Exploring Aligned-Images Bounds: Robust Secure GDoF of 3-to-1 Interference Channel

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***Abstract*—Sum-set inequalities based on Aligned-Images bounds have been recently introduced as essential elements of converse proofs for asymptotic/approximate wireless network capacity characterizations under *robust* assumptions, i.e., assumptions that limit channel knowledge at the transmitters to finite precision. While these sum-set inequalities have produced robust Generalized Degrees of Freedom (GDoF) results for various wireless networks, their scope and limitations in general are not well understood. To explore these limitations, in this work we study the robust secure GDoF of a symmetric 3-user many-to-one interference channel. We identify regimes where existing sum-set inequalities are sufficient, settling the GDoF for those settings. For the remaining regime we conjecture the form of new sum-set inequalities that may be needed, whose validity remains an open problem for future work.**

***Index Terms*—Finite Precision CSIT, Aligned Image Set, Sum-set inequalities, Secure Communication, Interference Channel.**

I. Introduction

The past decade has seen significant advances in our understanding of the capacity limits of wireless networks through the pursuit of approximate and asymptotic metrics, such as Generalized Degrees of Freedom (GDoF) characterizations [1], [2]. Much of this progress has come about under idealized assumptions that are too fragile to be meaningful in practice. One of the biggest challenges that stands in the way of understanding the *robust* capacity limits of wireless networks is the difficulty of finding good information theoretic outer bounds that can capture the impact of channel uncertainty.

Wireless networks are generally interference limited. Unlike random noise, interference can be highly structured. The structure of interference is critical to the capacity of a wireless network because it determines the extent to which interference can be managed — e.g., aligned [3], neutralized [4], [5], decoded [6], or cancelled [7]. The ability to shape signals to achieve desired interference structures depends strongly on the amount of channel state information available to the transmitters (CSIT). Yet, classical information theoretic approaches are ill-equipped to capture this critical tension between structure and channel uncertainty. This is evident from the fact that some of the largest gaps between the best known information theoretic inner and outer bounds tend to arise under channel uncertainty [8]. From a bird's-eye view, the difficulty may be summarized as follows — random noise requires statistical thinking which is well represented in the elegant toolset of classical information theory; however, accounting for structured interference tends to require *combinatorial* reasoning that, despite its importance, is far less developed in network information theory, and is even more challenging under channel uncertainty. Advances in this direction are few and far in between. One of the most promising recent advances is the emergence of the so called *Aligned-Images Sum-set Inequalities*.

Introduced by Davoodi and Jafar in [9], [10], Aligned-Images Sum-set Inequalities (in short, AI bounds) are information theoretic GDoF bounds that compare the entropies of different linear combinations (sums) of transmitted signals, when the combining coefficients (channels) are known to the transmitters only to finite precision. In their generalized forms AI bounds allow various forms of partitioning of signals by power levels, antennas, and alignment chains. AI bounds are based on a combinatorial accounting of the number of codewords that can cast resolvable images [11] at one receiver while casting 'aligned images' at another receiver. These bounds have been instrumental in settling important conjectures [12], closing large GDoF gaps [9], establishing new DoF characterizations [13], [14], identifying new parameter regimes for optimality of robust schemes such as treating interference as noise and rate-splitting [15]–[18], shedding new light on the significance of network coherence times [19], and quantifying extremal behaviors [20], [21]. The significance of AI bounds is underscored by their surprisingly large advantage over the best known alternatives; for example AI bounds show that under finite precision CSIT a K user interference channel has no more than 1 DoF [9], but no alternative approach thus far has produced an outer bound under finite precision CSIT that is better (smaller) than the trivial $K/2$ DoF bound (which corresponds to perfect CSIT [22]). The lack of alternatives suggests that exploring and expanding the scope of AI bounds may be imperative to further progress. This is the motivation for our work in this paper.

Following the information-theoretic mindset of studying elemental models that bring fundamental issues into sharp focus, we seek the simplest settings that allow us to see both the utility and limitations of AI bounds. Our search brings us to the problem of secure GDoF characterization of a symmetric 3-to-1 (only the first user sees interference) interference channel (see Fig. 1) under finite precision CSIT (in short, $\text{SGDoF}^{f.p.}_{3\to1}$). This problem is the 'simplest' in the sense that any further simplification reduces it to a solved problem for which known AI bounds are sufficient. For example, if we remove the security constraint, then the GDoF are characterized in [10];

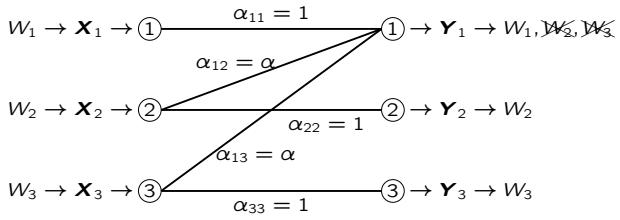

Fig. 1. A 3-to-1 Gaussian interference channel with secrecy constraint (2).

if we reduce the number of users then the secure GDoF of the remaining Z-channel are fully characterized in [23]. However, the known AI bounds appear to be insufficient to characterize $\text{SGDoF}^{f.p.}_{3\to1}$. Against this background, there are two main contributions of this work. First, we identify regimes where existing AI bounds are sufficient, settling $\text{SGDoF}^{f.p.}_{3\to1}$ for those settings. For the remaining regime we draw upon available insights to conjecture the form of new sum-set inequalities that may be needed, as well as the resulting $\text{SGDoF}^{f.p.}_{3\to1}$ characterization. The validity of the conjectured new sum-set inequality is currently an open problem.

Notations: For $m, n \in \mathbb{N}$ with $m \leq n$, denote $[n] = \{1, 2, \cdots, n\}$ and $[m:n] = \{m, m+1, \cdots, n\}$. For random variables A, B, C and $\mathcal{G}$, denote $H_{\mathcal{G}}(A \mid B) = H(A|B, \mathcal{G})$ and $I_{\mathcal{G}}(A; B \mid C) = I(A; B|C, \mathcal{G})$. If $\{X(t) : \ t \in [T]\}$ is well-defined for some $T \in \mathbb{N}$, then we denote this set as a bold $\boldsymbol{X}$. For two functions $f(x)$ and $g(x)$, denote $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$. Note that the values can be positive or negative. All logarithms are to the base 2.

II. Problem Formulation for $\text{SGDoF}^{f.p.}_{3\to1}$

We consider a 3-to-1 Gaussian interference channel (IC) as depicted in Fig. 1. The messages W_i are encoded into codewords $\boldsymbol{X}_i = E_i(W_i, \theta_i)$, where $i \in [3]$, $\boldsymbol{X}_i = \{X_i(t) : \ t \in [T]\}$ is a codeword spanning T channel uses, E_i is the encoding function, and θ_i is private randomness available to Transmitter i. The messages W_i and the randomness θ_i are independent of each other and independent across users. Codewords $\boldsymbol{X}_i$ are subject to a unit transmit power constraint; i.e., $\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[|X_i(t)|^2] \leq 1$ for $i \in [3]$. Following the GDoF framework, the received signals in the t-th channel use are defined as,

$$Y_i(t) = \sum_{k=1}^{3} G_{ik}(t)\sqrt{P^{\alpha_{ik}}}X_k(t) + N_i(t), \tag{1}$$

where $\alpha_{ik} = 0$ for $i \in \{2, 3\}, k \in [3]$ and $i \neq k$. $\bar{P}$ is a nominal *power* variable, which approaches infinity to define the GDoF metric. The $N_i(t)$ are the zero-mean unit-variance additive white Gaussian noise terms. All symbols are real-valued. Let $\mathcal{G}$ be a set of random variables that are drawn from bounded-density distributions (see Definition 4 in [10]), whose realizations are known to all the receivers and none of the transmitters. For ease of exposition[1] let us assume that elements of $\mathcal{G}$ are i.i.d. uniform in $(1-\delta, 1+\delta)$ for some small fixed $\delta > 0$, in order to model finite precision CSIT.

[1]Generalization to all *bounded-density* distributions [9] is immediate.

The random coefficients $G_{ik}(t)$ in (1) are distinct elements of $\mathcal{G}$, i.e., $G_{ik}(t) \in \mathcal{G}$ for all $i, k \in [3], t \in [T]$. Note that we have $I(\{W_i, \boldsymbol{X}_i : \ i \in [3]\}; \mathcal{G}) = 0$. We assume a symmetric setting with channel strength parameters $\alpha_{11} = \alpha_{22} = \alpha_{33} = 1$ and $\alpha_{12} = \alpha_{13} = \alpha$.

Remark 1: The parameters α_{ik} correspond to the approximate values of the point to point channel capacities in the original finite-SNR channel setting to which the GDoF framework is applied. To briefly summarize the intuition behind the GDoF metric, note that the capacity of the point-to-point channel from Transmitter k to Receiver i in the GDoF model (1) is $\approx \frac{1}{2}\alpha_{ik}\log(P)$. Thus, the GDoF framework scales the original capacity of each link, α_{ik}, by the same nominal scaling factor, $\frac{1}{2}\log(P)$. Intuitively, this scales the capacity of the network approximately by the same scaling factor of $\frac{1}{2}\log(P)$ as well, so that normalizing all rates by $\frac{1}{2}\log(P)$ as is indeed done in the definition of GDoF (see equation (3)) provides an approximation to the original network capacity. Indeed the deterministic models of [2], which have been essential to various capacity approximations over the past decade, are specializations of the GDoF framework under perfect CSIT.

A rate tuple $(R_1, R_2, R_3) \in \mathbb{R}^3_+$ is achievable if, $\forall \epsilon > 0$, $\exists\ T > 0$, such that (i) each message is comprised of TR_i i.i.d. uniform bits, (ii) the average decoding error probability of each user is no larger than ϵ, and (iii) the following secrecy constraint is satisfied,

$$I_{\mathcal{G}}(W_{-i}; \boldsymbol{Y}_i) \leq T\epsilon, \quad \forall i \in [3] \tag{2}$$

where $W_{-i} = \{W_j : \ j \in [3], j \neq i\}$. The secure capacity region $\mathcal{C}_P$ is the closure of the set of all achievable secure rate tuples. Finally, the secure GDoF (SGDoF) region of the 3-to-1 interference channel under finite precision CSIT is defined as

$$\text{SGDoF}^{f.p.}_{3\to1} = \left\{ (d_1, d_2, d_3) \;\middle|\; \begin{array}{l} d_i = \lim\limits_{P\to\infty} \dfrac{R_i}{\frac{1}{2}\log P}, \\ \forall i \in [3], \\ (R_1, R_2, R_3) \in \mathcal{C}_P \end{array} \right\}. \tag{3}$$

III. Definitions

The following definitions are inherited from [10].

Definition 1 (Power levels):

$$\mathcal{X}_\lambda \triangleq \{0, 1, 2, \cdots, \bar{P}^\lambda - 1\}, \tag{4}$$

where $\lambda \geq 0$ and $\bar{P}^\lambda = \left\lfloor \sqrt{\bar{P}}^\lambda \right\rfloor$. We refer to λ as power level, and abbreviate $\bar{P}^1 = \bar{P}$.

Definition 2 (Segments): For non-negative real numbers X, λ_1 and λ_2 with $\lambda_1 \leq \lambda_2$, define

$$(X)^{\lambda_2}_{\lambda_1} \triangleq \left\lfloor \frac{\left(X - \bar{P}^{\lambda_2}\left\lfloor \frac{X}{\bar{P}^{\lambda_2}} \right\rfloor\right)}{\bar{P}^{\lambda_1}} \right\rfloor \tag{5}$$

as the segment of X between power levels λ_2 and λ_1. Intuitively, this segment is analogous to expressing X in $\bar{P}$-ary symbols and keeping only the segment of symbols from the λ_2^{th} most-significant symbol to the λ_1^{th} most significant symbol. Similarly, let us define $(\boldsymbol{X})^{\lambda_2}_{\lambda_1} \triangleq \left\{(X)^{\lambda_2}_{\lambda_1} : \ X \in \boldsymbol{X}\right\}$. For $X \in \mathcal{X}_\lambda$ and with $\lambda \geq \mu \geq 0$, the segment $(X)^{\lambda}_{\lambda-\mu}$ appears frequently in this work, so we denote it as $(X)^{\mu}$. It

can be intuitively interpreted as the μ most significant symbols in the $\bar{P}$-ary expansion of X. Similarly, we define $(\boldsymbol{X})^\mu = \{(X)^\mu :\ X \in \boldsymbol{X}\}$.

IV. Results

In Section IV-A, we identify channel regimes where the SGDoF region can be established with the known sum-set inequalities from [10]. For the remaining parameter regime where the SGDoF characterization is still open, we conjecture that new sum-set inequalities may be needed. Section IV-B formalizes this conjecture.

A. Utilizing existing AI bounds

The following theorem characterizes the SGDoF region for certain channel regimes where the known sum-set inequalities are shown to be sufficient.

Theorem 1: For all values of α except $1.5 < \alpha < 2$, the SGDoF are characterized as follows.

1) Case 1: $0 \le \alpha \le 1$

$$\text{SGDoF}^{f.p.}_{3\to 1} = \{(d_1,d_2,d_3) \mid (6),(7),(8) \text{ are satisfied.}\}$$

$$0 \le d_i \le 1, \qquad \forall i \in [3] \tag{6}$$
$$d_1 + d_j \le 2 - \alpha, \qquad \forall j = 2,3 \tag{7}$$
$$d_1 + d_2 + d_3 \le 3 - \alpha. \tag{8}$$

2) Case 2: $1 < \alpha \le 1.5$ or $\alpha \ge 2$

$$\text{SGDoF}^{f.p.}_{3\to 1} = \{(d_1,d_2,d_3) \mid (9),(10),(11) \text{ are satisfied.}\}$$

$$0 \le d_i \le 1, \qquad \forall i \in [3] \tag{9}$$
$$d_1 + d_j \le 1, \qquad \forall j = 2,3 \tag{10}$$
$$d_1 + \frac{1}{3 - \min\{\alpha, 2\}}(d_2 + d_3) \le 1. \tag{11}$$

Remark 2: The two-user bounds in (10) are not implied by the bounds for the 2-user Gaussian Z channel found in [23]. This is because even if we set the rate for a user to zero, that user's transmitter remains available as a potential helper (jammer) to enhance security for others, and that user's receiver remains active as an eavesdropping threat. Thus, setting the rate of a user to zero is not the same as removing that user, and the 2-user bounds in this work are indeed different from those found in [23].

B. Exploring new AI bounds

Despite our efforts in this direction, as noted in Theorem 1, the SGDoF region remains open when $1.5 < \alpha < 2$. Based on the insights available to us from these efforts, we currently believe that for this regime new AI bounds may be needed. Going one step further, we conjecture what these new AI bounds may look like.

Conjecture 1: For $t \in [T]$ and $k \in [3]$, let $X_k(t) \in \mathcal{X}_1$, and

$$Y(t) = \sum_{k=1}^{3} \lfloor G_k(t) X_k(t) \rfloor,\ Z(t) = \sum_{k=1}^{3} \lfloor H_k(t) X_k(t) \rfloor \tag{12}$$

where $\mathcal{G} \triangleq \{G_k(t), H_k(t) :\ t \in [T], k \in [3]\}$ are a set of i.i.d. uniform random variables in $(1-\delta, 1+\delta)$ for a given $\delta > 0$. For $\ell \in [n]$ and $t \in [T]$, we define $\lambda_\ell = 1 - \frac{\ell}{n}$, and

$$X_{k\ell}(t) = (X_k(t))_{\lambda_\ell}^{\lambda_{\ell-1}},\ Z_\ell(t) = (Z(t))_{\lambda_\ell}^{\lambda_{\ell-1}}, \tag{13}$$

$$L_\ell(t) = \sum_{l=1}^{\ell} \sum_{k=1}^{3} \left\lfloor h_{kl}^{\ell}(t) X_{kl}(t) \right\rfloor, \tag{14}$$

where h_{kl}^{ℓ} are arbitrary constants for all $k \in [3], l \in [\ell]$ and $\ell \in [n]$. We collect $Y(t), L_\ell(t)$, and $Z_\ell(t)$ for all $t \in [T]$ respectively as $\boldsymbol{Y}, \boldsymbol{L}_\ell$ and $\boldsymbol{Z}_\ell$, and denote $\boldsymbol{L}_\mathcal{A} = \{\boldsymbol{L}_\ell :\ \ell \in \mathcal{A}\}$ and $\boldsymbol{Z}_\mathcal{A} = \{\boldsymbol{Z}_\ell :\ \ell \in \mathcal{A}\}$, where $\mathcal{A} \subseteq [n]$. Then

$$H_\mathcal{G}(\boldsymbol{Y}|\mathcal{W}) \ge H_\mathcal{G}(\boldsymbol{L}_\mathcal{U}, \boldsymbol{Z}_\mathcal{V}|\mathcal{W}) + To(\log \bar{P}), \tag{15}$$

for any set of random variables $\mathcal{W}$ such that $I(\mathbf{X}, \mathcal{W}; \mathcal{G}) = 0$, and $\{\mathcal{U}, \mathcal{V}\}$ forms a partition of $[n]$.

Remark 3: Inequality (15) has a similar form to the existing sum-set inequalities, and is a generalization of Theorem 4 of [10] under the same setting, i.e., the number of inputs, the number of the power level partitions and their heights. This can be seen by setting $\mathcal{U} = [n]$ and $\mathcal{V} = \emptyset$ in (15), which leads to $H_\mathcal{G}(\boldsymbol{Y}|\mathcal{W}) \ge H(\boldsymbol{L}_1, \boldsymbol{L}_2, \cdots, \boldsymbol{L}_n|\mathcal{W})$.

The essence of this conjecture lies in the ability to choose arbitrary combinations of input segments as in $\boldsymbol{L}_\mathcal{U}$, or output segments as in $\boldsymbol{Z}_\mathcal{V}$ for each of the n power level partitions. Various special cases of this conjectured sum-set inequality have already been proved previously. At one extreme, if we always choose output segments, i.e., $\mathcal{U} = \emptyset$ and $\mathcal{V} = [n]$, then such a choice leads to the bound $H_\mathcal{G}(\boldsymbol{Y}|\mathcal{W}) \ge H_\mathcal{G}(\boldsymbol{Z}|\mathcal{W}) + To(\log \bar{P})$, which is implied by Theorem 4 of [10]. At the other extreme, if we always choose input segments, i.e., $\mathcal{U} = [n]$ and $\mathcal{V} = \emptyset$, then we have Theorem 4 of [10], as noted in Remark 3. Other special cases of the conjectured bound that are already covered by existing sum-set inequalities of [10], include the case where $\mathcal{V} = [m]$, $\mathcal{U} = [m+1:n]$, and $h_{i,\ell-m}^{\ell} = 1$ for $\ell \in [m+1:n]$ and zero otherwise. This leads to the bound

$$H_\mathcal{G}(\boldsymbol{Y}|\mathcal{W}) \ge H_\mathcal{G}((\boldsymbol{Z})^{\lambda_m}, {\boldsymbol{X}_i}^{1-\lambda_m}|\mathcal{W}) \tag{16}$$

which can also be obtained from [10].

Finally, while we omit the details for this observation, let us note that if Conjecture 1 is true, then the bounds (9), (10), (11) also describe the GDoF region for $1.5 < \alpha < 2$, i.e., Case 2 covers all $\alpha > 1$.

V. Proof

Case 1 is already proved in [16], so in this section we focus on the proof for Case 2.

A. Achievability

In robust settings (finite precision CSIT), the main challenge is to prove the information theoretic outer bounds, while the optimal achievable schemes are relatively quite straightforward. That is also the case here. Given that achievability is rather trivial, we will only briefly summarize it here. When $\alpha \ge 2$, the SGDoF region is reduced to a tetrahedron with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$. The first tuple is trivial.

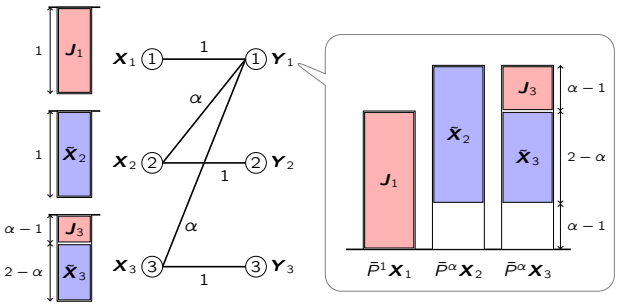

Fig. 2. Achievability of the SGDoF tuple $(d_1, d_2, d_3) = (0, 1, 2-\alpha)$.

For the tuple $(0, 1, 0)$, User 2 achieves 1 DoF by encoding the message W_i into $\boldsymbol{X}_i$ with a wiretap codebook, while Transmitter 3 transmits a jamming signal to prevent Receiver 1 from eavesdropping. The tuple $(0, 0, 1)$ is similarly achieved and time-sharing covers the entire SGDoF region.

On the other hand, when $1 < \alpha \leq 1.5$, the SGDoF region is a polyhedron with vertices $(0, 1, 0), (0, 0, 1), (1, 0, 0), (0, 1, 2-\alpha)$ and $(0, 2-\alpha, 1)$. It suffices to show the achievability of $(0, 1, 2-\alpha)$ as the other tuples either follow from symmetry or can be achieved as described previously. As illustrated in Fig. 2, User 2 achieves his maximum 1 DoF with a wiretap codebook. To prevent Receiver 1 from eavesdropping, Transmitter 1 sends a noise signal that jams everything up to power level 1, while Transmitter 3 sends a jamming signal that hides the remaining top $\alpha - 1$ power levels from Receiver 1. This leaves Transmitter 3 with the bottom $1-(\alpha-1) = 2-\alpha$ power levels that it can use to communicate with his desired receiver, thus achieving $d_3 = 2 - \alpha$.

B. Converse

The single-user bounds (9) are trivial. To show the other bounds (10) and (11), we need the following deterministic model.

1) Deterministic model: We associate the Gaussian model (1) with the following deterministic model:

$$\bar{Y}_1(t) = \sum_{k=1}^{3} \left\lfloor G_{1k}(t)\bar{X}_k(t) \right\rfloor, \tag{17}$$

$$\bar{Y}_i(t) = \left\lfloor G_{ii}(t)\bar{P}^{1-\alpha}\bar{X}_i(t) \right\rfloor, i = 2, 3 \tag{18}$$

where $t \in [T]$, and the new inputs $\bar{X}_1(t) = \bar{E}_{1,t}(W_1, \theta_1) \in \mathcal{X}_1$, and $\bar{X}_i(t) = \bar{E}_{i,t}(W_i, \theta_i) \in \mathcal{X}_\alpha$ $(i = 2, 3)$ are codewords independent of $\mathcal{G}$. As shown in Lemma 5.1 of [9], the SGDoF region of this deterministic model forms an outer bound of the SGDoF region of the original Gaussian model (1).

2) The Bounds $d_1 + d_j \leq 1$ $(j = 2, 3)$, for $\alpha \geq 1$: Here we take $j = 2$ as an example and case $j = 3$ can be similarly shown. We start from bounding $H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1|W_1)$ from below. Terms of the form $To(\log \bar{P})$, which are inconsequential for GDoF, are omitted for brevity.

$$H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1|W_1) \geq H_{\mathcal{G}}(\bar{\boldsymbol{Z}}|W_1) \tag{19}$$

$$= H_{\mathcal{G}}(\bar{\boldsymbol{Z}}) \tag{20}$$

$$= H_{\mathcal{G}}(\bar{\boldsymbol{Z}}, W_2) \tag{21}$$

$$\geq TR_2 + H_{\mathcal{G}}(\bar{\boldsymbol{Z}}|W_2) \tag{22}$$

$$\geq TR_2 + H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{\alpha-1}|W_2) \tag{23}$$

$$= TR_2 + H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{\alpha-1}), \tag{24}$$

where $\bar{\boldsymbol{Z}} = \{\bar{Z}(t) : t \in [T]\}$ and $\bar{Z}(t) = \left\lfloor G_2(t)\bar{X}_2(t) \right\rfloor + \left\lfloor G_3(t)(\bar{X}_3(t))^{\alpha-1} \right\rfloor$, and $G_1(t), G_2(t) \in \mathcal{G}$ for all $t \in [T]$. Inequality (19) follows from Theorem 4 of [10]. The independence of $\bar{\boldsymbol{Z}}$ and W_1 gives us (20). Note that $\bar{\boldsymbol{X}}_2$ appears in $\bar{\boldsymbol{Z}}$ with strength α whereas $\bar{\boldsymbol{X}}_3$ appears in $\bar{\boldsymbol{Z}}$ with reduced strength $\alpha-1$, thus the top 1 power levels of $\bar{\boldsymbol{Z}}$ expose $(\bar{\boldsymbol{X}}_2)^1$. In other words, from $\bar{\boldsymbol{Z}}$ one can obtain $(\bar{\boldsymbol{X}}_2)^1$ within bounded distortion, from which W_2 can be decoded. This gives us (21). By the chain rule of entropy we have (22). Then we apply Theorem 4 of [10] once again to obtain (23). Finally, (24) follows from the security constraint (2). Next we apply Fano's inequality on User 1, and get

$$TR_1 \leq I_{\mathcal{G}}(\boldsymbol{Y}_1; W_1) \tag{25}$$

$$\leq H_{\mathcal{G}}(\boldsymbol{Y}_1) - H_{\mathcal{G}}((\boldsymbol{Y}_1)^{\alpha-1}) - TR_2 \tag{26}$$

$$\leq H_{\mathcal{G}}(\boldsymbol{Y}_1 \mid (\boldsymbol{Y}_1)^{\alpha-1}) - TR_2 \tag{27}$$

$$\leq T \log \bar{P} - TR_2, \tag{28}$$

where inequality (26) is obtained by expanding mutual information as a difference of entropies and substituting from (24). In the next step, (27) follows from the identity $H(A) - H(B) \leq H(A \mid B)$. Next, (28) follows because after the conditioning on its top $\alpha - 1$ power levels, the remaining 1 power level of $\bar{\boldsymbol{Y}}_1$ cannot carry more than 1 GDoF. Rearranging (28), normalizing both sides by $T \log \bar{P} = \frac{1}{2} T \log(P)$, and applying the GDoF limit, we get the desired bound (10), i.e., $d_1 + d_2 \leq 1$, which concludes the proof.

3) The Bound $d_1 + d_2 + d_3 \leq 1$ for $\alpha \geq 2$. : Similar to the procedure in Section V-B2, we start by bounding $H_{\mathcal{G}}(\boldsymbol{Y}_1|W_1)$ from below, as follows.

$$H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1|W_1)$$

$$\geq H_{\mathcal{G}}((\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{Y}}_1)^{\alpha-1}|W_1) \tag{29}$$

$$= H_{\mathcal{G}}((\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1, (\bar{\boldsymbol{Y}}_1)^{\alpha-1}|W_1) \tag{30}$$

$$= H_{\mathcal{G}}((\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1, (\bar{\boldsymbol{Y}}_1)^{\alpha-1}) \tag{31}$$

$$\geq T(R_2 + R_3) + H_{\mathcal{G}}((\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1, (\bar{\boldsymbol{Y}}_1)^{\alpha-1}|W_2, W_3) \tag{32}$$

$$\geq T(R_2 + R_3) + H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{\alpha-1}|W_2, W_3) \tag{33}$$

$$= T(R_2 + R_3) + H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{\alpha-1}). \tag{34}$$

First we apply[2] Theorem 4 of [10] to obtain inequality (29). Then (30) follows because one can cancel $(\bar{\boldsymbol{X}}_2)^1$ from $(\bar{\boldsymbol{Y}}_1)^{\alpha-1}$ and extract $(\bar{\boldsymbol{X}}_3)^1$ within bounded distortion. Such an operation is possible since $\alpha - 1 \geq 1$. The next step, (31) drops the conditioning on W_1 because none of the terms contain $\bar{\boldsymbol{X}}_1$, and therefore all terms are independent of W_1 (up to the GDoF approximation). Note that $\bar{\boldsymbol{X}}_1$ appears only in the bottom 1 power level of $\bar{\boldsymbol{Y}}_1$, which has zero overlap with the top $\alpha - 1$ power levels of $\bar{\boldsymbol{Y}}_1$ because $\alpha \geq 2$. Next, (32) holds because we can decode W_2 and W_3 respectively

[2]Using the notations in [10], we let $l = 2, \lambda_1 = 1, \lambda_2 = \alpha - 1, I_1 = I_2 = \{2\}$, and choose $\vec{\gamma_1}, \vec{\gamma_2}, \vec{\delta_1}, \vec{\delta_1}$ properly, so that $Z_1(t) = \left\lfloor G_{12}(t)(X_2(t))^{\alpha-1} \right\rfloor + \left\lfloor G_{13}(t)(X_3(t))^{\alpha-1} \right\rfloor$, and $Z_2(t) = (X_2(t))^1$ for all $t \in [T]$.

from $(\bar{\boldsymbol{X}}_2)^1$ and $(\bar{\boldsymbol{X}}_3)^1$. Then we get (33) by dropping off $(\bar{\boldsymbol{X}}_2)^1$ and $(\bar{\boldsymbol{X}}_3)^1$, and arrive at (34) by applying the security constraint (2). Now, we again apply Fano's inequality to User 1, and get

$$\begin{aligned} TR_1 &\le I_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1; W_1) && (35)\\ &\le H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1) - H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{\alpha-1}) - T(R_2+R_3) && (36)\\ &\le T\log\bar{P} - T(R_2+R_3), && (37)\end{aligned}$$

where we apply the definition of mutual information and substitute from (34) to obtain (36). The next step uses the same reasoning as was used to get (28) from (26). Rearranging terms and applying the GDoF limit to (37) we obtain the desired bound (11) for $\alpha \ge 2$, i.e., $d_1+d_2+d_3 \le 1$, which concludes the proof.

4) The weighted sum bound for $1 < \alpha \le 1.5$: In this section we show the weighted sum bound (11) when $1 < \alpha \le 1.5$. First we apply Fano's inequality to User 1 and get

$$\begin{aligned} TR_1 &\le I_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1; W_1) && (38)\\ &\le I_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1; W_1 | (\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1}) && (39)\\ &\le T\log\bar{P} - H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | W_1, (\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1}). && (40)\end{aligned}$$

Inequality (39) holds because $(\bar{\boldsymbol{X}}_2)^{\alpha-1}$ and $(\bar{\boldsymbol{X}}_3)^{\alpha-1}$ are independent of W_1, and the identity that $I(A;B) \le I(A; B \mid C)$ holds whenever A is independent of C. Then (40) follows by applying the definition of mutual information and bounding the first entropy term.

To find a lower bound for $H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | W_1, (\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1})$ in (40), we make use of the following lemma, whose proof is relegated to Sec. V-C

Lemma 1: Let r be a rational number satisfying $0 < r \le \alpha - 1$. Then

$$\begin{aligned} &H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | W_1, (\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1})\\ &\ge \frac{T}{2-r}\left(R_2+R_3+(r+1-\alpha)\log\bar{P}\right) + To(\log\bar{P}) \quad (41)\end{aligned}$$

Let $\{r_\ell \in \mathbb{Q} :\ \ell \in \mathbb{N}, 0 < r \le \alpha - 1\}$ be a non-decreasing sequence with $\lim_{\ell\to\infty} r_\ell = \alpha - 1$. [3] Then by Lemma 1, for all $\ell \in \mathbb{N}$, we have (with $To(\log\bar{P})$ terms omitted as usual)

$$\begin{aligned} &H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | W_1, (\bar{\boldsymbol{X}}_2))^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1})\\ &\ge \frac{1}{2-r_\ell} T(R_2+R_3) + \left(\frac{3-\alpha}{2-r_\ell} - 1\right) T\log\bar{P} \quad (42)\end{aligned}$$

Since the right-hand side is a continuous function of r_ℓ in the neighborhood of $\alpha - 1$, and $r_\ell \to \alpha - 1$ as $\ell \to \infty$, we have

$$\begin{aligned} &\lim_{P\to\infty} \frac{1}{T\log\bar{P}} H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | W_1, (\bar{\boldsymbol{X}}_2))^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1})\\ &\ge \lim_{\ell\to\infty} \frac{1}{2-r_\ell}(d_2+d_2) + \left(\frac{3-\alpha}{2-r_\ell} - 1\right) \quad (43)\\ &= \frac{1}{3-\alpha}(d_2+d_3) \quad (44)\end{aligned}$$

By plugging (44) back into (40) and applying (3), we get the desired bound (11) for $1 < \alpha \le 1.5$. This concludes the proof.

[3] For example, choose $r_\ell = \left\lfloor (\alpha-1)\times 10^{\ell+\ell^*} \right\rfloor / 10^{\ell+\ell^*}$, where $\ell^* = \min\{\ell :\ \lfloor (\alpha-1)\times 10^{\ell}\rfloor \ne 0\}$.

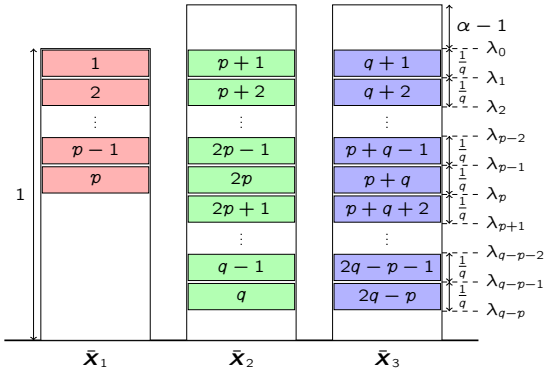

Fig. 3. The mapping between $\bar{\boldsymbol{U}}_\ell$ and segments of $\bar{\boldsymbol{X}}_k$ defined in (45)

C. Proof of Lemma 1

Since r is rational, we let $r = \frac{p}{q}$, where $p \ne 0, q$ are coprime integers. Also we define $\lambda_\ell = 1 - \frac{\ell}{q}$, and

$$\bar{\boldsymbol{U}}_\ell = \begin{cases} (\bar{\boldsymbol{X}}_1)_{\lambda_\ell}^{\lambda_{\ell-1}} & \ell \in [1:p]\\ (\bar{\boldsymbol{X}}_2)_{\lambda_{\ell-p}}^{\lambda_{\ell-p-1}} & \ell \in [p+1:q]\\ (\bar{\boldsymbol{X}}_3)_{\lambda_{\ell-q}}^{\lambda_{\ell-q-1}} & \ell \in [q+1:2q-p]. \end{cases} \quad (45)$$

Fig. 3 depicts the mapping between index ℓ and the segments of $\bar{\boldsymbol{X}}_k$, where $k \in [3]$. Then by applying Theorem 4 of [10], for $\ell = 1, 2, \cdots, 2q-p$ we have

$$H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | \mathcal{W}) \ge H(\bar{\boldsymbol{U}}_\ell, \bar{\boldsymbol{U}}_{\ell+1}, \cdots, \bar{\boldsymbol{U}}_{\ell+q-1} | \mathcal{W}), \quad (46)$$

where modulo-$(2q-p)$ is implicitly applied on the indices, and $\mathcal{W} = \{W_1, (\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1}\}$. On the right hand side $\mathcal{G}$ is removed because it is independent of $\{\bar{\boldsymbol{U}}_\ell :\ \ell \in [2q-p]\}$.

Next we sum up the $2q-p$ inequalities in (46), and get

$$\begin{aligned} &(2q-p) H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1 | \mathcal{W})\\ &\ge \sum_{\ell=1}^{2q-p} H(\bar{\boldsymbol{U}}_\ell, \bar{\boldsymbol{U}}_{\ell+1}, \cdots, \bar{\boldsymbol{U}}_{\ell+q-1} | \mathcal{W}) \quad (47)\\ &\ge qH(\bar{\boldsymbol{U}}_1, \bar{\boldsymbol{U}}_2, \cdots, \bar{\boldsymbol{U}}_{2q-p} | \mathcal{W}) \quad (48)\\ &\ge qH((\bar{\boldsymbol{X}}_1)^{\frac{p}{q}}, (\bar{\boldsymbol{X}}_2)_{\frac{p}{q}}^1, (\bar{\boldsymbol{X}}_3)_{\frac{p}{q}}^1 | \mathcal{W}) \quad (49)\\ &= qH((\bar{\boldsymbol{X}}_1)^r, (\bar{\boldsymbol{X}}_2)_r^1, (\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)_r^1, (\bar{\boldsymbol{X}}_3)^{\alpha-1} | W_1)\\ &\qquad - qH((\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1} | W_1) \quad (50)\\ &\ge qH((\bar{\boldsymbol{X}}_1)^r, (\bar{\boldsymbol{X}}_2)_{\alpha-1}^{\alpha}, (\bar{\boldsymbol{X}}_3)_{\alpha-1}^{\alpha} | W_1)\\ &\qquad - qH((\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1} | W_1) \quad (51)\\ &\ge qH((\bar{\boldsymbol{X}}_1)^{\alpha-1}, (\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1 | W_1) - q(\alpha-1-r)T\log\bar{P}\\ &\qquad - qH((\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1} | W_1) \quad (52)\end{aligned}$$

Inequality (48) holds due to the submodularity of entropy. The next inequality, (49) holds because we can recover $(\bar{\boldsymbol{X}}_1)^{p/q}$ from $\{\bar{\boldsymbol{U}}_\ell :\ \ell \in [p]\}$ within bounded distortion, and likewise $(\bar{\boldsymbol{X}}_2)_{p/q}^1$ is obtained from $\{\bar{\boldsymbol{U}}_\ell :\ \ell \in [p+1:q]\}$ and $(\bar{\boldsymbol{X}}_3)_{p/q}^1$ from $\{\bar{\boldsymbol{U}}_\ell :\ \ell \in [q+1:2q-p]\}$. Then (50) follows by the chain rule of entropy and $r = \frac{p}{q}$. Note that one can recover $(\bar{\boldsymbol{X}}_k)_{\alpha-1}^{\alpha}$ from $(\bar{\boldsymbol{X}}_k)^{\alpha-1}$ and $(\bar{\boldsymbol{X}}_k)_r^1$ within bounded distortion for $k = 2, 3$; so we have inequality (51). Finally in

inequality (52), we apply the uniform bound on the entropy of $(\bar{\boldsymbol{X}}_1)_{2-\alpha}^{1-r}$ and recognize that $(\bar{\boldsymbol{X}}_k)_{\alpha-1}^{\alpha} = (\bar{\boldsymbol{X}}_k)^1$, for $k = 2, 3$.

Then we bound $H((\bar{\boldsymbol{X}}_1)^{\alpha-1}, (\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1|W_1)$ in (52) further from below as follows

$$\begin{aligned}
&H((\bar{\boldsymbol{X}}_1)^{\alpha-1}, (\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1|W_1) \\
&\geq H((\bar{\boldsymbol{X}}_1)^{\alpha-1}, (\bar{\boldsymbol{X}}_2)^1, (\bar{\boldsymbol{X}}_3)^1|W_1, W_2, W_3) + T(R_2 + R_3) && (53)\\
&\geq H((\bar{\boldsymbol{X}}_1)^{\alpha-1}, (\bar{\boldsymbol{X}}_2)^{2(\alpha-1)}, (\bar{\boldsymbol{X}}_3)^{2(\alpha-1)}|W_1, W_2, W_3) \\
&\qquad + T(R_2 + R_3) && (54)\\
&\geq H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{2(\alpha-1)}|W_1, W_2, W_3) + T(R_2 + R_3) && (55)\\
&= H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{2(\alpha-1)}|W_1) + T(R_2 + R_3) && (56)
\end{aligned}$$

Inequality (53) holds because W_2 and W_3 can be decoded respectively from $(\bar{\boldsymbol{X}}_2)^1$ and $(\bar{\boldsymbol{X}}_3)^1$, and then by the chain rule of entropy. Next we note that $(\bar{\boldsymbol{X}}_k)^{2(\alpha-1)}$ is a function of $(\bar{\boldsymbol{X}}_k)^1$ due to the fact that $2(\alpha-1) \leq 1$, where $k = 2, 3$, so we have (54). Next, (55) holds because $\mathcal{G}$ is independent of $\bar{\boldsymbol{X}}_k$ ($k \in [3]$), and $(\bar{\boldsymbol{Y}}_1)^{2(\alpha-1)}$ can be recovered with $\mathcal{G}$, $(\bar{\boldsymbol{X}})^{\alpha-1}$ and $(\bar{\boldsymbol{X}}_k)^{2(\alpha-1)}$ with $k = 2, 3$ within bounded distortion. Finally, we apply the secrecy constraint (2) to obtain (56).

Plugging (56) back into (52), we get

$$\begin{aligned}
H_{\mathcal{G}}(\bar{\boldsymbol{Y}}_1|\mathcal{W}) \geq \frac{1}{2-r}\Big[&T(R_2 + R_3) - (\alpha - 1 - r)T\log\bar{P} \\
&+ H_{\mathcal{G}}((\bar{\boldsymbol{Y}}_1)^{2(\alpha-1)}|W_1) \\
&- H((\bar{\boldsymbol{X}}_2)^{\alpha-1}, (\bar{\boldsymbol{X}}_3)^{\alpha-1}|W_1)\Big] && (57)\\
= \frac{T}{2-r}&\left(R_2 + R_3 + (r + 1 - \alpha)\log\bar{P}\right), && (58)
\end{aligned}$$

which is the desired lower bound of Lemma 1. We apply Theorem 4 of [10] to the first entropy term of (57) to obtain (58). This concludes the proof.

VI. Conclusion

Taking the problem of characterizing the secure GDoF of a 3-to-1 interference channel under finite precision CSIT as an instance, we explore the utility and limitations of existing sum-set inequalities. On one hand we identify parameter regimes in which the known sum-set inequalities are sufficient, and settle the GDoF for these regimes as a byproduct. On the other hand, for the remaining regime we posit that the existing sum-set inequalities might not be enough to obtain tight GDoF bounds. Based on this regime, we conjecture a generalized sum-set inequality whose validity is currently an open problem for our ongoing work.

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