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Structural Engineering
Mechanics and Materials

SEMM Graduate Program Primer: 2020

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Welcome to the students

Dear students,
This primer has been prepared to help you in your transition into the SEMM graduate program at Berkeley. You are entering our program from a wide variety of schools that in our estimation have prepared you well for a successful graduate career. However, because of this diversity of backgrounds, there is no uniform base of knowledge especially when it comes to smaller technical points. Thus to help everyone engage with their new courses with essentially the same background knowledge, we have prepared these notes as a common and easy reference source for a collection of fundamental concepts that will arise during your studies at Berkeley. Should you have the opportunity, the best course of action is to familiarize yourself with the contents before classes start. If not, then this primer can serve as a reference for quick study if you come across unfamiliar fundamentals during the course of your studies in our program.

Many of the topics you will have seen and remember well, others you will only vaguely recall, and some may even be new to you. If you are confident in the topics, enjoy the refresher. If you only vaguely recall them, then take the opportunity to re-engage with them. If the topics are new, take a little extra time to digest them and maybe even follow up with some of the given references – familiarity and freshness in your mind are the most important issues, so that when you encounter them in class you have confidence in your knowledge and can benefit from the discussions in class.

Some of the sections also have *check-your-understanding* questions, so that you can independently test your knowledge. Answers are not given, but we will have posted office hours with some of our current graduate students and you can discuss your solutions with them.

As a guide to reading, the relevance of the material in the chapters that follow can be mapped to the introductory Fall semester courses as shown in Table 1.

	CE122	CE124	CE193	CE220	CE225	CE231	CE240	CE244
Chapter 1			H	H	M	H	L	
Chapter 2			M	M	H	H	L	
Chapter 3	L	L	H	L	L	M		M
Chapter 4						M	L	
Chapter 5	M	M	M	H	H	M	M	M
Chapter 6	H	H		L	L			H
Chapter 7							H	

Table 1: Relevance of chapters to the introductory Fall semester courses: High (H), Medium (M), Low (L), Not Applicable (blank).

Contents

Chapter 1

Elements of linear algebra

1.1 Introduction

The response of discrete models for structural and mechanical systems involves relations between vectors and matrices. Concepts and methods of linear algebra are an indispensable help for the understanding of the relations between response variables. The following sections of this chapter give a summary-like introduction to the most important concepts of linear algebra. For further information on any of the topics, the reader is referred to the textbook of Strang [2016].

1.2 Matrix and vector operations

Matrices and vectors are commonly used to describe physical quantities. Examples of matrices that are frequently encountered are stiffness matrices, stress matrices(tensors), and moment of inertia matrices(tensors). Vectors are also encountered often, for example to describe displacements, velocities, and forces. At their most elementary level matrices and vectors are just convenient ways to organize and manipulate information.

Matrices are two dimensional arrays of numbers with a certain number of rows and columns:

$$\boldsymbol{A}_{n\times m} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{2n} & \cdots & A_{nm} \end{bmatrix} .$$

In the example above there are n rows and m columns, and each entry is a number. Vectors can be thought of as matrices with just one row or one column. For the most part in structural engineering we use column vectors:

$$\boldsymbol{v}_{n\times 1} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} .$$

To effectively use matrices and vectors, it is important to be familiar with a number of *operations*. Below, we enumerate some basic ones.

1.2.1 Transpose

The transpose of a matrix is a new matrix that swaps the rows and columns of the original matrix. Operationally, the transpose of a matrix $\boldsymbol{A}$ is the matrix $\boldsymbol{A}^T$ where the entry A^T_{ij} takes on the value A_{ji}. Note that if $\boldsymbol{A}$ has n columns and m rows, then $\boldsymbol{A}^T$ will have m columns and n rows.

Example 1.1 (Transpose of a 2×3 matrix)**.** Suppose we are given the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 7 \end{bmatrix}.$$

Its transpose will then be given by

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 2 \\ 5 & 3 \\ 6 & 7 \end{bmatrix}.$$

Symmetric matrices

If a matrix is equal to its transpose, $\boldsymbol{A} = \boldsymbol{A}^T$, then we say that the matrix $\boldsymbol{A}$ is *symmetric*. Symmetry of matrices is very common in physical problems and implies a number of convenient properties. Common symmetric matrices are stiffness matrices and stress matrices. Note also that a matrix must be square for it to possibly be symmetric.

Skew-symmetric matrices

If a matrix is equal to the negative of its transpose, $\boldsymbol{A} = -\boldsymbol{A}^T$, then we say that the matrix $\boldsymbol{A}$ is *skew-symmetric*. Skew-symmetry of matrices is less common in physical problems but one encounters them when dealing with rotational motion. Note also that a matrix must be square for it to possibly be skew-symmetric.

Example 1.2 (Skew-symmetric matrix)**.** The matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 5 & -6 \\ -5 & 0 & 7 \\ 6 & -7 & 0 \end{bmatrix}$$

is skew-symmetric. This follows, since its transpose is given by

$$\boldsymbol{A}^T = \begin{bmatrix} 0 & -5 & 6 \\ 5 & 0 & -7 \\ -6 & 7 & 0 \end{bmatrix},$$

showing that $\boldsymbol{A} = -\boldsymbol{A}^T$. Note that the diagonal of a skew-symmetric matrix is always zero, and the entries on opposites sides of the diagonal differ by their algebraic sign.

1.2.2 Vector products

Vector products or dot products are products of two vectors that produce a number (scalar). They are sometimes also called scalar products; sometimes they are called inner products. Given two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, their vector product is defined as

$$\boldsymbol{v} \cdot \boldsymbol{w} \stackrel{\text{def}}{=} \sum_{i=1}^{n} v_i w_i = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \,.$$

Note that for this definition to make sense, the two vectors need to have the same number of entries(components). Operationally, one takes corresponding entries, multiplies them together, and then adds up all the products.

Example 1.3 (Dot product between two 2×1 vectors)**.** Consider the two vectors

$$\boldsymbol{f} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{g} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}.$$

Their dot product is given by

$$\boldsymbol{f} \cdot \boldsymbol{g} = 5 \times (-1) + (-1) \times (-5) = -5 + 5 = 0\,.$$

Whenever the dot product of two vectors is equal to zero, we say that the vectors are orthogonal. In regular three-dimensional space, this tells us that the vectors span an angle of 90 degrees in their plane.

1.2.3 Vector norms

The norm of a vector represents its length or magnitude. It is defined via the dot product as

$$\|\boldsymbol{v}\| \stackrel{\text{def}}{=} \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{\sum_{i=1}^{n} (v_i)^2 = (v_1)^2 + (v_2)^2 + \cdots + (v_n)^2}\,.$$

It is useful to note that the dot product of vectors in three-dimensional space is equal to the product of the vector norms times the cosine of the angle between them:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \, \|\boldsymbol{w}\| \cos \theta_{vw} \,,$$

where θ_{vw} is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

1.2.4 Matrix vector multiplication

Matrix vector products can occur in many different contexts. The product of a matrix and a vector generates a new vector. This operation is defined as

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{2n} & \cdots & A_{nm} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

where

$$c_i \stackrel{\text{def}}{=} \sum_{j=1}^{m} A_{ij} b_j \,.$$

Operationally the i^{th} entry of $\boldsymbol{c}$ is computed by taking the dot product of the i^{th} row of $\boldsymbol{A}$ with the vector $\boldsymbol{b}$. In compact form, one can write

$$\boldsymbol{c} = \boldsymbol{A}\boldsymbol{b} \,. \tag{1.1}$$

Note that the number of columns of $\boldsymbol{A}$ must match the number of rows of $\boldsymbol{b}$ for the definition to make sense. The number of rows of the resulting vector $\boldsymbol{c}$ matches the number of rows of $\boldsymbol{A}$.

Example 1.4 (Product of a matrix and a vector)**.** Consider the following matrix and vector

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}.$$

Their product will be a new vector

$$\boldsymbol{c} = \boldsymbol{A}\boldsymbol{b} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \times 6 + 2 \times 7 \\ 3 \times 6 + 4 \times 7 \end{pmatrix} = \begin{pmatrix} 20 \\ 46 \end{pmatrix}.$$

Another way of thinking of the matrix-vector product in (1.1) is that the vector $\boldsymbol{c}$ is the linear combination of the columns of matrix $\boldsymbol{A}$ with the components of the vector $\boldsymbol{b}$ serving as factors. All possible linear combinations of the columns of $\boldsymbol{A}$ form the *column space* of matrix $\boldsymbol{A}$. By the same argument all possible combinations of the rows of $\boldsymbol{A}$ form the *row space* of matrix $\boldsymbol{A}$. Further details may be found in Strang [2016, Section 3.1].

Positive definite

A physically important class of matrices are those that are *positive definite*. A positive definite matrix is a square matrix $\boldsymbol{A}$ such that

$$\boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{v} > 0$$

for all non-zero vectors $\boldsymbol{v}$. The mass matrix of a structure is an example of a positive definite matrix.

1.2.5 Matrix matrix products

It is also possible to multiply two matrices. The product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is a third matrix $\boldsymbol{C}$:

$$\boldsymbol{C}_{n\times p} = \boldsymbol{A}_{n\times q}\boldsymbol{B}_{q\times p}\,,$$

where the entry in the i^{th} row and j^{th} column of $\boldsymbol{C}$ is defined as

$$C_{ij} \stackrel{\text{def}}{=} \sum_{k=1}^{q} A_{ik}B_{kj}\,.$$

For this definition to make sense, the number of columns of $\boldsymbol{A}$ must equal the number of rows of $\boldsymbol{B}$. Operationally, the C_{ij} entry of $\boldsymbol{C}$ is the dot product of the i^{th} row of $\boldsymbol{A}$ with the j^{th} column of $\boldsymbol{B}$.

Example 1.5 (Product of two matrices)**.** Consider the following two matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

Their product will be a new matrix

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 2\times 3 & 1\times 4 + 2\times 2 \\ 3\times 1 + 4\times 3 & 3\times 4 + 4\times 2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 15 & 20 \end{bmatrix}.$$

1.2.6 Matrix inverse

Given a square matrix $\boldsymbol{A}$, its inverse $\boldsymbol{A}^{-1}$ is the matrix with the property that

$$\boldsymbol{A}^{-1}_{n\times n}\boldsymbol{A}_{n\times n} = \boldsymbol{I}_{n\times n}\,,$$

where $\boldsymbol{I}$ is the identity matrix. All the entries of $\boldsymbol{I}$ are zero except the diagonal entries, which are unity (equal to one). Note that the identity has the special property that $\boldsymbol{I}\boldsymbol{v} = \boldsymbol{v}$ for every vector $\boldsymbol{v}$. Further note that not all matrices have inverses. To have an inverse, a matrix must be *non-singular*. One possible test for checking whether or not a square matrix is singular is to compute its determinant. If $\det\boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is non-singular and it has an inverse.

1.3 Eigenvalue problems

Multiplying a vector by a matrix produces a new vector. If that new vector is a scaled version of the original vector, then we call such a vector an *eigenvector* and the scaling factor the corresponding *eigenvalue*. Note the original vector and the new vector need to have the same number of components for this to make sense. Thus, only square matrices can have eigenvectors and eigenvalues. In equation form, if

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}\,, \tag{1.2}$$

then $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ and λ is the eigenvalue. Eigenvalues and eigenvectors have many uses, for example they correspond to principal directions and stresses; they are intimately related to vibrational frequencies and modes of vibration.

To find the eigenvalues, one can rearrange (1.2) to read

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\,\boldsymbol{v} = \boldsymbol{0}\,,$$

where $\boldsymbol{0}$ is the vector of all zeros. Assuming $\boldsymbol{A}$ is an $n \times n$ matrix, this equation represents a system of n equations for the components of $\boldsymbol{v}$. Since the right-hand size is zero, this is a system of homogeneous equations (see Sec. 1.4.1) and can only have non-trivial (non-zero) solutions when

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0\,. \tag{1.3}$$

Computing the determinant, will generate an n^{th} order polynomial in λ. The roots of this polynomial (the characteristic polynomial of $\boldsymbol{A}$) provide the eigenvalues. Thus there will be n possible eigenvalues. Corresponding to each eigenvalue, there will be a solution to (1.2) for the related eigenvector (leaving aside some technical details associated with repeated eigenvalues).

The details of the computations for the eigenvalues and eigenvectors are generally not too important as the operations are normally not executed by hand, but rather by use of specialized software. Also observe that, if a vector is an eigenvector, then so is any scalar multiple of it. Thus when eigenvectors are reported, they are usually normalized to have unit length, $\|\boldsymbol{v}\| = 1$. Further, it is useful to know that symmetric matrices have real-valued eigenvalues and their eigenvectors are mutually orthogonal.

1.3.1 Generalized eigenvalue problem

The eigenvalue problem described above is the classic eigenvalue problem. In structural engineering, one also encounters a slight variant known as the generalized eigenvalue problem. In the generalized case, one starts with two square matrices of the same size, say, $\boldsymbol{K}$ and $\boldsymbol{M}$. Then one looks for vectors, that after multiplication by both $\boldsymbol{K}$ and $\boldsymbol{M}$, point in the same direction. Thus the defining equation in the generalized case is

$$\boldsymbol{K}\boldsymbol{v} = \lambda \boldsymbol{M}\boldsymbol{v}\,.$$

The eigenvalues are then found similarly to the classic case as the roots of an n^{th} order polynomial in λ,

$$\det(\boldsymbol{K} - \lambda \boldsymbol{M}) = 0\,.$$

In the generalized case the eigenvectors are usually normalized so that $\boldsymbol{v} \cdot \boldsymbol{M}\boldsymbol{v} = 1$. Further, when $\boldsymbol{K}$ and $\boldsymbol{M}$ are both symmetric, then the eigenvectors possess the property that they are *mass orthogonal*, which means if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors for different eigenvalues, then $\boldsymbol{v} \cdot \boldsymbol{M}\boldsymbol{w} = 0$. This situation arises in structural vibrations.

1.4 Systems of linear equations

The linear response of discrete models for structural and mechanical systems is described by relations of the form (1.1), rewritten here as

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\,. \tag{1.4}$$

Here we seek the solution $\boldsymbol{x}$ of the linear system of equations in (1.4) for a given matrix $\boldsymbol{A}$ and given vector $\boldsymbol{b}$. The general solution of this linear system depends on the properties of the matrix $\boldsymbol{A}$, which may not be square.

For the case that the matrix $\boldsymbol{A}$ is square, a unique solution exists if the rank of the matrix is equal to the number of rows m (and the number of columns $n = m$). Such a matrix is said to have *full rank*. A square matrix of full rank has *linearly independent* rows and columns, meaning that it is not possible to express a row or column as the linear combination of the other rows or columns. It is possible to establish the rank of a matrix by transforming it to *reduced row-echelon form* and counting the number of non-zero pivots in the process, as discussed in Strang [2016, Section 3.2].

Example 1.6 (Solution of system of equations)**.** Consider the following linear system of equations

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 0.3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

It can be established that the coefficient matrix has full rank. This would not be the case if the coefficient matrix were for example

$$\begin{bmatrix} 1 & 0.8 \\ 0.5 & 0.4 \end{bmatrix} ,$$

since in this case the first row is a multiple of the second row by the factor 2.

Returning to our original problem, we can use Gauss elimination to get the solution, multiplying the first row by 0.5 and subtracting it from the seecond to get

$$\begin{pmatrix} 3 \\ 2.5 \end{pmatrix} = \begin{bmatrix} 1 & 0.8 \\ 0 & -0.1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

which gives $x_2 = -25$ from the second equation and $x_1 = 23$ after substitution into the first.

It is easy to check that

$$\begin{bmatrix} 1 & 0.8 \\ 0.5 & 0.3 \end{bmatrix} \begin{pmatrix} 23 \\ -25 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} .$$

1.4.1 Under determined systems

We turn now our attention to the solution of a linear system with a coefficient matrix $\boldsymbol{A}$ having more columns n than rows m, i.e. with more unknowns n than available equations m.

Under the assumption that the coefficient matrix $\boldsymbol{A}$ has full rank, meaning that the rows are linearly independent, the general solution of the linear system of equations in (1.4) consists of the superposition of a particular solution $\boldsymbol{x}_p$ and a homogeneous solution $\boldsymbol{x}_h$. The particular solution is a vector, while the homogeneous solution consists of any linear combination of a set of $n - m$ linearly independent column vectors, known as the *nullspace* basis. For the particular solution $\boldsymbol{x}_p$ we set $n - m$ unknowns equal to zero after making sure that the rank of the matrix without the corresponding columns is still full. The homogeneous solution $\boldsymbol{x}_h$ is the general solution of the system of equations

$$\mathbf{0} = \boldsymbol{A}\boldsymbol{x}_h . \tag{1.5}$$

All possible linear combinations of the columns of the homogeneous solution $\boldsymbol{x}_h$ form the *nullspace* of the matrix $\boldsymbol{A}$.

Example 1.7 (Solution of under determined system of equations)**.** Consider the following linear system of equations

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{bmatrix} 1 & 0.8 & 0 & 0.8 \\ 0 & 0.6 & 1 & -0.6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} .$$

One can establish that the rows of the coefficient matrix are linearly independent, so that its rank is equal 2. For determining the particular solution we select $4 - 2 = 2$ unknowns and set them equal to zero. The most convenient choice is the selection of x_2 and x_4, since this leaves the identity matrix as the coefficient matrix for unknowns x_1 and x_3. x_2 and x_4 are called the *free variables* of the problem. This selection gives

$$\boldsymbol{x}_p = \begin{pmatrix} x_{p1} \\ x_{p2} \\ x_{p3} \\ x_{p4} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

as the particular solution.

The homogeneous solution results from the solution of the linear system of equations

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 0.8 & 0 & 0.8 \\ 0 & 0.6 & 1 & -0.6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} .$$

The most convenient way of accomplishing this is to set each free variable in turn equal to 1, with the other free variables left as 0, and then determine the remaining unknowns. Starting with $x_2 = 1$ and $x_4 = 0$ we get

$$\begin{aligned} x_1 &= -0.8 \\ x_3 &= -0.6 \, , \end{aligned}$$

and for $x_2 = 0$ and $x_4 = 1$ we get

$$\begin{aligned} x_1 &= -0.8 \\ x_3 &= 0.6 \, . \end{aligned}$$

The homogeneous solution then takes the general form

$$\boldsymbol{x}_h = \begin{pmatrix} -0.8 \\ 1 \\ -0.6 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} -0.8 \\ 0 \\ 0.6 \\ 1 \end{pmatrix} \beta$$

noting that x_2 and x_4 can assume any value, as represented by the *free variables* α and β. It is straightforward to check that $\boldsymbol{x}_h$ in the last expression satisfies (1.5) for

$$A = \begin{bmatrix} 1 & 0.8 & 0 & 0.8 \\ 0 & 0.6 & 1 & -0.6 \end{bmatrix} .$$

The complete solution is given by $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_h$ with two free parameters α and β (assuming any value) for the homogeneous solution.

1.4.2 Over determined systems

Finally, we turn our attention to the solution of a linear system with a coefficient matrix $\boldsymbol{A}$ having more rows m than columns n, i.e. with more equations m than unknowns n.

Under the assumption that the matrix $\boldsymbol{A}$ has full rank a solution to the linear system of equations in (1.4) exists *if and only if* the vector $\boldsymbol{b}$ is *orthogonal to the nullspace of the transpose* of the matrix $\boldsymbol{A}$. We demonstrate this necessary and sufficient condition with the following example.

Example 1.8 (Solution of over determined system of equations)**.** Consider the following linear system of equations

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.6 \\ 0 & 1 \\ 0.8 & -0.6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with the coefficient matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.6 \\ 0 & 1 \\ 0.8 & -0.6 \end{bmatrix} .$$

We recognize the coefficient matrix as the transpose of the coefficient matrix from the preceding example, so it is of full rank. Solving the first and third equation for x_1 and x_2 gives

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_3 . \end{aligned}$$

Substituting into the second and fourth equation gives

$$\begin{aligned} b_2 &= 0.8x_1 + 0.6x_2 \\ b_4 &= 0.8x_1 - 0.6x_2 \end{aligned} \quad \rightarrow \quad \begin{aligned} b_2 &= 0.8b_1 + 0.6b_3 \\ b_4 &= 0.8b_1 - 0.6b_3 . \end{aligned}$$

Rewriting the last two equations in terms of all components of the vector $\boldsymbol{b}$ gives

$$\begin{aligned} -0.8b_1 + (1)b_2 - 0.6b_3 + (0)b_4 &= 0 \\ -0.8b_1 + (0)b_2 + 0.6b_3 + (1)b_4 &= 0 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} -0.8 & 1 & -0.6 & 0 \\ -0.8 & 0 & 0.6 & 1 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \boldsymbol{0} .$$

As long as the vector $\boldsymbol{b}$ satisfies this property, all of the equations are satisfied and we have a solution to our system of equations. We recognize that the rows of the coefficient matrix for the last equation correspond to the columns of the homogeneous solution $\boldsymbol{x}_h$ from the preceding example. Thus we can interpret the last equation as requiring $\boldsymbol{b}$ to be orthogonal to any element of the nullspace of $\boldsymbol{A}^T$. (Note the matrix of Example 1.7 is the transpose of our current $\boldsymbol{A}$.) In conclusion, if the vector $\boldsymbol{b}$ is *orthogonal to the nullspace* of the transpose of the coefficient matrix $\boldsymbol{A}$, a solution to the over determined system of equations $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$ exists and is equal to

$$\boldsymbol{x} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}$$

for the problem at hand. If this condition is not satisfied, then there is no solution to the over determined system of equations.

1.5 Matrix decompositions

1.5.1 LU decompostion

There are a number of important matrix decompositions. In relation to solving linear equations the most common decomposition is the LU decomposition, pronounced "ell-you". Here a matrix $\boldsymbol{A}$ is written as a product

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} \, ,$$

where $\boldsymbol{L}$ is a lower triangular matrix and $\boldsymbol{U}$ is an upper triangular matrix. These matrices are the result of a Gaussian elimination process. They allow for the easy solution of linear equations once they are computed.

1.5.2 Cholesky decomposition

In the special case where $\boldsymbol{A}$ is symmetric positive definite then one also has the Cholesky decomposition

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T \, .$$

In the 3×3 case

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix} \, ,$$

where

$$\boldsymbol{L} = \begin{bmatrix} \sqrt{A_{11}} & 0 & 0 \\ A_{21}/L_{11} & \sqrt{A_{22} - L_{21}^2} & 0 \\ A_{31}/L_{11} & (A_{31} - L_{31}L_{21})/L_{32} & \sqrt{A_{33} - L_{31}^2 - L_{32}^2} \end{bmatrix} \, .$$

The components of $\boldsymbol{L}$ can be computed row by row or column by column. In general the entries of $\boldsymbol{L}$ are given by

$$L_{jj} = \sqrt{A_{jj} - \sum_{k=1}^{j-1} L_{jk}^2}$$

$$L_{ij} = \frac{1}{L_{jj}} \left(A_{ij} - \sum_{k=1}^{j-1} L_{ik}L_{jk} \right) \qquad i > j \, .$$

1.6 Check your understanding problems

1. Consider the following matrices and vectors:

$$\boldsymbol{A} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] , \quad \boldsymbol{B} = \left[\begin{array}{cc} 6 & 2 \\ 7 & 8 \end{array} \right] , \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \quad \boldsymbol{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \, .$$

(a) Compute $\boldsymbol{A}^T$.

(b) Compute $\boldsymbol{A}^T\boldsymbol{B}$.

(c) Compute $\boldsymbol{A}^T\boldsymbol{A}$ and determine if it is symmetric or not.

(d) Compute $\boldsymbol{A}\boldsymbol{v}$.

(e) Compute $\boldsymbol{w} \cdot \boldsymbol{A}\boldsymbol{v}$.

(f) Compute $\|\boldsymbol{w}\|$.

(g) Find $\boldsymbol{B}^{-1}$ by hand and check your answer using a software package, say, Matlab. There are many procedures for finding the inverse of a matrix. If you do not recall any, do not worry about it. It is pretty rare that one would do that by hand. Knowing how to get a software system (or a graphing calculator) to do it is more important.

(h) Using a graphing calculator or an other software system, find the eigenvalues and eigenvectors of $\boldsymbol{B}$. Then check that the vectors given to you are actually the eigenvectors of the matrix by multiplying them by $\boldsymbol{B}$ and showing that the resulting vector is just a scalar multiple of the original.

2. Consider the system of linear equations from Example 1.7.

(a) Determine the particular solution and the homogeneous solutions for the selection of x_1 and x_2 as free variables.

(b) Show that it is possible to select two suitable sets of coefficients for the homogeneous solutions, to reproduce the homogeneous solutions from Example 1.7. This shows that the *nullspace of the coefficient matrix is unique*.

Chapter 2

Ordinary differential equations

2.1 Introduction

Ordinary differential equations (ODEs) describe numerous phenomena relevant to Structural Engineering, Mechanics and Materials (SEMM). For example, recall the equation which describes the shape of a suspension bridge cable:

$$\frac{d^2y}{dx^2} = \frac{w(x)}{H}\,, \tag{2.1}$$

where H is the horizontal reaction at the supports (and also the tension of the cable when it is horizontal) and $w(x)$ is the vertical load per unit length at position x along the cable.

A similar equation describes the internal bending moment $M(x)$ of a horizontal beam at a horizontal location x:

$$\frac{d^2M(X)}{dx^2} = w(x)\,. \tag{2.2}$$

Equations (2.1) and (2.2) are obviously related; the shape of a cable that only acts in pure tension effectively provides an equivalent "moment" capacity that relates to the bending capacity of a beam.

Euler-Bernoulli beam theory assumes that the curvature of the beam is proportional to the moment according to another differential equation:

$$M(x) = EI\frac{d^2v}{dx^2}\,, \tag{2.3}$$

where v is the vertical deflection of a horizontal beam. Equation (2.3) can be re-written as a fourth order differential equation:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x)\,. \tag{2.4}$$

For uniform beam cross sections, (2.4) becomes:

$$EI\frac{d^4v}{dx^4} = w(x)\,. \tag{2.5}$$

These differential equations that are fundamental to structural engineering are all separable. They can be solved by separating the variables and integrating. Boundary conditions can then be used to solve for integration constants.

However, more generally, differential equation may not be separable. For example, consider the dynamic response of a mass and spring system which has the following equation of motion:

$$m\ddot{x}(t) + kx(t) = f(t)\,, \tag{2.6}$$

where t represents the time, $x(t)$ represents the position of a mass, m is the mass, k is the stiffness, and $f(t)$ is a function that does not include $x(t)$ or its derivatives, and is sometimes referred to as the forcing function. If m and k are constants, (2.6) is a linear second order ODE. How one determines m, k and $f(t)$ for a real system are unimportant for the present discussion.

For dynamic systems with velocity dependent damping, an additional term is added:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)\,, \tag{2.7}$$

where c is a constant related to the damping in the system. Again, the derivation of this equation, and its physical meaning, are not important here. From a mathematical perspective, (2.7) is a linear second order ODE.

This chapter focuses on methods to solve linear second order differential equations such as (2.7).

2.2 Homogeneous solutions

A differential equation is said to be homogeneous if all the terms involve the unknown x, or a derivative of x. In other words, the second order differential equation takes the following form:

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = 0\,, \tag{2.8}$$

where A, B, and C are constants. With reference to a dynamical system, a homogeneous equation is equivalent to specifying that there is no forcing function, i.e. $f(t) = 0$ in (2.7). The solution to a homogeneous second order linear differential equation is sometimes referred to as the homogeneous solution or the complementary function and can be written in various forms.

The nature of the derivatives of an exponential function make it a promising “guess” for the homogeneous solution. In particular, assume the homogeneous solution is of the form:

$$x(t) = e^{rt}\,. \tag{2.9}$$

Plugging this exponential function and its derivatives back into (2.8) yields:

$$Ar^2 + Br + C = 0\,. \tag{2.10}$$

This is often referred to as the characteristic equation of a linear second order ODE. Now, solving for the roots of (2.10) yields:

$$r_1, r_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\,. \tag{2.11}$$

Generally, three possible scenarios result: 1) two distinct real roots when $B^2 > 4AC$, 2) two distinct roots that are complex conjugates of each other when $B^2 < 4AC$, or 3) one repeated root when $B^2 = 4AC$. Each of these scenarios has a different general solution to the homogeneous ODE in (2.8).

Scenario 1: When the roots are distinct and real, the general solution to the homogeneous ODE can be written as a linear superposition of the solution related to each root:

$$x_c(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}\,, \tag{2.12}$$

where the subscript c stands for the complementary solution, and the unknown constants c_1 and c_2 can be determined from the initial conditions at time $t = 0$, i.e. $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Scenario 2: When the roots are distinct complex conjugates, the roots take the form:

$$r_1, r_2 = \lambda \pm i\mu\,. \tag{2.13}$$

The general solution to the homogeneous ODE can again be written as a linear superposition of the solution related to each root:

$$x_c(t) = c_1 e^{(\lambda+i\mu)t} + c_2 e^{(\lambda-i\mu)t}\,. \tag{2.14}$$

Rearranging:

$$x_c(t) = c_1 e^{\lambda t} e^{i\mu t} + c_2 e^{\lambda t} e^{-i\mu t}\,. \tag{2.15}$$

Recall that:

$$e^{i\mu t} = \cos(\mu t) + i\sin(\mu t)\,. \tag{2.16}$$

Substituting (2.16) into (2.15) yields:

$$x_c(t) = e^{\lambda t}(c_3\cos(\mu t) + c_4\sin(\mu t))\,. \tag{2.17}$$

The unknown constants c_3 and c_4 can again be determined from the initial conditions at time $t = 0$, i.e. $x(0) = x_0$ and $\dot{x}(0) = v_0$. Note that if $B = 0$ in (2.8), then $\lambda = 0$, and (2.17) simplifies to:

$$x_c(t) = c_3\cos(\mu t) + c_4\sin(\mu t)\,. \tag{2.18}$$

This is the relevant solution for the dynamical system in (2.6) which describes a system with no damping (i.e. $c = 0$ in (2.7)).

Scenario 3: When there is one repeated root:

$$r = r_1 = r_2 = \frac{-B}{2A}\,.$$

Using linear superposition, the solution could be:

$$x_c(t) = (c_1 + c_2)e^{rt} = c_3 e^{rt}\,. \tag{2.19}$$

This is a potential solution, but (2.19) is not sufficiently general to solve with two initial conditions (x_0, v_0). Instead, a more general "guess" for the initial solution is required:

$$x_c(t) = d(t)e^{rt} \,. \tag{2.20}$$

Plugging (2.20) and its derivatives back into (2.8) yields:

$$e^{rt}(\ddot{d}(t)) = 0 \,.$$

Since e^{rt} is not zero:

$$\ddot{d}(t) = 0 \,.$$

Integrating twice results in $d(t) = c_4 + c_5 t$. Plugging back into (2.20) yields:

$$x_c(t) = (c_4 + c_5 t)e^{rt} \,. \tag{2.21}$$

The unknown constants c_4 and c_5 can again be determined from the initial conditions at time $t = 0$, i.e. $x(0) = x_0$ and $\dot{x}(0) = v_0$.

2.3 Particular solutions

In most dynamical systems of interest, the forcing function ($f(t)$ in (2.7)) is not zero. In this case, the ODE is said to be inhomogeneous. In generic form, an inhomogeneous second order ODE with constant coefficients can be written as:

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = f(t) \,, \tag{2.22}$$

where $f(t)$ is a generic forcing function. The solution to inhomogeneous ODEs is the summation of the homogeneous solution (i.e. the complementary function) and the particular solution:

$$x(t) = x_c(t) + x_p(t) \,, \tag{2.23}$$

where the subscript p denotes the particular solution. In general, the particular solution can be found using the method of variation of parameters. However, for many forcing functions, it is easier to determine $x_p(t)$ using an educated guess for the particular solution, in a manner similar to the educated guess in (2.9) in Section 2.2. This educated guess method is used in the following sections.

2.3.1 Exponential forcing

Assume the inhomogeneous term is an exponential function and that the ODE takes the form:

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = De^{\alpha t} \,, \tag{2.24}$$

where D and α are constants. An educated guess for the particular solution preserves the exponent as follows:

$$x_p(t) = C_1 e^{\alpha t} \,, \tag{2.25}$$

where C_1 is a constant that can be determined by plugging (2.25) and its derivatives back into (2.24). For the generic function considered here:

$$A(C_1\alpha^2 e^{\alpha t}) + B(C_1\alpha e^{\alpha t}) + C(C_1 e^{\alpha t}) = De^{\alpha t}\,. \tag{2.26}$$

Solving (2.26) for C_1 yields:

$$C_1 = \frac{D}{A\alpha^2 + B\alpha + C}\,. \tag{2.27}$$

The complete solution is obtained by plugging (2.25) and (2.27) into (2.23):

$$x(t) = x_c(t) + x_p(t) = x_c(t) + \frac{D}{A\alpha^2 + B\alpha + C}e^{\alpha t}\,, \tag{2.28}$$

where the complimentary solution $x_c(t)$ results from one of the scenarios in Section 2.2.

Example 2.1. Solve $\ddot{x}(t) - 4\dot{x}(t) + 6x(t) = 2e^{4t}$, subjected to the following initial conditions: $x(0) = x_0 = 0$ and $\dot{x}(0) = v_0 = 0$.

Solution: The solution takes the form of (2.28)

$$x(t) = x_c(t) + x_p(t) = x_c(t) + \frac{D}{A\alpha^2 + B\alpha + C}e^{\alpha t}\,.$$

The roots of the characteristic equation are:

$$r_1, r_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = 2 \pm i2\sqrt{5} = \lambda \pm i\mu\,.$$

The complete solution is:

$$x(t) = x_c(t) + x_p(t) = e^{2t}\left(c_1\cos(2\sqrt{5}t) + c_2\sin(2\sqrt{5}t)\right) + \frac{1}{3}e^{4t}\,.$$

Now solve for the constants c_1 and c_2 using the initial conditions. Note that the initial conditions must be satisfied for $x(t)$, not $x_c(t)$.

$$x(0) = 0 \quad \Rightarrow \quad c_1 = -\frac{1}{3}\,,$$

$$\dot{x}(0) = 0 \quad \Rightarrow \quad c_2 = -\frac{1}{2\sqrt{5}}\,.$$

Plugging c_1 and c_2 into the complete solution yields:

$$x(t) = e^{2t}\left(-\frac{1}{3}\cos(2\sqrt{5}t) - \frac{1}{2\sqrt{5}}\sin(2\sqrt{5}t)\right) + \frac{1}{3}e^{4t}\,.$$

2.3.2 Polynomial forcing

Assume instead that the inhomogeneous term is a polynomial function, so the ODE takes the form:

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = Dt^{\beta}\,. \tag{2.29}$$

An educated guess for the particular solution is a complete polynomial of order β:

$$x_p(t) = C_1 + C_2 t + C_3 t^2 + \cdots + C_{\beta+1}t^{\beta}\,. \tag{2.30}$$

The constants C_1 to $C_{\beta+1}$can again be determined by plugging (2.30) and its derivatives back into (2.29). In this case, a general particular equation is cumbersome. The following example illustrates the process.

Example 2.2. Solve $\ddot{x}(t) - 2\dot{x}(t) - 8x(t) = 2t^2 - 1$, subjected to the following initial conditions: $x(0) = x_0 = 0$ and $\dot{x}(0) = v_0 = 1$.

Solution: An educated guess for the particular solution according to (2.30) is a complete second order polynomial:

$$x_p(t) = C_1 + C_2 t + C_3 t^2\,.$$

Plug this particular solution back into the ODE:

$$(2C_3) - 2(C_2 + 2C_3 t) - 8(C_1 + C_2 t + C_3 t^2) = 2t^2 - 1\,.$$

Collect terms of similar order:

$$2C_3 - 2C_2 - 8C_1 = -1\,,$$

$$-4C_3 t - 8C_2 t = 0\,,$$

$$-8C_3 t^2 = 2t^2\,.$$

Solving these equations simultaneously yields $C_3 = -\frac{1}{4}$, $C_2 = \frac{1}{8}$, $C_1 = \frac{1}{32}$, and the particular solution becomes:

$$x_p(t) = \frac{1}{32} + \frac{1}{8}t - \frac{1}{4}t^2\,.$$

The roots of the characteristic equation are:

$$r_1, r_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = 4,\ -2\,.$$

Applying (2.12) for the complementary function, the complete solution takes the form:

$$x(t) = x_c(t) + x_p(t) = c_1 e^{4t} + c_2 e^{-2t} + \left(\frac{1}{32} + \frac{1}{8} t - \frac{1}{4} t^2 \right) .$$

Now solve for the constants c_1 and c_2 using the initial conditions:

$$x(0) = 0 \quad \Rightarrow \quad c_1 + c_2 + \frac{1}{32} = 0 ,$$

$$\dot{x}(0) = 1 \quad \Rightarrow \quad 4c_1 - 2c_2 + \frac{1}{8} = 1 .$$

Solving for c_1 and c_2 yields the final solution:

$$x(t) = \frac{15}{32} e^{4t} - \frac{1}{2} e^{-2t} + \left(\frac{1}{32} + \frac{1}{8} t - \frac{1}{4} t^2 \right) .$$

2.3.3 Sine or cosine forcing

Assume instead that the inhomogeneous term (i.e. forcing function) is a sine or cosine function. A sine function will be considered here, but the same procedure can be followed if the inhomogeneous term is a cosine function or a combination of sine and cosine functions. Assume the ODE takes the form:

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = D \sin(\alpha t) . \tag{2.31}$$

An educated guess for the particular solution is a combination of sine and cosine functions with the same period:

$$x_p(t) = C_1 \cos(\alpha t) + C_2 \sin(\alpha t) . \tag{2.32}$$

The constants C_1 to C_2 can again be determined by plugging (2.32) and its derivatives back into (2.31). The following example illustrates the process.

NOTE: Recall from (2.16) that $D \sin(\alpha t) = \text{Im}\{De^{i\alpha t}\}$. Therefore, the sine forcing function in (2.31) could be transformed to an exponential forcing function, and the ODE could be solved in a similar fashion to Section 2.3.1, but this will not be demonstrated here.

Example 2.3. Solve: $\ddot{x}(t) + \dot{x}(t) - 8x(t) = \sin(2t)$, subjected to the following initial conditions: $x(0) = x_0 = 1$ and $\dot{x}(0) = v_0 = 0$.

<u>Solution:</u> Guess that the particular solution takes the form of (2.32):

$$x_p(t) = C_1 \cos(2t) + C_2 \sin(2t) .$$

Plug this equation back into the ODE:

$$(-2C_1 \sin(2t) + 2C_2 \cos(2t)) + (C_1 \cos(2t) + C_2 \sin(2t)) = \sin(2t)\,.$$

Collect similar terms:

$$-2C_1 \sin(2t) + C_2 \sin(2t) = \sin(2t)\,,$$

$$2C_2 \cos(2t) + C_1 \cos(2t) = 0\,.$$

Solving these equations for C_1 and C_2 and plugging back into the particular solution yields:

$$x_p(t) = \frac{2}{3}\cos(2t) - \frac{1}{3}\sin(2t)\,.$$

The roots of the characteristic equation are:

$$r_1, r_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \pm i\,.$$

Using (2.18) with $\mu = 1$, the complete solution takes the form:

$$x(t) = x_c(t) + x_p(t) = c_3 \cos(t) + c_4 \sin(t) + \frac{2}{3}\cos(2t) - \frac{1}{3}\sin(2t)\,.$$

Now solve for the constants c_3 and c_4 using the initial conditions:

$$x(0) = 1 \quad \Rightarrow \quad c_3 + \frac{2}{3} = 1\,,$$

$$\dot{x}(0) = 0 \quad \Rightarrow \quad c_4 - \frac{2}{3} = 0\,.$$

Solving for c_3 and c_4 yields the final solution:

$$x(t) = \frac{1}{3}\cos(t) + \frac{2}{3}\sin(t) + \frac{2}{3}\cos(2t) - \frac{1}{3}\sin(2t)\,.$$

2.4 Reduction of Order

The methods above can be used to directly solve linear second order ODEs. Alternatively, it is possible to reduce the order of a differential equation prior to solving. This is often convenient when solving differential equations numerically, e.g. solving using Python or MATLAB.

For example, assume that $x(t)$ represents the displacement of a mass, and that the equation of motion for the mass is of the form:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)\,. \tag{2.33}$$

Now introduce a new set of variables:

$$y_1(t) = x(t)\,,$$
$$y_2(t) = \dot{x}(t)\,.$$

The second order ODE in (2.33) can now be re-written as a system of coupled first order ODEs as follows:

$$\dot{y_1}(t) = y_2(t)\,,$$
$$\dot{y_2}(t) = \frac{f(t) - cy_2(t) + ky_1(t)}{m}\,.$$

This enables alternative solution procedures, which are not detailed here.

2.5 Check your understanding problems

1. Solve $2\ddot{x}(t) - 4\dot{x}(t) + 2x(t) = 0$, subjected to the following initial conditions: $x(0) = x_0 = 0$ and $\dot{x}(0) = v_0 = 1$.
2. Solve $2\ddot{x}(t) - 2\dot{x}(t) + x(t) = 2\cos(4t)$, subjected to the following initial conditions: $x(0) = x_0 = 1$ and $\dot{x}(0) = v_0 = 1$.
3. Solve $\ddot{x}(t) - 2\dot{x}(t) - 3x(t) = 4t$, subjected to the following initial conditions: $x(0) = x_0 = 0$ and $\dot{x}(0) = v_0 = 0$.

Chapter 3

Elements of integral calculus

3.1 Introduction

The solution to many engineering problems involves integration of a function of one or more variables, commonly called the integrand. Depending on the form of the integrand, an analytical solution may or may not exist. If an analytical solution is not available, numerical integration may be used. There are many numerical integration rules used in software commonly used by engineers, including MATLAB and Python. In such applications, sometimes attention should be paid in how the integrand is formulated to avoid numerical issues that are associated with finite precision arithmetic. Further, for bounded multi-fold integrals, careful attention should be paid to the limits of integration. Examples are given below.

3.1.1 Integration by parts

Consider the single-fold integral over the interval $[a, b]$

$$I = \int_a^b f(x)\, dx.$$

If $F(x)$ is the integral of $f(x)$, i.e., if $f(x) = dF(x)/dx$, then $I = F(b) - F(a)$. One useful formula for integration occurs when the integrand can be written in the form $f(x) = u(x)v'(x)$, where $v'(x)$ is the derivative of a known function $v(x)$. Then

$$\int_a^b u(x)\, v'(x)\, dx = [u(x)\, v(x)]_a^b - \int_a^b u'(x)\, v(x)\, dx\, .$$

Known as integration by parts, this formula is useful if the integral of the integrand $u'(x)\, v(x)$ has a known solution.

Example 3.1 (Integration by parts)**.** Let $f(x) = x\ \sin(x)$. We know that $\sin(x)$ is the derivative of $-\cos(x)$. Thus,

$$\int_a^b x\sin(x)\,dx = [-x\cos(x)]_a^b + \int_a^b 1 \times \cos(x)\,dx$$
$$= a\cos(a) - b\cos(b) + \sin(b) - \sin(a)\,.$$

3.1.2 Leibnitz rule

On occasion it is required to take the derivative of an integral expression with respect to a variable that appears in the limits of integration and/or in the integrand. Under fairly general conditions, this is accomplished using the rule of Leibnitz:

$$\frac{d}{d\alpha}\int_{h(\alpha)}^{g(\alpha)} f(x,\alpha)\,dx = f(g(\alpha),\alpha)\frac{dg}{d\alpha} - f(h(\alpha),\alpha)\frac{dh}{d\alpha} + \int_{h(\alpha)}^{g(\alpha)} \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx\,.$$

Example 3.2 (Leibnitz rule)**.** Consider the integral

$$I(\alpha) = \int_1^\alpha \ln(x)\exp(\alpha x)\,dx$$

and compute its derivative with respect to α. To compute, we can directly apply the Leibnitz rule to see that

$$\frac{dI}{d\alpha} = \ln(\alpha)\exp(\alpha) + \int_1^\alpha \alpha\ln(x)\exp(\alpha x)\,dx\,.$$

3.1.3 Numerical integration

The simplest numerical integration scheme is the rectangular rule (Riemann sum), which divides the interval $[a,b]$ into n equally spaced discrete points x_i, $i = 1,\ldots,n$, with $x_1 = a$, $x_n = b$ and $\Delta x = (a-b)/n$, and approximates the integral by

$$I \cong \sum_{i=1}^{n} f(x_i)\,\Delta x\,.$$

This approach requires computation of the function values at the discrete points x_i. Care should be taken in formulating the function so that these values remain within a computable range. More sophisticated methods with better accuracy include the trapezoidal rule, Simpson's rule, and Gauss rules.

Example 3.3 (Numerical integration)**.** Suppose we wish to numerically integrate the function $f(x) = x^m\exp(-x)$ over the interval $[0,\infty]$. If we are using a numerical scheme, we obviously need to replace the unbounded domain with a finite domain $[0, b]$, where b is sufficiently large to achieve accuracy. An additional issue to pay attention to is, if m is large, then evaluating the function in its given form can lead to numerical overflow when using finite precision arithmetic (as computers do) since x^m can be extremely large as x approaches b (which is supposed to be large). Note, however, that at the same time $\exp(-x)$ approaches zero and thus the product that forms the integrand is well behaved. To avoid working with the large numbers produced by x^m, we can rewrite the integrand in the form $f(x) = \exp[-x + m\ \ln(x)]$. Observe that the exponent $-x + m\ \ln(x)$ now is a much smaller number as x approaches b; it is order b.

This example demonstrates that one needs to be careful in formulating the integrand in software that performs numerical integration tasks. One should always look for possible cases of underflow and overflow in sub-functions of the integrand that the computer must evaluate. If detected, the integrand should be rewritten to avoid these numerical pathologies.

3.1.4 Complex functions

Special attention should be paid to integrands that have zero values outside an interval. The following example demonstrates this issue.

Example 3.4 (Probability density for the sum of two random variables)**.** Consider two random variables X_1 and X_2 having uniform distributions within the interval $[0, 1]$. Their probability density functions are given by

$$f_{X_i}(x_i) = \begin{cases} 1, & 0 \le x_i \le 1\,, i = 1, 2 \\ 0, & \text{elsewhere.} \end{cases}$$

One can show that the probability density function of $Y = X_1 + X_2$ is given by

$$f_Y(y) = \int_0^1 f_{X_1}(y - x_2) f_{X_2}(x_2)\, dx_2.$$

Obviously, $f_Y(y)$ is non-zero in the interval $[0, 2]$.

Observe that the function $f_{X_1}(y - x_2)$ is non-zero when its argument lies within the interval $[0, 1]$. Otherwise, that function has a zero value. This affects the limits of the integration. Since we must have $0 \le y - x_2 \le 1$, it follows that for any y we must have $x_2 \le y$ and $y - 1 \le x_2$. Observe further that the second function, $f_{X_2}(x_2)$, is zero outside the interval $[0, 1]$. Considering these limits, the above integral can be written with adjusted limits as follows:

$$f_Y(y) = \int_{\max(0, y-1)}^{\min(1, y)} f_{X_1}(y - x_2) f_{X_2}(x_2)\, dx_2.$$

We see that the limits depend on the selected value of y. The demarcation point clearly is $y = 1$. Hence, we evaluate the integral as follows:

For $0 \le y \le 1$:

$$f_Y(y) = \int_0^y 1 \times 1 \times dx = y \,.$$

For $1 \le y \le 2$:

$$f_Y(y) = \int_{y-1}^1 1 \times 1 \times dx = 2 - y \,.$$

The complete solution then is

$$f_Y(y) = \begin{cases} y, & \text{for } 0 \le y \le 1 \,, \\ 2 - y, & \text{for } 1 \le y \le 2 \,. \end{cases}$$

3.2 Check your understanding problems

1. Use integration by parts to evaluate the following integral. Note that you may have to use the rule repeatedly to come to the final expression.

$$I(x) = \int_0^x y^2 \exp(-y)\, dy \,.$$

2. Consider $Y = X_1 + X_2 + X_3$, where X_1, X_2, and X_3 are uniformly distributed random variables within the interval $[0, 1]$ with probability density functions as in Example 3.4 above. It can be shown that the probability density function of Y is given by

$$f_Y(y) = \int_0^1 \int_0^1 f_{X_1}(y - x_2 - x_1) f_{X_2}(x_2) f_{X_3}(x_3)\, dx_2\, dx_3 \,.$$

Derive an expression for $f_Y(y)$.

3. Consider the integral

$$I(n, \alpha) = \int_0^\infty e^{-\alpha x^2} x^n \, dx \,,$$

where $n \ge 0$.

(a) Use the Leibnitz rule to show that

$$I(n, \alpha) = -\frac{\partial I}{\partial \alpha}(n - 2, \alpha) \,.$$

(b) Noting that $I(0, \alpha) = \frac{1}{2}\sqrt{\pi}\alpha^{-\frac{1}{2}}$, compute $I(2, \alpha)$.

Chapter 4

Partial differential equations

4.1 Introduction

A number of theories in structural engineering are best described using partial differential equations. Partial differential equations differ from ordinary differential equations in that the unknowns depend on more than one independent variable. Common examples, among many, include vibrational motion of a beam which depends on time and a spatial coordinate, and the deflection of a plate which depends on (x, y) location on the plate.

4.2 Partial derivatives

Partial differential equations naturally make use of partial derivatives. Given a function of more than one variable, say, $f(x, y)$, then the partial derivative of f with respect to x is denoted

$$\frac{\partial f}{\partial x}(x, y) \, .$$

The meaning of the notation is that one takes the derivative of $f(x, y)$ with respect to x while holding y constant. Similarly,

$$\frac{\partial f}{\partial y}(x, y) \, ,$$

means the derivative of $f(x, y)$ with respect to y while holding x constant.

Example 4.1 (Taking partial derivatives)**.** Consider the function $f(x, y, z) = x^2y + yz + z$.

This is a function of three variables and has three first partial derivatives.

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y,z) &= 2xy \\ \frac{\partial f}{\partial y}(x,y,z) &= x^2 + z \\ \frac{\partial f}{\partial z}(x,y,z) &= y + 1\,.\end{aligned}$$

Higher order partial derivatives follow by taking derivatives of lower order derivative expressions. It is useful to note that in physically motivated problems the order of partial differentiation usually does not matter – *mixed partial derivatives are equal to each other* –

$$\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{\frac{\partial^2 f}{\partial y \partial x}} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{\frac{\partial^2 f}{\partial x \partial y}}\,.$$

4.3 Partial differential equations

Partial differential equations describe a number of important physical phenomena as mentioned above. Unfortunately, their solution tends to be rather complicated when done by hand, and thus they are usually solved using numerical techniques, such as the finite element method. Notwithstanding, it is instructive to have knowledge of how hand solutions are performed for the purposes of being able to better interpret numerical solutions and to have an intuitive understanding of how physical systems behave. For more extensive discussions on the solution of partial differential equations than provided here see for example Hildebrand [1976, Chaps. 8 and 9].

In the case of ordinary differential equations of order n, the general solution is constructed using a linear combination of n *fixed functions* and n arbitrary coefficients, where in the final solution the arbitrary coefficients are determined from the initial/boundary conditions. Most importantly with ordinary differential equations is the fact that one can determine the fixed functions with knowledge of the form of the differential equations; see Chap. 2.

In the case of partial differential equations we are not so lucky. The general solutions involve *arbitrary functions* of *fixed functions* of the independent variables. As a basic illustration, consider the partial differential equation

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0\,, \tag{4.1}$$

where the unknown to be solved for is $u(x,t)$ – with x and t being the independent variables. Without detailing the solution steps, it is noted that the general solution is given in terms of an arbitrary function f and the fixed function $s(x,t) = 2t - x$. In particular

$$u(x,t) = f(s) = f(2t - x)$$

is a solution of (4.1) for any function f.[1] The arbitrary function f, in the complete solution, is found from the initial/boundary conditions. In relation to the solution of ordinary differential equations, the function f takes the place of the arbitrary coefficients. The function $s(x,t)$ plays the role of the fixed functions and is determined from knowledge of the form of the differential equation. This style of solution of partial differential equations is known as a characteristics solution – a.k.a., the method of characteristics. A lot can be learned about the behavior of a physical system from a characteristics solution, as it gives the general solution. More often, however, we are interested in particular solutions to specific initial boundary value problems. In that case, the partial differential equations that arise are often in a format that admits solutions using the method of separation of variables.

4.3.1 Separation of variables

In the method of separation of variables we assume that the solution is composed of products of functions, each being only a function of one of the independent variables. This special assumption works for a large number of partial differential equations that occur in physical problems. While the assumption seems rather special, if one can satisfy all the boundary/initial conditions, then one is assured that one has the complete solution due to the fact that the solutions to many physical problems are unique. The description of the method is best illustrated via a concrete example.

Before starting, it should be noted that overall the methodology is straightforward, but can be in practice very cumbersome. The payoff for the effort is an analytic solution to the problem that can examined for its functional properties. In most practical cases, the method can not be carried out to completion and one must resort to numerical solution methods (like the finite element method); however, for a certain class of problems it is very effective. Below we give one example with the details of most of the steps.

Example 4.2 (Separation of variables for Laplace's equation)**.** Consider the partial differential equation (Laplace's equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.2}$$

over the square domain $0 < x < L$ and $0 < y < L$. Further assume the boundary conditions $u(x,0) = u(x,L) = u(L,y) = 0$ and $u(0,y) = \sin(4\pi y/L)$.

The method of separation of variables begins with *assuming* that the solution can be written in the form

$$u(x,y) = X(x)Y(y)\,,$$

[1] We can verify this by plugging this form of the solution into (4.1):

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = \frac{df}{ds}\frac{\partial s}{\partial t} + 2\frac{df}{ds}\frac{\partial s}{\partial x} = \frac{df}{ds} \times (2 - 2 \times 1) = 0\,.$$

where X and Y are functions of only x and y, respectively. The validity of this assumption can only be verified by showing that a solution to Laplace's equation of this form is possible and that the boundary conditions can also be satisfied.

If we plug the our assumption into (4.2), then we find that

$$X''Y + XY'' = 0\,,$$

where primes have been used to denote derivatives of functions of one variable (with respect to that variable). Rearranging gives

$$\frac{X''}{X} = -\frac{Y''}{Y}\,. \tag{4.3}$$

Examining (4.3), one observes that the left-hand side is a function of x only and the right-hand side is a function of y only. The only way for this to be true is if both sides are equal to the same constant. Let us call that constant k, so that

$$\frac{X''}{X} = -\frac{Y''}{Y} = k \quad \Rightarrow \quad \begin{cases} X'' - kX = 0 \\ Y'' + kY = 0\,. \end{cases}$$

These two ordinary differential equations have the general solutions:

$$\begin{aligned} X(x) &= A\exp(-\sqrt{k}x) + B\exp(\sqrt{k}x) \\ Y(y) &= C\cos(\sqrt{k}y) + D\sin(\sqrt{k}y)\,, \end{aligned}$$

where A, B, C and D are arbitrary constants. If we now employ the boundary condition that $u(x,0) = 0$, then we see that $C = 0$. Using the boundary condition that $u(x,L) = 0$, we see that either $D = 0$, which will give the trivial solution $u = XY = X \times 0 = 0$ which we do not want, or $\sin(\sqrt{k}L) = 0$. From this later condition we see that the *separation* constant k is not arbitrary, rather we must have

$$\sqrt{k}L = n\pi\,,$$

where $n = 1, 2, 3, \ldots$.

Taking stock of where we are, we see that any functions of the form

$$(A_n \exp(-n\pi x/L) + B_n \exp(n\pi x/L))\sin(n\pi y/L)$$

will satisfy our partial differential equation and the boundary conditions for $y = 0$ and $y = L$. Thus the most general solution we can construct from our starting assumption looks like

$$u(x,y) = \sum_{n=1}^{\infty} (A_n \exp(-n\pi x/L) + B_n \exp(n\pi x/L))\sin(n\pi y/L)\,.$$

Note that A_n and B_n are arbitrary constants. We will use the remaining two boundary conditions to try and determine them.

The boundary condition at $x = L$ tells us that

$$0 = \sum_{n=1}^{\infty} \left(A_n \exp(-n\pi) + B_n \exp(n\pi)\right) \sin(n\pi y/L)\,.$$

For this to be universally true, we must have

$$A_n = -B_n \exp(2n\pi)\,,$$

implying

$$u(x,y) = \sum_{n=1}^{\infty} B_n \left(-\exp(2n\pi)\exp(-n\pi x/L) + \exp(n\pi x/L)\right) \sin(n\pi y/L)\,.$$

The last remaining boundary condition allows us to determine the only remaining unknown, B_n. At $x = 0$ we have the condition that

$$\sin(4\pi y/L) = \sum_{n=1}^{\infty} B_n \left(-\exp(2n\pi) + 1\right) \sin(n\pi y/L)\,. \tag{4.4}$$

This indicates that $B_n\left(-\exp(2n\pi) + 1\right)$ are the Fourier coefficients for a sine series representation of $\sin(4\pi y/L)$. To determine the needed expressions for the coefficients, we note the properties that

$$\begin{aligned} L/2 &= \int_0^L \sin^2(n\pi y/L)\,dy \\ 0 &= \int_0^L \sin(n\pi y/L)\sin(m\pi y/L)\,dy \qquad (m \neq n)\,. \end{aligned}$$

To exploit these properties we multiply both sides of (4.4) by $\sin(m\pi y/L)$ and integrate from 0 to L. This gives

$$\int_0^L \sin(4\pi y/L)\sin(m\pi y/L)\,dy = B_m\left(-\exp(2m\pi) + 1\right) L/2\,.$$

Solving the integral and rearranging gives

$$B_4 = \frac{1}{1 - \exp(8\pi)}$$

and all other $B_m = 0$.

The final solution is then

$$u(x,y)=\frac{1}{1-\exp(8\pi)}\left(\exp(4\pi x/L)-\exp(8\pi)\exp(-4\pi x/L)\right)\sin(4\pi y/L)\,.$$

This function satisfies the partial differential equation and all the boundary conditions.

It should be noted that normally when one computes a solution via separation of variables, the final expression is usually in the form of an infinite series. Here we did not obtain a series solution because the Fourier representation of the last boundary condition only involved one term – which is a rather special situation.

4.4 Check your understanding problems

1. Consider the function $f(x,y,z)=x^4y^2z+yz$ and compute
   (a) $\partial f/\partial x$.
   (b) $\partial f/\partial y$.
   (c) $\partial f/\partial z$.
   (d) $\partial^2 f/\partial x\partial y$.
   (e) $\partial^3 f/\partial x^2\partial z$.

2. Consider the partial differential equation, defined for $y>0$,

$$\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0\,.$$

   Verify, by plugging into the given partial differential equation, that the general solution is of the form $u(x,y)=f(s)$ where $s(x,y)=x-\ln(y)$, and f is an arbitrary function.

3. Consider the same problem as in Example 4.2, except that the boundary conditions are now $u(x,0)=u(x,L)=u(L,y)=0$ and $u(0,y)=100$. Solve, using a separation of variables method, for $u(x,y)$. In the solution process, you should ignore the incompatibility in the boundary conditions at the points $(0,0)$ and $(0,L)$. Your final answer will be in the form of an infinite series.

Chapter 5

Elements of statics and dynamics

5.1 Introduction

The derivation of the force-deformation relations for the constituent elements of discrete structural models is based on the kinematics and the equilibrium of a typical element under different deformation and stress states. The relation between the force and deformation variables results from the constitutive material response, which is here assumed to be linear elastic. This chapter provides an overview of these relations for a few simple deformation states. Furthermore, the chapter offers a very brief review of Newton's second law and of the concepts of linear and angular momentum, as well as the concepts of potential and kinetic energy. For further details the reader can consult Govindjee [2013] or Lubliner and Papadopoulos [2014], and Gross et al. [2014].

5.2 Common deformation modes

There are three fundamental modes of deformation of structural members. Axial motion, torsional motion, and bending motion. The theories used to describe these modes of deformation all consist of a set of kinematic relations, a set of equilibrium relations, and a set of constitutive relations. Each triple comprises a complete set of equations to describe the element response. For each deformation mode, there are also additional relations that allow one to infer detailed response quantities for the structural members. Below we review the main equations without derivation. For a more detailed discussion please consult, for example, Govindjee [2013, Chapters 2, 7, and 8].

5.2.1 Axial extension

The axial mode of deformation occurs for example in truss elements, reinforcing bars, piers, and piles. The fundamental assumption, independent of the complexity of the member, is that plane sections remain plane.

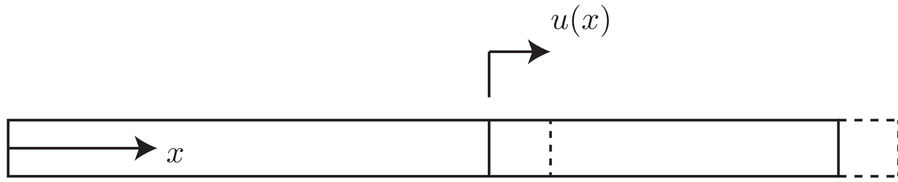


Figure 5.1: Bar motion and coordinates. Axial coordinate x, cross-sectional displacement $u(x)$.

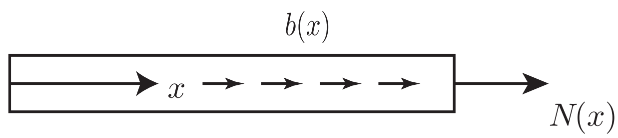


Figure 5.2: Bar resultants and loads. Distributed load $b(x)$, internal force $N(x)$.

Kinematics

Referring to Fig. 5.1, the motion of the bar is given by a function $u(x)$ that assigns to each cross-section an axial displacement. The strain in the bar is the normal strain in the x-direction

$$\varepsilon = \frac{du}{dx}\,. \tag{5.1}$$

Equilibrium

The important stress resultant in axial deformation is the internal force in the x-direction given by a function $N(x)$; see Fig. 5.2. Assuming that the bar is subjected to distributed loads $b(x)$ (force per unit length in the x-direction), then the equilibrium equation for the bar is given as

$$\frac{dN}{dx} + b = 0 \tag{5.2}$$

in the static case. In the dynamic case, the zero on the right-hand side is replaced by $\rho A\, \partial^2 u/\partial t^2$, where ρ is the mass density and A is the cross-sectional area.

Material response

Assuming for simplicity a homogeneous cross-section, the system of equations is completed by a constitutive relation connecting the internal force to the strain

$$N = EA\varepsilon\,, \tag{5.3}$$

where it is assumed that the material is linear elastic with Young modulus E.

Equations (5.1)–(5.3) describe the complete response of a homogeneous elastic bar under axial deformation. With slight modifications a non-prismatic bar with variable cross-section $A(x)$ can be treated with virtually the same equations, if it is noted that the internal force is related to the axial stress in the bar via

$$N(x) = \int_{A(x)} \sigma \, dA \,,$$

where σ is the normal stress on the cross-section $A(x)$ located at x. This relation also allows one to correctly treat the case of composite cross-sections.

5.2.2 Torsion: Circular sections

The fundamental assumption of torsional motion for circular bars, independent of the complexity of the member, is that plane sections remain plane and simply rotate.

Kinematics

Referring to Fig. 5.3, the torsional motion of the bar is given by a function $\varphi(x)$ that assigns to each cross-section a rotation about the x-axis. The primary strain in the bar is the shear strain in the $x\theta$-plane

$$\gamma_{x\theta} = \gamma = r\frac{d\varphi}{dx} \,. \tag{5.4}$$

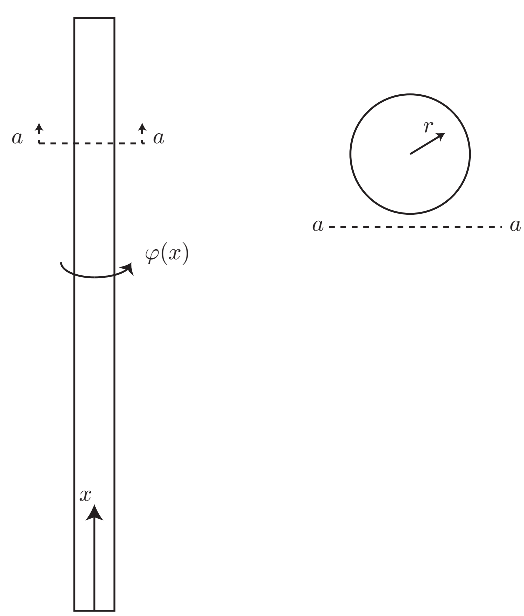


Figure 5.3: Torsion bar motion and coordinates. Axial coordinate x, rotation of cross-sections $\varphi(x)$, radial coordinate r.

Equilibrium

The important stress resultant in torsion is the internal torque about the x-axis given by a function $T(x)$. Assuming that the bar is subjected to distributed torsional loads $t(x)$ (torque per unit length in the x-direction), then the equilibrium equation for the bar is given as

$$\frac{dT}{dx} + t = 0 \tag{5.5}$$

in the static case.

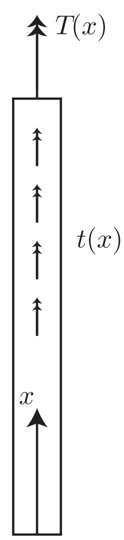


Figure 5.4: Torsion bar resultants and loads. Distributed load $t(x)$, internal torque $T(x)$. Double-headed arrows indicate toques according to right-hand rule with thumb aligned with arrows.

Material response

Assuming for simplicity a homogeneous cross-section, the system of equations is completed by a constitutive relation connecting the internal force to the angle of twist per unit length

$$T = GJ\frac{d\varphi}{dx}\,, \tag{5.6}$$

where it is assumed that the material is linear elastic with shear modulus G and the polar moment of inertia is $J = \int_A r^2\,dA$.

Equations (5.4)–(5.6) describe the complete response of a homogeneous circular elastic bar under torsional deformation. With slight modifications a non-prismatic shaft can be treated with virtually the same equations, if it is noted that the internal torque is related to the shear stress over the cross section with

$$T(x) = \int_{A(x)} \tau \, dA \,,$$

where τ is the shear stress in the θ-direction of the cross-section, $\sigma_{x\theta}$. This latter relation also allows for the treatment of composite cross-sections.

5.2.3 Bending

With regard to the bending of beams we review here the relations appropriate for describing the bending of a beam about a single axis. Further we only consider the case where the given axis is a principal axis of the beam cross-section. As with axial deformation and torsional deformation, the fundamental assumption of bending motion, independent of the complexity of the beam's cross-section, is that plane sections remain plane – only displacing and rotating.

Kinematics

Referring to Fig. 5.5, bending deformations are characterized by two functions that describe (1) the deflection of the *neutral axis* $v(x)$ and (2) the rotation of the beam's cross-sections $\theta(x)$. Within the assumptions of Bernoulli-Euler beam theory these two functions are not independent of each other. They are connected by the relation

$$\theta = \frac{dv}{dx} \,. \tag{5.7}$$

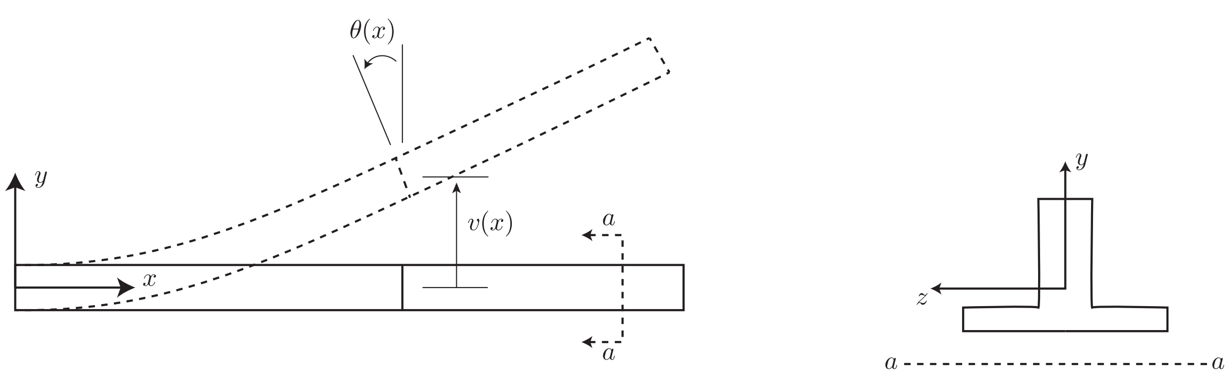


Figure 5.5: Bending motion and coordinates. Axial coordinate x, vertical displacement of cross-sections $v(x)$, rotation of cross-sections $\theta(x)$, cross-sectional coordinates y, z with origin determine by axial force equilibrium.

The primary strain in the beam is the normal strain in the x-direction given by

$$\varepsilon = -y\kappa \,, \tag{5.8}$$

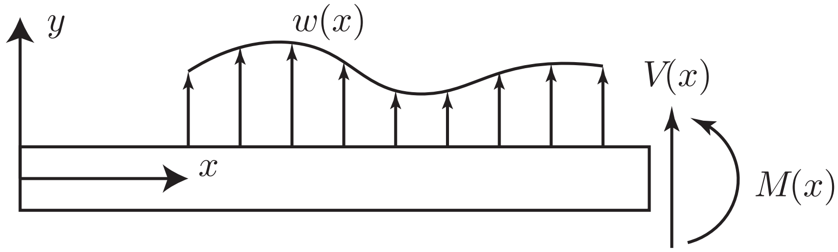


Figure 5.6: Beam resultants and loads. Distributed load $w(x)$, internal shear force $V(x)$, internal moment $M(x)$.

where κ is the beam curvature

$$\kappa \approx d\theta/dx\,. \tag{5.9}$$

with the approximation holding for small angles θ.

Equilibrium

In beams there are two equilibrium relations to observe. First, the moment equilibrium about the z-axis requires that

$$\frac{dM}{dx} + V = 0\,, \tag{5.10}$$

where $M(x)$ is the internal moment and $V(x)$ is the internal shear. Second, the force equilibrium in the y-direction requires that

$$\frac{dV}{dx} + w = 0\,, \tag{5.11}$$

where $w(x)$ represents any distributed loads on the beam in the y-direction; see Fig. 5.6.

Material response

Assuming for simplicity a homogeneous cross-section, the system of equations is completed by a constitutive relation connecting the internal moment to the curvature. To arrive at this relation we start with the definition of the bending moment M as

$$M = \int_A -y\sigma\, dA\,. \tag{5.12}$$

Under the assumption of linear elastic material response with Young modulus E the normal stress $\sigma = \sigma_{xx}$ is a linear function of the strain ε so that

$$M = \int_A -y(E\varepsilon)\, dA\,. \tag{5.13}$$

Substituting the kinematic relation from (5.8) gives

$$M = E\kappa \int_A y^2 \, dA \,, \tag{5.14}$$

if the modulus E does not depend on the area coordinates, as is the case for a homogeneous section. In any case the curvature κ is only a function of x, so that it can be taken outside the integral. With the definition of the second moment of area about the z-axis (also known as bending moment of inertia about the z-axis) $I = \int_A y^2 \, dA$ the last equation gives the desired relation between the bending moment and the curvature of the section at x

$$M(x) = EI\kappa(x) \,. \tag{5.15}$$

Equations (5.7)–(5.15) describe the complete response of a homogeneous elastic beam undergoing a bending deformation. With slight modifications a non-uniform beam can be treated with virtually the same relations, if it is noted that the internal moment is related to the normal stress σ by the generalization of (5.12)

$$M(x) = \int_{A(x)} -y\sigma \, dA \,.$$

It should also be observed that the location of the neutral axis, the location from which the y coordinate is measured, is determined from force equilibrium in the x-direction:

$$N(x) = \int_A \sigma \, dA \,,$$

which leads to the relation

$$N(x) = \int_A E(-y\kappa) \, dA = -E\kappa(x) \int_A y \, dA \,,$$

assuming again homogeneity of the material for simplicity. Because $N(x) = 0$ in the absence of axial loads this requires that

$$\int_A y \, dA = 0 \,,$$

which defines the location of the centroid of the cross-section. As long as the y-axis is measured from the centroid, the resultant of the normal stresses under purely flexural deformation is zero. In conclusion, *the neutral axis under pure flexure with linear elastic material response coincides with the centroid of the cross section for homogeneous cross-sections*. (For non-homogeneous material properties, these concepts need to be adjusted by leaving E underneath the integral signs.)

5.3 Section properties

The generalization of the kinematic and the equilibrium relations from the preceding section to the case of biaxial bending for a general section without axes of symmetry leads us to the general definition of the centroid, the first and second moments of area and the determination of the principal axes for the section.

5.3.1 Centroid

The assumption of plane sections remaining plane under biaxial bending leads to the following expression for the normal strain

$$\varepsilon = -y\kappa_z + z\kappa_y \,, \tag{5.16}$$

where κ_z is the curvature about the z-axis and κ_y the curvature about the y-axis for the section in Fig. 5.7. Note that the orthogonal y-z axis orientation assumes that the x-axis of the beam is normal to the yz-plane and points towards the viewer in accordance with the right hand rule. Equation (5.16) assumes that the plane passes through the point a in Fig. 5.7, which coincides with the centroid of the cross section (for a homogeneous section with no axial load), as will be shown in the following. The sign convention for (5.16) uses the right hand rule for the curvatures (which are rotations per unit of length) and accounts for the fact that a positive curvature about the z-axis gives rise to a negative strain at point m in Fig. 5.7, located in the positive quadrant of the section coordinate system, while a positive curvature about the y-axis gives rise to a positive strain at the same point.

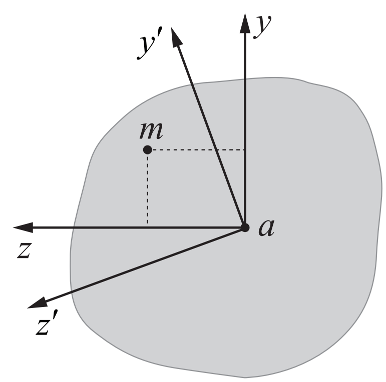


Figure 5.7: General cross section with principal axes y'-z'.

Under linear elastic material response with Young modulus E the normal stress $\sigma = \sigma_{xx}$ is given by $\sigma = E\varepsilon$ and generates bending moments M_z and M_y about the section axes according to

$$\begin{aligned} M_z &= \int_A -y\sigma \, dA \\ M_y &= \int_A \quad z\sigma \, dA \,. \end{aligned} \tag{5.17}$$

To ensure that the normal force $N(x)$ is zero requires that

$$N(x) = \int_A \sigma \, dA = 0 \quad \rightarrow \quad -E\kappa_z \int_A y \, dA + E\kappa_y \int_A z \, dA = 0 \,.$$

For the last equation to be satisfied for any curvature values requires that

$$\int_A y \, dA = 0 \qquad \text{and} \qquad \int_A z \, dA = 0 \,. \tag{5.18}$$

Consequently, a point a of the cross section in Fig. 5.7 that satisfies the conditions in (5.18) coincides with the centroid. Because $\sigma = 0$ at $y = 0$, $z = 0$ for the strain distribution in (5.16) the neutral axis of the section under biaxial bending passes through point a.

5.3.2 Moments of inertia and principal axes

Returning to the determination of the bending moments M_z and M_y in (5.17) and substituting the normal stress σ in terms of the normal strain ε and then the curvatures κ_z and κ_y according to (5.16) gives

$$\begin{aligned} M_z &= \int_A -yE(-y\kappa_z + z\kappa_y)\,dA \\ M_y &= \int_A \;\; zE(-y\kappa_z + z\kappa_y)\,dA \end{aligned} \quad \rightarrow \quad \begin{aligned} M_z &= \;\;\; E\left(\int_A y^2\,dA\right)\kappa_z - E\left(\int_A yz\,dA\right)\kappa_y \\ M_y &= -E\left(\int_A yz\,dA\right)\kappa_z + E\left(\int_A z^2\,dA\right)\kappa_y\,. \end{aligned}$$

With the definition of the second moments of area for the cross section

$$I_z = \int_A y^2\,dA \qquad I_y = \int_A z^2\,dA \qquad I_{zy} = I_{yz} = \int_A yz\,dA\,, \tag{5.19}$$

the last set of equations can be written in compact form as

$$\begin{pmatrix} M_z \\ M_y \end{pmatrix} = \begin{bmatrix} I_z & -I_{zy} \\ -I_{yz} & I_y \end{bmatrix} \begin{pmatrix} \kappa_z \\ \kappa_y \end{pmatrix}\,, \tag{5.20}$$

which furnishes the relation between the curvatures and the resulting bending moments under biaxial flexure. Equation (5.20) shows that for a general section with an arbitrary set of orthogonal reference axes y and z, as in Fig. 5.7, a curvature about the z-axis gives rise to moments about both section axes y and z. It is possible, however, to select a pair of orthogonal axes y'-z' for the section so that a curvature about one axis gives rise only to a moment about the same axis, i.e. that the flexural response about one axis is *uncoupled* from the response about the axis orthogonal to it. The determination of such a pair of axes results from the solution of a standard eigenvalue problem of the form (1.2) with

$$\boldsymbol{A} = \begin{bmatrix} I_z & -I_{zy} \\ -I_{yz} & I_y \end{bmatrix}\,.$$

Because $\boldsymbol{A}$ is symmetric, the two eigenvectors of this problem are orthogonal and give the directions y' and z', which are called the principal axes of the cross-section as schematically shown in Fig. 5.7. The corresponding eigenvalues λ_1 and λ_2 are equal to the principal moments of inertia $I_{z'}$ and $I_{y'}$, respectively.

5.4 Shear and moment diagrams

It is possible to solve the kinematic relation (5.9) separately from the equilibrium relations. After supplying the necessary boundary conditions for a beam segment under flexure it leads to the *curvature-area* method for determining the deformed shape of frame elements from the curvature distribution $\kappa(x)$. This will be discussed in the course on structural analysis.

The solution of the differential equilibrium relations in (5.10) and (5.11) leads to the determination of the bending moment distribution $M(x)$ of frame elements under flexure after satisfying the boundary conditions of the problem with the end moments of the beam segment. A common way of presenting the solution of (5.10) and (5.11) is in the form of bending moment and shear diagrams, as will be discussed next.

The differentiation of (5.10) leads to the following equation after the substitution of (5.11)

$$\frac{d^2M}{dx^2} - w(x) = 0\,. \tag{5.21}$$

Its solution consists of the homogeneous term with integration constants to be determined from the boundary conditions and the particular term, which depends on the form of $w(x)$.

We establish each term separately.

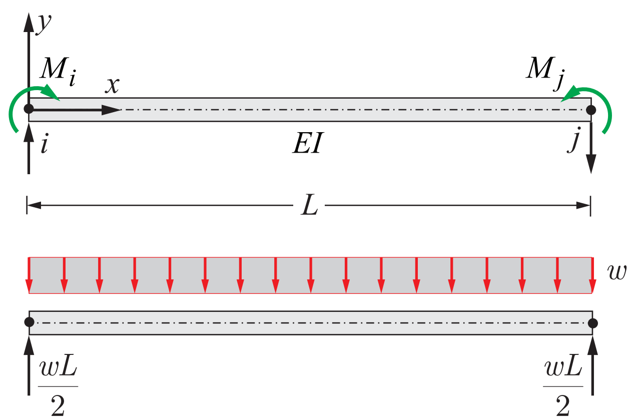


Figure 5.8: Homogeneous and particular solution of beam segment.

The homogeneous solution is

$$\frac{d^2M_h}{dx^2} = 0 \quad \rightarrow \quad M_h(x) = C_1\,x + C_2\,,$$

with two integration constants C_1 and C_2. These result from the moments at the boundary of the beam segment of length L in Fig. 5.8. Denoting the end at $x = 0$ with i and the opposite end with j we have

$$\begin{aligned} M_h(x=0) &= M_i = C_1(0) + C_2 \\ M_h(x=L) &= M_j = C_1(L) + C_2 \end{aligned} \quad \rightarrow \quad \begin{aligned} C_2 &= M_i \\ C_1 &= \frac{M_j - M_i}{L}\,, \end{aligned}$$

so that the homogeneous solution becomes

$$M_h(x) = M_i\left(1 - \frac{x}{L}\right) + M_j\frac{x}{L}\,. \tag{5.22}$$

In the absence of transverse loads over the beam segment the moment distribution is *linear*.

The general form of the particular solution under a uniformly distributed load w in Fig. 5.8 is

$$\frac{d^2 M_p}{dx^2} = -w \quad \rightarrow \quad M_p(x) = D_1\, x + D_2 - \frac{1}{2} w x^2 \,,$$

noting that the uniformly distributed load w in Fig. 5.8 is negative. Note that this form of the particular solution also includes $D_1 x + D_2$ – the form of the homogeneous solution. We do this here because our convention in structural analysis is to apply our boundary conditions directly to the homogeneous solution before determining the particular solution; cf. Chapter 2 where we applied boundary/initial conditions after adding the homogeneous and particular solutions together (either procedure is valid). Since we have made the homogeneous solution satisfy the boundary conditions, the particular solution needs to satisfy the homogeneous boundary conditions $M_i = 0$ and $M_j = 0$, which gives

$$M_p(x) = \frac{wL^2}{2}\left(1 - \frac{x}{L}\right)\frac{x}{L} \,.$$

The complete moment distribution under the uniform element load w then is

$$M(x) = M_h(x) + M_p(x) = M_i\left(1 - \frac{x}{L}\right) + M_j \frac{x}{L} + \frac{wL^2}{2}\left(1 - \frac{x}{L}\right)\frac{x}{L} \,,$$

i.e. a parabolic distribution.

The shear force can be similarly separated into the homogeneous and the particular solution with the following result

$$\begin{aligned} V_h(x) &= -\frac{dM_h}{dx} = -C_1 = \frac{M_i - M_j}{L} \\ V_p(x) &= -\frac{dM_p}{dx} \qquad\;\; = \frac{wL}{2}\left(1 - \frac{2x}{L}\right) \end{aligned} \,.$$

Note that the shear force is constant for the homogeneous solution with end values equal to

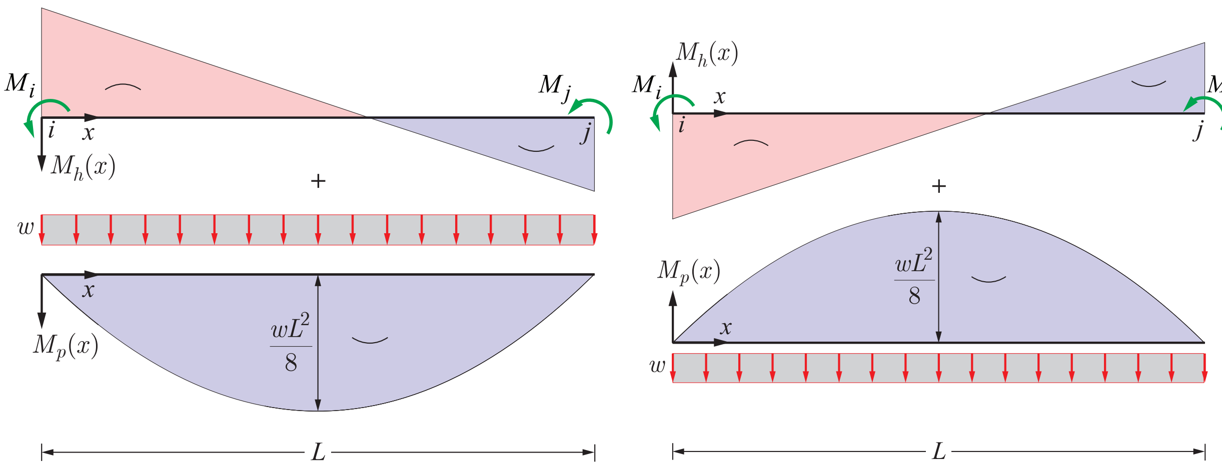
$$V_{hi} = V_{hj} = \frac{M_i - M_j}{L} \,,$$

while it is linear for the particular solution with end values equal to

$$V_{pj} = -V_{pi} = \frac{wL}{2} \,,$$

as shown in the lower half of Fig. 5.8.

The preceding derivations show that it is possible to determine the shear and bending moment distribution of any beam segment *from the moments at the ends of the beam segment and the transverse loads in the segment*. While the loading is given at the start of any analysis, the moments at the ends of the beam segments can only be determined with methods of structural analysis, which require the simultaneous satisfaction of the structure equilibrium, the structure kinematics, and the



(a) Moment diagram convention A (b) Moment diagram convention B

Figure 5.9: Two conventions for displaying the moment diagram of a beam segment

force-deformation relations of the constituent elements of the structural model. Such methods are discussed in courses of structural analysis.

Once the moments at the ends of each beam segment are established, it is convenient to display the results for the moment and shear distribution in terms of bending moment and shear diagrams, to facilitate the response evaluation and the subsequent design of the structural members.

Whereas the sign of the bending moments and shears is uniquely defined from (5.9) and (5.10), there are two ways for displaying the moment diagram of a beam segment. Fig. 5.9, which displays the homogeneous solution separately from the particular solution under a uniformly distributed load w, assumes that the moment M_i is negative while the moment M_j is positive.

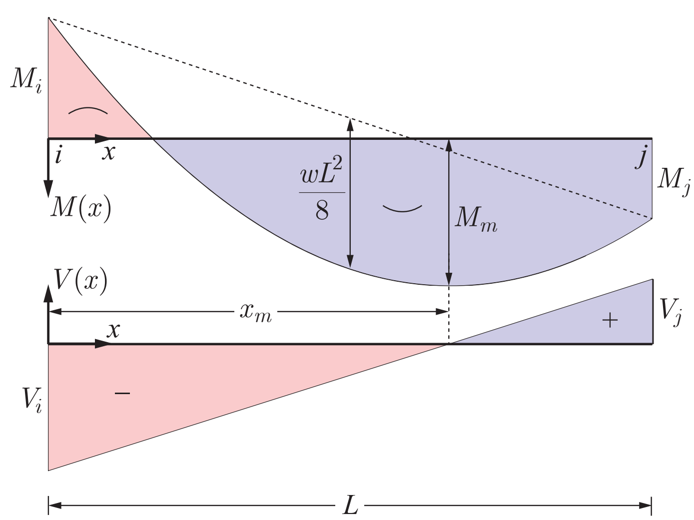


Figure 5.10: Combined bending moment and shear diagram for the beam segment in Fig. 5.9(a).

For convention A in Fig. 5.9(a) the positive moment axis *points downward*, so that the moment is drawn *on the tension side of the beam*.

For convention B in in Fig. 5.9(b) the positive moment axis points upward, consistent with the definition of a positive y-axis in Fig. 5.6.

It is not important which convention is used to display the moment diagrams, as long as the diagram is accompanied by an indication of *which way the beam bends* over the corresponding portion of the beam, as illustrated in Fig. 5.9.

The following examples adopt the convention A for the bending moments, as shown in Fig. 5.10, which displays the diagram for the total bending moment $M(x) = M_h(x) + M_p(x)$ for Fig. 5.9(a) along with the shear force diagram, which is displayed with the positive shear axis pointing upward, consistent with the positive y-axis in Fig. 5.6. This way of displaying the moment and shear diagram has the advantage that the shear force value corresponds ”visually” to the slope of the bending moment diagram if one disregards the way the the moment axis points in Fig. 5.10. This way the reversal of the moment axis for the slope evaluation accounts for the negative sign of the moment slope in the equilibrium equation (5.9).

According to (5.9) the bending moment has a local extremum at the location of zero shear. This allows us to determine the moment M_m in Fig. 5.10 by first determining the location x_m of zero shear. To this end we note that

$$x_m = \frac{|V_i|}{w} ,$$

and that the change ΔM of the bending moment between the end i and the section m is equal to the area under the shear force diagram between these two sections, i.e.

$$\Delta M = \frac{1}{2} |V_i| \, x_m = \frac{|V_i|^2}{2w} ,$$

so that the maximum moment M_m at section m is

$$M_m = -|M_i| + \Delta M = -|M_i| + \frac{|V_i|^2}{2w} .$$

We demonstrate these concepts with two simple structural models for which the internal force state depends only on equilibrium. Such structural models are called *statically determinate*.

Example 5.1 (Moment diagram for simply supported beam with overhang)**.** The purpose of this example is the determination of the moment distribution in the simply supported beam with overhang in Fig. 5.11. The beam is subjected to a uniformly distributed load $w = -5$ and a concentrated force of 20 units at the tip of the overhang. Using virtual cuts at the supports and at the middle of the free span we separate the structure into 3 free bodies a, b and c. The three cuts form three free body slices, which are shown separately below the free bodies a, b, and c. From the vertical force equilibrium of the middle slice we conclude that the transverse force is the same at both faces of the slice. We denote it with V_1. From moment equilibrium of the middle and the right slice we conclude that the bending moment at both faces of each slice is the same. We denote these with M_1 for the middle slice and M_2 for the right slice.

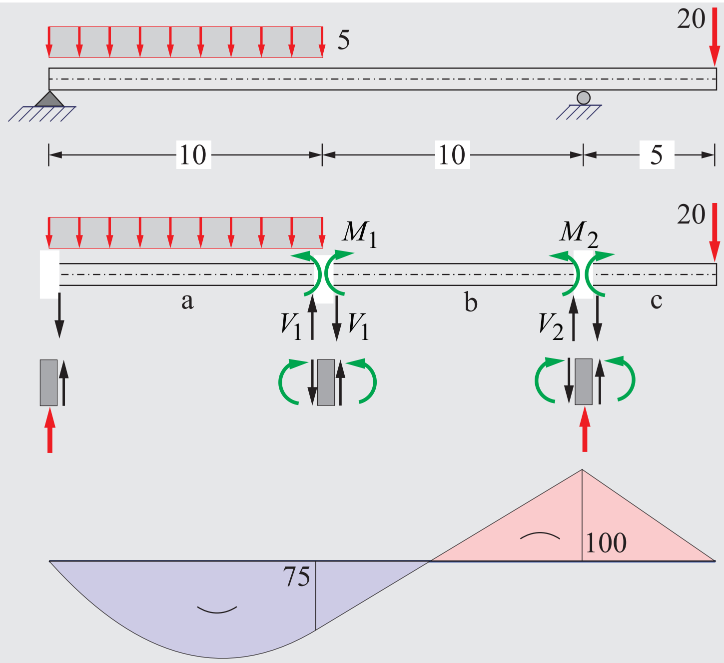


Figure 5.11: Simply supported beam with overhang.

The moment equilibrium of the free body c about the left end gives

$$M_2 = -20(5) = -100\,.$$

The moment equilibrium of the free body a about the left end gives

$$V_1(10) + M_1 + w(10)\frac{10}{2} = 0\,.$$

The moment equilibrium of the free body b about the right end gives

$$V_1(10) - M_1 + M_2 = 0\,.$$

Adding up the last two equations gives

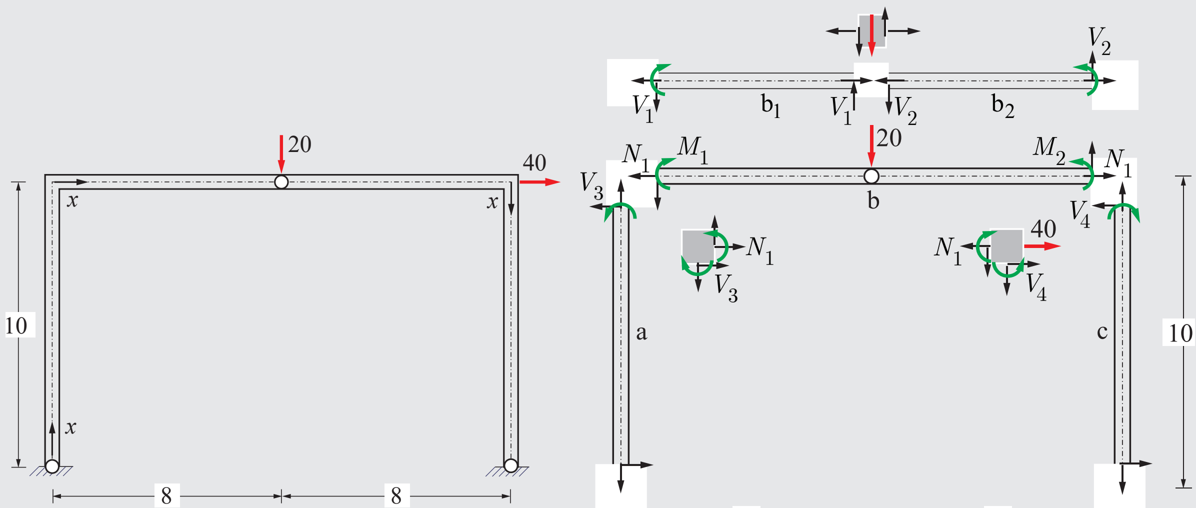
$$V_1(20) + w(10)\frac{10}{2} + M_2 = 0 \quad \to \quad V_1 = -w(2.5) + \frac{100}{20} = 17.5\,,$$

which when substituted into one of the two initial equations gives

$$M_1 = V_1(10) + M_2 = 75\,.$$

Because $w = 0$ for the free bodies b and c the moment distribution in these is linear. With a constant $w = -5$ along the free body a the moment distribution is parabolic. Fig. 5.11 shows the resulting bending moment diagram. The determination of the shear diagram and the value of the maximum bending moment is left as an exercise for the reader.

Example 5.2 (Moment diagram of three hinge frame)**.** The aim is the determination of the moment diagram for the three hinge frame in Fig. 5.12(a) under the action of a vertical force of 20 units and a horizontal force of 40 units. The figure also shows the orientation of the x-axis along the frame members. Suitable virtual cuts in Fig. 5.12(b) isolate free bodies for the columns and the girder and the two beam-column joints. A further virtual cut separates the girder into two free bodies of equal length with an additional joint free body at mid-span. At the cuts internal forces arise consisting of the normal force N, the shear force V, and the bending moment M. Only those internal forces that figure in the subsequent calculations are labeled. The repetition of labels is kept to a minimum to reduce clutter, since the relation between the internal forces in the figure is clear from the consideration of free body equilibrium.



(a) Structure (b) Element and joint free bodies

Figure 5.12: Equilibrium for three hinge frame.

$$-V_1 - 20 + V_2 = 0\,.$$

The moment equilibrium of the left girder half free body about either end gives

$$V_1(8) - M_1 = 0\,.$$

The moment equilibrium of the right girder half free body about either end gives

$$V_2(8) + M_2 = 0\,.$$

Using the last two equations to substitute for the shear forces in the first equation gives

$$-\frac{M_1}{8} - \frac{M_2}{8} - 20 = 0 \quad \to \quad M_1 + M_2 = -160\,.$$

From the horizontal force equilibrium of the entire girder free body we conclude that the axial force N_1 is the same at either end. The horizontal force equilibrium of the two beam-column joint free bodies gives

$$\begin{aligned} N_1 + V_3 &= 0 \\ 40 + V_4 - N_1 &= 0 \end{aligned} \quad \to \quad 40 + V_3 + V_4 = 0\,.$$

The moment equilibrium of the left column free body about either end gives

$$V_3(10) + M_1 = 0\,.$$

The moment equilibrium of the right column free body about either end gives

$$V_4(10) - M_2 = 0\,.$$

Using the last two equations to substitute for the shear forces V_3 and V_4 in the first equation gives

$$40 + \frac{M_2}{10} - \frac{M_1}{10} = 0 \quad \to \quad M_2 - M_1 = -400\,.$$

We are left with two equations for the moments M_1 and M_2:

$$\begin{aligned} M_1 + M_2 &= -160 \\ M_2 - M_1 &= -400\,. \end{aligned}$$

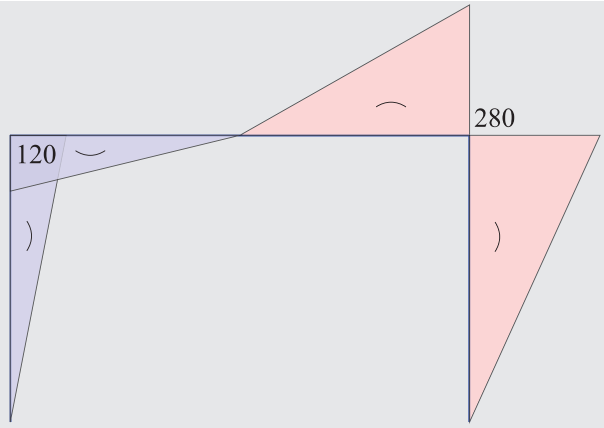


Figure 5.13: Moment diagram for three hinge frame under loading in Fig. 5.12(a).

The first moment equation reflects the *gravity load resistance* of the three hinge frame and the second *the lateral load resistance*. The solution of these two equations gives $M_1 = 120$ and $M_2 = -280$. Fig. 5.13 shows the bending moment diagram. Note that the bending moment is positive in the left column and the left half of the girder, but negative in the right half of the girder and in the right column, on account of the direction of the x-axis. Both columns, however, bend as shown in the figure. This example demonstrates clearly the importance of clearly identifying the x-axis of each element, which determines the positive shear and moment direction. Finally, it is best to communicate the results by sketching the way members bend.

The determination of the shear diagram is left as an exercise for the reader.

5.5 Beam stiffness coefficients

Combining the kinematic relations in (5.7) and (5.9) leads to

$$\kappa = \frac{d^2 v}{dx^2}\,. \tag{5.23}$$

Substituting into the bending moment-curvature relation for linear elastic material response in (5.15) gives

$$M(x) = EI(x)\frac{d^2 v}{dx^2}\,. \tag{5.24}$$

Finally, substituting this into the equilibrium equation in (5.21) with $w(x) = 0$ gives the differential equation (DE) for beam flexure

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 v}{dx^2}\right] = 0\,, \tag{5.25}$$

whose solution with appropriate boundary conditions furnishes the relation between the end forces of a beam segment and the corresponding kinematic variables.

Under the assumption that the beam element is homogeneous and prismatic, i.e. that $EI(x) = EI$ the DE in (5.25) becomes

$$EI\frac{d^4v}{dx^4} = 0\,, \tag{5.26}$$

whose general solution is a cubic polynomial

$$v(x) = C_1x^3 + C_2x^2 + C_3x + C_4\,. \tag{5.27}$$

For determining the integration constants $C_1, \ldots, C_4$ we need to impose boundary conditions on the function or its derivatives. For a beam element of length L with both ends i and j continuous we can imposed conditions on the translation v and on the rotation θ at the ends, i.e.

$$\begin{aligned}
v(x=0) &= v_i\,,\\
\left.\frac{dv}{dx}\right|_{x=0} &= \theta_i\,,\\
v(x=L) &= v_j\,,\\
\left.\frac{dv}{dx}\right|_{x=L} &= \theta_j\,.
\end{aligned}$$

The kinematic variables v_i, v_j, θ_i and θ_j are the *degrees-of-freedom* of the beam element. It is convenient to collect these in a *displacement* vector $\boldsymbol{u}$ with four components according to Fig. 5.14.

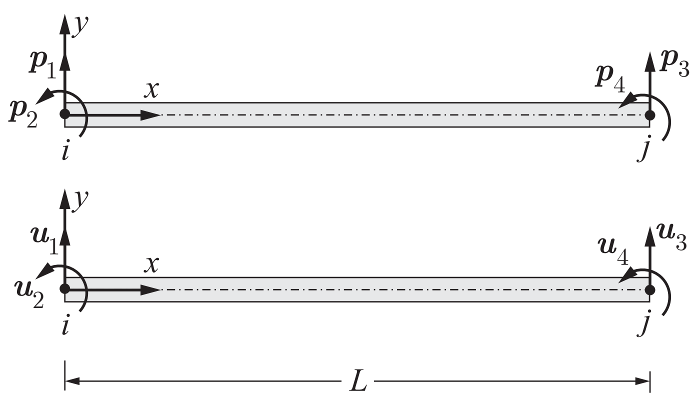
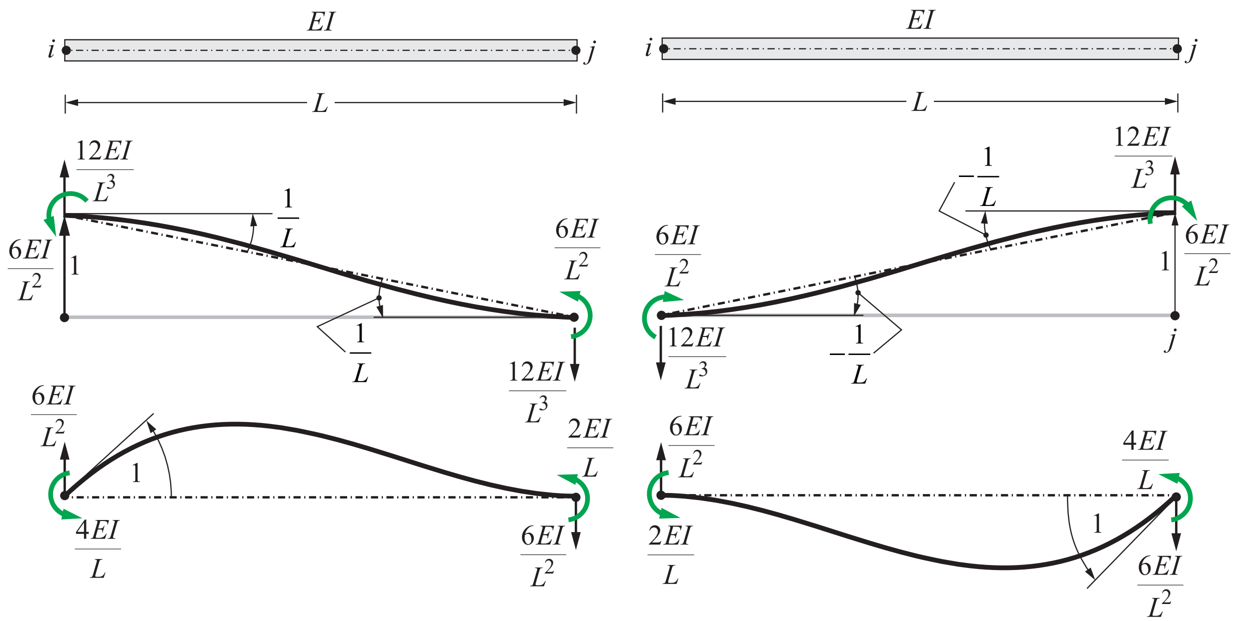


Figure 5.14: Degrees-of-freedom and corresponding force variables for beam element.

Expressing the boundary conditions in terms of the general solution in (5.27) gives

$$\begin{aligned}
v(x=0) = v_i &= C_1(0) + C_2(0) + C_3(0) + C_4\,,\\
\left.\frac{dv}{dx}\right|_{x=0} = \theta_i &= 3C_1(0) + 2C_2(0) + C_3\,,\\
v(x=L) = v_j &= C_1(L^3) + C_2(L^2) + C_3(L) + C_4\,,\\
\left.\frac{dv}{dx}\right|_{x=L} = \theta_j &= 3C_1(L^2) + 2C_2(L) + C_3\,.
\end{aligned}$$



(a) Stiffness coefficients for $u_i = 1, \theta_i = 1$ (b) Stiffness coefficients for $u_j = 1, \theta_j = 1$

Figure 5.15: Stiffness coefficients for beam element with continuous ends

The solution of the above equations gives the four integration constants $C_1, \ldots, C_4$ in terms of the components of $\boldsymbol{u}$. Substituting these into (5.27) gives the solution of the DE that satisfies the boundary conditions of the beam element with continuous ends. It is

$$v(x) = \left(2\xi^3 - 3\xi^2 + 1\right) u_i + \left(\xi^3 - 2\xi^2 + \xi\right) \theta_i L + \left(-2\xi^3 + 3\xi^2\right) u_j + \left(\xi^3 - \xi^2\right) \theta_j L \,, \tag{5.28}$$

where $\xi = x/L$. We can now express the *generalized end forces* $\boldsymbol{p}$ in Fig. 5.14 in terms of the general solution in (5.28) by noting that

$$\begin{aligned} M(x=0) &= EI \left.\frac{d^2v}{dx^2}\right|_{x=0} = -\boldsymbol{p}_2 \\ M(x=L) &= EI \left.\frac{d^2v}{dx^2}\right|_{x=L} = \quad \boldsymbol{p}_4 \,, \end{aligned}$$

noting the difference in the positive sign convention for $\boldsymbol{p}_2$ relative to the corresponding bending moment $M(x=0)$. Substituting the solution from (5.28) gives

$$\begin{aligned} \boldsymbol{p}_2 &= \frac{6EI}{L^2} v_i + \frac{4EI}{L}\theta_i - \frac{6EI}{L^2} v_j + \frac{2EI}{L}\theta_j \\ \boldsymbol{p}_4 &= \frac{6EI}{L^2} v_i + \frac{2EI}{L}\theta_i - \frac{6EI}{L^2} v_j + \frac{4EI}{L}\theta_j \,. \end{aligned} \tag{5.29}$$

The end forces $\boldsymbol{p}_2$ and $\boldsymbol{p}_4$ are shown for a unit value of each beam degree-of-freedom in Fig. 5.15. Denoting the rotation of the line connecting the end points with β and noting that this *chord rotation* is given by

$$\beta = \frac{v_j - v_i}{L}$$

simplifies the force-displacement relations in (5.29) to

$$\begin{aligned} \boldsymbol{p}_2 &= \frac{2EI}{L}\left(2\theta_i + \theta_j - 3\beta\right) \\ \boldsymbol{p}_4 &= \frac{2EI}{L}\left(\theta_i + 2\theta_j - 3\beta\right) . \end{aligned} \tag{5.30}$$

The relations in (5.30) are known as *slope-deflection equations*. Note that a positive chord rotation β is consistently defined as counter-clockwise like all rotations in the plane of the beam.

The most convenient way for establishing the remaining end forces $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ is to satisfy the moment equilibrium of the free body for the beam element in Fig. 5.14. It gives

$$\begin{aligned} \boldsymbol{p}_1 &= \quad \frac{\boldsymbol{p}_2 + \boldsymbol{p}_4}{L} \\ \boldsymbol{p}_3 &= -\frac{\boldsymbol{p}_2 + \boldsymbol{p}_4}{L} . \end{aligned}$$

Substituting the expressions for $\boldsymbol{p}_2$ and $\boldsymbol{p}_4$ from (5.29) gives

$$\begin{aligned} \boldsymbol{p}_1 &= \quad \frac{12EI}{L^3}v_i + \frac{6EI}{L^2}\theta_i - \frac{12EI}{L^3}v_j + \frac{6EI}{L^2}\theta_j \\ \boldsymbol{p}_3 &= -\frac{12EI}{L^3}v_i - \frac{6EI}{L^2}\theta_i + \frac{12EI}{L^3}v_j - \frac{6EI}{L^2}\theta_j . \end{aligned} \tag{5.31}$$

These forces are shown for a unit value of each beam degree-of-freedom in Fig. 5.15.

After introducing the components of the displacement vector $\boldsymbol{u}$ for v_i, θ_i, v_j and θ_j and combining the results from (5.29) and (5.31) gives

$$\begin{pmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \\ \boldsymbol{p}_4 \end{pmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{pmatrix} ,$$

which can be written in compact form as

$$\boldsymbol{p} = \boldsymbol{k}\boldsymbol{u} \tag{5.32}$$

with $\boldsymbol{k}$ *the stiffness matrix of the homogeneous, prismatic beam element*. The components of the stiffness matrix are known as stiffness coefficients.

The force-displacement relation of a beam element with only one end continuous, because of the presence of a flexural hinge at the other end, can be established by a similar process, noting that in this case the boundary conditions are

$$\begin{aligned} v(x=0) &= v_i \\ \left.\frac{dv}{dx}\right|_{x=0} &= \theta_i \\ v(x=L) &= v_j \\ \left.\frac{d^2v}{dx^2}\right|_{x=L} &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} v(x=0) &= v_i \\ \left.\frac{d^2v}{dx^2}\right|_{x=0} &= 0 \\ v(x=L) &= v_j \\ \left.\frac{dv}{dx}\right|_{x=L} &= \theta_j \,. \end{aligned}$$

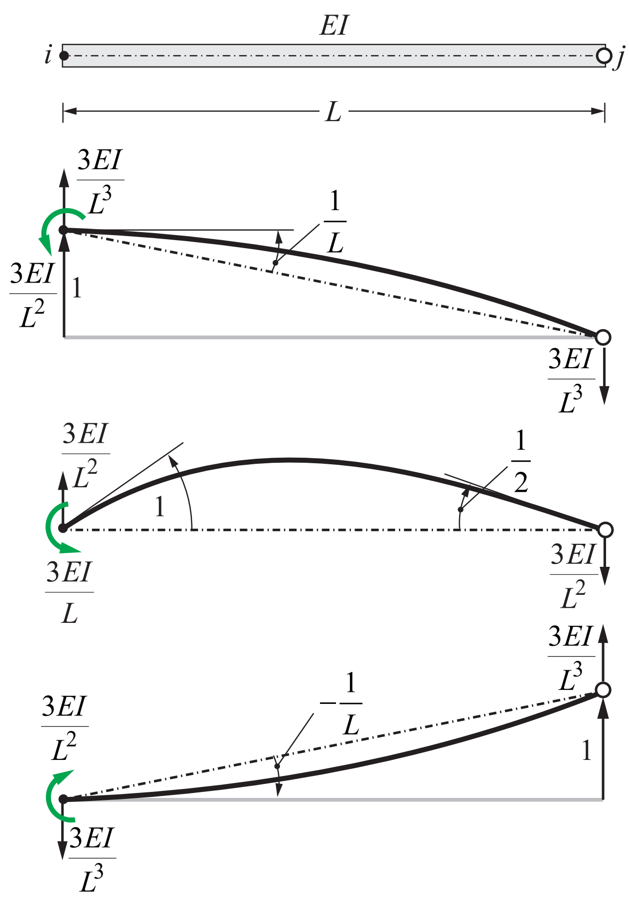


Figure 5.16: Stiffness coefficients for beam element with continuous end at i and hinge at j.

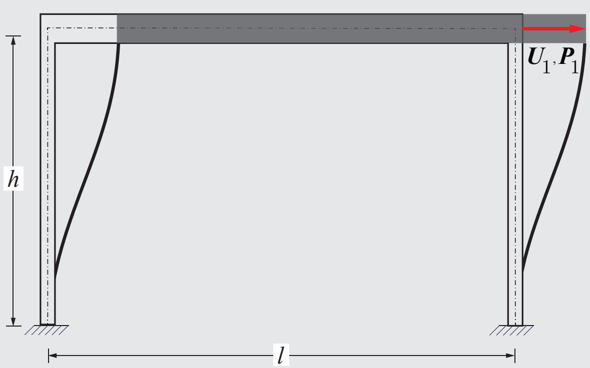
We do not pursue the details of the solution in the following and offer it instead as an exercise for the reader. The resulting force-displacement relations for a beam element with a continuous end at i and a flexural hinge at j are

$$\begin{pmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \\ \boldsymbol{p}_4 \end{pmatrix} = \begin{bmatrix} \dfrac{3EI}{L^3} & \dfrac{3EI}{L^2} & -\dfrac{3EI}{L^3} & 0 \\ \dfrac{3EI}{L^2} & \dfrac{3EI}{L} & -\dfrac{3EI}{L^2} & 0 \\ -\dfrac{3EI}{L^3} & -\dfrac{3EI}{L^2} & \dfrac{3EI}{L^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \end{pmatrix} . \tag{5.33}$$

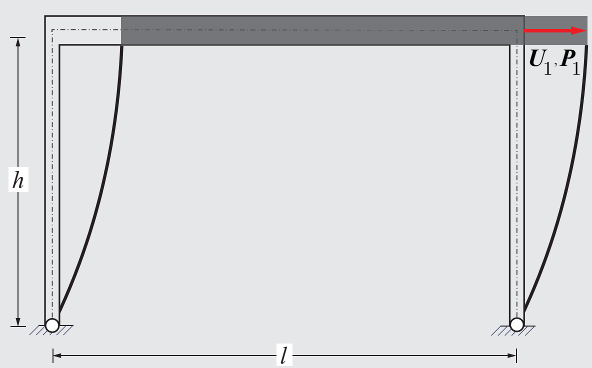
These forces are shown for a unit value of each beam degree-of-freedom in Fig. 5.16. The force-displacement relations for a beam element with a continuous end at j and a flexural hinge at i can be obtained by exchanging the second and the fourth rows as well as the second and the fourth columns of the stiffness matrix in (5.33).

It is clear from the results in (5.33) that the rotation at end j does not affect the element response and that the moment at end i is zero.

Example 5.3 (Lateral stiffness of one story frame models)**.** Fig. 5.17 shows two structural models approximating one-story frame response. In both models the girder is assumed to be infinitely rigid in flexure.



(a) Shear beam model for one-story frame



(b) Rigid girder and pinned columns at base

Figure 5.17: Two models for one story frame.

In the first model in Fig. 5.17(a) the columns are fixed at the base. This model is known as *shear beam model*. Under a lateral translation $\boldsymbol{U}_1$ of the girder the column rotation is zero at the top and at the base, so that the deformed shape of the columns corresponds to the upper deformed shape of Fig. 5.15(a). The lateral force $\boldsymbol{P}_1$ in this case is

$$\boldsymbol{P}_1 = \frac{12EI}{h^3}\boldsymbol{U}_1 ,$$

and the bending moment at the top and at the base of each column is

$$|M_t| = |M_b| = \frac{6EI}{h^2}\boldsymbol{U}_1 \,.$$

In the second model in Fig. 5.17(b) the columns are pinned at the base. In this case the deformed shape of the columns corresponds to the topmost deformed shape in Fig. 5.16. The lateral force $\boldsymbol{P}_1$ in this case is

$$\boldsymbol{P}_1 = \frac{3EI}{h^3}\boldsymbol{U}_1 \,,$$

and the bending moment at the top of each column is

$$|M_t| = |M_b| = \frac{3EI}{h^2}\boldsymbol{U}_1 \,.$$

5.6 Rotational motion

5.6.1 Newton's 2^{nd} law

Newton's second law states that the net force on an object is equal to the time rate of change of its linear momentum, i.e. the mass of the object multiplied by the acceleration of the object. For translational motion, this can be written in vector form as:

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{x}} \,,$$

where $\mathbf{F}$ is a translational force vector, m is the translational inertia (i.e. mass) and is assumed to be constant, and $\mathbf{a} = \ddot{\mathbf{x}}$ is a translational acceleration. For rotational motion of rigid bodies, the Cauchy-Euler extension of Newton's second law, *principle of angular momentum*, is:

$$T = J\alpha = J\ddot{\theta} \,, \tag{5.34}$$

where T is a torque (component) about a given axis, J is the rotational inertia (i.e. mass moment of inertia) about the same axis and $\alpha = \ddot{\theta}$ is the rotational acceleration (component) oriented along the axis. In these notes we only consider the simplest setting of rotation about a fixed axis. In general J is a matrix/tensor and the rotational acceleration α is a vector; see e.g. Gross et al. [2014, Section 3.4].

5.6.2 Rotational inertia

The general equation for rotational inertia is:

$$J = \int r^2 dM \,, \tag{5.35}$$

where r is the shortest distance of a small mass dM from the axis of rotation.

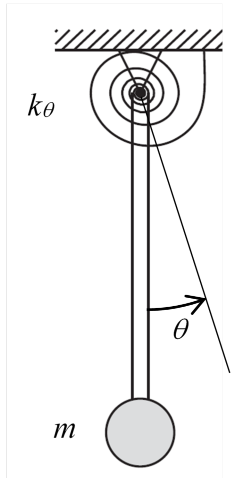


Figure 5.18: Pendulum consisting of a sphere connected to a rigid rod with a rotational spring at the pivot point.

For example, in Fig. 5.18 a sphere with a translational inertia (i.e. mass) of m is attached to the end of a rigid rod of length L. The sphere is assumed to move in the plane of the paper. Assume the rod is weightless. A real rod obviously cannot be weightless, but it might be appropriate to assume it is weightless if the weight of the sphere is considerably larger than the weight of the rod. In this case, if the mass of rod can be assumed to be zero, this means that the rod also has no translational or rotational inertia (i.e. assume $m_{rod} = J_{rod} = 0$).

The rotational inertia of the sphere is dependent on the considered axis of rotation. For example, the rotational inertia of a sphere about its own centroid can be calculated using (5.35) as $J_c = (2/5)mR^2$, where R is the radius of the sphere. However, the rotational inertia of the sphere about the rod pivot point in Fig. 5.18 is instead:

$$J_A = J_c + mL^2 \,. \tag{5.36}$$

Equation (5.36) is known as the parallel axis theorem.

5.6.3 Rotational springs

For a linear translational spring, the spring force F_s is:

$$F_s = k\delta \,, \tag{5.37}$$

where k is the spring stiffness and δ is the displacement of the spring from the zero force position where the spring is completely unloaded. For a linear rotational spring, as shown in Fig. 5.18, the rotational spring force (i.e. the torque applied by the spring) is similarly defined as:

$$T_s = k_\theta\theta \,, \tag{5.38}$$

where θ is the rotation of the spring from the zero torque position where the spring is completely unloaded.

Consider again the mass spring pendulum in Fig. 5.18 subjected to gravity acting downwards in the plane of the paper. Assume the spring is unstretched when $\theta = 0$. At any instant in time, i.e. at any rotation θ, the principle of angular momentum must hold. Therefore, assuming planar motion, (5.34) for any rotation θ, yields:

$$T_g + T_s = J_A \ddot{\theta}\,, \tag{5.39}$$

where T_s is the clockwise torque about the pivot point provided by the rotational spring, $\ddot{\theta}$ is the clockwise rotational acceleration, and T_g is the clockwise torque about the pivot point due to gravity, which can be written as:

$$T_g = mgL \sin\theta\,. \tag{5.40}$$

Using equations (5.40) and (5.38), (5.39) can be written:

$$mgL \sin\theta + k_\theta \theta = \left(J_c + mL^2\right) \ddot{\theta}\,, \tag{5.41}$$

where J_c is the mass moment of inertia of the sphere about its centroid. Equation (5.41) is the equation of motion for the system.

5.7 Momentum and energy

This section provides a very brief reminder of some fundamental equations of momentum and energy. It is expected that you will be familiar with these concepts and be able to apply these equations.

5.7.1 Conservation of momentum for impact

At impact, both energy and momentum are conserved. If an impact is elastic, conservation of energy and momentum equations can be easily applied. However, if a significant amount energy is dissipated at impact, i.e. transformed into heat or sound, the impact can not be considered elastic. In this case, conservation of energy can be difficult to apply but conservation of momentum is still straightforward. For example, assume two objects are travelling along the same axis but at different speeds. When these objects impact each other, conservation of momentum can be written as:

$$m_1 v_1^- + m_2 v_2^- = m_1 v_1^+ + m_2 v_2^+\,, \tag{5.42}$$

where m_1 and m_2 are the masses of the two objects, v_1 and v_2 are the velocities of the two masses, and the superscripts $^-$ and $^+$ indicate velocities before and after impact, respectively. If an impact is perfectly plastic, meaning that the two objects remain in contact after impact, (5.42) can be written as:

$$m_1 v_1^- + m_2 v_2^- = \left(m_1 + m_2\right) v_3^+\,,$$

where v_3^+ is the velocity of both objects after impact.

5.7.2 Potential and kinetic energy

The basic equation of gravitational potential energy is $PE = mgh$. For a linear translational spring with a spring force described by (5.37), the potential energy can be written as:

$$PE = \frac{1}{2} k_s \delta^2 ,$$

where k_s is again the translational spring stiffness. Similarly, for a rotational spring such as that in Fig. 5.18 whose torque is described by (5.38), the potential energy is:

$$PE = \frac{1}{2} k_\theta \theta^2 .$$

The well-known equation for translational kinetic energy is $KE = \frac{1}{2} mv^2$. In rotational form, the kinetic energy is:

$$KE = \frac{1}{2} J \dot{\theta}^2 ,$$

where J is again the mass moment of inertia about a given axis and $\dot{\theta}$ is the rotational velocity.

5.8 Check your understanding problems

1. Determine the shear force distribution for Example 5.1 and draw the shear diagram.
2. Determine the location and magnitude of the maximum bending moment along the simple span of the beam for Example 5.1.
3. Derive the force-deformation relation for the beam with one end continuous and a flexural hinge at the opposite end in (5.33).
4. Determine the shear force diagram for the three hinge frame of Example 5.2. Define the positive shear direction consistent with the selection of the x-axis orientation in the example, but communicate the results by sketching a free body slice in the middle of the girder and the columns with the shear forces on it.
5. Determine the normal force in the columns and in the girder for the three hinge frame of Example 5.2.
6. Determine the support reactions for the three hinge frame of Example 5.2 and check global equilibrium.
7. Draw the bending moment diagrams for the two structural models in Fig. 5.17.

Chapter 6

Elements of design

6.1 Introduction

Previous sections of this primer addressed elements of structural mechanics and structural analysis, including some of the underlying mathematical methods. This chapter shifts the discussion to structural design.

For any facility design, the structural engineer generally will work as part of a team engaged in the design of the facility. The structural engineer's tasks are oriented toward developing a structural system that fits within the functional space of the facility and that provides an efficient load path for both vertical and lateral loads. The design and analysis tasks may include the following:

- *Identify a concept for a structural system* that will be capable of efficiently providing a load path. For the pedestrian bridge crossing a freeway as shown in Fig. 6.1, the concept is of an arch spanning across a freeway, with tension hangers that support the suspended walkway. For the building of Fig. 6.2, the concept is of a beam-column frame that resists vertical and lateral forces.

- *Estimate the loads*. Once the general concept is developed, preliminary member sizes can be estimated using experience or, where experience is lacking, by making an educated guess. Given the preliminary member sizes, design loads associated with self-weight can be estimated. Other loads due to roadway surfaces, cladding, traffic and live loads, and wind and earthquake loads can also be estimated, either from first principles or, more likely, using specifications from bridge and building codes. As the loads depend on the member sizes, some iteration may be required.

- *Analyze the structure*. Given a structural idealization and design loads, the structure can be analyzed to determine the structure support reactions, the member internal forces and moments, and deflections of individual members of the entire structure.

- *Develop final member/structure proportions*. Now that member internal forces and moments are determined by analysis, the members and their connections to one another can be designed. This step is usually driven by considerations of safety. However, serviceability must

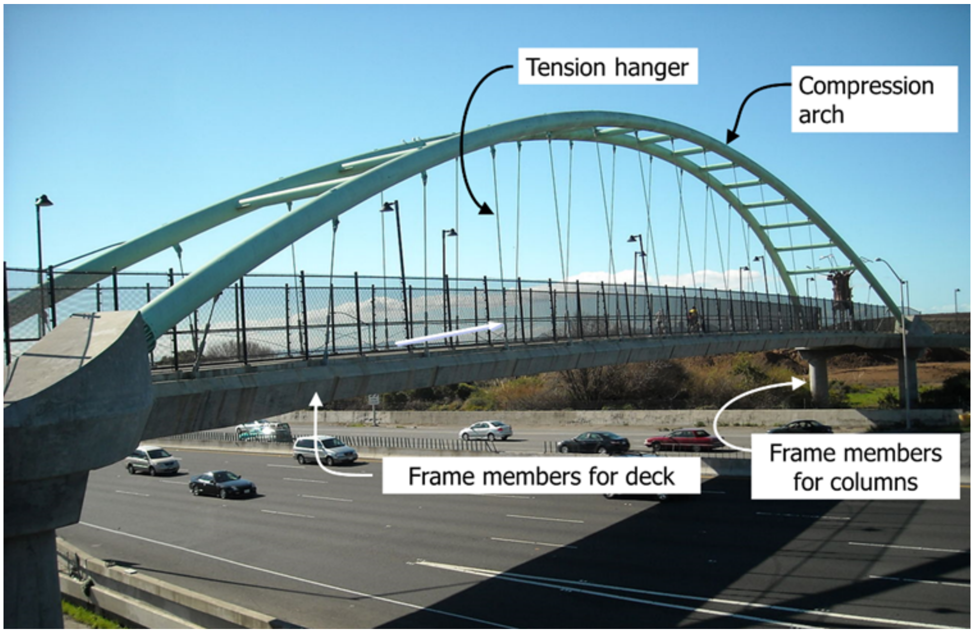


Figure 6.1: Berkeley I-80 pedestrian bridge. Photo by Daniel Ramirez from Honolulu, USA - Uploaded by Kurpfalzbilder.de. Licensed under CC BY 2.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Berkeley_I-80_bridge_02.jpg#mediaviewer/File:Berkeley_I-80_bridge_02.jpg.

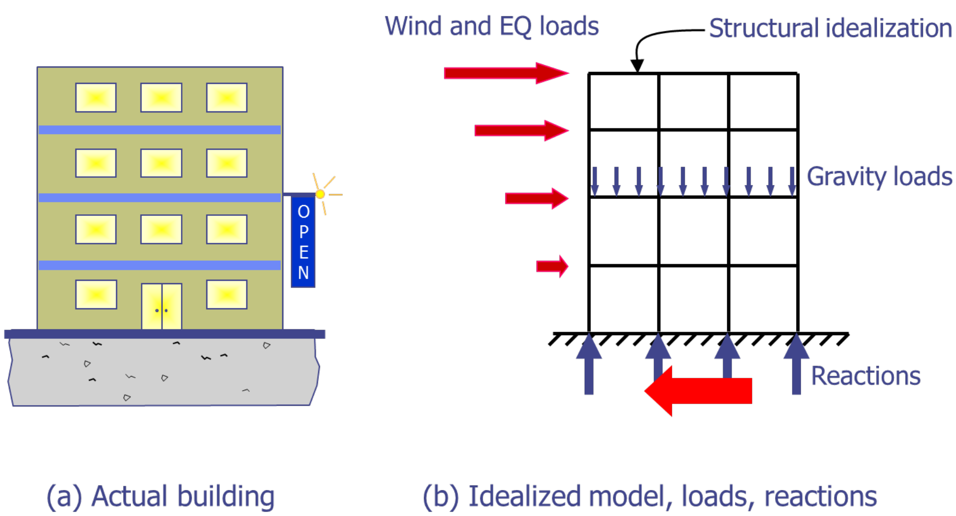


Figure 6.2: Idealized model, loads and reactions for a building concept.

also be considered. A trend is toward including environmental impacts, sustainability, and resilience by design.

- *Specify the design*. The design intent must be conveyed via design and construction documents. The *design documents* contain the calculations used to demonstrate safety and serviceability of the structure. The *construction documents* contain information on how to build the structure, documents such as detailed specifications for materials and components, and detailed structural drawings that convey unambiguously the required dimensions, member sizes, member connections, and any other required details.

The process outlined above involves both elements of structural design and structural analysis. *Structural analysis* (the third bullet) involves the determination of the reactions, internal actions, and deformations/deflections of the structure under the design loads. *Structural design* is a much broader endeavor, involving development of a structural concept, determination of loads, structural analysis, proportioning of the elements and their connections, assessing structural performance and its acceptability, and specifying the design. In this regard, structural analysis is an essential tool in the broader endeavor of structural design.

In the SEMM graduate program we will cover many aspects of structural analysis and structural design, including methods for determining design loads, effects of loads on behavior of structural components and connections, and overall acceptability of structural performance. You will benefit from having some basic knowledge of design loads and methods for design given those loads. The following subsections provide a brief review of these subjects.

6.2 Design loads

6.2.1 Load types

Loads on structures can be either *externally applied forces* (e.g., self-weight, live loads, wind loads) or *imposed deformations* (e.g., expansion due to temperature change or foundation settlement). In some documents, loads are referred to by the term *actions*.

Building codes classify loads based on their origin. This is convenient because some loads are determined by the structure itself, some by its occupancy, and some by the environment in which the structure is located. The different load types have different variability, duration, and directionality effects that may need to be considered in design. The main load types that are considered in ASCE 7 are:

D = dead load

E = earthquake load

F = load due to fluids with well-defined pressures and maximum heights

F_a = flood load

H = load due to lateral earth pressure, ground water pressure, or pressure of bulk materials

L = live load

L_r = roof live load

R = rain load

S = snow load

T = self-straining load

W = wind load

Among these, the following merit additional discussion:

Dead load (D) – These are loads due to self-weight and items that are permanently attached to a structure, such as floor finishes, HVAC (heating, ventilation, and air conditioning). Dead loads are constant in magnitude, direction, and position in the structure.

Live load (L) – These are loads due to occupancy and use, such as occupants, furnishings, and traffic. Some live loads may be relatively long-term, such as books in a library stack. However, live loads are usually considered to be short term loadings that are not constant in magnitude or location. Buildings must be designed to resist the maximum loads they are likely to be subjected to during some reference period T, frequently taken as 50 years. Consequently, the live loads specified in codes are usually much higher than the floor loads occurring at any point in time. Furthermore, while it is possible to crowd many items into a small floor area, thereby producing a large live load, it is unlikely that the same loading will occur everywhere in a structure. Therefore, design live loads are specified for a nominal influence area (around 400 ft^2), with live load reductions for larger tributary areas. There are, of course, exceptions for which live load does not reduce with increasing area (for example, a parking structure or warehouse loading).

Snow load (S), Rain load (R), Wind load (W), and Earthquake Load (E) – These are loads attributed to the environment and are generally of short duration. You'll hear more about earthquake loading during your time at Berkeley.

6.2.2 Load placement

The design must consider the possibility that live loads will be placed in patterns that produce the maximum load effects. For the propped cantilever shown below, dead load must be distributed wherever it occurs, while live load needs to be considered in different loading patterns to identify the worst effects. The second loading pattern will produce the most positive moment (bottom in tension) at point b, while the third loading pattern will produce the least positive moment, or perhaps a negative moment, at b.

You may have been introduced to influence lines in a structural analysis course to help figure out where to place loads. While useful in analysis, you do not need to study influence lines if you have not previously seen them in your courses.

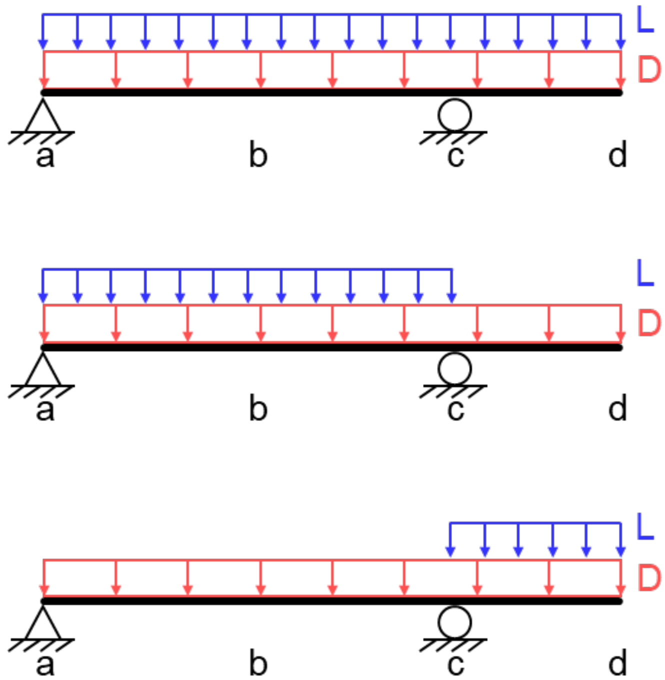


Figure 6.3: Design must consider different loading patterns to identify worst effects that might occur.

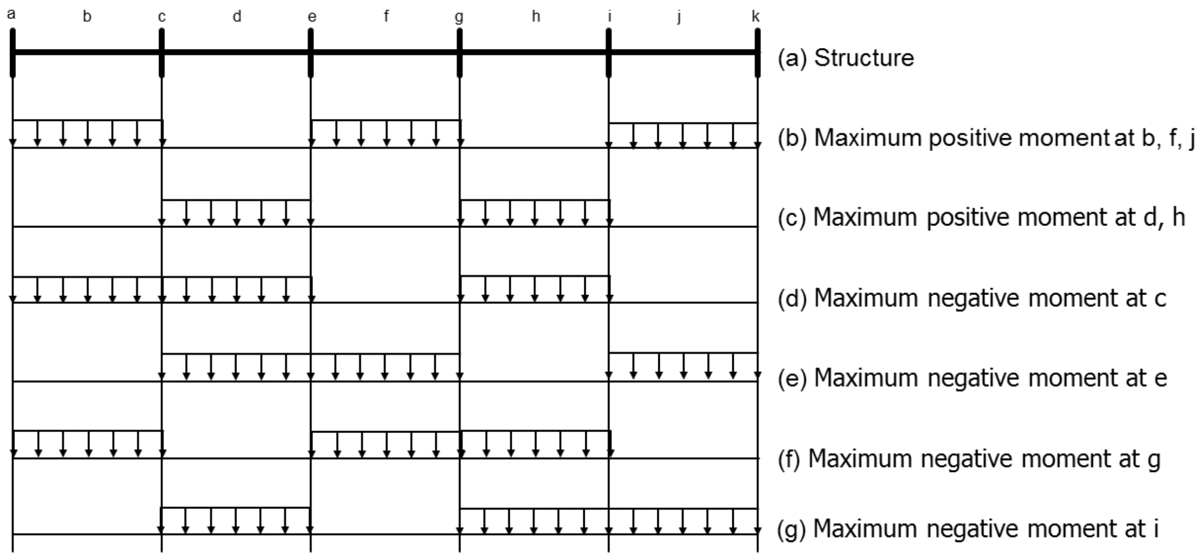


Figure 6.4: Pattern loadings to determine worst negative and positive moments in a floor system.

The term *pattern load* describes a load being positioned in a pattern that may produce maximum load effects. For example, storage loads can be placed in alternate bays, with the bays between those storage loads being unloaded so as to form corridors (b and c in Fig. 6.4). This loading will produce maximum positive moments in the loaded bays. Alternatively, two adjacent bays can be fully loaded with the next bays unloaded. This loading, along with alternate bays also being loaded, will produce the maximum negative moments at supports (d, e, f, and g in Fig. 6.4).

If you are unfamiliar with these concepts, you might wish to ask your instructors in design classes about them during the first week of classes.

6.2.3 Load paths

Roof and floor systems commonly are constructed using a series of surface structural elements supported by larger elements capable of spanning greater distances to the supporting columns or walls. For example, consider the framing system shown in Fig. 6.5. Floor load is applied to surface elements (which could be wood planks, plywood, or concrete slab). Although these elements are continuous in EW and NS directions, the shortest and, hence, stiffest load path is in the NS direction, where they are supported by joists. The joists support the reactions from the surface elements plus their own weight, and span EW to supporting beams. The beams support the joist loads plus self-weight and span those loads NS to girders. The girders in turn span EW to supporting columns, which transmit loads through axial forces to the foundations or other supporting elements.

The structural elements need not be stacked atop one another as implied by the exploded diagram in Fig. 6.5. Greater economy in construction and operations can sometimes be achieved by framing structural members into one another such that they have the same top elevation as shown in Fig. 6.6. Regardless, the conceptualization of the load path is the same as depicted in Fig. 6.5.

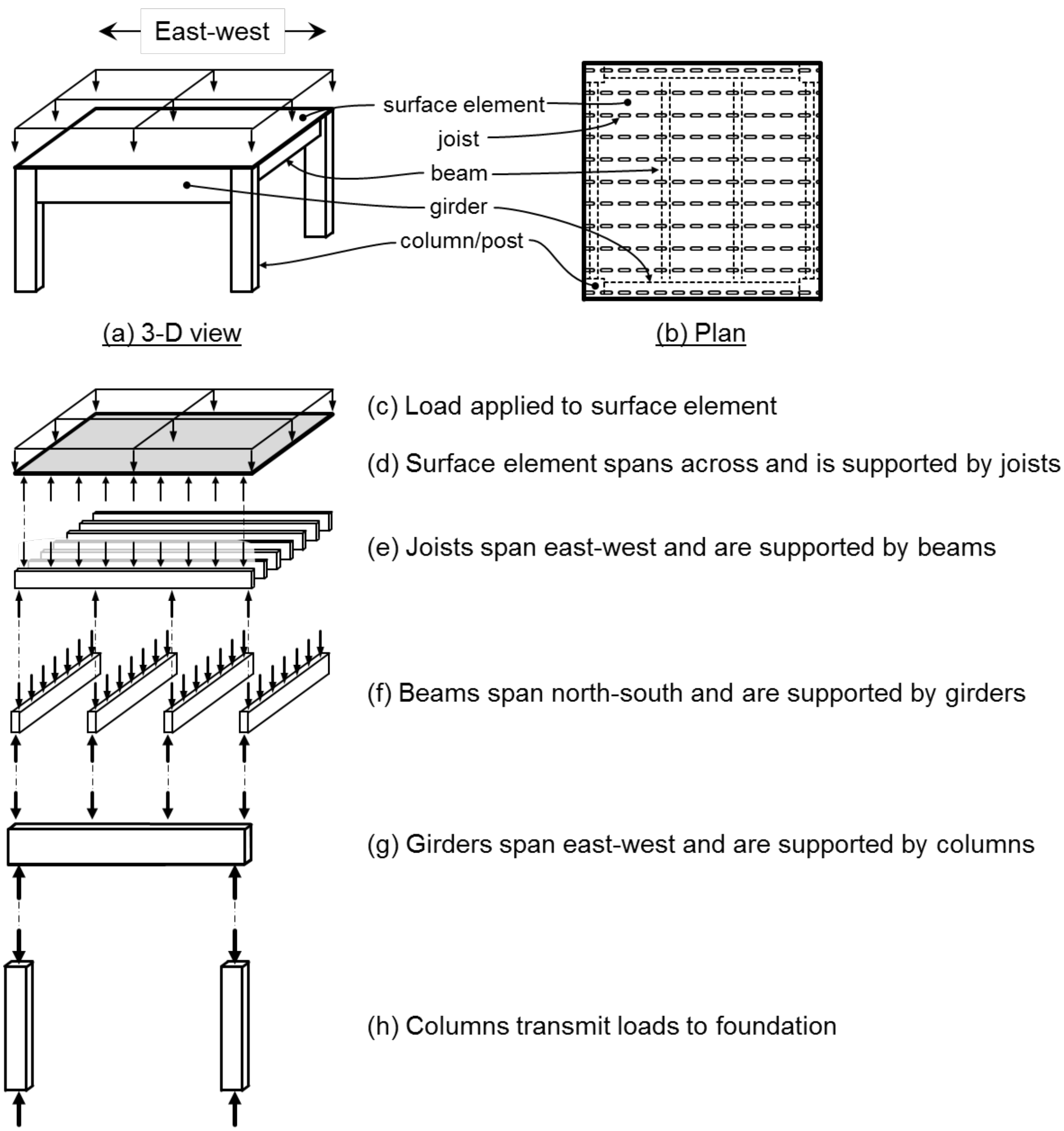


Figure 6.5: Load path for gravity loads in a floor framing system.

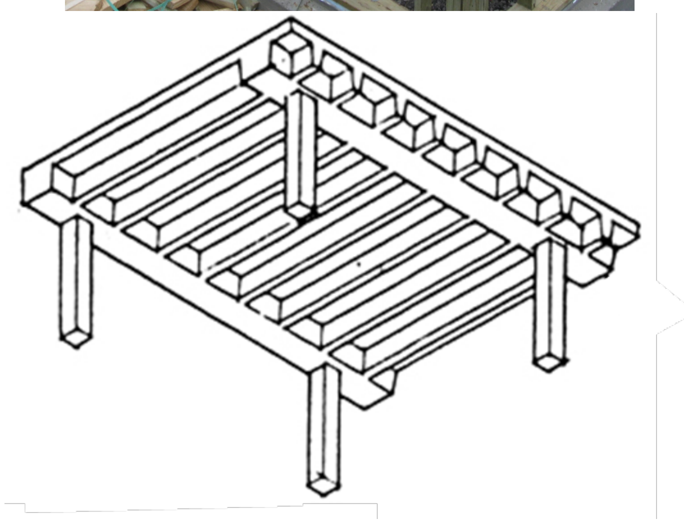


Figure 6.6: Joists framed into beams so as to have the same top elevation. (a) Wood framing (Southern Forest Products Association); (b) Reinforced concrete framing (Idees Deco Maison).

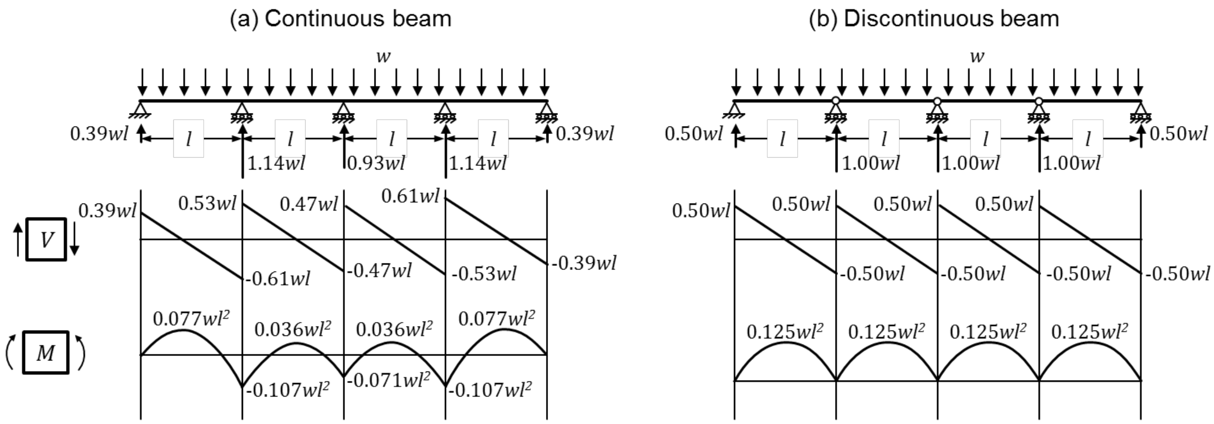


Figure 6.7: Comparison of reactions, shears, and moments for continuous and discontinuous beams.

6.2.4 Tributary width and tributary area

The tributary width or tributary area concept is an approximate analysis method used for estimating the load path in structural systems. To develop the basis for the method, consider the continuous and discontinuous beams supporting uniformly distributed loads shown in Fig. 6.7. The continuous beam was analyzed using the computer software RISA 2D, while the discontinuous beam was analyzed by hand. From the results we can observe the following:

- The reactions for the continuous beam are similar to the reactions from the discontinuous beam. The exterior reactions in the continuous beam are conservatively estimated by the results from the discontinuous beam, while the first interior reaction is underestimated by 14%.
- The shear diagrams for the continuous and discontinuous beams are also similar.
- The moment diagrams for the two beams are markedly different.

From the preceding observations, we conclude that reactions can be reasonably approximated by modeling the beam as a discontinuous beam. Moments, however, are strongly affected by continuity and cannot be accurately estimated by considering the beam to be discontinuous.

We can obtain the same reactions as above using the tributary width concept. According to the *tributary width* concept, the load transferred to a beam support is equal to the load acting within the tributary width, where the tributary width is a width extending halfway to each of the adjacent supports (see Fig. 6.8). This method works very well where loads are uniformly distributed. Where loads are not uniformly distributed, it is preferable to treat the beam as a discontinuous beam and calculate the reactions using equilibrium, or, alternatively, to analyze it as a continuous beam. Treating the beam as a discontinuous beam, for a concentrated load halfway between two supports, half the concentrated load would be transferred to one support and half to the other. If

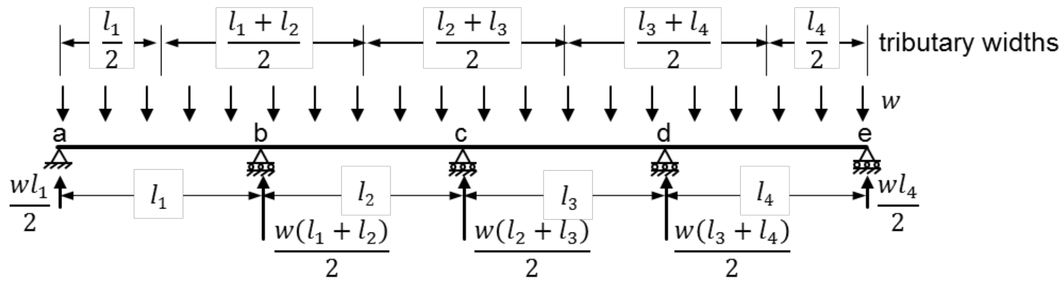


Figure 6.8: Tributary widths for a continuous beam.

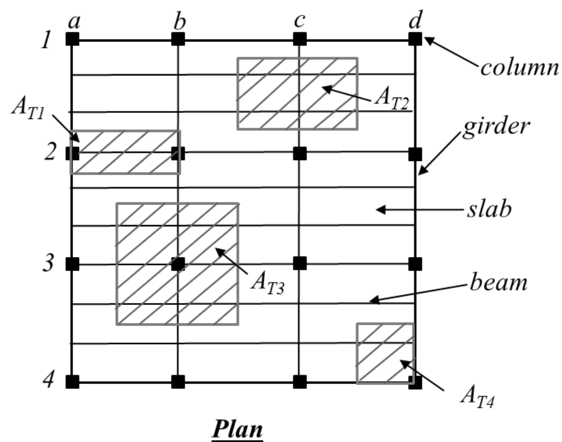
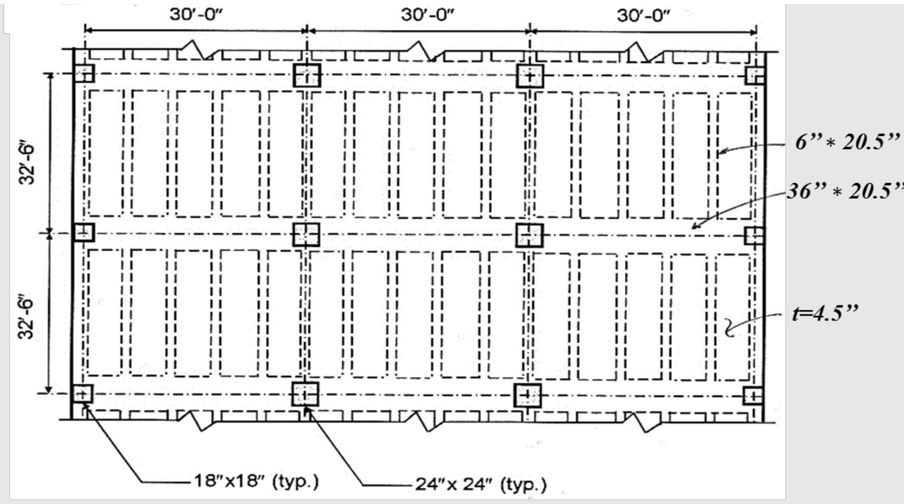


Figure 6.9: Tributary areas for a floor system.

the concentrated load was positioned three-quarters of the way along the support, three quarters of the load would go to the closer support with the remainder going to the more distant support.

The concept can be expanded to tributary areas, as depicted in Fig. 6.9. For a beam along axis 2 between axes a and b, the tributary area is A_{T1}. For the girder along axis c between axes 1 and 2, the tributary area A_{T2} is the area from two beams supported by the girder. We could also add the small area immediately above the girder, but this is too detailed for the approximate nature of the calculation. For the interior column at the intersection of axes 3 and b, the area is A_{T3}. A similar approach is used for the corner column at 4d.

Example 6.1 (Tributary area and live load reduction)**.** For the floor system shown in Fig. 6.10, determine the design gravity loads for a typical (a) slab, (b) interior joist, (c) interior beam, and (d) interior column supporting a single floor. The solution is shown in the figure.



<u>Design Data:</u>
<u>Materials</u>: Normal-weight concrete, 150 pcf

<u>Loads:</u>

Superimposed: w_D = 30 psf , w_L = 100 psf

Slab: $w = \left(\frac{4.5}{12} ft\right)(150\, pcf) = 56$ psf

Joist: $w = \left(\frac{6}{12} ft\right)\left(\frac{20.5-4.5}{12} ft\right)(150\, pcf) = 100$ plf

Beam: $w = \left(\frac{36}{12} ft\right)\left(\frac{20.5-4.5}{12} ft\right)(150\, pcf) = 600$ plf

Figure 6.10: Concrete floor system.

6.3 Load and Resistance Factor Design (Strength Design)

The *load and resistance factor design* (*LRFD*) method is a common method used to design for the ultimate limit state. There are a variety of different forms for the LRFD method. Here we adopt the form commonly used in the United States.

6.3.1 General approach

The LRFD method can be expressed generically through the following expression:

$$\phi S_n \geq U \tag{6.1}$$

in which ϕS_n is referred to as the design strength, ϕ = strength reduction factor, S_n = nominal strength, and U = factored load effect. In practice, expression (6.1) is applied to internal member forces such as shear and moment, as in

$$\phi V_n \geq V_u \tag{6.2}$$

$$\phi M_n \geq M_u \tag{6.3}$$

in which V_n = nominal shear strength, M_n = nominal moment strength, V_u = shear due to factored loads, and M_u = moment due to factored loads. Nominal strengths are strengths that are calculated using methods specified in the building codes.

Although the LRFD method refers to an ultimate limit state approaching the failure or collapse state, structural analysis for the limit state is usually done using assumptions of linear-elastic behavior. Thus, the ultimate limit state for the structural system as a whole is presumed to be reached for the loading that first causes a member cross section to reach the design strength ϕS_n.

Load and resistance factors for the LRFD method are established considering variability and uncertainty in different load effects and material properties, the accuracy and variability of nominal strengths, the brittleness of different failure modes, and the consequences of failure. For buildings assigned to Risk Category II of ASCE 7, the intended annual probabilities of failure for load conditions that do not include earthquake are 3 x 10^{-5}/yr for failure that is not sudden and does not lead to wide-spread progression of damage, 3 x 10^{-6}/yr for failure that is either sudden or leads to widespread progression of damage, and 7 x 10^{-7}/yr for failure that is sudden and results in widespread progression of damage (ASCE 7).

6.3.2 ASCE 7 factored load combinations

The factored load effect is represented by U in expression (6.1). In practice, the quantity U is the maximum (or minimum) load effect determined through a series of load combinations. Each load combination considers one or more load cases, whose load factors have been adjusted to achieve approximately uniform reliability.

The main load cases are listed below, and refer to the load itself or to its effect on internal moments and forces:

D = dead load

E = earthquake load

F = load due to fluids with well-defined pressures and maximum heights

H = load due to lateral earth pressure, ground water pressure, or pressure of bulk materials

L = live load

L_r = roof live load

S = snow load

W = wind load

The basic *load combinations* consider different combinations of the load cases, as follows:

1. $1.4D$

2. $1.2D + 1.6L + 0.5(L_r \text{ or } S \text{ or } R)$

3. $1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (\alpha_L L \text{ or } 0.5W)$

4. $1.2D + 1.0W + \alpha_L L + 0.5(L_r \text{ or } S \text{ or } R)$
5. $1.2D + 1.0E + \alpha_L L + 0.2S$
6. $0.9D + 1.0W$
7. $0.9D + 1.0E$

In combinations 3, 4, and 5, the factor $\alpha_L L$ applied to L is equal to 1.0 for garages, for areas occupied as places of public assembly, and for any occupancies in which $L \geq 100$ psf $(4.8$ kPa$)$. Otherwise, $\alpha_L = 0.5$.

Special load combinations are used where fluid loads F or earth pressure loads H are present. We need not concern ourselves with these in this primer.

In any of the load combinations, effects of one or more loads not acting, or effects of loads acting in the opposite direction (where possible) are to be investigated. The most unfavorable effects from both wind and earthquake loads are to be investigated, where appropriate, but they need not be considered to act simultaneously. Additional effects of flood, atmospheric ice loads, and self-restraining loads are not covered in this reader. See ASCE 7 for additional details.

For earthquake-resistant design, the engineer must consider the effects of earthquake directionality. In general, this includes effects of earthquake loads in two principal horizontal directions plus vertical earthquake shaking effects. Effects of over-strength on design loads must also be considered in some special cases. These details are not covered in this primer but may be covered in some of your classes at Berkeley.

Figure 6.11 illustrates the application of the load combinations for a planar system considering the load cases D, L, and E. Basic load combinations 1 and 2 consider only D and combined D and L. In this illustration, both D and L are taken at their full intensities. To obtain the worst shear at beam mid-span, however, L should be placed on only half of the beam span. The building code requires that this latter loading case also be considered.

Diagrams 5a and 5b in the figure illustrate ASCE 7 load combination 5; note that E must be considered both from left to right and from right to left. Illustrations 7a and 7b in the figure illustrate load combination 7. In a typical structure, load combination 5 results in higher axial compression in columns while load combination 7 results in higher axial tension in columns. Both load combinations must be considered in design. Not shown in these diagrams is the effect of vertical earthquake loads, which must be considered in accordance with ASCE 7.

To keep your calculations efficiently organized, it is worthwhile clarifying the difference between load cases (e.g., D, L, and E) versus load combinations. Usually it is preferred to analyze separately for each of the load cases. Then, the principle of linear superposition allows the load combinations to be calculated as linear combinations of the load cases.

6.3.3 Resistance factors, ϕ

In expression (6.1), the term ϕS_n is referred to as the design strength, which is the product of strength reduction factor ϕ and nominal strength S_n. Nominal strength is determined using nominal strength equations (which are covered in structural design courses). The strength reduction

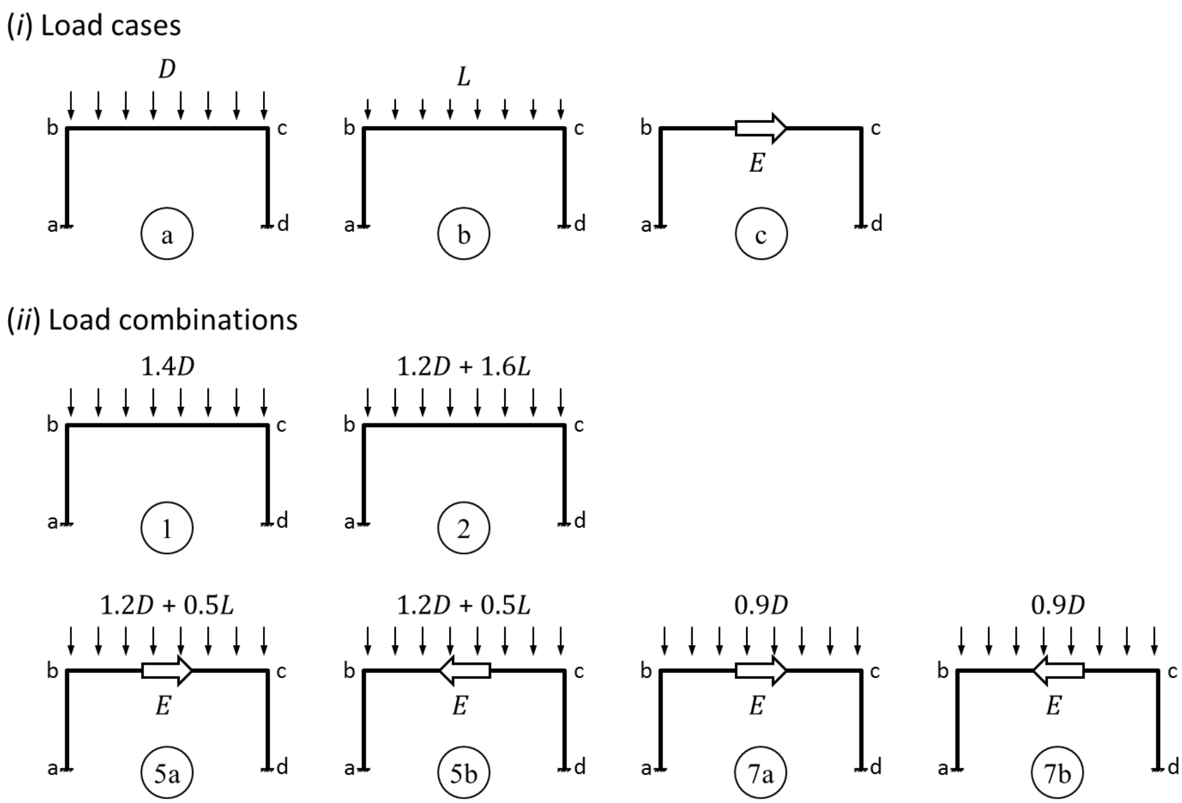


Figure 6.11: Load cases and load combinations in load and resistance factor design.

factors have numerical values less than 1.0, and are provided (1) to allow for the possibility of under-strength members due to variations in material strengths and dimensions, (2) to allow for inaccuracies in the design equations, (3) to reflect the available ductility and required reliability of the member under the load effects being considered, and (4) to reflect the importance of the member in the structure. You may see material-specific strength reduction factors in some of your design classes at Berkeley.

Example 6.2 (Single-bay, single-story, frame)**.** A weightless, one-bay, one-story frame has configuration and loading shown in Fig. 6.12. Dead load D is 3 klf (44 kN/m), live load L is 1.8 klf (26 kN/m), and earthquake load E is 45 kips (200 kN). Use the LRFD method to determine the required beam moment strengths at the faces of the beams (Sections 1 and 2).

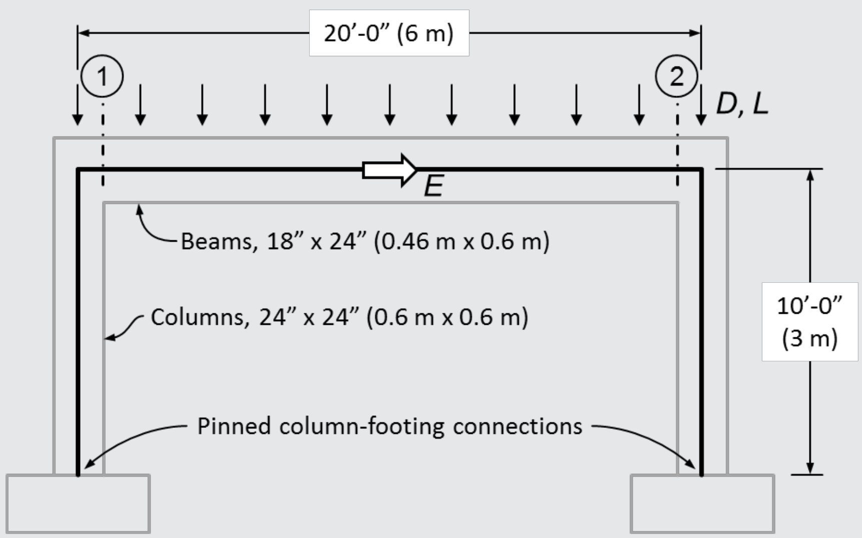


Figure 6.12: Single-bay, single-story, frame.

Solution: The load cases and load combinations are shown in Fig. 6.11. The structure is modeled using flexural stiffness equal to $0.3EI_g$ for beams and columns and analyzed for the load cases using computer software for structural analysis. The results of the load cases are then combined using the load combinations. Calculated moments at sections 1 and 2 are tabulated below.

Table 6.1: Load cases and combinations.

	Moments, k-ft (kN-m)	
Load case	Section 1	Section 2
D	-51.6 (-70.2)	-51.6 (-70.2)
L	-31.0 (-42.1)	-31.0 (-42.1)
E	203 (275)	-203 (-275)
Load Combination		
$1.4D$	-72.2 (-98.2)	-72.2 (-98.2)
$1.2D + 1.6L$	-111 (-152)	-111 (-152)
$1.2D + 0.5L + E$	125 (170)	-280 (-381)
$1.2D + 0.5L - E$	-280 (-381)	125 (170)
$0.9D + E$	156 (212)	-249 (-339)
$0.9D - E$	-249 (-339)	156 (212)
Minimum	**-280 (-381)**	**-280 (-381)**
Maximum	**156 (212)**	**156 (212)**

Example 6.3 (Required nominal moment strengths)**.** Determine the required nominal moment strengths of the beam at section 1 considering the loading of Example 6.2. Assume the strength reduction factor is $\phi = 0.9$ for beam moment strength.

Solution: From Example 6.2, the required moment strengths are M_u = -280 k-ft and +156 k-ft. Thus, the required nominal moment strengths are $M_n = M_u/\phi$ = -311 k-ft and +173 k-ft. The beams would need to be designed to provide at least these nominal strengths.

Chapter 7

Elements of structural materials

7.1 Introduction

In the primer we will not cover structural materials. The introductory graduate course treating structural materials will focus on development, behavior, durability design, and failure characteristics of high performance civil engineering materials such as High Performance Concrete (including High Performance fiber reinforced cement-based composites and Ultra-high performance fiber reinforced composites), High Performance Steel (ductile/brittle transitions, fatigue behavior, fracture and forensic analysis) and Polymeric Materials that are being utilized both for new and for retrofitted structures. The introductory course dealing primarily with structural materials will begin with a review of the properties, performance, and fracture characteristics of conventional civil engineering materials in the early parts of the semester in order to bring all students up to an equal level in terms of background knowledge.

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