_1)+1} \\ &= \psi_{\delta(k)} \circ \ldots \circ \psi_{d(k)+1} \circ \psi_{d(k)} \circ \ldots \circ \ldots \circ \psi_{d(j_1)+1} \\ &= id_{(\mathbb{Z}/p\mathbb{Z})/\{0\}} \circ \psi_{d(k)} \ldots \circ \psi_{d(j_1)+1} \\ &= \psi_{d(k)} \ldots \circ \psi_{d(j_1)+1} \\ &= \Psi^m(d(j_1) \to d(k)) \\ &= (\Psi^m \circ d)(j_1 \to j_2)\end{aligned}$$

- If $j_1 = k$ and $j_2 \geq k+1$, then

$$\begin{aligned}(\Psi^m \circ \delta)(k \to j_2) &= \psi_{\delta(j_2)} \circ \psi_{\delta(j_2)-1} \circ \ldots \circ \psi_{\delta(k)+1} \\ &= \psi_{d(j_2)} \circ \psi_{d(j_2)-1} \ldots \circ \psi_{\delta(k)+2} \circ \psi_{\delta(k)+1} \\ &= \psi_{d(j_2)} \circ \psi_{d(j_2)-1} \ldots \circ \psi_{\delta(k)+2} \circ \psi_{\delta(k)+1} \circ id \\ &= \psi_{d(j_2)} \circ \psi_{d(j_2)-1} \ldots \circ \psi_{\delta(k)+2} \circ \psi_{\delta(k)+1} \circ \psi_{\delta(k)} \circ \ldots \circ \psi_{d(k)+1} \\ &= \psi_{d(j_2)} \circ \psi_{d(j_2)-1} \ldots \circ \psi_{d(k)+(\delta(k)-d(k))+2} \circ \psi_{d(k)+(\delta(k)-d(k))+1} \\ &\quad \circ \psi_{d(k)+(\delta(k)-d(k))} \circ \ldots \circ \psi_{d(k)+1} \\ &= \Psi^m(d(k) \to d(j_2)) \\ &= (\Psi^m \circ d)(j_1 \to j_2)\end{aligned}$$

- If $j_1 = j_2 = k$, then

$$(\Psi^m \circ \delta)(k \to k) = (\Psi^m \circ \delta)(id_k) = \Psi^m(id_{\delta(k)}) = id_{\Psi^m(\delta(k))}$$

and

$$(\Psi^m \circ d)(k \to k) = (\Psi^m \circ d)(id_k) = \Psi^m(id_{d(k)}) = id_{(\Psi^m(d(k))}$$

But we know that

$$id = \psi_{\delta(k)} \circ \ldots \circ \psi_{d(k)+1} : \Psi^m(d(k)) \to \Psi^m(\delta(k))$$

and so $\Psi^m(\delta(k)) = \Psi^m(d(k))$.

(b) Assume that $d_*, \delta_* : \Phi^n \to \Psi^m$ are morphisms in $\mathcal{CC}$ with the given dependent definitions. I will show that

$$d_* = (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1})_* \circ (\gamma)_*$$

and

$$\delta_* = (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} d_{d(k)})_* \circ (\gamma)_*$$

for a Δ-map γ that I will define below. Then since

$$(d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1})_* \approx (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)})_*$$

we'll have $d_* \sim \delta_*$. This idea is summarized in the following diagram:

$$\begin{tikzcd}
\Phi^n \arrow[r, "\gamma_*"] & \Omega^{m-(\delta(k)-d(k))} \arrow[rr, shift left, "(d_{\delta(k)-1}\circ\ldots\circ d_{d(k)})_*"] \arrow[rr, shift right, "(d_{\delta(k)-1}\circ\ldots\circ d_{d(k)+1}\circ d_{d(k)+1})_*"'] \arrow[dr, shift left, "(d_{d(k)+1})_*"] \arrow[dr, shift right, "(d_{d(k)})_*"'] & & \Psi^m \\
& & \Sigma^{m-(\delta(k)-d(k))+1} \arrow[ul, "(s_{d(k)})_*"] \arrow[ur, "(d_{\delta(k)-1}\ldots d_{d(k)+1})_*"'] &
\end{tikzcd}$$

Figure 2.15: Scoot Lemma proof intuition.

where:

$$\Omega = {}^0\psi_1{}^1 \ldots {}^{d(k)-1}\psi_{d(k)}{}^{d(k)}\psi_{\delta(k)+1}{}^{d(k)+1}\psi_{\delta(k)+2}{}^{d(k)+2} \ldots {}^{m-(\delta(k)-d(k))+1}\psi_m{}^{m-(\delta(k)-d(k))}$$

and

$$\Sigma = {}^0\psi_1{}^1 \ldots {}^{d(k)-1}\psi_{d(k)}{}^{d(k)}id{}^{d(k)+1}\psi_{\delta(k)+1}{}^{d(k)+2}\psi_{\delta(k)+2}{}^{d(k)+3} \ldots {}^{m-(\delta(k)-d(k))}\psi_m{}^{m-(\delta(k)-d(k)+1)}$$

There are several checks to make this argument valid:

(1) Show that $\Omega^{m-(\delta(k)-d(k))}$ is actually an object in $\mathcal{CC}$.

(2) Show that there exists a map $\Phi^n \xrightarrow{\gamma_*} \Omega^{m-(\delta(k)-d(k))}$ in $\mathcal{CC}$ such that

$$d = d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)+1} \circ \gamma$$

and that

$$\delta = d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} d_{d(k)} \circ \gamma$$

(3) Show that

$$(d_{\delta(k)-1} \circ ... d_{d(k)+1} \circ d_{d(k)})_*$$

and

$$(d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)+1})_*$$

are morphisms in $\mathcal{CC}$.

Proof of (1): If $m = (\delta(k) - d(k))$, then the functor

$$\Omega^{m-(\delta(k)-d(k))} : [0] \to \mathcal{C}$$

has dimension 0, and is thus in $\mathcal{CC}$.

If $m \geq (\delta(k) - d(k)) + 1$, then all of the $\mathcal{C}$-morphisms in the chain are non-identities, and we just need to make sure that $\psi_{\delta(k)+1} \circ \psi_{d(k)}$ is well-defined. Since

$$id = \psi_{\delta(k)} \circ ... \circ \psi_{d(k)} : \Psi^m(d(k)) \to \Psi^m(\delta(k))$$

we know that

$$\Psi^m(\delta(k)) = \Psi^m(d(k))$$

Then the morphisms

$$\psi_{\delta(k)+1} : \Psi^m(\delta(k)) \to \Psi^m(\delta(k)+1)$$

and

$$\psi_{d(k)} : \Psi^m(d(k)-1) \to \Psi^m(d(k))$$

are composable.

Proof of (2): First note that if such a δ exists, then $d(k) \leq m - (\delta(k) - d(k))$ (intuitively, there must be room to "scoot" and define $\delta(k)$). Additionally, if $k < n$, then $d(k+1) = \delta(k+1) > \delta(k) = d(k) + (\delta(k) - d(k))$, and so $d(k+1) \geq \delta(k) + 1$.

Define the functor $\gamma : [n] \to [m - (\delta(k) - d(k))]$ by

$$\gamma(j) = \begin{cases} d(j) & j \leq k \\ d(j) - (\delta(k) - d(k)) & j \geq k+1 \end{cases}$$

We need to show that γ is in fact an order-preserving map:

- If $j_1 < j_2 \leq k$, then $\gamma(j_1) = d(j_1) < d(j_2) = \gamma(j_2)$
- If $j_1 \leq k$ and $j_2 \geq k+1$, then

  $\gamma(j_1) = d(j_1) \leq d(k) < d(k+1) - (\delta(k) - d(k)) \leq d(j_2) - (\delta(k) - d(k)) = \gamma(j_2)$
- If $k+1 \leq j_1 < j_2$, then $\gamma(j_1) = d(j_1) - (\delta(k) - d(k)) \leq d(j_2) - (\delta(k) - d(k)) = \delta(j_2)$

Now, we would like to show that γ_* is a map in $\mathcal{CC}$, so, given that $\Psi^m \circ d = \Phi^n$, we need to show that $\Omega^{m-(\delta(k)-d(k))} \circ \gamma = \Phi^n$. Consider the morphism $j_1 \to j_2$ in $[m - (\delta(k) - d(k)]$.

<u>First</u>, if $j_2 \leq l$, then

$$\Omega^{m-(\delta(k)-d(k))}(j_1 \to j_2) = \psi_{j_2} \circ \ldots \circ \psi_{j_1+1} = \Psi^m(j_1 \to j_2)$$

Second, if $j_1 \leq l$ and $j_2 \geq l+1$, then

$$\Omega^{m-(\delta(k)-d(k))}(j_1 \to j_2) = \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{\delta(k)+1} \circ \psi_{d(k)+1} \circ \psi_{d(k)} \circ \ldots \circ \psi_{j_1+1}$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{\delta(k)+1} \circ id \circ \psi_{d(k)+1} \circ \psi_{d(k)} \circ \ldots \circ \psi_{j_1+1}$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{\delta(k)+1} \circ \psi_{\delta(k)} \circ \ldots \circ \psi_{d(k)+1} \circ \psi_{d(k)} \circ \ldots \circ \psi_{j_1+1}$$

$$= \Psi^m(j_1 \to j_2 + (\delta(k) - d(k)))$$

Third, if $j_1 \geq d(k)+1$, then

$$\Omega(j_1 \to j_2) = \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{(j_1+1)+(\delta(k)-d(k))}$$

$$= \Psi^m(j_1 + (\delta(k)-d(k)) \to j_2 + (\delta(k)-d(k)))$$

Let $j_1 \to j_2$ be a morphism in $[n]$.

- if $j_2 \leq k$, then $d(j_2) \leq d(k)$, so

$$\Omega^{m-(\delta(k)-d(k))}(\gamma(j_1 \to j_2)) = \Psi^m(d(j_1 \to j_2)) = \Phi^n(j_1 \to j_2)$$

- if $j_1 \leq k$ and $j_2 \geq k+1$, then $d(j_1) \geq \delta(k)+1$, and

$$\Omega^{m-(\delta(k)-d(k))}(\gamma(j_1 \to j_2)) = \Omega^{m-2}(d(j_1) \to d(j_2) - (\delta(k)-d(k)))$$

$$= \Psi^m((d(j_1) \to (d(j_2) - (\delta(k)-d(k))) + (\delta(k)-d(k)))$$

$$= \Psi^m(d(j_1 \to j_2))$$

$$= \Phi^n(j_1 \to j_2)$$

- if $j_1 \geq k+1$, then $d(j_1) \geq \delta(k)+1$, and

$$\Omega(\gamma(j_1 \to j_2)) = \Omega(d(j_1) - (\delta(k)-d(k)) \to d(j_2) - (\delta(k)-d(k)))$$

$$= \Psi^m((d(j_1) - (\delta(k)-d(k))) + (\delta(k)-d(k)) \to (d(j_2)-w)+w)$$

$$= \Psi^m(d(j_1 \to j_2))$$

It remains to show that:

$$d = d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1} \circ \gamma$$

and

$$\delta = d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)} \circ \gamma$$

Note that since γ is in Δ, the maps

$$d_{\delta(k)-1} \circ \ldots d_{d(k)+1} \circ d_{d(k)+1} \circ \gamma$$

and

$$d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} d_{d(k)} \circ \gamma$$

are in Δ. Additionally, because for any Δ-map d we have $d(j_1 \to j_2) = d(j_1) \to d(j_2)$ for $j_1 \to j_2$ a morphism in $[n]$, once we have equality on objects, we also have equality on morphisms.

For the map d, let's look at two cases:

- if $j \leq k$, then $d(j) \leq d(k)$, so:

$$(d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} d_{d(k)+1}) \circ (\gamma(j)) = (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1})(d(j)) = d(j)$$

- if $j \geq k+1$, then $d(j) \geq d(k+1) \geq \delta(k) + 1$, so

$$\begin{aligned}
&(d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1}) \circ \gamma(j) \\
&\qquad = (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1})(d(j) - (\delta(k) - d(k))) \\
&\qquad = d(j) - (\delta(k) - d(k)) + (\delta(k) - d(k)) \\
&\qquad = d(j)
\end{aligned}$$

i.e., $d(j)$ moves one to the right $((\delta(k) - d(k)) - 1) + 1$ times.

Now, if $j_1 \to j_2$ is a morphism in $[n]$,

$$(d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)+1} \circ \gamma)(j_1 \to j_2)$$

$$= (d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)+1} \circ \gamma)(j_1)$$

$$\to (d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)+1} \circ \gamma)(j_2)$$

$$= d(j_1) \to d(j_2)$$

Let's look at three cases:

- if $j \leq k-1$, then $\delta(j) \leq \delta(k-1) = d(k-1) \leq d(k)-1$, so

$$\begin{aligned}(d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)} \circ \gamma)(j) &= (d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)})(d(j)) \\ &= d(j) \\ &= \delta(j)\end{aligned}$$

- if $j = k$, then

$$\begin{aligned}(d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} d_{d(k)} \circ \gamma)(k) &= (d_{\delta(k)-1} \circ ... \circ d_{d(k)+1} \circ d_{d(k)})(d(k)) \\ &= (d_{\delta(k)-1} ... d_{d(k)+1} \circ d_{d(k)})(d(k)) \\ &= \delta(k)\end{aligned}$$

- if $j \geq k+1$, then $d(j) \geq d(k+1) \geq \delta(k)+1$, so

$$(d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)})(\gamma(j))$$

$$= (d_{\delta(k)-1} \circ \ldots \circ d_{\delta(k)+1} \circ d_{d(k)})(d(j) - (\delta(k) - d(k)))$$

$$= (d(j) - (\delta(k) - d(k))) + (\delta(k) - d(k))$$

$$= d(j)$$

$$= \delta(j)$$

Proof of (3): For the Δ-map $d_{\delta(k)-1} \circ \ldots \circ d_{d(k)}$, we need to show that

$$\Psi^m \circ (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)}) = \Omega^{m-(\delta(k)-d(k))}$$

Let $j_1 \to j_2$ be a morphism in the category $[m-2]$.

- If $j_2 \leq l-1$, then

$$(\Psi^m \circ (d_{\delta(k)-1} \ldots \circ d_{d(k)}))(j_1 \to j_2)$$

$$= \Psi^m(j_1 \to j_2)$$

$$= \psi_{j_2} \circ \ldots \circ \psi_{j_1+1}$$

$$= \Omega^{m-(\delta(k)-d(k))}(j_1 \to j_2)$$

- If $j_1 \leq d(k)-1$ and $j_2 \geq l$, then

$$(\Psi^m \circ (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)}))(j_1 \to j_2) = \Psi^m(j_1 \to j_2 + (\delta(k) - d(k)))$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{j_1+1}$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{\delta(k)+1} \circ id \circ \psi_{d(k)} \circ \ldots \circ \psi_{j_1+1}$$

$$= \Omega^{m-(\delta(k)-d(k))}(j_1 \to j_2)$$

• If $j_1 \geq d(k)$, then

$$(\Psi^m \circ (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)}))(j_1 \to j_2)$$

$$= \Psi^m(j_1 + (\delta(k) - d(k)) \to j_2 + (\delta(k) - d(k)))$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{j_1+(\delta(k)-d(k))+1}$$

$$= \Omega^{m-2}(j_1 \to j_2)$$

For the Δ-map $d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} d_{d(k)+1}$, we need to show that

$$\Psi^m \circ (d_{\delta(k)-1} \ldots \circ d_{d(k)+1} \circ d_{d(k)+1}) = \Omega^{m-(\delta(k)-d(k))}$$

Let $j_1 \to j_2$ be a morphism in the category $[m-2]$.

• If $j_2 \leq d(k)$, then

$$(\Psi^m \circ (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1}))(j_1 \to j_2) = \Psi^m(j_1 \to j_2)$$

$$= \psi_{j_2} \circ \ldots \circ \psi_{j_1+1}$$

$$= \Omega^{m-(\delta(k)-d(k))}(j_1 \to j_2)$$

• If $j_1 \leq d(k)$ and $j_2 \geq d(k)+1$, then

$$(\Psi^m \circ (d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1}))(j_1 \to j_2) = \Psi^m(j_1 \to j_2 + (\delta(k) - d(k)))$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{j_1+1}$$

$$= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{\delta(k)+1} \circ id \circ \psi_{d(k)} \circ \ldots \circ \psi_{j_1+1}$$

$$= \Omega^{m-(\delta(k)-d(k))}(j_1 \to j_2)$$

• If $j_1 \geq d(k) + 1$, then

$$\begin{aligned}
(\Psi^m \circ ((d_{\delta(k)-1} \circ \ldots \circ d_{d(k)+1} \circ d_{d(k)+1})))(j_1 \to j_2) \\
&= \Psi^m(j_1 + (\delta(k) - d(k)) \to j_2 + (\delta(k) - d(k))) \\
&= \psi_{j_2+(\delta(k)-d(k))} \circ \ldots \circ \psi_{j_1+(\delta(k)-d(k))+1} \\
&= \Omega^{m-2}(j_1 \to j_2)
\end{aligned}$$

□

Example 2.8.3. Let $\mathcal{C}$ be a small category. Suppose $d : [3] \to [6]$ is a morphism in Δ and $d_* : \Phi^3 \to \Psi^6$ is a morphism in $\mathfrak{C}\mathcal{C}$ such that $d(0) = 0$, $d(1) = 2$, $d(2) = 5$, and $d(3) = 6$. Further, assume $\psi_4 \circ \psi_3 : c_2 \to c_4 = id_{c_2}$ (implying that $c_4 = c_2$). Then we can define the map d' in Δ by $d'(x) = d(x)$ if $x \neq 1$ and $d'(1) = 4$.

$$\begin{tikzcd}
d: & 0 \arrow[r] \arrow[d, mapsto] & 1 \arrow[r] \arrow[dr, mapsto] & 2 \arrow[r] \arrow[drrr, mapsto] & 3 \arrow[drrr, mapsto] & & & \\
\Psi^6: & 0 \arrow[r] \arrow[d, mapsto] & 1 \arrow[r] \arrow[d, mapsto] & 2 \arrow[r] \arrow[d, mapsto] & 3 \arrow[r] \arrow[d, mapsto] & 4 \arrow[r] \arrow[d, mapsto] & 5 \arrow[r] \arrow[d, mapsto] & 6 \arrow[d, Rightarrow] \\
& c_0 \arrow[r] & c_1 \arrow[r] & c_2 \arrow[r, "\psi_3"] & c_3 \arrow[r, "\psi_4"] & c_4 \arrow[r] & c_5 \arrow[r] & c_6 \\
d': & 0 \arrow[r] \arrow[d, mapsto] & 1 \arrow[r] \arrow[drrr, mapsto] & 2 \arrow[r] \arrow[drrr, mapsto] & 3 \arrow[drrr, mapsto] & & & \\
\Psi^6: & 0 \arrow[r] \arrow[d, Rightarrow] & 1 \arrow[r] \arrow[d, Rightarrow] & 2 \arrow[r] \arrow[d, Rightarrow] & 3 \arrow[r] \arrow[d, Rightarrow] & 4 \arrow[r] \arrow[d, Rightarrow] & 5 \arrow[r] \arrow[d, Rightarrow] & 6 \arrow[d, Rightarrow] \\
& c_0 \arrow[r] & c_1 \arrow[r] & c_2 \arrow[r, "\psi_3"] & c_3 \arrow[r, "\psi_4"] & c_4 \arrow[r] & c_5 \arrow[r] & c_6
\end{tikzcd}$$

Figure 2.16: Morphisms in the pre-subdivision.

We would like to see that d'_* is a morphism in $\mathfrak{C}\mathcal{C}$, i.e., that $\Psi^6 \circ d' = \Phi^3$. In fact, I claim that $\Psi^6 \circ d' = \Psi^6 \circ d$. Let's look at a few calculations:

$(\Psi^6 \circ d')(0 \to 1) = \Psi^6(0 \to 4) = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1 = id \circ \psi_2 \circ \psi_1 = \Psi^6(0 \to 2) = (\Psi^6 \circ d)(0 \to 1)$

$$(\Psi^6 \circ d')(1 \to 3) = \Psi^6(4 \to 6) = \psi_6 \circ \psi_5 \circ id = \psi_6 \circ \psi_5 \circ \psi_4 \circ \psi_3 = \Psi^6(2 \to 6) = (\Psi^6 \circ d)(1 \to 3)$$

Recall that in general if d is a morphism in Δ and $j_1 \to j_2$ is a morphism in $[n]$, then $d(j_1 \to j_2) = d(j_1) \to d(j_2)$ (see Remark 2.2.2).

$$(\Psi^6 \circ d')(0 \to 2) = \Psi^6(d'(0) \to d'(2)) = \Psi^6(d(0) \to d(2)) = (\Psi^6 \circ d)(0 \to 2)$$

$$(\Psi^6 \circ d')(2 \to 3) = \Psi^6(d'(2) \to d'(3)) = \Psi^6(d(2) \to d(3)) = (\Psi^6 \circ d)(0 \to 2)$$

These are not all of the calculations necessary to prove the claim, but the computations should give a sense of what is required. Lemma 2.8.2(a) proves this in general using similar techniques. △

Example 2.8.4. Now, let's look at an example to see how d_* and d'_* are equivalent (Lemma 2.8.2(b) proves this in general). Let's find all of the maps in $\mathrm{Sd}\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$ between

$${}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

First, let's list out the maps in $\mathcal{C}\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$ and indicate the insertions:

$$(d_5 \circ d_4 \circ d_3)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_5 \circ d_1 \circ d_0)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_5 \circ d_3 \circ d_2)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_5 \circ d_2 \circ d_1)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_5 \circ d_4 \circ d_0)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_2 \circ d_1 \circ d_0)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_4 \circ d_3 \circ d_0)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$(d_3 \circ d_2 \circ d_0)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

Now, I claim that the first four morphisms in the list are equivalent and the second four are equivalent. To see the "scoot" explicitly, let's examine why the the first is equivalent to the third:

$$\text{first:}\quad (d_5 \circ d_4 \circ d_3)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

$$\text{third:}\quad (d_5 \circ d_3 \circ d_2)_* : {}^0\alpha_1{}^1\alpha_1{}^2 \longrightarrow {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$$

We can get from ${}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$ to ${}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5$ by "scooting" 2 to 4. The reason we are able to perform this "scoot" is because we are "scooting" past $\alpha_1\alpha_1$ and $\alpha_1 \circ \alpha_1 = id_a$. So $(d_5 \circ d_4 \circ d_3)_* \sim (d_5 \circ d_3 \circ d_2)_*$.

To see how this intuition corresponds with May's formal definition, let's use degenerate chains to see that $(d_5 \circ d_4 \circ d_3)_* = (d_5 \circ d_3 \circ d_3)_* \approx (d_5 \circ d_3 \circ d_2)_*$. Note that

$$({}^0\alpha_1{}^1\alpha_1{}^2) \circ s_2 = ({}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4) \circ d_3 = {}^0\alpha_1{}^1\alpha_1{}^2 id_a{}^3,$$

as we can see in Figure 2.17.

$$\begin{tikzcd}
{}^0\alpha_1{}^1\alpha_1{}^2 \arrow[r, "(d_3)_*"] \arrow[dr, bend left=10, "(d_3)_*"] \arrow[dr, bend right=20, "(d_2)_*"'] & {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3 \arrow[r, "(d_4)_*"] & {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4 \arrow[r, "(d_5)_*"] & {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5 \\
& {}^0\alpha_1{}^1\alpha_1{}^2 id^3 \arrow[ul, "(s_2)_*"] \arrow[ur, "(d_3)_*"'] & &
\end{tikzcd}$$

Figure 2.17: May's definition and the Scoot Lemma.

In fact, we can see all four of the first morphisms by extending this diagram. The identities in the degenerate chains show where we can scoot.

$$
\begin{tikzcd}
{}^0\alpha_1{}^1\alpha_1{}^2 \arrow[r, shift left, "(d_3)_*"] \arrow[r, shift right, "(d_0)_*"'] & {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3 \arrow[r, shift left, "(d_4)_*"] \arrow[r, shift right, "(d_0)_*"'] & {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4 \arrow[r, "(d_5)_*"] & {}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4\alpha_1{}^5 \\
& {}^0\alpha_1{}^1\alpha_1{}^2id^3 \arrow[ul, "{(d_2)_*,(s_2)_*,(d_3)_*}"'] \arrow[ur, bend right, "(d_3)_*"'] & & \\
& {}^0\alpha_1{}^1id^2\alpha_1{}^2 \arrow[uul, bend left, "{(d_1)_*,(s_1)_*,(d_2)_*}"'] \arrow[uur, bend right, "(d_2)_*"'] & & \\
& {}^0id^1\alpha_1{}^2\alpha_1{}^3 \arrow[uuul, bend left, "{(d_0)_*,(s_0)_*,(d_1)_*}"'] \arrow[uuur, bend right, "(d_1)_*"'] & &
\end{tikzcd}
$$

Figure 2.18: The Scoot Lemma and equivalent morphisms.

Note that

$$ {}^0\alpha_1{}^1\alpha_1{}^2id^3 = ({}^0\alpha_1{}^1\alpha_1{}^2) \circ s_2 = ({}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4) \circ d_3 $$

$$ {}^0\alpha_1{}^1id^2\alpha_1{}^3 = ({}^0\alpha_1{}^1\alpha_1{}^2) \circ s_1 = ({}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4) \circ d_2 $$

$$ {}^0id^1\alpha_1{}^2\alpha_1{}^3 = ({}^0\alpha_1{}^1\alpha_1{}^2) \circ s_0 = ({}^0\alpha_1{}^1\alpha_1{}^2\alpha_1{}^3\alpha_1{}^4) \circ d_1 $$

Observe that the first and fifth morphisms are *not* equivalent because there is no way to change one to the other by scooting past an identity; in the category $\mathcal{O}_{\mathbb{Z}/2\mathbb{Z}}$, we are only allowed to "scoot" past an even number of α_1's.

Looking ahead, this begins to motivate the definition of a functor that sorts the morphisms of $\mathcal{CO}_{\mathbb{Z}/2\mathbb{Z}}$ into equivalence classes based on which morphisms may be "identity-scooted" into others. △

Chapter 3

The Weak Equivalence for Abelian Groups

In this chapter, assume $(G,+)$ is a finite abelian group. I will show that in this case, there is a weak equivalence from $\mathrm{Sd}\mathcal{O}_G$ to $\mathcal{L}_G$. This weak equivalence is based on the inf functor, defined in Section 3.1, a contravariant functor I created when I found that del Hoyo's sup functor [2] suppresses too much information about the objects in the subdivision of the orbit category. In Section ??, I use this functor to define the functor $\ell\,\mathrm{inf}$ from the pre-subdivision of the orbit category to the link orbit category. I prove that this functor is compatible with the equivalence relation required to pass to the subdivision of the orbit category, which allows me to define the induced functor $[\ell\,\mathrm{inf}]$. I prove that each left fiber of this functor is filtered in Section 3.3, and a result from Quillen tells us that filtered categories are contractible [8]. Consequently, $[\ell\,\mathrm{inf}]$ is a weak equivalence by Quillen's Theorem A.

3.1 The inf Functor

Definition 3.1.1. Define the contravariant functor $\mathrm{inf} : \mathcal{CO}_G \to \mathcal{O}_G$ as follows:

`on objects:` $\mathrm{inf}(\Phi) \coloneqq \Phi(0)$ (the first object in the chain Φ)

`on morphisms:` Suppose $d_* : \Phi \to \Psi$. Then define $\mathrm{inf}(d_*) : \mathrm{inf}(\Psi) \to \mathrm{inf}(\Phi)$ as $\mathrm{inf}(\Phi \xrightarrow{d_*} \Psi) \coloneqq \Psi(0 \to d(0))$ (the composition of the first $d(0)$ morphisms in Ψ).

Example 3.1.2. Let $G = \mathbb{Z}/4\mathbb{Z}$, and see Figure 3.1 below for two diagrams of $\mathcal{O}_{\mathbb{Z}/4\mathbb{Z}}$ (where $p = 2$), one in the definitional notation and one in an α-notation. Consider the objects

$$\Phi^2 = {}^0\omega_0{}^1\theta_1{}^2 \quad \text{and} \quad \Psi^5 = {}^0\alpha_1{}^1\alpha_2{}^2\omega_1{}^3\theta_1{}^4\theta_1{}^5$$

in $\mathcal{CO}_{\mathbb{Z}/4\mathbb{Z}}$. Further, note that $d_* : \Phi^2 \to \Psi^5$ defined by $d(0) = 2$, $d(1) = 4$, and $d(2) = 5$ is a morphism in $\mathcal{CO}_{\mathbb{Z}/4\mathbb{Z}}$ since $\omega_0 = \theta_1 \circ \omega_1$. Then we have the following calculations:

$$\mathrm{inf}(\Phi^2) = \Phi^2(0) = \langle 0 \rangle$$

and

$$\mathrm{inf}(d_*) = \Psi^5(0 \to d(0)) = \Psi^4(0 \to 2) = \alpha_2 \circ \alpha_1 = \alpha_3 \qquad \triangle$$

$$\begin{tikzcd}
\langle 0 \rangle \arrow[loop left, "\gamma_0^{\mathcal{O}},\dots,\gamma_{p^2-1}^{\mathcal{O}}"] \arrow[rr, "\gamma_0^{\mathcal{O}}"] \arrow[dr, "\gamma_0^{\mathcal{O}},\dots,\gamma_{p-1}^{\mathcal{O}}"'] & & \langle 1 \rangle \arrow[loop above, "\gamma_0^{\mathcal{O}}"] \\
& \langle p \rangle \arrow[loop below, "\gamma_0^{\mathcal{O}},\dots,\gamma_{p-1}^{\mathcal{O}}"] \arrow[ur, "\gamma_0^{\mathcal{O}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle 0 \rangle \arrow[loop left, "\alpha_0,\dots,\alpha_{p^2-1}"] \arrow[rr, "f_a"] \arrow[dr, "\omega_0,\dots,\omega_{p-1}"'] & & \langle 1 \rangle \arrow[loop above, "\beta_0"] \\
& \langle p \rangle \arrow[loop below, "\theta_0,\dots,\theta_{p-1}"] \arrow[ur, "f_c"'] &
\end{tikzcd}$$

Figure 3.1: The orbit category for $\mathbb{Z}/p^2\mathbb{Z}$

Remark 3.1.3. If $d(0) \geq 1$, then $\inf(d_*) = \psi_{d(0)} \circ \dots \circ \psi_1 : \Psi(0) \to \Psi(d(0))$. (Recall that $\Psi \circ d = \Phi$.) If $d(0) = 0$ then $\inf(d_*) = id_{\Psi(0)}$.

Lemma 3.1.4. As defined in Definition 3.1.1, inf defined previously is in fact a contravariant functor.

Proof. Let $\Phi \xrightarrow{d_*} \Psi \xrightarrow{\delta_*} \Omega$ be a composition of maps in $\mathcal{CO}_G$. Note that $d(0) \geq 0$ so $(\delta \circ d)(0) \geq \delta(0)$, and that $\Omega \circ \delta = \Psi$ (Definition 2.7.5). Then:

$$\begin{aligned}
\inf(\delta_* \circ d_*) &= \inf((\delta \circ d)_*) \\
&= \Omega(0 \to (\delta \circ d)(0)) \\
&= \Omega[(\delta(0) \to (\delta \circ d)(0)) \circ (0 \to \delta(0))] \\
&= \Omega[(\delta(0) \to (\delta \circ d)(0))] \circ \Omega[0 \to \delta(0)] \\
&= (\Omega \circ \delta)(0 \to d(0)) \circ \Omega(0 \to \delta(0)) \\
&= \Psi(0 \to d(0)) \circ \Omega(0 \to \delta(0)) \\
&= \inf(d_*) \circ \inf(\delta_*).
\end{aligned}$$

Further,

$$\begin{aligned}\inf(\Phi \xrightarrow{id_*} \Phi) &= \Phi(0 \to id(0))\\ &= \Phi(0 \to 0)\\ &= id_{\Phi(0)}.\end{aligned}$$

□

Our eventual goal is to define a functor on the subdivision of the orbit category that is based on the inf functor. In order to do this, we need to know that inf behaves well with the equivalence relation on morphisms inherent to $\mathrm{Sd}\mathcal{O}_G$. We assert this and a faithfulness property in the following Lemmas.

Lemma 3.1.5. Suppose $\delta_*, d_* : \Phi \to \Psi$ are morphisms in $\mathcal{C}\mathcal{O}_G$ such that $\delta_* \approx d_*$. Then $\inf(d_*) = \inf(\delta_*)$.

Proof. Assume $\delta(0) \geq d(0)$. We know from May's definition of the subdivision (Definition 2.8.1) that if $d_* \approx \delta_*$, then there exist maps σ, α, β, and ν in Δ such that σ is a surjection with right inverses α and β, $d = \nu \circ \alpha$, $\delta = \nu \circ \beta$, and $\Phi \circ \sigma = \Psi \circ \nu$.

First, let's look at the difference between insertions (this is the "identity scoot" we see in the Scoot Lemma 2.8.2, so we expect this to be an identity):

$$
\begin{aligned}
\Psi(d(0) \to \delta(0)) &= \Psi((\nu \circ \alpha)(0) \to (\nu \circ \beta)(0)) \\
&= (\Psi \circ \nu)(\alpha(0) \to \beta(0)) \\
&= (\Phi \circ \sigma)(\alpha(0) \to \beta(0)) \\
&= \Phi((\sigma \circ \alpha)(0) \to (\sigma \circ \beta)(0)) \\
&= \Phi(0 \to 0) \\
&= id_{\Phi(0)}.
\end{aligned}
$$

Now we have:

$$
\begin{aligned}
\inf(\delta_*) &= \Psi(0 \to \delta(0)) \\
&= \Psi((d(0) \to \delta(0)) \circ (0 \to d(0))) \\
&= \Psi((d(0) \to \delta(0))) \circ \Psi((0 \to d(0))) \\
&= id_{\Phi(0)} \circ \inf(d_*) \\
&= \inf(d_*).
\end{aligned}
$$

□

Lemma 3.1.6. Suppose $\delta_*, d_* : \Phi \to \Psi$ are morphisms in $\mathcal{CO}_G$ such that $\delta_* \sim d_*$. Then $\inf(d_*) = \inf(\delta_*)$.

Proof. See Lemmas 2.2.4 and 3.1.5. □

Lemma 3.1.7. Suppose $d_*, \delta_* : \Phi^n \to \Psi^m$ are in $\mathcal{CO}_G$. If $\inf(d_*) = \inf(\delta_*)$, then $d_* \sim \delta_*$.

Remark: I have not defined $[\inf]$ explicitly, but Lemma 3.1.7 tells us that

$$[\inf] : \mathrm{Sd}\mathcal{O}_G \to \mathcal{O}_G$$

is faithful.

Proof. Without loss of generality, assume $\delta(0) \geq d(0)$. The goal of this proof is to show that we can "scoot" d_* into δ_* using repeated applications of the Scoot Lemma 2.8.2. The first scoot of d_* becomes d_*', the second scoot becomes d_*'', and so on until we obtain $d_*^{(n+1)} = \delta_*$.

Since $\inf(d_*) = \inf(\delta_*)$, we have

$$\Psi^m(0 \to d(0)) = \Psi^m(0 \to \delta(0)) \text{ (the beginning differences are the same)}$$

Using the functoriality of Ψ^m, we also have

$$\Psi^m(0 \to \delta(0)) = \Psi^m(d(0) \to \delta(0)) \circ \Psi^m(0 \to d(0))$$

Combining these, we get

$$\Psi^m(0 \to d(0)) = \Psi^m(d(0) \to \delta(0)) \circ \Psi^m(0 \to d(0))$$

as in Figure 3.2 for some $H, K \leq G$.

$$\begin{tikzcd}
G/H \arrow[rr, "\Psi(0\to d(0))"] \arrow[dr, "\Psi(0\to d(0))"'] & & G/K \\
& G/K \arrow[ur, "\Psi(d(0)\to\delta(0))"'] &
\end{tikzcd}$$

Figure 3.2: Commutative maps in the orbit category.

The map $\Psi^m(0 \to d(0)) : G/H \to G/K$ is equivariant and so must be of the form $gH \mapsto g\gamma K$ for some $\gamma \in G$. Then by Lemma 2.1.12,

$$\Psi^m(d(0) \to \delta(0))(gK) = \gamma^{-1}\gamma gK = gK$$

and so

$$\Psi^m(d(0) \to \delta(0)) = \gamma_0^{\mathcal{O}} : G/K \to G/K$$

which is the identity.

Step 1:

- If $d(0) = \delta(0)$, define $d' := d$, and so $d_* \sim d'_*$, or
- if $\delta(0) > d(0)$, then define

$$d'(k) := \begin{cases} \delta(0) & k = 0 \\ d(k) & k \geq 1 \end{cases}$$

so $d_* \sim d'_*$ by the Scoot Lemma.

Step 2:

Note that because we must have an allowable insertion, we know that

$$\Psi^m(d'(0) \to d'(1)) = \Psi^m(\delta(0) \to \delta(1))$$

We can obtain this by going through the definitions carefully:

$$\begin{aligned} \Psi^m(d'(0) \to d'(1)) &= (\Psi^m \circ d')(0 \to 1) \\ &= \Phi^n(0 \to 1) \\ &= (\Psi^m \circ \delta)(0 \to 1) \\ &= \Psi^m(\delta(0) \to \delta(1)) \end{aligned}$$

Further,

$$\Psi^m(d'(0) \to d'(1)) = \Psi^m(\delta(0) \to d'(1))$$

so

$$\Psi^m(\delta(0) \to d'(1)) = \Psi^m(\delta(0) \to \delta(1))$$

- If $d'(1) = \delta(1)$, then define $d'' := d'$, so $d' \sim d''$.

- If $d'(1) < \delta(1)$, then

$$\Psi^m(d'(1) \to \delta(1)) \circ \Psi^m(\delta(0) \to d'(1)) = \Psi^m(\delta(0) \to \delta(1)) = \Psi^m(\delta(0) \to d'(1))$$

By Lemma 2.1.12,

$$\Psi^m(d(0) \to \delta(0)) = \alpha_0, \theta_0, \text{ or } \beta_0$$

Define

$$d''(k) := \begin{cases} \delta(k) & k = 0, 1 \\ d'(k) & k \geq 2 \end{cases}$$

and $d' \sim d''_*$.

- If $d'(1) > \delta(1)$, then

$$\Psi^m(\delta(1) \to d'(1)) \circ \Psi^m(\delta(0) \to d'(1)) = \Psi^m(\delta(0) \to \delta(1)) = \Psi^m(\delta(0) \to d'(1))$$

By Lemma 2.1.12,

$$\Psi^m(d(0) \to \delta(0)) = \alpha_0, \theta_0, \text{ or } \beta_0$$

Define

$$d''(k) := \begin{cases} \delta(k) & k = 0, 1 \\ d'(k) & k \geq 2 \end{cases}$$

and $d' \sim d''_*$.

Proceed similarly for Step 3 to Step $n+1$. These steps will show that

$$d_* \sim d'_* \sim d''_* \sim \ldots \sim d_*^{(n+1)} = \delta_*$$

(There are potentially $n+1$ scoots to transform d_* into δ_*.) $\square$

3.2 A Functor to the Link Orbit Category

Recall that G is a finite abelian group throughout this chapter.

Definition 3.2.1. Define the functor $\ell\inf : \mathcal{CO}_G \to \mathcal{L}_G$ as follows

`on objects:` Let

$$\Phi^n = H_0 \xrightarrow{\varphi_1} H_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_n} H_n$$

be an object in $\mathcal{CO}_G$. We can view Φ as a path in $\mathcal{O}_G$ passing through the objects $H_0, \ldots, H_n$. Then define

$$\ell\inf(\Phi^n) := H_0 < H'_1 < \ldots < H'_q$$

where $H_0, H'_1, \ldots, H'_q$ are the *distinct* objects that Φ^n passes through in increasing order.

`on morphisms:` Let $\Phi^n \xrightarrow{d_*} \Psi^m$ be fixed in $\mathcal{CO}_G$ (so $K_0 \leq H_0$). Let

$$\ell : \mathrm{Hom}_{\mathcal{O}_G}(\inf(\Psi^m), \inf(\Phi^n)) \to \mathrm{Hom}_{\mathcal{L}_G}(\ell\inf(\Phi^n), \ell\inf(\Psi^m))$$

be the function defined by $\ell(\gamma_i^{\mathcal{O}}) := \gamma_i^{\mathcal{L}}$. Then $\ell\inf(d_*) := \ell(\inf(d_*))$

We need to prove that $[\ell\inf]$ is in fact a well-defined functor. However, before doing that, let's look at an example showing the object image of $\mathrm{Sd}\mathcal{O}_{\mathbb{Z}/p^2\mathbb{Z}}$ in $\mathcal{L}_{\mathbb{Z}/p^2\mathbb{Z}}$.

Example 3.2.2. The link orbit category $\mathcal{L}_{\mathbb{Z}/4\mathbb{Z}}$ is given below in Figure 3.3 (an example of $G = \mathbb{Z}/p^2\mathbb{Z}$ where $p = 2$) with the object image of $[\ell\inf]$ in blue boxes. I have labeled the images by their general forms, e.g., $\alpha\omega\theta$ represents any path that includes at least one α_i, one ω_j, and one θ_k. (See Figure 3.1 for the orbit category labelings.) To see how $[\ell\inf]$ relates $\mathrm{Sd}\mathcal{O}_{\mathbb{Z}/4\mathbb{Z}}$ and $\mathcal{L}_{\mathbb{Z}/4\mathbb{Z}}$, observe that $[\ell\inf]$ maps the object ${}^0\alpha_1{}^1\alpha_2{}^2\alpha_1{}^3\omega_0{}^4$ to $\langle 0\rangle < \langle p\rangle$ because this path passes through exactly the objects $\langle 0\rangle$ and $\langle p\rangle$ in the orbit category. We have a morphism

$${}^0\omega_1{}^1 \xrightarrow{(d_3d_1d_0)_*} {}^0\alpha_1{}^1\alpha_2{}^2\alpha_1{}^3\omega_0{}^4$$

in the subdivision of the orbit category, and $[\ell\inf](d_3d_1d_0)_* = \alpha_3^{\mathcal{L}}$ to show that when we insert ${}^0\omega_1{}^1$, the beginning difference in the two paths is $\alpha_2 \circ \alpha_1 = \alpha_3$.

For another more general intuition into the morphisms, note that the morphisms

$$\omega_0^{\mathcal{L}}, \ldots, \omega_{p-1}^{\mathcal{L}} : \langle p\rangle \to \langle 0\rangle < \langle p\rangle$$

show what the possible beginning differences are when we insert paths of the forms c or θ into paths of the form ω, $\alpha\omega$, $\omega\theta$, or $\alpha\omega\theta$. (As mentioned in Remark 2.6.8, these maps can also be thought of as the self-equivariant maps of $(\mathbb{Z}/4\mathbb{Z})/\langle p\rangle$.) $\triangle$

Corollary 3.2.3. Suppose $\Phi \xrightarrow{d_*} \Psi$ is a fixed morphism in $\mathcal{C}\mathcal{O}_G$. Then the function

$$\ell : \mathrm{Hom}_{\mathcal{O}_G}(\inf(\Psi), \inf(\Phi)) \to \mathrm{Hom}_{\mathcal{L}_G}(\ell\inf(\Phi), \ell\inf(\Psi))$$
$$\ell(\inf(\Psi) \xrightarrow{\gamma_i^{\mathcal{O}}} \inf(\Phi)) \coloneqq \ell\inf(\Phi) \xrightarrow{\gamma_i^{\mathcal{L}}} \ell\inf(\Psi)$$

is well-defined and bijective.

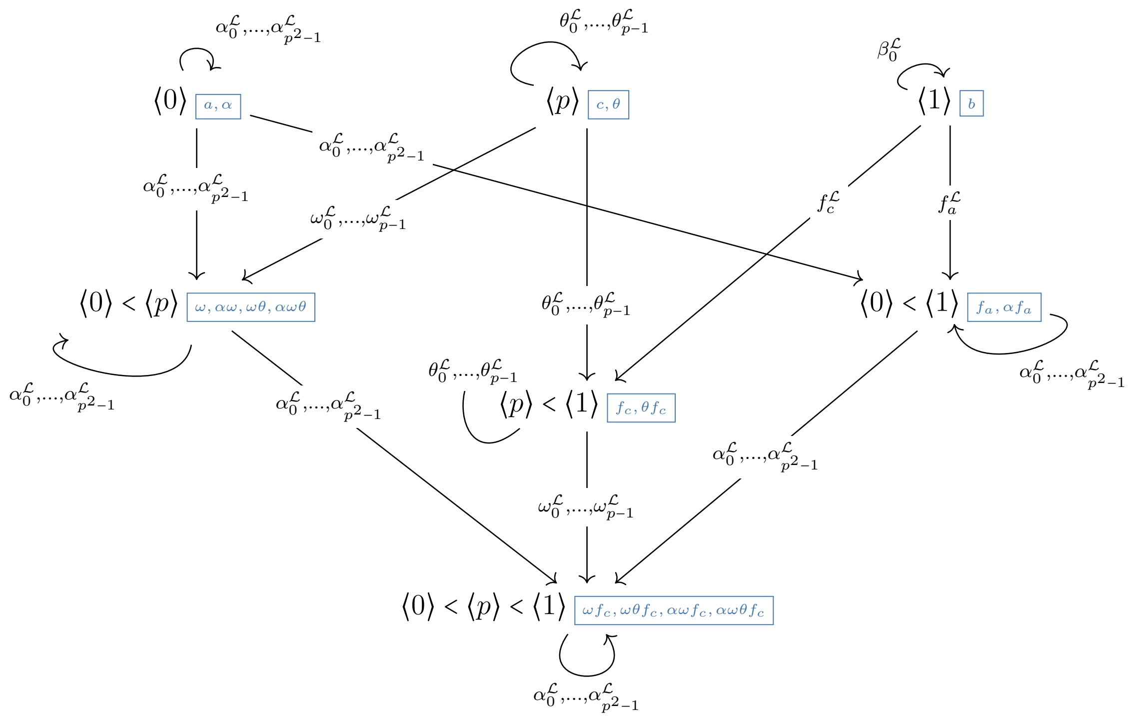

Figure 3.3: The category $\mathcal{L}_{\mathbb{Z}/p^2\mathbb{Z}}$ with the object image of $\ell\,\mathrm{inf}$.

Proof. Let $\mathrm{inf}(\Psi) = H$, and let $\mathrm{inf}(\Phi) = K$ (where H and K are subgroups of G).

Well-defined: Suppose that $\gamma_i^{\mathcal{O}} = \gamma_j^{\mathcal{O}} : H \to K$. We need to show that $\gamma_i^{\mathcal{L}}, \gamma_j^{\mathcal{L}} : \ell\,\mathrm{inf}(\Phi) \to \ell\,\mathrm{inf}(\Psi)$ are the same map in $\mathcal{L}_G$. By Corollary 2.6.5, we need to show that $R_j^0 = R_i^0$. Since $\gamma_i^{\mathcal{O}} = \gamma_j^{\mathcal{O}}$, we know that $i, j \in iK = jK$ by Lemma 2.1.10. So for $g \in G$, $R_i^0(gK) = giK = gjK = R_j^0(gK)$ and so $\gamma_i^{\mathcal{L}} = \gamma_j^{\mathcal{L}}$.

Injective: Suppose $\ell(\gamma_i^{\mathcal{O}}) = \ell(\gamma_j^{\mathcal{O}})$. Then $\gamma_i^{\mathcal{L}} = \gamma_j^{\mathcal{L}}$, and so $giK = gjK$, which implies that $iK = jK$. By Lemma 2.1.10, this implies that $\gamma_i^{\mathcal{O}} = \gamma_j^{\mathcal{O}}$.

Surjective: For any $\gamma_i^{\mathcal{L}} : \ell\,\mathrm{inf}(\Phi) \to \ell\,\mathrm{inf}(\Psi)$, we have $\ell(\gamma_i^{\mathcal{O}}) = \gamma_i^{\mathcal{L}}$. $\square$

Lemma 3.2.4. The functor $\ell\inf$ in Definition 3.2.1 is in fact a functor.

Proof. Let $\Phi \xrightarrow{d_*} \Psi \xrightarrow{\delta^*} \Omega$ be a composition of morphisms in $\mathcal{C}\mathcal{O}_G$ such that $\inf(\delta_*) = \gamma_j^{\mathcal{O}}$ and $\inf(d_*) = \gamma_i^{\mathcal{O}}$. Note that since inf is contravariant,

$$\inf(\delta_* \circ d_*) = \inf(d_*) \circ \inf(\delta_*) = \gamma_i^{\mathcal{O}} \circ \gamma_j^{\mathcal{O}} : \Omega(0) \to \Phi(0)$$

Then:

$$\begin{aligned}
\ell\inf(\delta_*) \circ \ell\inf(d_*) = \ell(\inf(\delta_*)) \circ \ell(\inf(d_*)) = \ell(\gamma_j^{\mathcal{O}}) \circ \ell(\gamma_i^{\mathcal{O}}) = \gamma_j^{\mathcal{L}} \circ \gamma_i^{\mathcal{L}} \\
= \gamma_{i+j}^{\mathcal{L}} \text{ because of how composition is defined in } \mathcal{L} \\
= \ell(\gamma_{i+j}^{\mathcal{O}}) \text{ because } \ell \text{ is surjective} \\
= \ell(\gamma_{j+i}^{\mathcal{O}}) \text{ because } G \text{ is abelian} \\
= \ell(\gamma_i^{\mathcal{O}} \circ \gamma_j^{\mathcal{O}}) \text{ because of how composition is defined in } \mathcal{O}_G \\
= \ell\inf(\delta_* \circ d_*)
\end{aligned}$$

Additionally,

$$\ell\inf(\Phi \xrightarrow{id_*} \Phi) = \ell(\inf(id_*)) = \ell(id_{\Phi(0)}) = \ell(\gamma_0^{\mathcal{O}}) = \gamma_0^{\mathcal{L}} = id_{\ell\inf(\Phi)}$$ □

Remark 3.2.5. In the proof of Lemma 3.2.4, we need G to be abelian in order to write the composition of morphisms in $\mathcal{O}_G$ properly. The flipped order of composition in $\mathcal{O}_G$ combined with the contravariant functor inf create the covariant functor $\ell\inf$, but $\mathcal{L}_G$ also flips the order of composition. It seems that $[\ell\inf]$ would need to be redefined substantially in order for this approach to suit the nonabelian case. Unfortunately, the covariant sup functor does not seem to work well because it loses information about the intial maps.

As with inf, we want to verify that $\ell\inf$ respects the equivalence relation on morphisms. Then we will be able to define $[\ell\inf]$ on $\mathrm{Sd}\mathcal{O}_G$.

Lemma 3.2.6. If $d_* \approx \delta_* : \Phi^n \to \Psi^m$ in $\mathcal{C}\mathcal{O}_G$, then $\ell\inf(d_*) = \ell\inf(\delta_*)$.

Proof. We have $\ell\inf(d_*) = \ell(\inf(d_*)) = \ell(\inf(\delta_*)) = \ell\inf(\delta_*)$ by Lemma 3.1.5. □

Lemma 3.2.7. Suppose $d_*, \delta_* : \Phi^n \to \Psi^m$ are morphisms in $\mathcal{C}\mathcal{O}_G$. If $d_* \sim \delta_*$, then $\ell\inf(d_*) = \ell\inf(\delta_*)$, i.e., we can define a functor $[\ell\inf]$ on the subdivision in a well-defined manner.

Proof. See Lemmas 2.2.4 and 3.2.6. □

Definition 3.2.8. Define the *link inf functor* $[\ell\inf] : \mathrm{Sd}\mathcal{O}_G \to \mathcal{L}_G$

`on objects:` $[\ell\inf](\Phi^n) := \ell\inf(\Phi^n)$

`on morphisms:` $[\ell\inf]([d_*]_\sim) = \ell\inf(d_*)$

$$\begin{tikzcd}
\mathcal{C}\mathcal{O}_G \arrow[r, "\ell\inf"] \arrow[dr, "\text{quotient on morphisms}"'] & \mathcal{L}_G \\
& \mathrm{Sd}\mathcal{O}_G \arrow[u, "{[\ell\inf]}"']
\end{tikzcd}$$

Figure 3.4: Defining $\ell\inf$ on equivalence classes of morphisms.

Lemma 3.2.9. The functor $[\ell\inf]$ is faithful, i.e., if $d_*, \delta_* \in \mathrm{Hom}(\Phi^n, \Psi^m)$ and $[\ell\inf](d_*) = [\ell\inf](\delta_*)$, then $d_* \sim \delta_*$.

Proof. We have $\ell(\inf(d_*)) = \ell(\inf(\delta_*))$ implies $\inf(d_*) = \inf(\delta_*)$ because ℓ is injective, and so $d_* \sim \delta_*$ by Lemma 3.1.7. □

3.3 Proving Weak Equivalence

We can rewrite Quillen's Theorem A [8] for $[\ell \inf]$. This gives us a strategy for proving that $[\ell \inf]$ is a weak equivalence.

Theorem 3.3.1. (Quillen's Theorem A 2.4.8 with the Link Inf Functor) The functor $[\ell \inf] : \mathrm{Sd}\mathcal{O}_G \to \mathcal{L}_G$ is a weak equivalence if the left fiber $[\ell \inf]/T$ is contractible for every object T of $\mathcal{L}_G$.

Definition 3.3.2. Let T be an object in $\mathcal{L}_G$. Using Definition 2.4.7, the *left fiber* $[\ell \inf]/T$ is the category with the following data:

`objects:` the pairs $(\Phi^n, \lambda^{\mathcal{L}})$ with Φ^n an object of $\mathrm{Sd}\mathcal{O}_G$ and $\lambda^{\mathcal{L}} : [\ell \inf]\Phi^n \to T$ a morphism of $\mathcal{L}_G$. The *dimension* of an object $(\Phi^n, \lambda^{\mathcal{L}})$ is n.

`morphisms:` There exists a morphism $(\Phi^n, \lambda_1^{\mathcal{L}}) \xrightarrow{d_*} (\Psi^m, \lambda_2^{\mathcal{L}})$ if there exists a map $\Phi^n \xrightarrow{d_*} \Psi^m$ in $\mathrm{Sd}\mathcal{O}_G$ such that $\lambda_2^{\mathcal{L}} \circ [\ell \inf] d_* = \lambda_1^{\mathcal{L}}$ in $\mathcal{L}_G$, i.e., the diagram in Figure 3.5 commutes.

$$\begin{tikzcd}
{[\ell\,\mathrm{inf}]\Phi^n} \arrow[r, "\lambda_1^{\mathcal{L}}"] \arrow[d, "{[\ell\,\mathrm{inf}]d_*}"'] & T \\
{[\ell\,\mathrm{inf}]\Psi^m} \arrow[ur, "\lambda_2^{\mathcal{L}}"'] &
\end{tikzcd}$$

Figure 3.5: Commutative diagram for morphisms in the left fiber $[\ell\,\mathrm{inf}]/T$

Example 3.3.3. Let $G = \mathbb{Z}/p^2\mathbb{Z}$. Let's explore the objects in the left fiber $[\ell\,\mathrm{inf}]/(\langle 0\rangle < \langle p\rangle)$. I will use the same shorthand for objects as used in Example 3.2.2. See Figure 3.6 below.

`objects:`

$(a, \alpha_i^{\mathcal{L}} : \langle 0\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p^2 - 1$

$(\alpha, \alpha_i^{\mathcal{L}} : \langle 0\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p^2 - 1$

$(c, \omega_i^{\mathcal{L}} : \langle p\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p - 1$

$(\theta, \omega_i^{\mathcal{L}} : \langle p\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p - 1$

$(\omega, \alpha_i^{\mathcal{L}} : \langle 0\rangle < \langle p\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p^2 - 1$

$(\alpha\omega, \alpha_i^{\mathcal{L}} : \langle 0\rangle < \langle p\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p^2 - 1$

$(\omega\theta, \alpha_i^{\mathcal{L}} : \langle 0\rangle < \langle p\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p^2 - 1$

$(\alpha\omega\theta, \alpha_i^{\mathcal{L}} : \langle 0\rangle < \langle p\rangle \to \langle 0\rangle < \langle p\rangle)$ for $0 \le i \le p^2 - 1$ $\triangle$

Lemma 3.3.4. There is at most one morphism between any two objects in $[\ell\,\mathrm{inf}]/T$ for all objects T in $\mathcal{L}_G$.

$$
\begin{tikzcd}
\langle 0\rangle\ \boxed{a,\alpha} \arrow[loop left, "\alpha_0^{\mathcal{L}},\dots,\alpha_{p^2-1}^{\mathcal{L}}"] \arrow[d, "\alpha_0^{\mathcal{L}},\dots,\alpha_{p^2-1}^{\mathcal{L}}"] & \langle p\rangle\ \boxed{c,\theta} \arrow[loop above, "\theta_0^{\mathcal{L}},\dots,\theta_{p-1}^{\mathcal{L}}"] \arrow[dl, "\omega_0^{\mathcal{L}},\dots,\omega_{p-1}^{\mathcal{L}}"'] \\
\langle 0\rangle < \langle p\rangle\ \boxed{\omega,\alpha\omega,\omega\theta,\alpha\omega\theta} \arrow[loop below, "\alpha_0^{\mathcal{L}},\dots,\alpha_{p^2-1}^{\mathcal{L}}"] &
\end{tikzcd}
$$

Figure 3.6: A portion of $\mathcal{L}_{\mathbb{Z}/p^2\mathbb{Z}}$ useful for defining the left fiber over $\langle 0\rangle < \langle p\rangle$

Proof. Let $(\Phi^n, \lambda_1^{\mathcal{L}})$ and $(\Psi^m, \lambda_2^{\mathcal{L}})$ be morphisms in $[\ell\inf]/T$. By definition, there exists an arrow $(\Phi^n, \lambda_1^{\mathcal{L}}) \xrightarrow{d_*} (\Psi^m, \lambda_2^{\mathcal{L}})$ if there exists a map $d_* : \Phi^n \to \Psi^m$ in $\mathrm{Sd}\mathcal{O}_G$ such that $\lambda_2^{\mathcal{L}} \circ [\ell\inf] d_* = \lambda_1^{\mathcal{L}}$ in $\mathcal{L}_G$.

In the case that no such d_* exists, there are no maps between the objects, and we are done. So assume that at least one such d_* exists. In this instance, I claim that d_* is unique. To this end, note that by Lemma 2.6.10, $\lambda_2^{\mathcal{L}} \circ [\ell\inf] d_* = \lambda_1^{\mathcal{L}}$ implies that $[\ell\inf] d_*$ is a unique map entirely determined by $\lambda_2^{\mathcal{L}}$ and $\lambda_1^{\mathcal{L}}$. By Lemma 3.2.9, $[\ell\inf]$ is faithful, so the map d_* must be unique. □

Corollary 3.3.5. Every left fiber $[\ell\inf]/T$ of $\mathcal{L}_G$ satisfies the second condition of being a filtered category (in Definition 2.2.5.)

Proof. See Remark 2.2.6 and Lemma 3.3.4. □

It remains to show that for each left fiber, the first condition of being a filtered category (Definition 2.2.5) is satisfied, i.e., for any two objects $(\Phi, \gamma_i^{\mathcal{L}})$ and $(\Psi, \gamma_j^{\mathcal{L}})$ in $[\ell \inf]/T$, we have an object $(\Omega, \gamma_k^{\mathcal{L}})$ and morphisms $d_\star : (\Phi, \gamma_i^{\mathcal{L}}) \to (\Omega, \gamma_k^{\mathcal{L}})$ and $\delta_\star : (\Psi, \gamma_j^{\mathcal{L}}) \to (\Omega, \gamma_k^{\mathcal{L}})$. We break the proof into a series of lemmas, and then describe the full argument using these lemmas in Corollary 3.3.10.

$$\begin{tikzcd}
(\Phi, \gamma_i^{\mathcal{L}}) \arrow[dr, "d_\star"] & & (\Psi, \gamma_j^{\mathcal{L}}) \arrow[dl, "\delta_\star"'] \\
& (\Omega, \gamma_k^{\mathcal{L}}) &
\end{tikzcd}$$

Figure 3.7: The first condition of being a filtered category applied to a left fiber.

Lemma 3.3.6. Suppose $(\Phi, \gamma_i^{\mathcal{L}})$ is an object in $[\ell \inf]/T$, Γ is an object in $\mathrm{Sd}\mathcal{O}_G$ such that $[\ell \inf]\Gamma$ is a subchain of T, and there exists a morphism $\Phi \xrightarrow{d_\star} \Gamma$ in $\mathrm{Sd}\mathcal{O}_G$ where $\inf(d_\star) = \gamma_i^{\mathcal{O}} : \inf(\Gamma) \to \inf(\Phi)$. Then $(\Phi, \gamma_i^{\mathcal{L}}) \xrightarrow{d_\star} (\Gamma, \gamma_0^{\mathcal{L}})$ is a morphism in $[\ell \inf]/T$.

Proof. First, since $[\ell \inf]\Gamma$ is a subchain of T, we know that $\gamma_0^{\mathcal{L}} : [\ell \inf]\Gamma \to T$ is a morphism in $\mathcal{L}_G$, and so $(\Gamma, \gamma_0^{\mathcal{L}})$ is an object in $[\ell \inf]/T$. Next, using the criterion for morphisms in Definition 3.3.2, we will check that $d_\star$ is valid:

$$\gamma_0^{\mathcal{L}} \circ [\ell \inf] d_\star = \gamma_0^{\mathcal{L}} \circ \ell(\gamma_i^{\mathcal{O}}) = \gamma_0^{\mathcal{L}} \circ \gamma_i^{\mathcal{L}} = \gamma_i^{\mathcal{L}}. \qquad \square$$

Lemma 3.3.7. Let G be a finite abelian group, and let $T = T_0 < \dots < T_S$ be an object in $\mathcal{L}_G$. Suppose $(\Phi, \gamma_i^{\mathcal{L}})$ is an object in the left fiber $[\ell \inf]/T$ such that $[\ell \inf]\Phi = \mathbb{H} = H_0 < \dots < H_N$ and $H_0 \neq G$. there exists a morphism $d_\star : (\Phi, \gamma_i^{\mathcal{L}}) \to (\Gamma_\Phi, \gamma_0^{\mathcal{L}})$ in $[\ell \inf]/T$.

Notation for this Lemma and Lemma 3.3.8: To avoid too many levels of indices in certain portions of this discussion, let $\gamma^{\mathcal{O}}(j) \coloneqq \gamma^{\mathcal{O}}_j$. Keep in mind that $\gamma^{\mathcal{O}}(j)$ is the map in $\mathcal{O}_G$ while j is the group element in G, and every group element induces an equivariant map $G/H \to G/K$ where distinct maps are identified with K-cosets (Lemma 2.1.10). So $\gamma^{\mathcal{O}}(j) : H \to K$ may not have an inverse map, but j has an inverse in G, and if $\gamma^{\mathcal{O}}(j) : H \to K$ and $\gamma^{\mathcal{O}}(-j) : K \to L$, then

$$(\gamma^{\mathcal{O}}(-j) \circ \gamma^{\mathcal{O}}(j))(g + H) = g + j + (-j) + L = g + L$$

i.e., $\gamma^{\mathcal{O}}(-j) \circ \gamma^{\mathcal{O}}(j) = \gamma^{\mathcal{O}}(0) : H \to L$.

Proof. Write Φ as:

$$\begin{aligned}
\Phi = & H_0 \xrightarrow{\gamma^{\mathcal{O}}(h_{10})} \cdots \xrightarrow{\gamma^{\mathcal{O}}(h_{n_0 0})} H_0 \\
& \xrightarrow{\gamma^{\mathcal{O}}(h_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(h_{11})} \ldots \xrightarrow{h_{n_1 1}} H_1 \\
& \cdots \\
& \xrightarrow{\gamma^{\mathcal{O}}(h_N)} H_N \xrightarrow{\gamma^{\mathcal{O}}(h_{1N})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(h_{n_N N})} H_N
\end{aligned}$$

Construct Γ_Φ to be the following object in $\mathrm{Sd}\mathcal{O}_G$:

$$\begin{aligned}
\Gamma_\Phi \coloneqq\ & H_0 \xrightarrow{\gamma^{\mathcal{O}}_i} H_0 \xrightarrow{\gamma^{\mathcal{O}}(h_{10})} \cdots \xrightarrow{\gamma^{\mathcal{O}}(h_{n_0 0})} H_0 \xrightarrow{\gamma^{\mathcal{O}}(\eta_1)} H_0 \\
& \xrightarrow{\gamma^{\mathcal{O}}_0} H_1 \xrightarrow{\gamma^{\mathcal{O}}(-\eta_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(h_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(h_{11})} \cdots \xrightarrow{\gamma^{\mathcal{O}}(h_{n_1 1})} H_1 \xrightarrow{\gamma^{\mathcal{O}}(\eta_2)} H_1 \\
& \xrightarrow{\gamma^{\mathcal{O}}_0} H_2 \xrightarrow{\gamma^{\mathcal{O}}(-\eta_2)} H_2 \xrightarrow{\gamma^{\mathcal{O}}(h_2)} H_2 \xrightarrow{\gamma^{\mathcal{O}}(h_{12})} \cdots \xrightarrow{\gamma^{\mathcal{O}}(h_{n_2 2})} H_2 \xrightarrow{\gamma^{\mathcal{O}}(\eta_3)} H_2 \\
& \xrightarrow{\gamma^{\mathcal{O}}_0} H_3 \xrightarrow{\gamma^{\mathcal{O}}(-\eta_3)} H_3 \xrightarrow{\gamma^{\mathcal{O}}(h_3)} \quad \cdots \\
& \qquad \ldots \xrightarrow{\gamma^{\mathcal{O}}(\eta_N)} H_{N-1} \\
& \xrightarrow{\gamma^{\mathcal{O}}_0} H_N \xrightarrow{\gamma^{\mathcal{O}}(-\eta_N)} H_N \xrightarrow{\gamma^{\mathcal{O}}(h_N)} H_N \xrightarrow{\gamma^{\mathcal{O}}(h_{1N})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(h_{n_N N})} H_N \xrightarrow{\gamma^{\mathcal{O}}(\eta_{N+1})} H_N
\end{aligned}$$

and define $\eta_j \in G$ recursively such that:

$$\eta_1 + h_{n_0 0} + \ldots + h_{10} + i = 0$$

$$\eta_{j+1} + h_{n_j j} + \ldots + h_{1j} + h_j + (-\eta_j) = 0 \text{ if } 1 \le j \le N$$

The map $\gamma^{\mathcal{O}}(\eta_j)$ is unique by Lemma 2.1.12. Further, note that if $\eta_j = 0$, then

$$\eta_{j+1} + h_{n_j j} + \ldots + h_{1j} + h_j = 0$$

To ensure that Γ is in $\mathrm{Sd}\mathcal{O}_G$ and does not contain any identity maps,

- if $i = 0$, i.e., $\gamma_i^{\mathcal{L}} = \gamma_0^{\mathcal{L}}$, delete the first map, $H_0 \xrightarrow{\gamma_i^{\mathcal{O}}} H_0$,
- if the composition defining η_j forces $\gamma^{\mathcal{O}}(\eta_j) = \gamma_0^{\mathcal{O}} : H_j \to H_j$, delete the maps $\gamma^{\mathcal{O}}(\eta_j)$ and $\gamma^{\mathcal{O}}(-\eta_j)$ from Γ, and
- if $\gamma^{\mathcal{O}}(h_j) = \gamma_0^{\mathcal{O}} : H_{j-1} \to H_j$ in Φ, delete $\gamma^{\mathcal{O}}(h_j) : H_j \to H_j$ from Γ.

Note that $(\Gamma, \gamma_0^{\mathcal{L}})$ is an object in the left fiber $[\ell\inf]/T$ with $[\ell\inf]\Gamma = \mathbb{H}$. Using Lemma 3.3.6, there exists a morphism $d_* : (\Phi, \gamma_i^{\mathcal{L}}) \to (\Gamma_\Phi, \gamma_0^{\mathcal{L}})$ in the left fiber where d_* is defined by inserting:

$$H_j \xrightarrow{\gamma^{\mathcal{O}}(h_{lj})} H_j \text{ into } H_j \xrightarrow{\gamma^{\mathcal{O}}(h_{lj})} H_j, \text{ and}$$
$$H_{j-1} \xrightarrow{\gamma^{\mathcal{O}}(h_j)} H_j \text{ into } H_{j-1} \xrightarrow{\gamma^{\mathcal{O}}(\eta_j)} H_{j-1} \xrightarrow{\gamma_0^{\mathcal{O}}} H_j \xrightarrow{\gamma^{\mathcal{O}}(-\eta_j)} H_j \xrightarrow{\gamma^{\mathcal{O}}(h_j)} H_j.$$

Most of the remainder of the proof is implied by the construction of Γ_Φ. We will comment on the requirements laid out in Lemma 3.3.6. Observe that Γ_Φ is a path passing through the same objects as Φ, and so $[\ell\inf]\Phi = [\ell\inf]\Gamma_\Phi$, and so $[\ell\inf]\Gamma$ is a subchain of T. Now, d_* is a valid map in $\mathrm{Sd}\mathcal{O}_G$ due to how we inserted Φ. Finally, if we insert Φ as described, $\inf(d_*) : \inf(\Gamma) \to \inf(\Phi)$ is $\inf(d_*) = H_0 \xrightarrow{\gamma_i^{\mathcal{O}}} H_0$ and we recover the correct beginning difference. □

Lemma 3.3.8. This is in essence a continuation of Lemma 3.3.7, and we will continue to use the set-up presented there. In addition, suppose $(\Psi, \gamma_q^{\mathcal{L}})$ is an object in $[\ell\inf]/T$ such that $[\ell\inf]\Psi = \mathbb{K} = K_0 < \ldots < K_M$, $K_0 \neq G$, and

$$\begin{aligned}\Psi = &K_0 \xrightarrow{\gamma^{\mathcal{O}}(k_{10})} \cdots \xrightarrow{\gamma^{\mathcal{O}}(k_{m_0 0})} K_0\\ &\xrightarrow{\gamma^{\mathcal{O}}(k_1)} K_1 \xrightarrow{\gamma^{\mathcal{O}}(k_{11})} \ldots\\ &\xrightarrow{\gamma^{\mathcal{O}}(k_M)} K_M \xrightarrow{\gamma^{\mathcal{O}}(k_{1M})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(k_{m_M M})} K_M\end{aligned}$$

Define Γ_Φ and Γ_Ψ as in Lemma 3.3.7. Without loss of generality, suppose $H_0 \leq K_0$. For convenience, write these as:

$$\begin{aligned}\Gamma_\Phi = &H_0 \xrightarrow{\gamma^{\mathcal{O}}(x_{10})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(x_{s_0 0})} H_0\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} H_1 \xrightarrow{\gamma^{\mathcal{O}}(x_{11})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(x_{s_1 1})} H_1\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} \ldots \xrightarrow{\gamma_0^{\mathcal{O}}} H_Q \xrightarrow{\gamma^{\mathcal{O}}(x_{1Q})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(x_{s_Q Q})} H_Q\end{aligned}$$

and

$$\begin{aligned}\Gamma_\Psi = &K_0 \xrightarrow{\gamma^{\mathcal{O}}(y_{10})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(y_{t_0 0})} K_0\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} K_1 \xrightarrow{\gamma^{\mathcal{O}}(y_{11})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(y_{t_1 1})} K_1\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} \ldots \xrightarrow{\gamma_0^{\mathcal{O}}} K_R \xrightarrow{\gamma^{\mathcal{O}}(y_{1R})} \ldots \xrightarrow{\gamma^{\mathcal{O}}(y_{t_R R})} K_R\end{aligned}$$

Without loss of generality, suppose $H_0 \leq K_0$. Let Ω be an object in $\mathrm{Sd}_{\mathcal{O}_G}$ defined as:

$$\begin{aligned}\Omega = &T_0 \xrightarrow{\gamma_0^{\mathcal{O}}} T_1 \xrightarrow{\gamma_0^{\mathcal{O}}} \ldots\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} H_0 \xrightarrow{x_{10}} \ldots \xrightarrow{x_{s_0 0}} H_0 \xrightarrow{y_{10}} \ldots \xrightarrow{y_{t_0 0}} H_0 \xrightarrow{\gamma_0^{\mathcal{O}}} \ldots\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} T_j \xrightarrow{x_{1j}} \ldots \xrightarrow{x_{s_j j}} T_j \xrightarrow{y_{1j}} \ldots \xrightarrow{y_{t_j j}} T_j\\ &\xrightarrow{\gamma_0^{\mathcal{O}}} \ldots \xrightarrow{\gamma_0^{\mathcal{O}}} \ldots T_S\end{aligned}$$

To summarize the construction of Ω, use $\gamma_0^{\mathcal{O}}$ to transition between different subgroups; for each subgroup in T, insert all the $T_j \to T_j$ morphisms from Γ_Φ followed by all of the $T_j \to T_j$ morphisms from Γ_Ψ. (Note that $[\ell\inf]\Omega = T$.)

Then:

There exist morphisms in the left fiber

$$(\Gamma_\Phi, \gamma_0^{\mathcal{O}}) \to (\Omega, \gamma_0^{\mathcal{O}}) \text{ and } (\Gamma_\Psi, \gamma_0^{\mathcal{O}}) \to (\Omega, \gamma_0^{\mathcal{O}})$$

Proof. Let's address the hypotheses of Lemma 3.3.6. First, $[\ell\inf]\Omega = T$ and so is certainly a subchain. We define the insertions of of Γ_Φ into Ω and Γ_Ψ into Ω by inserting $H_j \xrightarrow{x_{lj}} H_j$ into $H_j \xrightarrow{x_{lj}} H_j$ and $K_j \xrightarrow{y_{lj}} K_j$ into $K_j \xrightarrow{y_{lj}} K_j$, respectively. Because of how we defined Γ_Φ and Γ_Ψ, each T_j-"row" of Ω (as formatted previously) composes to $\gamma_0^{\mathcal{O}}$, so the $\gamma_0^{\mathcal{O}}$-transition maps in Γ_Φ and Γ_Ψ insert into potentially long compositions of maps, but these maps do compose to $\gamma_0^{\mathcal{O}}$. Thus no matter where Γ_Φ or Γ_Ψ first appear in the insertion, we get the correct beginning difference, $\gamma_0^{\mathcal{O}}$. □

Example 3.3.9. Let G be a finite abelian group, let T be an object in $\mathcal{L}_G$, and let Φ and Ψ be objects in $\mathrm{Sd}\mathcal{O}_G$ such that

$$T = H_0 < K_0 < H_1 < K_1$$

$$\Phi = H_0 \xrightarrow{\gamma^{\mathcal{O}}(h_1)} H_1$$

$$\Psi = H_0 \xrightarrow{\gamma^{\mathcal{O}}(k_1)} K_0 \xrightarrow{\gamma^{\mathcal{O}}(k_2)} K_1$$

Since $[\ell\inf]\Phi$ and $[\ell\inf]\Psi$ are subchains of T, there exists $\gamma_i^{\mathcal{L}}$ and $\gamma_j^{\mathcal{L}}$ such that $(\Phi, \gamma_i^{\mathcal{O}})$ and $(\Psi, \gamma_j^{\mathcal{O}})$ are objects in $[\ell\inf]/T$. Now, using the construction in Lemmas

3.3.7 and 3.3.7,

$$\begin{aligned}\Gamma_\Phi = & H_0 \xrightarrow{\gamma_i^{\mathcal{O}}} H_0 \xrightarrow{\gamma^{\mathcal{O}}(\eta_1)} H_0 \\ & \xrightarrow{\gamma_0^{\mathcal{O}}} H_1 \xrightarrow{\gamma^{\mathcal{O}}(-\eta_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(h_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(\eta_2)} H_1\end{aligned}$$

where $\eta_1 + i = 0$ if $i \neq 0$ and $\eta_2 + h_1 + (-\eta_1) = 0$ as long as η_2 is not forced to be 0.

- If $i = 0$, then the first row of Γ_Φ is deleted entirely, we keep the transition map $H_0 \xrightarrow{\gamma_0^{\mathcal{O}}} H_1$, and we delete $\gamma^{\mathcal{O}}(-\eta_1)$ from the second row.
- If η_2 is forced to be 0, then delete $\gamma^{\mathcal{O}}(\eta_2)$ from the second row.
- If $h_1 = 0$, then delete $\gamma^{\mathcal{O}}(h_1)$ from the second row.

$$\begin{aligned}\Gamma_\Psi = & H_0 \xrightarrow{\gamma_j^{\mathcal{O}}} H_0 \xrightarrow{\gamma^{\mathcal{O}}(\epsilon_1)} H_0 \\ & \xrightarrow{\gamma_0^{\mathcal{O}}} K_0 \xrightarrow{\gamma^{\mathcal{O}}(-\epsilon_1)} K_0 \xrightarrow{\gamma^{\mathcal{O}}(k_1)} K_0 \xrightarrow{\gamma^{\mathcal{O}}(\epsilon_2)} K_0 \\ & \xrightarrow{\gamma_0^{\mathcal{O}}} K_1 \xrightarrow{\gamma^{\mathcal{O}}(-\epsilon_2)} K_1 \xrightarrow{\gamma^{\mathcal{O}}(k_2)} K_1 \xrightarrow{\gamma^{\mathcal{O}}(\epsilon_3)} K_1\end{aligned}$$

$$\begin{aligned}\Omega = & H_0 \xrightarrow{\gamma_i^{\mathcal{O}}} H_0 \xrightarrow{\gamma^{\mathcal{O}}(\eta_1)} H_0 \xrightarrow{\gamma_j^{\mathcal{O}}} H_0 \xrightarrow{\gamma^{\mathcal{O}}(\epsilon_1)} H_0 \\ & \xrightarrow{\gamma_0^{\mathcal{O}}} K_0 \xrightarrow{\gamma^{\mathcal{O}}(-\epsilon_1)} K_0 \xrightarrow{\gamma^{\mathcal{O}}(k_1)} K_0 \xrightarrow{\gamma^{\mathcal{O}}(\epsilon_2)} K_0 \\ & \xrightarrow{\gamma_0^{\mathcal{O}}} H_1 \xrightarrow{\gamma^{\mathcal{O}}(-\eta_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(h_1)} H_1 \xrightarrow{\gamma^{\mathcal{O}}(\eta_2)} H_1 \\ & \xrightarrow{\gamma_0^{\mathcal{O}}} K_1 \xrightarrow{\gamma^{\mathcal{O}}(-\epsilon_2)} K_1 \xrightarrow{\gamma^{\mathcal{O}}(k_2)} K_1 \xrightarrow{\gamma^{\mathcal{O}}(\epsilon_3)} K_1\end{aligned}$$

$\triangle$

Corollary 3.3.10. If T is an object in $\mathcal{L}_G$ and G is abelian, then the left fiber $[\ell \inf]/T$ is a filtered category.

Proof. We have already proved the second condition of being filtered in Corollary 3.3.5. It remains to prove the first condition of being filtered. In each case, we will satisfy the requirements of Lemma 3.3.6.

First, if $T = G$ (implying that $H_0 = G$), then the left fiber $[\ell \inf]/T$ consists of the objects $(\Phi, \gamma_i^{\mathcal{L}})$ where we have the map $[\ell \inf]\Phi \xrightarrow{\gamma_i^{\mathcal{L}}} T$. This means that if $\mathbb{H} = [\ell \inf]\Phi$, then $H_0 = G$, and so $\mathbb{H}$ must be a sub-chain of T, implying that $\mathbb{H} = G$. So $\gamma_i^{\mathcal{L}}$ is a map from T to T. Further, $\gamma_i^{\mathcal{L}}$ corresponds to a self-equivariant map of G/G (Remark **??**), and since there is only one such map, namely the identity on G/G, we know that $\gamma_i^{\mathcal{L}} = \gamma_0^{\mathcal{L}}$. As for Φ, we know that $[\ell \inf]\Phi = G$, and so Φ must be a path through the orbit category that only passes through G, i.e., Φ must consist solely of equivariant self-maps of G/G. Therefore $\Phi = \overset{0}{G}$ is the object in the subdivision represented by $[0] \Rightarrow \mathcal{O}_G$ defined by $0 \mapsto G$. So if $T = G$, then $[\ell \inf]/T$ is just the category with one object, $(\overset{0}{G}, \gamma_0^{\mathcal{L}})$ and the identity morphism on this object (by Lemma 3.3.4 there are no other self-morphisms on $(\overset{0}{G}, \gamma_0^{\mathcal{L}})$). Thus the first condition is satisified for $T = G$.

Secondly, if $T \neq G$, $[\ell \inf]\Phi \neq G$, and $[\ell \inf]\Psi \neq G$ then we can use Lemmas 3.3.7 and 3.3.8 to construct morphisms $(\Phi, \gamma_i^{\mathcal{L}}) \to (\Gamma_\Phi, \gamma_0^{\mathcal{L}}) \to (\Omega, \gamma_0^{\mathcal{L}})$ and $(\Psi, \gamma_j^{\mathcal{L}}) \to (\Gamma_\Psi, \gamma_0^{\mathcal{L}}) \to (\Omega, \gamma_0^{\mathcal{L}})$, as in Figure 3.8.

Third, if $T \neq G$ but contains G and $\Phi = G$, then $\gamma_i^{\mathcal{O}} = \gamma_0^{\mathcal{O}}$. Let $\Psi \neq G$ be defined as in Lemma 3.3.8. Let

$$\Omega = K_1 \xrightarrow{\gamma_j^{\mathcal{O}}} K_1 \xrightarrow{\psi_{11}} \cdots \xrightarrow{\psi_{m_1 1}} K_1 \xrightarrow{\psi_2} K_2 \xrightarrow{\psi_{12}} \ldots \xrightarrow{\psi_M} K_M \xrightarrow{\psi_{1M}} \ldots \xrightarrow{\psi_{m_M M}} K_M \xrightarrow{\gamma_0^{\mathcal{O}}} G$$

$$\begin{tikzcd}[column sep=small]
(\Phi, \gamma_i^{\mathcal{L}}) \arrow[d, "\text{Lemma 3.3.7}"] & & (\Psi, \gamma_j^{\mathcal{L}}) \arrow[d, "\text{Lemma 3.3.7}"'] \\
(\Gamma_\Phi, \gamma_0^{\mathcal{L}}) \arrow[dr, "\text{Lemma 3.3.8}"'] & & (\Gamma_\Psi, \gamma_0^{\mathcal{L}}) \arrow[dl, "\text{Lemma 3.3.8}"] \\
& (\Omega, \gamma_0^{\mathcal{L}}) &
\end{tikzcd}$$

Figure 3.8: Proof outline of the left fibers having the first filtered condition

if $K_M \neq G$; otherwise, leave off the last map. Insert Φ into the last object in Ω (noting that the inf of this insertion will be modded by G) and insert Ψ starting at the second object.

Fourth, if $T \neq G$ but contains G and both $\Psi = \Phi = G$, then $i = j = 0$. Let $\Omega = T_1 \xrightarrow{\gamma_0^{\mathcal{O}}} \ldots \xrightarrow{\gamma_0^{\mathcal{O}}} G$, and insert both Φ and Ψ into the last object in Ω. □

Theorem 3.3.11. If G is a finite abelian group, then there exists a weak equivalence from the subdivision of the orbit category of G to the link orbit category of G.

Proof. By Corollary 3.3.10, each fiber of $[\ell \inf] : \mathrm{Sd}\mathcal{O}_G \Rightarrow \mathcal{L}_G$ is filtered and thus contractible by Theorem 2.4.6. Then by Quillen's Theorem A 2.4.8, $[\ell \inf]$ is a weak equivalence. □

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