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by

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Abstract

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Doctor of Philosophy in Mathematics

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Professor Martin Olsson, Co-Chair

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In this thesis, I study the geometry of algebraic stacks and some important cohomological invariants on them, motivated by moduli-theoretic problems. The language of stacks provides a natural framework for understanding moduli problems that cannot be captured by schemes alone.

The first main chapter is based on *Picard Groups of Stacky Curves* [32], and investigates the geometry and Picard groups of stacky curves and gerbes. I further develop the theory of rigidification and show that every stacky curve can be written as a gerbe over a stacky curve with trivial generic stabilizer. I calculate the Picard groups of tame stacky curves with trivial generic stabilizers and express the Picard groups of gerbes as an extension of two groups, which depend on the Brauer classes of the gerbes and the Picard groups of the base. These results together give a description of the Picard group of any tame stacky curve as an extension of two groups. I apply the theory to many examples, including some moduli stacks of interest.

The second main chapter is based on *The Brauer Group of BG and Gerbe Structures of Moduli Spaces* [33]. I calculate the Brauer group of the classifying stack BG for G a smooth geometrically connected linear algebraic group, and study Brauer–Severi stacks on BG. I calculate the Brauer classes of Brauer–Severi stacks coming from certain representations of the universal cover of G. These results are then applied to study the moduli stack of curves of genus g together with an order N automorphism. In particular, I look at curves whose quotients by the order N automorphism are genus 0, and completely determine the Brauer classes of these gerbes.

To my academic siblings

I hope that this manuscript can be of use to you.

Contents

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Chapter 1

Introduction

1.1 Algebraic Stacks and Moduli Theory

Algebraic stacks were invented in the 1960s by Alexander Grothendieck. Grothendieck developed a foundational change of perspective in algebraic geometry from viewing spaces as sets of points to viewing spaces through maps from other objects. Grothendieck develops the key technical preliminaries of algebraic stacks, including the theory of fibered categories and descent in [25]. In 1969, Deligne and Mumford apply these ideas to moduli to develop Deligne–Mumford stacks in [14]. They show that the moduli space of stable curves of genus g is a proper irreducible variety over any characteristic. In [21], Giraud further develops the theory of algebraic stacks and uses them to give geometric meaning to higher cohomology groups. Artin later generalizes Deligne–Mumford stacks to Artin stacks in [7], which are the algebraic stacks as we know them today. Jarod Alper has an excellent exposition of the history and motivation behind algebraic stacks in [4].

Algebraic stacks provide a natural framework for understanding moduli problems that cannot be captured by schemes alone. Moduli spaces occupy a central role in algebraic geometry: through their intersection theory, cohomology, and deformation theory, we gain insight into the families of geometric objects they classify. However, many of the most interesting moduli spaces are not representable by schemes but instead by stacks, making a deep understanding of the geometry of algebraic stacks indispensable.

In this thesis, we investigate the geometry of algebraic stacks, with particular attention to stacky curves, gerbes, Picard groups and Brauer groups of stacks, Brauer classes of gerbes, and moduli stacks of curves.

1.2 Picard Groups of Stacky Curves

In his famous 1965 paper *Picard Groups of Moduli Problems* [36], Mumford computes that the Picard group of the moduli stack of elliptic curves $\mathscr{M}_{1,1}$ over fields of characteristic different from 2 and 3 is $\mathbb{Z}/12\mathbb{Z}$. This work was subsequently revisited by Fulton and Olsson [18], who compute the Picard group of $\mathscr{M}_{1,1}$ and its compactification over general base schemes, and generalizations to the modular curves $\mathscr{Y}_0(2)$ and $\mathscr{Y}_0(3)$ were studied by Niles [37]. All of these works use the moduli interpretation of the stacks in question in essential ways. In Chapter 3, we put this work on $\mathscr{M}_{1,1}$ and other modular curves into a general context and give an exact sequence which describes the Picard group of an arbitrary tame stacky curve as the extension of two groups, which depend on the geometry of the curve rather than any moduli interpretation.

There are two key cases we need to consider. The first case is for a tame stacky curve $\mathscr{Y}$ with trivial generic stabilizer, in which we can compute the Picard group exactly in terms of the Picard group of its coarse moduli space and the stabilizer groups of the stacky points. The geometry of stacky curves with trivial generic stabilizers is studied in [8], [20], [38] and [47], and the Picard groups of such stacky curves are studied in [10, Theorem 5.4.6], [11, Corollary 3.2.1], and [12, Corollary 3.0.8, Theorem 3.2.3]. The exact sequence describing the Picard group of a stacky curve with trivial generic stabilizer is a special case of [11, Corollary 3.2.1], but in the following theorem, we also give an explicit pushout construction that computes the Picard group exactly.

Theorem 1.2.1. *Let $\mathscr{Y}$ be a tame stacky curve with trivial generic stabilizer and coarse moduli space $\pi : \mathscr{Y} \to X$. Let $x_1, \ldots, x_r$ be the stacky points of $\mathscr{Y}$ with stabilizer groups μ_{n_i}, cyclic of order n_i. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{Y}) \xrightarrow{\chi} \textstyle\prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} \longrightarrow 0,$$

where $\chi = (\chi_{x_1^*(-)}, \ldots, \chi_{x_r^*(-)})$ describes the representations of the inertia on a line bundle pulled back to the residual gerbes at the stacky points. Moreover, $\mathrm{Pic}(\mathscr{Y})$ *is the pushout of the diagram*

$$\begin{tikzcd} \mathbb{Z}^r \arrow[r, "(\cdot n_1,\ldots,\cdot n_r)"] \arrow[d, "\phi"] & \mathbb{Z}^r \arrow[d] \\ \mathrm{Pic}(X) \arrow[r, "\pi^*"] & \mathrm{Pic}(\mathscr{Y}), \end{tikzcd}$$

where $\phi : \mathbb{Z}^r \to \mathrm{Pic}(X)$ is given by $e_i \mapsto I_{\pi(x_i)}$, for $e_i = (0, \ldots, 1, \ldots, 0)$ the ith basis vector, and $I_{\pi(x_i)}$ the ideal sheaf of $\mathscr{O}_X$-modules of the point $\pi(x_i) \in X$.

The second case is for a stacky curve $\mathscr{X}$ which is a gerbe over a stacky curve with trivial generic stabilizer $\mathscr{Y}$. We give an exact sequence which describes the Picard group as an extension of two groups, one group being $\mathrm{Pic}(\mathscr{Y})$. In fact, the statement holds for $\mathscr{Y}$ a locally finitely presented algebraic stack over a locally noetherian base, so we present the theorem with this assumption.

Theorem 1.2.2. *Let $\mathscr{Y}$ be a locally finitely presented algebraic stack over a locally noetherian base, and let $\mathscr{H}$ be a tame locally constant finitely presented band on $\mathscr{Y}$. Let $p : \mathscr{X} \to \mathscr{Y}$ be* *a gerbe banded by $\mathscr{H}$, with cohomology class $[\mathscr{X}] \in H^2(\mathscr{Y}, \mathscr{H})$. We have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}(\mathscr{Y}) \xrightarrow{p^*} \operatorname{Pic}(\mathscr{X}) \xrightarrow{\chi} \operatorname{Hom}(\mathscr{H}, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{X}]} H^2(\mathscr{Y}, \mathbb{G}_m),$$

where χ describes the representation of $\mathscr{H}$ acting on the line bundle, and $(-)_*[\mathscr{X}]$ sends $(\epsilon : \mathscr{H} \to \mathbb{G}_m)$ to the image of $[\mathscr{X}]$ under the cohomology pushforward map

$$\epsilon_* : H^2(\mathscr{Y}, \mathscr{H}) \to H^2(\mathscr{Y}, \mathbb{G}_m).$$

Remark 1.2.3. A band is locally a sheaf of groups, with gluing data only well-defined up to conjugation. By a tame locally constant finitely presented band, we mean that locally, the band comes from a tame finite constant sheaf of groups. When $\mathscr{H}$ can not be identified with a sheaf of groups, ϵ_* is more complicated to define. We describe the stack of bands and the construction of ϵ_* in Sections 2.6 and 3.4.

To connect these two cases and describe the Picard group of a general stacky curve, we show that the geometry of a stacky curve comprises of two parts: 1) its rigidification, which is a stacky curve with trivial generic stabilizer, and 2) a gerbe over the rigidification. Rigidification is a process in which a subgroup G of the inertia $\mathscr{I}(\mathscr{X})$ is removed from $\mathscr{X}$ and results in a gerbe banded by G, $\mathscr{X} \to \mathscr{X} /\!\!/ G$. In Section 3.2, we further develop the theory of rigidification, which appears for general groups in [1, Appendix A], and in Section 3.3, we prove the following theorem which describes the geometry of stacky curves.

Theorem 1.2.4. *Let $\mathscr{X}$ be a stacky curve with inertia group $\mathscr{I}(\mathscr{X})$. Then there exists a finite flat subgroup stack $\mathscr{H} \leq \mathscr{I}(\mathscr{X})$ such that the rigidification of $\mathscr{X}$ with respect to $\mathscr{H}$ is a stacky curve with trivial generic stabilizer $\mathscr{Y} := \mathscr{X} /\!\!/ \mathscr{H}$, and the natural map $p : \mathscr{X} \to \mathscr{Y}$ is a gerbe banded by $\underline{\mathscr{H}}$, the band associated to $\mathscr{H}$. Moreover, if $\mathscr{X}$ is tame, then so is $\mathscr{Y}$.*

Remark 1.2.5. The proof of this proposition mainly requires finding the finite flat subgroup stack $\mathscr{H}$. The result that given a finite flat subgroup $\mathscr{H}$ of $\mathscr{I}(\mathscr{X})$, there exists a locally finitely presented algebraic stack $\mathscr{X} /\!\!/ \mathscr{H}$ and map $\mathscr{X} \to \mathscr{X} /\!\!/ \mathscr{H}$ such that $\mathscr{X}$ is a gerbe banded by $\underline{\mathscr{H}}$ over $\mathscr{X} /\!\!/ \mathscr{H}$ appears in [1].

Putting all of the pieces together, we have the following corollary.

Corollary 1.2.6. *Let $\mathscr{X}$ be a tame stacky curve with coarse moduli space $\pi : \mathscr{X} \to X$ and inertia stack $\mathscr{I}(\mathscr{X})$. Let $\mathscr{H} \leq \mathscr{I}(\mathscr{X})$ be as in Theorem 1.2.4, so that $\mathscr{Y} := \mathscr{X} /\!\!/ \mathscr{H}$ is a tame stacky curve with trivial generic stabilizer and $p : \mathscr{X} \to \mathscr{Y}$ is a gerbe banded by $\underline{\mathscr{H}}$. Let $[\mathscr{X}] \in H^2(\mathscr{Y}, \underline{\mathscr{H}})$ be the gerbe class of $\mathscr{X}$. We have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}(\mathscr{Y}) \xrightarrow{p^*} \operatorname{Pic}(\mathscr{X}) \xrightarrow{\chi} \operatorname{Hom}(\underline{\mathscr{H}}, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{X}]} H^2(\mathscr{Y}, \mathbb{G}_m) \ ,$$

where χ and and $(-)_*[\mathscr{X}]$ are as in Theorem 1.2.2, and $\operatorname{Pic}(\mathscr{Y})$ *is described as in Theorem 1.2.1.*

1.3 Brauer Group of BG

The Brauer group of a field k is an abelian group which classifies Morita equivalence classes of central simple algebras over k with multiplication given by tensor product. The Brauer group has been generalized to schemes and established as a cohomological invariant of a scheme by Grothendieck in [22, 23, 24]. Brauer groups have been extended to stacks more recently, and various work has been done computing Brauer groups of stacks, for example, in [2, 3, 5, 9, 16, 15, 27, 45].

In [2, Section 5], the Leray spectral sequence is used to compute the Brauer group of BG for a finite linearly reductive group G, in [34, Section 6], the authors calculate the Brauer group of BG for a finite étale group scheme G, and in [35, Theorem 3.1], the author calculates the Brauer group of BG for smooth connected linear algebraic groups G in the context where BG is viewed as a simplicial space. We calculate the cohomological Brauer group of BG in the following theorem.

Theorem 1.3.1. *Let G be a smooth connected semisimple linear algebraic group over an algebraically closed field k. Let $\widetilde{G}$ be the universal cover of G, and B the fundamental group of G, which fit into an exact sequence $1 \to B \to \widetilde{G} \to G \to 1$. Let $X(B) := \mathrm{Hom}(B, \mathbb{G}_m)$ be the character group of B. Then*

$$H^i(BG, \mathbb{G}_m) = \begin{cases} k^* & i = 0 \\ 0 & i = 1 \\ X(B) & i = 2. \end{cases}$$

For $i = 2$, an isomorphism $X(B) \to H^2(BG, \mathbb{G}_m)$ is given by $(\chi : B \to \mathbb{G}_m) \mapsto \chi_*(B\widetilde{G})$, where χ_* is the pushforward map $\chi_* : H^2(BG, B) \to H^2(BG, \mathbb{G}_m)$.

With the understanding of the Brauer group of BG, we may also understand the classes of Brauer-Severi varieties over BG, which we call *Brauer-Severi stacks* (see Section 4.2). Classically, a Brauer-Severi variety of dimension n over a field k is a variety X over a field k that is isomorphic to $\mathbb{P}^n_{\overline{k}}$ after base change to an algebraic closure of k. Its obstruction to being isomorphic to $\mathbb{P}^n_k$ itself is the existence or non-existence of a line bundle of degree 1 on X, which would then be isomorphic to $\mathscr{O}_{\mathbb{P}^n_k}(1)$. More generally, a *Brauer-Severi stack of relative dimension n over S* is a smooth proper morphism $\mathscr{X} \to S$ which is étale locally isomorphic to $\mathbb{P}^n_S$, where S is allowed to be a scheme or a stack. The existence of a line bundle of degree 1 on $\mathscr{X}$ no longer guarantees that $\mathscr{X}$ is globally isomorphic to $\mathbb{P}^n_S$, but rather only guarantees the existence of a rank $n+1$ vector bundle V on S such that $\mathscr{X} \cong \mathbb{P}V$.

There is an equivalence of categories between Brauer-Severi stacks $\mathscr{X}$ of relative dimension n over S and PGL_{n+1}-torsors over S. The image $\delta([\mathscr{X}]) \in H^2(S, \mathbb{G}_m)$ of the class of $\mathscr{X}$ in $H^1(S, \mathrm{PGL}_{n+1})$ via the short exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1$ is equal to the class of the $\mathbb{G}_m$-gerbe of degree 1 line bundles on $\mathscr{X}$, which we call the *Brauer class of the Brauer-Severi stack X*. It is trivial if and only if there is a degree 1 line bundle on $\mathscr{X}$.

We study Brauer-Severi stacks of the form $[\mathbb{P}V/G] \to BG$, and prove the following theorem which calculates their Brauer classes.

Theorem 1.3.2. *Let V be a representation of $\widetilde{G}$, acting via $\rho : \widetilde{G} \to \mathrm{GL}(V)$ such that $B \subseteq \widetilde{G}$ acts by a character $\chi : B \to \mathbb{G}_m$. Then we get an induced action of G on $\mathbb{P}V$, and $[\mathbb{P}V/G] \to BG$ is a Brauer-Severi stack over BG. Let $\mathscr{G}$ be the $\mathbb{G}_m$-gerbe of degree 1 line bundles on $[\mathbb{P}V/G]$ over BG. Then $\chi_*^{-1}(B\widetilde{G}) = \mathscr{G}$, and thus the class of $\mathscr{G}$, which is the Brauer class of $[\mathbb{P}V/G] \to BG$ in $H^2(BG, \mathbb{G}_m) = X(B)$, is given by χ^{-1}.*

The initial motivation for studying the Brauer group and Brauer-Severi stacks on BG was to understand the gerbe structure of the stack of genus g curves with an automorphism of order N, which we now explain.

Let $\mathscr{M}_g$ be the moduli stack of smooth curves of genus $g > 1$. The inertia stack $I(\mathscr{M}_g)$ is the stack with objects pairs $(X \in \mathscr{M}_g, \alpha \in \mathrm{Aut}(X))$ and morphisms are isomorphisms $\phi : X' \to X$ such that $\phi \circ \alpha' = \alpha \circ \phi$. The inertia $I(\mathscr{M}_g)$ is a disjoint union

$$I(\mathscr{M}_g) = \coprod_{N \in \mathbb{N} > 0} I_N(\mathscr{M}_g),$$

such that for $(X, \alpha) \in I_N(\mathscr{M}_g)$, α has order N. We call the substacks $I_N(\mathscr{M}_g)$ the *Nth twisted sectors* of the inertia stack.

Since every object of $I_N(\mathscr{M}_g)$ has an order N automorphism, this stack is a μ_N-gerbe, and corresponds to some class $[I_N(\mathscr{M}_g)] \in H^2(\mu_N)$. We look at the components of $I_N(\mathscr{M}_g)$ consisting of curves whose quotients by the order N automorphism α has genus 0 and completely determine the *Brauer classes* of these gerbes. For a μ_N-gerbe $\mathscr{G} \to S$ with class $[\mathscr{G}] \in H^2(S, \mu_n)$, its pushout to $\mathbb{G}_m$ under the inclusion $\mu_N \hookrightarrow \mathbb{G}_m$ is the $\mathbb{G}_m$-gerbe $\mathscr{G}^{\wedge \mathbb{G}_m} \to S$ and its *Brauer class* is given by $[\mathscr{G}^{\wedge \mathbb{G}_m}] \in H^2(S, \mathbb{G}_m)$.

Given an object (X, α) of $I_N(\mathscr{M}_g)$, we may take the quotient of X by the action of μ_N induced by α to get a Galois cover with group μ_N, $\pi : X \to C = X/\mu_N$. The quotient curve C can take many possible genera with varying ramification divisors. Following [41], using ideas from [42], to specify which genera and degrees of ramification divisors are allowed on C, we define a *g-admissible datum* to be an $N+1$-tuple $A = (g', N, d_1, \ldots, d_{N-1})$ such that $0 \le g' \le g$, $N > 1$, and $d_i \ge 0$, which satisfies Riemann-Hurwitz

$$2g - 2 = N(2g' - 2) + \sum_i d_i(N - \gcd(i, N)) \tag{1.3.2.1}$$

and the structural equation of abelian covers

$$\sum_i i d_i = 0 \pmod{N}. \tag{1.3.2.2}$$

Define the stack $\mathscr{M}_A$ to be the stack with objects

$$\mathscr{M}_A(S) = \{(f : C \to S, D_1, \ldots D_{N-1}, \mathscr{L}, \phi)\}$$

where $f : C \to S$ is a smooth curve of genus g', the D_i are disjoint divisors of degree d_i, that is, sections of $(\mathrm{Sym}^{d_i}_S C \setminus \Delta_{d_i}) \to S$, where Δ_{d_i} is the big diagonal, $\mathscr{L}$ is a line bundle on C and $\phi : \mathscr{L}^{\otimes N} \to \mathscr{O}_C(\sum iD_i)$ is an isomorphism. Morphisms will be defined in Section 4.3.

Every object of $\mathscr{M}_A$ corresponds to a μ_N-cover $X \to C$ of a smooth genus g curve over C, ramified over the D_i. Conversely, every μ_N-cover with the covering curve having genus g corresponds to some object of $\mathscr{M}_A$ for some g-admissible datum A. We roughly explain this correspondence in Section 4.3. Let $\mathscr{M}'_A$ be the substack of $\mathscr{M}_A$ such that the corresponding X is connected. The following theorem relates $I_N(\mathscr{M}_g)$ to the $\mathscr{M}'_A$.

Theorem 1.3.3. *[41, 2.2, Corollary 1], [42, Theorem 2.1, Proposition 2.1] Fix $g, N > 1$. Then $I_N(\mathscr{M}_g) = \coprod_A \mathscr{M}'_A$, where the disjoint union ranges over all g-admissible $A = (g', N, d_1, \ldots, d_{N-1})$.*

When $g' = 0$, either $\mathscr{M}'_A = \mathscr{M}_A$ or $\mathscr{M}'_A = \emptyset$ [41, 2.2, Remark 3]. Since we only study the case when $g' = 0$ in this paper, it suffices to study the $\mathscr{M}_A$ as defined above, and then only a subset of such $\mathscr{M}_A$ will correspond to components of $I_N(\mathscr{M}_g)$ To study the $\mathscr{M}_A$, we define a moduli stack $\overline{\mathscr{M}_A}$ that contains $\mathscr{M}_A$ as an open substack, namely, we allow the divisors to coincide.

When $g' = 0$, $\overline{\mathscr{M}_A}$, can be realized as a μ_N-gerbe over a stack $\overline{\mathscr{N}_A}$, which is a composition of Brauer-Severi stacks over the moduli space of genus 0 curves, $B\mathrm{PGL}_2$. Using Theorems 1.3.1 and 1.3.2, and a stacky version of a theorem of Gabber [19, Theorem 2, page 193], which relates the Brauer group of a Brauer-Severi variety to that of its base, we compute the Brauer group of $\overline{\mathscr{N}_A}$ and calculate the class of $\overline{\mathscr{M}_A}$ in the Brauer group of $\overline{\mathscr{N}_A}$ in the following theorems.

Theorem 1.3.4. *Let $N, d_1, \ldots, d_{N-1} \in \mathbb{Z}$ such that $N | d := \sum id_i$ and*

$$g := \frac{1}{2}\left(-2N + \sum_i id_i(N - \gcd(i, N)) + 2\right)$$

is an integer greater than 1. Then $A = (0, N, d_1, \ldots, d_{N-1})$ is a g-admissible datum. If all d_i are even, $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$, and if any d_i is odd, $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) = 0$.

Theorem 1.3.5. *Let $A = (0, N, d_1, \ldots, d_{N-1})$ be as above. Then*

$$[\overline{\mathscr{M}_A}^{\wedge \mathbb{G}_m}] = [\gamma^* \mathscr{G}_{d/N}] \in H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m),$$

where $\mathscr{G}_{d/N}$ is the $\mathbb{G}_m$-gerbe over $B\mathrm{PGL}_2$ of degree d/N line bundles on genus 0 curves, and $\gamma : \overline{\mathscr{N}_A} \to B\mathrm{PGL}_2$ is the natural forgetful map. In particular, if all d_i are even and d/N is odd, then $\overline{\mathscr{M}_A}^{\wedge \mathbb{G}_m}$ is nontrivial of order 2, otherwise, $\overline{\mathscr{M}_A}^{\wedge \mathbb{G}_m}$ is trivial. Moreover, since the restriction map on Brauer groups is injective, the same holds for $\mathscr{M}_A^{\wedge \mathbb{G}_m}$ where the divisors are disjoint.

1.4 Outline of the Thesis

Chapter 2 contains background on the theory of algebraic stacks. Chapter 3 is based on [32] and studies the geometry and Picard groups of stacky curves and gerbes. Chapter 4 is based on [33] and calculates the Brauer group of BG for G smooth connected linear algebraic groups, calculates the Brauer classes of certain Brauer-Severi stacks on BG, and calculates Brauer classes of some examples of moduli stacks of curves.

Chapter 2

Background

In this chapter, we give some relevant definitions and results on algebraic stacks. Everything in this chapter is well-known. We discuss effective descent, stacks, the inertia stack, Deligne–Mumford stacks, the topological space of points on stacks, stacks with trivial generic stabilizers, root stacks, abelian and non-abelian gerbes and bands, and stacky curves. Much of the material in this section was learned from [39]. We also reference [21],[25], [28], [30], [38], and [46].

2.1 Algebraic Stacks

We first define the descent condition for fibered categories, and then define stacks and algebraic stacks, following [39]. Since we will only be working with stacks defined over a base scheme S, we write all of our definitions in terms of fibered categories over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

Let $p : F \to (\mathrm{Sch}/S)_{\mathrm{fppf}}$ be a fibered category. For any S-scheme $Y \to S$ and morphism of S-schemes $f : X \to Y$, we define the category $F(f : X \to Y)$ as follows. Objects in $F(f : X \to Y)$ are pairs (x, σ) where x is an object of $F(X)$ and $\sigma : \mathrm{pr}_1^* x \to \mathrm{pr}_2^* x$ is an isomorphism in $F(X \times_Y X)$ satisfying the cocycle condition. The $\mathrm{pr}_i : X \times_Y X \to X$ for $i = 1, 2$ are the projection maps, and the cocycle condition states that

$$\mathrm{pr}_{23}^* \sigma \circ \mathrm{pr}_{12}^* \sigma = \mathrm{pr}_{13}^* \sigma : \mathrm{pr}_{12}^* \mathrm{pr}_1^* x \to \mathrm{pr}_{23}^* \mathrm{pr}_2^* x$$

in $F(X \times_Y X \times_Y X)$, where $\mathrm{pr}_{ij} : X \times_Y X \times_Y X \to X \times_Y X$ for $i, j \in \{1, 2, 3\}$ are the projection maps. Morphisms in $F(X \to Y)$ between objects (x, σ) and (x', σ') are morphisms $g : x \to x'$ in $F(X)$ such that the diagram

$$\begin{tikzcd} \mathrm{pr}_1^* x \arrow[r, "\mathrm{pr}_1^* g"] \arrow[d, "\sigma"] & \mathrm{pr}_1^* x' \arrow[d, "\sigma'"] \\ \mathrm{pr}_2^* x \arrow[r, "\mathrm{pr}_2^* g"] & \mathrm{pr}_2^* x' \end{tikzcd}$$

commutes.

There is a functor

$$F(Y) \to F(f : X \to Y) \tag{2.1.0.1}$$
$$y \mapsto (f^*y, \sigma = \sigma_{\text{can}} : \text{pr}_1^* f^* y \to \text{pr}_2^* f^* y).$$

Definition 2.1.1. We say $f : X \to Y$ is an *effective descent morphism for F* if the functor 2.1.0.1 is an equivalence of categories.

Many fibered categories over $(\text{Sch}/S)_{\text{fppf}}$ satisfy effective descent for the fppf topology, by which we mean that every fppf morphism is an effective descent morphism for the fibered category. Some categories that satisfy effective descent include closed subschemes, open embeddings, quasi-coherent sheaves, and affine morphisms [39, Section 4.3, 4.4].

Definition 2.1.2. A *stack $\mathscr{X}$ over S* is a category fibered in groupoids $p : \mathscr{X} \to (\text{Sch}/S)_{\text{fppf}}$ such that for all S-schemes $T \to S$ and all coverings $T' \to T$, the functor

$$\mathscr{X}(T) \to \mathscr{X}(T' \to T)$$

defined in 2.1.0.1 is an equivalence of categories.

A stack $\mathscr{X}/S$ has the property that for all S-schemes T and all $x, y \in \mathscr{X}(T)$, the presheaf

$$\underline{\text{Isom}}_T(x, y) : (\text{Sch}/S)_{\text{fppf}}|_T^{\text{op}} \to \text{Set}, \tag{2.1.2.1}$$
$$(f : T' \to T) \mapsto \text{Isom}_{T'}(f^*x, f^*y)$$

is a sheaf.

Definition 2.1.3. An *algebraic stack $\mathscr{X}$ over S*, is a stack over S, $p : \mathscr{X} \to (\text{Sch}/S)_{\text{fppf}}$ whose diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by algebraic spaces, and such that there exists a smooth surjective covering $X \to \mathscr{X}$ by a scheme X.

An algebraic stack $\mathscr{X}/S$ has the property that for all S-schemes T and all $x, y \in \mathscr{X}(T)$, the presheaf 2.1.2.1 is an algebraic space. In particular, $\underline{\text{Aut}}_T(x) := \underline{\text{Isom}}_T(x, x)$ is an algebraic space as well. We also have an action of $\underline{\text{Aut}}_T(x)$ on $\underline{\text{Isom}}_T(x, y)$, where $\alpha \in \underline{\text{Aut}}_T(x)$ acts on $\sigma \in \underline{\text{Isom}}_T(x, y)$ by $\alpha \cdot \sigma = \sigma \circ \alpha$. Similarly, $\beta \in \underline{\text{Aut}}_T(y)$ acts by $\beta \cdot \sigma = \beta \circ \sigma$. This action makes $\underline{\text{Isom}}_T(x, y)$ into both an $\underline{\text{Aut}}_T(x)$-torsor and an $\underline{\text{Aut}}_T(y)$-torsor. In particular, if $\underline{\text{Isom}}_T(x, y)$ has a section, then $\underline{\text{Aut}}_T(x) \cong \underline{\text{Isom}}_T(x, y) \cong \underline{\text{Aut}}_T(y)$, where the isomorphisms are determined by choosing a section of $\underline{\text{Isom}}_T(x, y)$.

Definition 2.1.4. For any geometric point $x : \text{Spec } k \to \mathscr{X}$ of $\mathscr{X}$, the sheaf of groups $G_x := \underline{\text{Aut}}_k(x)$ is called the *stabilizer group* of x, and if G_x is nontrivial, then we call x a *stacky point*. If $\mathscr{X}$ is defined over k, then x factors through the inclusion $BG_x \hookrightarrow \mathscr{X}$ of $BG_x = [\text{Spec } k/G_x]$ as a strictly full substack of $\mathscr{X}$, and we call BG_x the *residual gerbe* at x [46, 06ML, 06UH].

Some Notational Remarks

1. In this thesis, by a stack, we mean an algebraic stack.

2. For a sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$ or fibered category $p : F \to (\mathrm{Sch}/S)_{\mathrm{fppf}}$, we will occasionally write $x \in F$, in which case we mean $x \in F(T)$ for some S-scheme $T \to S$.

3. When defining a stack $p : \mathscr{X} \to (\mathrm{Sch}/S)_{\mathrm{fppf}}$ by its objects and morphisms, we will define morphisms only between objects in the same fiber. Morphisms between objects $x' \in \mathscr{X}(T')$ and $x \in \mathscr{X}(T)$ in different fibers are given by a morphism of S-schemes $t : T' \to T$ and a morphism $\sigma : x' \to t^*x$ in $\mathscr{X}(T')$. Moreover, we assume that we have chosen a splitting of the fibered category, so that every object of $\mathscr{X}$ has a unique pullback.

4. By a stack $\mathscr{X}$ over a base S, we mean $p : \mathscr{X} \to (\mathrm{Sch}/S)_{\mathrm{fppf}}$, but we will write $p : \mathscr{X} \to S$.

2.2 The Inertia Stack

Let $\mathscr{X}/S$ be an algebraic stack. The inertia stack $\mathscr{I}(\mathscr{X})$ is the fiber product of the following diagram,

$$\begin{tikzcd} \mathscr{I}(\mathscr{X}) \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\Delta"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X}. \end{tikzcd}$$

Objects of $\mathscr{I}(\mathscr{X})$ are tuples

$$(x_1 : T \to \mathscr{X}, x_2 : T \to \mathscr{X}, (\sigma_1, \sigma_2) : (x_1, x_1) \cong (x_2, x_2)),$$

and morphisms from $(x_1, x_2, (\sigma_1, \sigma_2))$ to $(x_1', x_2', (\sigma_1', \sigma_2'))$ are pairs of morphisms

$$(\alpha_1 : x_1 \to x_1', \alpha_2 : x_2 \to x_2')$$

such that the following diagram commutes,

$$\begin{tikzcd} (x_1, x_1) \arrow[r, "(\sigma_1,\sigma_2)"] \arrow[d, "(\alpha_1,\alpha_1)"'] & (x_2, x_2) \arrow[d, "(\alpha_2,\alpha_2)"] \\ (x_1', x_1') \arrow[r, "(\sigma_1',\sigma_2')"] & (x_2', x_2'). \end{tikzcd}$$

Note that any object $(x_1, x_2, (\sigma_1, \sigma_2))$ is isomorphic to $(x_1, x_1, (\mathrm{id}, \sigma_1^{-1} \circ \sigma_2))$ via the isomorphism $(\mathrm{id}, \sigma_1^{-1})$. Therefore, an equivalent, and simpler, presentation of $\mathscr{I}(\mathscr{X})$ is given by having objects pairs $(x : T \to \mathscr{X}, g : x \cong x)$, and morphisms from (x, g) to (x', g') given by morphisms $\alpha : x \to x'$ such that the following diagram commutes,

$$\begin{tikzcd} (x,x) \arrow[r, "(\mathrm{id},g)"] \arrow[d, "(\alpha,\alpha)"'] & (x,x) \arrow[d, "(\alpha,\alpha)"] \\ (x',x') \arrow[r, "(\mathrm{id},g')"] & (x',x'), \end{tikzcd}$$

or equivalently, that

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "\alpha"'] & x \arrow[d, "\alpha"] \\ x' \arrow[r, "g'"] & x' \end{tikzcd}$$

commutes.

For any object $t : T \to \mathscr{X}$, we have the following fiber product square,

$$\begin{tikzcd} \underline{\mathrm{Aut}}_T(t) \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{X}) \arrow[d] \\ T \arrow[r, "t"] & \mathscr{X}, \end{tikzcd}$$

which can be seen by the fact that every element of the fiber product $((t', g), \sigma : t' \cong t)$ is isomorphic to $((t, g'), \mathrm{id}_t : t \cong t)$ via the isomorphism σ, where $g' = \sigma^{-1} g \sigma$.

Inertia when $\mathscr{X}$ has finite diagonal

Let $\mathscr{X}$ be an algebraic stack with finite diagonal. Then $\phi : \mathscr{I}(\mathscr{X}) \to \mathscr{X}$ is finite, so it is affine, and we can write $\mathscr{I}(\mathscr{X}) = \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$, where $\mathscr{A} = \phi_* \mathscr{O}_{\mathscr{I}(\mathscr{X})}$ is a quasi-coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras on $\mathscr{X}$, under the equivalence of categories [39, Theorem 10.2.4]

$$\underline{\mathrm{Spec}}_{\mathscr{X}}(-) : (\text{sheaves of algebras in } \mathrm{Qcoh}(\mathscr{X}))^{\mathrm{op}} \to (\text{affine morphisms } f : \mathscr{Y} \to \mathscr{X}),$$

where $\mathscr{A} \mapsto \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}) \to \mathscr{X}$, and the inverse functor is given by

$$(f : \mathscr{Y} \to \mathscr{X}) \mapsto f_* \mathscr{O}_{\mathscr{Y}}.$$

Moreover, the functor restricts to an equivalence between coherent sheaves and finite morphisms, so for an algebraic stack with finite diagonal $\mathscr{X}$, there is a coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras $\mathscr{A}$ such that $\mathscr{I}(\mathscr{X}) = \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$.

Remark 2.2.1. Since the diagonal is finite, the diagonal is in fact representable by affine schemes, since for any affine scheme Spec A mapping into $\mathscr{X} \times_S \mathscr{X}$, the fiber product with the diagonal morphism will be an affine scheme Spec B, with B a finite A-algebra.

Inertia of a Quotient Stack

If $\mathscr{X}/S$ is a quotient stack, then the inertia $\mathscr{I}(\mathscr{X})$ has a nice description, also as a quotient stack.

Proposition 2.2.2. *Let X a scheme and let G be a smooth group scheme acting on X. Then $\mathscr{I}([X/G]) = [F/G]$, where $F \subseteq X \times G$ is the fiber product of the diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \times G \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X, \end{tikzcd} \tag{2.2.2.1}$$

where the right vertical map is given by $(x, g) \mapsto (x, gx)$. The action of G on $X \times G$ is $h \cdot (x, g) = (hx, hgh^{-1})$.

Proof. First, we can describe $\mathscr{I}([X/G])$ as the stack having objects $x : T \to [X/G]$ together with an automorphism $g : x \to x$, which we can describe as a G-torsor over T, $P \to T$, together with a G-equivariant morphism $\sigma : P \to X$, and an automorphism η of P respecting σ. For all $h \in G$ and all $p \in P$, σ satisfies $\sigma(hp) = h\sigma(p)$ and η satisfies $\eta(hp) = h\eta(p)$. Morphisms of $\mathscr{I}([X/G])$ are isomorphisms $\alpha : x \to x'$ such that $g' = \alpha g \alpha^{-1}$. That is, α is an isomorphism of G-torsors $\alpha : P \to P'$ such that the diagram

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, "\sigma"'] & & P' \arrow[dl, "\sigma'"] \\ & X & \end{tikzcd}$$

commutes, $\eta' = \alpha\eta\alpha^{-1}$, and for all $h \in G$ and $p \in P$, $\alpha(hp) = h\alpha(p)$.

Objects of $[F/G]$, $y : T \to [F/G]$, are pairs of G-torsors $P \to T$ and G-equivariant morphisms $\tau : P \to X \times G$ with image in F. For all $h \in G$ and all $p \in P$, $\tau = (\tau_X, \tau_G)$ satisfies $(\tau_X(hp), \tau_G(hp)) = (h\tau_X(p), h\tau_G(p)h^{-1})$. Morphisms $\alpha : y \to y$ are isomorphisms of torsors $\alpha : P \to P'$ such that the diagram

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, "\tau"'] & & P' \arrow[dl, "\tau'"] \\ & X \times G & \end{tikzcd}$$

commutes and $\alpha(hp) = h\alpha(p)$ for all $h \in G$ and $p \in P$. Observe that for any $p \in P$, $\tau_G(p)$ determines τ_G on all of P. Indeed, G acts transitively on P, and $\tau_G(hp) = h\tau_G(p)h^{-1}$.

We define a map $\mathscr{I}([X/G]) \to [F/G]$ as follows. On objects, we map

$$(P \to T, \sigma : P \to X, \eta : P \to P) \mapsto (P \to T, \tau : P \to X \times G)$$

where $\tau = (\sigma, \tau_G)$, and $\tau_G(p) = g_p$, where g_p is determined by $\eta(p) = g_p \cdot p$. We must check that τ lands in F and is G-equivariant. Let $p \in P$. Then

$$\tau_G(p) \cdot \tau_X(p) = g_p\sigma(p) = \sigma(g_p p) = \sigma(\eta(p)) = \sigma(p) = \tau_X(p).$$

For all $h \in G$ and $p \in P$, $\tau_X(hp) = h\tau_X(p)$ since $\tau_X = \sigma$. For $g_{hp} = \tau_G(hp)$ and $g_p = \tau_G(p)$,

$$g_{hp}hp = \eta(hp) = h\eta(p) = hg_p p,$$

so $g_{hp}h = hg_p$, and $\tau_G(hp) = h\tau_G(p)h^{-1}$.

On morphisms, we map

$$(\alpha : (P, \sigma, \eta) \to (P'\sigma', \eta')) \mapsto (\alpha : (P, \tau) \to (P', \tau')).$$

We need to check that α is a morphism in $[F/G]$. First we have $\tau_{X'}\alpha = \tau_X$ since $\sigma'\alpha = \sigma$, and $\alpha(hp) = h\alpha(p)$ since α is a morphism in $\mathscr{I}([X/G])$. Finally, for $g_p = \tau_G(p)$ and $g_{\alpha(p)} = \tau'_G(\alpha(p))$, since $\eta'\alpha = \alpha\eta$,

$$g_{\alpha(p)}\alpha(p) = \eta'(\alpha(p)) = \alpha(\eta(p)) = \alpha(g_p p) = g_p\alpha(p),$$

so $\tau'_G(\alpha(p)) = \tau_G(p)$.

We define an inverse functor $[F/G] \to \mathscr{I}([X/G])$ by

$$(P \to T, \tau : P \to X \times G) \mapsto (P \to T, \tau_X : P \to X, \eta : P \to P),$$

where $\eta(p) = \tau_G(p) \cdot p$. We check that

$$\eta(hp) = \tau_G(hp)hp = h\tau_G(p)h^{-1}hp = h\tau_G(p)p = h\eta(p),$$

and

$$\sigma(\eta(p)) = \sigma(\tau_G(p)p) = \tau_G(p)\sigma(p) = \tau_G(p)\tau_X(p) = \tau_X(p) = \sigma(p).$$

On morphisms, the inverse is given by $\alpha \mapsto \alpha$. We also need to check that α is a morphism in $\mathscr{I}([X/G])$, which is similar to the checks for $[F/G]$.

These constructions are inverses to each other, so we indeed get the desired equivalence $\mathscr{I}([X/G]) \cong [F/G]$.

□

Examples

Example 2.2.3. We compute the inertia stack of $\mathscr{M}_{1,1}$ using the presentation of its compactification $\overline{\mathscr{M}_{1,1}}$ as a weighted projective stack,

$$\overline{\mathscr{M}_{1,1}} \cong \mathscr{P}(4,6) = [(\mathrm{Spec}\ k[x,y] \setminus \{(0,0)\})/_{4,6}\mathbb{G}_m],$$

where the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ is given by $\lambda \cdot (x, y) = (\lambda^4 x, \lambda^6 y)$. We simplify the calculation by computing the inertia on two open substacks, namely the opens $\mathscr{U}_0$, that removes $[0 : 1]$, a point with stabilizer μ_6, and $\mathscr{U}_1$, that removes $[1 : 0]$, a point with stabilizer μ_4. First,

$$\mathscr{U}_0 = [(\mathbb{G}_m \times \mathbb{A}^1)/_{4,6}\mathbb{G}_m] \cong [\mathbb{A}^1/_6\mu_4],$$

where μ_4 acts on $\mathbb{A}^1$ by $\zeta \cdot z = \zeta^6 z$. An equivalence $[\mathbb{A}^1/_6\mu_4] \to [(\mathbb{G}_m \times A^1)/_{4.6}\mathbb{G}_m]$ is given by $z \mapsto [1 : z]$. For any $\zeta \in \mu_4$, $\zeta^6 z \mapsto [1 : \zeta^6 z] = [\zeta^4 : \zeta^6 z] = [1 : z]$. For any $(x, y) \in \mathbb{G}_m \times \mathbb{A}^1$, there exists $\lambda \in \mathbb{G}_m$ satisfying $\lambda^4 = \frac{1}{x}$. Then $[x : y] = [\lambda^4 x : \lambda^6 y] = [1 : \lambda^6 y]$, so $\lambda^6 y \mapsto [x : y]$. Finally, if $z_1, z_2 \mapsto [1 : z_1] = [1 : z_2]$, then $[1 : z_2] = [\lambda^4 : \lambda^6 z_1]$ for some $\lambda \in \mathbb{G}_m$, but since $\lambda^4 = 1$, $\lambda \in \mu_4$, so $z_1 = z_2$ in $[\mathbb{A}^1/_6\mu_4]$.

We now compute the inertia of $\mathscr{U}_0 = [\mathbb{A}^1/_6\mu_4]$, which is given by $[F_0/\mu_4]$ where

$$F_0 = \{(z, \zeta) \in \mathbb{A}^1 \times \mu_4 \mid z = \zeta^6 z\},$$

and μ_4 acts on $\mathbb{A}^1$ as before and trivially on μ_4 since μ_4 is abelian. For all $z \in \mathbb{A}^1$ and all $\zeta = \pm 1$, $z = \zeta^6 z$. For $z = 0$ and all $\zeta \in \mu_4$, $z = \zeta^6 z$. Thus

$$F_0 = (\mathbb{A}^1 \times \mu_2) \coprod \{(0, i)\} \coprod \{(0, -i)\},$$

and

$$\begin{aligned} \mathscr{I}([\mathbb{A}^1/_6\mu_4]) &= [F_0/\mu_4] \\ &= [(\mathbb{A}^1 \times \mu_2)/\mu_4] \coprod [\{(0, i)\}/\mu_4] \coprod [\{(0, -i)\}/\mu_4] \\ &= \big([\mathbb{A}^1/_6\mu_4] \times \mu_2\big) \coprod B\mu_4 \coprod B\mu_4. \end{aligned}$$

We repeat this calculation for $\mathscr{U}_1$, where

$$\mathscr{U}_1 = [(\mathbb{A}^1 \times \mathbb{G}_m)/_{4,6}\mathbb{G}_m] \cong [\mathbb{A}^1/_4\mu_6],$$

where μ_6 acts on $\mathbb{A}^1$ by $\zeta \cdot z = \zeta^4 z$. By a similar calculation,

$$\mathscr{I}([\mathbb{A}^1/_4\mu_6]) = \big([\mathbb{A}^1/_4\mu_6] \times \mu_2\big) \coprod B\mu_6 \coprod B\mu_6 \coprod B\mu_6 \coprod B\mu_6.$$

Putting these together, we get

$$\mathscr{I}(\overline{\mathscr{M}_{1,1}}) = (\overline{\mathscr{M}_{1,1}} \times \mu_2) \coprod \left(\coprod_2 B\mu_4\right) \coprod \left(\coprod_4 B\mu_6\right),$$

and restricting to the open substack $\mathscr{M}_{1,1} \subseteq \overline{\mathscr{M}_{1,1}}$,

$$\mathscr{I}(\mathscr{M}_{1,1}) = (\mathscr{M}_{1,1} \times \mu_2) \coprod \left(\coprod_2 B\mu_4\right) \coprod \left(\coprod_4 B\mu_6\right).$$

Example 2.2.4. We compute the inertia stack of $BS_3 = [\text{Spec } k/S_3]$. Since S_3 acts trivially on Spec k, we have $\mathscr{I}(BS_3) = [S_3/S_3]$, where S_3 acts on S_3 by conjugation. The conjugacy classes of S_3 are $\{(1)\}, \{(12),(23),(13)\}, \{(123),(132)\}$. The stabilizer group of (1) is all of S_3, the stabilizer group of (12) is $\langle(12)\rangle \cong \mathbb{Z}/2\mathbb{Z}$, and similarly for (23) and (13), and the stabilizer group of (123) is $\langle(123)\rangle \cong \mathbb{Z}/3\mathbb{Z}$, and similarly for (132). Thus,

$$\mathscr{I}(BS_3) = BS_3 \coprod B\mathbb{Z}/2\mathbb{Z} \coprod B\mathbb{Z}/3\mathbb{Z}.$$

Example 2.2.5. The inertia stack of $B(\mathbb{Z}/6\mathbb{Z}) = [\text{Spec } k/(\mathbb{Z}/6\mathbb{Z})]$ is

$$\mathscr{I}(B(\mathbb{Z}/6\mathbb{Z})) = [(\mathbb{Z}/6\mathbb{Z})/(\mathbb{Z}/6\mathbb{Z})] = B(\mathbb{Z}/6\mathbb{Z}) \times \mathbb{Z}/6\mathbb{Z}$$

since $\mathbb{Z}/6\mathbb{Z}$ is abelian.

2.3 Deligne–Mumford Stacks and Tame Stacks

Definition 2.3.1. A *Deligne–Mumford stack* is an algebraic stack such that there exists an étale surjection $U \to \mathscr{X}$ from a scheme U.

Lemma 2.3.2. *Let $\mathscr{X}$ be a finitely-presented Deligne–Mumford stack over a scheme S. Then for all geometric points $x : \text{Spec } k \to \mathscr{X}$, G_x is a finite constant group scheme, and the diagonal has finite discrete fibers.*

Proof. A Deligne–Mumford stack has formally unramified diagonal [39, Theorem 8.3.3]. Since $\mathscr{X}$ is finitely-presented, the diagonal is finitely-presented, and thus the diagonal is unramified. Then all G_x are unramified group schemes over algebraically closed fields. Since each G_x is of finite-type, it is étale if and only if it is flat and unramified. It is trivially flat over a field, so every G_x is étale, thus is a finite constant group scheme over k, and a discrete set of points. Additionally, all G_x have diagonal morphism an open immersion [25, Proposition I.3.1], and thus all G_x are reduced, and therefore étale. The fibers of the diagonal are $\underline{\text{Isom}}_k(x_1, x_2)$ for a geometric point $(x_1, x_2) : \text{Spec } k \to \mathscr{X} \times_S \mathscr{X}$, which is either empty or isomorphic to the finite discrete $\underline{\text{Aut}}(x_1) = G_{x_1}$. □

Definition 2.3.3. A Deligne–Mumford stack $\mathscr{X}$ is *tame* if for all geometric points $x :$ Spec $k \to \mathscr{X}$, G_x is linearly reductive, or equivalently, the characteristic of k is relatively prime to the order of G_x.

Coarse Moduli Space

Definition 2.3.4. Let $\mathscr{X}$ be an algebraic stack over a base scheme S. A *coarse moduli space* for $\mathscr{X}$ is a morphism $\pi : \mathscr{X} \to X$, where X is an algebraic space over S and π satisfies

1. For any morphism $g : \mathscr{X} \to Z$ from $\mathscr{X}$ to an algebraic space Z, there exists a unique morphism $f : X \to Z$ such that $g = f \circ \pi$.

2. For any algebraically closed field k, the map $|\mathscr{X}(k)| \to X(k)$ is bijective, where $|\mathscr{X}(k)|$ denotes the set of isomorphism classes in $\mathscr{X}(k)$.

Theorem 2.3.5. *[28, Theorem 1.1], [39, Theorem 11.1.2] For any algebraic stack $\mathscr{X}$ locally of finite presentation with finite diagonal over a locally noetherian base S, there exists a coarse moduli space $\pi : \mathscr{X} \to X$. Additionally, π is proper, the map $\mathscr{O}_X \to \pi_* \mathscr{O}_{\mathscr{X}}$ is an isomorphism, and for any flat morphism of algebraic spaces $X' \to X$, $\pi' : \mathscr{X}' := \mathscr{X} \times_X X' \to X'$ is a coarse moduli space for $\mathscr{X}'$.*

Local Structure Theorem

For Deligne–Mumford stacks with a coarse moduli space, we have a very nice local structure theorem.

Theorem 2.3.6. *[39, Theorem 11.3.1] Let $\mathscr{X}$ be a Deligne–Mumford stack over a locally noetherian scheme S, locally of finite type and with finite diagonal. Let $\pi : \mathscr{X} \to X$ be its coarse moduli space. Let $x : \mathrm{Spec}\ k \to \mathscr{X}$ be a geometric point with stabilizer group G_x. Then there exists an étale neighborhood $U \to X$ of the image of x and a finite U-scheme $V \to U$ with an action of G_x such that $\mathscr{X} \times_X U \cong [V/G_x]$.*

Remark 2.3.7. Theorem 2.3.6 will be quite useful to us, as it implies that any property of such a Deligne–Mumford stack that can be checked locally on the coarse space can be checked on a stack quotient of a scheme by a finite group. Moreover, we can take the scheme to be affine. Indeed, for any affine open neighborhood $\mathrm{Spec}\ A \subset X$, of the image of a geometric point x, $\mathrm{Spec}\ A \times_X \mathscr{X} = [\mathrm{Spec}\ B/G_x]$, where $A \to B$ is a finite ring map. Also, $[\mathrm{Spec}\ B/G_x] \to \mathrm{Spec}\ A$ is a coarse moduli space, so $A = B^{G_x}$. We will often take $A = \widehat{\mathscr{O}}_{X,\pi(x)}$, the complete local ring of X at x, in which case $B = \widehat{\mathscr{O}}_{\mathscr{X},x}$.

2.4 Stacks with Trivial Generic Stabilizer

In this section, we discuss stacks with trivial generic stabilizer, that is, stacks which have a dense open algebraic space. For this, we need to first define the topological space of points on a stack.

Definition 2.4.1. A *point* of an algebraic stack $\mathscr{X}$ is an equivalence class of morphisms from spectra of fields to $\mathscr{X}$. For two fields k and k', the morphisms $x : \mathrm{Spec}\ k \to \mathscr{X}$ and $x' : \mathrm{Spec}\ k' \to \mathscr{X}$ are *equivalent* if there exists a field K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\ K \arrow[r] \arrow[d] & \mathrm{Spec}\ k \arrow[d, "x"] \\ \mathrm{Spec}\ k' \arrow[r, "x'"] & \mathscr{X}. \end{tikzcd}$$

This notion of equivalence induces an equivalence relation on the set of morphism from spectra of fields [46, 04XE]. Because any point $x : \operatorname{Spec} k \to \mathscr{X}$ is equivalent to the point given by composing x with $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$, we can assume all points are geometric points. The set of points of $\mathscr{X}$ is denoted $|\mathscr{X}|$.

Lemma 2.4.2. *[[46],04XL] There exists a unique topology on the sets of points of algebraic stacks with the following properties:*

1. *For every morphism of algebraic stacks $\mathscr{X} \to \mathscr{Y}$, the map $|\mathscr{X}| \to |\mathscr{Y}|$ is continuous.*
2. *For every morphism $U \to \mathscr{X}$ which is flat and locally of finite presentation with U an algebraic space the map of topological spaces $|U| \to |\mathscr{X}|$ is continuous and open.*

Definition 2.4.3. The underlying topology on an algebraic stack $\mathscr{X}$ is the topology on $|\mathscr{X}|$ defined in Lemma 2.4.2.

We may also define a topology on an algebraic stack where open substacks are open immersions, and these two notions agree [46, 06FJ].

Definition 2.4.4. We say that an algebraic stack $\mathscr{X}$ has *trivial generic stabilizer* if it has a dense open substack that is an algebraic space.

In the case where $\mathscr{X}$ is an algebraic stack with finite diagonal, we can explicitly describe an open algebraic space $\mathscr{U}$ such that $\mathscr{X}$ has trivial generic stabilizer if and only if $\mathscr{U}$ is dense in $\mathscr{X}$.

Proposition 2.4.5. *Let $\mathscr{X}$ be an algebraic stack with finite diagonal. Then the subcategory $\mathscr{U} \hookrightarrow \mathscr{X}$ defined by*

$$\mathscr{U}(T) = \{t : T \to \mathscr{X} \mid \underline{\operatorname{Aut}}_T(t) \cong T\}$$

is an open algebraic space in $\mathscr{X}$.

Proof. Since $\mathscr{X}$ has finite diagonal, $\mathscr{I}(\mathscr{X}) \to \mathscr{X}$ is finite, and so $\mathscr{I}(\mathscr{X}) = \underline{\operatorname{Spec}}_{\mathscr{X}}(\mathscr{A})$ for some coherent sheaf of algebras $\mathscr{A}$ on $\mathscr{X}$ [39, Theorem 10.2.4]. Let $t : T \to \mathscr{X}$ be an object of $\mathscr{X}$. Pulling back to T gives the following diagram,

$$\begin{tikzcd}
\underline{\operatorname{Aut}}_t(t) = \underline{\operatorname{Spec}}_T(t^*\mathscr{A}) \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{X}) = \underline{\operatorname{Spec}}_{\mathscr{X}}(\mathscr{A}) \arrow[d] \\
T \arrow[r, "t"] & \mathscr{X}.
\end{tikzcd}$$

Then we can describe $\mathscr{U}$ as follows,

$$\mathscr{U}(T) = \{t : T \to \mathscr{X} \mid \dim_{\mathscr{O}_T}(t^*\mathscr{A}) = 1\}.$$

Let $\pi : X \to \mathscr{X}$ be a smooth cover from a scheme, and let $U \subseteq X$ be the pullback of $\mathscr{U}$ to X. Then we can describe U as follows,

$$U = \{x \in X : \dim_{\kappa(x)}(\mathscr{A}_x \otimes_{\mathscr{O}_{X,x}} \kappa(x)) = 1\},$$

where $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ is the residue field of x.

Then U is an open subscheme in X by semi-continuity [26, II. Exercise 5.8]. By descent for open immersions, we have an equivalence of categories between open immersions in $\mathscr{X}$ and pairs (U, σ) consisting of an open set $U \subseteq X$ and an isomorphism $\sigma : \mathrm{pr}_1^*U \cong \mathrm{pr}_2^*U$, where pr_i are the projections $X \times_{\mathscr{X}} X \to X$, such that σ satisfies the cocycle condition in $X \times_{\mathscr{X}} X \times_{\mathscr{X}} X$. For U defined above, we have that $\mathrm{pr}_1^*U = \mathrm{pr}_2^*U$. Indeed, pr_i^*U for $i = 1, 2$ is given by

$$\mathrm{pr}_i^*U = \{(x_1, x_2, \sigma) : x_1, x_2 \in X, x_i \in U, \sigma : \pi(x_1) \cong \pi(x_2)\}.$$

For any two points $x, y \in X$ such that there is an isomorphism $\sigma : \pi(x) \cong \pi(y)$ in $\mathscr{X}$, we get an isomorphism $\underline{\mathrm{Aut}}(\pi(x)) \cong \underline{\mathrm{Aut}}(\pi(y))$ given by $\alpha \mapsto \sigma\alpha\sigma^{-1}$. Thus, if $x \in U$ and $y \in X$ such that $\pi(x) \cong \pi(y)$, then $y \in U$ as well and we have an equality $\mathrm{pr}_1^*U = \mathrm{pr}_2^*U$. The equality $\mathrm{pr}_1^*U = \mathrm{pr}_2^*U$ clearly satisfies the cocycle condition, so descent gives us an open substack $\mathscr{U} \hookrightarrow \mathscr{X}$ characterized by the property that for any T-point $t : T \to \mathscr{X}$, t factors through $\mathscr{U}$ if and only if $\underline{\mathrm{Aut}}_T(t) \cong \{\mathrm{id}_t\} \times T$.

Finally, this substack $\mathscr{U}$ is an algebraic space since it has trivial stabilizer groups. $\square$

2.5 Root Stacks

In this section, we describe the root stack construction, following [39, Section 10.3]. Essentially, the nth root stack of a scheme X along a divisor D on X is the universal object $\mathscr{X}_n$ mapping to X such that there is an nth root of D pulled back to $\mathscr{X}_n$ on $\mathscr{X}_n$. In general, for a scheme X and an effective Cartier divisor D on X, it may not be possible to take an nth root of D on X. Whenever this is not possible, stacky structure appears in the root stack.

Definition 2.5.1. Let X be a scheme. A *generalized effective Cartier divisor* on X is a pair (L, ρ), where L is a line bundle on X and $\rho : L \to \mathscr{O}_X$ is a morphism of $\mathscr{O}_X$-modules. An isomorphism $\sigma : (L', \rho') \cong (L, \rho)$ of generalized effective Cartier divisors on X is an isomorphism of line bundles $\sigma : L' \cong L$ such that the diagram

$$\begin{tikzcd} L' \arrow[rr, "\sigma"] \arrow[dr, "\rho'"'] & & L \arrow[dl, "\rho"] \\ & \mathscr{O}_X & \end{tikzcd}$$

commutes. A product structure on generalized effective Cartier divisors is defined by

$$(L, \rho) \cdot (L', \rho') = (L \otimes L', \rho \otimes \rho'),$$

where $\rho \otimes \rho' : L \otimes L' \to \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \mathscr{O}_X$. For a morphism of schemes $g : Y \to X$ and a generalized effective Cartier divisor (L, ρ) on X, $(g^*L, g^*\rho)$ is a generalized effective Cartier divisor on Y, where $g^*\rho : g^*L \to g^*\mathscr{O}_X = \mathscr{O}_Y$. Note that we cannot always pull back effective Cartier divisors, but we can always pull back generalized effective Cartier divisors.

We define a fibered category $\mathscr{D}$ over the category of schemes with objects pairs $(T, (L, \rho))$, where T is a scheme and (L, ρ) is a generalized effective Cartier divisor on T. Morphisms in $\mathscr{D}(T)$ are isomorphisms of generalized effective Cartier divisors on T. This fibered category is a stack because line bundles and morphisms of line bundles satisfy effective descent.

Lemma 2.5.2. *[39, Proposition 10.3.7] The stack $\mathscr{D}$ is isomorphic to the stack quotient $[\mathbb{A}^1/\mathbb{G}_m]$, where $u \in \mathbb{G}_m$ acts by left multiplication on $t \in \mathbb{A}^1$.*

Consider the morphisms $\mathbb{A}^1 \to \mathbb{A}^1$ by $f \mapsto f^n$ and $\mathbb{G}_m \to \mathbb{G}_m$ by $u \mapsto u^n$. This defines a morphism of stacks $p_n : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$, which corresponds to the morphism $p_n : \mathscr{D} \to \mathscr{D}$, $(T, (L, \rho)) \mapsto (T, (L^{\otimes n}, \rho^{\otimes n}))$.

Let X be a scheme, (L, ρ) a generalized effective Cartier divisor on X, and consider the fiber product diagram

$$\begin{tikzcd} \mathscr{X}_n \arrow[r] \arrow[d, "\pi"] & \mathscr{D} \arrow[d, "p_n"] \\ X \arrow[r, "(L,\rho)"] & \mathscr{D}, \end{tikzcd}$$

where $(L, \rho) : X \to \mathscr{D}$ sends an X-scheme $f : T \to X$ to $(T, (f^*L, f^*\rho))$.

Definition 2.5.3. We define $\mathscr{X}_n$ to be the *nth root stack of X associated to* (L, ρ), and it is the stack with objects

$$(f : T \to X, (M, \lambda), \sigma : (M^{\otimes n}, \lambda^{\otimes n}) \cong (f^*L, f^*\rho)),$$

where $f : T \to X$ is an X-scheme, (M, λ) is a generalized effective Cartier divisor on T, and σ is an isomorphism of generalized effective Cartier divisors. Morphisms $((M', \lambda'), \sigma') \to ((M, \lambda), \sigma)$ are isomorphisms $\tau : (M', \lambda') \to (M, \lambda)$ of generalized effective Cartier divisors on T, such that the diagram

$$\begin{tikzcd} M'^{\otimes n} \arrow[rr, "\tau^{\otimes n}"] \arrow[dr, "\sigma'"'] & & M^{\otimes n} \arrow[dl, "\sigma"] \\ & f^*L & \end{tikzcd}$$

commutes.

Using the Yoneda lemma, the data $(f : T \to X, (M, \lambda), \sigma)$ as an object of $\mathscr{X}_n$ corresponds to a morphism $T \to \mathscr{X}_n$. In particular, the identity map $\mathscr{X}_n \to \mathscr{X}_n$ corresponds to the data $(\pi : \mathscr{X}_n \to X, (\mathscr{L}_n, \lambda_n), \sigma)$, where $(\mathscr{L}_n, \lambda_n)$ is a generalized effective Cartier divisor on $\mathscr{X}_n$, whose nth power is isomorphic to $(\pi^*L, \pi^*\rho)$. That is, we get a line bundle on $\mathscr{X}_n$ whose nth power is isomorphic to the pullback of L.

Example 2.5.4. Let X be a scheme and let $(L,\rho) = (\mathscr{O}_X, \cdot f)$, we have an isomorphism [39, 10.3.10(ii)]

$$\mathscr{X}_n \cong \left[\underline{\mathrm{Spec}}_X \left(\frac{\mathscr{O}_X[z]}{z^n - f}\right) \Big/ \mu_n\right], \tag{2.5.4.1}$$

where μ_n acts trivially on $\mathscr{O}_X$ and by left multiplication on z. We also have an isomorphism of $\mathscr{X}_n$ with X over the open set of X where f is invertible. Moreover,

$$(\mathscr{O}_X, \cdot f) \cong (I_f, \iota_f : I_f \hookrightarrow \mathscr{O}_X),$$

where I_f is the ideal sheaf of f in $\mathscr{O}_X$, and ι_f is the standard inclusion, via the diagram

$$\begin{tikzcd} I_f \arrow[rr, "\cdot 1/f"] \arrow[dr, "\iota_f"'] & & \mathscr{O}_X \arrow[dl, "\cdot f"] \\ & \mathscr{O}_X. & \end{tikzcd}$$

Thus, we get a line bundle on $\mathscr{X}_n$ whose nth power is isomorphic to the pullback of I_f.

Example 2.5.5. Consider the effective Cartier divisor (x) on $\mathbb{A}^1_x = \mathrm{Spec}\ k[x]$. We would like to take the nth root stack of $\mathbb{A}^1_x$ with respect to (x). The corresponding generalized effective Cartier divisor is (I_x, ι), where $\iota_x : I_x \to \mathscr{O}_{\mathbb{A}^1}$ is the inclusion of the ideal sheaf at (x). At the level of rings, (I_x, ι_x) is $((x), (x) \subset k[x])$, and the isomorphism $(I_x, \iota_x) \cong (\mathscr{O}_{\mathbb{A}^1}, \cdot x)$ is due to the diagram

$$\begin{tikzcd} (x) \arrow[rr, "\cdot 1/x"] \arrow[dr, "\iota_x"'] & & k[x] \arrow[dl, "\cdot x"] \\ & k[x]. & \end{tikzcd}$$

Then the nth root stack of $\mathbb{A}^1_x$ by the divisor (x) is

$$\mathscr{X}_n \cong \left[\mathrm{Spec}\ \left(\frac{k[x][z]}{z^n - x}\right) \Big/ \mu_n\right] \cong [\mathrm{Spec}\ k[z]/\mu_n] = [\mathbb{A}^1_z/\mu_n]$$

where $\zeta \in \mu_n$ acts by left multiplication on $z \in \mathbb{A}^1_z$. Furthermore, over $\mathbb{A}^1_x \setminus \{0\}$, the subset of $\mathbb{A}^1_x$ where x is invertible, $[\mathbb{A}^1_z/\mu_n]$ is isomorphic to $\mathbb{A}^1_x$.

2.6 Gerbes

In this section, we review gerbes, following [21, Chapter IV] and [39, Section 12.2]. A general gerbe has an associated band, rather than an associated sheaf of groups, and we discuss this notion as well.

Definition 2.6.1. Let S be a scheme or algebraic stack. A *gerbe over* S is a stack $\pi : \mathscr{G} \to S$ such that

(G1) For any $T \to S$, there exists a covering $\{T_i \to T\}_{i\in I}$ such that $\mathscr{G}(T_i)$ is nonempty for each $i \in I$.

(G2) For any two objects $t, t' \in \mathscr{G}(T)$, there exists a covering $\{f_i : T_i \to T\}_{i\in I}$ such that f_i^*t and f_i^*t' are isomorphic in $\mathscr{G}(T_i)$ for all $i \in I$.

For any gerbe $\pi : \mathscr{G} \to S$ and S-scheme $T \to S$, if $\sigma : t \to t'$ is an isomorphism in $\mathscr{G}(T)$, then we get an induced isomorphism

$$\Phi_\sigma : \underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(t'),\ \alpha \mapsto \sigma\alpha\sigma^{-1}, \tag{2.6.1.1}$$

where $\underline{\mathrm{Aut}}_T(t)$ is the sheaf of groups on T, given by

$$(f : U \to T) \mapsto \mathrm{Aut}_U(f^*t).$$

To be more precise, over $f : U \to T$, Φ_σ is given by

$$\mathrm{Aut}_U(f^*t) \to \mathrm{Aut}_U(f^*t'),\ \alpha \mapsto (f^*\sigma)\alpha(f^*\sigma)^{-1},$$

where $f^*\sigma : f^*t \to f^*t'$ is an isomorphism in $\mathscr{G}(U)$.

The isomorphism Φ_σ in 2.6.1.1 is not canonical, but depends on the choice of isomorphism σ. Indeed, given a different choice $\sigma' : t \to t'$, the map $\Phi_{\sigma'}$ is given by $\alpha \mapsto \sigma'\alpha\sigma'^{-1}$, which differs from Φ_σ by conjugation by $\sigma\sigma'^{-1} \in \underline{\mathrm{Aut}}_T(t)$. In the case where $\underline{\mathrm{Aut}}_T(t)$ is abelian for every $T \to S$ and $t \in \mathscr{G}(T)$, the isomorphism Φ is canonical.

Definition 2.6.2. Let $\pi : \mathscr{G} \to S$ be a gerbe. We say it is an *abelian gerbe* if for all $T \to S$ and all $t \in \mathscr{G}(T)$, the automorphism group sheaf $\underline{\mathrm{Aut}}_T(t)$ is abelian. Otherwise, we say that it is a *non-abelian gerbe*.

If $\mathscr{G}$ is an abelian gerbe, we get a sheaf of abelian groups A on S and a canonical isomorphism $A_T \to \underline{\mathrm{Aut}}_T(t)$ for every $t \in \mathscr{G}(T)$. In this case, we have the following definition.

Definition 2.6.3. Let A be a sheaf of abelian groups on S. An A-*gerbe over* S is a gerbe over S, $\pi : \mathscr{G} \to S$ together with an isomorphism of sheaves of groups on T,

$$\iota_t : A_T \to \underline{\mathrm{Aut}}_T(t),$$

for every $T \to S$ and $t \in \mathscr{G}(T)$ such that

(G3) For every $T \to S$ and isomorphism $\sigma : t \to t'$ in $\mathscr{G}(T)$, the diagram

$$\begin{tikzcd} & A_T \arrow[dl, "\iota_t"'] \arrow[dr, "\iota_{t'}"] & \\ \underline{\mathrm{Aut}}_T(t) \arrow[rr, "\Phi_\sigma"] & & \underline{\mathrm{Aut}}_T(t') \end{tikzcd}$$

commutes.

For any gerbe $\pi : \mathscr{G} \to S$, we would like to associate a sheaf of groups G on S, which locally is given by $\underline{\mathrm{Aut}}_T(t)$ for $T \to S$ and $t \in \mathscr{G}(T)$. In the abelian case, due to the canonical isomorphisms of the automorphism sheaves $\underline{\mathrm{Aut}}_T(t)$, these sheaves of groups defined locally satisfy descent and glue to a sheaf of groups on S. In general, the data is not descent data for the category of sheaves of groups, but is rather descent data for the category of bands, of which the category of sheaves of groups is a subcategory. We first define the sheaf of groups on S via descent data in the case where we have an abelian gerbe, and then we define the stack of bands, and define the band on S via descent data for the category of bands in the case of a non-abelian gerbe.

Let $\pi : \mathscr{G} \to S$ be an abelian gerbe. The sheaf of groups on S is given by the following descent data. Let $T \to S$ be a cover where $\mathscr{G}(T) \neq \emptyset$, let $t \in \mathscr{G}(T)$. Then

$$(\underline{\mathrm{Aut}}_T(t), \eta : \mathrm{pr}_1^*\underline{\mathrm{Aut}}_T(t) \to \mathrm{pr}_2^*\underline{\mathrm{Aut}}_T(t))$$

is descent data that defines a sheaf of groups on S. The isomorphism η is also defined using descent data, as we explain below.

We have that $\mathrm{pr}_i^*\underline{\mathrm{Aut}}_T(t) \cong \underline{\mathrm{Aut}}_T(t) \times_S T \cong \underline{\mathrm{Aut}}_{T\times_S T}(\mathrm{pr}_i^* t)$ for $i = 1, 2$. Indeed, these are the fiber product of the diagram

$$\begin{tikzcd}
\underline{\mathrm{Aut}}_{T\times_S T}(\mathrm{pr}_i^* t) \arrow[r] \arrow[d] & \underline{\mathrm{Aut}}_T(t) \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{G}) \arrow[d] \\
T \times_S T \arrow[r, "\mathrm{pr}_i"] & T \arrow[r, "t"] & \mathscr{G}.
\end{tikzcd}$$

Since $\mathrm{pr}_1^* t$ may not be isomorphic to $\mathrm{pr}_2^* t$ on $T \times_S T$, we may need to take a cover $T' \to T \times_S T$ for these to become isomorphic. The isomorphism η still exists on $T \times_S T$ due to the following lemma.

Lemma 2.6.4. *Let $\pi : \mathscr{G} \to S$ be an abelian gerbe. For any $T \to S$ and $t_1, t_2 : T \to \mathscr{G}$, there exists a canonical isomorphism of sheaves of groups $\underline{\mathrm{Aut}}_T(t_1) \cong \underline{\mathrm{Aut}}_T(t_2)$.*

Proof. Let $f : T' \to T$ be a cover where $f^* t_1 \cong f^* t_2$. Any choice of isomorphism gives a canonical isomorphism $\Phi : \underline{\mathrm{Aut}}_{T'}(f^* t_1) \cong \underline{\mathrm{Aut}}_{T'}(f^* t_2)$. We have

$$\mathrm{pr}_1^*\underline{\mathrm{Aut}}_{T'}(f^* t_i) \cong \underline{\mathrm{Aut}}_{T'\times_T T'}(\mathrm{pr}_1^* f^* t_i) \cong \underline{\mathrm{Aut}}_{T'\times_T T'}(\mathrm{pr}_2^* f^* t_i) \cong \mathrm{pr}_2^*\underline{\mathrm{Aut}}_{T'}(f^* t_i)$$

for $i = 1, 2$, so the following diagram commutes,

$$\begin{tikzcd}
\mathrm{pr}_1^*\underline{\mathrm{Aut}}_{T'}(f^* t_1) \arrow[r, "\mathrm{pr}_1^*\Phi"] \arrow[d] & \mathrm{pr}_1^*\underline{\mathrm{Aut}}_{T'}(f^* t_2) \arrow[d] \\
\mathrm{pr}_2^*\underline{\mathrm{Aut}}_{T'}(f^* t_1) \arrow[r, "\mathrm{pr}_2^*\Phi"] & \mathrm{pr}_2^*\underline{\mathrm{Aut}}_{T'}(f^* t_2)
\end{tikzcd}$$

and together with Φ, this defines descent data for an isomorphism of group sheaves on T, $\underline{\mathrm{Aut}}_T(t_1) \cong \underline{\mathrm{Aut}}_T(t_2)$. □

Lemma 2.6.4 also shows that for a different choice of $t \in \mathscr{G}(T)$, we get isomorphic descent data.

In the case where $\mathscr{G}$ is a non-abelian gerbe, η is only well-defined up to conjugation, and a choice of η may not satisfy the cocycle condition. We do however get descent data for the category of bands, which we now define.

The Stack of Bands

Let GroupSheaves = GroupSheaves/S be the category of sheaves of groups on S. Write the set of morphisms from objects F and G in GroupSheaves as $\mathrm{Hom}(F, G)$. We may also view a morphism as a global section of the sheaf

$$\mathscr{H}om(F, G) : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$$

$$(T \to S) \mapsto \mathrm{Hom}(F_T, G_T).$$

Definition 2.6.5. Let F and G be objects in GroupSheaves. Define

$$\mathscr{H}ex(F, G) := G\backslash\mathscr{H}om(F, G)/F$$

where for $\varphi \in \mathscr{H}om(F, G)$, $g \in G$, $f \in F$, $x \in F$, the action of G on the left and F on the right on $\mathscr{H}om(F, G)$ are given by

$$(g \cdot \varphi)(x) = g\varphi(x)g^{-1},$$

$$(\varphi \cdot f)(x) = \varphi(fxf^{-1}),$$

respectively. Let $\mathrm{Hex}(F, G)$ be the set of global sections of $\mathscr{H}ex(F, G)$.

Lemma 2.6.6. *We have isomorphisms*

$$G\backslash\mathscr{H}om(F, G)/F \cong G\backslash\mathscr{H}om(F, G) \cong \mathrm{Int}(G)\backslash\mathscr{H}om(F, G),$$

where $\mathrm{Int}(G) = G/Z(G)$.

Proof. Clearly $G\backslash\mathscr{H}om(F, G)$ surjects onto $G\backslash\mathscr{H}om(F, G)/F$. Let $\varphi, \varphi' \in \mathscr{H}om(F, G)$ such that for some $f \in F, g \in G$, and all $x \in F$, $\varphi'(x) = g\varphi(fxf^{-1})g^{-1}$. Then

$$\varphi'(x) = (g\varphi(f))\varphi(x)(g\varphi(f))^{-1},$$

so φ and φ' represent the same class in $G\backslash\mathscr{H}om(F, G)$. Finally, for $g \in Z(G)$, and any $\varphi \in \mathscr{H}om(F, G)$, $g \cdot \varphi = \varphi$, giving the second isomorphism. □

Given two sheaves of groups F and G with G abelian, $\mathscr{H}ex(F, G) \cong \mathscr{H}om(F, G)$.

Definition 2.6.7. Define the *category of pre-bands on* S, $\mathrm{Band}^{\mathrm{pre}} = \mathrm{Band}^{\mathrm{pre}}/S$, to be the category whose objects are sheaves of groups on S, and for objects F and G, morphisms are given by global sections of $\mathscr{H}ex(F, G)$.

We have a natural functor

$$\Pi : \mathrm{GroupSheaves} \to \mathrm{Band}^{\mathrm{pre}} \tag{2.6.7.1}$$

sending a sheaf of groups G to itself, and a morphism $\varphi : F \to G$ to its image in $\mathscr{H}ex(F, G)$. This functor is essentially surjective and full, but not faithful. When restricted to the category of sheaves of abelian groups, Π is fully faithful. We will write the image of $G \in$ GroupSheaves under Π by $\underline{G}$.

Definition 2.6.8. Define the *category of bands on S*, Band $=$ Band$/S$, to be the stackification of the category Band$^{\mathrm{pre}}$.

We have a functor

$$\Pi' : \mathrm{GroupSheaves} \to \mathrm{Band} \tag{2.6.8.1}$$

which is given by Π composed with the stackification functor. This functor is no longer essentially surjective, but is locally essentially surjective. Again when restricted to the category of sheaves of abelian groups, Π' is fully faithful. Bands can be constructed locally and satisfy decent, and a band can locally be lifted to a sheaf of groups.

The Band Associated to a Non-abelian Gerbe

For any non-abelian gerbe $\pi : \mathscr{G} \to S$, we get a band H on S and a canonical isomorphism $\iota_t : H_T \to \underline{\underline{\mathrm{Aut}_T(t)}}$ for every $t \in \mathscr{G}(T)$, where $\underline{\underline{\mathrm{Aut}_T(t)}}$ is the band on T associated to $\underline{\mathrm{Aut}}_T(t)$. We say that the gerbe $\pi : \mathscr{G} \to S$ is *banded by H*.

The band H on S is given by the following descent data. Let $T \to S$ be a cover where $\mathscr{G}(T) \neq \emptyset$, and let $t \in \mathscr{G}(T)$. Then

$$(\underline{\mathrm{Aut}}_T(t), \overline{\eta} \in \mathrm{Hex}(\underline{\mathrm{Aut}}_{T\times_S T}(\mathrm{pr}_1^* t), \underline{\mathrm{Aut}}_{T\times_S T}(\mathrm{pr}_2^* t))$$

is descent data that defines a band on S, where $\overline{\eta}$ exists due to an argument similar to Lemma 2.6.4, and is canonically defined as a morphism of bands.

Definition 2.6.9. Let H be a band on S. A *gerbe banded by H over S* is a gerbe over S, $\pi : \mathscr{G} \to S$ together with an isomorphism of bands on T,

$$\iota_t : H_T \to \underline{\underline{\mathrm{Aut}_T(t)}},$$

for every $T \to S$ and $t \in \mathscr{G}(T)$ such that

(G3') For every $T \to S$ and isomorphism $\sigma : t \to t'$ in $\mathscr{G}(T)$, the diagram

$$\begin{tikzcd} & H_T \arrow[dl, "\iota_t"'] \arrow[dr, "\iota_{t'}"] & \\ \underline{\underline{\mathrm{Aut}_T(t)}} \arrow[rr, "\Phi_\sigma"] & & \underline{\underline{\mathrm{Aut}_T(t')}} \end{tikzcd}$$

commutes.

Morphisms of Gerbes

Definition 2.6.10. Let A be a sheaf of abelian groups on S. A *morphism of A-gerbes* $f : (\pi' : \mathscr{G}' \to S, \{\iota_{t'}\}) \to (\pi : \mathscr{G} \to S, \{\iota_t\})$ is a morphism of stacks $f : \mathscr{G}' \to \mathscr{G}$ over S such that for every object $t : T \to \mathscr{G}'$, the diagram

$$\begin{tikzcd} & A \arrow[dl, "\iota_t"'] \arrow[dr, "\iota_{f(t)}"] & \\ \underline{\mathrm{Aut}}_T(t) \arrow[rr, "f_*"] & & \underline{\mathrm{Aut}}_T(f(t)) \end{tikzcd}$$

commutes. Here f_* is the natural pushforward of morphisms in $\mathscr{G}'$ coming from the morphism of stacks.

Definition 2.6.11. Let $u : A \to B$ be morphism a of sheaves of abelian groups on S. A *u-morphism of gerbes* $f : (\pi' : \mathscr{G}' \to S, \{\iota_{t'}\}) \to (\pi : \mathscr{G} \to S, \{\iota_t\})$ from an A-gerbe $\mathscr{G}'$ to a B-gerbe $\mathscr{G}$ is a morphism of stacks $f : \mathscr{G}' \to \mathscr{G}$ with morphism of stabilizers given by u, that is, such that for every $t : T \to \mathscr{G}'$, the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\iota_t"'] & B \arrow[d, "\iota_{f(t)}"] \\ \underline{\mathrm{Aut}}_T(t) \arrow[r, "f_*"] & \underline{\mathrm{Aut}}_T(f(t)) \end{tikzcd}$$

commutes.

Remark 2.6.12. We have analogous definitions for a morphism of non-abelian gerbes banded by H, and a u-morphism of gerbes where $u : G \to H$ is a morphism of bands, by requiring the diagrams to commute after passing to the category of bands.

Gerbes are Classified by H^2

Let A be a sheaf of abelian groups on S. We define a map

$$H^2(S, A) \to \{\text{isomorphism classes of } A\text{-gerbes over } S\}.$$

First, choose an inclusion $a : A \to I$ of sheaves of abelian groups over S with I injective. Let $K = I/A$ be the quotient. We have a short exact sequence

$$0 \longrightarrow A \xrightarrow{a} I \xrightarrow{b} K \longrightarrow 0.$$

Since I is injective, the boundary map $\delta : H^1(S, K) \to H^2(S, A)$ is an isomorphism. Then any $p \in H^2(S, A)$ corresponds to a K-torsor $P \to S$ with class $[P] = \delta^{-1}(p) \in H^1(S, K)$. Define $\mathscr{G}_P$ to be the stack over S with objects

$$(f : T \to S,\ Q \to T,\ \epsilon : b_* Q \cong P),$$

where T is an S-scheme, Q is an I-torsor, and ϵ is an isomorphism of K-torsors. Morphisms $(Q', \epsilon') \to (Q, \epsilon)$ are isomorphisms $\alpha : Q' \to Q$ of I-torsors such that the diagram

$$\begin{tikzcd} b_*Q' \arrow[rr, "b_*\alpha"] \arrow[dr, "\epsilon'"'] & & b_*Q \arrow[dl, "\epsilon"] \\ & P & \end{tikzcd}$$

commutes. Then $\mathscr{G}_P$ is an A-gerbe, and we map $p \mapsto [\mathscr{G}_P]$. This defines the bijection in the following theorem.

Theorem 2.6.13. *[21, IV. Theorem 3.4.2], [39, Theorem 12.2.8] Let A be a sheaf of abelian groups on S. We have a bijection*

$$H^2(S, A) \to \{\textit{isomorphism classes of } A\textit{-gerbes over } S\}.$$

Definition 2.6.14. Let G be a band on S, and define $H^2(S, G)$ to be the set of isomorphism classes of gerbes banded by G on S.

For a morphism of sheaves of abelian groups $u : A \to B$, we have a map

$$u_* : H^2(S, A) \to H^2(S, B)$$

given by the usual cohomology map. We would like to have a map like this for non-abelian gerbes too. Let $u : G \to H$ be a morphism of bands on S. Then we define a correspondence between $H^2(S, G)$ and $H^2(S, H)$ [21, Definition IV.3.1.4], where $p \in H^2(S, G)$ corresponds to $q \in H^2(S, H)$ if there is a gerbe P in the class of p and Q in the class of q and a u-morphism $f : P \to Q$. It follows from the definition that the correspondence is transitive for two morphisms of bands $u : G \to H$, $v : H \to K$.

The correspondence that Giraud defines does not always result in a map, but if H is abelian, then we get a morphism

$$u^{(2)} : H^2(S, G) \to H^2(S, H),$$

defined by the correspondence above [21, IV. Proposition 3.1.5, Corollary 3.1.8].

Proposition 2.6.15. *[39, Exercise 12.F(v)] Let $u : A \to B$ be a morphism of sheaves of abelian groups on S. Then the map $u^{(2)} : H^2(S, A) \to H^2(S, B)$ defined by the correspondence agrees with the cohomology morphism $u_* : H^2(S, A) \to H^2(S, B)$.*

In accordance with Proposition 2.6.15, we will write u_* for the correspondence $u^{(2)}$ whenever it is a morphism.

The Inertia of Gerbes

In this subsection, let S be a scheme or algebraic space, let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic stacks over S, and let G be a band on $\mathscr{Y}$. We show that if $p : \mathscr{X} \to \mathscr{Y}$ is gerbe banded by G, then there is a group G' on $\mathscr{X}$, given by the relative inertia of $\mathscr{X}$ over $\mathscr{Y}$, $G' = \mathscr{I}(\mathscr{X}/\mathscr{Y})$, such that $\underline{G'} \cong p^*G$ as bands. We describe this group G' in terms of the absolute inertia of $\mathscr{X}$ and $\mathscr{Y}$.

Lemma 2.6.16. *Let $p : \mathscr{X} \to \mathscr{Y}$ be a gerbe banded by G, a flat band on $\mathscr{Y}$. Then any section $s : \mathscr{Y} \to \mathscr{X}$ is faithfully flat.*

Proof. Let $\widetilde{G} = \underline{\mathrm{Aut}}_{\mathscr{Y}}(s)$, a sheaf of groups on $\mathscr{Y}$. Since p has a section, we have $\mathscr{X} \cong B_{\mathscr{Y}}\widetilde{G}$ by Proposition 2.6.21, and we can choose an isomorphism where s maps to the standard section $\mathscr{Y} \to B_{\mathscr{Y}}\widetilde{G}$. Moreover, $\underline{\widetilde{G}} \cong G$ as bands, and $\widetilde{G}$ is also flat. We can check that the standard section is faithfully flat after pulling back by the standard section,

$$\begin{tikzcd} \mathscr{Y} \times \widetilde{G} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{Y} \arrow[r] & B_{\mathscr{Y}}\widetilde{G}, \end{tikzcd}$$

which is true since $\widetilde{G}$ is flat. $\square$

Proposition 2.6.17. *Let S be a scheme or algebraic space and let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic stacks over S. Let $p : \mathscr{X} \to \mathscr{Y}$ be a gerbe banded by G. Then there is a sheaf of groups on $\mathscr{X}$, $G' \cong \mathscr{I}(\mathscr{X}/\mathscr{Y})$, such that for all $x : T \to \mathscr{X}$, G'_x is the kernel of the map*

$$\underline{\mathrm{Aut}}_T(x) \to \underline{\mathrm{Aut}}_T(p(x)). \tag{2.6.17.1}$$

If G is flat on $\mathscr{Y}$, then G' is flat on $\mathscr{X}$ and 2.6.17.1 is surjective. Moreover, $p_*\underline{G'} \cong G$ as bands on $\mathscr{Y}$, and $p^*G \cong \underline{G'}$ as bands on $\mathscr{X}$. This lifts to an isomorphism $p^*G \cong G'$ of groups on $\mathscr{X}$ if and only if G' is abelian.

Proof. For a T-point $x \in \mathscr{X}(T)$, we have a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Aut}}_T(x) \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{X}) \arrow[d] \\ T \arrow[r, "x"] & \mathscr{X}. \end{tikzcd}$$

Similarly, we have a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Aut}}_T(p(x)) \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{Y}) \arrow[d] \\ T \arrow[r, "x"] & \mathscr{X} \arrow[r, "p"] & \mathscr{Y}. \end{tikzcd}$$

From the commutative diagram

$$\begin{tikzcd}
\mathscr{I}(\mathscr{X}) \arrow[r] \arrow[d] & \mathscr{I}(\mathscr{Y}) \arrow[d] \\
\mathscr{X} \arrow[r, "p"] & \mathscr{Y}
\end{tikzcd}$$

we get a morphism of algebraic stacks

$$\mathscr{I}(\mathscr{X}) \to \mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X},$$

$$(x, g) \mapsto ((p(x), p(g)), x, \mathrm{id}_{p(x)} : p(x) \to p(x)).$$

Thus, we have the following diagram with two cartesian squares,

$$\begin{tikzcd}
\underline{\mathrm{Aut}}_T(x) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{I}(\mathscr{X}) \arrow[dr] \arrow[dd] & \\
& \underline{\mathrm{Aut}}_T(p(x)) \arrow[rr] \arrow[dl] & & \mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X} \arrow[dl] \\
T \arrow[rr, "x"] & & \mathscr{X} &
\end{tikzcd}.$$

To show that $\underline{\mathrm{Aut}}_T(x) \to \underline{\mathrm{Aut}}_T(p(x))$ is surjective for all $x \in \mathscr{X}(T)$ in the case that G is flat, we will directly show that $\mathscr{I}(\mathscr{X}) \to \mathscr{I}(\mathscr{Y}) \times_Y \mathscr{X}$ is surjective.

Let $\mathscr{P}$ be the fiber product $(\mathscr{X} \times \mathscr{X}) \times_{\mathscr{Y} \times \mathscr{Y}} \mathscr{Y}$, and let $\tau : \mathscr{X} \to \mathscr{P}$ be the morphism induced by $p : \mathscr{X} \to \mathscr{Y}$ and $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$. Write $p^*\mathscr{I}(\mathscr{Y}) = \mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X}$. We have the following commutative diagram,

$$\begin{tikzcd}
\mathscr{I}(\mathscr{X}) \arrow[r] \arrow[d] & p^*\mathscr{I}(\mathscr{Y}) \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\Delta"] \\
\mathscr{X} \arrow[r, "\tau"] \arrow[dr, "p"] & \mathscr{P} \arrow[r] \arrow[d] & \mathscr{X} \times \mathscr{X} \arrow[d] \\
& \mathscr{Y} \arrow[r, "\Delta"] & \mathscr{Y} \times \mathscr{Y},
\end{tikzcd}$$

and thus if $\tau : \mathscr{X} \to \mathscr{P}$ is surjective, then $\mathscr{I}(\mathscr{X}) \to p^*\mathscr{I}(\mathscr{Y})$ will be as well.

Since $\mathscr{X} \times \mathscr{X} \to \mathscr{Y} \times \mathscr{Y}$ is a $G \times G$-gerbe, so is $\mathscr{P} \to \mathscr{Y}$, and so is $\mathscr{Q} = \mathscr{X} \times_{\mathscr{Y}} \mathscr{P} \to \mathscr{X}$. Moreover, we have a section $s : \mathscr{X} \to \mathscr{Q}$ induced by τ. Then by Lemma 2.6.16, s is faithfully flat, and thus surjective. Then since $\mathscr{Q} \to \mathscr{P}$ is a gerbe, the composition $\tau : \mathscr{X} \to \mathscr{Q} \to \mathscr{P}$ is also surjective, which proves that $\mathscr{I}(\mathscr{X}) \to p^*\mathscr{I}(\mathscr{Y})$ is surjective.

Define G' to be the kernel of $\mathscr{I}(\mathscr{X}) \to \mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X}$. The zero section of the inertia stack $\mathscr{I}(\mathscr{X})$ consists of elements (x, id_x), and the zero section of $\mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X}$ consists of elements $((y, \mathrm{id}_y), x, \sigma : y \to p(x))$. Thus, the kernel G' consists of elements (x, g) such that $p(g)$ is the identity automorphism of $p(x)$, which agrees with $\mathscr{I}(\mathscr{X}/\mathscr{Y})$. $\square$

Examples of Gerbes

Example 2.6.18. The trivial gerbe BG

Definition 2.6.19. Let S be a scheme or algebraic stack and let G be a sheaf of groups on S. Define the *trivial gerbe BG over S* to be the stack quotient $[S/G]$ where G acts trivially on S.

BG is the stack with objects $(f : T \to S,\ P \to T)$, where T is an S-scheme and P is an G-torsor on T. Morphisms over T are isomorphisms of G-torsors on T.

Proposition 2.6.20. *Let G be a sheaf of abelian groups on S. The class of BG in $H^2(S, G)$ is 0.*

Proof. Let $a : G \to I$ be an inclusion of sheaves of abelian groups on S with I injective and let $K = I/G$ be the quotient, as in the discussion above Theorem 2.6.13. We have a short exact sequence

$$0 \longrightarrow G \xrightarrow{a} I \xrightarrow{b} K \longrightarrow 0.$$

Then the G-gerbe corresponding to $0 \in H^2(S, G)$ is the G-gerbe $\mathscr{G}$ associated to $0 \in H^1(S, K)$, with objects

$$(f : T \to S,\ Q \to T,\ \epsilon : b_*Q \cong T \times K),$$

where T is an S-scheme, Q is an I-torsor, and ϵ is an isomorphism of K-torsors. Morphisms are isomorphisms $\alpha : Q' \to Q$ such that the diagram

$$\begin{tikzcd} b_*Q' \arrow[rr, "b_*\alpha"] \arrow[dr, "\epsilon'"'] & & b_*Q \arrow[dl, "\epsilon"] \\ & T \times K & \end{tikzcd}$$

commutes.

Given $(Q, \epsilon) \in \mathscr{G}$, define the sheaf $P : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$ by

$$(f : T \to S) \mapsto \{\sigma : Q|_{T'} \cong T' \times I \mid b_*\sigma = \epsilon : b_*Q|_{T'} \cong T' \times K\}.$$

Then P is a G-torsor over T. Given a morphism $\alpha : (Q'\epsilon') \to (Q, \epsilon)$ in $\mathscr{G}$, we get a morphism $P' \to P$ of the associated G-torsors, given by $\sigma \mapsto \sigma \circ \alpha^{-1}$. The association $(Q, \epsilon) \mapsto P$ defines a morphism of G-gerbes, so is an isomorphism, $\mathscr{G} \cong BG$. $\square$

We have a section of $BG \to S$, given by the trivial G-torsor $S \times G$ on S, and this corresponds to the natural quotient map $S \to [S/G]$ and is the universal G-torsor on BG. Let $P \to T$ be a G-torsor. Consider the fiber product

$$\begin{tikzcd} S \times_{[S/G]} T \arrow[r] \arrow[d] & S \arrow[d] \\ T \arrow[r, "P"] & {[S/G],} \end{tikzcd}$$

which has objects over $T' \to T$ given by isomorphisms $\sigma : G \times T' \cong P|_{T'}$. We have an isomorphism $P \cong S \times_{[S/G]} T$ given by $(x : T' \to P) \mapsto (\sigma : G \times T' \to P|_{T'},\ e_G \mapsto x)$.

Proposition 2.6.21. *A gerbe $\mathscr{G}$ over S is trivial, that is, isomorphic to BG for some group G on S, if and only if it has a section $s : S \to \mathscr{G}$.*

Proof. Let $G = \underline{\mathrm{Aut}}_S(s)$, a sheaf of groups on S. We use the section s to define a map $\mathscr{G} \to BG$. Let $t : T \to \mathscr{G}$ be an object of $\mathscr{G}$ over $f : T \to S$. We have two sections of the pullback $\mathscr{G} \times_S T \to T$, namely f^*s and the section induced by t, which we will also write as t. We define a G-torsor P_t over T by

$$P_t : (T' \to T) \mapsto \{\sigma : t \cong f^*s\}.$$

Given a morphism $\alpha : t' \to t$ in $\mathscr{G}$, we get a morphism $P_{t'} \to P_t$ of the corresponding G-torsors by $\sigma \mapsto \sigma \circ \alpha^{-1}$. The association $t \mapsto P_t$ defines a morphism of G-gerbes $\mathscr{G} \to BG$, and thus it is an isomorphism of G-gerbes. $\square$

Example 2.6.22. The gerbe of nth roots of a line bundle

Let S be a scheme or algebraic stack and let $\mathscr{L}$ be a line bundle on S. Let $\mathscr{G}$ be the stack with objects

$$(f : T \to S, (\mathscr{M}, \epsilon : \mathscr{M}^{\otimes n} \to f^*\mathscr{L})),$$

where T is an S-scheme, $\mathscr{M}$ is a line bundle on T, and ϵ is isomorphism of line bundles on T. Morphisms $(\mathscr{M}', \epsilon') \to (\mathscr{M}, \epsilon)$ in $\mathscr{G}(T)$ are isomorphisms of line bundles on T, $\sigma : \mathscr{M}' \to \mathscr{M}$ such that the diagram

$$\begin{tikzcd} \mathscr{M}'^{\otimes n} \arrow[rr, "\sigma^{\otimes n}"] \arrow[dr, "\epsilon'"'] & & \mathscr{M}^{\otimes n} \arrow[dl, "t^*\epsilon"] \\ & f^*\mathscr{L} & \end{tikzcd}$$

commutes.

We show that $\mathscr{G}$ is a μ_n-gerbe over S. For any $f : T \to S$, let $g : T' \to T$ be a covering where $\mathscr{L}$ is trivial. Then the trivial line bundle on T' together with the multiplication morphism $\mathscr{O}_{T'}^{\otimes n} \to \mathscr{O}_{T'}$ gives a section $\mathscr{G}(T')$.

Let $(\mathscr{M}', \epsilon'), (\mathscr{M}, \epsilon) \in \mathscr{G}(f : T \to S)$. Again we choose a trivializing cover so that we may assume $\mathscr{M} = \mathscr{M}' = \mathscr{O}_T$. The isomorphism in the top line of the diagram

$$\begin{tikzcd} \mathscr{O}_T \cong \mathscr{O}_T^{\otimes n} \arrow[rr] \arrow[dr, "\epsilon'"'] & & \mathscr{O}_T^{\otimes n} \cong \mathscr{O}_T \arrow[dl, "\epsilon"] \\ & \mathscr{O}_T & \end{tikzcd}$$

is given by multiplication by some $u \in \mathscr{O}_T^*$. Étale locally, u has an nth root, and on this cover, $(\mathscr{M}', \epsilon') \cong (\mathscr{M}, \epsilon)$ via the root of u.

Finally, for any $(\mathscr{M},\epsilon) \in \mathscr{P}(f : T \to S)$, $\underline{\mathrm{Aut}}_T(\mathscr{M},\epsilon)$ is the set of automorphisms $\sigma : \mathscr{M} \to \mathscr{M}$ such that $\epsilon = \epsilon \circ \sigma^{\otimes n} : \mathscr{M}^{\otimes n} \to f^*\mathscr{L}$, so $\underline{\mathrm{Aut}}_T(\mathscr{M},\epsilon) = \mu_n$.

Thus, $\mathscr{G} \to S$ is a μ_n-gerbe, which is isomorphic to $B_S\mu_n$ if and only if there exists an nth root of $\mathscr{L}$ on S. Let $[\mathscr{G}]$ be the class of the gerbe $\mathscr{G}$ in $H^2(S,\mu_n)$.

Consider the short exact sequence

$$1 \longrightarrow \mu_n \xrightarrow{\alpha} \mathbb{G}_m \xrightarrow{\beta} \mathbb{G}_m \longrightarrow 1,$$

where α is the inclusion and β is multiplication by n. We get a long exact sequence

$$H^1(S,\mathbb{G}_m) \xrightarrow{\beta_*} H^1(S,\mathbb{G}_m) \xrightarrow{\delta} H^2(S,\mu_n) \xrightarrow{\alpha_*} H^2(S,\mathbb{G}_m).$$

Lemma 2.6.23. *The class $[\mathscr{G}] \in H^2(S,\mu_n)$ is the image of $[\mathscr{L}] \in H^1(S,\mathbb{G}_m)$ under δ.*

Proof. The $\mathbb{G}_m$-torsor over S associated to $\mathscr{L}$ is the principal homogeneous space $\mathbb{L}$, where

$$\mathbb{L} = \underline{\mathrm{Spec}}_S(\oplus_{i\in\mathbb{Z}}\mathscr{L}^{\otimes i}),$$

with action of $\mathbb{G}_m$ given as follows.

Write

$$\begin{aligned}\mathbb{L}(f : T \to S) &= \{\rho : f^*(\oplus_{i\in\mathbb{Z}}\mathscr{L}^{\otimes i}) \to \mathscr{O}_T\} \\ &= \{\oplus_{i\in\mathbb{Z}}\rho_i : f^*\mathscr{L}^{\otimes i} \to \mathscr{O}_T\}.\end{aligned}$$

Then $u \in \mathbb{G}_m(T)$ acts on $\rho \in \mathbb{L}(T)$ by $u \cdot \rho = \oplus_{i\in\mathbb{Z}}u^i\rho_i$.

This defines a $\mathbb{G}_m$-torsor. For any $f : T \to S$, take a covering $T' \to T$ where $\mathscr{L}$ is trivial. Then the direct sum of the standard multiplication maps $\mathscr{O}_{T'}^{\otimes i} \to \mathscr{O}_{T'}$ define an element of $\mathbb{L}_P(T')$, which is thus nonempty.

Given two elements $\rho,\rho' : \oplus_{i\in\mathbb{Z}}f^*\mathscr{L}^{\otimes i} \to \mathscr{O}_T$ in $\mathbb{L}(f : T \to S)$, there exists a unique $u \in \mathbb{G}_m(T)$ such that $\rho_1 = u\rho'_1 : f^*\mathscr{L} \to \mathscr{O}_T$, and this is compatible with the tensor product structure, so also $\rho = u \cdot \rho' : \oplus_{i\in\mathbb{Z}}f^*\mathscr{L}^{\otimes i} \to \mathscr{O}_T$.

Now let $\mathscr{Y}$ be the image of $\mathbb{L}$ under the boundary map δ. This is the μ_n-gerbe with objects

$$(f : T \to S,\ P,\ \sigma : \beta_*P \to \mathbb{L}|_T),$$

where T is an S-scheme, P is a $\mathbb{G}_m$-torsor on T, and σ is an isomorphism of $\mathbb{G}_m$-torsors on T. Morphisms $(P',\sigma') \to (P,\sigma)$ in $\mathscr{Y}(T)$ are isomorphisms of $\mathbb{G}_m$-torsors $\tau : P' \to P$ such that the diagram

$$\begin{tikzcd} \beta_*P' \arrow[rr, "\beta_*\tau"] \arrow[dr, "\sigma'"'] & & \beta_*P \arrow[dl, "\sigma"] \\ & \mathbb{L}|_T & \end{tikzcd}$$

commutes [39, Section 12.2]. We define a morphism of μ_n-gerbes $\phi : \mathscr{G} \to \mathscr{Y}$ to conclude that $\mathscr{G} \cong \mathscr{Y}$ [39, Lemma 12.2.4].

Let ϕ be the map given by

$$(f : T \to S, (\mathscr{M}, \epsilon)) \mapsto (f : T \to S,\ \mathbb{M},\ \sigma_\epsilon : \beta_*\mathbb{M} \cong \mathbb{L}|_T),$$

where $\mathbb{M} = \underline{\mathrm{Spec}}_T(\oplus_{i\in\mathbb{Z}}\mathscr{M}^{\otimes i})$, with objects

$$\mathbb{M}(h : U \to T) = \{\lambda : h^*(\oplus_{i\in\mathbb{Z}}\mathscr{M}^{\otimes i}) \to \mathscr{O}_U\}$$

and $\mathbb{G}_m$ acts analogously as on $\mathbb{L}$. To understand σ_ϵ, we must first understand $\beta_*\mathbb{M}$. Since β is surjective, $\beta_*\mathbb{M}$ is given by $\mathbb{M}/\mu_2$, with action of $\mathbb{G}_m$ given by $u \cdot \overline{\lambda} = \overline{u^{1/n}\lambda}$. Note that in the quotient, the choice of root is unambiguous. Taking the quotient of $\mathbb{M}$ by the μ_n-action makes any element $\lambda : h^*(\oplus_{i\in\mathbb{Z}}\mathscr{M}^{\otimes i}) \to \mathscr{O}_U$ well-defined only up to $i \in n\mathbb{Z}$, so we can write elements of $\mathbb{M}$ as $\lambda' : h^*(\oplus_{i\in\mathbb{Z}}\mathscr{M}^{\otimes ni}) \to \mathscr{O}_U$, with $\mathbb{G}_m$-action given by $u \cdot \lambda = \oplus_{i\in\mathbb{Z}} u^i\lambda'_i : \oplus_{i\in\mathbb{Z}} h^*\mathscr{M}^{\otimes ni} \to \mathscr{O}_U$. Then σ_ϵ is simply induced by the isomorphism of line bundles $\epsilon : \mathscr{M}^{\otimes n} \to f^*\mathscr{L}$, and σ_ϵ is an isomorphism of $\mathbb{G}_m$-torsors as it is clearly $\mathbb{G}_m$-equivariant.

A morphism $\sigma : \mathscr{M}' \to \mathscr{M}$ in $\mathscr{G}(T)$ maps to the morphism $\tau : \mathbb{M} \to \mathbb{M}'$ induced by σ. This is a morphism of μ_n-gerbes, so is an isomorphism. □

Finally, the Yoneda lemma gives us a universal line bundle $\mathscr{M}^u$ on $\mathscr{G}$ and an isomorphism $\mathscr{M}^{u\otimes n} \cong \mathscr{L}$.

Example 2.6.24. The gerbe of line bundles of degree 1 on a Brauer-Severi variety
This is example will be studied thoroughly in Chapter 4. Let X be a Brauer-Severi variety over a field k, so $X_{\overline{k}} \cong \mathbb{P}^n_{\overline{k}}$. for some n.

Let $\mathscr{G}$ be the stack with objects $(f : T \to \mathrm{Spec}\ k, \mathscr{L})$, where T is an k-scheme and $\mathscr{L}$ is a line bundle on X_T of degree 1. Morphisms in $\mathscr{G}(T)$ are isomorphisms of line bundles $\sigma : \mathscr{L}' \to \mathscr{L}$.

Then $\mathscr{G}$ is a $\mathbb{G}_m$-gerbe over Spec k and $\mathscr{G} \cong B\mathbb{G}_m$ if and only if there is a degree 1 line bundle on X. Consider the short exact sequence,

$$1 \to \mathbb{G}_m \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1,$$

and the resulting long exact sequence

$$H^1(k, \mathrm{GL}_{n+1}) \to H^1(k, \mathrm{PGL}_{n+1}) \to H^2(k, \mathbb{G}_m).$$

It will be explained in Chapter 4 that Brauer-Severi varieties of dimension n are in bijection with PGL_{n+1}-torsors, and that $[\mathscr{G}] \in H^2(k, \mathbb{G}_m)$ is the image of $[X] \in H^1(k, \mathrm{PGL}_{n+1})$. In the case when $\mathscr{G} \cong B\mathbb{G}_m$, there exists a rank-$n+1$ vector bundle V such that the image of $[V] \in H^1(k, \mathrm{GL}_{n+1})$ is $[X]$, and $X \cong \mathbb{P}V$.

Again the Yoneda Lemma gives us a universal line bundle $\mathscr{L}^u$ of degree 1 on $\mathscr{G}$.

2.7 Stacky Curves

Definition 2.7.1. A *stacky curve over a field* k is a smooth, separated, geometrically connected, locally finitely presented Deligne–Mumford stack of dimension 1 over k.

Lemma 2.7.2. *Let $\mathscr{X}$ be a stacky curve. Then the diagonal is finite, and $\mathscr{X}$ has a coarse moduli space $\pi : \mathscr{X} \to X$ with X a scheme.*

Proof. Since $\mathscr{X}$ is separated, the diagonal is proper and thus the diagonal is separated, of finite type, and universally closed. Since the diagonal is separated, the diagonal is a closed immersion, and thus is finite.

Alternatively, the diagonal is locally quasi-finite if and only if its fibers are discrete [46, 06RW, 04XB]. Lemma 2.3.2 shows that the fibers are discrete, so the diagonal is locally quasi-finite. Being finite type implies being quasi-compact, so the diagonal is quasi-finite. Since the diagonal is also proper, it is finite [46, 02LS, 0418, 03HA].

Since $\mathscr{X}$ is locally of finite type with finite diagonal, $\mathscr{X}$ has a coarse moduli space $\pi : \mathscr{X} \to X$. Since $\mathscr{X}$ is smooth, it is normal, so X is normal of dimension 1 and thus smooth. The coarse space is also separated [39, Theorem 11.1.2 (i)], [28, Corollary 1.3] and 1-dimensional over a field, so X is a scheme [30, Theorem V.4.9]. □

Moreover, the result of Lemma 2.3.2, that for all geometric points $x : \operatorname{Spec} k \to \mathscr{X}$, G_x is a finite constant group scheme, holds for any stacky curve $\mathscr{X}$.

Proposition 2.7.3. *Let $\mathscr{X}$ be a stacky curve with trivial generic stabilizer. Then the dense open $\mathscr{U} \hookrightarrow \mathscr{X}$ in Proposition 2.4.5 is a dense open subscheme in $\mathscr{X}$.*

Proof. The dense open $\mathscr{U}$ is a smooth, separated, 1-dimensional algebraic space over a field, so is a scheme [30, Theorem V.4.9]. □

Proposition 2.7.4. *[38, Proposition 2.2 (i)] Let $\mathscr{X}$ be a tame stacky curve with trivial generic stabilizer and $\pi : \mathscr{X} \to X$ its coarse space, and let $x : \operatorname{Spec} k \to X$ be a geometric point. Then*

$$\operatorname{Spec} \mathscr{O}_{X,x} \times_X \mathscr{X} \cong \left[\operatorname{Spec}\left(\frac{\mathscr{O}_{X,x}[z]}{z^n - f}\right) \Big/ \mu_n\right]$$

for some $n \in \mathbb{Z}$, where $\mathscr{O}_{X,x}$ is the étale local ring of x, $f \in \mathscr{O}_{X,x}$ is a local coordinate at x, generating the maximal ideal of $\mathscr{O}_{X,x}$, and μ_n acts trivially on $\mathscr{O}_{X,x}$ and acts by multiplication by a primitive nth root of unity on z. In particular, the stabilizer group of x is a finite cyclic group μ_n for some $n \in \mathbb{Z}$.

We will see that stacky curves with trivial generic stabilizers are root stacks over their coarse spaces in Proposition 3.1.9, and more generally, stacky curves are gerbes over stacky curves with trivial generic stabilizers in Theorem 3.3.1.

Chapter 3

Picard Groups of Stacky Curves

In this chapter, we give a general description of the Picard group of a stacky curve. We begin in Section 3.1 with the case of a stacky curve with trivial generic stabilizer. Then in Section 3.3 we show that every stacky curve can be written as a gerbe over a stacky curve with trivial generic stabilizer, using results from Section 3.2 on rigidification. Finally, in Section 3.4, we give an exact sequence which describes the Picard group of a gerbe over its base, and apply this to the case of stacky curves. We conclude this chapter with examples in Section 3.5.

3.1 Picard Groups of Stacky Curves with Trivial Generic Stabilizer

The key to calculating the Picard groups of stacky curves with trivial generic stabilizers is to understand the Picard groups at the stacky points, and more generally, in étale neighborhoods of the stacky points. In the first subsection, we describe the line bundles on the classifying stack $BG = [\mathrm{Spec}\ k/G]$ for a field k and finite group G acting trivially on k, and more generally, quasi-coherent sheaves on stack quotients of the form $[\mathrm{Spec}\ R/G]$ for a ring R and finite group G acting on R.

Quasi-coherent Sheaves on Affine Stack Quotients

Proposition 3.1.1. *Let G be a finite group scheme over a field k. We have an equivalence of categories between line bundles $\mathscr{L}$ on BG and rank one representations $\chi_{\mathscr{L}} : G \to \mathbb{G}_m$ of G.*

Proof. Consider the map $\mathrm{Spec}\ k \to BG$, which is an étale covering of BG, since G is étale. The fiber product $\mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k$ is isomorphic to G, and the two projections

$$p_1, p_2 : \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k \to \mathrm{Spec}\ k$$

are the same. The triple fiber product Spec $k \times_{BG}$ Spec $k \times_{BG}$ Spec k is isomorphic to $G \times G$ with projections

$$\mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13} : \text{Spec } k \times_{BG} \text{Spec } k \times_{BG} \text{Spec } k \to \text{Spec } k \times_{BG} \text{Spec } k$$

given by

$$\mathrm{pr}_1, \mathrm{pr}_2, m : G \times G \to G,$$

respectively, where $\mathrm{pr}_1(g,h) = g, \mathrm{pr}_2(g,h) = h$, and $m(g,h) = gh$. Since $\mathscr{P}ic$, the fibered category of line bundles over stacks, satisfies effective descent,

$$\mathscr{P}ic(BG) = \mathscr{P}ic(\text{Spec } k \to BG),$$

where $\mathscr{P}ic(\text{Spec } k \to BG)$ is the category consisting of line bundles $\mathscr{L}$ on Spec k and isomorphisms $\sigma : p_1^*\mathscr{L} \cong p_2^*\mathscr{L}$ which satisfy the cocycle condition,

$$m^*\sigma = \mathrm{pr}_1^*\sigma \circ \mathrm{pr}_2^*\sigma : \mathrm{pr}_1^*p_1^*\mathscr{L} \cong \mathrm{pr}_2^*p_2^*\mathscr{L}.$$

A line bundle $\mathscr{L}$ on Spec k is a one-dimensional k-vector space L, and $p_1^*\mathscr{L} = p_2^*\mathscr{L} = \prod_{g\in G} L$. Then an isomorphism $\sigma : p_1^*\mathscr{L} \to p_2^*\mathscr{L}$ is an isomorphism $\sigma_g : L \cong L$ for each $g \in G$. Write the data of the isomorphism as a map $\rho : G \to \mathrm{Aut}(L) = L^\times$, where $\rho(g) = \sigma_g$. Since G acts trivially on k, we also have $\mathrm{pr}_1^*p_1^*\mathscr{L} = \mathrm{pr}_2^*p_2^*\mathscr{L} = \prod_{(g,h)\in G\times G} L$. Each of $\eta = \mathrm{pr}_1^*\sigma, \mathrm{pr}_2^*\sigma, m^*\sigma$ is an isomorphism $\eta_{(g,h)} : L \cong L$ for each $(g,h) \in G \times G$ given by $(\mathrm{pr}_1^*\sigma)_{(g,h)} = \sigma_g, (\mathrm{pr}_2^*\sigma)_{(g,h)} = \sigma_h$, and $(m^*\sigma)_{(g,h)} = \sigma_{gh}$. Then the cocycle condition holds if for all $(g,h) \in G \times G$, $\rho(gh) = \rho(g)\rho(h)$, that is, if ρ is a homomorphism. Therefore, giving a line bundle on BG is equivalent to giving a one-dimensional representation of G. □

Proposition 3.1.1 above is a special case of the following proposition, describing quasi-coherent sheaves on affine stack quotients.

Proposition 3.1.2. *Let G be a finite group acting on a ring R. We have an equivalence of categories between quasi-coherent sheaves on* $[\text{Spec } R/G]$ *and* Mod_R^G*, the category of R-modules with R-semi-linear G-action, that is, a G-action satisfying $g \cdot (rm) = (g \cdot r)(g \cdot m)$ for all $g \in G, r \in R, m \in M$.*

Proof. Let $X = \text{Spec } R$ and $\mathscr{X} = [\text{Spec } R/G]$. Again, $X \to \mathscr{X}$ is an étale covering and quasi-coherent sheaves on $\mathscr{X}$ can be described as quasi-coherent sheaves M on X together with an isomorphism $\sigma : \mathrm{pr}_1^*M \cong \mathrm{pr}_2^*M$ on $X \times_{\mathscr{X}} X$ satisfying the cocycle condition on $X \times_{\mathscr{X}} X \times_{\mathscr{X}} X$. To understand the isomorphism and the cocycle condition, we use the isomorphism $G \times X \cong X \times_{\mathscr{X}} X$, given by $(g,x) \mapsto (gx, x)$. This isomorphism changes the projection maps

$$\mathrm{pr}_1 : X \times_{\mathscr{X}} X \to X, \quad (x_1, x_2) \mapsto x_1,$$

$$\mathrm{pr}_2 : X \times_{\mathscr{X}} X \to X, \quad (x_1, x_2) \mapsto x_2,$$

to

$$\mathrm{pr}_1 : G \times X \to X, \quad (g, x) \mapsto gx,$$
$$\mathrm{pr}_2 : G \times X \to X, \quad (g, x) \mapsto x.$$

For the triple intersection, note that we have an isomorphism

$$X \times_{\mathscr{X}} X \times_{\mathscr{X}} X \cong (X \times_{\mathscr{X}} X) \times_{\mathrm{pr}_2, X, \mathrm{pr}_1} (X \times_{\mathscr{X}} X),$$

given by $(x_1, x_2, x_3) \mapsto ((x_1, x_2), (x_2, x_3))$. Under this isomorphism, the projection maps $\mathrm{pr}_{12}, \mathrm{pr}_{13}, \mathrm{pr}_{23} : X \times_{\mathscr{X}} X \times_{\mathscr{X}} X \to X \times_{\mathscr{X}} X$ are given by

$$\mathrm{pr}_{12} = (\mathrm{pr}_1, \mathrm{pr}_1) : (X \times_{\mathscr{X}} X) \times_{\mathrm{pr}_2, X, \mathrm{pr}_1} (X \times_{\mathscr{X}} X) \to X \times_{\mathscr{X}} X, \quad ((x_1, x_2), (x_2, x_3)) \mapsto (x_1, x_2),$$

$$\mathrm{pr}_{13} = (\mathrm{pr}_1, \mathrm{pr}_2) : (X \times_{\mathscr{X}} X) \times_{\mathrm{pr}_2, X, \mathrm{pr}_1} (X \times_{\mathscr{X}} X) \to X \times_{\mathscr{X}} X, \quad ((x_1, x_2), (x_2, x_3)) \mapsto (x_1, x_3),$$
$$\mathrm{pr}_{23} = (\mathrm{pr}_2, \mathrm{pr}_2) : (X \times_{\mathscr{X}} X) \times_{\mathrm{pr}_2, X, \mathrm{pr}_1} (X \times_{\mathscr{X}} X) \to X \times_{\mathscr{X}} X, \quad ((x_1, x_2), (x_2, x_3)) \mapsto (x_2, x_3).$$

Similarly, we have an isomorphism

$$G \times G \times X \cong (G \times X)_{\mathrm{pr}_2, X, \mathrm{pr}_1} (G \times X),$$

given by $(g, h, x) \mapsto ((g, hx), (h, x))$. Then the projection maps $\mathrm{pr}_{ij} : G \times G \times X \to G \times X$ are determined by the composition

$$G \times G \times X \cong (G \times X)_{\mathrm{pr}_2, X, \mathrm{pr}_1} (G \times X) \cong (X \times_{\mathscr{X}} X) \times_{\mathrm{pr}_2, X, \mathrm{pr}_1} (X \times_{\mathscr{X}} X)$$

$$\xrightarrow{\mathrm{pr}_{ij}} X \times_{\mathscr{X}} X \cong G \times X \ ,$$

and are given by

$$\mathrm{pr}_{12} : G \times G \times X \to G \times X, \quad (g, h, x) \mapsto (g, hx),$$
$$\mathrm{pr}_{13} : G \times G \times X \to G \times X, \quad (g, h, x) \mapsto (gh, x),$$
$$\mathrm{pr}_{23} : G \times G \times X \to G \times X, \quad (g, h, x) \mapsto (h, x).$$

A quasi-coherent sheaf on X is given by an R-module M. We will show that an isomorphism $\sigma : \mathrm{pr}_1^* M \cong \mathrm{pr}_2^* M$ satisfying the cocycle condition is given by an R-semi-linear action of G. Let $\Delta : R \to R \times \cdots \times R$ be the diagonal inclusion, and $\rho : R \to R \times \cdots \times R$ the action map, $r \mapsto (\rho_g(r))_{g \in G} := (gr)_{g \in G}$. Then

$$\mathrm{pr}_1^* M = M \otimes_{R, \rho} (R \times \cdots \times R) \cong \prod_{g \in G} M \otimes_{R, \rho_g} R,$$

and

$$\mathrm{pr}_2^* M = M \otimes_{R, \Delta} (R \times \cdots \times R) = \prod_{g \in G} M.$$

To give an isomorphism $\sigma : \mathrm{pr}_1^* M \to \mathrm{pr}_2^* M$, we must give R-module isomorphisms

$$\sigma_g : M \otimes_{R,\rho_g} R \to M$$

for all $g \in G$. We next show that giving an R-module isomorphism $\sigma_g : M \otimes_{R,\rho_g} R \to M$ is equivalent to giving an R-semi-linear isomorphism $\alpha_g : M \to M$, that is, an isomorphism such that $\alpha_g(rm) = g(r)\alpha_g(m)$ for all $r \in R$ and $m \in M$.

Let $\sigma_g : M \otimes_{R,\rho_g} R \to M$ be an R-module isomorphism. Then for all $r, a \in A$ and $m \in M$, $\sigma_g(m \otimes ra) = r\sigma_g(m \otimes a)$. Define $\alpha_g : M \to M$ by $\alpha_g(m) = \sigma_g(m \otimes 1)$. Then

$$\alpha_g(rm) = \sigma_g(rm \otimes 1) = \sigma_g(m \otimes gr) = gr\sigma_g(m \otimes 1) = gr\alpha_g(m),$$

so α_g is R-semi-linear. Conversely, given $\alpha_g : M \to M$ an R-semi-linear isomorphism, define $\sigma_g(m \otimes 1) = \alpha_g(m)$. Then

$$\sigma_g(m \otimes a) = \sigma_g(g^{-1}(a)m \otimes 1) = \alpha_g(g^{-1}(a)m) = a\alpha_g(m),$$

and so

$$\sigma_g(m \otimes ra) = ra\alpha_g(m) = r\sigma_g(m \otimes a),$$

and σ_g is R-linear.

Finally, we show that the cocycle condition is equivalent to this collection of action maps σ_g defining a group action of G on M. We must check that

$$\mathrm{pr}_{23}^*\sigma \circ \mathrm{pr}_{12}^*\sigma = \mathrm{pr}_{13}^*\sigma : \mathrm{pr}_{12}^*\mathrm{pr}_1^* M \to \mathrm{pr}_{23}^*\mathrm{pr}_2^* M.$$

First, we calculate $\mathrm{pr}_{12}^*\sigma : \mathrm{pr}_{12}^*\mathrm{pr}_1^* M \to \mathrm{pr}_{12}^*\mathrm{pr}_2^* M$. We have

$$\mathrm{pr}_1 \circ \mathrm{pr}_{12} : G \times G \times X \to X, \quad (g,h,x) \mapsto ghx,$$

$$\mathrm{pr}_2 \circ \mathrm{pr}_{12} : G \times G \times X \to X, \quad (g,h,x) \mapsto hx,$$

and thus

$$\mathrm{pr}_{12}^*\mathrm{pr}_1^* M = \prod_{(g,h)\in G\times G} M \otimes_{R,\rho_{gh}} R,$$

$$\mathrm{pr}_{12}^*\mathrm{pr}_2^* M = \prod_{(g,h)\in G\times G} M \otimes_{R,\rho_h} R,$$

and so $\mathrm{pr}_{12}^*\sigma$ is given by

$$(\mathrm{pr}_{12}^*\sigma)_{(g,h)} = \sigma_g \otimes_{R,\rho_h} R : M \otimes_{R,\rho_{gh}} R \cong (M \otimes_{R,\rho_g} R) \otimes_{R,\rho_h} R \to M \otimes_{R,\rho_h} R.$$

We have $\mathrm{pr}_1 \circ \mathrm{pr}_{23} = \mathrm{pr}_2 \circ \mathrm{pr}_{12}$, and

$$\mathrm{pr}_2 \circ \mathrm{pr}_{23} : G \times G \times X \to X, \quad (g,h,x) \mapsto x,$$

so

$$\mathrm{pr}_{23}^*\mathrm{pr}_2^* M = \prod_{(g,h)\in G\times G} M,$$

and $\mathrm{pr}_{23}^*\sigma$ is given by

$$(\mathrm{pr}_{23}^*\sigma)_{(g,h)} = \sigma_h : M \otimes_{R,\rho_h} R \to M.$$

Finally, we have $\mathrm{pr}_1 \circ \mathrm{pr}_{13} = \mathrm{pr}_1 \circ \mathrm{pr}_{12}$ and $\mathrm{pr}_2 \circ \mathrm{pr}_{13} = \mathrm{pr}_2 \circ \mathrm{pr}_{23}$, and $\mathrm{pr}_{13}^*\sigma$ is given by

$$(\mathrm{pr}_{13}^*\sigma)_{(g,h)} = \sigma_{gh} : M \otimes_{R,\rho_{gh}} R \to M.$$

Therefore, the cocycle condition is satisfied if and only if $\sigma_{gh} = \sigma_h \circ (\sigma_g \otimes_{R,\rho_h} R)$ for all $g, h \in G$, that is, the σ_g form an action on M.

□

Let $\pi : [\mathrm{Spec}\ R/G] \to \mathrm{Spec}\ R^G$ be the coarse moduli space morphism. We have the following pushforward and pullback maps of quasi-coherent sheaves,

$$\pi_* : \mathrm{Qcoh}([\mathrm{Spec}\ R/G]) \to \mathrm{Qcoh}(\mathrm{Spec}\ R^G),$$

$M \mapsto M^G$, where the R^G-module structure on M^G is induced by the inclusion of R^G into R, and

$$\pi^* : \mathrm{Qcoh}(\mathrm{Spec}\ R^G) \to \mathrm{Qcoh}([\mathrm{Spec}\ R/G]), \tag{3.1.2.1}$$

$N \mapsto N \otimes_{R^G} R$, where the R-module structure on $N \otimes_{R^G} R$ is given by

$$s \cdot (n \otimes r) = n \otimes sr$$

for $s, r \in R$ and $n \in N$, and the G-action is given by

$$g \cdot (n \otimes r) = n \otimes g \cdot r$$

for $g \in G, n \in N, r \in R$. Then if $s \in R^G$, we have $n \otimes sr = ns \otimes r$, so we recover the R^G-module structure on N, and

$$g \cdot (s \cdot (n \otimes r)) = n \otimes g \cdot (sr) = n \otimes (g \cdot s)(g \cdot r) = (g \cdot s)(g \cdot (n \otimes r))$$

for $g \in G, s, r \in R, n \in N$.

The next proposition is the local version of Lemma 3.1.13 ([40, Proposition 6.1]) extended to locally free sheaves of finite rank. The proof is essentially the same.

Proposition 3.1.3. *Let R be a ring and let G be a linearly reductive group scheme acting on R. The pullback functor*

$$\pi^* : \mathrm{Qcoh}(\mathrm{Spec}\ R^G) \to \mathrm{Qcoh}([\mathrm{Spec}\ R/G]),$$

as defined in 3.1.2.1 induces an equivalence of categories between locally free sheaves of finite rank on $\mathrm{Spec}\ R^G$ *and locally free sheaves M of finite rank on* $[\mathrm{Spec}\ R/G]$ *such that for every field k which is a quotient of R, the action of G on $M \otimes_R k$ induced by the action on M is trivial.*

Proof. We first show that for $N \in \mathrm{Qcoh}(\mathrm{Spec}\ R^G)$, the natural adjunction map $N \to \pi_*\pi^*N$ is an isomorphism. We have $\pi_*\pi^*N = (N \otimes_{R^G} R)^G$, and the map is given by $n \mapsto n \otimes 1$. Then $n \otimes r \in (N \otimes_{R^G} R)^G$ if and only if $g(n \otimes r) = n \otimes gr = n \otimes r$ if and only if $r \in R^G$. Thus, $(N \otimes_{R^G} R)^G = N \otimes_{R^G} R^G = N$.

Now, we show that given $M \in \mathrm{Qcoh}([\mathrm{Spec}\ R/G])$ with the above property, the natural adjunction map $\pi^*\pi_*M \to M$ is an isomorphism. We can prove this locally, so we may assume that R is a local ring. We have $\pi^*\pi_*M = M^G \otimes_{R^G} R$, so we show that there is a basis for M as an R-module which is invariant under the action of G, so that we can write

$$M = Re_1 \oplus \cdots \oplus Re_n,$$

and then

$$M^G \otimes_{R^G} R = (R^G e_1 \oplus \cdots \oplus R^G e_n) \otimes_{R^G} R = M.$$

Given that R is a local ring, the only quotient of R is $R \to k = R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. We get a surjection of R-modules $M \to M \otimes_R k$, and by assumption, the action of G on $M \otimes_R k$ induced by the action of G on M is trivial. Since G is linearly reductive, the map remains surjective after taking G-invariants, $M^G \twoheadrightarrow (M \otimes_R k)^G$.

We have the following commutative diagram

$$\begin{tikzcd} M \arrow[r] & M \otimes_R k \arrow[d, equal] \\ M^G \arrow[u, hook] \arrow[r, two heads] & (M \otimes_R k)^G \end{tikzcd}$$

Write $M \otimes_R k = k\bar{e_1} \oplus \cdots \oplus k\bar{e_n}$ as a k-module with $\bar{e_i} \in (M \otimes_R k)^G = M \otimes_R k$. Then by Nakayama's lemma, the $\bar{e_i}$ lift to $e_i \in M^G$ which generate M^G as an R^G-module. Viewing the e_i in M via the inclusion $M^G \hookrightarrow M$, we see that the e_i map to the $\bar{e_i}$ under $M \to M \otimes_R k$ since the diagram commutes. Because the $\bar{e_i}$ form a basis of $M \otimes_R k$, the e_i also generate M as an R-module. $\square$

The ideal sheaf of a stacky point

Let $\mathscr{X}$ be an algebraic stack and $x : \mathrm{Spec}\ k \to \mathscr{X}$ a geometric point with residual gerbe BG_x. In this subsection, we describe the ideal sheaf of $\mathscr{X}$ at BG_x. In particular, in the stacky curve case, where $\mathscr{X}$ has a coarse moduli space $\pi : \mathscr{X} \to X$ and $G_x = \mu_n$ is cyclic of order n for some n, we show that the ideal sheaf at BG_x is an nth root of the ideal sheaf of X at the scheme-theoretic image of $\pi(x)$.

Definition 3.1.4. Let $\mathscr{X}$ be an algebraic stack and let $x : \mathrm{Spec}\ k \to \mathscr{X}$ be a geometric point. Let $I_x = \ker(\mathscr{O}_{\mathscr{X}} \to x_*\mathscr{O}_{\mathrm{Spec}\ k})$, and Z_x the associated closed substack. We call I_x the *ideal sheaf of the point* x and Z_x the *scheme-theoretic image of* x.

Alternatively, we may define the ideal sheaf of x on an étale cover of $\mathscr{X}$. Let $U \to \mathscr{X}$ be an étale cover, and take U_x to be the fiber product

$$\begin{tikzcd} U_x \arrow[r, "f"] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} k \arrow[r, "x"] & \mathscr{X}. \end{tikzcd}$$

Then $U_x = \coprod_{g \in G_x} \operatorname{Spec} k$. The scheme-theoretic image of U_x in U is given by $\coprod_{g \in G_x} \operatorname{Spec} k_g$, a disjoint union of closed points in U. The ideal sheaf of this closed subscheme descends to an ideal sheaf on $\mathscr{X}$, which coincides with I_x.

Lemma 3.1.5. *Let $\mathscr{X}$ be an algebraic stack and $x : \operatorname{Spec} k \to \mathscr{X}$ a geometric point. The scheme-theoretic image of x is given by $Z_x = BG_x$.*

Proof. We have the following cartesian diagram,

$$\begin{tikzcd} U_x \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} k \arrow[r] & BG_x \arrow[r] & \mathscr{X}. \end{tikzcd}$$

The scheme-theoretic image of U_x in U must factor through $\operatorname{Spec} k$ as in the diagram above, so $\coprod_{g \in G_x} \operatorname{Spec} k_g = \operatorname{Spec} k$. Moreover, this descends to Z_x, the scheme-theoretic image of x in $\mathscr{X}$, so $Z_x = BG_x$. $\square$

Now suppose that $\mathscr{X}$ has a coarse moduli space, $\pi : \mathscr{X} \to X$. The image of x in X is the geometric point $\pi(x)$ of X. We have the following commutative diagram,

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[dr, "\pi(x)"'] & \mathscr{X} \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Definition 3.1.6. The *ideal sheaf of the point* $\pi(x)$ is given by

$$I_{\pi(x)} = \ker(\mathscr{O}_X \to \pi(x)_* \mathscr{O}_{\operatorname{Spec} k}),$$

and the associated closed subscheme, $Z_{\pi(x)}$, is the *scheme-theoretic image of* $\pi(x)$.

We have some exact sequence describing the situation. The following sequences are left-exact, by definition:

$$0 \to I_{\pi(x)} \to \mathscr{O}_X \to \pi(x)_* \mathscr{O}_{\operatorname{Spec} k}, \tag{3.1.6.1}$$

$$0 \to I_x \to \mathscr{O}_{\mathscr{X}} \to x_* \mathscr{O}_{\operatorname{Spec} k}. \tag{3.1.6.2}$$

On the étale cover $U \to \mathscr{X}$, Sequence 3.1.6.2 is given by

$$0 \to I \to \mathscr{O}_U \to f_* \mathscr{O}_{U_x}.$$

Applying π_* to Sequence 3.1.6.2 above gives us Sequence 3.1.6.1. Therefore, $\pi_* I_x = I_{\pi(x)}$. However, applying π^* to Sequence 3.1.6.1 does not give us Sequence 3.1.6.2, and $\pi^* I_{\pi(x)} \neq I_x$ in general, as shown by the following proposition.

Proposition 3.1.7. *Let $\mathscr{X}$ be a tame stacky curve with trivial generic stabilizer and let $x : \operatorname{Spec} k \to \mathscr{X}$ be a stacky point with stabilizer group μ_n. Let $\pi : \mathscr{X} \to X$ be the coarse moduli space. Then $\pi^* I_{\pi(x)} \cong I_x^{\otimes n}$.*

Proof. By Proposition 2.7.4, the stabilizer group of x is a finite cyclic group μ_n for some $n \in \mathbb{Z}$, and

$$\operatorname{Spec} \mathscr{O}_{X,x} \times_X \mathscr{X} \cong \left[\operatorname{Spec} \left(\frac{\mathscr{O}_{X,x}[z]}{z^n - a} \right) \Big/ \mu_n \right],$$

where $\mathscr{O}_{X,x}$ is the étale local ring of x, $a \in \mathscr{O}_{X,x}$ is a local coordinate at x, generating the maximal ideal of $\mathscr{O}_{X,x}$, and μ_n acts trivially on $\mathscr{O}_{X,x}$ and by multiplication on z.

Then we have the following cartesian diagram,

$$\begin{tikzcd}
U_x = \mu_n \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[r] \arrow[d] & \operatorname{Spec} \frac{k[z]}{z^n - a} \arrow[r] \arrow[d] & \operatorname{Spec} \frac{\mathscr{O}_{X,x}[z]}{z^n - a} \arrow[r] \arrow[d] & U \arrow[d, "u"] \\
\operatorname{Spec} k \arrow[r] \arrow[drr, "="'] & B\mu_n \arrow[r] & \left[\operatorname{Spec} \frac{k[z]}{z^n-a} \Big/ \mu_n\right] \arrow[r] \arrow[d] & \left[\operatorname{Spec} \frac{\mathscr{O}_{X,x}[z]}{z^n-a} \Big/ \mu_n\right] \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\pi"] \\
 & & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} \mathscr{O}_{X,x} \arrow[r] & X.
\end{tikzcd}$$

The composition of the middle horizontal arrows is the geometric point x. We will show that $u^* \pi^* I_{\pi(x)} = u^* I_x^{\otimes n}$ locally on the étale cover, that is, on the cover $\operatorname{Spec} \mathscr{O}_{X,x}[z]/(z^n - a)$. The composition of the first three horizontal arrows in the middle line, $x : \operatorname{Spec} k \to \left[\operatorname{Spec} \dfrac{\mathscr{O}_{X,x}[z]}{z^n - a} \Big/ \mu_n \right]$ corresponds to the trivial torsor $\mu_n \times \operatorname{Spec} k \to \operatorname{Spec} k$ together with the μ_n-equivariant map $f : \mu_n \times \operatorname{Spec} k \to \operatorname{Spec} \dfrac{\mathscr{O}_{X,x}[z]}{z^n - a}$ given by $z \mapsto 0$ on each component of μ_n, and which is the composition of the first three horizontal arrows in the top row of the diagram.

Then

$$u^* I_x = \ker \left(\frac{\mathscr{O}_{X,x}[z]}{z^n - a} \to k \times \cdots \times k,\ z \mapsto (0, \dots, 0) \right) = (z),$$

with action of μ_n on (z) given by multiplication by ζ_n, and

$$u^* \pi^* I_{\pi(x)} = \ker \left(\frac{\mathscr{O}_{X,x}[z]}{z^n - a} \to \frac{k[z]}{z^n - a},\ a \mapsto 0 \right) = (z^n).$$

Thus $\pi^* I_{\pi(x)} \cong I_x^{\otimes n}$.

$\square$

Example 3.1.8. Let $k = \overline{k}$, and $\mathscr{X} = [\text{Spec } k[z]/\mu_n]$ where μ_n acts trivially on k and by multiplication on z. The coarse moduli space of $\mathscr{X}$ is given by $X = \text{Spec } k[t]$, where $t = z^n$. An étale cover of $\mathscr{X}$ is given by $U = \text{Spec } k[z] \to \mathscr{X}$, the quotient map. The map $U \to X$ is the following composition

$$U = \text{Spec } k[z] \to \mathscr{X} = \left[\text{Spec } \left(\frac{k[t][z]}{z^n - t}\right) \middle/ \mu_n \right] \to X = \text{Spec } k[t]$$

and is given by $t \mapsto z^n$, the degree-n cover $\mathbb{A}^1 \to \mathbb{A}^1$ ramified at the origin.

Sequence 3.1.6.1 is given by

$$0 \to (t) \to k[t] \to k \to 0,$$

which happens to be right-exact, and Sequence 3.1.6.2 pulled back to the étale cover is given by

$$0 \to (z) \to k[z] \to k \times \cdots \times k.$$

Also pulling back Sequence 3.1.6.1 to the étale cover gives

$$0 \to (z^n) \to k[z] \to k[z]/z^n \to 0.$$

Equivalently, this sequence can be obtained by tensoring $0 \to (t) \to k[t] \to k \to 0$ with $k[z,t]/(z^n - t)$ over $k[t]$. Again, we see that $u^*\pi^* I_{\pi(x)} \cong (z^n) \cong u^* I_x^{\otimes n}$, and thus $\pi^* I_{\pi(x)} \cong I_x^{\otimes n}$.

Stacky curves with trivial generic stabilizers are root stacks

Proposition 3.1.9. *Let $\mathscr{X}$ be a tame stacky curve with trivial generic stabilizer, coarse space $\pi : \mathscr{X} \to X$, and stacky geometric points $x_1, \dots, x_r : \text{Spec } k \to \mathscr{X}$ with stabilizer groups μ_{n_i}. Let*

$$\mathscr{X}_{\mathbf{n}} = \mathscr{X}_{n_1} \times_X \cdots \times_X \mathscr{X}_{n_r}$$

where $\mathscr{X}_{n_i}$ is the n_ith root stack of X at $\pi(x_i)$. Then $\mathscr{X} \cong \mathscr{X}_{\mathbf{n}}$ over X.

Proof. The statement follows from [20, Theorem 1], but we construct the isomorphism in the case where $\mathscr{X}$ is a stacky curve. To give a map $\mathscr{X} \to \mathscr{X}_{\mathbf{n}}$, we must give a map $\mathscr{X} \to \mathscr{X}_{n_i}$ for every i. We take the map which corresponds to the data $(\pi : \mathscr{X} \to X, (I_{x_i}, \iota), \sigma)$, where $\iota : I_{x_i} \to \mathscr{O}_{\mathscr{X}}$ is the inclusion of the ideal sheaf of the stacky point x_i, and $\sigma : I_{x_i}^{\otimes n} \to \pi^* I_{\pi(x_i)}$ is the natural isomorphism constructed above in Proposition 3.1.7.

Thus, we get a map $\mathscr{X} \to \mathscr{X}_{\mathbf{n}}$ which we can check is an isomorphism locally. Over the open set of X away from the points $\pi(x_1), \dots, \pi(x_r)$, $\mathscr{X}$ and $\mathscr{X}_{\mathbf{n}}$ are both isomorphic to X, and over the étale local rings of each $\pi(x_i)$, both $\mathscr{X}$ and $\mathscr{X}_{\mathbf{n}}$ are isomorphic to $\left[\text{Spec } \frac{\mathscr{O}_{X,x_i}[z]}{z^{n_i} - a} \middle/ \mu_{n_i}\right]$, where $a \in \mathscr{O}_{X,x_i}$ generates the maximal ideal of $\mathscr{O}_{X,x_i}$, by Proposition 2.7.4 and Equation 2.5.4.1, respectively. $\square$

Remark 3.1.10. Under the isomorphism $\mathscr{X} \cong \mathscr{X}_{\mathbf{n}}$ defined in the proof of Proposition 3.1.9, the ideal sheaf I_{x_i} of x_i on $\mathscr{X}$ corresponds to the universal line bundle on $\mathscr{X}_{n_i}$ obtained by the root stack construction.

Remark 3.1.11. A stacky curve with r stacky points may be the root stack at less than r distinct points on the coarse space if multiple geometric points have the same scheme-theoretic images in X. For instance, see Example 3.5.2.

Calculation of the Picard group

In this subsection, we calculate the Picard group of a tame stacky curve with trivial generic stabilizer. The exact sequence in Theorem 3.1.12 below is also studied in [11, Corollary 3.2.1], [10, Theorem 5.4.6], and [12, Corollary 3.0.8, Theorem 3.2.3].

Let $\mathscr{Y}$ be a tame stacky curve with trivial generic stabilizer and coarse moduli space $\pi : \mathscr{Y} \to X$. As in Proposition 3.1.9 and Remark 3.1.11, write $\mathscr{Y}$ as the root stack of X of order n_i at $\pi(x_i)$ where $x_i : \operatorname{Spec} k \to \mathscr{Y}$ are the stacky geometric points of $\mathscr{Y}$ with stabilizer groups μ_{n_i} such that the scheme-theoretic images of $\pi(x_i)$, for $i = 1, \ldots, r$ in X are distinct.

For each stacky point $x_i : \operatorname{Spec} k \to \mathscr{Y}$, we have a morphism

$$\chi_i : \operatorname{Pic}(\mathscr{Y}) \to \mathbb{Z}/n_i\mathbb{Z}$$

defined as follows. Let $x_i : B\mu_{n_i} \hookrightarrow \mathscr{Y}$ also denote the inclusion of the residual gerbe at x_i into $\mathscr{Y}$. Then given a line bundle $\mathscr{L}$ on $\mathscr{Y}$, $x_i^*\mathscr{L}$ is a line bundle on $B\mu_{n_i}$. As described by Proposition 3.1.1, this line bundle comes with an action $\chi_{x_i^*\mathscr{L}} : \mu_{n_i} \to \mathbb{G}_m$ of the stabilizer group μ_{n_i}. Let χ_i be the map $\mathscr{L} \mapsto \chi_{x_i^*\mathscr{L}} \in \operatorname{Hom}(\mu_{n_i}, \mathbb{G}_m) \cong \mathbb{Z}/n_i\mathbb{Z}$.

Putting these together for each $i = 1, \ldots, r$, we get a morphism

$$\chi := (\chi_1, \ldots, \chi_r) : \operatorname{Pic}(\mathscr{Y}) \to \prod_{i=1}^{r} \mathbb{Z}/n_i\mathbb{Z}. \tag{3.1.11.1}$$

Theorem 3.1.12. *Let $\mathscr{Y}$ be a tame stacky curve with trivial generic stabilizer and coarse moduli space $\pi : \mathscr{Y} \to X$. Let $x_1, \ldots, x_r$ be the stacky points of $\mathscr{Y}$ with stabilizer groups μ_{n_i} such that $\mathscr{Y}$ is the root stack of X of order n_i at $\pi(x_i)$ for $i = 1, \ldots, r$. Then we have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}(X) \xrightarrow{\pi^*} \operatorname{Pic}(\mathscr{Y}) \xrightarrow{\chi} \textstyle\prod_{i=1}^{r} \mathbb{Z}/n_i\mathbb{Z} \longrightarrow 0,$$

where χ is described in 3.1.11.1 above. Moreover, $\operatorname{Pic}(\mathscr{Y})$ *is the pushout of the diagram*

$$\begin{tikzcd} \mathbb{Z}^r \arrow[r, "(\cdot n_1, \ldots, \cdot n_r)"] \arrow[d, "\phi"] & \mathbb{Z}^r \arrow[d] \\ \operatorname{Pic}(X) \arrow[r, "\pi^*"] & \operatorname{Pic}(\mathscr{Y}), \end{tikzcd}$$

where $\phi : \mathbb{Z}^r \to \operatorname{Pic}(X)$ is given by $e_i \mapsto I_{\pi(x_i)}$, for $e_i = (0, \ldots, 1, \ldots, 0)$ the ith basis vector, and $I_{\pi(x_i)}$ the ideal sheaf of $\mathscr{O}_X$-modules of the point $\pi(x_i) \in X$.

Proof. The following lemma shows that the sequence is exact at the left and in the middle.

Lemma 3.1.13. *[40, Proposition 6.1] Let S be a scheme, $\mathscr{X}$ a tame stack over S, with coarse moduli space $\pi : \mathscr{X} \to X$. The pullback functor*

$$\pi^* : (\textit{line bundles on } X) \to (\textit{line bundles on } \mathscr{X})$$

induces an equivalence of categories between line bundles on X and line bundles $\mathscr{L}$ on $\mathscr{X}$ such that for all geometric points $x : \mathrm{Spec}\ k \to \mathscr{X}$, the representation of G_x corresponding to $x^*\mathscr{L}$ is trivial.

Locally, where $\mathscr{X} = [\mathrm{Spec}\ R/G]$ and $X = \mathrm{Spec}\ R^G$ for R a ring and G a linearly reductive group scheme, a line bundle $\mathscr{L}$ on $\mathscr{X}$ is a rank one R-module M with R-semi-linear G-action. Then at each geometric point $x : \mathrm{Spec}\ k \to \mathscr{X}$ of $\mathscr{X}$, the representation of G corresponding to $x^*\mathscr{L}$ is just the action of G on $M \otimes_R k \cong k$.

By the lemma, we have exactness on the left, and we have that the image of π^* is contained in the kernel of χ. To show that the kernel of χ is contained in the image of π^*, we must show that for any two stacky points x_1 and x_2 of $\mathscr{Y}$ such that the scheme-theoretic images of $\pi(x_1)$ and $\pi(x_2)$ agree, a line bundle $\mathscr{L}$ on $\mathscr{Y}$ has the same representation at $x_1^*\mathscr{L}$ and $x_2^*\mathscr{L}$. Indeed, write $\pi(x)$ for the scheme-theoretic image of $\pi(x_1)$ and $\pi(x_2)$. Then locally at $\pi(x)$, $\mathscr{Y}$ is described by $\left[\mathrm{Spec}\ \frac{\mathscr{O}_{X,\pi(x)}[z]}{z^n-a}\Big/\mu_n\right]$ where $a \in \mathscr{O}_{X,\pi(x)}$ generates the maximal ideal of $\mathscr{O}_{X,\pi(x)}$, and $n = n_1 = n_2$ is the order of the stabilizer groups at x_1 and x_2. A line bundle on $\mathscr{Y}$ is locally given by an $\mathscr{O}_{X,\pi(x)}[z]/(z^n - a)$-module together with a semi-linear μ_n-action. Upon pulling back by x_1 and x_2, we get a rank one k-vector space with the same action of μ_n. Therefore, any line bundle on $\mathscr{Y}$ that has trivial stabilizer action at x_i for $i = 1, \ldots, r$ has trivial stabilizer action at every stacky point.

By Proposition 3.1.1, only the finitely many stacky points of $\mathscr{Y}$ can have interesting Picard groups at their residual gerbes $B\mu_{n_i}$. Thus, for a line bundle $\mathscr{L}$ on $\mathscr{Y}$ to come from a line bundle on X, as in Lemma 3.1.13, it is enough to require that the representations at the finitely many stacky points are trivial. In fact, by what we just showed, it is enough to require that the representation at the finitely many stacky points with distinct scheme-theoretic images in X are trivial. This implies that the cokernel of the pullback map is finite. We now show that the sequence above is exact on the right as well.

Let $\mathscr{L} = I_{x_i}$. Locally at $\pi(x_i)$, we have the local structure as in the diagram below,

$$\begin{array}{ccc} \mathscr{U}_i = \left[\mathrm{Spec}\left(\dfrac{\mathscr{O}_{X,\pi(x_i)}[z]}{z^{n_i} - a}\right)\Big/\mu_{n_i}\right] & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ \mathscr{O}_{X,\pi(x_i)} & \longrightarrow & X, \end{array}$$

and I_x is the $A := \mathscr{O}_{X,\pi(x_i)}[z]/(z^{n_i} - a)$-module $A \cdot (z)$ generated by (z) with μ_{n_i}-action multiplication by ζ_{n_i} on z.

We have $A \cdot (z) \otimes_A k = k \cdot (z)$, with μ_{n_i} still acting by multiplication on z, so $x_i^*\mathscr{L}$ corresponds to the standard representation. At the other stacky points x_j, $j \neq i$, $x_j^*\mathscr{L}$ is trivial, thus so are the corresponding representations, and so the images of $I_{x_i} \in \operatorname{Pic}(\mathscr{Y})$ for $i = 1, \ldots, r$ generate $\prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$. Therefore, the sequence is exact.

Now, let E be the pushout of the diagram

$$\begin{tikzcd} e_i \arrow[d, mapsto] & \mathbb{Z}^r \arrow[r, "(\cdot n_1, \ldots, \cdot n_r)"] \arrow[d, "\phi"] & \mathbb{Z}^r \arrow[d] \\ I_{\pi(x_i)} & \operatorname{Pic}(X) \arrow[r] & E. \end{tikzcd}$$

That is, E is given by

$$\begin{aligned} E &= \operatorname{coker}(\mathbb{Z}^r \to \mathbb{Z}^r \oplus \operatorname{Pic}(X), e_i \mapsto (n_i e_i, -I_{\pi(x_i)})) \\ &= (\mathbb{Z}^r \oplus \operatorname{Pic}(X))/\langle (n_i e_i, -I_{\pi(x_i)}) \rangle_{i=1,\ldots,r}. \end{aligned}$$

The vertical map $\mathbb{Z}^r \to E$ is given by $e_i \mapsto (e_i, \mathscr{O}_X)$ for all i, and the horizontal map $\operatorname{Pic}(X) \to E$ is given by $L \mapsto (0, L)$.

Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^r \arrow[r, "(\cdot n_1, \ldots, \cdot n_r)"] \arrow[d, "\phi"] & \mathbb{Z}^r \arrow[r] \arrow[d] & \prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Pic}(X) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \prod_{i=1}^r \mathbb{Z}/n_i Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Pic}(X) \arrow[r, "\pi^*"] & \operatorname{Pic}(\mathscr{Y}) \arrow[r, "\chi"] & \prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

where the top and bottom sequences are exact, and the map $E \to \operatorname{Pic}(\mathscr{Y})$ is the unique map coming from the universal property of pushout corresponding to the diagram

$$\begin{tikzcd} e_i \arrow[d, mapsto] & \mathbb{Z}^r \arrow[r, "(\cdot n_1, \ldots, \cdot n_r)"] \arrow[d, "\phi"] & \mathbb{Z}^r \arrow[d] & e_i \arrow[d, mapsto] \\ I_{\pi(x_i)} & \operatorname{Pic}(X) \arrow[r, "\pi^*"] & \operatorname{Pic}(\mathscr{Y}) & I_{x_i}. \end{tikzcd}$$

We now describe the map $E \to \prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$, by first describing the map $E \to \operatorname{Pic}(\mathscr{Y})$, and then composing this with the map $\chi : \operatorname{Pic}(\mathscr{Y}) \to \prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$.

For each equivalence class $((a_i)_i, L)$ in E, write $a_i = n_i b_i + c_i$, where b_i, c_i are integers and $0 \leq c_i < n_i$. Then

$$((a_i)_i, L) = ((c_i)_i, L \otimes (\otimes_{i=1}^r I_{\pi(x_i)}^{\otimes b_i})) =: ((c_i)_i, L'),$$

and write

$$((c_i)_i, L') = (0, L') + ((c_i)_i, \mathscr{O}_X).$$

Since $(0, L')$ maps to $\pi^* L' \in \mathrm{Pic}(\mathscr{Y})$, and $((c_i)_i, \mathscr{O}_X)$ maps to $\otimes_{i=1}^r I_{x_i}^{\otimes c_i} \in \mathrm{Pic}(\mathscr{Y})$, we have

$$((a_i)_i, L) \mapsto \pi^* L' \otimes (\otimes_{i=1}^r I_{x_i}^{\otimes c_i}).$$

Then the map $E \to \prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$ is given by

$$((a_i)_i, L) \mapsto (c_i)_i.$$

We can now check directly that the middle sequence is also exact, so by the 5-lemma, $E \cong \mathrm{Pic}(\mathscr{Y})$. □

3.2 Rigidification

In this section, we discuss rigidification following [1, Appendix A], in which a subgroup of the inertia can be removed from certain stacks, so that we can realize these stacks as gerbes over their *rigidifications*, the stacks obtained by the quotient procedure. Let S be a scheme or an algebraic space, and let $\mathscr{X} \to S$ be a locally finitely presented algebraic stack. For an S-scheme T and $t \in \mathscr{X}(T)$, $\underline{\mathrm{Aut}}_T(t) = T \times_{\mathscr{X}} \mathscr{I}(\mathscr{X})$ is a locally finitely presented group algebraic space. Furthermore, for a morphism $t' \to t$ over an S-morphism $T' \to T$, we have an isomorphism $\underline{\mathrm{Aut}}_{T'}(t') \cong T' \times_T \underline{\mathrm{Aut}}_T(t)$.

Let $G \subseteq \mathscr{I}(\mathscr{X})$ be a flat finitely presented subgroup stack. The pullback of G to T by t is a flat finitely presented subgroup algebraic space $G_t \subseteq \underline{\mathrm{Aut}}_T(t)$, and again $G_{t'} \cong T' \times_T G_t$ for any morphism $t' \to t$ over any S-morphism $T' \to T$.

Conversely, given a subgroup algebraic space $G_t \subseteq \underline{\mathrm{Aut}}_T(t)$ for each object $t : T \to \mathscr{X}$ which is flat and finitely presented over T, such that for each morphism $t' \to t$ over any S-morphism $T' \to T$, the isomorphism $\underline{\mathrm{Aut}}_{T'}(t') \cong T' \times_T \underline{\mathrm{Aut}}_T(t)$ carries $G_{t'}$ to $T' \times_T G_t$, we get a unique flat finitely presented subgroup stack $G \subseteq \mathscr{I}(\mathscr{X})$ such that for any $t : T \to \mathscr{X}$, the pullback of G to T by t is G_t. This condition implies that each subgroup G_t is normal in $\underline{\mathrm{Aut}}_T(t)$.

Theorem 3.2.1. *[1, Theorem A.1] Let S be a scheme or algebraic space, and let $\mathscr{X} \to S$ be a locally finitely presented algebraic stack with inertia stack $\mathscr{I}(\mathscr{X})$. Let G be a flat finitely presented subgroup stack of $\mathscr{I}(\mathscr{X})$. Then there exists a locally finitely presented algebraic stack $\mathscr{X} /\!\!/ G$ over S and a morphism $p : \mathscr{X} \to \mathscr{X} /\!\!/ G$ such that $\mathscr{X}$ is an fppf-gerbe over $\mathscr{X} /\!\!/ G$ banded by $\underline{G}$ and for all S-schemes T and $t \in \mathscr{X}(T)$, the natural morphism*

$$\underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(p(t))$$

is surjective with kernel G_t. Furthermore, if G is finite over $\mathscr{X}$, then p is proper, and if G is étale, then p is étale.

Remark 3.2.2. The band $\underline{G}$ associated to the group G on $\mathscr{X}$ descends to $\mathscr{X} /\!\!/ G$. Let $T \to \mathscr{X} /\!\!/ G$ be a covering of $\mathscr{X} /\!\!/ G$ with a lifting to $\mathscr{X}$, $t : T \to \mathscr{X}$. The band $\underline{G}$ on $\mathscr{X} /\!\!/ G$ is given by the descent data

$$(G_t, \overline{\sigma} \in \mathrm{Hex}(\mathrm{pr}_1^* G_t, \mathrm{pr}_2^* G_t))$$

where $G_t \leq \underline{\mathrm{Aut}}_T(t)$, $\mathrm{pr}_i^* G_t = G_{\mathrm{pr}_i^* t} \leq \underline{\mathrm{Aut}}_{T \times_{X /\!\!/ G} T}(\mathrm{pr}_i^* t) = \mathrm{pr}_i^* \underline{\mathrm{Aut}}_T(t)$ for $i = 1, 2$, and $\overline{\sigma}$ is the canonical isomorphism of bands $\underline{\mathrm{pr}_1^* \underline{\mathrm{Aut}}_T(t)} \to \underline{\mathrm{pr}_2^* \underline{\mathrm{Aut}}_T(t)}$ restricted to $\mathrm{pr}_1^* G_t$.

Proposition 3.2.3. *Let S be a scheme or algebraic space, $\mathscr{X} \to S$ a locally finitely presented algebraic stack with inertia stack $\mathscr{I}(\mathscr{X})$, and G a flat finitely presented subgroup stack of $\mathscr{I}(\mathscr{X})$. Suppose $\mathscr{Y}$ is an algebraic stack over S with a morphism $p : \mathscr{X} \to \mathscr{Y}$ such that $\mathscr{X}$ is an fppf-gerbe over $\mathscr{Y}$ and for all S-schemes T and $t \in \mathscr{X}(T)$, the natural morphism*

$$\underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(p(t))$$

is surjective with kernel G_t. Then $\mathscr{Y}$ is equivalent to $\mathscr{X} /\!\!/ G$, and thus p is banded by $\underline{G}$ and $\mathscr{Y}$ is locally finitely presented over S.

Proof. For any $t, t' : T \to \mathscr{X}$, we have a right action of $\underline{\mathrm{Aut}}_T(t)$, and a left action of $\underline{\mathrm{Aut}}_T(t')$, on $\underline{\mathrm{Isom}}_T(t, t')$ by composition. These actions commute, and restrict to actions of G_t and $G_{t'}$, respectively. We claim that the action of $G_{t'}$ on the quotient $\underline{\mathrm{Isom}}_T(t, t')/G_t$ is trivial. Indeed, let $(f : t \to t') \in \underline{\mathrm{Isom}}_T(t, t')$ and $(\sigma : t' \to t') \in G_{t'}$ be isomorphisms. We need to find an isomorphism $(\alpha : t \to t) \in G_t$ such that $f \cdot \alpha = \alpha \circ f$ agrees with $\sigma \cdot f = \sigma \circ f$. According to the diagram

$$\begin{tikzcd} t \arrow[r, "f"] \arrow[d, "\alpha"'] & t' \arrow[d, "\sigma"] \\ t \arrow[r, "f"] & t' \end{tikzcd},$$

we may take $\alpha = f^{-1} \sigma f$. By the property of G that any isomorphism $t \to t'$ induces an isomorphism $\underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(t')$ which maps G_t to $G_{t'}$, $\alpha \in G_t$.

As in the proof of Theorem 3.2.1 [1, Theorem A.1], we define $(\mathscr{X} /\!\!/ G)^{\mathrm{pre}}$ to be the stack with objects of $\mathscr{X}$, and for any S-scheme T, and $t, t' \in \mathscr{X}(T)$, $\mathrm{Hom}_{(\mathscr{X} /\!\!/ G)^{\mathrm{pre}}}(t, t')$ is the set of global sections of the quotient sheaf $\underline{\mathrm{Isom}}_T(t, t')/G_t$.

Let $\pi^{\mathrm{pre}} : \mathscr{X} \to (\mathscr{X} /\!\!/ G)^{\mathrm{pre}}$ be the natural morphism of prestacks which is the identity map on objects, and is the quotient map on morphisms. We define a morphism of prestacks $\phi : (\mathscr{X} /\!\!/ G)^{\mathrm{pre}} \to \mathscr{Y}$, which on objects is given by $t \mapsto p(t)$. For morphisms, we need a map

$$\mathrm{Hom}_{(\mathscr{X} /\!\!/ G)^{\mathrm{pre}}}(t, t') \to \mathrm{Hom}_{\mathscr{Y}}(p(t), p(t')).$$

From the morphism of stacks $p : \mathscr{X} \to \mathscr{Y}$, we have

$$\underline{\mathrm{Isom}}_T(t, t') \to \underline{\mathrm{Isom}}_T(p(t), p(t')).$$

We would like to show that this factors through the quotient $\underline{\mathrm{Isom}}_T(t,t')/G_t$. Since

$$0 \to G_t \to \underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(p(t)) \to 0$$

is exact, G_t acts trivially on $\underline{\mathrm{Isom}}_T(p(t),p(t'))$, and the morphism indeed factors through the quotient, and the resulting morphism

$$\underline{\mathrm{Isom}}_T(t,t')/G_t \to \underline{\mathrm{Isom}}_T(p(t),p(t'))$$

is an isomorphism since it is a morphism of $\underline{\mathrm{Aut}}_T(t)/G_t \cong \underline{\mathrm{Aut}}_T(p(t))$-torsors.

We also have a natural morphism $\mathrm{st} : (\mathscr{X} /\!\!/ G)^{\mathrm{pre}} \to \mathscr{X} /\!\!/ G$, the stackification morphism, and thus we get a unique morphism $\psi : \mathscr{X} /\!\!/ G \to \mathscr{Y}$ making the diagram of prestacks

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "p"] \arrow[d, "\pi^{\mathrm{pre}}"'] & \mathscr{Y} \\
(\mathscr{X} /\!\!/ G)^{\mathrm{pre}} \arrow[r, "\mathrm{st}"] \arrow[ur, "\phi"] & \mathscr{X} /\!\!/ G \arrow[u, "\psi"']
\end{tikzcd}$$

commute. Denote the composition $\mathscr{X} \to \mathscr{X} /\!\!/ G$ by π. We show that $\psi : \mathscr{X} /\!\!/ G \to \mathscr{Y}$ is an isomorphism of stacks.

First, we show that ψ is fully faithful, so we check that for all $z, z' : T \to \mathscr{X} /\!\!/ G$,

$$\mathrm{Hom}_{\mathscr{X} /\!\!/ G}(z,z') \cong \mathrm{Hom}_{\mathscr{Y}}(\psi(z),\psi(z')),$$

so we can check that we have an isomorphism of sheaves,

$$\underline{\mathrm{Isom}}_T(z,z') \cong \underline{\mathrm{Isom}}_T(\psi(z),\psi(z')),$$

which we can check locally. Let $T' \to T$ be a cover where z and z' come from $\mathscr{X}$, say $z = \pi(t)$, $z' = \pi(t')$. Then

$$\underline{\mathrm{Isom}}_{T'}(z,z') \cong \underline{\mathrm{Isom}}_{T'}(t,t')/G_t \cong \underline{\mathrm{Isom}}_{T'}(p(t),p(t')) \cong \underline{\mathrm{Isom}}_{T'}(\psi(z),\psi(z')),$$

since the Isom sheaf on $\mathscr{X} /\!\!/ G$ agrees with that of the prestack $(\mathscr{X} /\!\!/ G)^{\mathrm{pre}}$, which by what we have shown above, agrees with that of $\mathscr{Y}$.

For essential surjectivity, since $p : \mathscr{X} \to \mathscr{Y}$ is a gerbe, for all S-schemes T, and $y \in \mathscr{Y}(T)$, there exists a cover $\{f_i : T_i \to T\}$ such that $f_i^* y \in \mathscr{Y}(T_i)$ lifts to $\mathscr{X}(T_i)$. The image of this lift in $\mathscr{X} /\!\!/ G$ maps to $f_i^* y$ under ψ. $\square$

Proposition 3.2.4. *Let S be a scheme or algebraic space, $\mathscr{Y}$ a locally finitely presented algebraic stack over S, G a flat finitely presented band on $\mathscr{Y}$, and $p : \mathscr{X} \to \mathscr{Y}$ a gerbe banded by G. Then $\mathscr{X}$ is locally finitely presented over S and there exists a flat finitely presented sheaf of groups $G' \subseteq \mathscr{I}(\mathscr{X})$ such that $\mathscr{X} /\!\!/ G' \cong \mathscr{Y}$, and $p_* \underline{G'} \cong G$ is an isomorphism of bands on $\mathscr{Y}$.*

Proof. To check that $\mathscr{X}$ is locally finitely presented over S, we may assume that $p : \mathscr{X} \to \mathscr{Y}$ has a section $s : \mathscr{Y} \to \mathscr{X}$ and $\mathscr{X} \cong B_{\mathscr{Y}}\widetilde{G}$ is the trivial gerbe, where $\widetilde{G} \cong \underline{\mathrm{Aut}}_{\mathscr{Y}}(s)$ and $\underline{\widetilde{G}} \cong G$ by Proposition 2.6.21. Then the section $s : \mathscr{Y} \to B_{\mathscr{Y}}\widetilde{G}$ is a covering of $B_{\mathscr{Y}}\widetilde{G}$. Consider the fiber product diagram

$$\begin{tikzcd} \widetilde{G} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{Y} \arrow[r] & B_{\mathscr{Y}}\widetilde{G}. \end{tikzcd}$$

Since $G \to \mathscr{Y}$ is finitely presented, $\mathscr{Y} \to B_{\mathscr{Y}}\widetilde{G}$ is locally finitely presented, so $B_{\mathscr{Y}}\widetilde{G}$ has a locally finitely presented covering by a locally finitely presented algebraic stack, so $B_{\mathscr{Y}}\widetilde{G}$ is also a locally finitely presented algebraic stack.

As in the proof of Proposition 2.6.17, let

$$G' = \ker(\mathscr{I}(\mathscr{X}) \to \mathscr{I}(\mathscr{Y}) \times_{\mathscr{Y}} \mathscr{X}.$$

In order to use Proposition 3.2.3, we must show that G' is flat and finitely presented, and for all $t : T \to \mathscr{X}$, $\underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(p(t))$ is surjective with kernel G_t. This is exactly the statement of Proposition 2.6.17, and G' is finitely presented since G is. $\square$

Corollary 3.2.5. *Let S be a scheme or algebraic space, and let $\mathscr{X} \to S$ be a locally finitely presented algebraic stack. Let $H \subseteq G \subseteq \mathscr{I}(\mathscr{X})$ be flat finitely presented subgroup stacks of $\mathscr{I}(\mathscr{X})$. Then G/H is a flat finitely presented subgroup stack of $\mathscr{I}(\mathscr{X} /\!\!/ H)$, $(\mathscr{X} /\!\!/ H) /\!\!/ (G/H) \cong \mathscr{X} /\!\!/ G$, and we have a commutative diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\pi_H"] \arrow[dr, "\pi_G"'] & \mathscr{X} /\!\!/ H \arrow[d, "\pi_{G/H}"] \\ & \mathscr{X} /\!\!/ G. \end{tikzcd}$$

Proof. By Theorem 3.2.1, we get two locally finitely presented algebraic stacks $\mathscr{X} /\!\!/ G$ and $\mathscr{X} /\!\!/ H$ and morphisms $\pi_G : \mathscr{X} \to \mathscr{X} /\!\!/ G$ and $\pi_H : \mathscr{X} \to \mathscr{X} /\!\!/ H$ such that π_G and π_H are gerbes banded by $\underline{G}$ and $\underline{H}$ respectively, and for all S-schemes T and T-points $t \in \mathscr{X}(T)$, $\underline{\mathrm{Aut}}_T(t)/G_t \cong \underline{\mathrm{Aut}}_T(\pi_G(t))$ and $\underline{\mathrm{Aut}}_T(t)/H_t \cong \underline{\mathrm{Aut}}_T(\pi_H(t))$.

Since G_t and H_t are normal flat finitely presented subgroup algebraic spaces of $\underline{\mathrm{Aut}}_T(t)$, G_t/H_t is a flat finitely presented subgroup algebraic space of $\underline{\mathrm{Aut}}_T(t)/H_t \cong \underline{\mathrm{Aut}}_T(\pi_H(t))$. Indeed, we can check that G_t/H_t is flat and finitely presented locally, and the cokernel of a morphism of finitely presented modules is finitely presented, and any quotient of a flat module is flat.

Moreover, the construction of the G_t/H_t satisfies the condition that for any morphism $t' \to t$ over an S-morphism $T' \to T$, the isomorphism

$$\underline{\mathrm{Aut}}_{T'}(\pi_H(t')) \cong T' \times_T \underline{\mathrm{Aut}}_T(\pi_H(t))$$

carries $G_{t'}/H_{t'}$ to $T' \times_T G_t/H_t$. Thus, G/H is a flat finitely presented subgroup stack of $\mathscr{I}(\mathscr{X} \mathbin{/\mkern-6mu/} H)$. Thus, again by Theorem 3.2.1, we have a locally finitely presented algebraic stack $(\mathscr{X} \mathbin{/\mkern-6mu/} H) \mathbin{/\mkern-6mu/} (G/H)$ over S and a morphism $\pi_{G/H} : \mathscr{X} \mathbin{/\mkern-6mu/} H \to (\mathscr{X} \mathbin{/\mkern-6mu/} H) \mathbin{/\mkern-6mu/} (G/H)$ which is an fppf-gerbe banded by $\underline{G/H}$, such that for every S-scheme T and $s \in (\mathscr{X} \mathbin{/\mkern-6mu/} H)(T)$, the morphism $\underline{\mathrm{Aut}}_T(s) \to \underline{\mathrm{Aut}}_T(\pi_{G/H}(s))$ is surjective with kernel $(G/H)_s$.

Clearly, the composition

$$\pi_{G/H} \circ \pi_H : \mathscr{X} \to \mathscr{X} \mathbin{/\mkern-6mu/} H \to (\mathscr{X} \mathbin{/\mkern-6mu/} H) \mathbin{/\mkern-6mu/} (G/H)$$

is an fppf-gerbe, and the composition

$$\underline{\mathrm{Aut}}_T(t) \to \underline{\mathrm{Aut}}_T(\pi_H(t)) \to \underline{\mathrm{Aut}}_T(\pi_{G/H} \circ \pi_H(t))$$

is surjective with kernel G_t. Therefore, by Proposition 3.2.3, $(\mathscr{X} \mathbin{/\mkern-6mu/} H) \mathbin{/\mkern-6mu/} (G/H) \cong \mathscr{X} \mathbin{/\mkern-6mu/} G$. $\square$

3.3 Stacky Curves as Gerbes

The purpose of this section is to prove the following theorem, which shows that stacky curves can be realized as gerbes over stacky curves with trivial generic stabilizers.

Theorem 3.3.1. *Let $\mathscr{X}$ be a stacky curve with inertia stack $\mathscr{I}(\mathscr{X})$. Then there exists a finite flat étale subgroup stack $\mathscr{H} \leq \mathscr{I}(\mathscr{X})$ locally constant on $\mathscr{X}$ such that the rigidification of $\mathscr{X}$ with respect to $\mathscr{H}$ is a stacky curve with trivial generic stabilizer $\mathscr{Y} := \mathscr{X} \mathbin{/\mkern-6mu/} \mathscr{H}$, and the natural map $p : \mathscr{X} \to \mathscr{Y}$ is an étale gerbe banded by $\underline{\mathscr{H}}$. Moreover, if $\mathscr{X}$ is tame, then so is $\mathscr{Y}$.*

Proof. Since $\mathscr{X}$ has finite diagonal, we can write $\mathscr{I}(\mathscr{X}) = \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A})$, where $\mathscr{A}$ is a coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras on $\mathscr{X}$. Let $j : \mathscr{U} \to \mathscr{X}$ be the inclusion of a dense open substack where the inertia group $\mathscr{I}(\mathscr{U})$ is finite flat over $\mathscr{U}$, and thus $\mathscr{U}$ is a gerbe over its coarse space. This exists by [46, 06RB]. We have the following cartesian diagram

$$\begin{tikzcd} \mathscr{I}(\mathscr{U}) = \underline{\mathrm{Spec}}_{\mathscr{U}}(j^*\mathscr{A}) \arrow[r, "j'"] \arrow[d, "\phi'"'] & \mathscr{I}(\mathscr{X}) = \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}) \arrow[d, "\phi"] \\ \mathscr{U} \arrow[r, "j"] & \mathscr{X}. \end{tikzcd} \tag{3.3.1.1}$$

Since ϕ' is finite flat, $j^*\mathscr{A}$ is a locally free sheaf of $\mathscr{O}_{\mathscr{U}}$-modules of finite rank on $\mathscr{U}$. The morphism $j' : \mathscr{I}(\mathscr{U}) \to \mathscr{I}(\mathscr{X})$ is an embedding with image in $\mathscr{I}(\mathscr{X})$ given by $j'(\mathscr{I}(\mathscr{U})) = \underline{\mathrm{Spec}}_{\mathscr{X}}(j_*j^*\mathscr{A})$, and the inclusion of the image $j'(\mathscr{I}(\mathscr{U})) \hookrightarrow \mathscr{I}(\mathscr{X})$ is induced by the natural adjunction map $\mathscr{A} \to j_*j^*\mathscr{A}$.

Let $\mathscr{H} = \underline{\mathrm{Spec}}_{\mathscr{X}}(\overline{\mathscr{A}})$, where $\overline{\mathscr{A}} = \mathscr{A}/\ker(\mathscr{A} \to j_*j^*\mathscr{A})$, be the closure of $j'(\mathscr{I}(\mathscr{U}))$ in $\mathscr{I}(\mathscr{X})$. For $\mathscr{H}$ to be finite flat over $\mathscr{X}$, we must have that $\overline{\mathscr{A}}$ is a locally free sheaf of $\mathscr{O}_{\mathscr{X}}$-modules of finite rank on $\mathscr{X}$.

We have the following commutative diagram,

$$\begin{tikzcd} \mathscr{I}(\mathscr{U}) \arrow[r] \arrow[d] & j'(\mathscr{I}(\mathscr{U})) \arrow[r] \arrow[drr] & \mathscr{H} \arrow[r] \arrow[dr] & \mathscr{I}(\mathscr{X}) \arrow[d, "\phi"] \\ \mathscr{U} \arrow[rrr, "j"] & & & \mathscr{X}. \end{tikzcd}$$

In particular, we have the following commutative diagram of stacks finite over $\mathscr{X}$,

$$\begin{tikzcd} & \mathscr{H} = \underline{\mathrm{Spec}}_{\mathscr{X}}(\overline{\mathscr{A}}) \arrow[dr] & \\ j'(\mathscr{I}(\mathscr{U})) = \underline{\mathrm{Spec}}_{\mathscr{X}}(j_*j^*\mathscr{A}) \arrow[ur] \arrow[rr] & & \mathscr{I}(\mathscr{X}) = \underline{\mathrm{Spec}}_{\mathscr{X}}(\mathscr{A}), \end{tikzcd}$$

induced from the following commutative diagram of coherent $\mathscr{O}_{\mathscr{X}}$-algebras on $\mathscr{X}$,

$$\begin{tikzcd} & \overline{\mathscr{A}} = \mathscr{A}/\ker(\mathscr{A} \to j_*j^*\mathscr{A}) \arrow[dl] & \\ j_*j^*\mathscr{A} & & \mathscr{A}. \arrow[ll] \arrow[ul] \end{tikzcd}$$

We check that $\mathscr{H}$ is flat after base change to Spec $\mathscr{O}_{\mathscr{X},x}$, the étale local ring of a geometric point $x \in \mathscr{X}$. Write $\mathscr{O}_{\mathscr{X},x} = R$, a regular local ring of dimension 1. Let $\mathfrak{m}$ be the maximal ideal of R. Then locally, $\mathscr{A}$ is given by a finite R-algebra $M = \oplus_{i=1}^{n} R/\mathfrak{m}^{e_i} \oplus R^r$ for some $e_i, n, r \in \mathbb{Z}$. It also suffices to assume that the pullback of $\mathscr{U}$ to Spec R is Spec K, where K is the field of fractions of R. Then locally $j^*\mathscr{A}$ is the K-algebra $M \otimes_R K = K^r$, $j_*j^*\mathscr{A}$ is the R-algebra K^r, and $\overline{\mathscr{A}}$ is the R-algebra R^r. Clearly, R^r is a finite free R-module, so $\mathscr{H}$ is finite flat over $\mathscr{X}$.

Next, we show that $\mathscr{H}$ is locally constant on $\mathscr{X}$. To do this, we describe the local picture surrounding the construction of $\mathscr{H}$ more explicitly. As in Remark 2.3.7, we can express $\mathscr{X}$ as a stack quotient locally, and we have the following cartesian diagram with $\pi : \mathscr{X} \to X$ the coarse space morphism,

$$\begin{tikzcd} {[\mathrm{Spec}\ R/G]} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\pi"] \\ \mathrm{Spec}\ R^G \arrow[r] & X, \end{tikzcd}$$

where R is a local ring and G is a group scheme.

Base changing the inertia stack diagram 3.3.1.1 with $[\mathrm{Spec}\ R/G] \to \mathscr{X}$ yields the diagram,

$$\begin{tikzcd} \mathscr{I}_K := \mathscr{I}([\mathrm{Spec}\ K/G]) \arrow[r] \arrow[d] & \mathscr{I}_R := \mathscr{I}([\mathrm{Spec}\ R/G]) \arrow[d] \\ {[\mathrm{Spec}\ K/G]} \arrow[r, hook, "j"] & {[\mathrm{Spec}\ R/G],} \end{tikzcd} \tag{3.3.1.2}$$

where $\mathscr{I}_K$ and $\mathscr{I}_R$ are described by Proposition 2.2.2. Recall that for $\mathscr{X} = [X/G]$, $\mathscr{I}(\mathscr{X}) = [F/G]$, where

$$F = \{(x,g) \mid x = gx\} \subseteq X \times G,$$

and G acts on F by $h \cdot (x,g) = (h \cdot x, hgh^{-1})$. For $g \in G$, let $X^g = \{x \in X \mid x = gx\}$. Then we can write $F = \coprod_{g\in G} X^g$. Note that for $x \in X^g$, and $h \in G$, $h \cdot x \in X^{hgh^{-1}}$. Write $\mathscr{I}_K = [F_K/G]$ and $\mathscr{I}_R = [F_R/G]$.

Further base changing the diagram 3.3.1.2 with Spec $R \to$ [Spec R/G] gives

$$\begin{tikzcd} F_K \arrow[r] \arrow[d] & F_R \arrow[d] \\ \operatorname{Spec} K \arrow[r, hook, "j"] & \operatorname{Spec} R. \end{tikzcd}$$

Note that for all $g \in G$, $(\operatorname{Spec} K)^g = \operatorname{Spec} K$ if and only if $(\operatorname{Spec} R)^g = \operatorname{Spec} R$, otherwise $(\operatorname{Spec} K)^g = \emptyset$. Indeed, $(\operatorname{Spec} R)^g = \operatorname{Spec} R$ implies $(\operatorname{Spec} K)^g = \operatorname{Spec} K$, and if $(\operatorname{Spec} K)^g = \operatorname{Spec} K$, then we have $\operatorname{Spec} K \subseteq (\operatorname{Spec} R)^g \subseteq \operatorname{Spec} R$. But because $(\operatorname{Spec} R)^g$ is the fiber product of the diagram

$$\begin{tikzcd} (\operatorname{Spec} R)^g \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "(\mathrm{id},g)"] \\ \operatorname{Spec} R \arrow[r, "\Delta"] & \operatorname{Spec} R \times \operatorname{Spec} R, \end{tikzcd}$$

and $\mathscr{X}$ is separated, so the diagonal is closed, we have $(\operatorname{Spec} R)^g$ closed in Spec R, so $(\operatorname{Spec} R)^g = \operatorname{Spec} R$.

Let $H = \{g \in G : (\operatorname{Spec} R)^g = \operatorname{Spec} R\}$, a subgroup scheme of G. We see that

$$H = \{g \in G : g = \mathrm{id}_R \in \operatorname{Aut}(R)\} = \ker(G \to \operatorname{Aut}(R)),$$

so H is normal in G. We have that

$$F_K = \coprod_{g\in G} (\operatorname{Spec} K)^g = \coprod_{h\in H} \operatorname{Spec} K \cong H,$$

and the closure of the image of F_K in $F_R \subseteq \operatorname{Spec} R \times G$ is $\operatorname{Spec} R \times H$. Thus, $\mathscr{H}$ is locally constant. Note also that

$$\mathscr{I}_K = \left[\coprod_{h\in H} \operatorname{Spec} K \middle/ G \right] = [H/G],$$

and the closure $\mathscr{H}$ of the image of $\mathscr{I}_K$ in $\mathscr{I}_R$ is

$$\mathscr{H} = \left[\coprod_{h\in H} \operatorname{Spec} R \middle/ G \right] = [(\operatorname{Spec} R \times H)/G],$$

where G acts on H by conjugation in both quotients, so if G is abelian, $\mathscr{H}$ is constant on $[\mathrm{Spec}\ R/G]$. Moreover, since $\mathscr{X}$ is Deligne–Mumford, G is étale, so H, and thus also $\mathscr{H}$, is étale.

Now, we can apply Theorem 3.2.1 to get a locally finitely presented algebraic stack $\mathscr{Y} := \mathscr{X} /\!\!/ \mathscr{H}$ and a morphism $\pi : \mathscr{X} \to \mathscr{Y}$ which is an étale-gerbe banded by $\underline{\mathscr{H}}$. Since $\mathscr{H}$ is finite, π is proper. Since $\mathscr{X}$ is smooth, proper, and geometrically connected, so is $\mathscr{Y}$. An étale cover $U \to \mathscr{X}$ gives us an étale cover of $\mathscr{Y}$ after composing with π, thus $\mathscr{Y}$ is also Deligne–Mumford. Moreover, if $\mathscr{X}$ is tame, then so is $\mathscr{Y}$, since the stabilizer groups of $\mathscr{Y}$ are quotients of stabilizer groups of $\mathscr{X}$. We show that $\mathscr{Y}$ is a stacky curve with trivial generic stabilizer.

We must find a dense open algebraic space in $\mathscr{Y}$. Using the same inclusion $j : \mathscr{U} \hookrightarrow \mathscr{X}$, we get an induced commutative diagram

$$\begin{tikzcd} \mathscr{U} \arrow[r, "j"] \arrow[d] & \mathscr{X} \arrow[d] \\ \mathscr{U} /\!\!/ j^*\mathscr{H} \arrow[r] & \mathscr{X} /\!\!/ \mathscr{H}, \end{tikzcd}$$

and we will show that the diagram above is cartesian. Let $\mathscr{U} \to \mathscr{X} \times_{\mathscr{X} /\!\!/ \mathscr{H}} (\mathscr{U} /\!\!/ j^*H)$ be the induced map to the fiber product. Since $\mathscr{X} \to \mathscr{X} /\!\!/ \mathscr{H}$ is an $\underline{\mathscr{H}}$-gerbe, so is the pullback to $\mathscr{U} /\!\!/ j^*\mathscr{H}$, and so is $\mathscr{U} \to \mathscr{U} /\!\!/ j^*\mathscr{H}$. Thus the induced map is a morphism of $\underline{\mathscr{H}}$-gerbes, and thus is an isomorphism, so the diagram above is cartesian. By descent, we have an equivalence of categories between opens of $\mathscr{X} /\!\!/ \mathscr{H}$ and opens of $\mathscr{X}$. The open substack $j : \mathscr{U} \hookrightarrow \mathscr{X}$ descends to an open of $\mathscr{X} /\!\!/ \mathscr{H}$, and the cartesian diagram above shows that it descends to $\mathscr{U} /\!\!/ j^*\mathscr{H}$.

Since $\mathscr{X} \to \mathscr{X} /\!\!/ \mathscr{H}$ is a gerbe, we have $|\mathscr{X}| = |\mathscr{X} /\!\!/ \mathscr{H}|$, and similarly $|\mathscr{U}| = |\mathscr{U} /\!\!/ j^*\mathscr{H}|$. Then since $\mathscr{U}$ is dense in $\mathscr{X}$, $\mathscr{U} /\!\!/ j^*\mathscr{H}$ is dense in $\mathscr{X} /\!\!/ \mathscr{H}$.

Note also that $j^*\overline{\mathscr{A}} \cong j^*\mathscr{A}$ since $j^*\mathscr{A} \to j^*j_*j^*\mathscr{A} \cong j^*\mathscr{A}$ factors as $j^*\mathscr{A} \to j^*\overline{\mathscr{A}} \to j^*j_*j^*\mathscr{A}$. Thus, $j^*\mathscr{H} \cong \mathscr{I}(\mathscr{U})$, and we have $\mathscr{I}(\mathscr{U} /\!\!/ j^*\mathscr{H}) \cong \mathscr{U} /\!\!/ j^*\mathscr{H}$, so $\mathscr{X} /\!\!/ \mathscr{H}$ has trivial generic stabilizer. □

3.4 Picard Groups of Gerbes

In this section, we construct an exact sequence that describes the Picard groups of stacky curves which are allowed to have a generic stabilizer. In fact, the exact sequence holds for more general gerbes. Let S be a locally noetherian scheme, and let $\mathscr{Y}$ be a locally finitely presented algebraic stack over S. Let $\mathscr{H}$ be a tame locally constant finitely presented band on $\mathscr{Y}$. Let $\pi : \mathscr{X} \to \mathscr{Y}$ be a gerbe banded by $\mathscr{H}$, and let $[\mathscr{X}] \in H^2(\mathscr{Y}, \mathscr{H})$ be the associated cohomology class.

We define a map $\chi : \mathrm{Pic}(\mathscr{X}) \to \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m)$ as follows. Let L be a line bundle on $\mathscr{X}$. For each S-scheme T and T-point $T \to \mathscr{Y}$, we must define a morphism $\mathscr{H}(T) \to \mathbb{G}_m(T)$. We define each such morphism locally, on a cover $T' \to T$ such that there is a lift $t : T' \to \mathscr{X}$.

Pulling back along t, we get a line bundle t^*L on T', which comes equipped with the inertial action of $\mathscr{H}(T') \subseteq \underline{\mathrm{Aut}}_{T'}(t)$. That is, we get an isomorphism $\chi_L(a) : t^*L \to t^*L$ for each $a \in \underline{\mathrm{Aut}}_{T'}(t)$, which defines a character $\chi_L : \underline{\mathrm{Aut}}_{T'}(t) \to \mathrm{GL}(t^*L))$ that restricts to $\mathscr{H}(T')$. As described in Section 2.6, these morphisms $\mathscr{H}(T') \to \mathbb{G}_m(T')$ glue together only as morphisms of bands, and we get a morphism of bands $\mathscr{H} \to \mathbb{G}_m$.

For two line bundles L and M on $\mathscr{X}$, the character corresponding to $L \otimes M$ is

$$\chi_{L\otimes M}(a) : t^*L \otimes t^*M \to t^*L \otimes t^*M$$

by $l \otimes m \mapsto \chi_L(a)(l) \otimes \chi_M(a)(m) = \chi_L \cdot \chi_M(a)(l \otimes m)$. Thus, we have a homomorphism $\chi : \mathrm{Pic}(\mathscr{X}) \to \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m)$.

Next, we define a map $(-)_*[\mathscr{X}] : \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m) \to H^2(\mathscr{Y}, \mathbb{G}_m)$ by

$$(\epsilon : \mathscr{H} \to \mathbb{G}_m) \mapsto \epsilon_*([\mathscr{X}]),$$

as described in Section 2.6, with ϵ_* given by Giraud's correspondence relation $\epsilon^{(2)}$ [21, Definition IV.3.1.4].

This gives us a sequence

$$0 \longrightarrow \mathrm{Pic}(\mathscr{Y}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{X}) \xrightarrow{\chi} \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{X}]} H^2(\mathscr{Y}, \mathbb{G}_m), \qquad (3.4.0.1)$$

which we will show is exact in Theorem 3.4.3.

Proposition 3.4.1. *Let S be a locally noetherian scheme, $\mathscr{Y}$ a locally finitely presented stack over S, and $\mathscr{H}$ be a tame locally constant band on $\mathscr{Y}$. Let $\pi : \mathscr{X} \to \mathscr{Y}$ be a gerbe banded by $\mathscr{H}$. The pullback functor*

$$\pi^* : (\text{line bundles on } \mathscr{Y}) \to (\text{line bundles on } \mathscr{X})$$

induces an equivalence of categories between line bundles on $\mathscr{Y}$ and line bundles $\mathscr{L}$ on $\mathscr{X}$ such that for all geometric points $x : \mathrm{Spec}\ k \to \mathscr{X}$, the representation of $\mathscr{H}_x \subseteq \underline{\mathrm{Aut}}_k(x)$ on $x^*\mathscr{L}$ is trivial.

Proof. Using ideas from the proofs of Lemma 3.1.13 [40, Proposition 6.1] and Proposition 3.1.3, we will check that for a line bundle L on $\mathscr{Y}$, the natural adjunction map $L \to \pi_*\pi^*L$ is an isomorphism, and for a line bundle $\mathscr{L}$ on $\mathscr{X}$ as above, the natural adjunction $\pi^*\pi_*\mathscr{L} \to \mathscr{L}$ is an isomorphism. We can check this locally, so we can assume that $\mathscr{X} = B_{\mathscr{Y}}\mathscr{H}$, $\mathscr{Y} = \mathrm{Spec}\ R$ for some local ring R, and $\mathscr{H} = H$ is a constant linearly reductive group.

A line bundle on $\mathscr{Y} = \mathrm{Spec}\ R$ is then a 1-dimensional R-module M and a line bundle on $\mathscr{X} = B_RH = [\mathrm{Spec}\ R/H]$ is a 1-dimensional R-module M with semi-linear H-action, with H acting trivially on R. The pullback map is given by $M \mapsto M$, where the output M is equipped with the trivial H-action, and the pushforward map is given by $M \mapsto M$, where the H-action is forgotten on the output. It is then straightforward that $\pi_*\pi^*L \cong L$.

Let $\mathfrak{m}_R$ be the maximal ideal of R and write $\mathscr{L}$ as a 1-dimensional R-module $M = R \cdot e$, with H-action $h{\cdot}m$. By the assumption of $\mathscr{L}$, the action satisfies $h{\cdot}\overline{m} = \overline{m}$, for $\overline{m} \in M/\mathfrak{m}_R M$. Since H is linearly reductive, the surjection $M \to M/\mathfrak{m}_R M$ remains surjective after taking invariants under H, so we have a surjection

$$M^H \twoheadrightarrow (M/\mathfrak{m}_R M)^H \cong M/\mathfrak{m}_R M.$$

Thus, we can lift the basis element $\overline{e}$ of $M/\mathfrak{m}_R M$ to M^H so that we can write M with basis element e invariant under H. Then, any $m \in M$ can be written $r \cdot e$ for some $r \in R$, so that the action is given by $h \cdot (r \cdot e) = (h \cdot r)(h \cdot e) = r \cdot e$, since H acts trivially on R and on e. Therefore $\pi^*\pi_*\mathscr{L} \cong \mathscr{L}$, and the result follows. □

Next, we describe the map

$$(-)_*[\mathscr{X}] : \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m) \to H^2(\mathscr{Y}, \mathbb{G}_m)$$

more explicitly. Locally, $\mathscr{H}$ is given by a constant sheaf of groups, and the sub-band $\mathscr{H}^{\mathrm{der}}$ is given locally by the corresponding derived subgroup. Since $\mathscr{H}$ is a band, these glue together only up to conjugation, and we get a band $\mathscr{H}^{\mathrm{der}}$. The band $\mathscr{H}^{\mathrm{ab}}$ is constructed similarly, but is locally given by the abelianizations of the groups. Since all of these groups are abelian, they glue together canonically, forming a sheaf of groups $\mathscr{H}^{\mathrm{ab}}$.

Let $\epsilon : \mathscr{H} \to \mathbb{G}_m$ be a morphism of bands, and let $\phi : \mathscr{H} \to \mathscr{H}^{\mathrm{ab}}$ be the morphism of bands corresponding the abelianization morphism of groups. There exists a unique morphism of bands $\epsilon^{\mathrm{ab}} : \mathscr{H}^{\mathrm{ab}} \to \mathbb{G}_m$ such that $\epsilon = \epsilon^{\mathrm{ab}} \circ \phi$. Recall that since $\mathscr{H}^{\mathrm{ab}}$ and $\mathbb{G}_m$ are both sheaves of abelian groups, ϵ^{ab} corresponds to a unique morphism of groups. Then

$$\epsilon_* : H^2(\mathscr{Y}, \mathscr{H}) \to H^2(\mathscr{Y}, \mathbb{G}_m)$$

is the composition of

$$H^2(\mathscr{Y}, \mathscr{H}) \xrightarrow{\phi_*} H^2(\mathscr{Y}, \mathscr{H}^{\mathrm{ab}}) \xrightarrow{\epsilon_*^{\mathrm{ab}}} H^2(\mathscr{Y}, \mathbb{G}_m),$$

and ϵ_*^{ab} is given by the usual cohomology pushforward map for abelian groups, but ϕ_* is only defined by the correspondence relation $\phi^{(2)}$. Lemma 3.4.2 below explicitly describes this map ϕ_*.

Lemma 3.4.2. *Let S be a scheme or algebraic space, and let $\mathscr{Y}$ be a locally finitely presented algebraic stack over S. Let $\mathscr{H}$ be a locally constant finitely presented band on $\mathscr{Y}$. Let $\phi : \mathscr{H} \to \mathscr{H}^{\mathrm{ab}}$ be the abelianization morphism. Let $G \cong \mathscr{I}(\mathscr{X}/\mathscr{Y}) \subseteq \mathscr{I}(\mathscr{X})$ be the group as in Proposition 3.2.4 such that $\mathscr{X} /\!\!/ G \cong \mathscr{Y}$. Then the correspondence*

$$\phi_* : H^2(\mathscr{Y}, \mathscr{H}) \to H^2(\mathscr{Y}, \mathscr{H}^{\mathrm{ab}})$$

is given by $[\mathscr{X}] \mapsto [\mathscr{X} /\!\!/ G^{\mathrm{der}}]$.

Proof. First, we show that $\mathscr{H}$ and $\mathscr{H}^{\mathrm{der}}$ are flat and finitely presented. Since these properties may be checked locally and $\mathscr{H}$ is locally constant, we may assume that $\mathscr{H}$, and thus also $\mathscr{H}^{\mathrm{der}}$, is a finite constant group scheme, and both $\mathscr{H}$ and $\mathscr{H}^{\mathrm{der}}$ are flat and finitely presented.

Let $p : \mathscr{X} \to \mathscr{Y}$ be a gerbe banded by $\mathscr{H}$. Then Proposition 3.2.4 implies that $\mathscr{X}$ is locally finitely presented, and there exists a flat finitely presented subgroup stack $G \subseteq \mathscr{I}(\mathscr{X})$ such that $\mathscr{X} /\!\!/ G \cong \mathscr{Y}$ and $p_*\underline{G} \cong \mathscr{H}$. Consider $G^{\mathrm{der}} \subseteq G \subseteq \mathscr{I}(\mathscr{X})$. Clearly $p_*\underline{G^{\mathrm{der}}} \cong \mathscr{H}^{\mathrm{der}}$, so G^{der} is also flat finitely presented.

Then Corollary 3.2.5 implies that $G/G^{\mathrm{der}} \cong G^{\mathrm{ab}}$ is a flat finitely presented subgroup stack of $\mathscr{I}(\mathscr{X} /\!\!/ G^{\mathrm{der}})$ and we have the following commutative diagram,

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\pi_1"] \arrow[dr, "p"'] & \mathscr{X} /\!\!/ G^{\mathrm{der}} \arrow[d, "\pi_2"] \\ & \mathscr{Y}, \end{tikzcd} \tag{3.4.2.1}$$

such that $\pi_2 : \mathscr{X} /\!\!/ G^{\mathrm{der}} \to \mathscr{Y}$ is an $\mathscr{H}^{\mathrm{ab}}$-gerbe, and $\pi_1 : \mathscr{X} \to \mathscr{X} /\!\!/ G^{\mathrm{der}}$ is a $\pi_2^*\mathscr{H}^{\mathrm{der}}$-gerbe, and a ϕ-morphism. $\square$

Theorem 3.4.3. *Let S be a locally noetherian scheme, let $\mathscr{Y}$ be a locally finitely presented algebraic stack over S, and let $\mathscr{H}$ be a tame locally constant finitely presented band on $\mathscr{Y}$. Let $p : \mathscr{X} \to \mathscr{Y}$ be a gerbe banded by $\mathscr{H}$, with cohomology class $[\mathscr{X}] \in H^2(\mathscr{Y}, \mathscr{H})$. We have an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(\mathscr{Y}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{X}) \xrightarrow{\chi} \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{X}]} H^2(\mathscr{Y}, \mathbb{G}_m),$$

where χ describes the representation of $\mathscr{H}$ acting on the line bundle, and $(-)_*[\mathscr{X}]$ sends $(\epsilon : \mathscr{H} \to \mathbb{G}_m)$ to the image of $[\mathscr{X}]$ under the cohomology pushforward map

$$\epsilon_* : H^2(\mathscr{Y}, \mathscr{H}) \to H^2(\mathscr{Y}, \mathbb{G}_m).$$

Proof. We first reduce to the case when $\mathscr{H}$ is abelian. Using notation from Lemma 3.4.2 and its proof, consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(\mathscr{Y}) \arrow[r, "\pi_2^*"] \arrow[d, equal] & \mathrm{Pic}(\mathscr{X} /\!\!/ G^{\mathrm{der}}) \arrow[r, "\chi"] \arrow[d, "\pi_1^*"] & \mathrm{Hom}(\mathscr{H}^{\mathrm{ab}}, \mathbb{G}_m) \arrow[r, "(-)_*{[\mathscr{X} /\!\!/ G^{\mathrm{der}}]}"] \arrow[d, "(-)\circ\phi"] & H^2(\mathscr{Y}, \mathbb{G}_m) \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Pic}(\mathscr{Y}) \arrow[r, "p^*"] & \mathrm{Pic}(\mathscr{X}) \arrow[r, "\chi"] & \mathrm{Hom}(\mathscr{H}, \mathbb{G}_m) \arrow[r, "(-)_*{[\mathscr{X}]}"] & H^2(\mathscr{Y}, \mathbb{G}_m). \end{tikzcd} \tag{3.4.3.1}$$

The first square commutes by the diagram 3.4.2.1 above, and the third squares commutes by the discussion describing $(-)_*[\mathscr{X}]$. We show that the second square also commutes. Let L be a line bundle on $\mathscr{X} /\!\!/ G^{\mathrm{der}}$. To define $\chi_{\pi_1^* L}$, we have, for every $t : T \to \mathscr{X}$ and automorphism

$a \in \underline{\mathrm{Aut}}_T(t)$, an automorphism $\chi_{\pi_1^*L}(a) : t^*\pi_1^*L \cong t^*\pi_1^*L$, defining $\underline{\mathrm{Aut}}_T(t) \to \mathbb{G}_m$ locally on T, which restricts to $\chi_{\pi_1^*L} : \mathscr{H} \to \mathbb{G}_m$.

To define χ_L, we have, for every $s : T \to \mathscr{X} /\!\!/ G^{\mathrm{der}}$ and automorphism $a \in \underline{\mathrm{Aut}}_T(s)$, an automorphism $\chi_L(a) : s^*L \cong s^*L$, defining $\underline{\mathrm{Aut}}_T(s) \to \mathbb{G}_m$ locally, which restricts to $\chi_L : \mathscr{H}^{\mathrm{ab}} \to \mathbb{G}_m$. Since $\pi_1 : \mathscr{X} \to \mathscr{X} /\!\!/ G^{\mathrm{der}}$ is a gerbe, there exists a covering $f : T' \to T$ such that f^*s lifts to $\mathscr{X}$, and call the lift t'. We have $f^*s = \pi_1(t')$, as in the diagram below,

$$\begin{tikzcd} & & \mathscr{X} \arrow[d, "\pi_1"] \\ T' \arrow[r, "f"'] \arrow[urr, "t'"] & T \arrow[r, "s"'] & \mathscr{X} /\!\!/ G^{\mathrm{der}}. \end{tikzcd}$$

Precomposing the character χ_L with $\phi : \mathscr{H} \to \mathscr{H}^{\mathrm{ab}}$ corresponds to precomposing with $\underline{\mathrm{Aut}}_{T'}(t') \to \underline{\mathrm{Aut}}_{T'}(f^*s) = \underline{\mathrm{Aut}}_{T'}(\pi_1 \circ t')$, letting automorphisms $a \in \underline{\mathrm{Aut}}_{T'}(t')$ in $\mathscr{I}(\mathscr{X}/\mathscr{Y})$ act on $t'^*\pi_1^*L$. Therefore $\chi_L \circ \phi$ is $\chi_{\pi_1^*L}$, and the second square in diagram 3.4.3.1 commutes.

Next consider the map $\pi_1^* : \mathrm{Pic}(\mathscr{X} /\!\!/ G^{\mathrm{der}}) \to \mathrm{Pic}(\mathscr{X})$. For any $\mathscr{L} \in \mathrm{Pic}(\mathscr{X})$ and geometric point $x : \mathrm{Spec}\ k \to \mathscr{X}$, we have an action of $\underline{\mathrm{Aut}}_k(x)$ on $x^*\mathscr{L}$ inducing the character $\chi_{x^*\mathscr{L}} : \mathscr{H}_x^{\mathrm{der}} \subseteq \underline{\mathrm{Aut}}_k(x) \to \mathbb{G}_m$ which must be the zero map since $\mathbb{G}_m$ is abelian. Thus, $\mathscr{L}$ satisfies the condition that for all geometric points $x : \mathrm{Spec}\ k \to \mathscr{X}$, the representation of $\mathscr{H}_x^{\mathrm{der}}$ on $x^*\mathscr{L}$ is trivial, so by Proposition 3.4.1, π_1^* is an equivalence $\mathrm{Pic}(\mathscr{X} /\!\!/ G^{\mathrm{der}}) \cong \mathrm{Pic}(\mathscr{X})$.

Now every vertical arrow in 3.4.3.1 is an isomorphism, so we have reduced to the case where $\mathscr{H}$ is a sheaf of abelian groups on $\mathscr{Y}$.

Proposition 3.4.1 also implies that sequence 3.4.0.1 is exact on the left and in the middle. For exactness at $\mathrm{Hom}(\mathscr{H}, \mathbb{G}_m)$, we need to show that the image of χ consists of characters $\epsilon : \mathscr{H} \to \mathbb{G}_m$ such that $\epsilon_*[\mathscr{X}] = [B_{\mathscr{Y}}\mathbb{G}_m]$.

Let $\mathscr{L}$ be a line bundle in $\mathrm{Pic}(\mathscr{X})$ with character $\chi_{\mathscr{L}}$. To have $\chi_{\mathscr{L}*}[\mathscr{X}] = [B_{\mathscr{Y}}\mathbb{G}_m]$ means that there exists a morphism of stacks $\mathscr{X} \to B_{\mathscr{Y}}\mathbb{G}_m$, which is a $\chi_{\mathscr{L}}$-morphism of gerbes, as in Definition 2.6.11. That is, the morphism of stabilizer groups of $\mathscr{X} \to B_{\mathscr{Y}}\mathbb{G}_m$ over $\mathscr{Y}$ is given by $\chi_{\mathscr{L}} : \mathscr{H} \to \mathbb{G}_m$. Let $\mathscr{X} \to B_{\mathscr{Y}}\mathbb{G}_m$ be the map corresponding to the line bundle $\mathscr{L}$ on $\mathscr{X}$. We check that its morphism on stabilizers is indeed given by $\chi_{\mathscr{L}}$. On objects, this is the map that sends $t : T \to \mathscr{X}$ to $t^*\mathscr{L} : T \to B_{\mathscr{Y}}\mathbb{G}_m$ and on morphisms, $\alpha \in \underline{\mathrm{Aut}}_T(t)$ maps to $\chi_{\mathscr{L}}(\alpha) \in \underline{\mathrm{Aut}}_T(t^*\mathscr{L})$.

Conversely, suppose that $\epsilon : \mathscr{H} \to \mathbb{G}_m$ satisfies $\epsilon_*[\mathscr{X}] = [B_{\mathscr{Y}}\mathbb{G}_m]$. Then there is an ϵ-morphism of gerbes $\mathscr{X} \to B_{\mathscr{Y}}\mathbb{G}_m$. This morphism corresponds to a line bundle $\mathscr{L}$ on $\mathscr{X}$, and the morphism of stabilizer groups is given by $\chi_{\mathscr{L}}$. Thus $\epsilon = \chi_{\mathscr{L}}$ for some line bundle $\mathscr{L}$ on $\mathscr{X}$. □

We conclude this section with a corollary that combines the calculations of Picard groups of stacky curves which are root stacks over their coarse spaces in Theorem 3.1.12, and stacky curves which are gerbes in Theorem 3.4.3. These cases can be put together due to the geometric result in Theorem 3.3.1 that every stacky curve can be written as a gerbe over a stacky curve with trivial generic stabilizer.

Corollary 3.4.4. *Let $\mathscr{X}$ be a tame stacky curve with coarse moduli space $\pi : \mathscr{X} \to X$ and inertia stack $\mathscr{I}(\mathscr{X})$. Let $\mathscr{H} \le \mathscr{I}(\mathscr{X})$ be as in Theorem 3.3.1, so that $\mathscr{Y} := \mathscr{X} \,/\!\!/\, \mathscr{H}$ is a tame stacky curve with trivial generic stabilizer and $p : \mathscr{X} \to \mathscr{Y}$ is a gerbe banded by $\underline{\mathscr{H}}$. Let $[\mathscr{X}] \in H^2(\mathscr{Y}, \underline{\mathscr{H}})$ be the gerbe class of $\mathscr{X}$. We have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}(\mathscr{Y}) \xrightarrow{p^*} \operatorname{Pic}(\mathscr{X}) \xrightarrow{\chi} \operatorname{Hom}(\underline{\mathscr{H}}, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{X}]} H^2(\mathscr{Y}, \mathbb{G}_m) \ ,$$

where χ and and $(-)_*[\mathscr{X}]$ are as in Theorem 3.4.3, and $\operatorname{Pic}(\mathscr{Y})$ *is described as in Theorem 3.1.12.*

Proof. We can apply Theorem 3.4.3 to $p : \mathscr{X} \to \mathscr{Y}$ to get the desired exact sequence. Moreover, since $\mathscr{X}$ is tame, $\mathscr{Y}$ is also tame, and we can apply Theorem 3.1.12 to realize $\operatorname{Pic}(\mathscr{Y})$ as an extension of two explicit groups. □

3.5 Examples

We conclude this chapter with some examples of stacky curves for which we can compute their Picard groups.

Example 3.5.1. Consider the moduli stack of elliptic curves, $\mathscr{M}_{1,1}$, over a field k of characteristic not equal to 2 or 3. Objects of $\mathscr{M}_{1,1}$ are pairs $(T, (f : E \to T, e : T \to E))$, where T is a k-scheme, and (E, e) is an elliptic curve over T, that is, f is a smooth proper morphism with a section e whose geometric fibers are smooth proper genus 1 curves with a point.

A morphism in $\mathscr{M}_{1,1}$, $(s, g) : (T', (E', e')) \to (T, (E, e))$ is a pair (s, g) where $s : T' \to T$ is a k-morphism, and $g : E' \to E$ such that the diagram

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ T' \arrow[r, "s"] & T \end{tikzcd}$$

is cartesian and $g \circ e' = e \circ s$.

The compactification $\overline{\mathscr{M}_{1,1}}$ of $\mathscr{M}_{1,1}$ can be described as the weighted projective stack $\mathscr{P}(4, 6)$, and $\mathscr{M}_{1,1}$ is open inside of it. That is,

$$\overline{\mathscr{M}_{1,1}} \cong \mathscr{P}(4,6) = [\operatorname{Spec} k[a,b] \setminus \{(0,0)\}/_{4,6}\mathbb{G}_m],$$

where $\lambda \in \mathbb{G}_m$ acts by $\lambda \cdot (a, b) = (\lambda^4 a, \lambda^6 b)$. The point $[a : b] \in \mathscr{P}(4, 6)$ corresponds to the elliptic curve $y^2 = x^3 + ax + b$ in $\overline{\mathscr{M}_{1,1}}$. Elliptic curves are classified by their j-invariants. An elliptic curve $E : y^2 = x^3 + ax + b$ has j-invariant $j = 1728\dfrac{4a^3}{4a^3 + 27b^2}$. The open substack $\mathscr{M}_{1,1} \subseteq \overline{\mathscr{M}_{1,1}}$ is determined by $\Delta = 4a^3 + 27b^2 \neq 0$. Note that the weighted action of $\mathbb{G}_m$

on Spec $k[a,b] \setminus \{(0,0)\}$ does not affect the j-invariant, that is, $E : y^2 = x^3 + ax + b$ is isomorphic to $E' : y^2 = x^3 + \lambda^4 ax + \lambda^6 b$ because

$$j(E) = 1728\frac{4a^3}{4a^3+27b^2} = 1728\frac{4(\lambda^4 a)^3}{4(\lambda^4 a)^3 + 27(\lambda^6 b)^2} = j(E').$$

The two special points of $\mathscr{M}_{1,1}$ with larger stabilizer groups are $[0:1]$ and $[1:0]$. The point $[0:1]$ has stabilizer group μ_6, and corresponds to the elliptic curve $y^2 = x^3+1$ with j-invariant $j=0$. The other point $[1:0]$ has stabilizer group μ_4 and corresponds to the elliptic curve $y^2 = x^3 + x$ with j-invariant $j = 1728$.

We now describe the map to the coarse moduli space of $\mathscr{M}_{1,1}$. First, on $\overline{\mathscr{M}_{1,1}}$, we have a map

$$\overline{\mathscr{M}_{1,1}} = [\text{Spec } k[x,y] \setminus \{(0,0)\}/\mathbb{G}_m] \to \mathbb{P}^1 = [\text{Spec } k[s,t] \setminus \{(0,0)\}/\mathbb{G}_m]$$

given by $[a:b] \mapsto [a^3 : b^2]$, induced by the ring map

$$\text{Spec } k[s,t] \to \text{Spec } k[a,b],$$

where $s \mapsto a^3$ and $t \mapsto b^2$. To get the standard morphism to the coarse space $\mathscr{M}_{1,1} \to \mathbb{A}^1_j$ that sends an elliptic curve to its j-invariant, we compose this with the map $\mathbb{P}^1 \to \mathbb{A}^1_j$, $[s:t] \mapsto 1728\dfrac{4s}{4s+27t}$, defined everywhere except where $4s+27t=0$, which is exactly the open whose preimage in $\overline{\mathscr{M}_{1,1}}$ is $\mathscr{M}_{1,1}$.

Next, we describe the rigidification of $\mathscr{M}_{1,1}$. Define

$$\overline{\mathscr{N}_{1,1}} = \mathscr{P}(2,3) = [\text{Spec } k[a,b] \setminus \{(0,0)\}/_{2,3}\mathbb{G}_m],$$

where $\lambda \cdot (a,b) = (\lambda^2 a, \lambda^3 b)$. The map $\overline{\mathscr{M}_{1,1}} \to \overline{\mathscr{N}_{1,1}}$ is induced by the identity map on Spec $k[a,b]$, and the squaring map $\beta : \mathbb{G}_m \to \mathbb{G}_m$, $\lambda \mapsto \lambda^2$. The map $\mathscr{M}_{1,1} \to \mathscr{N}_{1,1}$ is given by pulling back to the open that excludes the point where $\Delta = 0$ as before.

Then $\overline{\mathscr{M}_{1,1}} \to \overline{\mathscr{N}_{1,1}}$ is a μ_2-gerbe, and since the sequence

$$1 \longrightarrow \mu_2 \xrightarrow{\alpha} \mathbb{G}_m \xrightarrow{\beta} \mathbb{G}_m \longrightarrow 1 \tag{3.5.1.1}$$

describes $\mathbb{G}_m$ as the nontrivial extension of μ_2 and $\mathbb{G}_m$, this shows that $\overline{\mathscr{M}_{1,1}} \to \overline{\mathscr{N}_{1,1}}$ is a nontrivial μ_2-gerbe. Indeed, the trivial μ_2-gerbe over $\overline{\mathscr{N}_{1,1}}$ is

$$B_{\overline{\mathscr{N}_{1,1}}}\mu_2 = [\text{Spec } k[a,b] \setminus \{(0,0)\}/(\mathbb{G}_m \times \mu_2)],$$

where $\lambda \in \mathbb{G}_m$ acts by $\lambda \cdot (a,b) = (\lambda^2 a, \lambda^3 b)$ and μ_2 acts trivially. But $\overline{\mathscr{M}_{1,1}} \ncong B_{\overline{\mathscr{N}_{1,1}}}\mu_2$, because on $B_{\overline{\mathscr{N}_{1,1}}}\mu_2$, $[1:0]$ has stabilizer group $\mu_2 \times \mu_2$, but on $\overline{\mathscr{M}_{1,1}}$, $[1:0]$ has stabilizer group μ_4. This argument also shows that $\mathscr{M}_{1,1} \to \mathscr{N}_{1,1}$ is a nontrivial μ_2-gerbe, because removing the point where $\Delta = 0$ does not change the fact that $\overline{\mathscr{M}_{1,1}} \ncong B_{\overline{\mathscr{N}_{1,1}}}\mu_2$.

This describes the coarse space map $\mathscr{M}_{1,1} \to \mathbb{A}^1_j$ and the rigidification $\mathscr{M}_{1,1} \to \mathscr{N}_{1,1}$. The map $\mathscr{N}_{1,1} \to \mathbb{A}^1_j$ puts $\mathscr{N}_{1,1}$ as a root stack over $\mathbb{A}^1_j$ of order 3 at $j = 0$ and order 2 at $j = 1728$.

In Example 2.2.3, we computed that

$$\mathscr{I}(\mathscr{M}_{1,1}) = (\mathscr{M}_{1,1} \times \mu_2) \coprod \left(\coprod_2 B\mu_4 \right) \coprod \left(\coprod_4 B\mu_6 \right).$$

From this, it is also clear that the dense open where $\mathscr{M}_{1,1}$ is a gerbe over its coarse space as described in the proof of Theorem 3.3.1 is $\mathscr{U} = \mathscr{M}_{1,1} \setminus \{E_0, E_{1728}\}$, and $\mathscr{I}(\mathscr{U}) = \mathscr{U} \times \mu_2$. Then $\mathscr{H} = \mathscr{M}_{1,1} \times \mu_2$ is the closure of the image of $\mathscr{I}(\mathscr{U})$ in $\mathscr{I}(\mathscr{M}_{1,1})$, and $\mathscr{H}$ is a finite flat constant subgroup. Rigidifying $\mathscr{M}_{1,1}$ by $\mathscr{H}$ gives $\mathscr{N}_{1,1}$. Moreover, on the open $\mathscr{U}$, we have that $\mathscr{U} \cong B_U(\mu_2)$, where $U = \mathbb{A}^1_j \setminus \{0, 1728\}$. Indeed, in Example 2.2.3, we already computed that on the open U_0 of $\overline{\mathscr{M}_{1,1}}$ that removes $[0:1]$, $U_0 \cong [\mathbb{A}^1/_6\mu_4]$, and on U_1, which removes $[1:0]$, $U_1 = [\mathbb{A}^1/_4\mu_6]$, and we can show that $U_0 \cap U_1 = [\mathbb{A}^1 \setminus \{0\}/\mu_2]$ with μ_2 acting trivially. This of course still holds after removing $\Delta = 0$. Another way to see that $\mathscr{U} \cong B_U(\mu_2)$ is that we have a section $U \to \mathscr{U}$, which is given by the family of elliptic curves with j-invariant j, for $j \neq 0, 1728$,

$$y^2 = x^3 - \frac{27j}{4(j-1728)}x - \frac{27j}{4(j-1728)}.$$

By Theorem 3.4.3, we get an exact sequence

$$0 \longrightarrow \operatorname{Pic}(\mathscr{N}_{1,1}) \xrightarrow{\pi^*} \operatorname{Pic}(\mathscr{M}_{1,1}) \xrightarrow{\chi} \operatorname{Hom}(\mu_2, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{M}_{1,1}]} H^2(\mathscr{N}_{1,1}, \mathbb{G}_m).$$

The pushout of $\mathscr{M}_{1,1}$ under the inclusion $\mu_2 \to \mathbb{G}_m$ is a trivial gerbe. We have the following commutative diagram

$$\begin{tikzcd}
H^2(\mathscr{N}_{1,1}, \mu_2) \arrow[r, "\alpha_*"] \arrow[d] & H^2(\mathscr{N}_{1,1}, \mathbb{G}_m) \arrow[d] \\
H^2(U, \mu_2) \arrow[r, "\alpha_*"] & H^2(U, \mathbb{G}_m)
\end{tikzcd}$$

where the horizontal arrows are pushouts along the inclusion and the vertical arrows are the restriction maps. The right vertical arrow is an inclusion. Then $[\mathscr{M}_{1,1}] \mapsto [B_U\mu_2] \mapsto [B_U\mathbb{G}_m]$ along the bottom left, so we must have $\alpha_*[\mathscr{M}_{1,1}] = [B\mathbb{G}_m] \in H^2(\mathscr{N}_{1,1}, \mathbb{G}_m)$.

Since $[\mathscr{M}_{1,1}] \in H^2(\mathscr{N}_{1,1}, \mu_2)$ pushes forward to $[B\mathbb{G}_m] \in H^2(\mathscr{N}_{1,1}, \mathbb{G}_m)$ under both the trivial map and the standard inclusion $\mu_2 \to \mathbb{G}_m$, we have that $(-)_*[\mathscr{M}_{1,1}]$ is the zero map, and we have an exact sequence

$$0 \longrightarrow \operatorname{Pic}(\mathscr{N}_{1,1}) \xrightarrow{\pi^*} \operatorname{Pic}(\mathscr{M}_{1,1}) \longrightarrow \operatorname{Hom}(\mu_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

By Theorem 3.1.12, $\mathrm{Pic}(\mathscr{N}_{1,1})$ can be described as the pushout of the following diagram,

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, "(\cdot 2, \cdot 3)"] \arrow[d, "\phi"] & \mathbb{Z}^2 \arrow[d] \\ \mathrm{Pic}(\mathbb{A}^1_j) \arrow[r, "\pi^*"] & \mathrm{Pic}(\mathscr{N}_{1,1}), \end{tikzcd}$$

so $\mathrm{Pic}(\mathscr{N}_{1,1}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, since $\mathrm{Pic}(\mathbb{A}^1_j) = 0$. Thus, $\mathrm{Pic}(\mathscr{M}_{1,1})$ is described by

$$0 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathrm{Pic}(\mathscr{M}_{1,1}) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Moreover, we can deduce that the extension is non-split using the fact that the class of $\mathscr{M}_{1,1}$ is a nontrivial gerbe. Indeed, since $[\mathscr{M}_{1,1}]$ maps to the trivial class in $H^2(\mathscr{N}_{1,1}, \mathbb{G}_m)$, it must come from a class in $H^1(\mathscr{N}_{1,1}, \mathbb{G}_m)$, thus $\mathscr{M}_{1,1}$ can be identified as the gerbe of square roots of a line bundle $\mathscr{L}$ on $\mathscr{N}_{1,1}$. Moreover, since the class of $\mathscr{M}_{1,1}$ is nontrivial, $\mathscr{L}$ does not have a square root on $\mathscr{N}_{1,1}$. The line bundle $\mathscr{I}_0$ of order 3 has a square root, indeed $(\mathscr{I}_0^{\otimes 2})^{\otimes 2} = \mathscr{I}_0$. Thus, we take the gerbe of square roots of the line bundle of order 2, $\mathscr{I}_{1728}$, which produces a line bundle of order 4 on $\mathscr{M}_{1,1}$, and we conclude $\mathrm{Pic}(\mathscr{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}$.

Example 3.5.2. The author thanks Santiago Arango-Piñeros for suggesting the following example. Consider the compactification of the modular curve over $\mathbb{Q}$ that parameterizes elliptic curves together with a cyclic subgroup of order 5, $\mathscr{X} := \mathscr{X}_0(5)$. We have the following presentation of $\mathscr{X}$ [6, Example 2.5]. Let

$$S = \mathrm{Spec}\ \frac{\mathbb{Q}[x,y,z]}{x^2+y^2=z^4} \setminus \{(0,0,0)\} \subseteq \mathbb{A}^3_{\mathbb{Q}},$$

and let $\mathbb{G}_m$ act on S via $\lambda\cdot(x,y,z) = (\lambda^4 x, \lambda^4 y, \lambda^2 z)$. Then $\mathscr{X} = [S/_{4,4,2}\mathbb{G}_m]$. Let $\mathbb{G}_m$ also act on S via $\lambda\cdot(x,y,z) = (\lambda^2 x, \lambda^2 y, \lambda z)$. The rigidification $\mathscr{Z} := \mathscr{Z}_0(5)$ of $\mathscr{X}$ is $\mathscr{Z} = [S/_{2,2,1}\mathbb{G}_m]$. The map $\mathscr{X} \to \mathscr{Z}$ is induced by the squaring map $\mathbb{G}_m \to \mathbb{G}_m$ and makes $\mathscr{X}$ a μ_2-gerbe over $\mathscr{Z}$. We first describe the Picard group of the rigidification. Let $p = [x:y:z]$ be a geometric point of $\mathscr{Z}$. Then p has trivial stabilizer group unless $z = 0$, in which case the stabilizer group is μ_2. There are two such geometric points, $x_1 = [1:i:0]$ and $x_2 = [1:-i:0]$.

The coarse moduli space is given by $X := X_0(5) = V(x^2+y^2=w^2) \subseteq \mathbb{P}^2_{\mathbb{Q}}$, and the map $\mathscr{Z} \to X$ is induced by $S \to X$, where $(x,y,z) \mapsto [x:y:z^2]$. This map is $\mathbb{G}_m$-invariant, so we get a map $\pi : \mathscr{Z} \to X$ which is the coarse space morphism.

Next we show that the scheme-theoretic images of $\pi(x_1)$ and $\pi(x_2)$ in X are both the closed subscheme corresponding to the sheaf of ideals of (w), $I_{w=0}$, in X. Both are given by

$$I_{\pi(x_i)} = \ker(\mathscr{O}_X \to \pi(x_i)_*\mathbb{C}),$$

or

$$I_{\pi(x_i)} = \ker(\mathbb{Q}[x,y,w]/(x^2+y^2=w^2) \to \mathbb{C}, (x,y,w) \mapsto (1, \pm i, 0)),$$

so $I_{\pi(x_i)} = I_{w=0}$ for $i = 1, 2$.

By Proposition 3.1.9, $\mathscr{Z}$ is the root stack of X or order 2 at $w = 0$. From the construction of the root stack of X, we get a line bundle $\mathscr{L}$ on $\mathscr{Z}$ that does not come from a line bundle on X, but whose square is the ideal sheaf of $w = 0$.

Now, since X is a degree 2 plane curve that has a rational point, $X \cong \mathbb{P}^1$. We can write the degree 2 morphism

$$\mathbb{P}^1 \to \mathbb{P}^2, \quad [s : t] \mapsto [s^2 - t^2 : 2st : s^2 + t^2],$$

with image $X \subseteq \mathbb{P}^2$. The ideal sheaf of $w = 0$ on $\mathbb{P}^2$ is $\mathscr{O}(-1)$, so its pullback under the degree 2 inclusion is $\mathscr{O}(-2)$ on $\mathbb{P}^1$.

By Theorem 3.1.12, we have an exact sequence

$$0 \to \operatorname{Pic}(X) \cong \mathbb{Z} \to \operatorname{Pic}(\mathscr{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Finally, $\operatorname{Pic}(\mathscr{Z})$ is the pushout of the following diagram:

$$\begin{tikzcd} 1 \arrow[d, mapsto] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "\phi"] & \mathbb{Z} \arrow[d] \\ I_{w=0} & \operatorname{Pic}(X) = \mathbb{Z} \arrow[r] & \operatorname{Pic}(\mathscr{Z}). \end{tikzcd}$$

Thus, $\operatorname{Pic}(\mathscr{Z}) = (\mathbb{Z} \times \mathbb{Z})/\langle(2, 2)\rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The line bundle of order 2 on $\mathscr{Z}$ is $\pi^*\mathscr{O}(1) \otimes \mathscr{L}$, because $\mathscr{L}^{\otimes 2} = \pi^* I_{w=0} = \pi^*\mathscr{O}(-2)$.

As in the previous example, since $\mathbb{G}_m$ is a nontrivial extension of μ_2 and $\mathbb{G}_m$, the μ_2-gerbe $\mathscr{X} \to \mathscr{Z}$ is nontrivial. The stacky points $[1 : i : 0]$ and $[1 : -i : 0]$ on $\mathscr{X}$ have stabilizer group μ_4, but on $B_{\mathscr{Y}}\mu_2$, they have stabilizer group $\mu_2 \times \mu_2$. Also similarly to the previous example, after we remove the stacky points with larger stabilizer groups, $\mathscr{X} \to \mathscr{Z}$ becomes the trivial μ_2-gerbe. Removing the points where $z = 0$, $\mathscr{X} \to \mathscr{Z}$ is given by

$$\left[\operatorname{Spec} \frac{\mathbb{Q}[x, y, z^{\pm}]}{x^2 + y^2 = z^4} \Big/ {}_{4,4,2}\mathbb{G}_m\right] = \left[\operatorname{Spec} \frac{\mathbb{Q}[x, y]}{x^2 + y^2 = 1} \Big/ \mu_2\right] \to \operatorname{Spec} \frac{\mathbb{Q}[x, y]}{x^2 + y^2 = 1},$$

with μ_2 acting trivially. Thus, we conclude that $\mathscr{X}^{\wedge\mathbb{G}_m}$ is a trivial $\mathbb{G}_m$-gerbe over $\mathscr{Z}$. We have the following short exact sequence,

$$0 \to \operatorname{Pic}(\mathscr{Z}) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \operatorname{Pic}(\mathscr{X}) \to \operatorname{Hom}(\mu_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z} \to 0.$$

As before, since $\mathscr{X}^{\wedge\mathbb{G}_m}$ is a trivial $\mathbb{G}_m$-gerbe over $\mathscr{Z}$ and $\mathscr{X}$ is a nontrivial μ_2-gerbe over $\mathscr{Z}$, we know that the class of $\mathscr{X}$ in $H^2(\mathscr{Z}, \mu_2)$ is given by the class of the gerbe of square roots of a line bundle $\mathscr{L}$ on $\mathscr{Z}$, which does not have a square root on $\mathscr{Z}$ itself. Thus, $\operatorname{Pic}(\mathscr{X})$ is a nontrivial extension of the sequence above, but now there are two ways to extend the sequence, corresponding to two line bundles on $\mathscr{Z}$ for which we could take the gerbe of square roots of, one of infinite order, and one of order 2. To determine the correct class, consider the following cartesian diagram,

$$\begin{array}{ccc} \mathscr{X} = \left[\mathrm{Spec}\ \dfrac{\mathbb{Q}[x,y,z]}{x^2+y^2=z^4}\Big/_{4,4,2}\mathbb{G}_m\right] & \longrightarrow & \mathscr{Z} = \left[\mathrm{Spec}\ \dfrac{\mathbb{Q}[x,y,z]}{x^2+y^2=z^4}\Big/_{2,2,1}\mathbb{G}_m\right] \\ \downarrow & & \downarrow \\ \mathscr{Y} := \left[\mathrm{Spec}\ \dfrac{\mathbb{Q}[x,y,w]}{x^2+y^2=w^2}\Big/_{2,2,2}\mathbb{G}_m\right] & \longrightarrow & \mathbb{P}^1 \cong \left[\mathrm{Spec}\ \dfrac{\mathbb{Q}[x,y,w]}{x^2+y^2=w^2}\Big/_{1,1,1}\mathbb{G}_m\right]. \end{array}$$

Similarly to the map $\mathscr{X} \to \mathscr{Z}$, $\mathscr{Y} \to \mathbb{P}^1$ is a nontrivial μ_2-gerbe whose pushout to $\mathbb{G}_m$ is trivial, so its class in $H^2(\mathbb{P}^1, \mu_2)$ is the class of a gerbe of square roots of a line bundle on $\mathbb{P}^1$ which does not have a square root on $\mathbb{P}^1$ itself. Thus $\mathscr{Y}$, and thus also $\mathscr{X}$, can be described as the gerbe of square roots of $\mathscr{O}(1)$ on $\mathbb{P}^1$. We conclude that the Picard group of $\mathscr{X}$ is $\mathrm{Pic}(\mathscr{X}) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by a square root of $\mathscr{O}(1)$ and the line bundle of order 2 on $\mathscr{Z}$.

Example 3.5.3. Let P be the vanishing locus of the equation $x^2+y^2+z^2$ inside $\mathbb{P}^2_{\mathbb{R}}$, that is,

$$P = \mathrm{Proj}\ (\mathbb{R}[x,y,z]/(x^2+y^2+z^2))$$

or

$$P = \left[\mathrm{Spec}\ \frac{(\mathbb{R}[x,y,z])\backslash\{(0,0,0)\}}{x^2+y^2+z^2}\Big/\mathbb{G}_m\right].$$

This curve has no $\mathbb{R}$-points, and thus $P \ncong \mathbb{P}^1_{\mathbb{R}}$, but $P_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$, with inclusion into $\mathbb{P}^2_{\mathbb{C}}$ corresponding to a line bundle of degree 2. We have the following cartesian diagram,

$$\begin{array}{ccc} P_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & P \\ \downarrow & & \downarrow \\ \mathbb{P}^2_{\mathbb{C}} & \longrightarrow & \mathbb{P}^2_{\mathbb{R}}, \end{array}$$

and there exists a line bundle of degree 2 on P defining the inclusion of P into $\mathbb{P}^2_{\mathbb{R}}$, call it $\mathscr{L}_P$.

Let $\pi : \mathscr{P} \to P$ be the μ_2-gerbe of square roots of $\mathscr{L}$ over P and let $[\mathscr{P}]$ be its class in $H^2(P, \mu_2)$. Since $\mathscr{L}_P$ does not have a global square root on P, this gerbe is nontrivial.

By Theorem 3.4.3, we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(P) \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{P}) \xrightarrow{\chi} \mathrm{Hom}(\mu_2, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{P}]} H^2(P, \mathbb{G}_m)$$

Since $[\mathscr{P}]$ is the image of $[\mathscr{L}_P] \in H^1(P, \mathbb{G}_m)$, the pushout of $\mathscr{P}$ to $\mathbb{G}_m$ is $B_P\mathbb{G}_m$, so $(-)_*[\mathscr{P}]$ is the zero map, and the sequence

$$0 \longrightarrow \mathrm{Pic}(P) \cong \mathbb{Z} \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{P}) \xrightarrow{\chi} \mathrm{Hom}(\mu_2, \mathbb{G}_m) \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact.

Moreover, by the Yoneda lemma, we get a universal line bundle $\mathscr{L}_{\mathscr{P}}$ on $\mathscr{P}$ which is a square root of $\mathscr{L}_P$. Thus, we have a line bundle on $\mathscr{P}$ whose square is the generator of $\mathrm{Pic}(P)$. Thus $\mathrm{Pic}(\mathscr{P}) \cong \mathbb{Z}$ is generated by $\mathscr{L}_{\mathscr{P}}$.

Example 3.5.4. Using the same set-up as in the previous example, now let $\mathscr{P}' = B_P(\mu_2)$ be the trivial μ_2-gerbe over P. We have the same short exact sequence

$$0 \longrightarrow \mathrm{Pic}(P) \cong \mathbb{Z} \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{P}') \xrightarrow{\chi} \mathrm{Hom}(\mu_2, \mathbb{G}_m) \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

but $\mathrm{Pic}(\mathscr{P}') \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Indeed, the section $P \to \mathscr{P}' = B_p(\mu_2)$ gives a section of the sequence $\mathrm{Pic}(\mathscr{P}') \to \mathrm{Pic}(P)$, so the extension is split.

Remark 3.5.5. The stacky curves in the last two examples have different Picard groups, but the Picard groups are given by an extension of the same two groups. This exemplifies the fact that the Picard group of a stacky curve depends on its structure as a gerbe.

Example 3.5.6. Let (E, e) be an elliptic curve over a field k with structure morphism $f : E \to \mathrm{Spec}\, k$ and section $e : \mathrm{Spec}\, k \to E$. Using the sequence 3.5.1.1, we construct an example of a stacky curve $\mathscr{X}$ which is a μ_2-gerbe over E with $\chi : \mathrm{Pic}(\mathscr{X}) \to \mathrm{Hom}(\mu_2, \mathbb{G}_m)$ not surjective, unlike the previous examples.

The section e induces a pullback map $e^* : H^2(E, \mathbb{G}_m) \to H^2(k, \mathbb{G}_m)$ which splits the pullback of the structure morphism $f^* : H^2(k, \mathbb{G}_m) \to H^2(E, \mathbb{G}_m)$, and similarly for H^3.

We can use the Leray Spectral Sequence

$$E_2^{p,q} = H^p(k, R^q f_* \mathbb{G}_m) \Rightarrow H^{p+q}(E, \mathbb{G}_m)$$

to compute $H^2(E, \mathbb{G}_m)$. We have $R^0 f_* \mathbb{G}_m = \mathbb{G}_m$, $R^1 f_* \mathbb{G}_m = \mathrm{Pic}_{E/k}$, the relative Picard scheme of E over k, and $R^i f_* \mathbb{G}_m = 0$ for all $i \geq 2$ by Tsen's Theorem. Then the spectral sequence results in an exact sequence describing $H^2(E, \mathbb{G}_m)$,

$$\mathrm{Pic}_{E/k}(k) \xrightarrow{\phi} H^2(k, \mathbb{G}_m) \xrightarrow{f^*} H^2(E, \mathbb{G}_m) \longrightarrow H^1(k, \mathrm{Pic}_{E/k}) \xrightarrow{\phi'} H^3(k, \mathbb{G}_m),$$

and a filtration

$$0 \longrightarrow H^2(k, \mathbb{G}_m)/\mathrm{im}(\phi) \longrightarrow H^2(E, \mathbb{G}_m) \longrightarrow \ker(\phi') \longrightarrow 0.$$

Similarly, for $H^3(E, \mathbb{G}_m)$, we have an exact sequence

$$H^1(k, \mathrm{Pic}_{E/k}) \xrightarrow{\phi'} H^3(k, \mathbb{G}_m) \xrightarrow{f^*} H^3(E, \mathbb{G}_m).$$

Putting the two sequences together, we have an exact sequence

$$\mathrm{Pic}_{E/k}(k) \xrightarrow{\phi} H^2(k, \mathbb{G}_m) \xrightarrow{f^*} H^2(E, \mathbb{G}_m)$$

$$\longrightarrow H^1(k, \mathrm{Pic}_{E/k}) \xrightarrow{\phi'} H^3(k, \mathbb{G}_m) \xrightarrow{f^*} H^3(E, \mathbb{G}_m).$$

The section $e : \mathrm{Spec}\, k \to E$ gives us two splittings of this sequence so that both f^* maps are injective. We conclude that $\phi = \phi' = 0$, and

$$0 \longrightarrow H^2(k, \mathbb{G}_m) \xrightarrow{f^*} H^2(E, \mathbb{G}_m) \longrightarrow H^1(k, \mathrm{Pic}_{E/k}) \longrightarrow 0$$

is a split exact sequence.

Let $k = \mathbb{R}$, so that $H^2(k, \mathbb{G}_m) = \mu_2$, with nontrivial element the class of the Hamiltonian algebra $\mathbb{H}$. Let $[\mathscr{G}_{\mathbb{H}}]$ be the image of $[\mathbb{H}]$ under the inclusion $f^* : H^2(k, \mathbb{G}_m) \to H^2(E, \mathbb{G}_m)$. Since $[\mathbb{H}]$ has order 2 in $H^2(k, \mathbb{G}_m)$, $\beta_*[\mathscr{G}_{\mathbb{H}}] = [B_E(\mathbb{G}_m)]$ in $H^2(E, \mathbb{G}_m)$ with β as in 3.5.1.1. Thus, $[\mathscr{G}_{\mathbb{H}}]$ came from an element $[\mathscr{X}] \in H^2(E, \mu_2)$. Then $\mathscr{X}$ is a stacky curve which is a μ_2-gerbe over E, and we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(E) \longrightarrow \mathrm{Pic}(\mathscr{X}) \xrightarrow{\chi} \mathrm{Hom}(\mu_2, \mathbb{G}_m) \xrightarrow{(-)_*[\mathscr{X}]} H^2(E, \mathbb{G}_m).$$

For the inclusion $\alpha : \mu_2 \to \mathbb{G}_m$, we have $\alpha_*[\mathscr{X}] = [\mathscr{G}_{\mathbb{H}}]$, so $(-)_*[\mathscr{X}]$ is injective, thus χ is the zero map, and $\mathrm{Pic}(\mathscr{X}) \cong \mathrm{Pic}(E)$. In this example, the gerbe structure of $\mathscr{X}$ did not contribute to the Picard group of $\mathscr{X}$ over E. We say that there does not exist any twisted sheaves on $\mathscr{X}$. The moduli of twisted sheaves is studied in [31].

Chapter 4

Brauer Group of BG and Brauer–Severi stacks on BG

In this chapter, we calculate the Brauer group of the classifying stack BG for G a smooth connected semisimple linear algebraic group over an algebraically closed field. We then study Brauer–Severi stacks over BG, and calculate their Brauer classes in the Brauer group. We apply these calculations to moduli stacks of curves of genus g together with an automorphism of order N. These moduli stacks are naturally μ_N-gerbes, and we explore what corresponding H^2-cohomology classes the components of this stack can have. In particular, we look at curves whose quotients by the order N automorphism are genus 0, and completely determine the Brauer classes of these gerbes.

4.1 The Brauer Group of BG

Let G a smooth connected semisimple linear algebraic group over an algebraically closed field k. Let $\widetilde{G}$ be the universal cover of G, and B the fundamental group of G, which exist by [17, Corollary 4.6]. Then $\widetilde{G}$ is also a smooth connected semisimple linear algebraic group, B is finite and diagonalizable, thus a finite abelian group, and we have an exact sequence

$$1 \to B \to \widetilde{G} \to G \to 1,$$

with the image of B in the center of $\widetilde{G}$.

Let

$$X(-) : (\text{algebraic groups}/k) \to \text{Sets}$$

be $H \mapsto \mathrm{Hom}(H, \mathbb{G}_m)$, where Hom refers to homomorphisms of algebraic groups.

We may view $k^* \subseteq H^0(H, \mathscr{O}_H^*) = H^0(H, \mathbb{G}_m)$ as constant functions, and we have a surjection $H^0(H, \mathbb{G}_m) \to X(H)$, by $f \mapsto f/f(e)$, which gives us a split exact sequence

$$0 \to k^* \to H^0(H, \mathbb{G}_m) \to X(H) \to 0,$$

and thus $H^0(H, \mathbb{G}_m) = k^* \oplus X(H)$ ([17, Corollary 2.2], [43, Theorem 3], [44, Lemma 6.5 (iii)]). Since G is semisimple, $X(G) = 0$.

We have $\widetilde{G} \to G$ a B-torsor, and $B\widetilde{G} \to BG$ a B-gerbe. Given a character $\chi : B \to \mathbb{G}_m$ of B, we get pushforward maps

$$\chi_* : H^1(G, B) \to H^1(G, \mathbb{G}_m), \quad \chi_* : H^2(BG, B) \to H^2(BG, \mathbb{G}_m).$$

Thus we get morphisms

$$X(B) \to H^1(G, \mathbb{G}_m) = \mathrm{Pic}(G), \quad X(B) \to H^2(BG, \mathbb{G}_m) = \mathrm{Br}(BG),$$

given by $(\chi : B \to \mathbb{G}_m) \mapsto \chi_*(\widetilde{G})$ and $(\chi : B \to \mathbb{G}_m) \mapsto \chi_*(B\widetilde{G})$, respectively.

It follows from [44, Corollary 6.11, Remark 6.11.3] that the first map is an isomorphism, also using the fact that $X(\widetilde{G}) = 0$ and $\mathrm{Pic}(\widetilde{G}) = 0$. That the second map is an isomorphism is the statement of the following theorem.

Theorem 4.1.1. *Let k be an algebraically closed field, G a smooth connected semisimple linear algebraic group over k, and $\widetilde{G}$ and B as above. Then*

$$H^i(BG, \mathbb{G}_m) = \begin{cases} k^* & i = 0 \\ 0 & i = 1 \\ \widehat{B} & i = 2. \end{cases}$$

For $i = 2$, an isomorphism $\widehat{B} \to H^2(BG, \mathbb{G}_m)$ is given by $(\chi : B \to \mathbb{G}_m) \mapsto \chi_*(B\widetilde{G})$, where χ_* is the pushforward map $\chi_* : H^2(BG, B) \to H^2(BG, \mathbb{G}_m)$.

Proof. We use the spectral sequence $E_1^{p,q} = H^q(U_p, \mathbb{G}_m) \Rightarrow H^{p+q}(BG, \mathbb{G}_m)$ of the standard simplicial cover $U_\bullet \to BG$ ([13, Equation 5.2.1.1]) coming from $\mathrm{Spec}\ k \to BG$.

Here, $U_0 = \mathrm{Spec}\ k$, and $U_p = U_{p-1} \times_{BG} \mathrm{Spec}\ k$ for all $p \geq 1$. We have $p+1$ projection maps $U_p \to U_{p-1}$, each of which project onto all but one copy of Spec k. For all $p \geq 1$, $U_p \cong G^p$. In particular, $U_1 = \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k \cong G$ and $U_2 \cong G \times G$. The two projections

$$q_1, q_2 : \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k \to \mathrm{Spec}\ k$$

are both given by the structure map $\pi : G \to \mathrm{Spec}\ k$. The three projections

$$p_{23}, p_{13}, p_{12} : \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k \to \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k$$

are given by

$$p_2, m, p_1 : G \times G \to G,$$

respectively. The simplicial cover $U_\bullet \to BG$ can then be written

$$U_\bullet : \quad \cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; G \times G \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; G \;\substack{\longrightarrow\\\longrightarrow}\; \mathrm{Spec}\ k.$$

We may apply the functors $H^q(-, \mathbb{G}_m)$ to $U_\bullet$ to get

$$H^q(U_\bullet, \mathbb{G}_m) : H^q(\text{Spec } k, \mathbb{G}_m) \xrightarrow{d^{0,q}} H^q(G, \mathbb{G}_m) \xrightarrow{d^{1,q}} H^q(G \times G, \mathbb{G}_m) \to \cdots$$

where the differentials $d^{p,q}$ are given by the alternating sums of the maps induced by the projection maps $U_{p+1} \to U_p$. The complexes $H^q(U_\bullet, \mathbb{G}_m)$ make up the rows of the first page of the spectral sequence.

We write out some low-degree terms on the E_1 page:

$$\begin{array}{c|ccccccc}
q & & & & & & & \\
\uparrow & 0 & \longrightarrow & \cdots & & & & \\
 & & & & & & & \\
 & 0 & \longrightarrow & \text{Pic}(G) & \longrightarrow & \text{Pic}(G \times G) & \longrightarrow & \cdots \\
 & & & & & & & \\
 & k^* & \longrightarrow & H^0(G, \mathbb{G}_m) & \longrightarrow & H^0(G \times G, \mathbb{G}_m) & \longrightarrow & \cdots \\
\hline
 & & & & & & & \rightarrow p
\end{array}$$

Note that since k is algebraically closed, $\text{Br}(k) = \text{Pic}(k) = 0$. Note also that for all q, $d^{0,q}$ is the alternating sum of two copies of $\pi^* : H^q(\text{Spec } k, \mathbb{G}_m) \to H^q(G, \mathbb{G}_m)$, and thus $d^{0,q} = 0$.

We first compute the cohomology of the bottom line, $H^p(E_1^{\bullet,0})$ to get the bottom line of the E_2 page. Using the fact that $H^0(G^p, \mathbb{G}_m) = k^* \oplus X(G^p) = k^*$ is the set of constant functions on G^p for all $p \geq 1$, we see that the pullback along any projection map $G^{p+1} \to G^p$ is the identity map $k^* \to k^*$. Thus, for all $p \geq 1$, $d^{p,0}$ is the alternating sum of $p + 2$ copies of $\text{id} : k^* \to k^*$, so is given by the identity map for p odd and the zero map for p even. For example, $d^{1,0} : H^0(G, \mathbb{G}_m) \to H^0(G \times G, \mathbb{G}_m)$ is given by $p_2^* - m^* + p_1^*$, which is the identity map $k^* \to k^*$, since each term in the sum is the identity map.

Thus $H^p(E_1^{\bullet,0})$ is the cohomology of

$$k^* \xrightarrow{0} k^* \xrightarrow{\text{id}} k^* \xrightarrow{0} \cdots,$$

so $H^0(E_1^{\bullet,0}) = k^*$, and $H^p(E_1^{\bullet,0}) = 0$ for $p \geq 1$.

Next, we compute the cohomology of the middle line, $H^p(E_1^{\bullet,1})$. We must understand the three pullback maps

$$p_1^*, p_2^*, m^* : \text{Pic}(G) = X(B) \to \text{Pic}(G \times G) = X(B \times B).$$

Let $\chi \in X(B)$ correspond to $\chi_* \widetilde{G} \in \text{Pic}(G)$. Write $\rho : \widetilde{G} \to G$ and $\rho' : \chi_* \widetilde{G} \to G$. Then in $\text{Pic}(G \times G)$,

$$p_1^*(\chi_* \widetilde{G}) = \chi_* \widetilde{G} \times_{G,p_1} (G \times G)$$
$$p_2^*(\chi_* \widetilde{G}) = \chi_* \widetilde{G} \times_{G,p_2} (G \times G),$$

$$m^*(\chi_*\widetilde{G}) = \chi_*\widetilde{G} \times_{G,m} (G \times G)$$

Each of the above is a $\mathbb{G}_m$-torsor over $G \times G$, which we can write $\chi_*\widetilde{G} \times G \to G \times G$ with the following structure maps,

$$p_1^*(\chi_*\widetilde{G}) = \chi_*\widetilde{G} \times G \to G \times G,\ (x, g) \mapsto (\rho'(x), g),$$

$$p_2^*(\chi_*\widetilde{G}) = \chi_*\widetilde{G} \times G \to G \times G,\ (x, g) \mapsto (g, \rho'(x)),$$

$$m^*(\chi_*\widetilde{G}) = \chi_*\widetilde{G} \times G \to G \times G,\ (x, g) \mapsto (g, g^{-1}\rho'(x)).$$

Let $p_1^*\chi, p_2^*\chi, m^*\chi \in X(B \times B)$ correspond to $p_1^*(\chi_*\widetilde{G}), p_2^*(\chi_*\widetilde{G}), m^*(\chi_*\widetilde{G})$, respectively. These maps are given as follows,

$$p_1^*\chi = \chi \circ p_1 : (b_1, b_2) \mapsto \chi(b_1),$$

$$p_2^*\chi = \chi \circ p_2 : (b_1, b_2) \mapsto \chi(b_2),$$

$$m^*\chi = \chi \circ m : (b_1, b_2) \mapsto \chi(b_1 b_2),$$

since they pushout the $B \times B$-torsor $\widetilde{G} \times \widetilde{G} \to G \times G$ to the $\mathbb{G}_m$-torsors $p_1^*(\chi_*\widetilde{G})$, $p_2^*(\chi_*\widetilde{G})$, and $m^*(\chi_*\widetilde{G})$, respectively.

Then $d^{1,1} : X(B) \to X(B \times B)$ is given by

$$d^{1,1} : \chi \mapsto p_2^*\chi - m^*\chi + p_1^*\chi : (b_1, b_2) \mapsto \chi(b_2) - \chi(b_1 b_2) + \chi(b_1) = 0,$$

since χ is a homomorphism. Thus $H^p(E_1^{\bullet,1})$ is given by the cohomology of the sequence

$$0 \longrightarrow X(B) \xrightarrow{0} X(B \times B)$$

This gives $H^0(E_1^{\bullet,1}) = 0$, $H^1(E_1^{\bullet,1}) = X(B)$.

We can now write out the E_2 page,

$$\begin{tikzcd}
q & & & & \\
0 & \cdots & & & \\
0 \arrow[drr] & X(B) \arrow[drr] & & & \\
k^* & 0 & 0 & 0 & p
\end{tikzcd}$$

from which we can read off $H^i(BG, \mathbb{G}_m)$ for $i = 0, 1, 2$. This concludes the proof of the first statement.

We now show that the map

$$X(B) \to H^2(BG, \mathbb{G}_m), \ \chi \mapsto \chi_*(B\widetilde{G})$$

is an isomorphism. Let $\mathscr{H} \to BG$ be a $\mathbb{G}_m$-gerbe. Take the fiber product with $B\widetilde{G} \to BG$ to get $\mathscr{H} \times_{BG} B\widetilde{G} \to B\widetilde{G}$, a $\mathbb{G}_m$-gerbe over $B\widetilde{G}$. By the first part of the proof, $H^2(B\widetilde{G}, \mathbb{G}_m) = 0$ since the fundamental group of $\widetilde{G}$ is 0. Thus, our $\mathbb{G}_m$-gerbe $\mathscr{H} \times_{BG} B\widetilde{G} \to B\widetilde{G}$ is trivial, and thus has a section. Composing the section with the projection to $\mathscr{H}$ gives a map $B\widetilde{G} \to \mathscr{H}$. The map on stabilizers over BG is a map $\chi : B \to \mathbb{G}_m$ such that $\chi_*(B\widetilde{G}) = \mathscr{H}$. Thus the map is surjective.

Now let $\chi_1 : B \to \mathbb{G}_m$ and $\chi_2 : B \to \mathbb{G}_m$ be two characters such that $\mathscr{H}_1 = \chi_{1*}(B\widetilde{G})$ is isomorphic to $\mathscr{H}_2 = \chi_{2*}(B\widetilde{G})$. That is, there is an isomorphism of $\mathbb{G}_m$-gerbes $\sigma : \mathscr{H}_1 \to \mathscr{H}_2$ on BG and an isomorphism

$$\tau : (\mathscr{H}_1 \to BG) \cong (\mathscr{H}_1 \xrightarrow{\sigma} \mathscr{H}_2 \to BG).$$

From χ_1 and χ_2, we get morphisms $\phi_1 : B\widetilde{G} \to \mathscr{H}_1$ and $\phi_2 : B\widetilde{G} \to \mathscr{H}_2$ whose morphisms on stabilizers are χ_1 and χ_2 respectively. This defines $\phi_1 \times \phi_2 : B\widetilde{G} \to \mathscr{H}_1 \times_{BG} \mathscr{H}_2$.

We have the following equivalence of categories,

$$\mathscr{H}_1 \times_{BG} \mathscr{H}_2 \to \mathscr{H}_1 \times B\mathbb{G}_m,$$

where an equivalence is given by sending

$$\Big(h_1 : S \to \mathscr{H}_1, h_2 : S \to \mathscr{H}_2, \alpha : (S \xrightarrow{h_1} \mathscr{H}_1 \to BG) \cong (S \xrightarrow{h_2} \mathscr{H}_2 \to BG)\Big)$$

to $\big(h_1, \underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1, h_2)\big)$, where $\underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1, h_2) : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Sets}$ is a $\mathbb{G}_m$-torsor defined as follows:

$$(f : S' \to S) \mapsto \left\{ \begin{array}{c} \eta : \Big(S' \xrightarrow{f} S \xrightarrow{h_1} \mathscr{H}_1 \xrightarrow{\sigma} \mathscr{H}_2\Big) \cong \Big(S' \xrightarrow{f} S \xrightarrow{h_2} \mathscr{H}_2\Big) \\ \text{such that } (\mathscr{H}_2 \to BG) \circ \eta = f^*(\alpha \circ h_1^* \tau^{-1}) \end{array} \right\},$$

where $\alpha \circ h_1^* \tau^{-1}$ is the composition of the maps

$$\Big(S \xrightarrow{h_1} \mathscr{H}_1 \xrightarrow{\sigma} \mathscr{H}_2 \to BG\Big) \xrightarrow{h_1^*\tau^{-1}} \Big(S \xrightarrow{h_1} \mathscr{H}_1 \to BG\Big) \xrightarrow{\alpha} \Big(S \xrightarrow{h_2} \mathscr{H}_2 \to BG\Big).$$

The $\mathbb{G}_m$-torsor $\underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1, h_2)$ can be thought of as the torsor of isomorphisms between h_1 and h_2 whose image in BG is given by α.

A morphism in $\mathscr{H}_1 \times_{BG} \mathscr{H}_2$ is a pair

$$(\beta_1, \beta_2) : (h_1, h_2, \alpha) \to (h_1', h_2', \alpha'),$$

where $\beta_1 : h_1 \to h_1'$ and $\beta_2 : h_2 \to h_2'$ are isomorphisms such that the diagram

$$\begin{tikzcd} \overline{h_1} \arrow[r, "\alpha"] \arrow[d, "\overline{\beta_1}"'] & \overline{h_2} \arrow[d, "\overline{\beta_2}"] \\ \overline{h_1'} \arrow[r, "\alpha'"] & \overline{h_2'} \end{tikzcd} \tag{4.1.1.1}$$

commutes, where the bar denotes composition with the appropriate map to BG.

A morphism (β_1, β_2) in $\mathscr{H}_1 \times_{BG} \mathscr{H}_2$ is sent to (β_1, γ) in $\mathscr{H}_1 \times B\mathbb{G}_m$, where

$$\gamma : \underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1, h_2) \to \underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1', h_2')$$

is given by

$$(\eta : f^*(\sigma \circ h_1) \cong f^* h_2) \mapsto \eta' := f^*\beta_2 \circ \eta \circ f^*(\sigma \circ \beta_1)^{-1}.$$

Note that

$$\begin{aligned} \overline{\eta'} &= f^*\overline{\beta_2} \circ \overline{\eta} \circ f^*(\overline{\sigma \circ \beta})^{-1} \\ &= f^*\overline{\beta_2} \circ f^*(\alpha \circ h_1^* \tau^{-1}) \circ f^*(\overline{\sigma \circ \beta})^{-1} \\ &= f^*\overline{\beta_2} \circ f^*\alpha \circ f^* h_1^* \tau^{-1} \circ f^*(\overline{\sigma \circ \beta})^{-1} \\ &= f^*\overline{\beta_2} \circ f^*\alpha \circ f^*\overline{\beta_1}^{-1} \circ f^* h_1'^* \tau^{-1} \\ &= f^*\alpha' \circ f^* h_1'^* \tau^{-1} \end{aligned}$$

due to the commutativity of the diagram 4.1.1.1, and functoriality of τ.

We now show that the map $\mathscr{H}_1 \times_{BG} \mathscr{H}_2 \to \mathscr{H}_1 \times B\mathbb{G}_m$ is fully faithful and essentially surjective. Let $(h_1, h_2, \alpha), (h_1', h_2', \alpha') \in \mathscr{H}_1 \times_{BG} \mathscr{H}_2$. Given a morphism $\beta : h_1 \to h_1'$ and $\gamma : \underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1, h_2) \to \underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1', h_2')$, we may set $\beta_1 = \beta$, and construct β_2 locally as follows. Choose a cover $f : S' \to S$ where $\underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ h_1, h_2)$ has a section $\eta : f^*(\sigma \circ h_1) \to f^* h_2$. We may define β_2 locally by $\gamma(\eta) \circ f^*(\sigma \circ \beta) \circ \eta^{-1}$. Then by descent on $\mathscr{H}_2$, there exists a unique morphism $\beta_2 : h_2 \to h_2'$. Thus the map is fully faithful.

For essential surjectivity, let $(h : S \to \mathscr{H}_1, T : S \to B\mathbb{G}_m) \in \mathscr{H}_1 \times B\mathbb{G}_m$. Let $f : S' \to S$ be a cover of S where T becomes the trivial torsor, $T|_{S'} = S' \times \mathbb{G}_m$. Then

$$\left(f^*h, f^*(\sigma \circ h), f^*h^*\tau : f^*\overline{h} \to f^*\overline{(\sigma \circ h)}\right) \mapsto (f^*h, S' \times \mathbb{G}_m)$$

since $\underline{\mathrm{Isom}}_{\mathscr{H}_2}(f^*(\sigma \circ h), f^*(\sigma \circ h))$ has a section, namely the identity map.

We may now use our equivalence of categories and the map $\phi_1 \times \phi_2 : B\widetilde{G} \to \mathscr{H}_1 \times_{BG} \mathscr{H}_2$ to get a map $B\widetilde{G} \to \mathscr{H}_1 \times B\mathbb{G}_m$. By the calculation in the first part of the proof, $H^1(B\widetilde{G}, \mathbb{G}_m) = 0$, so the map is given by some $\phi : B\widetilde{G} \to \mathscr{H}_1$ and the trivial torsor $B\widetilde{G} \times \mathbb{G}_m$. Since the image consists of the trivial torsor, this tells us that $\underline{\mathrm{Isom}}_{\mathscr{H}_2}(\sigma \circ \phi_1, \phi_2)$ has a section over $B\widetilde{G}$, say $\eta : (B\widetilde{G} \xrightarrow{\phi_1} \mathscr{H}_1 \xrightarrow{\sigma} \mathscr{H}_2) \to (B\widetilde{G} \xrightarrow{\phi_2} \mathscr{H}_2)$.

Recall that $\chi_1, \chi_2 : B \to \mathbb{G}_m$ describe the map on stabilizers over BG of $\phi_1 : B\widetilde{G} \to \mathscr{H}_1$ and $\phi_2 : B\widetilde{G} \to \mathscr{H}_2$, respectively. Let $b \in B$ and $T : S \to B\widetilde{G}$, and view $b : T \cong T$ as an isomorphism in $B\widetilde{G}$ over BG. Then $\chi_1(b) : \phi_1(T) \to \phi_1(T)$ and $\chi_2(b) : \phi_2(T) \to \phi_2(T)$ are isomorphisms in $\mathscr{H}_1$ and $\mathscr{H}_2$ over BG, respectively. The following diagram commutes,

$$\begin{tikzcd}
\phi_1(T) \arrow[r, "\sigma"] \arrow[d, "\chi_1(b)"'] & (\sigma \circ \phi_1)(T) \arrow[r, "\eta"] & \phi_2(T) \arrow[d, "\chi_2(b)"] \\
\phi_1(T) \arrow[r, "\sigma"] & (\sigma \circ \phi_1)(T) \arrow[r, "\eta"] & \phi_2(T),
\end{tikzcd}$$

with η and σ isomorphisms. Then, since $\mathbb{G}_m$ is abelian, we conclude that $\chi_1(b) = \chi_2(b)$, and thus our map $X(B) \to H^2(BG, \mathbb{G}_m)$ is injective. □

Remark 4.1.2. One could also formally use the spectral sequence to determine the isomorphism $X(B) \to H^2(BG, \mathbb{G}_m)$, but we have shown directly that the given map is an isomorphism.

4.2 Brauer–Severi stacks over BG

Definition 4.2.1. Let S be an algebraic stack. A *Brauer–Severi stack of relative dimension n over S* is a smooth, proper, morphism $\pi : \mathscr{X} \to S$ of relative dimension n such that there exists an étale cover $S' \to S$ where $\mathscr{X}_{S'} \cong \mathbb{P}^n_{S'}$.

Remark 4.2.2. A smooth, proper morphism $\pi : \mathscr{X} \to S$ such that for every geometric point $s : \mathrm{Spec}\ k \to S$, $\mathscr{X}_s \cong \mathbb{P}^n_k$, is a Brauer–Severi stack.

The following two propositions provide equivalent ways to view Brauer–Severi stacks in general, or Brauer–Severi stacks that are parameterized by a vector bundle.

Proposition 4.2.3. *There is an equivalence of categories between Brauer–Severi stacks of relative dimension n over S and PGL_{n+1}-torsors over S.*

Proof. Given a Brauer–Severi stack $\pi : \mathscr{X} \to S$, we define a PGL_{n+1}-torsor $I_{\mathscr{X}}$ on S to be

$$I_{\mathscr{X}}(T \to S) = \{\text{isomorphisms } \sigma : \mathscr{X} \times_S T \cong \mathbb{P}^n_T\},$$

where $\alpha \in \mathrm{PGL}_{n+1}(T) = \mathrm{Aut}(\mathbb{P}^n_T)$ acts by composition with σ. The equivalence is given by sending $\pi : \mathscr{X} \to S$ to $I_{\mathscr{X}}$, and an isomorphism of Brauer–Severi stacks $\phi : \mathscr{X} \to \mathscr{X}'$ is sent to $\phi_* : I_{\mathscr{X}} \to I_{\mathscr{X}'}$, given by

$$(\sigma : \mathscr{X} \times_S T \cong \mathbb{P}^n_T) \mapsto (\sigma \circ (\phi^{-1} \times \mathrm{id}) : \mathscr{X}' \times_S T \cong \mathscr{X} \times_S T \cong \mathbb{P}^n_T).$$

The map $\mathscr{X} \mapsto I_{\mathscr{X}}$ is fully faithful. Indeed, let $\mathscr{X}$ and $\mathscr{X}'$ be Brauer–Severi stacks of relative dimension n over S, and let $S' \to S$ be an étale covering where there exists isomorphisms $\mathscr{X} \times_S S' \cong \mathbb{P}^n_{S'}$ and $\mathscr{X}' \times_S S' \cong \mathbb{P}^n_{S'}$. Given a morphism of PGL_{n+1}-torsors $\tau : I_{\mathscr{X}} \to I_{\mathscr{X}'}$, choose an isomorphism $\sigma : \mathscr{X} \times_S S' \to \mathbb{P}^n_{S'}$, and let $\phi_{S'} : \mathscr{X} \times_S S' \to \mathscr{X}' \times_S S'$ be the composition $\tau(\sigma)^{-1} \circ \sigma$. Note that $\phi_{S'}$ does not depend on the choice of σ, since for any $\alpha \in \mathrm{PGL}_{n+1}(S') = \mathrm{Aut}(\mathbb{P}^n_{S'})$, $\tau(\alpha \circ \sigma) = \alpha \circ \tau(\sigma)$. Then $\phi_{S'}$ descends to a morphism $\phi : \mathscr{X} \to \mathscr{X}'$ which maps to τ.

Now, let $\phi, \phi' : \mathscr{X} \to \mathscr{X}'$ be two morphisms whose images in $\mathrm{Hom}(I_{\mathscr{X}}, I_{\mathscr{X}'})$ are equal. Then for all étale covers $S' \to S$ where there exists an isomorphism $\sigma : \mathscr{X} \times_S S' \to \mathbb{P}^n_{S'}$, $\sigma \circ (\phi^{-1} \times \mathrm{id}) = (\sigma \circ (\phi'^{-1} \times \mathrm{id})$, thus $\phi \times \mathrm{id} = \phi' \times \mathrm{id}$, so $\phi = \phi'$.

The map is also essentially surjective. Locally, any torsor is trivial, and clearly the Brauer–Severi stack $\pi : \mathbb{P}^n_S \to S$ maps to the trivial torsor. □

Example 4.2.4. For $n = 1$, this is the statement that the moduli stack of genus 0 curves $\mathscr{M}_0$ is isomorphic to $B\mathrm{PGL}_2$.

Proposition 4.2.5. *There is an equivalence of categories between rank $n+1$-vector bundles on S, GL_{n+1}-torsors over S, and Brauer–Severi stacks of relative dimension n together with a degree 1 line bundle over S.*

Proof. Given a rank $n+1$-vector bundle V on S, we define a GL_{n+1}-torsor I_V on S to be

$$I_V(T \to S) = \{\text{isomorphisms } \sigma : V|_T \cong \mathscr{O}_T^{\oplus n+1}\},$$

where $\alpha \in \mathrm{GL}_{n+1}(T) = \mathrm{Aut}(\mathscr{O}_T^{\oplus n+1})$ acts by composition with σ. The equivalences are given by sending V to I_V, and by sending V to $(\mathbb{P}V, \mathscr{O}_{\mathbb{P}V}(1))$. □

Definition 4.2.6. The image of the class of $\pi : \mathscr{X} \to S$ in $H^1(S, \mathrm{PGL}_{n+1})$ under the differential coming from the short exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1$$

is the class $\delta([\mathscr{X}]) \in H^2(S, \mathbb{G}_m)$, which we call the *Brauer class of the Brauer–Severi stack* $\mathscr{X} \to S$.

Proposition 4.2.7. *The Brauer class of the Brauer–Severi stack $\mathscr{X} \to S$ is given by the class $[\mathscr{G}] = \delta([\mathscr{X}]) \in H^2(S, \mathbb{G}_m)$ of the following $\mathbb{G}_m$-gerbe over S,*

$$\mathscr{G}(T \to S) = \{\textit{degree 1 line bundles on } \mathscr{X} \times_S T\}.$$

Morphisms in $\mathscr{G}$ are isomorphisms of line bundles. In particular, $\mathscr{G}$ is a trivial gerbe if and only if $\mathscr{X} \cong \mathbb{P}V$ for some vector bundle V on S.

Proof. The class $\delta([\mathscr{X}])$ is the $\mathbb{G}_m$-gerbe

$$\delta(I_{\mathscr{X}})(T \to S) = \{(P, \epsilon : P^{\wedge \mathrm{PGL}_{n+1}} \cong I_{\mathscr{X}}|_T\},$$

where P is a $\mathrm{GL}_{n+1}|_T$-torsor on T, and ϵ is an isomorphism of the pushout of P to PGL_{n+1} with $I_{\mathscr{X}}|_T$. Morphisms in $\delta(I_{\mathscr{X}})$ are isomorphisms of GL_{n+1}-torsors that are compatible with the isomorphisms with $I_{\mathscr{X}}$. By Proposition 4.2.5, we see that the data of a $\mathrm{GL}_{n+1}|_T$-torsor together with an isomorphism of its PGL_{n+1}-pushout to $I_{\mathscr{X}}$ is equivalent to the data of a rank $n+1$-vector bundle V on T together with an isomorphism $\mathbb{P}V \cong \mathscr{X} \times_S T$, which is equivalent to the data of a Brauer–Severi variety $Y \to T$ together with a degree 1 line bundle on Y and an isomorphism $Y \cong \mathscr{X} \times_S T$, which is equivalent to the data of a degree 1 line bundle on $\mathscr{X} \times_S T$.

Moreover, the following are equivalent:

(i) $\mathscr{G}$ is trivial,

(ii) $[\mathscr{G}] = \delta([\mathscr{X}]) = 0$,

(iii) there is a GL_{n+1} torsor over S whose pushout to PGL_{n+1} is $I_{\mathscr{X}} = [\mathscr{X}]$,

(iv) there is a vector bundle V on S such that $\mathscr{X} \cong \mathbb{P}V$.

□

Brauer–Severi stacks of the form $[\mathbb{P}V/G] \to BG$

From this point forward, we will only consider Brauer–Severi stacks over BG of the form $[\mathbb{P}V/G] \to BG$, where G is as in Section 4.1 and V is a representation of the universal cover $\widetilde{G}$. We will use Theorem 4.1.1 to describe their Brauer classes. Let V be a representation of $\widetilde{G}$, acting (on the left) via $\rho : \widetilde{G} \to \mathrm{GL}(V)$ such that $B \subseteq \widetilde{G}$ acts by a character $\chi : B \to \mathbb{G}_m$. Since B acts by a character, we get an induced action of G on $(V \setminus \{0\})/\mathbb{G}_m$. It will be useful to understand V and $(V \setminus \{0\})/\mathbb{G}_m$ functorially, so we introduce $\mathbb{A}_V$ and $\mathbb{P}V$ to represent these.

Let $\mathbb{A}_V$ represent the functor

$$\mathbb{A}_V : (\mathrm{Sch}/\mathrm{Spec}\ k)^{\mathrm{op}} \to \mathrm{Sets}$$

$$(T \to \mathrm{Spec}\ k) \mapsto \mathrm{Hom}_k(V \otimes_k \mathscr{O}_T, \mathscr{O}_T)$$

We have $\mathbb{A}_V(k) = \mathrm{Hom}_k(V, k) = V^\vee$, and $\mathbb{A}_V = \mathrm{Spec}\ (\mathrm{Sym}_k^\bullet V)$. Indeed,

$$\begin{aligned}(\mathrm{Spec}\ (\mathrm{Sym}_k^\bullet V))(T) &= \mathrm{Hom}(\mathrm{Sym}_k^\bullet V, \Gamma(T, \mathscr{O}_T)) \\ &= \mathrm{Hom}(\mathrm{Sym}_k^\bullet V \otimes_k \mathscr{O}_T, \mathscr{O}_T) \\ &= \mathrm{Hom}(V \otimes_k \mathscr{O}_T, \mathscr{O}_T) \\ &= \mathbb{A}_V(T).\end{aligned}$$

By 0 in $\mathbb{A}_V(k)$, we mean the zero map $V \to k$. More generally, by 0 in $\mathbb{A}_V(T)$ we mean the maps $V \otimes_k \mathscr{O}_T \to \mathscr{O}_T$ which are not surjective.

On k-points, the action of $\widetilde{G}$ on $V^\vee$ induced by the action of $\widetilde{G}$ on V is given by composition with the action map, and makes the action on $V^\vee$ into a right action. Explicitly, for $g \in \widetilde{G}$ and $f : V \to k$ in $V^\vee$, $g \cdot f : V \to k$, is given by $v \mapsto f(g \cdot v)$. For $b \in B$, $b \cdot f : V \to k$ is given by $v \mapsto f(b \cdot v) = f(\chi(b)v) = \chi(b)f(b)$, so B also acts by χ on $V^\vee$.

Remark 4.2.8. If one prefers only to work with left actions, then we can make the action of $\widetilde{G}$ on $V^\vee$ into a left action by composing with the inverse action map instead. Explicitly, $g \cdot f : V \to k$ is given by $v \mapsto f(g^{-1} \cdot v)$. In this case, B acts by χ^{-1}.

The action of $\widetilde{G}$ makes sence functorially, on $\mathbb{A}_V$, as well. Explicitly, $g \cdot (f : V \otimes_k \mathscr{O}_T \to \mathscr{O}_T)$ is the composition of f with

$$(v \mapsto g \cdot v) \otimes \mathrm{id} : V \otimes_k \mathscr{O}_T \to V \otimes_k \mathscr{O}_T.$$

Let $\mathbb{P}V$ represent the functor

$$\mathbb{P}V : (\mathrm{Sch}/\mathrm{Spec}\ k)^{\mathrm{op}} \to \mathrm{Sets}$$

$$(T \to \mathrm{Spec}\ k) \mapsto \{V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{L}\}/\sim$$

where two surjections $(V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{L}_1)$ and $(V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{L}_2)$ are equivalent if there is an isomorphism $s : \mathscr{L}_1 \to \mathscr{L}_2$ making the diagram

$$\begin{tikzcd} V \otimes_k \mathscr{O}_T \arrow[r] \arrow[dr] & \mathscr{L}_1 \arrow[d, "s"] \\ & \mathscr{L}_2 \end{tikzcd}$$

commute.

Note that $\mathbb{P}V$ is the sheafification of the functor

$$(\mathbb{P}V)^{\mathrm{pre}} : (\mathrm{Sch}/\mathrm{Spec}\ k)^{\mathrm{op}} \to \mathrm{Sets}$$

$$(T \to \mathrm{Spec}\ k) \mapsto \{V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{O}_T\}/\sim$$

with the same equivalence relation. That is, $\mathbb{P}V \cong (\mathbb{A}_V \setminus \{0\})/\mathbb{G}_m$.

The action of $\widetilde{G}$ on $\mathbb{P}V$ is similarly given by composition with the action on V. Since B acts by a character, B acts trivially on the quotient, and thus the action of $\widetilde{G}$ descends to an action of G on $\mathbb{P}V$.

Consider the stack quotient $[\mathbb{P}V/G] \to BG$, which is a Brauer–Severi stack over BG, since pulling back along $\mathrm{Spec}\ k \to BG$ gives $\mathbb{P}V \to \mathrm{Spec}\ k$. The Brauer class of $[\mathbb{P}V/G]$ is given by the $\mathbb{G}_m$-gerbe $\mathscr{G}$ of degree 1 line bundles on $[\mathbb{P}V/G]$ over BG,

$$\mathscr{G}(S \to BG) = \{\text{degree 1 line bundles on } S \times_{BG} [\mathbb{P}V/G] =: \mathbb{P}_S\}. \tag{4.2.8.1}$$

Lemma 4.2.9. *The fiber product $\mathbb{P}_S := S \times_{BG} [\mathbb{P}V/G]$ is given by $T \times^G \mathbb{P}V := (T \times \mathbb{P}V)/G$, where $T \to S$ is the G-torsor corresponding to $S \to BG$ and G acts on $T \times \mathbb{P}V$ via $g \cdot (t, v) = (g \cdot t, g \cdot v)$.*

Proof. The objects of the fiber product $\mathbb{P}_S$ can be described as tuples

$$(S' \to S,\ T' \to S',\ \varphi : T' \to \mathbb{P}V,\ \sigma : T \times_S S' \cong T')$$

where S' is an S-scheme, $T' \to S'$ is a G-torsor, φ is a G-equivariant morphism, and σ an isomorphism of G-torsors over S'.

Morphisms between two tuples $(S' \to S,\ T' \to S',\ \varphi,\ \sigma) \to (S'' \to S,\ T'' \to S'',\ \varphi'',\ \sigma'')$ are given by pairs (s, t), where $s : S' \to S''$ is a morphism of S-schemes and $t : T' \to T''$ is a G-equivariant morphism, such that the following diagrams commute,

$$\begin{tikzcd} T' \arrow[rr, "t"] \arrow[dr, "\varphi"'] & & T'' \arrow[dl, "\varphi''"] \\ & \mathbb{P}V, & \end{tikzcd} \qquad \begin{tikzcd} T \times_S S' \arrow[r, "\sigma"] \arrow[d, "\mathrm{id}\times s"'] & T' \arrow[d, "t"] \\ T \times_S S'' \arrow[r, "\sigma''"] & T''. \end{tikzcd}$$

Note that due to the second compatibility condition, t is completely determined by $t = \sigma'' \circ (\mathrm{id} \times s) \circ \sigma^{-1}$.

We can see that this data is equivalent to the stack with objects given by pairs

$$(S' \to S, \phi : T \times_S S' \to \mathbb{P}V)$$

where S' is an S-scheme and ϕ is a G-equivariant morphism, and morphisms between two tuples $(S' \to S,\ \phi) \to (S'' \to S, \phi'')$ are morphisms of S-schemes $s : S' \to S''$ such that the diagram commutes,

$$\begin{tikzcd} T \times_S S' \arrow[rr, "\mathrm{id}\times s"] \arrow[dr, "\phi"'] & & T \times_S S'' \arrow[dl, "\phi''"] \\ & \mathbb{P}V. & \end{tikzcd}$$

Indeed, an equivalence is given by sending $(S' \to S, T' \to S', \varphi, \sigma)$ to $(S' \to S, \varphi \circ \sigma)$, and on morphisms, $(s, t) \mapsto s$.

This is fully faithful, since t is determined as remarked above, and essentially surjective since $(S' \to S, T \times_S S' \to S', \phi, \mathrm{id}) \mapsto (S' \to S, \phi)$.

Now we define an equivalence between $T \times^G \mathbb{P}V$ and $\mathbb{P}_S$.

First, consider the G-torsor $T \times \mathbb{P}V \to T \times^G \mathbb{P}V$. With the action of G on $T \times \mathbb{P}V$ as described above, we get both projection maps, from $T \times \mathbb{P}V$ to T and $\mathbb{P}V$, to be G-equivariant morphisms. Taking the quotient of $T \times \mathbb{P}V \to T$ by G gives the following cartesian square,

$$\begin{tikzcd} T \times \mathbb{P}V \arrow[r] \arrow[d] & T \arrow[d] \\ T \times^G \mathbb{P}V \arrow[r] & S. \end{tikzcd}$$

Given a morphism $S' \to T \times^G \mathbb{P}V$, compose with the map to S to get $s : S' \to S$. The fiber product $S' \times_{T \times^G \mathbb{P}V} (T \times \mathbb{P}V)$ is given by $T \times_S S'$ due to the commutativity of the above square, and we get a G-equivariant map, $T \times_S S' \to T \times \mathbb{P}V \to \mathbb{P}V$. Indeed, since $T \times_S S' \to T$ and $T \times \mathbb{P}V \to T$ are both G-equivariant, $T \times_S S' \to T \times \mathbb{P}V$ is G-equivariant, and thus the composition to $\phi : T \times_S S' \to \mathbb{P}V$ is as well.

Conversely, given $s : S' \to S$ and $\phi : T \times_S S' \to \mathbb{P}V$, take the quotient of $\mathrm{pr}_1 \times \phi : T \times_S S' \to T \times \mathbb{P}V$ by the action of G to get $S' \to T \times^G \mathbb{P}V$. These are clearly inverses of each other. $\square$

Remark 4.2.10. One can think of the category of G-equivariant morphisms $T \to \mathbb{P}V$ as pairs (t, v) of input and output, which are well-defined up to the equivalence $(g \cdot t, g \cdot v) \sim (t, v)$, coinciding with the local description of $T \times^G \mathbb{P}V$.

Remark 4.2.11. The following commutative diagram can also be used as a justification of Lemma 4.2.9,

$$\begin{tikzcd}
T \times \mathbb{P}V \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{P}V \arrow[dr] \arrow[dd] & \\
& T \arrow[rr] \arrow[dd] & & \mathrm{Spec}\ k \arrow[dd] \\
\mathbb{P}_S \arrow[rr] \arrow[dr] & & {[\mathbb{P}V/G]} \arrow[dr] & \\
& S \arrow[rr] & & BG.
\end{tikzcd}$$

Lemma 4.2.12. *$\mathbb{A}_V \setminus \{0\} \to \mathbb{P}V$ is a $\mathbb{G}_m$-torsor, corresponding to the line bundle $\mathscr{O}_{\mathbb{P}V}(1)$ on $\mathbb{P}V$, that is,*

$$\mathbb{A}_V \setminus \{0\} = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}V}(1)) \setminus \{0\} := \underline{\mathrm{Spec}}_{\mathbb{P}V}(\mathrm{Sym}^\bullet_k \mathscr{O}_{\mathbb{P}V}(-1)) \setminus \{0\}.$$

Proof. The map $\mathbb{A}_V \setminus \{0\} \to \mathbb{P}V$ is given by

$$(V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{O}_T) \mapsto [V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{O}_T]$$

and objects of $\mathbb{A}_V \setminus \{0\}$ over a point $f : T \to \mathbb{P}V$, corresponding to $\varphi : V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{L}$ pulled back from the universal quotient $V \otimes_k \mathscr{O}_{\mathbb{P}V} \twoheadrightarrow \mathscr{O}_{\mathbb{P}V}(1)$, are given by

$$(\mathbb{A}_V \setminus \{0\})(T \to \mathbb{P}V) = \left\{ \phi : V \otimes_k \mathscr{O}_T \twoheadrightarrow \mathscr{O}_T | \exists s : \begin{tikzcd} V \otimes_k \mathscr{O}_T \arrow[r, "\phi"] \arrow[dr, "\varphi"'] & \mathscr{O}_T \arrow[d, "s"] \\ & \mathscr{L} \end{tikzcd} \text{ commutes} \right\}.$$

This data is equivalent to the set of isomorphisms $\{s : \mathscr{O}_T \cong \mathscr{L}\}$.

Objects of $\mathrm{Tot}(\mathscr{O}_{\mathbb{P}V}(1)) \setminus \{0\}$ over a point $f : T \to \mathbb{P}V$ are given by

$$\left(\underline{\mathrm{Spec}}_{\mathbb{P}V}(\mathrm{Sym}^\bullet_k \mathscr{O}_{\mathbb{P}V}(-1)) \setminus \{0\}\right)(T \to \mathbb{P}V)$$

$$= \{\mathrm{Sym}^\bullet \mathscr{L}^\vee \twoheadrightarrow \mathscr{O}_T\} \cong \{\mathscr{L}^\vee \twoheadrightarrow \mathscr{O}_T\} \cong \{\mathscr{O}_T \cong \mathscr{L}\}.$$

Thus $\mathbb{A}_V \setminus \{0\} = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}V}(1)) \setminus \{0\}$. □

Theorem 4.2.13. *Let $G, \widetilde{G}, B, V, \chi$ as in Section 4.2 and $\mathscr{G}$ as in Equation 4.2.8.1. Then $\chi_*^{-1}(B\widetilde{G}) = \mathscr{G}$, and thus by Theorem 4.1.1, the class of $\mathscr{G}$ in $H^2(BG, \mathbb{G}_m) = X(B)$ is given by χ^{-1}.*

Proof. We define a morphism of stacks $B\widetilde{G} \to \mathscr{G}$ over BG and show that the morphism on stabilizers is given by χ^{-1}. Given $S \to B\widetilde{G}$ corresponding to a $\widetilde{G}$-torsor $\widetilde{T} \to S$, we must assign this to a degree 1 line bundle on $\mathbb{P}_S = T \times^G \mathbb{P}V$, where T is the image of $\widetilde{T}$ in BG. Let its image in $\mathscr{G}$ be the line bundle corresponding to the $\mathbb{G}_m$-torsor

$$\widetilde{T} \times^{\widetilde{G}} \mathbb{A}_V \setminus \{0\} \to T \times^G \mathbb{P}V = \widetilde{T} \times^{\widetilde{G}} \mathbb{P}V.$$

Note that this map appears in the cartesian diagram

$$\begin{tikzcd}
\widetilde{T} \times^{\widetilde{G}} (\mathbb{A}_V \setminus \{0\}) \arrow[r] \arrow[d] & {[\mathbb{A}_V \setminus \{0\}/\widetilde{G}]} \arrow[d] & \\
T \times^G \mathbb{P}V = \widetilde{T} \times^{\widetilde{G}} \mathbb{P}V \arrow[r] \arrow[d] & {[\mathbb{P}V/\widetilde{G}]} \arrow[r] \arrow[d] & {[\mathbb{P}V/G]} \arrow[d] \\
S \arrow[r] & B\widetilde{G} \arrow[r] & BG.
\end{tikzcd}$$

A morphism $b : \widetilde{T} \to \widetilde{T}'$ of $\widetilde{G}$-torsors whose pushouts to G agree is sent to the morphism

$$\widetilde{T} \times^{\widetilde{G}} (\mathbb{A}_V \setminus \{0\}) \to \widetilde{T}' \times^{\widetilde{G}} (\mathbb{A}_V \setminus \{0\}),$$

which we describe on the cover by

$$\widetilde{T} \times (\mathbb{A}_V \setminus \{0\}) \to \widetilde{T}' \times (\mathbb{A}_V \setminus \{0\}), \quad (t, v) \mapsto (b \cdot t, v).$$

Upon taking the quotient by the $\widetilde{G}$-action, the map is $[(t, v)] \mapsto [(b \cdot t, v)] = [(t, b^{-1} \cdot v)] = [(t, \chi(b)^{-1} v)]$.

Thus, the morphism on automorphism groups $B \to \mathbb{G}_m$ for $B\widetilde{G} \to \mathscr{G}$ over BG is given by χ^{-1}, so $\chi_*^{-1}(B\widetilde{G}) = \mathscr{G}$. □

More generally, let $V_1, \ldots, V_r$ be representations of $\widetilde{G}$, acting via $\rho_i : \widetilde{G} \to \mathrm{GL}(V_i)$ such that B acts by characters χ_i, for $i = 1, \ldots, r$. Again the action descends to $\mathbb{P}V_i$. Consider

$$[(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r)/G] \to BG,$$

and the $\mathbb{G}_m$-gerbe over BG,

$$\mathscr{G}^{a_1,\ldots,a_r}(S \to BG) = \left\{ \begin{array}{c} \mathscr{L} \text{ on } S \times_{BG} [\mathbb{P}V_1 \times \cdots \mathbb{P}V_r/G] = \mathbb{P}_S \text{ such that} \\ \forall s : \mathrm{Spec}\, \overline{k} \to S,\ \mathscr{L}|_{\mathbb{P}_s} \cong \mathscr{O}(a_1, \ldots, a_r) \end{array} \right\}.$$

Here $\mathscr{O}(a_1, \dots, a_r)$ is the line bundle on $\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r$ given by the tensor product of the pullbacks of $\mathscr{O}(a_i)$ on $\mathbb{P}V_i$. To simplify notation, we will write $\mathscr{O}(a_1, \dots, a_r)$ on $\mathbb{P}_S$ for such a line bundle.

Corollary 4.2.14. *Let $\psi = \sum_{i=1}^r a_i \chi_i^{-1}$. Then $\psi_*(B\widetilde{G}) = \mathscr{G}^{a_1,\dots,a_r}$, so the class of $\mathscr{G}^{a_1,\dots,a_r}$ is given by $\psi \in \widehat{B}$.*

Proof. Using the result from Theorem 4.2.13, we only need to show that $\mathscr{G}^{a_1,\dots,a_r}$ is the product of $\mathbb{G}_m$-gerbes $\mathscr{G}^{a_1}, \dots, \mathscr{G}^{a_r}$, to be defined below, and each $\mathscr{G}^{a_i}$ is the product $a_i\mathscr{G}_i$, where $\mathscr{G}_i$ is the $\mathbb{G}_m$-gerbe

$$\mathscr{G}_i(S \to BG) = \{\mathscr{O}(1) \text{ on } S \times_{BG} [\mathbb{P}V_i/G]\}.$$

as in Theorem 4.2.13.

Let $\mathscr{G}^{a_i}$ be the following $\mathbb{G}_m$-gerbe over BG,

$$\mathscr{G}^{a_i}(S \to BG) = \{\mathscr{O}(a_i) \text{ on } S \times_{BG} [\mathbb{P}V_i/G]\}.$$

Note that

$$[\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r/G] \cong [\mathbb{P}V_1/G] \times_{BG} \cdots \times_{BG} [\mathbb{P}V_r/G],$$

and let $p_i : [\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r/G] \to [\mathbb{P}V_i/G]$ be the projection maps. To show that $\mathscr{G}^{a_1,\dots,a_r} \cong \mathscr{G}^{a_1} \cdots \mathscr{G}^{a_r}$, we define a morphism of gerbes

$$\mathscr{G}^{a_1} \times \cdots \times \mathscr{G}^{a_r} \to \mathscr{G}^{a_1,\dots,a_r}$$

such that the map on stabilizers is the product map $\mathbb{G}_m \times \cdots \times \mathbb{G}_m \to \mathbb{G}_m$. The map is given by

$$(\mathscr{L}_1, \dots, \mathscr{L}_r) \mapsto p_1^*\mathscr{L}_1 \otimes \cdots \otimes p_r^*\mathscr{L}_r.$$

Similarly, to show that $\mathscr{G}^{a_i} \cong a_i\mathscr{G}_i$, we define a morphism $\mathscr{G}_i \times \cdots \times \mathscr{G}_i \to \mathscr{G}^{a_i}$, with a_i copies of $\mathscr{G}_i$, such that the map on stabilizers is the product map. The map is given by

$$(\mathscr{L}_1, \dots, \mathscr{L}_{a_i}) \mapsto \mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_{a_i}. \qquad \square$$

Example 4.2.15. Let $G = \mathrm{PGL}_2$, so $\widetilde{G} = \mathrm{SL}_2$, and $B = \mu_2$. Let V be the standard representation of G, so $V = k^{\oplus 2}$, and SL_2 acts via $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}$. Then μ_2 acts by the standard character, and the action descends to an action of PGL_2 on $\mathbb{P}V$. The quotient $[\mathbb{P}V/\mathrm{PGL}_2] \to B\mathrm{PGL}_2$ is the universal curve over the moduli of genus 0 curves (See Proposition 4.3.3, (2)). Let $\mathscr{G}$ be the $\mathbb{G}_m$-gerbe of degree 1 line bundles on $[\mathbb{P}V/\mathrm{PGL}_2]$ over $B\mathrm{PGL}_2$,

$$\mathscr{G}(S \to B\mathrm{PGL}_2) = \{\text{degree 1 line bundles on } S \times_{B\mathrm{PGL}_2} [\mathbb{P}V/\mathrm{PGL}_2] =: \mathbb{P}_S\},$$

which we recognize as $B\mathrm{GL}_2$. Theorem 4.2.13 confirms that $\chi_*(B\mathrm{SL}_2) = \mathscr{G} \cong B\mathrm{GL}_2$, where $\chi : \mu_2 \to \mathbb{G}_m$ is the standard character ($\chi = \chi^{-1}$ here). Note that $B\mathrm{GL}_2$ is indeed a nontrivial gerbe, due to the existence of genus 0 curves over non-algebraically closed fields without line bundles of degree 1.

Example 4.2.16. The next simplest case to understand is $G = \mathrm{PGL}_2$, as above, and $V = \mathrm{Sym}^d(k^{\oplus 2}) = H^0(\mathbb{P}k^{\oplus 2}, \mathscr{O}(d)) \cong k^{\oplus d+1}$. We can think of $\mathbb{P}V$ as the space parameterizing effective degree d divisors on $\mathbb{P}k^{\oplus 2} \cong \mathbb{P}^1$, coming from the vanishing of homogeneous degree d polynomials in two variables, with $d+1$ coefficients, giving a unique polynomial up to scalar multiple. The action of μ_2 on V is induced by the action on $k^{\oplus 2}$ and is given by $(-1)^d$.

The action descends to an action of PGL_2 on $\mathbb{P}V$, and the quotient, $[\mathbb{P}V/\ \mathrm{PGL}_2] \to B\mathrm{PGL}_2$, is the moduli of degree d divisors on genus 0 curves (See Proposition 4.3.3 (4)). Let $\mathscr{G}_d$ be the $\mathbb{G}_m$-gerbe of degree 1 line bundle on $[\mathbb{P}V/\mathrm{PGL}_2]$ over $B\mathrm{PGL}_2$,

$$\begin{aligned}\mathscr{G}_d(S \to B\mathrm{PGL}_2) &= \{\text{degree 1 line bundles on } S \times_{B\mathrm{PGL}_2} [\mathbb{P}\mathrm{Sym}^d(k^{\oplus 2})/\mathrm{PGL}_2]\} \\ &\cong \{\text{degree } d \text{ line bundles on } S \times_{B\mathrm{PGL}_2} [\mathbb{P}k^{\oplus 2}/\mathrm{PGL}_2]\} \\ &\cong d\{\text{degree 1 line bundles on } S \times_{B\mathrm{PGL}_2} [\mathbb{P}k^{\oplus 2}/\mathrm{PGL}_2]\}.\end{aligned}$$

For the first isomorphism, note that $S \times_{B\mathrm{PGL}_2} [\mathbb{P}\mathrm{Sym}^d(k^{\oplus 2})/\mathrm{PGL}_2] = \mathrm{Sym}^d_{\mathbb{P}_S/S}$, where $\mathbb{P}_S = S \times_{B\mathrm{PGL}_2} [\mathbb{P}k^{\oplus 2}/\mathrm{PGL}_2]$. Thus $[\mathscr{G}_d] = d[\mathscr{G}] = [d] \in \mathbb{Z}/2\mathbb{Z}$, where $\mathscr{G}$ is as in the previous example. For d even, $\mathscr{G}_d$ is a trivial gerbe, corresponding to the trivial character of μ_2, and for d odd, $\mathscr{G}_d$ is a nontrivial gerbe, corresponding to the standard character of μ_2. This is equivalent to the statement that a genus 0 curve may not have a divisor of odd degree, but it will always have a divisor of even degree, equal to some power of the canonical divisor.

We conclude this section by recording a theorem of Gabber which also applies to algebraic stacks. While the proof is essentially the same as previously given, we could not find a reference for the statement generalized to stacks, so we include it here and give a sketch of its proof.

Theorem 4.2.17. *([19, Theorem 2, page 193], extended to stacks) Let S be a stack, $\pi : \mathscr{X} \to S$ a Brauer–Severi stack. The following sequence is exact,*

$$H^0_{\text{ét}}(S, \mathbb{Z}) \longrightarrow H^2(S, \mathbb{G}_m)_{\text{tors}} \xrightarrow{\pi^*} H^2(\mathscr{X}, \mathbb{G}_m)_{\text{tors}} \longrightarrow 0,$$

where the first maps sends $1 \mapsto -\delta([\mathscr{X}])$.

Proof. Gabber uses the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$ to obtain the exact sequence in the theorem. The calculations that $\pi_*\mathbb{G}_m = \mathbb{G}_m$, $R^1\pi_*\mathbb{G}_m = \mathbb{Z}$ and $R^2\pi_*\mathbb{G}_m = 0$ in Gabber's proof apply to Brauer–Severi stacks, because these are computed étale locally. $\square$

4.3 Application: The moduli stack of curves of genus g with an automorphism of order N

Let $A = (g, N, d_1, \ldots, d_{N-1})$ be an $N+1$-tuple such that $g, d_i \geq 0$, $N \mid d := \sum id_i$ and

$$\widetilde{g} := \frac{1}{2}\left(N(2g-2) + \sum_i d_i(N - \gcd(i, N)) + 2\right)$$

is an integer greater than 1. It is then a *$\widetilde{g}$-admissible datum* for some $\widetilde{g} > 1$, as defined in the introduction. Define the stack $\mathscr{M}_A$ to be the stack with objects

$$\mathscr{M}_A(S) = \{(f : C \to S, D_1, \dots D_{N-1}, \mathscr{L}, \phi)\}$$

where $f : C \to S$ is a smooth genus g curve, the D_i are disjoint divisors of degree d_i, sections of $(\mathrm{Sym}^{d_i}_{C/S} \setminus \Delta_{d_i}) \to S$, where Δ_{d_i} is the big diagonal, $\mathscr{L}$ is a line bundle on C and $\phi : \mathscr{L}^{\otimes N} \to \mathscr{O}_C(\sum iD_i)$ is an isomorphism.

A morphism

$$(\sigma, \tau) : (f : C \to S, D_i, \mathscr{L}, \phi) \to (f' : C' \to S, D'_i, \mathscr{L}', \phi')$$

is a pair of isomorphisms $\sigma : C \to C'$ and $\tau : \sigma^*\mathscr{L}' \to \mathscr{L}$ such that $\sigma^*(D'_i) = D_i$ and the following diagram commutes:

$$\begin{tikzcd}
\sigma^*(\mathscr{L}'^{\otimes N}) \arrow[r, "\sigma^*(\phi')"] \arrow[d, "\tau^{\otimes N}"] & \sigma^*\mathscr{O}_{C'}(\sum iD'_i) \arrow[d] \\
\mathscr{L}^{\otimes N} \arrow[r, "\phi"] & \mathscr{O}_C(\sum iD_i).
\end{tikzcd}$$

Following [42], we roughly explain the correspondence between μ_N-covers $\pi : X \to C$ and objects $(C, D_i, \mathscr{L}, \phi)$ of $\mathscr{M}_A$.

For any abelian cover $\pi : X \to Y$ of smooth complete varieties with group G, let D be the locus of Y where π is ramified. Then $R := \pi^{-1}(D)$ is the locus on X whose points are stabilized by a subgroup of G not equal to the identity. For a closed point T of R, let $H \leq G$ be its stabilizer. Let $\mathscr{O}_{X,T}$ be its local ring with maximal ideal $\mathfrak{m}$. Since H fixes T, we get a faithful representation ψ of H on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$. Now we can split D into disjoint components $D_{H,\psi}$ according to where $\pi^{-1}(D_{H,\psi})$ has stabilizer H and representation ψ. Since π is Galois, all points in the same fiber of π will have the same stabilizer group H and representation ψ.

In the case where X and Y are curves, $\mathfrak{m}/\mathfrak{m}^2$ has dimension 1. Thus, any representation of H is a character of H. Moreover, if additionally G is cyclic of order N, we have a bijection

$$\{1, \dots, N-1\} \to \{(H, \psi) : \langle 1 \rangle \neq H \leq G = \mu_N, \psi \in \widehat{H}\}$$

given by $m \mapsto \left(\langle \zeta_N^m \rangle \subseteq \mu_N, \psi : \zeta_N^m \mapsto \zeta_N^{\gcd(m,N)}\right)$. Because of this, we can label the divisors $D_1, \dots D_{N-1}$. Moreover, the pushforward $\pi_*\mathscr{O}_X$ of the structure sheaf can be broken into eigensheaves $\pi_*\mathscr{O}_X = \oplus_{i=0}^{N-1}\mathscr{L}_i^{-1}$, where the $\mathscr{L}_i^{-1}$ are line bundles such that μ_N acts on $\mathscr{L}_i^{-1}$ by ζ_N^i, and we have an isomorphism $\mathscr{L}_1^{\otimes N} \cong \mathscr{O}_Y(\sum iD_i)$.

Conversely, given $(C, D_1, \dots, D_{N-1}, \mathscr{L}, \phi)$, the cover X is given by $X = \underline{\mathrm{Spec}}_C(\mathscr{O}_C \oplus \mathscr{L}_1^{-1} \oplus \cdots \oplus \mathscr{L}_{N-1}^{-1})$, where $\mathscr{L}_1 = \mathscr{L}$ and the remaining $\mathscr{L}_i$ are determined by a formula involving the divisors, and the algebra structure of $\mathscr{O}_C \oplus \mathscr{L}_1^{-1} \oplus \cdots \oplus \mathscr{L}_{N-1}^{-1}$ is determined by ϕ. This is worked out in detail for μ_N-covers of curves in [41], using the general theory in [42].

Given a $\widetilde{g}$-admissible datum $(g, N, d_i, \ldots, d_{N-1})$, the condition for whether a corresponding covering curve is connected is as follows. Let k be the greatest common divisor of N and all of the i such that $d_i \neq 0$. Then the covering curve is connected if and only if $\mathscr{L}^{\otimes N/k} \otimes \mathscr{O}_C(-\sum \frac{i}{k} D_i)$ has exactly order k in the Picard group of C ([41, Remark 2.11]). Note that if C has genus 0, then this line bundle always has order 1, and thus when $k = 1$, all covers are connected and $\mathscr{M}'_A = \mathscr{M}_A$, and when $k > 1$, all covers are disconnected and $\mathscr{M}'_A = \emptyset$.

Thus, to study the $\mathscr{M}'_A$ with $g = 0$, it suffices to study the $\mathscr{M}_A$ with $k = 1$, where k is defined as above. However, this condition on k does not affect our study of the $\mathscr{M}_A$, so we ignore it.

Let $\overline{\mathscr{M}_A}$ be the stack with objects

$$\overline{\mathscr{M}_A}(S) = \{(f : C \to S, D_1, \ldots D_{N-1}, \mathscr{L}, \phi)\}$$

where $f : C \to S$ is a smooth genus g curve, the D_i are divisors of degree d_i, sections of $\mathrm{Sym}^{d_i}_{C/S} \to S$, $\mathscr{L}$ is a line bundle on C and $\phi : \mathscr{L}^{\otimes N} \to \mathscr{O}_C(\sum iD_i)$ is an isomorphism. That is, the divisors are no longer required to be disjoint. Morphisms are defined to be the same as for $\mathscr{M}_A$.

We next define a stack $\overline{\mathscr{N}_A}$, and show that it is the μ_N-rigidification of $\overline{\mathscr{M}_A}$. First we include a bit of background about the Picard stack and Picard scheme.

Picard stack and Picard scheme

Let S be a scheme and let $f : X \to S$ be flat and proper. The Picard stack ([7, Appendix]) is the stack $\mathscr{P}ic_{X/S}$ whose objects are pairs $(T \to S, \mathscr{L})$, where T is an S-scheme and $\mathscr{L}$ is a line bundle on $X_T = X \times_S T$. Morphisms are pairs $(t, \sigma) : (T \to S, \mathscr{L}) \to (T' \to S, \mathscr{L}')$ where $t : T \to T'$ is a morphism of S-schemes and $\sigma : t_X^* \mathscr{L}' \to \mathscr{L}$ is an isomorphism of line bundles, where $t_X : X_T \to X_{T'}$.

Let $g : X \to X'$ be an S-morphism, and $g_T : X_T \to X'_T$. Define

$$g^* : \mathscr{P}ic_{X'/S} \to \mathscr{P}ic_{X/S}$$

by $(T \to S, \mathscr{L}) \mapsto (T \to S, g_T^* \mathscr{L})$.

We now define the relative Picard functor and Picard scheme, following [29]. Let f : $X \to S$ be separated and of finite type.

The relative Picard functor is

$$\mathrm{Pic}^{\mathrm{pre}}_{X/S} : (\mathrm{Sch}/S) \to \mathrm{Groups}$$

$$(T \to S) \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T).$$

In some cases, this is representable, but in general it is not. Let $\mathrm{Pic}_{X/S}$ denote the fppf sheafification.

Then an object of $\mathrm{Pic}_{X/S}(T \to S)$ is represented by a pair $(T' \to T, \mathscr{L}')$ of an fppf covering $T' \to T$ and a line bundle $\mathscr{L}'$ on $X_{T'}$ such that there exists an fppf covering $T'' \to T' \times_T T'$ where the pullbacks of $\mathscr{L}'$ to T'' under the two projections are isomorphic. We will call such objects *isomorphism classes of line bundles on* X_T. Two representations $(T'_1 \to T, \mathscr{L}'_1)$ and $(T'_2 \to T, \mathscr{L}'_2)$ are equivalent if there exists an fppf covering $T' \to T'_1 \times_T T'_2$ such that $\mathscr{L}'_1$ and $\mathscr{L}'_2$ become isomorphic when pulled back to $X_{T'}$.

Let $g : X \to X'$ as above, and define

$$g^* : \mathrm{Pic}_{X'/S} \to \mathrm{Pic}_{X/S}$$

by $(T' \to T, \mathscr{L}') \mapsto (T' \to T, g^*_{T'}\mathscr{L}')$.

Now suppose that $f_*\mathscr{O}_X = \mathscr{O}_S$ and f is flat, proper, and cohomologically flat in dimension 0, that is, the formation of $f_*\mathscr{O}_X$ commutes with base change. When $f_*\mathscr{O}_X \cong \mathscr{O}_S$ holds universally and f has a section étale locally, the étale sheafification and the fppf sheafification agree. In particular, when f is smooth, f has an étale local section. We have a morphism $[-] : \mathscr{P}ic_{X/S} \to \mathrm{Pic}_{X/S}$ sending a line bundle $\mathscr{L}$ on X_T to the pair $(T \to T, \mathscr{L})$, which we denote by $[\mathscr{L}]$.

Lemma 4.3.1. *Let $f : X \to S$ be flat, proper, and such that $f_*\mathscr{O}_X = \mathscr{O}_S$ holds universally. Then $[-] : \mathscr{P}ic_{X/S} \to \mathrm{Pic}_{X/S}$ realizes the Picard stack as a $\mathbb{G}_m$-gerbe over the Picard scheme.*

Proof. By descent for $\mathscr{P}ic_{X/S}$, for any fppf covering $T' \to T$, the category $\mathscr{P}ic_{X/S}(T)$ is equivalent to $\mathscr{P}ic_{X/S}(T' \to T)$, which has objects pairs $(\mathscr{L}', \sigma : \mathrm{pr}_1^*\mathscr{L}' \cong \mathrm{pr}_2^*\mathscr{L}')$, where $\mathscr{L}'$ is a line bundle on $X_{T'}$ and σ is an isomorphism of line bundles on $X_{T' \times_T T'}$ which satisfies the cocycle condition.

Given $(T' \to T, \mathscr{L}') \in \mathrm{Pic}_{X/S}(T)$, we have some fppf cover $T'' \to T' \times_T T'$ such that the two pullbacks of $\mathscr{L}'$ to T'' are isomorphic. Let $(T' \times_T T'' \to T'', \mathscr{L}'')$ denote the pullback of $(T' \to T, \mathscr{L}')$ to T''. The two pullbacks of $\mathscr{L}''$ to $(T' \times_T T'') \times_{T''} (T' \times_T T'')$ agree, so descent gives a line bundle $\mathscr{L}$ on $X_{T''}$. Given $\mathscr{L}_1, \mathscr{L}_2 \in \mathscr{P}ic_{X/S}(T)$ such that $(T \to T, \mathscr{L}_1)$ and $(T \to T, \mathscr{L}_2)$ are equivalent in $\mathrm{Pic}_{X/S}(T)$, there is an fppf cover $T' \to T$ such that $\mathscr{L}_1|_{T'} \cong \mathscr{L}_2|_{T'}$. Since $f_*\mathscr{O}_X = \mathscr{O}_S$ holds universally, $H^0(X_T, \mathscr{O}_{X_T}^\times) = \mathscr{O}_T^\times$, and automorphisms of $\mathscr{L} \in \mathscr{P}ic_{X/S}(T)$ over $[\mathscr{L}]$ are given by $\mathbb{G}_{m,T}$. □

Let $\mathscr{P}ic^d_{X/S}$ be the open and closed substack of $\mathscr{P}ic_{X/S}$ consisting of line bundles of relative degree d, and similarly let $\mathrm{Pic}^d_{X/S}$ be the open and closed subscheme of $\mathrm{Pic}_{X/S}$ consisting of isomorphism classes of line bundles of relative degree d. We again have a $\mathbb{G}_m$-gerbe $[-] : \mathscr{P}ic^d_{X/S} \to \mathrm{Pic}^d_{X/S}$.

In what follows, we will denote elements $(T' \to T, \mathscr{L}')$ of $\mathrm{Pic}_{X/S}(T)$ by a single symbol, ℓ. By ℓ^N, we mean the element $(T' \to T, \mathscr{L}'^{\otimes N})$.

Rigidification of $\overline{\mathscr{M}_A}$

Define the stack $\overline{\mathscr{N}_A}$ to be the stack with objects

$$\overline{\mathscr{N}_A}(S) = \left\{ (f : C \to S, D_1, \ldots, D_{N-1}, \ell) \middle| \ell^N = \left[\mathscr{O}_C \left(\sum_i iD_i \right) \right] \right\}$$

where $f : C \to S$ and D_i are as in the definition of $\overline{\mathscr{M}_A}$, and $\ell \in \mathrm{Pic}^{d/N}_{C/S}(S)$.

A morphism

$$\sigma : (f : C \to S, D_i, \ell) \to (f' : C' \to S, D_i', \ell')$$

is an isomorphism $\sigma : C \to C'$ of S-schemes such that $\sigma^*(D_i') = D_i$ and $\sigma^*\ell' = \ell$.

We have a natural map $\overline{\mathscr{M}_A} \to \overline{\mathscr{N}_A}$ given by

$$(f : C \to S, D_i, \mathscr{L}, \phi) \mapsto (f : C \to S, D_i, [\mathscr{L}]).$$

Proposition 4.3.2. *$\overline{\mathscr{M}_A} \to \overline{\mathscr{N}_A}$ is a μ_N-gerbe and thus $\overline{\mathscr{N}_A}$ is the μ_N-rigidification of $\overline{\mathscr{M}_A}$.*

Proof. Let $(f : C \to S, D_i, \ell) \in \overline{\mathscr{N}_A}(S)$. Then ℓ is a pair $(S' \to S, \mathscr{L}')$ such that there is an fppf cover $S_1 \to S' \times_S S'$ where the pullbacks of $\mathscr{L}'$ agree. Moreover, that $\ell^N = [\mathscr{O}_C(\sum_i iD_i)]$ means there exists an fppf cover $S_2 \to S' \times_S S = S'$ where the pullbacks of $\mathscr{L}'^{\otimes N}$ and $\mathscr{O}_C(\sum_i iD_i)$ agree. Taking a mutual covering of S_1 and S_2, say $S'' = S_1 \times_S S_2$ we get a line bundle $\mathscr{L}$ on $C_{S''}$, which is the pullback of $\mathscr{L}'$, and an isomorphism $\phi : \mathscr{L}^{\otimes N} \to \mathscr{O}_{C_{S''}}(\sum_i iD_i)$.

Let $(f : C \to S, D_i, \mathscr{L}_1, \phi_1)$ and $(f : C \to S, D_i, \mathscr{L}_2, \phi_2)$ be two elements of $\overline{\mathscr{M}_A}(S)$ over $(f : C \to S, D_i, [\mathscr{L}_1] = [\mathscr{L}_2]) \in \overline{\mathscr{N}_A}(S)$. Then there exists an fppf cover $S' \to S$ such that $\mathscr{L}_1|_{S'} \cong \mathscr{L}_2|_{S'}$. Choose an isomorphism, say ϕ. Then $\phi_2 \circ \phi$ agrees with ϕ_1 up to multiplication by some $s \in \mathbb{G}_m(S')$, and fppf locally, we may take the Nth root of s, and compose with ϕ, so that fppf locally, our elements of $\overline{\mathscr{M}_A}(S)$ are isomorphic.

Finally, automorphisms of $(f : C \to S, D_i, \mathscr{L}, \phi)$ over $(f : C \to S, D_i, [\mathscr{L}])$ are given by $\mu_{N,S}$, since they consist of pairs (id, τ) such that $\tau : \mathscr{L} \cong \mathscr{L}$ and $\tau^{\otimes N} = \mathrm{id}$. □

We also define $\mathscr{N}_A$ to be the stack with objects and morphisms of $\overline{\mathscr{N}_A}$ having disjoint divisors, as in $\mathscr{M}_A$. Then $\mathscr{M}_A \to \mathscr{N}_A$ is also a μ_N-gerbe, and $\overline{\mathscr{M}_A} \times_{\overline{\mathscr{N}_A}} \mathscr{N}_A \cong \mathscr{M}_A$.

Let $\mathscr{C}_g$ be the universal curve of genus g and consider the stack $\mathrm{Pic}^{d/N}_{\mathscr{C}_g/\mathscr{M}_g}$, with objects

$$\mathrm{Pic}^{d/N}_{\mathscr{C}_g/\mathscr{M}_g}(S) = \left\{ \left(f : C \to S, \ell \in \mathrm{Pic}^{d/N}_{C/S}(S) \right) \right\},$$

and morphisms $\sigma : (f : C \to S, \ell) \to (f : C' \to S, \ell')$ given by isomorphisms $\sigma : C \to C'$ such that $\sigma^*\ell' = \ell$.

We have a natural map $\overline{\mathscr{N}_A} \to \mathrm{Pic}^{d/N}_{\mathscr{C}_g/\mathscr{M}_g}$ given by $(f : C \to S, D_i, \ell) \mapsto (f : C \to S, \ell)$. Note also that

$$\overline{\mathscr{N}_A} = \left(\mathrm{Sym}^{d_1}_{\mathscr{C}_g/\mathscr{M}_g} \times_{\mathscr{M}_g} \cdots \times_{\mathscr{M}_g} \mathrm{Sym}^{d_{N-1}}_{\mathscr{C}_g/\mathscr{M}_g} \right) \times_{\mathrm{Pic}^d_{\mathscr{C}_g/\mathscr{M}_g}} \mathrm{Pic}^{d/N}_{\mathscr{C}_g/\mathscr{M}_g},$$

where the fiber product is taken over the map that sends $(f : C \to S, \{D_i\})$ to $(f : C \to S, [\mathscr{O}_C(\sum_i iD_i)])$, and the map that sends $(f : C \to S, \ell)$ to $(f : C \to S, \ell^N)$. The map $\overline{\mathscr{N}_A} \to \mathrm{Pic}^{d/N}_{\mathscr{C}_g/\mathscr{M}_g}$ is projection onto the second factor.

Proposition 4.3.3. *When $g = 0$, we can make the following simplifications.*

1. $\mathscr{M}_0 = B\mathrm{PGL}_2$.
2. $\mathscr{C}_0 = [\mathbb{P}V/\mathrm{PGL}_2]$ *is the universal curve of genus 0, where* $V = k^{\oplus 2}$ *and* PGL_2 *acts on* $\mathbb{P}V$ *via the standard representation of* SL_2 *on* V*.*
3. $\mathrm{Pic}^d_{\mathscr{C}_0/\mathscr{M}_0} \cong B\mathrm{PGL}_2$ *for all* d*.*
4. $\mathrm{Sym}^d_{\mathscr{C}_0/\mathscr{M}_0} = [\mathbb{P}\mathrm{Sym}^d V/\mathrm{PGL}_2]$ *is the moduli of degree* d *divisors on genus 0 curves.*
5. $\mathrm{Sym}^{d_1}_{\mathscr{C}_0/\mathscr{M}_0} \times_{B\mathrm{PGL}_2} \cdots \times_{B\mathrm{PGL}_2} \mathrm{Sym}^{d_{N-1}}_{\mathscr{C}_0/\mathscr{M}_0} = [(\mathbb{P}\mathrm{Sym}^{d_1} V \times \cdots \times \mathbb{P}\mathrm{Sym}^{d_{N-1}} V)/\mathrm{PGL}_2]$.

Proof. (1) is Example 4.2.4, a special case of Proposition 4.2.3.

For (2), let V be the standard representation of SL_2, so $V = k^{\oplus 2}$, and SL_2 acts via $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}$, as in Example 4.2.15. Then μ_2 acts by the standard character, and the action descends to an action of PGL_2 on $\mathbb{P}V$. Let $f : X \to S$ be a genus 0 curve, and $S \to B\mathrm{PGL}_2$ be the torsor $I_X : (T \to S) \mapsto \{\sigma : X \times_S T \cong \mathbb{P}V_T = \mathbb{P}^1_T\}$. Due to the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{P}V \times I_X \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{P}V \arrow[dr] \arrow[dd] & \\
& I_X \arrow[rr] \arrow[dd] & & \mathrm{Spec}\ k \arrow[dd] \\
(\mathbb{P}V \times I_X)/\mathrm{PGL}_2 \arrow[rr] \arrow[dr] & & {[\mathbb{P}V/\mathrm{PGL}_2]} \arrow[dr] & \\
& S \arrow[rr] & & B\mathrm{PGL}_2,
\end{tikzcd}$$

we would like to show that $(\mathbb{P}V \times I_X)/\mathrm{PGL}_2 \cong X$. Indeed an isomorphism is given by sending $(x, \sigma) \in \mathbb{P}V \times I_X$ to $\sigma^{-1}(x)$, which descends to an isomorphism on the quotient.

For (3), we have $\mathrm{Pic}_{\mathscr{C}_0/\mathscr{M}_0} \cong \mathbb{Z} \times \mathscr{M}_0$, and for each d, $\mathrm{Pic}^d_{\mathscr{C}_0/\mathscr{M}_0} \cong \mathscr{M}_0$, represented by the class of $\mathscr{O}(d)$ on $\mathbb{P}V$.

For (4) and (5), we use the fact that for two SL_2 representations V_1 and V_2 that descend to actions of PGL_2 on $\mathbb{P}V_1$ and $\mathbb{P}V_2$, we have $[(\mathbb{P}V_1 \times \mathbb{P}V_2)/\mathrm{PGL}_2] \cong [\mathbb{P}V_1/\mathrm{PGL}_2] \times_{B\mathrm{PGL}_2} [\mathbb{P}V_2/\mathrm{PGL}_2]$. One can see this by taking the PGL_2 quotient of the cartesian diagram of $\mathbb{P}V_1 \times \mathbb{P}V_2$. Assuming statement (4), statement (5) follows. For (4), we may view $\mathbb{P}\mathrm{Sym}^d V$

as the space of polynomials of degree d in 2 variables up to multiplication by a scalar, and the zero locus of this polynomial is a degree d divisor on $\mathbb{P}V$. Thus, $\mathbb{P}V \cong \mathbb{P}\mathrm{Sym}^1 V$ and $\mathbb{P}\mathrm{Sym}^d V = (\mathbb{P}V \times \cdots \times \mathbb{P}V)/S_d$. Let $f : X \to S$ be a genus 0 curve. Using a similar argument as in the proof of (2), we see that

$$((\mathbb{P}V \times \cdots \times \mathbb{P}V)/S_d \times I_X)/\mathrm{PGL}_2 = (X \times_S \cdots \times_S X)/S_d,$$

which shows that $[\mathbb{P}\mathrm{Sym}^d V/\mathrm{PGL}_2]$ is the moduli of degree d divisors on genus 0 curves. Alternatively,

$$\begin{aligned}\mathrm{Sym}^d_{\mathscr{C}_0/\mathscr{M}_0} &= (\mathscr{C}_0 \times_{\mathscr{M}_0} \cdots \times_{\mathscr{M}_0} \mathscr{C}_0)/S_d = [(\mathbb{P}V \times \cdots \times \mathbb{P}V)/\mathrm{PGL}_2]/S_d \\ &= [((\mathbb{P}V \times \cdots \times \mathbb{P}V)/S_d)/\mathrm{PGL}_2] = [\mathbb{P}\mathrm{Sym}^d V/\mathrm{PGL}_2]. \qquad \square\end{aligned}$$

From the proposition, we can conclude that when $g = 0$,

$$\overline{\mathscr{N}_A} = [(\mathbb{P}\mathrm{Sym}^{d_1} V \times \cdots \times \mathbb{P}\mathrm{Sym}^{d_{N-1}} V)/\mathrm{PGL}_2] \to \mathrm{Pic}^{d/N}_{\mathscr{C}_0/\mathscr{M}_0} = B\mathrm{PGL}_2$$

is a composition of Brauer–Severi stacks. Indeed, write

$$\mathscr{X}_i := [\mathbb{P}\mathrm{Sym}^{d_1} V/\mathrm{PGL}_2] \times_{B\mathrm{PGL}_2} \cdots \times_{B\mathrm{PGL}_2} [\mathbb{P}\mathrm{Sym}^{d_i} V/\mathrm{PGL}_2],$$

and we factor γ by

$$\gamma : \overline{\mathscr{N}_A} = \mathscr{X}_{N-1} \xrightarrow{\gamma_{N-1}} \cdots \xrightarrow{\gamma_2} \mathscr{X}_1 \xrightarrow{\gamma_1} B\mathrm{PGL}_2, \tag{4.3.3.1}$$

where each γ_i is a projection onto the first $i-1$ factors. Then

$$\mathscr{X}_i = \mathscr{X}_{i-1} \times_{B\mathrm{PGL}_2} [\mathbb{P}\mathrm{Sym}^{d_i} V/\mathrm{PGL}_2],$$

so $\mathscr{X}_i \to \mathscr{X}_{i-1}$ is a Brauer–Severi stack, given by the pullback of $[\mathbb{P}\mathrm{Sym}^{d_i} V/\mathrm{PGL}_2] \to B\mathrm{PGL}_2$.

The Brauer group of $\overline{\mathscr{N}_A}$

Theorem 4.3.4. *Let $N, d_1, \ldots, d_{N-1} \in \mathbb{Z}$ such that $N | d := \sum i d_i$ and*

$$g := \frac{1}{2}\left(-2N + \sum_i i d_i (N - \gcd(i, N)) + 2\right)$$

is an integer greater than 1. Then $A = (0, N, d_1, \ldots, d_{N-1})$ is a g-admissible datum. If all d_i are even, $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$, and if any d_i is odd, $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) = 0$.

Proof. From the factorization 4.3.3.1 of $\gamma : \overline{\mathscr{N}_A} \to B\mathrm{PGL}_2$ as a composition of Brauer–Severi stacks, we get a factorization of $\gamma^* : H^2(B\mathrm{PGL}_2, \mathbb{G}_m) \to H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$ as a composition of the $\gamma_i^* : H^2(\mathscr{X}_{i-1}, \mathbb{G}_m) \to H^2(\mathscr{X}_i, \mathbb{G}_m)$, and by Gabber's Theorem (4.2.17), these are surjections such that $\delta([\mathscr{X}_i]) \mapsto 0$.

Recall that $\mathscr{X}_i = \mathscr{X}_{i-1} \times_{B\mathrm{PGL}_2} [\mathbb{P}\mathrm{Sym}^{d_i} V/\mathrm{PGL}_2]$ is the Brauer–Severi stack given by the pullback of $[\mathbb{P}\mathrm{Sym}^{d_i} V/\mathrm{PGL}_2] \to B\mathrm{PGL}_2$ to $\mathscr{X}_i$. We have

$$\delta([[\mathbb{P}\mathrm{Sym}^{d_i} V/\mathrm{PGL}_2]]) \in H^2(B\mathrm{PGL}_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$$

given by $[d_i] \in \mathbb{Z}/2\mathbb{Z}$ by Example 4.2.16, so $\delta([\mathscr{X}_i]) \in H^2(\mathscr{X}_{i-1}, \mathbb{G}_m)$ is just the image of $[d_i]$ under

$$H^2(B\mathrm{PGL}_2, \mathbb{G}_m) \twoheadrightarrow H^2(\mathscr{X}_{i-1}, \mathbb{G}_m).$$

Thus under $H^2(BG, \mathbb{G}_m) \twoheadrightarrow H^2(\mathscr{X}_i, \mathbb{G}_m)$, $[d_i] \mapsto 0$, and under $\gamma^* : H^2(BPGL_2, \mathbb{G}_m) \twoheadrightarrow H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$ and $[d_i] \mapsto 0$ for all $i = 1, \ldots, N-1$. Then if all d_i are even, $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$, and if any d_i is odd, $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) = 0$. □

The Brauer class of $\mathscr{M}_A$

Now, since $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$, depending on the parities of the d_i, the Brauer class of $\overline{\mathscr{M}_A}$ is either trivial or nontrivial of order 2. We now compute precisely what the Brauer class of $\overline{\mathscr{M}_A}$, and thus also $\mathscr{M}_A$, is.

Theorem 4.3.5. *Let $A = (0, N, d_1, \ldots, d_{N-1})$ be as above. Then $[\overline{\mathscr{M}_A}^{\wedge \mathbb{G}_m}] = [\gamma^* \mathscr{G}_{d/N}] \in H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$, where $\mathscr{G}_{d/N}$ is the $\mathbb{G}_m$-gerbe over $B\mathrm{PGL}_2$ of degree d/N line bundles on genus 0 curves, and $\gamma : \overline{\mathscr{N}_A} \to B\mathrm{PGL}_2$ is the natural forgetful map. In particular, if all d_i are even and d/N is odd, then $\overline{\mathscr{M}_A}^{\wedge \mathbb{G}_m}$ is nontrivial of order 2, otherwise, $\overline{\mathscr{M}_A}^{\wedge \mathbb{G}_m}$ is trivial. Moreover, since the restriction map on Brauer groups is injective, the same holds for $\mathscr{M}_A^{\wedge \mathbb{G}_m}$ where the divisors are disjoint.*

Proof. We work with the following diagram

$$\begin{tikzcd}
\mathbb{P}_{\overline{\mathscr{N}_A}} \arrow[r, "\gamma'"] \arrow[d, "\pi'"'] & \mathbb{P} \arrow[d, "\pi"] \\
\overline{\mathscr{N}_A} \arrow[r, "\gamma"] & B\mathrm{PGL}_2,
\end{tikzcd}$$

where $\mathbb{P} := [\mathbb{P}V/\mathrm{PGL}_2]$ is the universal curve of genus 0, as in Example 4.2.15.

Consider the $\mathbb{G}_m$-gerbe over $B\mathrm{PGL}_2$,

$$\mathscr{G}_{d/N}(S \to B\mathrm{PGL}_2) = \{\text{degree } d/N \text{ line bundles on } S \times_{B\mathrm{PGL}_2} \mathbb{P}\}.$$

We have $[\mathscr{G}_{d/N}] = [d/N] \in H^2(B\mathrm{PGL}_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$.

We will show that $[\overline{\mathscr{M}_A}]^{\wedge\mathbb{G}_m} = [\gamma^*\mathscr{G}_{d/N}]$, where $[\overline{\mathscr{M}_A}]^{\wedge\mathbb{G}_m}$ is the Brauer class of $\overline{\mathscr{M}_A}$, that is, the image of $[\overline{\mathscr{M}_A}]$ under the map $H^2(\overline{\mathscr{N}_A}, \mu_N) \to H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$ coming from the inclusion $\mu_N \to \mathbb{G}_m$, and $\gamma^*\mathscr{G}_{d/N}$ is the $\mathbb{G}_m$-gerbe over $\overline{\mathscr{N}_A}$ given by the cartesian product $\mathscr{G}_{d/N} \times_{\mathrm{BPGL}_2} \overline{\mathscr{N}_A}$. In particular,

$$\gamma^*\mathscr{G}_{d/N}(S \to \overline{\mathscr{N}_A}) = \{\text{degree } d/N \text{ line bundles on } S \times_{B\mathrm{PGL}_2} \mathbb{P}\}.$$

We define a morphism of stacks $\overline{\mathscr{M}_A} \to \gamma^*\mathscr{G}_{d/N}$ by

$$(f : C \to S, D_i, \mathscr{L}, \phi) \mapsto ((f : C \to S, D_i, [\mathscr{L}]), \mathscr{L}).$$

We can identify $S \times_{B\mathrm{PGL}_2} \mathbb{P}$ with the curve C, so to give an object of $\gamma^*\mathscr{G}_{d/N}$, we must give a degree d/N line bundle on C, namely $\mathscr{L}$. The map on morphisms is the identity map, thus the map on stabilizers is the inclusion $\mu_N \to \mathbb{G}_m$, so indeed $[\overline{\mathscr{M}_A}]^{\wedge\mathbb{G}_m} = [\gamma^*\mathscr{G}_{d/N}]$.

If d/N is even, then $\mathscr{G}_{d/N}$ is the trivial gerbe over $B\mathrm{PGL}_2$, and if d/N is odd, then it is nontrivial, isomorphic to $B\mathrm{GL}_2$. Together with the calculation of $H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$, the result follows. Finally, consider the open restriction map

$$\mathrm{res} : H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m) \to H^2(\mathscr{N}_A, \mathbb{G}_m),$$

which sends $[\overline{\mathscr{M}_A}^{\wedge\mathbb{G}_m}] \mapsto [\mathscr{M}_A^{\wedge\mathbb{G}_m}]$. Due to the injectivity of the restriction map on Brauer groups, we get the same result for $[\mathscr{M}_A^{\wedge\mathbb{G}_m}]$. □

Remark 4.3.6. In the case where d is even, we can actually write down the class of $\overline{\mathscr{M}_A}$ as a μ_N-gerbe over $\overline{\mathscr{N}_A}$, rather than just the class of its pushout to $\mathbb{G}_m$. However, as remarked in the introduction, since the restriction map is not injective on $H^2(\mu_N)$, this is not sufficient to compute the μ_N-class of the open substack $\mathscr{M}_A$, whose moduli we are interested in.

Our gerbe of interest, $\overline{\mathscr{M}_A} \to \overline{\mathscr{N}_A}$ can be rewritten as follows,

$$\overline{\mathscr{M}_A}(f : S \to \overline{\mathscr{N}_A}) = \{\mathscr{L}, \phi : \mathscr{L}^{\otimes N} \cong f'^*\mathscr{O}(\mathscr{D})\},$$

where $\mathscr{L}$ is a line bundle on $\mathbb{P}_S$ (which we have identified with C), $\mathscr{D} = \sum_i \mathscr{D}_i$ is the sum of the universal divisors on $\mathbb{P}_{\mathscr{N}_A}$, and $f' : \mathbb{P}_S \to \mathbb{P}_{\mathscr{N}_A}$.

Consider the $\mathbb{G}_m$-gerbes over $B\mathrm{PGL}_2$, using notation from Corollary 4.2.14,

$$\mathscr{G}_d(S \to B\mathrm{PGL}_2) = \{\mathscr{O}(d) \text{ on } S \times_{B\mathrm{PGL}_2} \mathbb{P}\},$$

$$\mathscr{G}^{1,\ldots,N-1}_{d_1,\ldots,d_{N-1}}(S \to B\mathrm{PGL}_2) = \left\{\mathscr{O}(1, \ldots, N-1) \text{ on } S \times_{B\mathrm{PGL}_2} \overline{\mathscr{N}_A}\right\}.$$

We have

$$[\mathscr{G}_d] = [d] \in H^2(B\mathrm{PGL}_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z},$$

$$\left[\mathscr{G}^{1,\ldots,N-1}_{d_1,\ldots,d_{N-1}}\right] = \left[\sum id_i\right] = [d] \in H^2(B\mathrm{PGL}_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z},$$

by Corollary 4.2.14 and Example 4.2.16. From this, we can conclude that $\mathscr{G}_d \cong \mathscr{G}^{1,\ldots,N-1}_{d_1,\ldots,d_{N-1}}$, and the isomorphism comes from the fact that, whenever such line bundles exist,

$$\mathscr{O}(\mathscr{D}) \cong \gamma'^*\mathscr{O}(d) \otimes \pi'^*\mathscr{O}(1,\ldots,N-1).$$

The existence of $\mathscr{O}(d)$ on $\mathbb{P}$ is equivalent to the existence of $\mathscr{O}(1,\ldots,N-1)$ on $\overline{\mathscr{N}_A}$, and both exist after pulling back to $\mathscr{G}_d$. If d is even, the class of $\mathscr{G}_d$ is trivial, meaning that $\mathscr{O}(d)$ exists on $\mathbb{P}$ and $\mathscr{O}(1,\ldots,N-1)$ exists on $\overline{\mathscr{N}_A}$.

Thus, if d is even, we have $\overline{\mathscr{M}_A} \cong \mathscr{M}' \otimes \mathscr{M}''$, where

$$\mathscr{M}'(f : S \to \overline{\mathscr{N}_A}) = \{\mathscr{L}', \phi' : \mathscr{L}'^{\otimes N} \cong f'^*\pi'^*\mathscr{O}(1,\ldots,N-1)\},$$

$$\mathscr{M}''(f : S \to \overline{\mathscr{N}_A}) = \{\mathscr{L}'', \phi'' : \mathscr{L}''^{\otimes N} \cong f'^*\gamma'^*\mathscr{O}(d)\}.$$

First, note that $\mathscr{M}'$ is isomorphic to the μ_N-gerbe of Nth roots of $\mathscr{O}(1,\ldots,N-1)$ on $\overline{\mathscr{N}_A}$,

$$\sqrt[N]{\mathscr{O}(1,\ldots,N-1)}\left(f : S \to \overline{\mathscr{N}_A}\right) = \{L, \phi : L^{\otimes N} \cong f^*\mathscr{O}(1,\ldots,N-1) \text{ on } S\}.$$

A map is given by sending (L,ϕ) to $(\pi^*L, \pi^*\phi)$. From the exact sequence $1 \to \mu_N \to \mathbb{G}_m \to \mathbb{G}_m \to 1$, we get

$$H^1(\overline{\mathscr{N}_A}, \mathbb{G}_m) \to H^2(\overline{\mathscr{N}_A}, \mu_N) \to H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m),$$

and the first map thus sends the line bundle $\mathscr{O}(1,\ldots,N-1)$ to $\mathscr{M}'$, so under the second map, $\mathscr{M}' \mapsto 0$. Thus, $[\mathscr{M}']^{\wedge\mathbb{G}_m} = [B\mathbb{G}_m] \in H^2(\overline{\mathscr{N}_A}, \mathbb{G}_m)$, and thus $[\mathscr{M}'']^{\wedge\mathbb{G}_m} = [\overline{\mathscr{M}_A}]^{\wedge\mathbb{G}_m}$.

Next, the same exact sequence gives

$$0 = H^1(B\mathrm{PGL}_2, \mathbb{G}_m) \to H^2(B\mathrm{PGL}_2, \mu_N) \to H^2(B\mathrm{PGL}_2, \mathbb{G}_m) \to H^2(B\mathrm{PGL}_2, \mathbb{G}_m)$$

with $H^2(B\mathrm{PGL}_2, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$. The rightmost map sends $[\mathscr{G}_{d/N}] \mapsto [\mathscr{G}_d] = 0$, so the class of $\mathscr{G}_{d/N}$ comes from the pushout of a μ_N-gerbe on $B\mathrm{PGL}_2$, which can be described as follows,

$$\mathscr{H}(f : S \to B\mathrm{PGL}_2) = \{\mathscr{L}, \phi : \mathscr{L}^{\otimes N} \cong f'^*\mathscr{O}(d) \text{ on } S \times_{B\mathrm{PGL}_2} \mathbb{P}\}.$$

Thus we see that $\mathscr{M}'' = \gamma^*\mathscr{H}$, and

$$\overline{\mathscr{M}_A} \cong \mathscr{M}' \otimes \mathscr{M}'' \cong \sqrt[N]{\mathscr{O}(1,\ldots,N-1)} \otimes \gamma^*\mathscr{H}.$$

Moreover, this confirms that

$$[\overline{\mathscr{M}_A}]^{\wedge\mathbb{G}_m} = [\mathscr{M}'']^{\wedge\mathbb{G}_m} = [\gamma^*\mathscr{H}]^{\wedge\mathbb{G}_m} = \gamma^*[\mathscr{G}_{d/N}].$$

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