Submitted to Advances in Water Resources

LBL-11550 c.2
Preprint

A NOTE ON THE FORMULATION OF SATURATED-UNSATURATED FLUID FLOW THROUGH DEFORMABLE POROUS MEDIA

J. Noorishad, M. Mehran, and T.N. Narasimhan

September 1980

Prepared for the U.S. Department of Energy
under Contract W-7405-ENG-48

DISCLAIMER

Submitted to
Advances in Water Resources

LBL-11550

A NOTE ON THE FORMULATION OF SATURATED-UNSATURATED FLUID FLOW THROUGH DEFORMABLE POROUS MEDIA

J. Noorishad, M. Mehran, and T. N. Narasimhan

Earth Sciences Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

September 1980

Prepared under the auspices of the U. S. Department of Energy under Contract W-7405-ENG-48.

INTRODUCTION

Biot's[2,3] fundamental relationships in the general theory of three-dimensional consolidation have provided a basis for the coupled stress—fluid flow analysis of saturated deformable porous media[1,4,5,6,8]. Extension of such analysis to the entire flow regime (saturated-unsaturated) has recently been attempted by Safai and Pinder[7]. In their development, similar to the saturated case, the contribution to change in storage in the unsaturated state is derived from the skeletal deformation and the fluid compressibility effects corrected for the effect of degree of saturation on the latter term only. However, the major contribution to the storage change in unsaturated state is from the actual dewatering of pores which does not appear in their governing equation. Ignoring the desaturation of pores, in view of its significance as compared to the skeletal deformation and the fluid compressibility effects, may lead to erroneous results.

In this paper, along the lines of Biot's development, a general formulation consisting of constitutive and governing relations for saturated-unsaturated flow in deformable porous media is presented.

FORMULATION

The basic equations governing the coupled stress and fluid flow behavior of saturated elastic porous media are introduced in Biot's[2,3] three-dimensional theory of consolidation. The constitutive relations and the governing equations are as follows:

$$\left.\begin{aligned} \tau_{ij} &= 2\mu e_{ij} + \lambda\delta_{ij}\delta_{k\ell}e_{k\ell} + \alpha\delta_{ij}\pi \\ \zeta &= -\alpha\delta_{ij}e_{ij} + \frac{1}{M}\pi \end{aligned}\right\} \qquad (1)$$

$$\frac{\partial w_i}{\partial t} = \frac{k}{\eta}\left(\pi_{,i} + \rho_f g z_{,i}\right) \qquad (2)$$

$$\frac{\partial \zeta}{\partial t} = \left(\frac{k}{\eta}(\pi_{,k} + \rho_f g z_{,i}) \right)_{,i} \tag{3}$$

$$\tau_{ij,j} + \rho_s f_i = 0 \tag{4}$$

where τ_{ij} are the components of the stress tensor, π is the fluid pressure, e_{ij} are the components of the solid strain tensor, and ζ is the fluid volumetric strain.

The first two equations are the constitutive laws relating the seven components of stress, τ_{ij} and π, to the seven components of strain, e_{ij} and ζ. Equation (2) is Darcy's generalized constitutive law and equations (3) and (4) are the fluid flow and static equilibrium laws, respectively. The above set of equations form the foundation for the analysis of hydromechanical phenomena in saturated deformable porous media.

Although the extension of such analysis to the entire flow regime (saturated-unsaturated) might appear to follow immediately from the previous equations, the complete formulation can only be obtained from the physical considerations of the prevailing effects.

Time rate of change in fluid mass content in a unit control volume of the unsaturated region consists of three different quantities of varying significance[5]. Starting with the least important quantity, the fluid compressibility in response to changes of the very limited range of suctions, accounts for a fluid mass quantity of $\rho_f n s_f \beta \frac{d\pi}{dt}$ or $\frac{\rho_f s_f}{M} \frac{d\pi}{dt}$, where n, s_f, and β are porosity, degree of saturation, and compressibility of fluid, respectively. The second quantity is derived from the skeletal deformation of the medium. Depending upon the degree of coupling, any volume dilatation of the bulk causes a change in the degree of saturation of the unsaturated medium in the following manner:

$$s_f' = \frac{n s_f}{n + \alpha \delta_{ij} e_{ij}} = \frac{s_f}{1 + \frac{\alpha}{n} \delta_{ij} e_{ij}} \approx s_f \left(1 - \frac{\alpha}{n} \delta_{ij} e_{ij} \right) \tag{5}$$

where s_f' is the modified value of the degree of saturation. Therefore, as a first approximation, the value of $\rho_f s_f \frac{d}{dt}(\alpha\delta_{ij}e_{ij})$ expresses the deformability effect. Finally, the most important quantity is the actual dewatering of the pores amounting to $\rho_f \frac{ds_f}{dt}$. This quantity, when written in terms of the dependent variable π, yields $\rho_f \frac{ds_f}{d\pi}\frac{d\pi}{dt}$. Using the above expressions for the effects of fluid compressibility, skeletal deformation, and desaturation of pores, the proper constitutive laws can be established. The complete set of relations for the analysis of the coupled stress and saturated-unsaturated fluid flow in deformable media can be written as follows:

$$\left.\begin{aligned} \tau_{ij} &= 2\mu e_{ij} + \lambda\delta_{ij}\delta_{k\ell}e_{k\ell} + s_f\alpha\delta_{ij}\pi \\ \zeta &= -s_f\alpha\delta_{ij}e_{ij} + \frac{s_f}{M}\pi + nc_f\pi \end{aligned}\right\} \quad (6)$$

$$\frac{\partial w_i}{\partial t} = \frac{kK_r}{\eta}(\pi_{,i} + \rho_f g z_{,i}) \quad (7)$$

$$\frac{\partial \xi}{\partial t} = \left(\frac{kK_r}{\eta}(\pi_{,i} + \rho_f g z_{,i})\right)_{,i} \quad (8)$$

$$\tau_{ij,j} + \rho_s f_i = 0 \quad (9)$$

where $c_f = \frac{ds_f}{d\pi}$, and K_r is the relative permeability. It is important to note that $s_f\alpha$ in equation (6) is equivalent to the χ value known in the soil mechanics literature as Bishop's Parameter. This parameter appears in the effective stress formulation appropriating the contribution of the fluid pressure to the total stress value.

It is clear that in the saturated state, where $s_f = 1$ and $\frac{ds_f}{d\pi} = 0$, equation (6) changes to equation (1). Therefore, equation (6) applies throughout the saturated-unsaturated flow region for the entire range of fluid pressure change. In unsaturated state, the third term of the second part of equation (6), $nc_f\pi$, expressing the desaturation of pores, plays a dominant role in fluid volume strain as compared to the other two terms. In the development given by Safai and Pinder[7], it appears that the term accounting for the desaturation has been neglected.

The set of equations developed above describes the constitutive and governing relations for saturated-unsaturated flow in deformable porous media.

NOMENCLATURE

f_i	components of the body force vector
g	acceleration of gravity
k	intrinsic permeability
M	Biot's fluid storativity coefficient
s_f	degree of saturation
w_i	fluid displacement vector
$z_{,i}$	elevation gradient
α	Biot's coupling coefficient
β	compressibility of fluid
δ_{ij}	Kronecker's delta
μ and λ	Lamé constants
η	fluid viscosity
$\pi_{,i}$	pressure gradient
ρ_f	fluid density
ρ_s	mass bulk density

REFERENCES

1. Ayatollahi, M. S., P. A. Witherspoon, and J. Noorishad. finite element method for stress - fluid flow analysis in fractured rock masses: Part I. Formulation and verification. Submitted to the Journal of Engineering Mechanics Div., ASE.

2. Biot, M. A. General theory of three-dimensional consolidation. Journal of Applied Physics, 1941, vol. 12, 155-164.

3. Biot, M. A. Theory of elasticity and consolidation for a porous anisotropic media. Journal of Applied Physics, 1955, vol. 26, 182-185.

4. Ghabousi, J. Dynamic stress analysis of porous elastic solids with compressible fluids. Ph.D. thesis, University of California, Berkeley, California, 1971.

5. Narasimhan, T. N. and P. A. Witherspoon. Numerical model for saturated-unsaturated flow in deformable porous media: 1. Theory. Water Resources Research, 1977, vol. 13, no. 3, 657-664.

6. Noorishad, J., P. A. Witherspoon, and M. S. Ayatollahi. A finite-element method for stress - fluid flow analysis in fractured rock masses: Part II. Generalization and application to a deep axisymmetric confined aquifer with a horizontal fracture, at the center intersecting the well. Submitted to Journal of Engineering Mechanics Div., ASCE.

7. Safai, N. M. and G. F. Pinder. Vertical and horizontal land deformation in a desaturating porous medium. Advances in Water Resources, 1979, vol. 2, 19-25.

8. Sandu, R. S. and E. L. Wilson. Finite-element analysis of seepage in elastic media. Journal of Engineering Mechanics Div., ASCE, 1969, vol. 95, no. EM3, 641-652.