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Authors
Honein, TE
O'Reilly, OM

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On the Gibbs-Appell Equations for the Dynamics of Rigid Bodies

Theresa E. Honein · Oliver M. O'Reilly

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Abstract In this technical brief, a simple concise derivation of the Gibbs-Appell equations for the dynamics of a constrained rigid body is presented.

Keywords Gibbs-Appell equations · Constraints · Constraints forces · Constraint moments

1 Introduction

Since their introduction in the early 20th century, the Gibbs-Appell equations have proven to be a remarkably popular and influential method to formulate the equations of motion of constrained rigid bodies (see, e.g., [1,2] for applications to robotics). In particular, when the coordinates and quasi-velocities are chosen appropriately, the resulting equations of motion are reactionless - even if the constraints on the system are non-holonomic.

Although derivations of the Gibbs-Appell equations using several methods, including Gauss's Principle of Least Constraint, can be found in the literature (see [3,4,5,6,7], their treatments of constraint forces and constraint moments are not transparent. Indeed, a discussion of why the Gibbs-Appell equations are equivalent to the Newton-Euler balance laws for constrained rigid bodies is surprisingly absent from the literature. In this brief note, we provide such a demonstration with the help of a recent treatment in [8] of constraint forces and constraint moments. Our developments are also applicable to Kane's equations of motion [9] but we do not pursue this matter in the interests of brevity.

Theresa E. Honein and Oliver M. O'Reilly
Department of Mechanical Engineering,
University of California at Berkeley,
Berkeley, CA 94720-1740
USA
E-mail: theresa_honein@berkeley.edu, oreilly@berkeley.edu

2 Background and Notation

In this paper, we will follow the notation used in [10]. Referring to Figure 1, the motion of a material point X on a rigid body can be described relative to the motion of the center of mass $\bar{X}$:

$$\mathbf{x} = \mathbf{Q}\left(\mathbf{X} - \bar{\mathbf{X}}\right) + \bar{\mathbf{x}}. \tag{1}$$

Here, $\mathbf{Q} = \mathbf{Q}(t)$ is the rotation tensor of the rigid body, $\mathbf{x}$ is the position vector of X, and $\bar{\mathbf{x}}$ is the position vector of $\bar{X}$ in the present configuration of the rigid body at time t. The vectors $\mathbf{X}$ and $\bar{\mathbf{X}}$ are the respective position vectors of X and $\bar{X}$ in a fixed reference configuration of the rigid body. The rotation tensor can be used to define a body-fixed basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$: $\mathbf{e}_i = \mathbf{Q}\mathbf{E}_i$ where $i = 1, 2, 3$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a fixed right-handed Cartesian basis for $\mathbb{E}^3$.

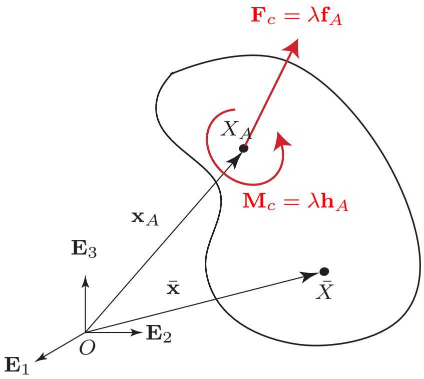

Fig. 1: *Schematic of the present configuration of the rigid body. In this figure, a constraint force* $\mathbf{F}_c = \lambda \mathbf{f}_A$ *acts at the material point* X_A *and a constraint moment* $\mathbf{M}_c = \lambda \mathbf{h}_A$.

Equation (1) can be differentiated to yield an equation relating the velocities of any pair of material points X_B and X_C, say, on a rigid body: $\mathbf{v}_B - \mathbf{v}_C = \boldsymbol{\omega} \times (\mathbf{x}_B - \mathbf{x}_C)$. Here, $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body: $\boldsymbol{\omega} \times \mathbf{a} = \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{a}$ for any vector $\mathbf{a}$. That is, $\boldsymbol{\omega}$ is the axial vector of $\dot{\mathbf{Q}}\mathbf{Q}^T$. We assume that the rigid body has a mass m and a moment of inertia tensor $\mathbf{J}$ relative to the center of mass.

In the sequel, we assume that a set of coordinates $\left(q^1, \ldots, q^6\right)$ have been chosen to parameterize the motion of the system. A set of quasi-velocities are also defined:

$$v^K = \sum_{L=1}^{6} A_{KL}\dot{q}^K + g^K. \tag{2}$$

where A_{KL} and g^K are functions of $q^1, \ldots, q^6$ and t. The matrix $[A_{KL}]$ is assumed to be invertible. For any function $f = f\left(q^1, \ldots, q^6, v^1, \ldots v^6, t\right)$:

$$\dot{f} = \sum_{K=1}^{6} \frac{\partial f}{\partial q^K}\dot{q}^K + \frac{\partial f}{\partial v^K}\dot{v}^K + \frac{\partial f}{\partial t}, \tag{3}$$

Thus,

$$\frac{\partial \dot{f}}{\partial \dot{v}^K} = \frac{\partial f}{\partial v^K}. \tag{4}$$

A velocity vector $\mathbf{v}_B$ and the angular velocity $\boldsymbol{\omega}$ are vector-valued examples of the function f:

$$\frac{\partial \dot{\mathbf{v}}_B}{\partial \dot{v}^K} = \frac{\partial \mathbf{v}_B}{\partial v^K}, \qquad \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \dot{v}^K} = \frac{\partial \boldsymbol{\omega}}{\partial v^K}. \tag{5}$$

For completeness, we note that $\frac{\partial \mathbf{x}_B}{\partial v^K} = \mathbf{0}$, $\frac{\partial \mathbf{J}}{\partial v^K} = \mathbf{0}$, and $\frac{\partial \mathbf{Q}}{\partial v^K} = \mathbf{0}$.

3 The Newton-Euler and Gibbs Appell Equations of Motion

The Gibbs-Appell function for the rigid body is defined as follows (see, e.g., [3,6]):

$$\begin{aligned} S &= \frac{1}{2}\int_{\mathcal{P}} \dot{\mathbf{v}} \cdot \dot{\mathbf{v}} \rho dv \\ &= \frac{1}{2} m \dot{\bar{\mathbf{v}}} \cdot \dot{\bar{\mathbf{v}}} + \frac{1}{2}\left(\dot{\boldsymbol{\omega}} \cdot (\mathbf{J}\dot{\boldsymbol{\omega}}) + 2\left(\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}\right) \cdot \dot{\boldsymbol{\omega}}\right), \end{aligned} \tag{6}$$

where $\bar{\mathbf{v}} = \dot{\bar{\mathbf{x}}}$ is the velocity vector of $\bar{X}$, ρ is the mass density of the rigid body, and dv is the volume element for the region of space $\mathcal{P} \in \mathbb{E}^3$ occupied by the body in its present configuration. The function S can be expressed as a function of $q^1, \ldots, q^6$, $v^1, \ldots, v^6$, and $\dot{v}^1, \ldots, \dot{v}^6$.

The balance laws for the rigid body are

$$m\dot{\bar{\mathbf{v}}} = \mathbf{F}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{J}\boldsymbol{\omega}\right) = \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) = \mathbf{M}, \tag{7}$$

where $\mathbf{F}$ is the resultant force acting on the body at $\bar{X}$ and $\mathbf{M}$ is the resultant moment relative to $\bar{X}$ acting on the rigid body.

In the absence of constraints, we can differentiate S with respect to a generalized acceleration, use the identities (5), and invoke the balance laws (7) to show that

$$\begin{aligned} \frac{\partial S}{\partial \dot{v}^K} &= m\dot{\bar{\mathbf{v}}} \cdot \frac{\partial \dot{\bar{\mathbf{v}}}}{\partial \dot{v}^K} + \left(\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}\right) \cdot \frac{\partial \dot{\boldsymbol{\omega}}}{\partial \dot{v}^K} \\ &= R_K, \qquad (K = 1, \ldots, 6), \end{aligned} \tag{8}$$

where the generalized force R_K is

$$R_K = \mathbf{F} \cdot \frac{\partial \bar{\mathbf{v}}}{\partial v^K} + \mathbf{M} \cdot \frac{\partial \boldsymbol{\omega}}{\partial v^K}. \tag{9}$$

Thus, as is the case with Lagrange's equations of motion, the Gibbs-Appell equations,

$$\frac{\partial S}{\partial \dot{v}^K} = R_K, \tag{10}$$

are equivalent to linear combinations of the components of $\mathbf{F} = m\dot{\bar{\mathbf{v}}}$ and $\mathbf{M} = \dot{\overline{\mathbf{J}\boldsymbol{\omega}}}$. Discussions of special cases, including Euler's equations, can be found in [3,6].

4 The Generalized Force R_K

Suppose that a system of R forces $\mathbf{F}_\Gamma$ each acting at a point X_Γ and a moment $\mathbf{M}_P$ act on the rigid body. Then,

$$\begin{aligned}\mathbf{F} &= \sum_{\Gamma=1}^{R} \mathbf{F}_\Gamma \text{ acting at } \bar{X},\\ \mathbf{M} &= \mathbf{M}_P + \sum_{\Gamma=1}^{R} (\mathbf{x}_\Gamma - \bar{\mathbf{x}}) \times \mathbf{F}_\Gamma. \end{aligned}\tag{11}$$

As $\mathbf{v}_\Gamma = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{x}_\Gamma - \bar{\mathbf{x}})$, it can be shown that

$$\mathbf{F}_\Gamma \cdot \frac{\partial \mathbf{v}_\Gamma}{\partial v^K} = \mathbf{F}_\Gamma \cdot \frac{\partial \bar{\mathbf{v}}}{\partial v^K} + ((\mathbf{x}_\Gamma - \bar{\mathbf{x}}) \times \mathbf{F}_\Gamma) \cdot \frac{\partial \boldsymbol{\omega}}{\partial v^K}. \tag{12}$$

Consequently, with the help of (9), the generalized force R_K can be shown to have the following equivalent representation:

$$R_K = \mathbf{M}_P \cdot \frac{\partial \boldsymbol{\omega}}{\partial v^K} + \sum_{\Gamma=1}^{R} \mathbf{F}_\Gamma \cdot \frac{\partial \mathbf{v}_\Gamma}{\partial v^K}. \tag{13}$$

The contributions of individual force and moments to the generalized forces R_K can be easily identified using this representation.

5 Lagrange's Prescription for the Constraint Force and Constraint Moment

In rigid body dynamics, kinematic constraints can be expressed in a canonical form $\pi_A = 0$, where

$$\pi_A = \mathbf{f}_A \cdot \mathbf{v}_A + \mathbf{h}_A \cdot \boldsymbol{\omega} + e_A. \tag{14}$$

Here, $\mathbf{v}_A = \dot{\mathbf{x}}_A$ is the velocity vector of a material point X_A on the body and the functions $\mathbf{f}_A$, $\mathbf{h}_A$, and e_A depend on $\mathbf{Q}$, $\bar{\mathbf{x}}$ and t. For example, if a rigid body is rolling with one point X_P in contact with a fixed surface or is free to rotate about a fixed point X_P then the rigid body is subject to three constraints: $\mathbf{v}_P \cdot \mathbf{E}_k = 0$ where $k = 1, 2, 3$.

Following [8,10] and referring to Figure 1, we define Lagrange's prescription for the constraint force $\mathbf{F}_c$ and the constraint $\mathbf{M}_c$ as

$$\begin{aligned}\mathbf{F}_c &= \lambda \frac{\partial \pi_A}{\partial \mathbf{v}_A} = \lambda \mathbf{f}_A \text{ acting at the point } X_A,\\ \mathbf{M}_c &= \lambda \frac{\partial \pi_A}{\partial \boldsymbol{\omega}} = \lambda \mathbf{h}_A, \end{aligned}\tag{15}$$

where λ is a function which is determined by the equations of motion. The extensive set of examples in [8,10] demonstrate that the prescription (15) is equivalent to the prescription used in analytical mechanics and presumes the absence of dynamic Coulomb friction.

6 The Gibbs-Appell Equations in the Presence of Constraints

We now suppose that the rigid body is subject to a constraint $\pi_A = 0$. As is standard in applications, we choose one of the quasi-velocities so the constraint function π_A can be expressed in a very simple manner:

$$\pi_A = v^6 - f_A, \tag{16}$$

We can impose the constraint $v^6 = f_A$ on S and compute a constrained Gibbs-Appell function $\tilde{S}$:

$$\tilde{S} = S\left(q^1, \ldots, q^6, v^1, \ldots v^5, v^6 = f_A, \dot{v}^1, \ldots \dot{v}^5, \dot{v}^6 = \dot{f}_A\right). \tag{17}$$

It is important to observe that

$$\begin{aligned} \frac{\partial \tilde{S}}{\partial \dot{v}^K} &= \frac{\partial S}{\partial \dot{v}^K}\Big|_{v^6 = f_A, \dot{v}^6 = \dot{f}_A}, \qquad (K = 1, \ldots, 5), \\ \frac{\partial \tilde{S}}{\partial \dot{v}^6} &= 0 \neq \frac{\partial S}{\partial \dot{v}^6}\Big|_{v^6 = f_A, \dot{v}^6 = \dot{f}_A}. \end{aligned} \tag{18}$$

These results have parallels in computing Lagrange's equations of motion also (see, e.g., [10, Chapter 3]).

A second consequence of the choice (16) pertains to the generalized forces R_K. First, we observe that

$$\delta^6_K = \frac{\partial \pi_A}{\partial v^K} = \mathbf{f}_A \cdot \frac{\partial \mathbf{v}_A}{\partial v^K} + \mathbf{h}_A \cdot \frac{\partial \boldsymbol{\omega}}{\partial v^K}, \tag{19}$$

where $\delta^L_K = 1$ if $L = K$ and is otherwise 0 is the Kronecker delta. Thus, after appealing to (13), (14), and (15), we will find that $\mathbf{F}_c$ and $\mathbf{M}_c$ will only contribute to one of the Gibbs-Appell equations:

$$\begin{aligned} \mathbf{M}_c \cdot \frac{\partial \boldsymbol{\omega}}{\partial v^K} + \mathbf{F}_c \cdot \frac{\partial \mathbf{v}_A}{\partial v^K} &= \lambda \left(\mathbf{h}_A \cdot \frac{\partial \boldsymbol{\omega}}{\partial v^K} + \mathbf{f}_A \cdot \frac{\partial \mathbf{v}_A}{\partial v^K} \right) \\ &= \lambda \frac{\partial \pi_A}{\partial v^K} \\ &= \lambda \delta^6_K, \end{aligned} \tag{20}$$

Consequently, if one seeks to find the reactionless form of the equations of motion, then it suffices to use $\tilde{S}$:

$$\frac{\partial \tilde{S}}{\partial \dot{v}^K} = R_K, \qquad (K = 1, \ldots, 5). \tag{21}$$

These are the Gibbs-Appell equations emphasized in the literature. If needed, S is computed and the Gibbs-Appell equation of motion $\frac{\partial S}{\partial \dot{v}^6} = R_6$ used to compute λ.

It is straightforward to extend the derivation of the Gibbs-Appell equations from the Newton-Euler balance laws we have just presented to the case of multiple constraints and (with the help of material in [10, Section 11.8]) to the case of systems of particles and rigid bodies.

7 An Illustrative Example

As an example, we consider a cylinder moving on a smooth horizontal surface shown in Figure 2. There are two constraints on the motion of the cylinder. First, its center of mass remains in a horizontal plane. Second, the cylinder does not rotate into the plane. Thus,

$$\bar{\mathbf{v}} \cdot \mathbf{E}_3 = 0, \qquad \boldsymbol{\omega} \cdot \mathbf{h} = 0, \tag{22}$$

where the unit vector $\mathbf{h} = \mathbf{e}_3 \times \mathbf{E}_3$. The cylinder has a mass m and a moment of inertia tensor

$$\mathbf{J} = J_t \left(\mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3 \right) + J_a \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{23}$$

Imposing constraints and using the fact that $\mathbf{E}_3 \perp \mathbf{e}_3$ for this system, we find that

$$\begin{gathered} \boldsymbol{\omega} = \Omega_3 \mathbf{E}_3 + \omega_3 \mathbf{e}_3, \qquad \dot{\boldsymbol{\omega}} = \dot{\Omega}_3 \mathbf{E}_3 + \dot{\omega}_3 \mathbf{e}_3 - \Omega_3 \omega_3 \mathbf{h}, \\ \bar{\mathbf{v}} = \dot{x} \mathbf{E}_1 + \dot{y} \mathbf{E}_2. \end{gathered} \tag{24}$$

For this problem, we choose

$$\begin{gathered} u^1 = \dot{x}, \qquad u^2 = \dot{y}, \qquad u^3 = \Omega_3, \\ u^4 = \omega_3, \qquad u^5 = \dot{z}, \qquad u^6 = \boldsymbol{\omega} \cdot \mathbf{h}. \end{gathered} \tag{25}$$

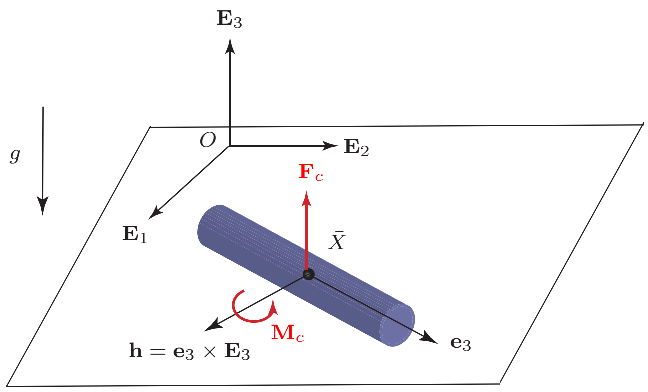

Fig. 2: *A circular cylinder sliding on a smooth horizontal surface.*

The constrained Gibbs-Appell function $\tilde{S}$ is computed with the help of (6):

$$\tilde{S} = \frac{m}{2} \left(\ddot{x}^2 + \ddot{y}^2 \right) + \frac{1}{2} \left(J_a \dot{\omega}_3^2 + J_t \dot{\Omega}_3^2 + \left(2J_a - J_t \right) \omega_3^2 \Omega_3^2 \right). \tag{26}$$

The resultant force and resultant moment relative to $\bar{X}$ acting on the system are obtained by applying Lagrange's prescription twice:

$$\mathbf{F} = \left(\lambda_1 - mg \right) \mathbf{E}_3 \text{ acting at } \bar{X}, \qquad \mathbf{M} = \lambda_2 \mathbf{h}. \tag{27}$$

As mentioned in [11], the constraint moment $\lambda_2\mathbf{h}$ can be interpreted as the moment relative to $\bar{X}$ of the distribution of normal forces along the line of contact of the cylinder and the plane and the normal force $\lambda_1\mathbf{E}_3$ can be interpreted as the resultant of the normal forces.

With the help of (13), we find that $\mathbf{F}$ and $\mathbf{M}$ will not contribute to the first four Gibbs-Appell equations:

$$\frac{\partial\tilde{S}}{\partial\ddot{x}} = m\ddot{x} = 0, \qquad \frac{\partial\tilde{S}}{\partial\ddot{y}} = m\ddot{y} = 0,$$
$$\frac{\partial\tilde{S}}{\partial\dot{\Omega}_3} = J_t\dot{\Omega}_3 = 0, \qquad \frac{\partial\tilde{S}}{\partial\dot{\omega}_3} = J_a\dot{\omega}_3 = 0. \tag{28}$$

As expected, these equations of motion imply that the center of mass moves at constant speed in a straight line and the angular speeds Ω_3 and ω_3 of the cylinder are constant. However, $\dot{\boldsymbol{\omega}} = -\omega_3\Omega_3\mathbf{h}$ is not necessarily constant.

While (28) determine $\bar{\mathbf{x}}(t)$, to determine $\mathbf{Q}(t)$ of the cylinder, a set of 3-2-3 (ψ, θ, ϕ) Euler angles can be defined where $\theta = 90^\circ$. Thus, $\boldsymbol{\omega} = \Omega_3\mathbf{E}_3 + \omega_3\mathbf{e}_3 = \dot{\psi}\mathbf{E}_3 + \dot{\phi}\mathbf{e}_3$ and the equations $\dot{\psi} = \Omega_3$ and $\dot{\phi} = \omega_3$ are integrated to determine $\mathbf{Q}(t)$.

The conclusions we have drawn on the motion of the cylinder are in complete agreement with those that can be inferred by formulating and analyzing Lagrange's equations of motion for this problem. Referring to [10,11] for the corresponding analysis, a set of Euler angles are required to establish these equations. The resulting Lagrange's equations of motion provide a set of four second-order differential equations for x, y, and two Euler angles to determine the motion of the cylinder.

Conflict of Interests Statement

The authors report no conflicts of interest.

Data Availability Statement

No data, models, or codes were generated or used for this paper.

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