UNIVERSITY OF CALIFORNIA SAN DIEGO

New Techniques in Linear Parameter-Varying Systems

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by

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The Dissertation of Amit Pandey is approved and is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To my Family.

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ABSTRACT OF THE DISSERTATION

New Techniques in Linear Parameter-Varying Systems

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Linear Parameter-Varying (LPV) techniques provide a convenient extension of linear systems theory to a rich class of systems - including uncertain, switched and non-linear systems. LPV systems theory also allows for the analysis of gain-scheduled controllers - where a controller is designed to perform over multiple operating points. The arrival of interior-point methods in the 1990s brought LPV systems and the analysis of LPV systems to the attention of many as a large subclass of LPV design conditions can be expressed as Linear Matrix Inequalities (LMIs).

This dissertation makes several contributions to LPV systems theory - both in terms of the analysis of this class of systems and new approaches for controller and filter design.

We start by revisiting the issue of quadratic gain-scheduled and robust state-feedback. The goal of this analysis is to explore to what extent solvability of certain LMIs for gain-

scheduled control also implies solvability of the corresponding robust control inequalities. One issue investigated in detail is the use of pre-filters to handle uncertainty appearing in the input matrix. We show that this technique is rarely productive in that the solvability of certain gain-scheduled control design problems for the original system augmented with a pre-filter often implies existence of a robust control for the original system.

Following this, we introduce new conditions for the H_∞ synthesis of discrete-time gain-scheduled state feedback controllers and LPV state estimators in the form of LMIs. A distinctive feature of the proposed conditions is the ability to handle time-variation in both the dynamics and the input or output matrices without resorting to pre-filtering or conservative iterative procedures. We show that these new conditions contain existing poly-quadratic conditions as a particular case and illustrate by way of numerical examples their superiority to many existing conditions.

To conclude, we introduce a strategy for combining these state-feedback and state-estimation conditions for the H_∞ synthesis of output feedback controllers. This strategy allows us to design output-feedback controllers where time-variation is present in the dynamics and the input or output matrices. To our knowledge, no techniques presently exist to solve this problem - even when the input and output matrices are held fixed.

Chapter 1

Introduction

1.1 Background and Motivation

The use of gain-scheduled based control laws is commonplace in both academic [3] and industrial applications [4, 5]. In gain-scheduled control, a scheduling variable is found which parameterizes the space of interest for the system to be controlled [6]. For this choice of scheduling variable, a family of models of the plant is found which covers its entire range of operation. A controller can then be designed to achieve the usual control objectives - be it stability, closed-loop tracking or performance. The difference however, is that this controller will be an explicit function of the scheduling variable and will be designed to meet these control objectives over the entire range of operation as opposed to at a single operating point.

When designing gain-scheduled controllers, a convenient theoretical construct is that of Linear Parameter Varying (LPV) systems. LPV techniques provide a convenient extension of linear systems theory to a rich class of systems - including switched systems, uncertain systems and non-linear systems. Further to this, LPV systems theory also allows for the analysis of gain-scheduled control design, where a controller is designed to perform over multiple operating points of a system.

Modeling of a plant as an LPV system is generally done in one of two ways [6]. The first way is to formulate an LPV representation of the system from available non-linear dynamic equations [7]. As an alternative to this, system identification techniques can be applied to a plant at a series of different operating points. In this case, a linear model will be identified at each of these operating points. Following this, we can combine these models together into a single model characterized by the scheduling parameter [8, 9].

Following the construction of a plant model, there are many strategies which can be used to design gain-scheduled controllers for the LPV plant model [10, 11, 12, 1, 2, 13]. The arrival of interior-point methods and other computational solutions for convex optimization problems in the 1990s resulted in a number of new techniques for gain-scheduled control design whose design conditions could be expressed as Linear Matrix Inequalities (LMIs) [10, 13]. These conditions have the advantage of being computationally tractable whilst also allowing for performance considerations - such as H_2 or H_∞ norm minimization to be considered [13]. For this reason, the focus of this dissertation will be on conditions which can be expressed as LMIs.

In this dissertation, we will make a number of contributions to LPV systems theory. Following a brief introduction of the relevant mathematical tools used in this work, we will start the dissertation by revisiting the issue of quadratic gain-scheduled and robust state-feedback. The goal of this analysis is to explore to what extent solvability of certain LMIs for gain-scheduled control also implies solvability of the corresponding robust control inequalities. One issue investigated in detail is that of using pre-filters to handle uncertainty appearing in the input matrix, $B(\xi(k))$ in the discrete-time case or $B(\xi(t))$ in the continuous-time case. We show that this technique is rarely productive in that the solvability of certain gain-scheduled control design problems for the original system augmented with a pre-filter often implies existence of a robust control for the original system, which we calculate explicitly.

Following this, we introduce new conditions for the H_∞ synthesis of discrete-time, gain-scheduled state-feedback controllers. A distinctive feature of the proposed conditions is the ability to handle variation in both the dynamics and the input matrices without resorting to dynamic augmentation or iterative procedures. We show that these new conditions contain existing parameter-dependent poly-quadratic state-feedback conditions as a particular case. We will also illustrate by way of numerical examples that our new conditions can out-perform comparable techniques from the literature.

The extension of the H_∞ synthesis conditions for state-feedback design to the problem of designing state estimators for LPV systems is not trivial. Before introducing such conditions, we provide a proof of necessity for one of the results in "W. M. H. Heemels, J. Daafouz, and G.

Millerioux, Observer-based control of discrete-time LPV systems with uncertain parameters," Automatic Control, IEEE Transactions on, vol. 55, no. 9, pp. 2130–2135, 2010. This result is essential to the design of discrete-time LPV filters using parameter-dependent poly-quadratic Lyapunov functions.

Following this proof, we are able to introduce new conditions for the H_∞ synthesis of discrete-time, LPV filters. As in the case of state-feedback, a distinctive feature of the proposed conditions is the ability to handle variation in both the dynamics and the output matrices without resorting to dynamic augmentation or iterative procedures. Again, we will also show that these new conditions contain the existing parameter-dependent poly-quadratic filtering conditions as a particular case.

We conclude this dissertation by showing how these new conditions can be utilized to design output-feedback controllers.

1.2 Continuous and Discrete-Time LPV models

For the purposes of this dissertation, we define an LPV system as a finite-dimensional, linear time-varying plant [6]. In this work, we will consider both continuous-time and discrete-time state-space LPV models. Input-output LPV models are not considered in this text but the interested reader can consult [14] for a in depth treatment. We define a continuous time LPV system with as the following

$$\begin{aligned} \dot{x}(t) &= A(\xi(t))x(t) + B(\xi(t))u(t) + E(\xi(t))w(t) \\ y(t) &= C(\xi(t))x(t) + D(\xi(t))u(t) + F(\xi(t))w(t), \end{aligned} \tag{1.1}$$

where $\xi(t)$ is the scheduling parameter which is generally assumed to be measurable and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^p$ and $w(t) \in \mathbb{R}^r$. In discrete time, we have following definition of a LPV system

$$\begin{aligned} x(k+1) &= A(\xi(k))x(k) + B(\xi(k))u(k) + E(\xi(k))w(k) \\ y(k) &= C(\xi(k))x(k) + D(\xi(k))u(k) + F(\xi(k))w(k), \end{aligned} \tag{1.2}$$

where $\xi(k)$ is the scheduling parameter which is generally assumed to be measurable and $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^q$, $u(k) \in \mathbb{R}^p$ and $w(k) \in \mathbb{R}^r$. In LPV analysis, additional assumptions are imposed regarding how the scheduling parameter $\xi(k)$ enters into the system definitions. For the purposes of this work, the matrices $A(\xi(k)), B(\xi(k)), C(\xi(k)), D(\xi(k)), E(\xi(k))$ and $F(\xi(k))$ are assumed to depend affinely on the time-varying parameter $\xi(k)$, which assumes values in the unit simplex,

$$\Xi = \left\{ \xi \in \mathbb{R}_+^N : \quad \sum_{i=1}^{N} \xi_i = 1 \right\}. \tag{1.3}$$

The affine assumption means that matrices $A(\xi(k))$, $B(\xi(k))$,$C(\xi(k))$, $D(\xi(k))$,$E(\xi(k))$ and $F(\xi(k))$ can be written as

$$\begin{bmatrix} A(\xi(k)) & B(\xi(k)) & E(\xi(k)) \\ C(\xi(k)) & D(\xi(k)) & F(\xi(k)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(k) \begin{bmatrix} A_i & B_i & E_i \\ C_i & D_i & F_i \end{bmatrix} \tag{1.4}$$

where we have indicated the construction in discrete time. This is identical for continuous time systems. Alternative assumptions on the construction of the matrices (1.4) exist, but will not be considered here.

1.3 Linear Matrix Inequalities

In this section we will briefly summarize some of the theory surrounding Linear Matrix Inequalities. Much of the content in this section appears in [15, 16]. A Linear Matrix Inequality (LMI) is a convex constraint Optimization problems with a convex objective function and LMI constraints can be readily solved using off-the-shelf software [15]. A LMI has the form:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{1.5}$$

where $x \in \mathbb{R}^m$, $F_i \in \mathbb{R}^{n \times n}$. The implication of the above inequality is that $F(x)$ is a positive definite matrix, that is

$$z^T F(x) z > 0, \qquad \forall z \neq 0, z \in \mathbb{R}^n.$$

$F(x)$ is an affine function of the elements of x.

1.3.1 Convexity

A set C is *convex* if:

$$\lambda x + (1-\lambda)x \in C,$$

for all $x, y \in C$ and $\lambda \in [0,1]$. An important property of LMIs is that the set $\{x|F(x) > 0\}$ is convex [15]. To see this, let x be a vector such that $F(x) > 0$ and y a vector such that $F(y) > 0$. Let $\lambda \in [0,1]$. Then

$$\begin{aligned} F(\lambda x + (1-\lambda)y)) &= F_0 + \sum_{i=1}^{m} (\lambda x_i + (1-\lambda)y_i)F_i \\ &= \lambda F(x) + (1-\lambda)F(y) \\ &> 0. \end{aligned}$$

1.3.2 Matrices as variables

Most of the problems posed as LMIs will have matrices as variables, e.g. the Lyapunov inequality

$$A^T P + PA \prec 0, \tag{1.6}$$

where $A \in \mathbb{R}^{n\times n}$ is given and $P \succ 0$ is a variable [16]. We can write (1.6) as (1.5). Let $P_1, \ldots, P_m$ be a basis for the set of symmetric $n \times n$ matrices where $m = n(n+1)/2$. Now, setting $F_0 = 0$ and $F_i = -A^T P_i - P_i A$ gives (1.5).

1.3.3 Multiple LMIs can be expressed as a single LMI

Consider a set of q LMIs given by

$$F^1(x) \succ 0; F^2(x) \succ 0; \ldots ; F^q(x) \succ 0.$$

Then, we have the following equivalent single LMI,

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i = \mathrm{diag}\{F^1(x), F^2(x), \ldots, F^q(x)\} \succ 0,$$

where

$$F_i = \mathrm{diag}\{F_i^1, F_i^2, \ldots, F_i^q\},$$

for all $i = 0, \ldots, m$.

1.3.4 The Schur complement lemma

The Schur complement lemma converts a convex nonlinear inequality into an LMI. The form of the convex nonlinear inequality is given by,

$$R(x) \succ 0, Q(x) - S(x)R(x)^{-1}S^T(x) \succ 0, \tag{1.7}$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend affinely on x. The Schur complement lemma converts into the equivalent statement,

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} \succ 0.$$

A proof of the Schur complement lemma can be found in [15].

1.4 Stability Theory

In this dissertation, many conditions will be proposed which guarantee the stability of an LPV system. The basis of all these conditions is the Lyapunov method [17, 18]. The idea behind the Lyapunov method is to search for a positive definite function whose time derivative is negative definite (in the case of a continuous-time system). In the case of a continuous-time linear system

$$\dot{x} = Ax,$$

a necessary and sufficient condition for stability is the existence of a Lyapunov function $V(x) = x^T Px$ where P is symmetric and positive definite such that the time derivative of $V(x)$ is negative, namely:

$$\begin{aligned}\frac{dV(x)}{dt} &= \dot{x}^T Px + x^T P\dot{x} \\ &= x^T(A^T P + PA)x < 0, \forall x \neq 0,\end{aligned}$$

which is equivalent to the following LMI,

$$A^T P + PA \prec 0.$$

In the case of a discrete-time linear system

$$x(k+1) = Ax(k),$$

a necessary and sufficient condition for stability is the existence of a Lyapunov function $V(x) = x^T Px$ where P is symmetric and positive definite such that the time difference of $V(x)$ is negative, namely:

$$V(x(k+1)) - V(x(k)) < 0,$$

for all $x(k) \neq 0$. This is equivalent to the following LMI,

$$A^T PA - P \prec 0.$$

1.4.1 Stability of LPV systems

In the case of an LPV system,

$$\dot{x} = A(\xi(t))x(t),$$

in continuous-time or

$$x(k+1) = A(\xi(k))x(k),$$

in discrete time, we use similar Lyapunov techniques to propose stability conditions. These issues will be dealt with in much more detail later in this dissertation. For the time being, the reader should be aware of the distinction between a parameter-independent Lyapunov funciton $V(x)$ which is not an explicit function of the scheduling parameter $\xi(k)$ or $\xi(t)$ and a parameter-dependant Lyapunov function $V(\xi, x)$ which is an explicit function of the scheduling parameter. Both classes of Lyapunov function will be used in this work.

1.5 Chapter Contents and Contributions

In Chapter 2, we discuss the issue of gain-scheduled versus robust state-feedback control with a focus on well-known Linear Matrix Inequalities (LMI) conditions for continuous-time LPV systems. It has been established that for continuous-time Linear Parameter-Varying (LPV) systems, gain-scheduled stabilizability implies robust stabilizability [19]. These results are however do not allow someone to construct a stabilizing robust controller from the corresponding stabilizing gain-scheduled controller, and the associated necessary and sufficient conditions are hard to verify even with low state dimensions. Providing proofs of such results via construction in a number of special cases is one contribution of this chapter.

A second goal of Chapter 2 to examine the popular technique, e.g. [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], of augmenting LPV systems via pre-filtering in case the input matrix of the original system depends on the scheduling parameter. The motivation behind this procedure is that standard LMI conditions for quadratic gain-scheduled control require the input matrix to be parameter-independent. By making the state-feedback controller dynamic, one can design a gain-scheduled controller using standard quadratic LMI conditions. As we demonstrate in this chapter, this strategy is never beneficial in terms of stabilizability or even performance in a variety of cases involving quadratic stability or quadratic performance requirements.

In Chapter 3, we will look at the problem of gain-scheduled vs robust control in the case of discrete-time LPV systems. We will show that in the case of quadratic stability, a similar result that was proved in Chapter 2 holds - that gain-scheduled control via pre-filtering yields no advantage in terms of stabilizability when compared with a robust control approach. In the case of a poly-quadratic parameter-dependant Lyapunov functions, no such result holds as we will indicate with a simple counter example.

In Chapter 4, we move away from the problem of gain-scheduled vs robust control in the case of LPV systems. We introduce new conditions for the H_∞ synthesis of discrete-time Linear Parameter Varying (LPV) systems in the form of Linear Matrix Inequalities (LMIs). A distinctive feature of the proposed conditions is the ability to handle variation in both the dynamics and the input matrices without resorting to dynamic augmentation or iterative procedures. We show that this new condition contains the poly-quadratic H_∞ synthesis result of Daafouz and Bernussou (2001) as a particular case. We also derive a corollary which shows improvement even in the stronger case of quadratic H_∞ synthesis. Additionally, we show that, surprisingly, a dynamic gain-scheduled quadratic H_∞ controller can result in inferior performance compared to a static robust controller. Numerical examples illustrate our results.

In Chapter 5, we move away from the state-feedback design problem and instead consider the state-filtering problem for discrete-time LPV systems. In Chapter 5, we are interested in H_∞ synthesis conditions for Linear Parameter Varying (LPV) filter design. We start by proving that an existing class of sufficient conditions for poly-quadratic H_∞ filter synthesis is in fact necessary as well. We also develop alternative conditions based on a new theory of duality for poly-quadratic Lyapunov functions. We use these results to show how existing poly-quadratic H_∞ state feedback synthesis conditions can be used for poly-quadratic H_∞ LPV filter synthesis. Numerical examples compare the different approaches to LPV filter design.

In Chapter 6, we introduce new H_∞ synthesis conditions for Linear Parameter Varying

(LPV) filter design in the form of Linear Matrix Inequalities (LMIs). A distinctive feature of the proposed conditions is the ability to handle variation in both the dynamics as well as the output matrices without resorting to dynamic augmentation or iterative procedures. We show that these new conditions contain the existing poly-quadratic filtering result as a particular case. We also derive a corollary which shows improvement even in the stronger case of quadratic H_∞ synthesis. Finally, we illustrate how the conditions proposed in this chapter can be combined with existing state-feedback conditions to design output feedback controllers where variation is permitted in all the state, input and output matrices. Numerical examples will be used throughout to illustrate all the results.

Chapter 2

Pre-Filtering in State-Feedback Control - The Continuous-time Case

2.1 Introduction

In the present chapter, we discuss the issue of gain-scheduled versus robust state-feedback control with a focus on well-known Linear Matrix Inequalities (LMI) conditions. It has been established that for continuous-time Linear Parameter-Varying (LPV) systems, gain-scheduled stabilizability implies robust stabilizability [19]. These results are however do not allow someone to construct a stabilizing robust controller from the corresponding stabilizing gain-scheduled controller, and the associated necessary and sufficient conditions are hard to verify even with low state dimensions. Providing proofs of such results via construction in a number of special cases is one contribution of this chapter. One goal is to explore to what extent solvability of certain LMIs for gain-scheduled control also implies solvability of related robust control matrix inequalities.

A second goal is to examine the popular technique, e.g. [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], of augmenting LPV systems via pre-filtering in case the input matrix of the original system depends on the scheduling parameter. The motivation behind this procedure is that standard LMI conditions for quadratic gain-scheduled control require the input matrix to be parameter-independent. By making the state-feedback controller dynamic, one can design a gain-scheduled controller using standard quadratic LMI conditions. As we demonstrate in this chapter, this strategy is never beneficial in terms of stabilizability or even performance in a variety of cases involving quadratic stability or quadratic performance requirements. Indeed, we show that whenever a gain-scheduled controller is obtained using pre-filtering and quadratic stability that a simpler robust controller, i.e. one that does not make use of the scheduling parameter, is also available with no degradation in performance.

Readers might also be interested in the work reported in [32, 33] in the context of output-feedback. It is worth pointing out the differences between the state-feedback results in the present chapter and the output-feedback results in [33]. Since it is not possible to solve a robust output-feedback control problem for structured polytopic uncertainty, the mentioned output-feedback results are limited to a reduction from a higher-order dynamic gain-scheduled output-feedback controller to a lower-order dynamic gain-scheduled output-feedback controller. Indeed, it is not possible to generate the strong implications proved in the present chapter in the output-feedback context, namely that the existence of a gain-scheduled controller implies the existence of a robust controller, not even if we restrict our attention to stabilizability alone.

This chapter is structured as follows. In Section 2.2, we give a brief overview on quadratic stability and stabilization using robust and gain-scheduled state-feedback controllers. We proceed by outlining the technique of state augmentation via pre-filters in order to design gain-scheduled rather than robust controllers in Section 2.3 and subsequently outline the flaws of this approach in Section 2.4. These results are extended to quadratic H_2 and H_∞ performance indices in Sections 2.5 and 2.6, respectively. We conclude with a short summary of the implications of the results in Section 2.7. Numerical examples are provided in the various sections.

2.2 Quadratic Stabilization

Consider continuous-time linear systems of the form

$$\dot{x}(t) = A(\xi(t))\, x(t) + B(\xi(t))\, u(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ and the matrices $A(\xi(t))$ and $B(\xi(t))$ are assumed to depend affinely on the time-varying parameter $\xi(t)$, which assumes values in the unit simplex

$$\Xi = \left\{ \xi(t) \in \mathbb{R}^N : \quad \xi_i(t) \geq 0, \quad \sum_{i=1}^{N} \xi_i(t) = 1 \right\}.$$

That is, matrices $A(\xi(t))$ and $B(\xi(t))$ can be written as

$$\begin{bmatrix} A(\xi(t)) & B(\xi(t)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(t) \begin{bmatrix} A_i & B_i \end{bmatrix}, \qquad \xi(t) \in \Xi.$$

The discussion throughout this chapter relies heavily on the notion of quadratic stability [34, 21].

Definition 1. *Matrix $A(\xi(t))$ with $\xi(t) \in \Xi$ is quadratically stable if there exist a matrix X such that*

$$A_i X + X A_i^T \prec 0, \quad X \succ 0 \tag{2.2}$$

for all $i = 1, \ldots, N$.

Quadratic stability of matrix $A(\xi(t))$ implies asymptotic stability of the system

$$\dot{x}(t) = A(\xi(t))\, x(t), \qquad \xi(t) \in \Xi,$$

as can be verified easily using the quadratic Lyapunov function $V(x) = x^T X x$, thus the term *quadratic stability.*

2.2.1 Robust Quadratic Stabilization

Quadratic stability can be used to design robust linear state-feedback controllers of the form $u(t) = K\, x(t)$ upon substitution of $A_i \rightarrow A_i + B_i\, K$ in (2.2) and the one-to-one change of variables $L = K\, X$ [35, 36]. This yields an LMI in the matrices X and L.

Lemma 1. *There exist matrices X and L such that*

$$A_i X + X A_i^T + B_i L + L^T B_i^T \prec 0, \quad X \succ 0, \tag{2.3}$$

for all $i = 1, \ldots, N$ if and only if there exists a robust state-feedback controller gain K such that $A(\xi(t)) + B(\xi(t))\, K$ is quadratically stable for all $\xi(t) \in \Xi$. In particular

$$u(t) = K\, x(t), \qquad K = L\, X^{-1} \tag{2.4}$$

is one such controller.

2.2.2 Gain-Scheduled Quadratic Stabilization

When the parameter $\xi(t)$ can be measured online, one might wonder whether a gain-scheduled controller of the form

$$u(t) = K(\xi(t))\, x(t), \tag{2.5}$$

where the gain $K(\xi(t))$ is an affine function of the parameter $\xi(t)$, can bring any advantage. It is known from [19] that the existence of a gain-scheduled controller can bring no advantage in terms of quadratic stabilizability over a robust controller. However, this question is still of interest since the statement in [19] is not verified via construction and does not require the corresponding robust controller to be linear or even static, such as (2.4). If a gain-scheduled controller of the form (2.5) exists and is able to quadratically stabilize the system (2.1), inequality

$$A_i X + X A_i^T + B_i L_i + L_i^T B_i^T \prec 0 \tag{2.6}$$

must necessarily hold for some $X \succ 0$. This is the case because inequality (2.2) must hold for some matrix $X \succ 0$ and $A_i \to A_i + B_i\, K_i$, from which (2.6) must hold with the same matrix X and $L_i = K_i\, X$. Unfortunately, inequality (2.6) is not a sufficient condition for quadratic stabilizability. The main obstacle is the construction of a suitable controller gain $K(\xi(t))$ from the solution X and L_i. For instance, the common choice

$$K(\xi(t)) = \sum_{k=1}^{N} \xi_i(t)\, K_i, \qquad K_i = L_i\, X^{-1} \tag{2.7}$$

results in quadratic instead of affine dependence of the closed-loop system matrix on the parameter $\xi(t)$, such that (2.2) does not imply closed-loop stability through the quadratic Lyapunov function $V(x) = x^T X x$. However, notice that this obstacle vanishes when $B_i = B$ for all $i = 1, \ldots, N$. In this case, the gain-scheduled controller (2.5) with gain (2.7) quadratically stabilizes the closed-loop system.

Lemma 2. *There exist matrices X and L_i such that*

$$A_i X + X A_i^T + B L_i + L_i^T B^T \prec 0, \quad X \succ 0, \tag{2.8}$$

for all $i = 1, \ldots, N$ if and only if there exist a gain-scheduled state-feedback controller $K(\xi(t))$ such that $A(\xi(t)) + B\,K(\xi(t))$ is quadratically stable for all $\xi(t) \in \Xi$. In particular,

$$u(t) = K(\xi(t))\,x(t), \quad K(\xi(t)) = \sum_{k=1}^{N} \xi_i(t)\,K_i, \quad K_i = L_i\,X^{-1} \tag{2.9}$$

is one such controller.

2.3 State Augmentation via Pre-Filtering

Lemma 2 motivates the use of a dynamic *pre-filter* as a way to handle the dependence of B on the parameter $\xi(t)$. One idea is to introduce an auxiliary control signal $\tilde{u}(t)$ which is related to $u(t)$ via integration,

$$u(t) = \int_0^t \tilde{u}(\tau)\,d\tau,$$

and solve the stabilizability problem for the augmented system realization

$$\dot{\tilde{x}}(t) = \tilde{A}(\xi(t))\,\tilde{x}(t) + \tilde{B}\,\tilde{u}(t), \qquad \tilde{x}(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \tag{2.10}$$

where the matrices $\tilde{A}(\xi(t))$ and $\tilde{B}$ have vertices

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B}_i = \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.11}$$

If we can find a state-feedback controller gain

$$\tilde{K}(\xi(t)) = \begin{bmatrix} K_x(\xi(t)) & K_u(\xi(t)) \end{bmatrix}, \tag{2.12}$$

then the resulting dynamic gain-scheduled state-feedback controller with realization

$$\begin{aligned} \dot{x}_c(t) &= K_u(\xi(t))\, x_c(t) + K_x(\xi(t))\, x(t), \\ u(t) &= x_c(t), \end{aligned} \tag{2.13}$$

does also stabilize the original LPV system (2.1). Even with the extra burden of implementing a dynamic controller, this approach is popular [20, 21]. In fact, it is easier to handle the dynamic implementation of the controller than working with the resulting non-convex counterpart of Lemma 2.

It is not hard to establish that if a gain-scheduled state-feedback controller with gain $K(\xi(t))$ can stabilize the original system, then a dynamic controller of the form (2.13) should also exist by letting $K_x(\xi(t)) = \rho K(\xi(t))$ and $K_u = -\rho I$ where ρ is sufficiently large. Indeed, in this case the resulting dynamic controller is simply the low-pass filter with transfer-function $\rho(s+\rho)^{-1}$ with the signal $r(t) = K_x(\xi(t))x(t)$ as its input. A solution to the closed-loop system with the resulting dynamic controller will therefore approach $u(t) \to r(t)$ as $\rho \to \infty$, which follows from standard singular-perturbation type analysis for time-varying systems [37]. That a finite value of ρ must exist is therefore a consequence of the fact that asymptotic stability implies robustness to small enough input perturbations.

A similar approach is to use an explicit filter, e.g. [21], in which case the matrices in the augmented realization (2.10) have vertices

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & A_u \end{bmatrix}, \qquad \tilde{B}_i = \tilde{B} = \begin{bmatrix} 0 \\ B_u \end{bmatrix},$$

and A_u, B_u are chosen *a priori*, for example as a low-pass filter with sufficiently high cutoff frequency so as not to interfere with the underlying stabilization problem. In fact, one might take a step further and work with an augmented system of the form (2.10) and vertices

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i \\ F_i & G_i \end{bmatrix}, \qquad \tilde{B}_i = \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2.14}$$

where the matrices F_i and G_i encompass all previously mentioned pre-filters. Notice that the choice of identity in $\tilde{B}$ is without loss of generality among all constant full-rank matrices of same dimension. Note also that the structure (2.14) is universal in the sense that it can also be applied to the case when the original $B_i = B$ for all $i = 1, \ldots, N$. Indeed, assuming without loss of generality that B is full rank, there exists a non-singular matrix T such that

$$\tilde{A}_i = T A_i T^{-1}, \qquad \tilde{B}_i = TB = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2.15}$$

which is of the form (2.14).

The rest of this chapter is dedicated to studying the behavior of various gain-scheduled and robust control design conditions available in the literature, such as Lemmas 1 and 2, with emphasis on the interplay with the pre-filtering techniques described in this section. We will show that such strategies for handling parameter-variation through the control matrix $B(\xi(t))$ are rarely effective in the sense that the resulting gain-scheduled controllers often come accompanied by an implicit robust controller that achieves the same control goal, be it stabilization or performance. This is a surprising and somewhat disappointing result, since one would normally expect that a gain-scheduled controller, which can use knowledge of the parameter, would perform much better than a robust controller which does not use that same information. Furthermore, we will parametrize underlying robust controllers for stabilizing gain-scheduled controllers in the various cases.

2.4 Quadratic Stabilizability via Pre-Filtering

The following result summarizes how gain-scheduled control via Lemma 2 applied to an augmented realization of system (2.1) equipped with a pre-filter does not yield any advantage in terms of stabilizability over robust control via Lemma 1.

Theorem 1. *If there exist matrices X, L, Y, F_i, G_i, U_i and V_i such that inequalities* (2.8) *hold* *with*

$$X \to \tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}, \quad L_i \to \tilde{L}_i = \begin{bmatrix} U_i & V_i \end{bmatrix},$$
$$A_i \to \tilde{A}_i = \begin{bmatrix} A_i & B_i \\ F_i & G_i \end{bmatrix}, \quad B \to \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2.16}$$

namely

$$\tilde{A}_i \tilde{X} + \tilde{X} \tilde{A}_i^T + \tilde{B} \tilde{L}_i + \tilde{L}_i^T \tilde{B}^T \prec 0,$$

then inequalities (2.3) *also hold with* X *and* L*.*

Proof. Define

$$\mathcal{R}_i = \begin{bmatrix} A_i X + B_i L & A_i L^T + B_i Y \\ F_i X + G_i L + U_i & F_i L^T + G_i Y + V_i \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{2.17}$$

If inequalities (2.8) hold with the substitutions (2.16), then

$$\mathcal{R}_i + \mathcal{R}_i^T \prec 0, \qquad \tilde{X} \succ 0.$$

Consequently,

$$\mathcal{B}(\mathcal{R}_i + \mathcal{R}_i^T)\mathcal{B}^T = A_i X + X A_i^T + B_i L + L^T B_i^T \prec 0.$$

To complete the proof, note that $\tilde{X} \succ 0 \implies X \succ 0$. □

This theorem shows that whenever $B_i \neq B_j$ for some $i \neq j$, there can be no advantage in designing a gain-scheduled control by adding a pre-filter as in (2.14) as far as quadratic stabilizability is concerned. Indeed, every dynamic, quadratically stabilizing gain-scheduled controller of the form (2.13) implicitly parameterizes a much simpler static robust controller of the form (2.4) which also quadratically stabilizes system (2.1). Remarkably, these implications hold independently of the choice of particular pre-filtering matrices F_i and G_i. Theorem 1 is further illustrated by the following numerical example.

Example 1. *Consider the following system, which is a slightly modified version of an example from [38]. The matrices* $A(\alpha)$ *and* $B(\beta)$ *are*

$$A(\alpha) = \begin{bmatrix} -2+\alpha & 1 & 1-\alpha & 1 \\ 3 & 0 & 0 & 2 \\ -1+\alpha & 0 & -2-\alpha & -3 \\ -2 & -1 & 2 & -1 \end{bmatrix}, \qquad B(\beta) = \begin{bmatrix} 0 \\ \beta \\ 0 \\ 1-\beta, \end{bmatrix},$$

where $0 \leq \beta \leq 1$ *and* $|\alpha| \leq \gamma$. *This system can be written in the form* (2.1) *with* 4 *vertices. We aim to find the largest possible value of* γ *such that the system is quadratically stabilizable. Because the input matrix* B *is parameter-varying, we augment the original system as in* (2.10)-(2.11) *to design a gain-scheduled controller. The largest possible value of* γ *for which we were able to satisfy conditions* (2.8) *in Lemma 2 with the substitutions*

$$A_i \to \tilde{A}_i, \qquad B \to \tilde{B}, \qquad X \to \tilde{X}, \qquad L_i \to \tilde{L}_i,$$

is $\gamma = 0.95$. *The corresponding matrices* $\tilde{X}$ *and* $\tilde{L}_i$ *are*

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}$$
$$= \left[\begin{array}{cccc|c} 5.70 & -5.83 & 9.17 & -1.27 & -19.59 \\ -5.83 & 11.88 & -16.13 & -1.26 & 9.87 \\ 9.17 & -16.13 & 29.75 & -6.33 & -32.53 \\ -1.27 & -1.26 & -6.33 & 10.45 & 15.63 \\ \hline -19.59 & 9.87 & -32.53 & 15.63 & 121.31 \end{array}\right]$$

and

$$\tilde{L}_1 = \begin{bmatrix} -44.44 & -93.81 & -50.04 & 51.38 & -20.81 \end{bmatrix},$$
$$\tilde{L}_2 = \begin{bmatrix} -19.84 & -93.81 & -25.45 & 51.38 & -20.81 \end{bmatrix},$$
$$\tilde{L}_3 = \begin{bmatrix} -44.44 & 27.49 & -50.04 & -69.92 & -20.81 \end{bmatrix},$$
$$\tilde{L}_4 = \begin{bmatrix} -19.84 & 27.49 & -25.45 & -69.92 & -20.81 \end{bmatrix}.$$

However, as a consequence of Theorem 1, the – much simpler – robust controller (2.4) *with gain*

$$K = LX^{-1} = \begin{bmatrix} -5.03 & -7.93 & -4.43 & -2.75 \end{bmatrix}$$

is also quadratic stabilizing.

We next state the following converse to Theorem 1.

Theorem 2. *If there exist matrices X and L such that inequalities* (2.3) *hold, i.e.,*

$$A_i X + X A_i^T + B_i L + L^T B_i^T \prec 0,$$

then there also exist matrices $\tilde{X}$ and $\tilde{L}_i = \tilde{L}$ such that inequalities (2.8) *hold with*

$$\begin{aligned} X &\to \tilde{X}, & L_i &\to \tilde{L}_i = \tilde{L}, \\ A_i &\to \tilde{A}_i = \begin{bmatrix} A_i & B_i \\ F_i & G_i \end{bmatrix}, & B &\to \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \end{aligned} \tag{2.18}$$

for any choice of the matrices F_i, G_i, namely

$$\tilde{A}_i \tilde{X} + \tilde{X} \tilde{A}_i^T + \tilde{B} \tilde{L}_i + \tilde{L}_i^T \tilde{B}^T \prec 0.$$

Proof. Assume (2.3) holds for some X and L. Define

$$Y = LX^{-1}L^T + \epsilon I,$$

which is positive definite for any $\epsilon > 0$. By the Schur Complement Lemma [39], the matrix

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}$$

is also positive definite for any $\epsilon > 0$. Now let $\tilde{A}_i$ and $\tilde{B}$ be as in (2.18) and

$$\tilde{L} = \begin{bmatrix} 0 & -(\alpha/2)I \end{bmatrix},$$

such that

$$\tilde{A}_i\tilde{X} + \tilde{X}\tilde{A}_i^T + \tilde{B}\tilde{L} + \tilde{L}^T\tilde{B}^T = \begin{bmatrix} \Phi_i & \Psi_i^T \\ \Psi_i & \Omega_i - \alpha I \end{bmatrix}$$

with matrices

$$\Phi_i = A_iX + XA_i^T + B_iL + L^TB_i^T,$$

$$\Psi_i = F_iX + LA_i^T + G_iL + YB_i^T,$$

$$\Omega_i = G_iY + YG_i^T + F_iL^T + LF_i^T.$$

The proof is complete because $\Phi_i \prec 0$ by (2.3) and there always exists a sufficiently large value of α for which

$$\alpha I \succ \Omega_i - \Psi_i\Phi_i^{-1}\Psi_i^T$$

for all $i = 1, \ldots, N$, namely $\alpha > \max_i \lambda_{\max}(\Omega_i - \Psi_i\Phi_i^{-1}\Psi_i^T)$, where $\lambda_{\max}(M)$ is the maximum eigenvalue of matrix M. We then have that,

$$\tilde{A}_i\tilde{X} + \tilde{X}\tilde{A}_i^T + \tilde{B}\tilde{L} + \tilde{L}^T\tilde{B}^T \prec 0$$

for all $i = 1, \ldots, N$. □

The meaning of Theorem 2 is that quadratic gain-scheduled and robust stabilizability are not only tied in that existence of a gain-scheduled controller implies existence of a corresponding robust controller, but that the LMIs for gain-scheduled quadratic stabilizability through augmentation must hold even if the matrices L_i are constrained to be identical over $i = 1, \ldots, N$. This idea has been used implicitly in works as early as [20] and seems to have been forgotten over time with the emergence of *cleaner* conditions in spirit of those in [36], such as those in Lemma 1. A consequence of Theorems 1 and 2 is the following corollary.

Corollary 1. *If $B_i = B$ for $i = 1, \ldots, N$ and B has full column rank, then there exist matrices X and L such that inequalities* (2.3) *hold if and only if there exist matrices X and L_i such that inequalities* (2.8) *hold.*

Proof. One direction is trivial: assume that there exist X and L such that (2.3) holds. Then (2.8) also holds for $L_i = L$ over $i = 1, \ldots, N$. The other direction is a consequence of Theorems 1 and 2. Suppose there exist X and L_i, $i = 1, \ldots, N$, such that (2.8) holds. Then find a full rank linear transformation T such that

$$\hat{A}_i = TA_iT^{-1} = \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{F}_i & \mathcal{G}_i \end{bmatrix}, \qquad \hat{B} = TB = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

where the existence of such a T is guaranteed as B is of full column rank. Consequently,

$$\hat{L}_i = L_iT^T, \qquad \hat{X} = TXT^T = \begin{bmatrix} \mathcal{X} & \mathcal{L}^T \\ \mathcal{L} & \mathcal{Y} \end{bmatrix} \succ 0$$

also satisfy inequalities (2.8) with the substitutions

$$A_i \to \hat{A}_i, \qquad B \to \hat{B}, \qquad X \to \hat{X}, \qquad L_i \to \hat{L}_i,$$

namely

$$\hat{A}_i \hat{X} + \hat{X} \hat{A}_i^T + \hat{B} \hat{L}_i + \hat{L}_i^T \hat{B}^T \prec 0.$$

Now apply Theorem 1 with

$$\tilde{A}_i \to \hat{A}_i, \qquad \tilde{B} \to \hat{B}, \qquad \tilde{X} \to \hat{X}, \qquad \tilde{L}_i \to \hat{L}_i$$

to conclude that the matrices $\mathcal{X}$ and $\mathcal{L}$ are such that

$$\mathcal{A}_i \mathcal{X} + \mathcal{X} \mathcal{A}_i^T + \mathcal{B}_i \mathcal{L} + \mathcal{L}^T \mathcal{B}_i^T \prec 0, \qquad \mathcal{X} \succ 0.$$

Next use Theorem 2 with

$$A_i \to \mathcal{A}_i, \qquad B_i \to \mathcal{B}_i, \qquad X \to \mathcal{X}, \qquad L \to \mathcal{L}$$

to show that there exist matrices $\tilde{X}$ and $\tilde{L}_i = \tilde{L}$ satisfying inequalities (2.3) with

$$A_i \to \hat{A}_i, \qquad B \to \hat{B}, \qquad X \to \tilde{X}, \qquad L_i \to \tilde{L},$$

namely

$$\hat{A}_i \tilde{X} + \tilde{X} \hat{A}_i^T + \hat{B} \tilde{L} + \tilde{L}^T \hat{B}^T \prec 0, \quad \tilde{X} \succ 0.$$

Finally, revert to the original coordinates via T to complete this proof. □

Such a strong converse result usually does not hold in more complicated problems. We will see that in the following sections when dealing with H_2 and H_∞ performance measures.

Similarly, if conditions not based on quadratic stability – such as those in [40, 41, 42] – are used in place of (2.3) and (2.8), there may be some advantage in using gain-scheduled control via state augmentation.

2.5 Quadratic H_2 Performance

For the remainder of this chapter, consider the system

$$\begin{aligned} \dot{x}(t) &= A(\xi(t))\,x(t) + B(\xi(t))\,u(t) + E(\xi(t))\,w(t), \\ z(t) &= C(\xi(t))\,x(t) + D(\xi(t))\,u(t), \end{aligned} \tag{2.19}$$

where $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$ and $z(t)$ is a performance output. We wish to compute robust and gain-scheduled controllers that minimize the upper bound

$$\mu > \lim_{t\to\infty} \mathbb{E}\left[z^T z\right] \tag{2.20}$$

on the closed-loop quadratic cost. In analogy with the previous sections, we can augment system (2.19) to the form

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}(\xi(t))\,\tilde{x}(t) + \tilde{B}\,\tilde{u}(t) + \tilde{E}(\xi(t))\,w(t), \\ z(t) &= \tilde{C}(\xi(t))\,\tilde{x}(t) + \tilde{D}\,\tilde{u}(t) \end{aligned} \tag{2.21}$$

where $\tilde{A}(\xi(t))$ and $\tilde{B}$ have vertices as in (2.14) and

$$\tilde{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \qquad \tilde{C}_i = \begin{bmatrix} C_i & D_i \end{bmatrix}, \qquad \tilde{D} = 0. \tag{2.22}$$

2.5.1 Controllability Gramian

The following results extend Lemmas 1 and 2 to guarantee closed-loop performance with respect to (2.20) via the Controllability Gramian.

Lemma 3 ([43]). *If there exist matrices X, L and Z_i such that*

$$A_i X + X A_i^T + B_i L + L^T B_i^T + E_i W E_i^T \prec 0, \tag{2.23}$$

$$\begin{bmatrix} Z_i & C_i X + D_i L \\ X C_i^T + L^T D_i^T & X \end{bmatrix} \succ 0, \tag{2.24}$$

$$\mathrm{trace}(Z_i) < \mu, \tag{2.25}$$

for all $i = 1, \ldots, N$, then the robust state-feedback controller

$$u(t) = K\, x(t), \qquad K = L\, X^{-1}$$

is such that $A(\xi(t)) + B(\xi(t))\, K$ is quadratically stable and (2.20) *holds for all $\xi(t) \in \Xi$.*

Lemma 4 ([44]). *If there exist matrices X, L_i and Z_i such that*

$$A_i X + X A_i^T + B L_i + L_i^T B^T + E_i W E_i^T \prec 0, \tag{2.26}$$

$$\begin{bmatrix} Z_i & C_i X + D L_i \\ X C_i^T + L_i^T D^T & X \end{bmatrix} \succ 0, \tag{2.27}$$

$$\mathrm{trace}(Z_i) < \mu \tag{2.28}$$

for all $i = 1, \ldots, N$, then the gain-scheduled state-feedback controller

$$u(t) = K(\xi(t))\, x(t), \qquad K(\xi(t)) = \sum_{k=1}^{N} \xi_i(t)\, K_i, \qquad K_i = L_i\, X^{-1}$$

is such that $A(\xi(t)) + B\, K(\xi(t))$ is quadratically stable and (2.20) *holds for all $\xi(t) \in \Xi$.*

The following theorem shows how there is no advantage in cost (2.20) when using gain-scheduled control via Lemma 4 for the augmented system (2.21).

Theorem 3. *If there exist matrices X, L, Y, F_i, G_i, U_i, V_i and Z_i such that inequalities* (2.26)-

(2.28) *hold with*

$$\begin{aligned} X &\to \tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}, \quad L_i \to \tilde{L}_i = \begin{bmatrix} U_i & V_i \end{bmatrix}, \\ A_i &\to \tilde{A}_i = \begin{bmatrix} A_i & B_i \\ F_i & G_i \end{bmatrix}, \quad B \to \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\ C_i &\to \tilde{C}_i = \begin{bmatrix} C_i & D_i \end{bmatrix}, \quad E_i \to \tilde{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \\ D &\to \tilde{D} = 0, \end{aligned} \tag{2.29}$$

then inequalities (2.23)-(2.25) *also hold with* X, L *and* Z_i.

Proof. Suppose (2.26)-(2.28) are satisfied with the substitutions (2.29). Proceed as in the proof of Theorem 1 and define $\mathcal{R}_i$, $\mathcal{B}$ as in (2.17) to show

$$\mathcal{R}_i + \mathcal{R}_i^T + \mathcal{W}_i \prec 0,$$

where

$$\mathcal{W}_i = \tilde{E}_i W \tilde{E}_i^T = \begin{bmatrix} E_i W E_i^T & 0 \\ 0 & 0 \end{bmatrix} \succeq 0. \tag{2.30}$$

Consequently,

$$\mathcal{B}(\mathcal{R}_i + \mathcal{R}_i^T + \mathcal{W}_i)\mathcal{B}^T \prec 0,$$

for all $i = 1, \ldots, N$, where the above inequality is the same as (2.23). Secondly, by $\tilde{D} = 0$, inequalities (2.27) with substitutions (2.29) can be expanded as

$$\begin{bmatrix} Z_i & C_i X + D_i L & C_i L^T + D_i Y \\ X C_i^T + L^T D_i^T & X & L^T \\ L C_i^T + Y D_i^T & L & Y \end{bmatrix} \succ 0$$

for all $i = 1, \ldots, N$, which after multiplication by

$$\mathcal{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

from the left and $\mathcal{C}^T$ from the right yields (2.24). Finally, note that the matrices Z_i are the same in inequalities (2.28) and (2.25). □

Example 2. *To illustrate the last result, consider again the system introduced in Example 1 with $\gamma = 0.95$ being the maximum possible value for which we could verify quadratic stabilizability. Further let*

$$C_1 = C_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \qquad C_3 = C_4 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$
$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

One can verify that the minimum upper bound for the cost (2.20) *computed by first building the augmented matrices* (2.14) *and* (2.22) *and then minimizing μ subject to inequalities* (2.26)-(2.28) *is $\mu \approx 161.5$. In fact, the matrices $\tilde{X}$ and $\tilde{L}_i$ and the gain-scheduled controller already shown in Example 1 are the ones that achieve this minimum cost. Because the structure of the partition in $\tilde{X}$ and associated projection in Theorems 1 and 3 are the same, the robust controller in Example 1 is also the one that minimizes the upper bound to μ in inequalities* (2.23)-(2.25)*.*

The reason why the same matrices from Example 1 yield the controller with the optimal performance is due to the fact that $\gamma = 0.95$ is at the boundary of the set of stabilizing controllers. To illustrate the case in which more freedom is bestowed to the controller design problem we solved the problem again for $\gamma = 0.9$. If we design a robust controller using Lemma 3, we *achieve* $\mu = 112.65$ *with*

$$X = \begin{bmatrix} 4.02 & -3.68 & 6.44 & -1.08 \\ -3.68 & 8.12 & -11.06 & -1.07 \\ 6.44 & -11.06 & 22.22 & -5.43 \\ -1.08 & -1.07 & -5.43 & 8.3 \end{bmatrix}, \tag{2.31}$$

and

$$L = \begin{bmatrix} -13.95 & 5.93 & -24.87 & 12.66 \end{bmatrix}, \tag{2.32}$$

for which the corresponding robust gain is

$$K = LX^{-1} = \begin{bmatrix} -4.05 & -6.55 & -3.78 & -2.32 \end{bmatrix}.$$

If we instead design a gain-scheduled controller using Lemma 4 after augmenting the original system, we achieve the same value of $\mu = 112.65$ *with*

$$X \to \tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}$$

where X *and* L *are as in equations* (2.31) *and* (2.32) *above and* $Y = 82.13$*. We also have the following* $\tilde{L}_i$ *matrices*

$$\tilde{L}_1 = \begin{bmatrix} -31.45 & -65.6 & -35.55 & 40.44 & -15.29 \end{bmatrix},$$
$$\tilde{L}_2 = \begin{bmatrix} -11.79 & -65.6 & -15.89 & 40.44 & -15.29 \end{bmatrix},$$
$$\tilde{L}_3 = \begin{bmatrix} -31.45 & 16.53 & -35.55 & -41.7 & -15.29 \end{bmatrix},$$
$$\tilde{L}_4 = \begin{bmatrix} -11.79 & 16.53 & -15.89 & -41.7 & -15.29 \end{bmatrix}.$$

The following result extends Theorem 2.

Theorem 4. *If there exist matrices X, L and Z_i such that inequalities* (2.23)-(2.25) *hold, then there also exist matrices $\tilde{X}$ and $\tilde{L}_i = \tilde{L}$ such that inequalities* (2.26)-(2.28) *hold with*

$$\begin{aligned}
&X \rightarrow \tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}, \quad L_i \rightarrow \tilde{L}_i = \tilde{L}, \\
&A_i \rightarrow \tilde{A}_i = \begin{bmatrix} A_i & B_i \\ F_i & G_i \end{bmatrix}, \quad B \rightarrow \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\
&C_i \rightarrow \tilde{C}_i = \begin{bmatrix} C_i & D_i \end{bmatrix}, \quad E_i \rightarrow \tilde{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \\
&D \rightarrow \tilde{D}_i = 0,
\end{aligned} \tag{2.33}$$

for any choice of the matrices F_i, G_i.

Proof. Assume (2.23)-(2.25) hold for some X, L and Z_i. From (2.24), note that there exists $\epsilon > 0$ sufficiently small such that

$$Z_i - (C_i X + D_i L) X^{-1} (C_i X + D_i L)^T \succ \epsilon D_i D_i^T,$$

which after a Schur complement is,

$$\begin{bmatrix} Z_i - \epsilon D_i D_i^T & C_i X + D_i L \\ X C_i^T + L^T D_i^T & X \end{bmatrix} \succ 0,$$

and hence

$$\begin{bmatrix} Z_i & C_i X + D_i L & \epsilon D_i \\ X C_i^T + L^T D_i^T & X & 0 \\ \epsilon D_i^T & 0 & \epsilon I \end{bmatrix} \succ 0,$$

for all $i = 1, \ldots, N$. Choosing

$$Y = L X^{-1} L^T + \epsilon I,$$

this is equivalent to

$$\begin{bmatrix} Z_i & C_iX + D_iL & D_i(Y - LX^{-1}L^T) \\ XC_i^T + L^TD_i^T & X & 0 \\ (Y - LX^{-1}L^T)D_i^T & 0 & Y - LX^{-1}L^T \end{bmatrix} \succ 0,$$

$i = 1, \ldots, N$. Multiplying by the matrix

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & LX^{-1} & I \end{bmatrix}$$

from the left and $\mathcal{T}^T$ from the right, one obtains

$$\begin{bmatrix} Z_i & \tilde{C}_i\tilde{X} \\ \tilde{X}\tilde{C}_i^T & \tilde{X} \end{bmatrix} \succ 0,$$

for all $i = 1, \ldots, N$, which is equivalent to (2.27) with the substitutions (2.33). Further notice that, after choosing

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}, \qquad \tilde{L} = \begin{bmatrix} 0 & -(\alpha/2)I \end{bmatrix},$$

the left-hand side of (2.26) with substitutions (2.33) reads

$$\tilde{A}_i\tilde{X} + \tilde{X}\tilde{A}_i^T + \tilde{B}\tilde{L} + \tilde{L}^T\tilde{B}^T + \tilde{E}_iW\tilde{E}_i^T = \begin{bmatrix} \Phi_i + E_iWE_i^T & \Psi_i^T \\ \Psi_i & \Omega_i - \alpha I \end{bmatrix},$$

where Φ_i, Ψ_i and Ω_i are as in the proof of Theorem 2 and $\Phi_i + E_iWE_i^T \prec 0$ is inequality (2.23). The proof is complete after noting that for any $\epsilon > 0$, there exists a sufficiently large α such that (2.26) is satisfied, namely $\alpha > \max_i \lambda_{\max}(\Omega_i - \Psi_i(\Phi_i + E_iWE_i^T)^{-1}\Psi_i^T)$ and that the matrices Z_i are identical in (2.25) and (2.28). □

Notice that a converse result in the sense of Corollary 1 cannot exist for Lemmas 3 and 4. The main obstruction is the fact that $\tilde{D} = 0$ in (2.22), which allows for $\tilde{L}$ to be chosen as in the above proof so as to satisfy the augmented version of the inequality (2.26) independently of the inequality (2.27). Indeed, the following is a numerical counter-example.

Example 3. *Consider a slightly modified version of the system in Example 1, with constant input matrix*

$$B_1 = B_2 = B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and vertices

$$C_1 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_1 = D_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

An upper bound $\mu_{GS} \approx 3.11$ *is obtained with the gain-scheduled controller that minimizes* μ *in Lemma 4, whereas the upper bound* $\mu_R \approx 3.35 > \mu_{GS}$ *is obtained with the robust controller that minimizes* μ *in Lemma 3.*

2.5.2 Observability Gramian

The properties established above for Lemmas 3 and 4 also hold for a dual characterization of H_2 performance in terms of an upper bound on the Observability Gramian, as stated through the following set of results.

Lemma 5 ([43]). *If there exist matrices X, L and Z_i such that*

$$\begin{bmatrix} A_i X + X A_i^T + B_i L + L^T B_i^T & X C_i^T + L^T D_i^T \\ C_i X + D_i L & -I \end{bmatrix} \prec 0, \tag{2.34}$$

$$\begin{bmatrix} Z_i & E_i^T \\ E_i & X \end{bmatrix} \succ 0, \tag{2.35}$$

$$\operatorname{trace}(Z_i W) < \mu \tag{2.36}$$

for all $i = 1, \ldots, N$, then the robust state-feedback controller

$$u(t) = K\,x(t), \qquad K = L\,X^{-1}$$

is such that $A(\xi(t)) + B(\xi(t))\,K$ is quadratically stable and (2.20) *holds for all $\xi(t) \in \Xi$.*

Lemma 6 ([44]). *If there exist matrices X, L_i and Z_i such that*

$$\begin{bmatrix} A_i X + X A_i^T + B L_i + L_i^T B^T & X C_i^T + L_i^T D^T \\ C_i X + D L_i & -I \end{bmatrix} \prec 0, \tag{2.37}$$

$$\begin{bmatrix} Z_i & E_i^T \\ E_i & X \end{bmatrix} \succ 0, \tag{2.38}$$

$$\operatorname{trace}(Z_i W) < \mu \tag{2.39}$$

for all $i = 1, \ldots, N$, then the gain-scheduled state-feedback controller

$$u(t) = K(\xi(t))\,x(t), \qquad K(\xi(t)) = \sum_{k=1}^{N} \xi_i(t)\,K_i, \qquad K_i = L_i\,X^{-1}$$

is such that $A(\xi(t)) + B\,K(\xi(t))$ is quadratically stable and (2.20) *holds for all $\xi(t) \in \Xi$.*

Theorem 5. *If there exist matrices X, L, Y, F_i, G_i, U_i, V_i and Z_i such that inequalities* (2.37)-(2.39) *hold with* (2.29), *then inequalities* (2.34)-(2.36) *also hold with X, L and Z_i.*

Proof. Suppose inequalities (2.37)-(2.39) are satisfied with the substitutions (2.29). Define $\mathcal{R}_i$

as in (2.17) in the proof of Theorem 1 and

$$\mathcal{P}_i = \begin{bmatrix} XC_i^T + L^T D_i^T \\ LC_i^T + Y D_i^T \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{2.40}$$

Inequalities (2.37) with (2.29) then read

$$\begin{bmatrix} \mathcal{R}_i + \mathcal{R}_i^T & \mathcal{P}_i \\ \mathcal{P}_i^T & -I \end{bmatrix} \prec 0,$$

such that

$$\mathcal{B} \begin{bmatrix} \mathcal{R}_i + \mathcal{R}_i^T & \mathcal{P}_i \\ \mathcal{P}_i^T & -I \end{bmatrix} \mathcal{B}^T \prec 0$$

for all $i = 1, \ldots, N$, which is (2.34). By $\tilde{D} = 0$, inequalities (2.38) with substitutions (2.29) can be expanded as

$$\begin{bmatrix} Z & E_i & 0 \\ E_i & X & L^T \\ 0 & L & Y \end{bmatrix} \succ 0$$

for all $i = 1, \ldots, N$, which after multiplication by

$$\mathcal{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

from the left and $\mathcal{C}^T$ from the right yields (2.35). Finally, notice that the matrices Z_i are identical in inequalities (2.39) and (2.36). □

Theorem 6. *If there exist matrices X, L and Z_i such that inequalities* (2.34)-(2.36) *hold, then there also exist matrices $\tilde{X}$ and $\tilde{L}_i = \tilde{L}$ such that inequalities* (2.37)-(2.39) *hold with* (2.33)*, for any choice of the matrices F_i, G_i.*

Proof. Assume (2.34)-(2.36) hold for some X, L and Z_i. From (2.35), choose

$$Y = LX^{-1}L^T + \Gamma I,$$

and note that there exists $\Gamma > 0$ sufficiently large such that

$$\begin{bmatrix} Z_i & E_i^T & E_i^T X^{-1} L^T \\ E_i & X & 0 \\ -LX^{-1}E_i & 0 & Y - LX^{-1}L^T \end{bmatrix} = \begin{bmatrix} Z_i & E_i^T & -E_i^T X^{-1} L^T \\ E_i & X & 0 \\ -LX^{-1}E_i & 0 & \Gamma I \end{bmatrix} \succ 0 \tag{2.41}$$

for all $i = 1, \ldots, N$. Now define

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & LX^{-1} & I \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}$$

and multiply (2.41) by $\mathcal{T}$ from the left and $\mathcal{T}^T$ from the right to recover (2.38) with (2.33). Now notice that, after choosing

$$\tilde{L} = \begin{bmatrix} 0 & -(\alpha/2)I \end{bmatrix},$$

the left-hand side of (2.37) with substitutions (2.33) reads

$$\begin{bmatrix} \tilde{A}_i\tilde{X} + \tilde{X}\tilde{A}_i^T + \tilde{B}\tilde{L} + \tilde{L}^T\tilde{B}^T & \tilde{X}\tilde{C}_i^T + \tilde{L}^T\tilde{D} \\ \tilde{C}_i\tilde{X} + \tilde{D}\tilde{L} & -I \end{bmatrix} = \begin{bmatrix} \Phi_i & \Psi_i^T & XC_i^T + L^T D_i^T \\ \Psi_i & \Omega_i^T - \alpha I & LC_i^T + YD_i^T \\ C_iX + D_iL & C_iL^T + D_iY & -I \end{bmatrix},$$

where Φ_i, Ψ_i and Ω_i are as in the proof of Theorem 2. Now choose α sufficiently large such that (2.37) holds, namely

$$\alpha > \max_i \lambda_{\max}\left(\Omega_i^T - \begin{bmatrix}\Psi_i & LC_i^T + YD_i^T\end{bmatrix}\begin{bmatrix}\Phi_i & XC_i^T + L^T D_i^T \\ C_i X + D_i L & -I\end{bmatrix}^{-1}\begin{bmatrix}\Psi_i^T \\ C_i L^T + D_i Y\end{bmatrix}\right),$$

and note that the terms in (2.36) and (2.39) are identical to complete the proof. □

2.6 Quadratic H_∞ Performance

In this section, we show how our findings for quadratic H_2 performance extend to quadratic H_∞ performance (see e.g. [36]). That is, we seek to determine state-feedback controllers so that the closed-loop upper bound on the L_2 gain,

$$\nu > \sup_{\|w\|_2=1} \|z\|_2 \tag{2.42}$$

holds over all parameter trajectories $\xi(t) \in \Xi$, where

$$\|v\|_2 = \int_0^\infty v(t)^T v(t)\, dt$$

denotes the L_2 norm of signal $v(t)$. The following conditions characterize robust and gain-scheduled controllers with H_∞ performance certificate.

Lemma 7 ([45]). *If there exist matrices $X \succ 0$ and L such that*

$$\begin{bmatrix} A_i X + X A_i^T + B_i L + L^T B_i^T + E_i E_i^T & XC_i^T + L^T D_i^T \\ C_i X + D_i L & -\nu^2 I \end{bmatrix} \prec 0 \tag{2.43}$$

for all $i = 1, \ldots, N$, then the robust state-feedback controller

$$u(t) = K\, x(t), \qquad K = L\, X^{-1}$$

is such that $A(\xi(t)) + B(\xi(t))\, K$ is quadratically stable and (2.42) holds for all $\xi(t) \in \Xi$.

Lemma 8 ([46]). *If there exist matrices $X \succ 0$ and L_i such that*

$$\begin{bmatrix} A_i X + X A_i^T + B L_i + L_i^T B^T + E_i E_i^T & X C_i^T + L_i^T D^T \\ C_i X + D L_i & -\nu^2 I \end{bmatrix} \prec 0 \tag{2.44}$$

for all $i = 1, \dots, N$, then the gain-scheduled state-feedback controller

$$u(t) = K(\xi(t))\, x(t), \qquad K(\xi(t)) = \sum_{k=1}^{N} \xi_i(t)\, K_i, \qquad K_i = L_i\, X^{-1}$$

is such that $A(\xi(t)) + B\, K(\xi(t))$ is quadratically stable and (2.42) *holds for all $\xi(t) \in \Xi$.*

Theorem 7. *If there exist matrices X, L, Y, F_i, G_i, U_i and V_i such that inequalities* (2.44) *hold with* (2.29) *and $\tilde{X} \succ 0$, then inequalities* (2.43) *also hold with $X \succ 0$ and L.*

Proof. Define $\mathcal{R}_i$ as in (2.17), $\mathcal{B}$, $\mathcal{P}_i$ as in (2.40), and $\mathcal{W}_i$ as in (2.30) with $W = I$. If inequalities (2.44) are satisfied with the substitutions (2.29) and $\tilde{X} \succ 0$, then

$$\begin{bmatrix} \mathcal{R}_i + \mathcal{R}_i^T + \mathcal{W}_i & \mathcal{P}_i \\ \mathcal{P}_i^T & -\nu^2 I \end{bmatrix} \prec 0$$

for all $i = 1, \dots, N$. Consequently

$$\mathcal{B} \begin{bmatrix} \mathcal{R}_i + \mathcal{R}_i^T + \mathcal{W}_i & \mathcal{P}_i \\ \mathcal{P}_i^T & -\nu^2 I \end{bmatrix} \mathcal{B}^T \prec 0$$

for all $i = 1, \dots, N$, which is (2.43). □

Theorem 8. *If there exist matrices $X \succ 0$ and L such that inequalities* (2.43) *hold, then there also exist matrices $\tilde{X} \succ 0$ and $\tilde{L}_i = \tilde{L}$ such that inequalities* (2.44) *hold with* (2.33), *for any choice of the matrices F_i, G_i.*

Proof. Assume inequalities (2.43) hold for some $X \succ 0$ and L. Define

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}, \qquad Y = LX^{-1}L^T + \epsilon I, \qquad \tilde{L} = \begin{bmatrix} 0 & -(\alpha/2)I \end{bmatrix},$$

and choose any $\epsilon > 0$ such that $\tilde{X} \succ 0$. Moreover, let Φ_i, Ψ_i and Ω_i be as in the proof of Theorem 2. Recall that $\tilde{D} = 0$ and proceed as in the second half of the proof of Theorem 6 to obtain

$$\begin{bmatrix} \tilde{A}_i\tilde{X} + \tilde{X}\tilde{A}_i^T + \tilde{B}\tilde{L} + \tilde{L}^T\tilde{B}^T + \tilde{E}_i\tilde{E}_i^T & \tilde{X}\tilde{C}_i^T + \tilde{L}^T\tilde{D}^T \\ \tilde{C}_i\tilde{X} + \tilde{D}\tilde{L} & -\nu^2 I \end{bmatrix} = \begin{bmatrix} \Phi_i + E_iE_i^T & \Psi_i^T & XC_i^T + L^TD_i^T \\ \Psi_i & \Omega_i^T - \alpha I & LC_i^T + YD_i^T \\ C_iX + D_iL & C_iL^T + D_iY & -\nu^2 I \end{bmatrix} \prec 0$$

for all $i = 1, \ldots, N$ and sufficiently large $\alpha > 0$. The calculation of the range of α necessary is similar to the one in the proof of Theorem 6 and is omitted for brevity. This is (2.37). □

Example 4. *To conclude, we will extend Example 2 to cover the quadratic H_∞ results in Theorems 7 and 8. We first utilize Lemma 8 to design a gain-scheduled controller. We choose $\gamma = 0.9$ and solve Lemma 8 with substitutions*

$$A_i \to \tilde{A}_i, \qquad B \to \tilde{B}, \qquad X \to \tilde{X}, \qquad L_i \to \tilde{L}_i.$$

The corresponding matrices $\tilde{X}$ and $\tilde{L}_i$ are

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}$$
$$= \left[\begin{array}{cccc|c} 314.72 & -321.63 & 272.35 & -118.08 & -66.92 \\ -321.63 & 336.4 & -290.4 & 114.89 & 41.15 \\ 272.35 & -290.4 & 258.13 & -101.16 & -41.33 \\ -118.08 & 114.89 & -101.16 & 66.87 & 9.68 \\ \hline -66.92 & 41.15 & -41.33 & 9.68 & 30.46 \times 10^4 \end{array}\right]$$

and

$$\tilde{L}_1 = \begin{bmatrix} 1.16 & -3.76 & 0.72 & -0.17 & -2861.24 \end{bmatrix} \times 10^5,$$
$$\tilde{L}_2 = \begin{bmatrix} 1.16 & -3.76 & 0.72 & -0.17 & -2860.68 \end{bmatrix} \times 10^5,$$
$$\tilde{L}_3 = \begin{bmatrix} 1.16 & -0.71 & 0.72 & -3.21 & -2861.88 \end{bmatrix} \times 10^5,$$
$$\tilde{L}_4 = \begin{bmatrix} 1.16 & -0.72 & 0.72 & -3.21 & -2861.36 \end{bmatrix} \times 10^5,$$

where we have $\nu = 13.24$. *However, as a consequence of Theorem 7, we also have the following robust controller,*

$$K = LX^{-1} = \begin{bmatrix} -88.55 & -135.66 & -71.53 & -31.49 \end{bmatrix} \times 10^2,$$

which also achieves $\nu = 13.24$.

2.7 Discussion

We have shown constructively that there is no advantage in gain-scheduled control via state augmentation and input filtering when concerned with state-feedback of continuous-time LPV systems with affine dependence on the scheduling parameter using quadratic stability. We have shown this for both stabilizability as well as H_2, H_∞ performance measures. The results are remarkable in that one would usually expect to improve stabilizability or at least achievable performance when granting the controller access to information on system parameter variation. A practical consequence is that state augmentation with the goal of accommodating variations on the input matrix is always counterproductive on a quadratic stability setup. As shown in the current chapter, the additional complexity of the resulting gain-scheduled controllers provides no benefit over simpler robust controllers even when performance is optimized. Another repercussion of these new insights is that even more recent non-quadratic conditions for gain-scheduled control found in the literature might in fact also be equivalent to corresponding conditions for robust control, and that the projection techniques introduced in this chapter could be used to verify such potential limitation.

2.8 Acknowledgments

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Chapter 3

Pre-Filtering in State-Feedback Control - The Discrete-time Case

3.1 Introduction

The purpose of this chapter is to discuss robust and gain-scheduled stabilizability for discrete-time systems. The results presented here are a distinction to those in Chapter 2 as simple counter-examples are known where discrete-time gain-scheduled stabilizability does not imply robust stabilizability [47]. However, we will show that the same kind of implications observed in the continuous-time case in Chapter 2 hold for quadratic discrete-time stabilizability.

3.2 Discrete-Time Quadratic Stabilizability

Consider discrete-time linear systems of the form

$$x(k+1) = A(\xi(k))x(k) + B(\xi(k))u(k), \tag{3.1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ and the matrices $A(\xi(k))$ and $B(\xi(k))$ are assumed to depend affinely on the time-varying parameter $\xi(k)$, which assumes values in the unit simplex

$$\Xi = \left\{ \xi(k) \in \mathbb{R}^N : \quad \xi_i(k) \geq 0, \quad \sum_{i=1}^{N} \xi_i(k) = 1 \right\}.$$

That is, matrices $A(\xi(k))$ and $B(\xi(k))$ can be written as

$$\begin{bmatrix} A(\xi(k)) & B(\xi(k)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(k) \begin{bmatrix} A_i & B_i \end{bmatrix}, \qquad \xi(k) \in \Xi.$$

The discussion throughout this chapter relies heavily on the notion of quadratic stability [34, 21]. Matrix $A(\xi(k))$ with $\xi(k) \in \Xi$ is quadratically stable if there exist a matrix X such that

$$A_i^T X A_i - X \prec 0, \tag{3.2}$$

for all $i = 1, \dots, N$. Quadratic stability of matrix $A(\xi(k))$ implies asymptotic stability of the system

$$x(k+1) = A(\xi(k))x(k), \xi(t) \in \Xi,$$

as can be verified easily using the quadratic Lyapunov function $V(x) = x^T X x$, thus the term *quadratic stability*.

3.2.1 Robust Quadratic Stabilization

Quadratic stability can be used to design robust linear state-feedback controllers of the form $u(k) = K\,x(k)$ upon substitution of $A_i \to A_i + B_i\,K$ in (3.2) and the one-to-one change of variables $L = K\,X$ [35, 36]. This yields an LMI in the matrices X and L.

Lemma 9. *If there exist matrices X and L such that*

$$\begin{bmatrix} X & A_i X + B_i L \\ X A_i^T + L^T B_i^T & X \end{bmatrix} \succ 0 \tag{3.3}$$

for all $i = 1, \dots, N$, then the robust controller

$$u(k) = K\,x(k), \qquad K = L\,X^{-1} \tag{3.4}$$

is such that $A(\xi) + B(\xi)K$ *is quadratically stable for all* $\xi \in \Xi$*.*

3.2.2 Gain-Scheduled Quadratic Stabilization

When the parameter $\xi(k)$ can be measured online, one might wonder whether a gain-scheduled controller of the form

$$u(k) = K(\xi(k))\, x(k), \tag{3.5}$$

can bring any advantage. As in the continuous-time case discussed in Chapter 2, we restrict ourselves to the case where $B_i = B$ for all $i = 1, \ldots, N$ to obtain the following lemma.

Lemma 10. *If there exist matrices* X *and* L_i *such that*

$$\begin{bmatrix} X & A_i X + BL_i \\ XA_i^T + L_i^T B^T & X \end{bmatrix} \succ 0 \tag{3.6}$$

for all $i = 1, \ldots, N$*, then the gain-scheduled controller* (3.5) *with gain*

$$K(\xi) = \sum_{k=1}^{N} \xi_i K_i, \quad K_i = L_i X^{-1} \tag{3.7}$$

is such that $A(\xi) + BK(\xi)$ *is quadratically stable for all* $\xi \in \Xi$*.*

3.3 State Augmentation via Pre-Filtering

As in the continuous-time case, we can introduce a dynamic *pre-filter* to handle the dependance of B on the parameter $\xi(k)$. In discrete-time, we have the following dynamic gain-scheduled controller,

$$\begin{aligned} r(k+1) &= K_u(\xi(k))\, r(k) + K_x(\xi(k))\, x(k), \\ u(k) &= r(k), \end{aligned} \tag{3.8}$$

where K_u and K_x are obtained from the augmented gain

$$\tilde{K}(\xi(k)) = \begin{bmatrix} K_x(\xi(k)) & K_u(\xi(k)) \end{bmatrix}. \tag{3.9}$$

where we solve the stabilizability problem for the augmented system realization

$$\tilde{x}(k+1) = \tilde{A}(\xi(k))\,\tilde{x}(k) + \tilde{B}\,\tilde{u}(k), \qquad \tilde{x}(k) = \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \tag{3.10}$$

where the matrices $\tilde{A}(\xi(k))$ and $\tilde{B}$ have vertices

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i \\ F_i & G_i \end{bmatrix}, \qquad \tilde{B}_i = \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{3.11}$$

As anticipated, we will now show that gain-scheduled control via pre-filtering and condition (3.6) yields no advantage in terms of stabilizability when compared with robust controllers designed via condition (3.3).

Theorem 9. *Let $\tilde{A}_i$ and $\tilde{B}$ be as in* (3.11)*. If there exist matrices F_i, G_i, $\tilde{L}_i$ and*

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix} \tag{3.12}$$

such that the inequalities (3.6) *in Lemma 10 are feasible with $A_i \to \tilde{A}_i$, $B \to \tilde{B}$, then the inequalities* (3.3) *in Lemma 9 are also feasible with the above matrices X and L.*

Proof. Define

$$\mathcal{R}_i = \begin{bmatrix} A_iX + B_iL & A_iL^T + B_iY \\ F_iX + G_iL + U_i & F_iL^T + G_iY + V_i \end{bmatrix} \tag{3.13}$$

and

$$\mathcal{B} = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}. \tag{3.14}$$

If inequality (3.6) holds for some F_i, G_i, $A_i \to \tilde{A}_i$, $B \to \tilde{B}$ as in (3.11), and $\tilde{X}$ with the structure as given in (3.12) and

$$\tilde{L}_i = \begin{bmatrix} U_i & V_i \end{bmatrix}, \tag{3.15}$$

then

$$\begin{bmatrix} \tilde{X} & \mathcal{R}_i \\ \mathcal{R}_i^T & \tilde{X} \end{bmatrix} \succ 0.$$

Consequently,

$$\mathcal{B}^T \begin{bmatrix} \tilde{X} & \mathcal{R}_i \\ \mathcal{R}_i^T & \tilde{X} \end{bmatrix} \mathcal{B} = \begin{bmatrix} X & A_i X + B_i L \\ X A_i^T + L^T B_i^T & X \end{bmatrix} \succ 0,$$

which completes the proof. □

This shows that there is no advantage in terms of quadratic stabilizability for discrete-time uncertain linear systems if one uses system augmentation to handle uncertainty in the input matrix B. This is even more surprising in the discrete-time context because simple counter-examples are known when discrete-time gain-scheduled stabilizability does not imply robust stabilizability [47].

Example 5. *Consider the following discrete-time example from [48]. Our goal is to determine* *the largest $\gamma > 0$ such that the discrete-time system given by*

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\alpha & -0.5\alpha & 0.2 & 0.03+\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B(\beta) = \begin{bmatrix} \beta \\ 0 \\ 1-\beta \\ 0 \end{bmatrix},$$

where $0 \leq \beta \leq 1$ can be quadratically stabilized for all $|\alpha| \leq \gamma$. This system can be put in the form (2.1) with 4 vertices. To design a gain-scheduled controller as prescribed in Lemma 10, we first augment the system as in (3.11). This time, just for illustration, we chose non the zero matrices F_i's and G_i's

$$F_1 = F_2 = F_3 = F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

and $G_1 = G_2 = G_3 = G_4 = 0$. We solve the LMIs for the augmented system given in Lemma 10 and found $\gamma = 0.5$ to be the largest possible value we could design a quadratically stabilizing gain-scheduled controller. The corresponding matrices $\tilde{X}$ and $\tilde{L}_i$'s are

$$\tilde{X} = \begin{bmatrix} X & L^T \\ L & Y \end{bmatrix}$$

$$= \left[\begin{array}{cccc|c} 1.596 & 0.988 & 0.174 & -0.616 & -0.026 \\ 0.988 & 2.497 & 0.037 & 0.202 & 0.036 \\ 0.174 & 0.037 & 0.214 & -0.158 & -0.103 \\ -0.616 & 0.202 & -0.158 & 0.642 & 0.046 \\ \hline -0.026 & 0.036 & -0.103 & 0.046 & 0.071 \end{array}\right],$$

and

$$\tilde{L}_1 = \begin{bmatrix} -2.165 & -3.635 & -0.287 & -0.085 & 0.063 \end{bmatrix},$$
$$\tilde{L}_2 = \begin{bmatrix} -2.073 & -3.774 & -0.307 & -0.059 & 0.066 \end{bmatrix},$$
$$\tilde{L}_3 = \begin{bmatrix} -2.148 & -3.656 & -0.234 & -0.109 & 0.025 \end{bmatrix},$$
$$\tilde{L}_4 = \begin{bmatrix} -2.057 & -3.794 & -0.254 & -0.084 & 0.030 \end{bmatrix}.$$

Using the above partition of $\tilde{X}$ *we compute the corresponding robust controller*

$$K = LX^{-1} = \begin{bmatrix} -0.007 & 0.031 & -0.538 & -0.077 \end{bmatrix}.$$

3.4 Discrete-Time Parameter-Dependent Stabilizability

The following sequence of results extends our observations to non-quadratic stabilizability based on the parameter-dependent conditions presented in [48], for the time-invariant case, and [10], for the time-varying case. Let us first consider the time-invariant conditions from [48].

We show that even in this case, stabilizability of uncertain discrete-time linear systems is not improved via gain-scheduled control design after pre-filtering as in (3.11). A robust parameter-dependent stabilizability result as in Lemma 9 was presented in [48] and allows for extension to a gain-scheduled condition as in Lemma 10. Notice that both conditions require the uncertain parameter ξ to be constant.

Lemma 11. *If there exist matrices* X_i*,* S *and* M *such that*

$$\begin{bmatrix} X_i & A_i S + B_i M \\ S^T A_i^T + M^T B_i^T & S + S^T - X_i \end{bmatrix} \succ 0 \tag{3.16}$$

for all $i = 1, \ldots, N$*, then the robust controller* (3.4) *with gain*

$$K = MS^{-1} \tag{3.17}$$

is such that the matrix $A(\xi) + B(\xi)K$ is robustly stable for all time-invariant $\xi \in \Xi$.

Lemma 12. *If there exist matrices X_i, S and M_i such that*

$$\begin{bmatrix} X_i & A_i S + B M_i \\ S^T A_i^T + M_i^T B^T & S + S^T - X_i \end{bmatrix} \succ 0 \tag{3.18}$$

for all $i = 1, \ldots, N$, then the gain-scheduled controller with gain

$$K(\xi) = \sum_{k=1}^{N} \xi_i K_i, \quad K_i = M_i S^{-1} \tag{3.19}$$

is such that the matrix $A(\xi) + BK(\xi)$ is stable for all time-invariant $\xi \in \Xi$.

As before, we prove that gain-scheduled control via augmentation and condition (3.18) yields no advantage in terms of stabilizability when compared with controllers designed via condition (3.16).

Theorem 10. *Let $\tilde{A}_i$ and $\tilde{B}$ be as in (3.11). If there exist matrices F_i, G_i, $\tilde{M}_i$ and*

$$\tilde{X}_i = \begin{bmatrix} X_i & L_i^T \\ L_i & Y_i \end{bmatrix}, \qquad \tilde{S} = \begin{bmatrix} S & R \\ M & Q \end{bmatrix} \tag{3.20}$$

such that the inequalities (3.18) in Lemma 12 are feasible with $A_i \to \tilde{A}_i$, $B \to \tilde{B}$, then the inequalities (3.16) in Lemma 11 are also feasible with the above matrices X_i, S and M.

Proof. Define

$$\mathcal{R}_i = \begin{bmatrix} A_i S + B_i M & A_i R + B_i Q \\ F_i S + G_i M + U_i & F_i R + G_i Q + V_i \end{bmatrix} \tag{3.21}$$

and $\mathcal{B}$ as in (3.14). If inequalities (3.6) hold for some F_i, G_i, $A_i \to \tilde{A}_i$, $B \to \tilde{B}$ as in (2.14), $\tilde{X}_i$, $\tilde{S}$ with structure as given in (3.20) and

$$\tilde{M}_i = \begin{bmatrix} U_i & V_i \end{bmatrix},$$

then

$$\begin{bmatrix} \tilde{X}_i & \mathcal{R}_i \\ \mathcal{R}_i^T & \tilde{S} + \tilde{S}^T - \tilde{X}_i \end{bmatrix} \succ 0.$$

Consequently,

$$\mathcal{B}^T \begin{bmatrix} \tilde{X}_i & \mathcal{R}_i \\ \mathcal{R}_i^T & \tilde{S} + \tilde{S}^T - \tilde{X}_i \end{bmatrix} \mathcal{B} = \begin{bmatrix} X_i & A_i S + B_i M \\ S^T A_i^T + M^T B_i^T & S + S^T - X_i \end{bmatrix} \succ 0,$$

which completes the proof. □

Now consider the time-varying stabilizability condition from [10].

Lemma 13. *If there exist matrices X_i, G_i and M_i such that*

$$\begin{bmatrix} X_j & A_i S_i + B M_i \\ S_i^T A_i^T + M_i^T B^T & S_i + S_i^T - X_i \end{bmatrix} \succ 0 \tag{3.22}$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$, then the gain-scheduled controller with gain

$$K(\xi) = \sum_{k=1}^{N} \xi_i K_i, \quad K_i = M_i S_i^{-1} \tag{3.23}$$

is such that the matrix $A(\xi) + BK(\xi)$ is robustly stable for all time-varying $\xi \in \Xi$.

Gain-scheduled controllers obtained from the conditions in Lemma 13 do not seem to reduce to robust controllers even in the case of system augmentation. In fact, it is possible to obtain a counter-example in the case of the following constrained version of Lemma 13.

Lemma 14. *If there exist matrices X_i, S and M such that*

$$\begin{bmatrix} X_j & A_i S + B_i M \\ S^T A_i^T + M^T B_i^T & S + S^T - X_i \end{bmatrix} \succ 0 \tag{3.24}$$

for all $i = 1, \ldots, N$*, then the robust controller with gain*

$$K = MS^{-1} \tag{3.25}$$

is such that the matrix $A(\xi) + B(\xi)K$ *is robustly stable for all time-varying* $\xi \in \Xi$*.*

It turns out that for this pair of conditions from [10], there is no result in parallel with the above theorems. That is, we can find gain-scheduled stabilizing controllers via pre-filtering when there does not exist a robustly stabilizing state-feedback controller. This is illustrated with the following example.

Example 6. *Consider again Example 5 with conditions* (3.24) *and* (3.22)*. For* $\gamma = 0.59$*, we can find a stabilizing gain-scheduled controller after augmenting the system using* (3.11) *and solving condition* (3.22) *in Lemma 13. However, condition* (3.24) *in Lemma 14 is not solvable for any* $\gamma > 0.56$.

Notice that the reason why the projections used before do not work in this case is due to the N matrices S_i's in Lemma 11.

3.5 Discussion

In this chapter, we have extended the results from Chapter 2 to discrete-time systems. In particular, we have shown that in the case of quadratic stabilizability, that there is no advantage for discrete-time time-varying linear systems if one uses system augmentation to handle uncertainty in the input matrix B. In the case of parameter-dependent Lyapunov functions, the same result holds for time-invariant conditions. A simple counter example illustrated that this is no longer true in the case of the time-varying parameter-dependant conditions from [10].

If one wants to use the conditions from [10] in the case of a time-varying $B(\xi(k))$, then the use of system augmentation is required to handle uncertainty in the input matrix $B(\xi(k))$. Even though our counter example suggests that this is can still bring advantages compared to simply using a robust controller, there are still drawbacks in using system augmentation. The most significant drawback is that the resulting controller is much more complex, being no longer static.

Given this drawback of system augmentation, it would be beneficial if LMI conditions existed to extend to conditions of [10] to the case where there is time variation in the input matrix $B(\xi(k))$ without having to resort to system augmentation. We will introduce these conditions in the following chapter.

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Chapter 4

Discrete-Time H_∞ Control of LPV Systems

4.1 Introduction and Motivation

Consider time-varying discrete-time linear systems of the form

$$\begin{aligned} x(k+1) &= A(\xi(k))x(k) + B(\xi(k))u(k) + E(\xi(k))w(k), \\ z(k) &= C(\xi(k))x(k) + D(\xi(k))u(k) + F(\xi(k))w(k), \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ and the matrices $A(\xi(k)), B(\xi(k)), C(\xi(k)), D(\xi(k)), E(\xi(k))$ and $F(\xi(k))$ are assumed to depend affinely on the time-varying parameter $\xi(k)$, which assumes values in the unit simplex,

$$\Xi = \left\{ \xi \in \mathbb{R}_+^N : \quad \sum_{i=1}^{N} \xi_i = 1 \right\}. \tag{4.2}$$

For the purposes of this chapter, it is assumed that $\xi(k)$ is measurable for all k. The affine assumption means that matrices $A(\xi(k)), B(\xi(k)), C(\xi(k))D(\xi(k)), E(\xi(k))$ and $F(\xi(k))$ can be written as

$$\begin{bmatrix} A(\xi(k)) & B(\xi(k)) & E(\xi(k)) \\ C(\xi(k)) & D(\xi(k)) & F(\xi(k)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(k) \begin{bmatrix} A_i & B_i & E_i \\ C_i & D_i & F_i \end{bmatrix}$$

We are concerned with stabilization by a gain-scheduled controller of the form:

$$u(k) = K(\xi(k))x(k), \tag{4.3}$$

with simultaneous minimization of the H_∞ cost criterion.

For the purpose of defining the H_∞ cost criterion, we consider the asymptotically stable open-loop dynamics of the form

$$\begin{aligned} x(k+1) &= \mathsf{A}(k)x(k) + \mathsf{E}(k)w(k), \\ z(k) &= \mathsf{C}(k)x(k) + \mathsf{F}(k)w(k), \end{aligned} \tag{4.4}$$

for which the H_∞ performance is defined by the l_2-to-l_2 gain:

$$\|H\|_\infty = \sup_{\|w(k)\|_2 \neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2},$$

where $w(k) \in l_2^r$ and $z(k) \in l_2^p$. An upper bound for the H_∞ norm can be characterized by the bounded real lemma in a way similar to [2, 49],

Lemma 15. *If there exists a bounded matrix sequence $P(k) = P(k)^T$ such that*

$$\begin{bmatrix} P(k+1) & P(k+1)\mathsf{A}(k) & 0 & P(k+1)\mathsf{E}(k) \\ \mathsf{A}^T(k)P(k+1) & P(k) & \mathsf{C}^T(k) & 0 \\ 0 & \mathsf{C}(k) & \eta I & \mathsf{F}(k) \\ \mathsf{E}^T(k)P(k+1) & 0 & \mathsf{F}^T(k) & \eta I \end{bmatrix} \succ 0, \tag{4.5}$$

for all $k = 0, 1, \cdots$, the time-varying discrete-time system (6.6) *is exponentially stable and*

$$\|H\|_\infty < \inf \eta.$$

Lemma 15 is equivalent to the existence of a Lyapunov function

$$V(x(k), \xi(k)) = x(k)^T P(k) x(k),$$

such that

$$V(x(k+1), \xi(k+1)) - V(x(k), \xi(k)) + \eta^{-1}\|z(k)\|^2 - \eta\|w(k)\|^2 < 0,$$

for all $k, k+1 = 0, 1, \ldots$ as was discussed in [50].

To the best of our knowledge, the most general stabilizability conditions for this class of time-varying systems that can still be expressed as LMIs are the ones from [50, 10], which we reproduce in the next lemma.

Lemma 16 ([50])**.** *The system* (4.1) *with* $B_i = B$ *and* $D_i = D$ *for all* $i = 1, \ldots, N$ *is poly-quadratically stabilizable with* H_∞ *performance bound* η *if and only if there exists* $Q_i = Q_i^T \succ 0$ *and* X_i, L_i *such that*

$$\begin{bmatrix} X_i + X_i^T - Q_i & 0 & X_i^T A_i^T + L_i^T B^T & X_i^T C_i^T + L_i^T D^T \\ 0 & \eta I & E_i^T & F_i^T \\ A_i X_i + B L_i & E_i & Q_j & 0 \\ C_i X_i + D L_i & F_i & 0 & \eta I \end{bmatrix} \succ 0, \tag{4.6}$$

for all $i, j = 1, \ldots, N$*. The control law is given by*

$$K(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) K_i, \qquad K_i = L_i X_i^{-1}, \tag{4.7}$$

The above lemma makes use of the notion of *poly-quadratic stability*, in which stability of the time-varying system (4.1) is proved by constructing an affine *parameter-dependent* Lyapunov function [51] of the form

$$V(x(k), \xi(k)) = x(k)^T P(\xi(k)) x(k), \qquad P(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) P_i \succ 0. \tag{4.8}$$

In Lemma 16, if inequalities (4.6) are feasible, then $P_i = Q_i^{-1}$ provides such a Lyapunov function. See [10] for details. Stabilizability conditions using higher order polynomial Lyapunov functions can be constructed based on the conditions of Lemma 16 using various devices such as in [1, 12]. Such extensions will be discussed later.

The main deficiency of the LMIs in Lemma 16 is the fact that the system cannot have variation in the input matrix B nor in the feed forward matrix D, hence the assumption $B(\xi(k)) = B$ and $D(\xi(k)) = D$.

We are not aware of any result in the literature that can simultaneously consider variation in both B's, D's and K's and still lead to convex problems in the form of LMIs, as in Lemma 16, without either introducing conservativeness or non-convexity constraints. This is the case even in the well studied context of quadratic stability.

The simplest of the techniques that allow for $B(\xi(k))$ and $D(\xi(k))$ to be parameter varying is to require the controller to be parameter independent. This approach have been discussed for instance in [52]. The sufficient conditions from [1, 12, 2] allow for variation in $B(\xi(k))$ and $D(\xi(k))$ with a gain-scheduled controller $K(\xi(k))$ but at the expense of more computationally intensive conditions. And finally, the popular concept of pre-filtering [20, 53, 54] which allows for variation in $B(\xi(k))$ and $D(\xi(k))$ but requires the implementation of a dynamic gain-scheduled controller.

The main contribution of this work is to present new LMI conditions that will be necessary and sufficient for the existence of a poly-quadratic function of the form (4.8) when B and D are parameter independent but that continue to be sufficient conditions for stabilizability and H_∞ performance when $B(\xi(k))$ and $D(\xi(k))$ are parameter dependent.

Throughout the chapter, we will use numerical examples to illustrate how our new results compare with these existing techniques. We will show that our conditions are able to achieve superior results compared to all existing techniques reviewed here. Additionally, as a side finding, our numerical example will show that the use of state-augmentation to overcome the limitation that $B(\xi(k)) = B$ and $D(\xi(k)) = D$ and implementing a gain-scheduled controller (4.3) can result in inferior performance compared to simply using a robust controller. This is the second contribution of this chapter and reveals a different behavior from what can happen in continuous-time [55].

4.2 Existing Approaches

In this section we will provide a more in-depth review of existing approaches that allow to incorporate time-variation in the input matrices B and D in gain-scheduled control design.

4.2.1 Robust Control

The first family of conditions that allows for controller design by LMIs when $B(\xi(k))$ and $D(\xi(k))$ depend on the time-varying parameter $\xi(k)$ make use of a potentially conservative robust controller, that is a controller of the form

$$u(k) = Kx(k), \tag{4.9}$$

where the feedback gain, K, is independent of the time-varying parameter $\xi(k)$. One such stabilizability condition is the one from [52], which uses the concept of poly-quadratic stability. We omit this result here and instead present the corresponding H_∞ performance result.

Lemma 17 ([50])**.** *The system* (4.1) *is poly-quadratically stabilizable with H_∞ performance bound η if and only if there exists $Q_i = Q_i^T \succ 0$ and X, L such that*

$$\begin{bmatrix} X + X^T - Q_i & 0 & X^T A_i^T + L^T B_i^T & X^T C_i^T + L^T D_i^T \\ 0 & \eta I & E_i^T & F_i^T \\ A_i X + B_i L & E_i & Q_j & 0 \\ C_i X + D_i L & F_i & 0 & \eta I \end{bmatrix} \succ 0, \tag{4.10}$$

for all $i, j = 1, \ldots, N$. The robust control law is given by $K = LX^{-1}$.

The above lemma reduces to the well known quadratic H_∞ performance criteria of [53] if $Q_i = X = X^T = Q$, $i = 1, \ldots, N$.

4.2.2 Bespoke Sufficient Conditions

There have been numerous attempts to overcome the limitations of conditions [10, 50] that do not fall back to a robust controller, e.g. [1, 12, 52, 2]. The conditions presented in [1, 52, 2] are LMI based but are sufficient only and may fail to produce a feasible controller even when [10, 50] succeed. Reference [12] are BMIs, hence non-convex, and are a generalization of the results from [1]. The conditions in [2] as posed relate to the output-feedback problem where there can be additional bounds on the rate of parameter variation. In the context of this chapter, we will consider them only in the case of state-feedback where there are no parameter bounds. For contrast, we repeat the simpler of those conditions, the ones from [1], in the following Lemma.

Lemma 18. *([1]) If there exist matrices $Q_i = Q_i^T \succ 0$, X_i and L_i such that,*

$$\begin{bmatrix} X_j + X_j^T - Q_j & 0 & T_{13} & T_{14} \\ \star & I & E_j^T & F_j^T \\ \star & \star & Q_i & 0 \\ \star & \star & \star & \eta I \end{bmatrix} \succ 0,$$

where

$$T_{13} = X_j^T A_j^T + L_j^T B_j^T, \qquad T_{14} = X_j^T C_j^T + L_j^T D_j^T,$$

and

$$\begin{bmatrix} R_{11} & 0 & R_{13} & R_{14} \\ \star & I & E_j^T + E_k^T & F_j^T + F_k^T \\ \star & \star & 2Q_i & 0 \\ \star & \star & \star & 2\eta I \end{bmatrix} \succ 0,$$

where

$$R_{11} = X_j + X_j^T + X_k + X_k^T - Q_j - Q_k,$$
$$R_{13} = X_j^T A_k^T + X_k^T A_j^T + L_j^T B_k^T + L_k^T B_j^T,$$
$$R_{14} = X_j^T C_k^T + X_k^T C_j^T + L_j^T D_k^T + L_k^T D_j^T,$$

for all $i = 1, \ldots, N, j = 1, \ldots, N-1$ and $K = j+1, \ldots, N$, then the gain-scheduled state-feedback controller,

$$K(\xi(k)) = L(\xi(k))X(\xi(k))^{-1},$$

poly-quadratically stabilizes the system (4.1) *and*

$$\|H\|_\infty < \sqrt{\eta}.$$

These conditions are much more complex than the ones from Lemma 16: there are now three loops involving indices i, j, and k. Furthermore, it is not clear whether the condition in Lemma 18 or in any of the references [1, 12, 2] are also necessary for stabilizability by a gain-scheduled controller (4.7) when $B(\xi(k)) = B$ and $D(\xi(k)) = D$. That is, they may fail even when Lemmas 16 succeed. This behavior will be displayed in our numerical example.

4.2.3 Pre-filtering

A popular way to handle variation in the input matrices is to introduce filters [20, 54] and work with augmented systems such as:

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\xi(k))\,\tilde{x}(k) + \tilde{B}\,u(k) + \tilde{E}(\xi(k))w(k), \\ z(k) &= \tilde{C}(\xi(k))\tilde{x}(k) + \tilde{D}\,u(k) + \tilde{F}(\xi(k))w(k), \end{aligned} \tag{4.11}$$

where the matrices $\tilde{A}(\xi(k)), \tilde{C}(\xi(k)), \tilde{E}(\xi(k))$ and $\tilde{F}(\xi(k))$ are as in

$$\begin{bmatrix} \tilde{A}(\xi(k)) & \tilde{E}(\xi(k)) \\ \tilde{C}(\xi(k)) & \tilde{F}(\xi(k)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(k) \left[\begin{array}{cc|c} A_i & B_i & E_i \\ 0 & 0 & 0 \\ \hline C_i & D_i & F_i \end{array}\right] \tag{4.12}$$

and $\tilde{B}$ and $\tilde{D}$ are

$$\tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \tilde{D} = 0, \tag{4.13}$$

which are independent of the parameter, $\xi(k)$. This approach remains popular with recent applications in spacecraft control [28, 27] and the design of active suspension systems [24, 25]. The main appeal of this approach is that a gain-scheduled controller for the auxiliary system (4.11) can now be designed using any of the existing LMI design procedures from the literature, for example Lemma 16. However, this flexibility comes with some serious drawbacks.

First, the controller resulting from applying Lemma 16 to the auxiliary system (4.11) is much more complex, being no longer static. Indeed, it is the dynamic gain-scheduled controller:

$$\begin{aligned} r(k+1) &= K_u(\xi(k))\, r(k) + K_x(\xi(k))\, x(k), \\ u(k) &= r(k), \end{aligned} \tag{4.14}$$

where K_u and K_x are obtained from the augmented gain

$$\tilde{K}(\xi(k)) = \begin{bmatrix} K_x(\xi(k)) & K_u(\xi(k)) \end{bmatrix}. \tag{4.15}$$

Second, as we have shown in [56], in the case quadratic stabilizability of continuous-time LPV systems, such augmentation is never advantageous, in the sense that quadratic stabilizability of the augmented system (4.11) by a gain-scheduled controller of the form (4.3) in fact implies the existence of a static robust controller of the form (4.9) which is also stabilizing. In [56], we have shown that a discrete-time counterpart to the stabilizability results also holds in the case of quadratic stabilizability. However, a counter-example provided in [56] where it was possible to design a dynamic gain scheduled controller designed for the augmented system (4.11) but not a static robust controller shows that the same property is not true for poly-quadratic stabilizability. In the present chapter we will show by means of a simple numerical example that employing a dynamic controller with a quadratic performance condition in conjunction with a pre-filter to overcome the limitation that $B(\xi(k)) = B$ and $D(\xi(k)) = D$ can in fact result in inferior performance when compared to a quadratic robust controller derived from Lemma 17. Our example also suggests, in line with the findings from [56], that this loss of performance may not occur when using poly-quadratic conditions. Moreover, our new proposed poly-quadratic conditions is able to outperform all such existing approaches.

4.2.4 Higher-order Polynomial Lyapunov Functions

The conditions presented thus far make use of Lyapunov functions that are quadratic in the state and linear in the time-varying parameter $\xi(k)$ as in (4.8). An immediate generalization is to consider Lyapunov functions which are polynomial in $\xi(k)$ as in

$$V(x(k), \xi(k)) = x(k)^T P(\xi(k)) x(k), \tag{4.16}$$

where $P(\xi(k)) \succ 0$ for all $\xi \in \Xi$ but $P(\xi(k))$ is no longer restricted to be affine in $\xi(k)$. Such polynomial Lyapunov functions have been recently explored in [57, 49, 58] among others.

When $\xi(k)$ is time-invariant but still uncertain, it is well known that the existence of a Lyapunov functions of the above class is both necessary and sufficient for robust stability [59]. This is no longer the case when $\xi(k)$ is time-varying [60], which requires a Lyapunov that is no longer quadratic in the state. Nevertheless, it is common to find works in which Lyapunov functions of the form (4.16) is also applied to time-varying stability analysis, e.g. [57, 49, 58]. It turns out that higher-order polynomial dependence is not effective in the time-varying case at all, as shown by the next lemma, a result which is simple but that we could not find anywhere else in the literature.

Lemma 19. *Consider the time-varying dynamics of the form*

$$x(k+1) = A(\xi(k)) x(k). \tag{4.17}$$

If there exists a Lyapunov functions of the form (4.16) *in which a bounded* $P(\xi(k)) \succ 0$ *proves stability of the time-varying linear system* (4.17) *for all* $\xi(k) \in \Xi$ *then there is also a Lyapunov functions of the form* (4.8) *which proves stability as well.*

Proof. If a bounded $P(\xi(k)) \succ 0$ proves stability of (4.17) then

$$A^T(\xi(k)) P(\xi(k+1)) A(\xi(k)) - P(\xi(k)) \prec 0,$$

for all $\xi(k), \xi(k+1) \in \Xi$. Evaluating P and A at each vertex of Ξ, one obtains

$$A_i^T P_j A_i - P_i \prec 0,$$

where $P_i = P(\xi_i)$, ξ_i being one of the N vertices of Ξ. But this implies that $P(\xi(k)) =$

$\sum_{i=1}^{N} \xi_i(k)P_i$ also proves that the system (4.17) is also poly-quadratically stable. □

A similar result holds in the case of stabilization by way of a robust controller or with a gain-scheduled controller in the case where $B(\xi(k)) = B$ for all k. The case of a constant B implies also that if a gain-scheduled controller $K(\xi(k))$ exists which is nonlinear in $\xi(k)$, then a linear one also does exist. The proof is along the same lines as the one of Lemma 19. See also [61] for a related statement about stability of time-varying systems.

Some researchers have also looked into the problem of gain-scheduled control when the time-varying parameter, $\xi(k)$, has known bounds on its rate of variation, e.g. [57, 58]. In this scenario, higher-order polynomial dependence on $\xi(k)$ might be of help. A full comparison with such results requires a modified version of our the results to be presented in this chapter and will be the subject of a future publication.

4.3 Main Results

We will only present the H_∞ performance results here, noting that the corresponding stabilizability conditions can be found in [62, 63].

Theorem 11. *Consider the time-varying discrete-time linear system of the form* (4.1)*. If there exists matrices* L_i, W_i, X_i, Y_i, Z_i *and* $Q_i = Q_i^T \succ 0$*,* $i = 1, \ldots, N$ *such that*

$$\begin{bmatrix} X_i + X_i^T - Q_i & X_i^T A_i^T & -L_i^T & X_i^T C_i^T & 0 \\ A_i X_i & Q_j - B_i Y_j - Y_j^T B_i^T & B_i Z_j - Y_j^T & -B_i W_j - Y_j^T D_i^T & E_i \\ -L_i & Z_j^T B_i^T - Y_j & Z_j + Z_j^T & Z_j^T D_i^T - W_j & 0 \\ C_i X_i & -D_i Y_j - W_j^T B_i^T & D_i Z_j - W_j^T & \eta I - D_i W_j - W_j^T D_i^T & F_i \\ 0 & E_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \quad (4.18)$$

for all $i, j = 1, \ldots, N$*, then the gain-scheduled state-feedback controller* (4.3) *with gain* $K(\xi(k))$ *as in* (4.7) *poly-quadratic stabilizes the system* (4.1) *and*

$$\|H\|_\infty < \inf \eta. \quad (4.19)$$

Furthermore, for $B_i = B$ and $D_i = D$ for all $i = 1, \ldots, N$, if there exists a gain-scheduled controller (4.3) *with gain $K(\xi(k))$ as in* (4.7) *which poly-quadratic stabilizes the system* (4.1) *with H_∞ performance as in* (4.19) *then there exists matrices L_i, W_i, X_i, Y_i, Z_i and $Q_i = Q_i^T \succ 0$ such that* (4.18) *holds for all $i, j = 1, \ldots, N$.*

The above condition provides an inclusive generalization of Lemma 16. As we will show in detail later, it is guaranteed to hold whenever the one in Lemma 16 holds. Namely, in the case of $B_i = B$ and $D_i = D$, Lemma 16 and Theorem 11 are equivalent.

Theorem 11 is so remarkable that even its specialization to the case of quadratic Lyapunov functions can bring advantage when compared to classic quadratic stabilizability conditions. This is a feat unheard of in the current literature. Setting $X_i = X_j = Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$ in (4.18) gives a quadratic version of Theorems 11. Remarkably, in this case, it is also possible to freeze the auxiliary variables W, Y and Z without losing necessity with respect to standard quadratic stabilizability conditions, such as the ones in [54]. This results is presented in the next corollary.

Corollary 2. *Consider the time-varying discrete-time linear system of the form* (4.1)*. If there exists matrices L_i, W, X, Y, Z and $Q = Q^T \succ 0$, $i = 1, \ldots, N$ such that*

$$\begin{bmatrix} Q & QA_i^T & -L_i^T & QC_i^T & 0 \\ A_iQ & Q - B_iY - Y^TB_i^T & B_iZ - Y^T & -B_iW - Y^TD_i^T & E_i \\ -L_i & Z^TB_i^T - Y & Z + Z^T & Z^TD_i^T - W & 0 \\ C_iQ & -D_iY - W^TB_i^T & D_iZ - W^T & \eta I - D_iW - W^TD_i^T & F_i \\ 0 & E_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \tag{4.20}$$

for all $i, j = 1, \ldots, N$, then the gain-scheduled state-feedback controller (4.3) *with gain*

$$K(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) K_i, \qquad K_i = L_i Q^{-1},$$

as in (4.7) *quadratically stabilizes the system* (4.1) *and*

$$\|H\|_\infty < \inf \eta. \tag{4.21}$$

Furthermore, if $B_i = B$ *and* $D_i = D$ *for all* $i = 1, \dots, N$*, then the converse also holds.*

We will postpone the proofs until Section 4.5.

Remark 1. *The* H_∞ *performance condition Theorem 11 does not introduce any further requirement in terms of stabilizability. Indeed, whenever a stabilizing controller can be obtained, the inequalities Theorem 11 will be feasible for some large enough* η *and* $W_i = 0$ *for all* $i = 1, \dots, N$*.*

4.4 Comparative Numerical Examples

We shall now introduce a series of numerical examples and use them to compare the performance of the existing approaches revisited in Section 4.2 with the new conditions that we have proposed in Section 4.3.

4.4.1 Example 1

Consider first, the following discrete-time example from [48] in which

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\,\alpha & -0.5\,\alpha & 0.2 & 0.03+\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B(\beta) = \begin{bmatrix} \beta \\ 0 \\ 1-\beta \\ 0 \end{bmatrix}, \tag{4.22}$$

$$C = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{4.23}$$

$$E = \begin{bmatrix} I & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{4.24}$$

where

$$0 \le \beta \le 1, \qquad |\alpha| \le \gamma.$$

This system can be easily put in the form (4.1) with 4 vertices. Our first goal is to determine the largest $\gamma > 0$ such that the closed-loop discrete-time system is stable. The results obtained from different existing conditions in the literature are summarized in Table 4.1. We have compared:

- Static Robust (SR), in which robust static state-feedback controllers are calculated using a quadratic condition [20] and a poly-quadratic condition [52];
- Dynamic Gain-Scheduled (DGS), in which dynamic gain-scheduled state-feedback controllers are calculated using the quadratic condition from [54] and the poly-quadratic condition from [10] after introducing a pre-filter as discussed in Section 4.2.3;
- Static Gain-Scheduled (SGS), in which a static gain-scheduled state-feedback controller is calculated from the poly-quadratic conditions from [1].
- Static Gain-Scheduled (SGS), in which a static gain-scheduled state-feedback controller is calculated from the homogenous polynomial conditions from [57] where for this comparison the matrix degree, $g = 1$, and the bound variation parameter, $b = 1$.
- Static Gain-Scheduled (SGS), in which a static gain-scheduled state-feedback controller is calculated from the poly-quadratic conditions from [2] where the parameter variation bound, $b = 1$.
- Static Gain-Scheduled (SGS) state-feedback controllers using quadratic condition (Corollary 2) and poly-quadratic condition (Theorem 11).

In the case of quadratic stability, we observe that the maximum possible value of γ is identical for both the static robust controller and the dynamic gain-scheduled controller (see [56] for more discussion on why this is the case). Remarkably, the quadratic conditions derived from Corollary 2 are able to reach higher maximum values of γ when compared to both static and dynamic quadratic designs. In the case of poly-quadratic stabilizability, Theorem 11 reaches a maximum value of γ that is superior to all competing designs.

Now, we minimize the corresponding H_∞ bound, η. As in the stabilization problem we have compared:

Table 4.1. Maximum γ from Example 1 for different control approaches.

Quadratic	SR	Geromel et al. [20]	0.53
	DGS	Apkarian et al. [54]	0.53
	SGS	Corollary 6	**0.55**
	SGS	Theorem 13*	**0.55**
Poly-Quadratic	SR	Mao [52]	0.56
	DGS	Daafouz & Bernussou [10]	0.59
	SGS	Montagner et al. [1]	0.59
	SGS	De Caigny et al. [2]	0.61
	SGS	Theorem 13	**0.64**
Polynomial	SGS	Oliveira & Peres [57]	0.60

* with $X_i = X_i^T = Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$.

Table 4.2. H_∞ performance bounds from Example 1 for different control approaches; '—' means no feasible solution.

			γ					
			0.1	0.2	0.3	0.4	0.5	0.6
Quad	SR	Geromel et al. [53]	6.33	8.01	11.4	20.4	94.8	—
	DGS	Apkarian et al. [54]	6.42	8.12	11.5	20.6	95.5	—
	SGS	Corollary 2	**6.16**	**7.56**	**10.4**	**17.5**	**56.8**	—
P-Quad	SR	Mao [52]	6.11	7.65	10.6	17.3	46.7	—
	DGS	Daafouz & Bernussou [10]	6.01	7.35	9.82	15.0	31.3	—
	SGS	Montagner et al. [1]	7.23	8.19	10.5	15.4	30.1	—
	SGS	De Caigny et al. [2]	**5.63**	**6.62**	**8.54**	**12.7**	25.5	456.4
	SGS	Theorem 11	**5.63**	6.74	8.74	**12.7**	**22.7**	**84.3**

- Static Robust (SR), quadratic from [53] and poly-quadratic from [50, 52];
- Dynamic Gain-Scheduled (DGS), quadratic from [54] and poly-quadratic from [50];
- Static Gain-Scheduled (SGS), poly-quadratic from [1] and [2].
- Static Gain-Scheduled (SGS) state-feedback controllers using quadratic condition (Corollary 2) and poly-quadratic condition (Theorem 11).

The performance bounds corresponding to particular values of γ are presented in Table 4.2. In Figure 1, we have plotted the η values for a selection of the best controllers from example 1.

Let us first consider the performance bounds corresponding to the quadratic conditions. The quadratic performance bounds computed in example 1 are somewhat counter intuitive. For quadratic based conditions, a dynamic gain-scheduled controller provides no advantage in terms of stabilizability [56] and has interior performance bounds compared to a static robust controller! The same is not true in for poly-quadratic controllers, with the dynamic gain-scheduled controller being able to achieve both a stabilizability advantage as well as superior performance bounds.

We shall offer an additional comment about the above behavior. In continuous-time, we have shown in [55] that the use of a dynamic gain-scheduled controllers is equivalent to a static robust controller for both stabilizability as well as performance. However, in discrete-time, augmentation necessarily comes with an additional hidden cost. Indeed, a dynamic controller of the form (4.14) necessarily introduces an additional delay in the feedback loop. The presence of this additional delay can explain the performance degradation experienced with dynamic controllers derived from quadratic conditions. This finding is one of the main motivations for the introduction of the new conditions in this chapter, since we expect that procedures that can directly handle time-variation in the input matrix $B(\xi(k))$ and feed-forward matrix $D(\xi(k))$ will lead to better closed-loop performance as compared with controllers obtained through augmentation.

Surprisingly, these advantages do not fully materialize with the conditions from [1]. Here, only for large values of γ does the proposed sufficient condition provides better performance compared to all other conditions, including quadratic. This is despite the fact that the condition in Lemma 18 results in a static gain-scheduled controller which is not hindered by the aforementioned delays associated with a dynamic controller.

Turning our attention to the conditions presented in this chapter, we see that in the quadratic case Corollary 2 is able to achieve lower performance bounds compared to all other quadratic conditions tested. In the case of poly-quadratic stability, the conditions from [2] and Theorem 11 are able to achieve the lowest performance bounds. As we saw previously, Theorem 11 is able to achieve the highest stability bound, γ, and as γ increases, the corresponding performance bound, η computed from Theorem 11 drops below that computed from [2].

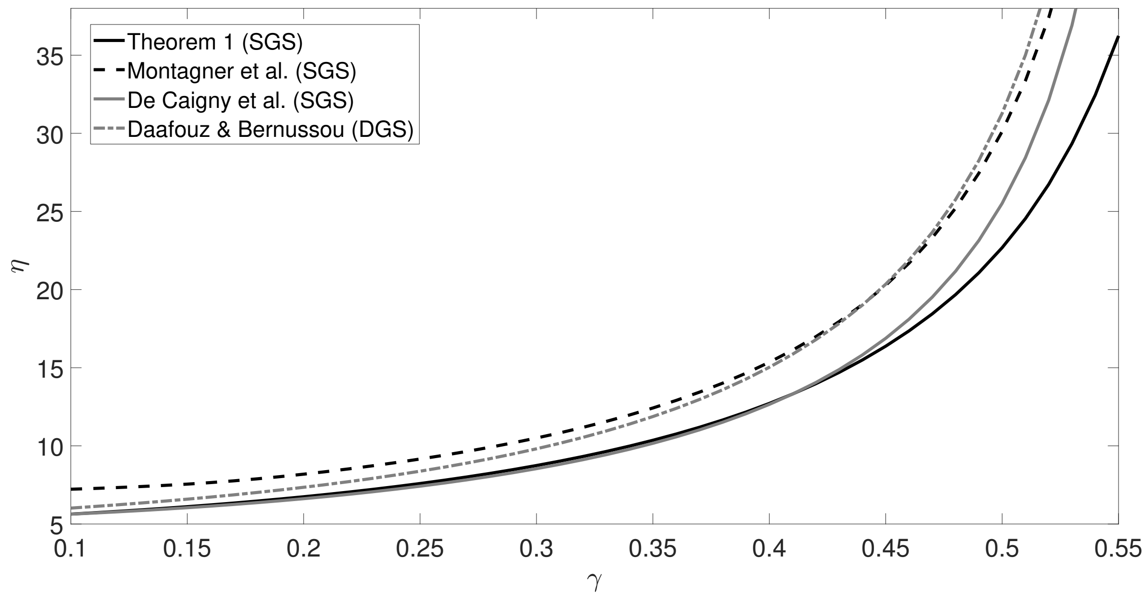

Figure 4.1. Performance bounds from Example 1 for Theorem 11, Lemma 18 ([1]), the state-feedback condition from [2] and the DGS controller derived from Lemma 16.

4.4.2 Example 2

Consider the discrete-time example from [1] where the vertices are as follows,

$$A_1 = \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad E_1 = E_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

Table 4.3. H_∞ performance bounds and numerical complexity from Example 2 for different control approaches.

			η	Time (seconds)	V	R
Poly-Quadratic	SGS	Montagner et al. [1]	4.16	2.25	17	36
		De Caigny et al. [2]	3.52	3.23	19	138
		Theorem 11	**2.52**	2.41	25	32
	DGS	Daafouz & Bernussou [50]	5.00	2.59	31	32

and $D_1 = D_2 = F_1 = F_2 = 0$. Our goal is to first determine the minimal H_∞ bound, η for a selection of the poly-quadratic conditions discussed in this chapter. We will also seek to quantify the numerical complexity of these methods. To quantify the numerical complexity, we will compute the execution time, the number of decision variables (V) and the number of LMI rows (R). We will perform these comparisons for the static gain-scheduled, poly-quadratic methods from [1], [2], Theorem 11 and for the dynamic gain-scheduled condition derived from [50]. The results are presented in Table 4.3. As indicated by the table, Theorem 11 is able to achieve the lowest H_∞ norm bound out of the conditions tested. Theorem 11 is roughly equivalent to the condition from [1] with respect to numerical complexity. The condition from [2] appears to be the most numerically involved.

4.4.3 Example 3

Consider the following discrete-time example with vertices as follows,

$$\left[\begin{array}{c|c|c|c} A_1 & A_2 & A_3 & A_4 \end{array}\right] = \alpha \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & -2 & 3 & 0 & -1 & -1 & 1 & 0 & -1 & 2 & -2 \\ 2 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 & 2 & 1 & 1 & 1 & 0 & -1 \end{array}\right],$$

$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\left[\begin{array}{c|c|c|c} C_1 & C_2 & C_3 & C_4 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right],$$

$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\left[\begin{array}{c|c|c|c} E_1 & E_2 & E_3 & E_4 \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} F_1 & F_2 & F_3 & F_4 \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 1 & 1 & -1 & 0 & 1 & 1 & -2 & -1 \\ 0 & -2 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right],$$

where α is fixed in the range $0.3 \leq \alpha \leq 0.7$. Our first goal is to determine the minimal H_∞ bound, η, across the indicated range of α for a selection of quadratic and poly-quadratic conditions discussed in this chapter. As with Example 2, we will also seek to quantify the numerical complexity of these methods. We will use the same metrics to quantify this as we did in Example 2 - the execution time, the number of decision variables (V) and the number of LMI rows (R). We will perform these comparisons for the static gain-scheduled, poly-quadratic methods from [1], [2] and Theorem 11. As $B_i = B$ and $D_i = D$ for all $i = 1, \ldots, N$, we will also incorporate the poly-quadratic condition from [50] where no pre-filter is necessary. Additionally, we will consider the quadratic condition in Corollary 2. Table 4.4 indicates the H_∞ performance bounds for each condition we tested for various values of α.

For this example, Theorem 11 achieves identical H_∞ bounds with the condition from [50]. As we have indicated above and will prove in the next section, Theorem 11 is equivalent to the condition from [50] when $B_i = B$ and $D_i = D$ for all $i = 1, \ldots, N$. This however, is not the case

Table 4.4. H_∞ performance bounds from Example 3 for different control approaches.

			α				
			0.3	0.4	0.5	0.6	0.7
Quad	SGS	Corollary 6	4.87	5.40	6.90	21.2	254.1
P-Quad	SGS	Montagner et al. [1]	4.87	5.47	6.54	21.0	225.2
	SGS	De Caigny et al. [2]	4.78	5.07	5.91	20.8	219.8
	SGS	Daafouz & Bernussou [50]	**4.77**	**5.06**	**5.83**	**19.8**	**208.4**
	SGS	Theorem 13	**4.77**	**5.06**	**5.83**	**19.8**	**208.4**

Table 4.5. Numerical complexity from Example 3 for different control approaches.

			Time (seconds)	V	R
Poly-Quadratic	SGS	Montagner et al. [1]	2.60	73	400
		De Caigny et al. [2]	20.7	85	4144
		Daafouz & Bernussou [50]	1.36	73	160
		Theorem 11	2.71	121	204

for the conditions from [1] and [2] who are unable to achieve as low a performance bound.

The numerical complexity for each of the poly-quadratic conditions tested in this example is shown in Table 4.5. As with Example 2, we see that the condition from [2] is the most numerically involved - taking approximately 8 times the time to compute compared to the other conditions tested.

4.5 Proofs

4.5.1 Poly-quadratic Conditions

In this section we will prove Theorem 11. We will make use of the following technical result.

Lemma 20. *If $X + X^T \succ Y \succ 0$ then X is nonsingular,*

$$X^T Y^{-1} X \succeq X + X^T - Y$$

and

$$Y^{-1} \succeq X^{-1} + X^{-T} - X^{-T} Y X^{-1}.$$

Furthermore, equality always holds for $X = Y$.

Proof. We have that $X + X^T \succ Y \succ 0$ which implies X is nonsingular. Thus, $(Y - X)^T Y^{-1} (Y - X) \succeq 0$ and, after rearranging,

$$X^T Y^{-1} X \succeq X + X^T - Y.$$

The second statement follows after multiplication by X^{-1} on the right and X^{-T} on the left. □

Proof of Theorem 11

We can now move on to proving the main results of this chapter. We start with the H_∞ performance result given in Theorem 11. We will prove that the feasibility of the inequalities (4.18) imply system (4.1) is poly-quadratically stable with a guaranteed H_∞ performance bound (4.19). To prove this, we will show that the inequalities (4.18) are sufficient for inequalities (4.5).

Assume that (4.18) holds. Because $X_i + X_i^T \succ Q_i \succ 0$ then X_i is nonsingular. Calculate $K_i = L_i X_i^{-1}$ and substitute to obtain

$$\begin{bmatrix} X_i + X_i^T - Q_i & X_i^T A_i^T & -X_i^T K_i^T & X_i^T C_i^T & 0 \\ A_i X_i & Q_j - B_i Y_j - Y_j^T B_i^T & B_i Z_j - Y_j^T & -B_i W_j - Y_j^T D_i^T & E_i \\ -K_i X_i & Z_j^T B_i^T - Y_j & Z_j + Z_j^T & Z_j^T D_i^T - W_j & 0 \\ C_i X_i & -D_i Y_j - W_j^T B_i^T & D_i Z_j - W_j^T & \eta I - D_i W_j - W_j^T D_i^T & F_i \\ 0 & E_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \quad (4.25)$$

for all $i, j = 1, \ldots, N$. Now use Lemma 20 with $X_i = X$ and $Y = Q_i$ to show that

$$X_i^T Q_i^{-1} X_i \succeq X_i + X_i^T - Q_i.$$

Hence, inequalities (4.25) imply

$$\begin{bmatrix} X_i^T Q_i^{-1} X_i & X_i^T A_i^T & -X_i^T K_i^T & X_i^T C_i^T & 0 \\ A_i X_i & Q_j - B_i Y_j - Y_j^T B_i^T & B_i Z_j - Y_j^T & -B_i W_j - Y_j^T D_i^T & E_i \\ -K_i X_i & Z_j^T B_i^T - Y_j & Z_j + Z_j^T & Z_j^T D_i^T - W_j & 0 \\ C_i X_i & -D_i Y_j - W_j^T B_i^T & D_i Z_j - W_j^T & \eta I - D_i W_j - W_j^T D_i^T & F_i \\ 0 & E_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \qquad (4.26)$$

for all $i, j = 1, \ldots, N$. Multiplying inequalities (4.26) by

$$T_i = \begin{bmatrix} X_i^{-T} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

on the left and its transpose on the right gives

$$\begin{bmatrix} Q_i^{-1} & A_i^T & -K_i^T & C_i^T & 0 \\ A_i & Q_j - B_i Y_j - Y_j^T B_i^T & B_i Z_j - Y_j^T & -B_i W_j - Y_j^T D_i^T & E_i \\ -K_i & Z_j^T B_i^T - Y_j & Z_j + Z_j^T & Z_j^T D_i^T - W_j & 0 \\ C_i & -D_i Y_j - W_j^T B_i^T & D_i Z_j - W_j^T & \eta I - D_i W_j - W_j^T D_i^T & F_i \\ 0 & E_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \qquad (4.27)$$

for all $i, j = 1, \ldots, N$. Since Q_i and Z_i are nonsingular, the definitions

$$P_i = Q_i^{-1}, \qquad H_i = Z_i^{-T}, \qquad M_i = P_i Y_i^T H_i, \qquad G_i = W_i^T H_i, \qquad \text{for all } i = 1, \ldots, N,$$

allow one to rewrite (4.27) in the form

$$\begin{bmatrix} P_i & \star & \star & \star & \star \\ A_i & P_j^{-1} - R_{i,j} & \star & \star & \star \\ -K_i & H_j^{-1}B_i^T - H_j^{-T}M_j^T P_j^{-1} & H_j^{-1} + H_j^{-T} & \star & \star \\ C_i & -G_j H_j^{-1} B_i^T - D_i H_j^{-T} M_j^T P_j^{-1} & D_i H_j^{-T} - G_j H_j^{-1} & \eta I - S_{i,j} & \star \\ 0 & E_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \tag{4.28}$$

where $R_{i,j} = B_i H_j^{-T} M_j^T P_j^{-1} + P_j^{-1} M_j H_j^{-1} B_i^T$ and $S_{i,j} = D_i H_j^{-T} G_j^T + G_j H_j^{-1} D_i^T$ for all $i, j = 1, \ldots, N$ and the $\star$ notation stand for symmetric blocks omitted for brevity. A congruence transformation, multiplying (4.28) by

$$T_j = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & H_j & 0 & 0 \\ 0 & P_j & M_j & 0 & 0 \\ 0 & 0 & G_j & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

on the left and its transpose on the right produces

$$\begin{bmatrix} P_i & -K_i^T H_j^T & A_i^T P_j - K_i^T M_j^T & C_i^T - K_i^T G_j^T & 0 \\ -H_j K_i & H_j + H_j^T & M_j^T + B_i^T P_j & D_i^T + G_j^T & 0 \\ -M_j K_i + P_j A_i & M_j + P_j B_i & P_j & 0 & P_j E_i \\ C_i - G_j K_i & D_i + G_j & 0 & \eta I & F_i \\ 0 & 0 & E_i^T P_j & F_i^T & \eta I \end{bmatrix} \succ 0, \tag{4.29}$$

for all $i, j = 1, \ldots, N$. For each i, multiply the corresponding $j = 1, \ldots, N$ inequalities by $\xi_j(k+1)$ and sum. Then, multiplying the resulting $i = 1, \ldots, N$ by $\xi_i(k)$ and sum to obtain

$$\begin{bmatrix} P(\xi(k)) & \star & \star & \star & \star \\ -H(\xi(k+1))K(\xi(k)) & H(\xi(k+1)) + H^T(\xi(k+1)) & \star & \star & \star \\ -M(\xi(k+1))K(\xi(k)) + P(\xi(k+1))A(\xi(k)) & M(\xi(k+1)) + P(\xi(k+1))B(\xi(k)) & P(\xi(k+1)) & \star & \star \\ C(\xi(k)) - G(\xi(k+1))K(\xi(k)) & D(\xi(k)) + G(\xi(k+1)) & 0 & \eta I & \star \\ 0 & 0 & E^T(\xi(k))P(\xi(k+1)) & F^T(\xi(k)) & \eta I \end{bmatrix} \succ 0, \tag{4.30}$$

for all $\xi(k), \xi(k+1) \in \Xi$. Multiplying (4.30) by

$$T(\xi(k)) = \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ I & K^T(\xi(k)) & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

on the left and by its transpose on the right yields

$$\begin{bmatrix} P((\xi(k+1)) & P((\xi(k+1))\mathsf{A}((\xi(k)) & 0 & P((\xi(k+1))E((\xi(k)) \\ \mathsf{A}^T((\xi(k))P((\xi(k+1)) & P(\xi(k)) & \mathsf{C}^T((\xi(k)) & 0 \\ 0 & \mathsf{C}((\xi(k)) & \eta I & F((\xi(k)) \\ E^T((\xi(k))P((\xi(k+1)) & 0 & F^T((\xi(k)) & \eta I \end{bmatrix} \succ 0,$$

where $\mathsf{A}(\xi(k)) = A(\xi(k)) + B(\xi(k))K(\xi(k))$ and $\mathsf{C}(\xi(k)) = C(\xi(k)) + D(\xi(k))K(\xi(k))$. This is (4.5).

The second matter is to show that in the case of constant $B_i = B$ and $D_i = D$ for all $i = 1, \ldots, N$, inequalities (4.18) are also necessary for poly-quadratic stabilizability of (4.1) with a guaranteed H_∞ performance bound (4.19). Assuming that (4.5) holds with $P(k)$ as in (4.8) and evaluating at its vertices gives,

$$\begin{bmatrix} P_j & P_j(A_i + BK_i) & 0 & P_jE_i \\ (A_i^T + K_i^TB^T)P_j & P_i & C_i^T + K_i^TD^T & 0 \\ 0 & C_i + DK_i & \eta I & F_i \\ E_i^TP_j & 0 & F_i^T & nI \end{bmatrix} \succ 0, \qquad (4.31)$$

for all $i, j = 1, \dots, N$. Now, let ρ be sufficiently large so that

$$\begin{bmatrix} P_j & P_j(A_i + BK_i) & 0 & P_j E_i & 0 \\ (A_i^T + K_i^T B^T)P_j & P_i & C_i^T + K_i^T D^T & 0 & -K_i^T \\ 0 & C_i + DK_i & \eta I & F_i & 0 \\ E_i^T P_j & 0 & F_i^T & \eta I & 0 \\ 0 & -K_i & 0 & 0 & \rho I \end{bmatrix} \succ 0, \tag{4.32}$$

for all $i, j = 1, \dots, N$. Define

$$Q_i = P_i^{-1}, \quad X_i = Q_i, \quad L_i = K_i Q_i, \quad Z = \frac{\rho}{2} I, \quad Y = -ZB^T, \quad W = -ZD^T. \tag{4.33}$$

Applying the congruence transform,

$$T_{i,j} = \begin{bmatrix} 0 & Q_i & 0 & 0 & 0 \\ Q_j & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & D \\ 0 & 0 & 0 & I & 0 \end{bmatrix},$$

to (4.32) gives (4.18) which completes the proof.

4.5.2 Quadratic Conditions

Following the proof of Theorems 11 and 11 one obtains immediately that the particular choice $X_i = X_i^T = Q_i = Q$, $i = 1, \dots N$, produces a quadratic Lyapunov function with $P(\xi) = P = Q^{-1}$. Necessity in the H_∞ synthesis case when $B_i = B$ is as follows. Assuming that (4.5) holds with $P(k) = P$ and evaluating at its vertices gives,

$$\begin{bmatrix} P & P(A_i + BK_i) & 0 & PE_i \\ (A_i^T + K_i^T B^T)P & P & C_i^T + K_i^T D^T & 0 \\ 0 & C_i + DK_i & \eta I & F_i \\ E_i^T P & 0 & F_i^T & nI \end{bmatrix} \succ 0,$$

for all $i, j = 1, \ldots, N$. The result then follows after defining

$$Q = P^{-1}, \qquad L_i = K_i Q, \qquad Z = \frac{\rho}{2} I, \qquad Y = -ZB^T, \qquad W = -ZD^T$$

similarly to (4.33) and following the same steps as in the poly-quadratic case.

4.6 Discussion

We have introduced a new LMI condition for the H_∞ synthesis of discrete-time linear parameter-varying systems. Contrary to some similar conditions existing in the literature, our condition allows for variation in the input matrix as well as in the dynamic matrix. We have shown that they include the poly-quadratic H_∞ synthesis condition of Daafouz and Bernussou [50] as a particular case. We have also derived a corollary which is also capable of improving performance even in the stronger case of quadratic H_∞ synthesis, e.g. [53, 54]. The improvements are obtained without resorting to auxiliary dynamic system augmentation, iterative procedures, or higher-order multipliers. A series of numerical examples shows improvement compared to the existing approaches from [1, 12, 52, 2].

It is tempting to speculate whether the new conditions are also necessary for poly-quadratic stabilizability in the presence of variation in the input matrices. The only difficulty seems to be the assignment (4.33), which does not hold when the input matrices are not constant. These issues shall be investigated in future work.

We additionally show by means of a numerical example that in the case of quadratic H_∞ performance, a static robust controller can outperform a dynamic gain-scheduled controller.

4.7 Acknowledgments

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Chapter 5

Discrete time H_∞ synthesis conditions for LPV filter design

5.1 Introduction

Consider time-varying discrete-time linear systems of the form

$$\begin{aligned} x(k+1) &= A(\xi(k))x(k) + B(\xi(k))w(k), \\ z(k) &= E(\xi(k))x(k) + F(\xi(k))w(k), \\ y(k) &= Cx(k) + Dw(k), \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state, $z \in \mathbb{R}^p$ is the signal one wishes to estimate, and $y \in \mathbb{R}^r$ is the available measurement. Matrices $A(\xi(k))$, $B(\xi(k))$, $E(\xi(k))$ and $F(\xi(k))$ of compatible dimensions are assumed to depend affinely on the uncertain time-varying parameter $\xi(k)$, which assumes values in the unit simplex,

$$\Xi = \left\{ \xi \in \mathbb{R}_+^N : \quad \sum_{i=1}^{N} \xi_i = 1 \right\},$$

and the matrices C and D are assumed to be constant. The affine assumption means that matrices $A(\xi(k))$, $B(\xi(k))$, $E(\xi(k))$ and $F(\xi(k))$ can be written as

$$\begin{bmatrix} A(\xi(k)) & B(\xi(k)) \\ E(\xi(k)) & F(\xi(k)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(k) \begin{bmatrix} A_i & B_i \\ E_i & F_i \end{bmatrix}.$$

We are interested in the design of an LPV filter:

$$\begin{aligned}\hat{x}(k+1) &= A(\xi(k))\hat{x}(k) + L(\xi(k))(y(k) - \hat{y}(k)),\\ \hat{z}(k) &= E(\xi(k))\hat{x}(k),\\ \hat{y}(k) &= C\hat{x}(k),\end{aligned} \tag{5.2}$$

where,

$$L(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) L_i. \tag{5.3}$$

We denote the state and output estimation errors as

$$e(k) = x(k) - \hat{x}(k), \qquad e_z(k) = z(k) - \hat{z}(k),$$

so that the goal is to choose an appropriate $L(\xi(k))$ such that the error system

$$\begin{aligned}e(k+1) &= (A(\xi(k)) - L(\xi(k))C)e(k)\\ &\quad + (B(\xi(k)) - L(\xi(k))D)w(k)\\ e_z(k) &= E(\xi(k))e(k) + F(\xi(k))w(k),\end{aligned} \tag{5.4}$$

is asymptotically stable and the l_2-to-l_2 gain

$$\sup_{\|w\|_2 \neq 0} \frac{\|e_z\|_2}{\|w\|_2} < \eta, \tag{5.5}$$

is minimized.

The chapter is organized as follows. We first introduce the notion of poly-quadratic H_∞ performance from [10]. We discuss this notion in the context of two different formulations for poly-quadratic H_∞ performance. The first was introduced in [10, 50] in the context of state feedback design. The second formulation was introduced in [6, 64] in the context of Linear Parameter Varying (LPV) filter design.

We initially consider the formulation in [6, 64] used for LPV filter design. We show that these sufficient conditions for poly-quadratic LPV H_∞ synthesis are in fact necessary as well. The existence of a proof of necessity allows for many other papers that use similar filter synthesis condition e.g. [65, Chapter 4], [6, 66, 67, 68, 69], to also claim necessity in their design conditions.

As an alternative to the LPV filter synthesis conditions in [6, 64], we derive properties of duality that hold with poly-quadratic Lyapunov functions and use the state feedback conditions from [10, 50] to design LPV filters. We develop the theory required to prove such a result by appealing to the properties of parameter independent Lyapunov functions that have been explored in [70, 60]. We then present LPV design conditions which are the dual of the state feedback conditions from [10, 50].

We conclude with a numerical example comparing the two alternative conditions for LPV filter design.

5.2 Evaluating H_∞ Performance

For the purpose of defining the H_∞ cost criterion, we consider the asymptotically stable open-loop system of the form

$$\begin{aligned} x(k+1) &= \mathsf{A}(k)x(k) + \mathsf{E}(k)w(k), \\ z(k) &= \mathsf{C}(k)x(k) + \mathsf{F}(k)w(k), \end{aligned} \tag{5.6}$$

for which the H_∞ performance is defined by the l_2-to-l_2 gain:

$$\|H\|_\infty = \sup_{\|w(k)\|_2 \neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2},$$

where $w(k) \in l_2^r$ and $z(k) \in l_2^p$. The notation $\|H\|_\infty$ is a slight abuse since (5.6) is time-varying. An upper bound for the H_∞ norm can be characterized by the bounded real lemma in a way similar to [2, 49],

Lemma 21. *If there exists a bounded matrix sequence $P(k) = P(k)^T$ such that*

$$\begin{bmatrix} P(k+1) & \star & \star & \star \\ \mathsf{A}^T(k)P(k+1) & P(k) & \star & \star \\ 0 & \mathsf{C}(k) & \eta I & \star \\ \mathsf{E}^T(k)P(k+1) & 0 & \mathsf{F}^T(k) & \eta I \end{bmatrix} \succ 0, \tag{5.7}$$

for all $k = 0, 1, \cdots$, the time-varying discrete-time system (5.6) *is asymptotically stable and*

$$\|H\|_\infty < \eta.$$

For the remainder of the chapter, we will deal with the special case where the system takes the form

$$\begin{aligned} x(k+1) &= \mathsf{A}(\xi(k))x(k) + \mathsf{E}(\xi(k))w(k), \\ z(k) &= \mathsf{C}(\xi(k))x(k) + \mathsf{F}(\xi(k))w(k), \end{aligned} \tag{5.8}$$

where A, E, C, and F are affine, as in the introduction, and $\xi(k) \in \Xi$. We will represent the system (5.8) with the quadruple $(A(\xi(k)), E(\xi(k)), C(\xi(k)), F(\xi(k))$. A sequence $P(k)$ satisfying Lemma 21 can be constructed by letting $P(k) = P(\xi(k))$. A surprising observation is that if $P(\xi)$ is a polynomial function of ξ, then $P(\xi)$ can be taken to be affine, that is

$$P(\xi) = \sum_{i=1}^{N} \xi_i P_i \succ 0. \tag{5.9}$$

This result is proved in the the following lemma.

Lemma 22. *The following are equivalent:*

a) There exists a polynomial $P(\xi) \succ 0$ such that (5.7) *holds for all $\xi \in \Xi$.*

b) There exists an affine $P(\xi)$ as in (5.9) *such that* (5.7) *holds for all $\xi \in \Xi$.*

c) There exists symmetric matrices $P_i \succ 0$ such that

$$\begin{bmatrix} P_j & P_j\mathsf{A}_i & 0 & P_j\mathsf{E}_i \\ \star & P_i & \mathsf{C}_i^T & 0 \\ \star & \star & \eta I & \mathsf{F}_i \\ \star & \star & \star & \eta I \end{bmatrix} \succ 0, \tag{5.10}$$

d) There exists square matrices X_i, $i = 1, \ldots, N$, and positive-definite matrices S_i, $i = 1, \ldots, N$, such that

$$\begin{bmatrix} X_i + X_i^T - S_i & 0 & X_i^T\mathsf{A}_i^T & X_i^T\mathsf{C}_i^T \\ 0 & \eta I & \mathsf{E}_i^T & \mathsf{F}_i^T \\ \mathsf{A}_i X_i & \mathsf{E}_i & S_j & 0 \\ \mathsf{C}_i X_i & \mathsf{F}_i & 0 & \eta I \end{bmatrix} \succ 0, \tag{5.11}$$

Proof. a) $\implies$ c): If a bounded, polynomial $P(\xi(k)) \succ 0$ satisfies the inequalities (5.7) for all $\xi(k), \xi(k+1) \in \Xi$ then evaluating the matrices P and $(\mathsf{A}, \mathsf{E}, \mathsf{C}, \mathsf{F})$ at each vertex of Ξ, one obtains (5.10) where $P_i = P(\xi_i)$ and ξ_i is one of the N vertices of Ξ.

c) $\implies$ b): In (5.10), for each i, multiply the corresponding $j = 1, \ldots, N$ inequalities by $\xi(k+1)$ and sum. Then, multiplying the resulting $i = 1, \ldots, N$ inequalities by $\xi(k)$ and sum to obtain (5.7) with open-loop system as in (5.8) and $P(k) = P(\xi(k))$ as in (5.9).

b) $\implies$ a) is trivial.

c) $\iff$ d) is proved in [50]. □

This is an extension of the result proved for stability in [13] and, as far as the authors know, have not appeared before in the literature.

The above lemma makes use of the notion of *poly-quadratic stability*, in which stability of the time-varying system (5.8) is proved by constructing an affine *parameter-dependent* Lyapunov function [51], such as (5.9). Indeed, Lemma 22 is ofter taken as a definition of poly-quadratic H_∞ performance. See [10] for more details.

In the sequel, if any of the conditions in Lemma 22 holds, then we say that system (5.8) is poly-quadratically stable with H_∞ performance η.

Unfortunately, Lemma 22, as presented, is not suited for filter synthesis because after substituting for the error system (5.4) and gain $L(\xi(k))$ as in (5.3), that is

$$\mathsf{A}_i \leftarrow A_i - L_i C, \tag{5.12}$$

and

$$\mathsf{E}_i \leftarrow B_i - L_i D, \tag{5.13}$$

it is not possible to apply the standard change-of-variables to rewrite (5.10) nor (5.11) as an Linear Matrix Inequality (LMI). In the next sections we will discuss alternative conditions which can be used for LPV filter synthesis.

5.3 Alternative conditions for poly-quadratic H_∞ performance

The notion of poly-quadratic stability was first introduced in [10]. In [50], the authors proposed poly-quadratic conditions for the design of LPV state-feedback H_∞ controllers. Their characterization of poly-quadratic stability is the one given by item d) in Lemma 22.

An alternative sufficient condition for poly-quadratic H_∞ performance which is better suited for filtering design was later provided in [64, 6], as given in the following Lemma.

Lemma 23. *If there exists symmetric matrices $P_i \succ 0$ and matrices X_i such that*

$$\begin{bmatrix} X_i + X_i^T - P_j & 0 & X_i\mathsf{A}_i & X_i\mathsf{E}_i \\ 0 & \eta I & \mathsf{C}_i & \mathsf{F}_i \\ \mathsf{A}_i^T X_i^T & \mathsf{C}_i^T & P_i & 0 \\ \mathsf{E}_i^T X_i^T & \mathsf{F}_i^T & 0 & \eta I \end{bmatrix} \succ 0, \tag{5.14}$$

for all $i, j = 1, \ldots, N$, then system (5.8) *is poly-quadratically stable with H_∞ performance η.*

Even though the LMI in the above Lemma 23 is sufficient for poly-quadratic H_∞ performance, it is not clear whether it is necessary as well. A proof of necessity for the stability only version of the above conditions was provided in [71]. In the present chapter, this result is extended to cover Lemma 23 as well. That is, we will prove the following converse theorem.

Theorem 12. *If the system* (5.8) *is poly-quadratically stable with H_∞ performance η then there exists symmetric matrices $P_i \succ 0$ and matrices X_i such that the inequalities* (5.14) *hold for all $i, j = 1, \ldots, N$.*

A proof of this result is a bit technical and can be found in Section 5.8.

We will show later in Section 5.5 how the above condition can be used for LPV filter design. Surprisingly, it is also possible to use the LMIs from Lemma 22 for filtering design after developing a duality theory for poly-quadratic Lyapunov functions. This will be the focus of the following section.

5.4 Duality for Poly-Quadratic H_∞ performance

At this point, a reader familiar with the analysis of H_∞ performance for time-invariant systems might be wondering why filters cannot be designed by *dualizing* the existing state-feedback conditions from [10, 50]. As it turns out, we are not aware of any results in the literature that can relate the poly-quadratic H_∞ performance of a system of the form (5.8) described by the matrices $(\mathsf{A}, \mathsf{C}, \mathsf{E}, \mathsf{F})$ with the poly-quadratic H_∞ performance of a system of the same form described by the dual matrices $(\mathsf{A}^T, \mathsf{E}^T, \mathsf{C}^T, \mathsf{F}^T)$. The goal of the present section is to develop such theory.

Stability

We can leverage the conjugacy relationship between a max of quadratics and the convex hull of quadratics to establish duality properties for poly-quadratic Lyapunov functions. To see this we first present the following fundamental result from [70, 60].

Lemma 24. *Let $\mathsf{A}(\xi(k)) = \sum_{i=1}^{N} \xi_i(k)\mathsf{A}_i$, where $\xi(k) \in \Xi$. The following statements are equivalent:*

a) The time-varying system

$$x(k+1) = \mathsf{A}(\xi(k))x(k),$$

is asymptotically stable for all $\xi(k) \in \Xi$.

b) There exists an integer $L > 0$*, matrices* $P_\ell = P_\ell^T \succ 0$ *and scalars* $\gamma_{ij\ell} \geq 0$, $i = 1, \ldots, N; j, \ell = 1, \ldots, L$ *satisfying* $\sum_{j=1}^{L} \gamma_{ij\ell} < 1$ *for all* i, ℓ *and*

$$\mathsf{A}_i^T P_\ell \mathsf{A}_i \prec \sum_{j=1}^{L} \gamma_{ij\ell} P_j \qquad \forall i, \ell. \tag{5.15}$$

c) There exists an integer $L > 0$ *and*

$$V(x) = \max_{\ell=1,\ldots,L} x^T P_\ell x, \tag{5.16}$$

where $P_\ell = P_\ell^T \succ 0$ *for all* $\ell = 1, \ldots, L$ *for which*

$$V(\mathsf{A}(\xi)x) < V(x) \tag{5.17}$$

for all $x \neq 0$ *and* $\xi(k) \in \Xi$.

d) The time-varying system

$$x(k+1) = \mathsf{A}(\xi(k))^T x(k),$$

is asymptotically stable.

e) There exists an integer $L > 0$ *and*

$$V^\sharp(x) = \min_{\beta \in \Xi} x^T \left(\sum_{\ell=1}^{L} \beta_\ell P_\ell^{-1} \right)^{-1} x. \tag{5.18}$$

where $P_\ell = P_\ell^T \succ 0$ *for all* $\ell = 1, \ldots, L$ *for which*

$$V^\sharp(\mathsf{A}(\xi)^T y) < V^\sharp(y) \tag{5.19}$$

for all $y \neq 0$ *and* $\xi(k) \in \Xi$.

In Lemma 24, statements $a) - c)$ relate to the stability of the primal system described by matrix A The remaining statements $d) - e)$ relate to the stability of the dual system described by matrix A^T. Statement $b)$ was introduced in [60]. The equivalence between $a) - c)$ and $d) - e)$ has been discussed in [70] and [72].

Lemma 24 essentially establishes that stability of a time-varying system and its dual are equivalent, a result which has been known since the 1990's and proved in [73] using polyhedral Lyapunov functions. In fact, one can substitute A for A^T in Lemma 24 to obtain equivalent characterizations based on the dual system. Note also that the matrices P's appearing in $b)$, $c)$, and $e)$ of Lemma 24 can be taken to be the same, that is, constructing a max of quadratics Lyapunov function for the primal system automatically constructs a convex-hull of quadratics for the dual system and vice-versa.

By leveraging Lemma 24 we can also directly prove that poly-quadratic stability of the dual system directly implies asymptotic stability of the primal system by directly constructing a max of quadratics Lyapunov function. This result will be later extended to cover H_∞ performance, essentially establishing a dual theory for poly-quadratic H_∞ performance.

Lemma 25. *If the system*

$$v(k+1) = \mathsf{A}(\xi(k))^T v(k) \tag{5.20}$$

where $\xi(k) \in \Xi$ *is poly-quadratically stable with* $P(\xi(k)) = \sum_{i=1}^N \xi_i P_i$, *then*

$$V^\#(\mathsf{A}(\xi)x) < V^\#(x) \tag{5.21}$$

for all $x \neq 0$ *and* $\xi \in \Xi$ *for*

$$V^{\#}(x) = \min_{\beta \in \Xi} x^T \left(\sum_{\ell=1}^{L} \beta_\ell P_\ell^{-1} \right)^{-1} x, \tag{5.22}$$

and $L = N$.

Proof. Assume that the system (5.20) is poly-quadratically stable. There exists $P_i \succ 0$ such that

$$\begin{bmatrix} P_k & P_k \mathsf{A}_i^T \\ \mathsf{A}_i P_k & P_i \end{bmatrix} \succ 0, \tag{5.23}$$

for all $i, k = 1, \ldots, N$ by [10]. Using Schur complement, inequality (5.23) implies

$$\mathsf{A}_i P_k \mathsf{A}_i^T \prec P_i \tag{5.24}$$

for all $i, k = 1, \ldots, N$. Defining

$$\gamma_{ij\ell} = \begin{cases} 1 - \epsilon, & j = i \\ 0, & \text{otherwise} \end{cases} \tag{5.25}$$

for some $1 > \epsilon > 0$ it follows that $\gamma_{ij\ell} \geq 0$ and $\sum_{j=1}^{N} \gamma_{ij\ell} = 1 - \epsilon < 1$ for all $i, j, \ell = 1, \ldots, N$. Furthermore, because inequality (5.24) is strict, there exists a small enough $1 > \epsilon > 0$ such that for $P_\ell = P_k$

$$\mathsf{A}_i P_\ell \mathsf{A}_i^T \prec (1 - \epsilon) P_i = \sum_{j=1}^{N} \gamma_{ij\ell} P_j \tag{5.26}$$

for all $i, \ell = 1, \ldots, N$. From Lemma 24 this implies there exists a max of quadratics Lyapunov function $V(v)$ for the dual system satisfying (5.17), therefore that there exists a convex-hull of quadratics $V^{\#}(x)$ satisfying (5.19) for the primal system. □

A corollary of the above lemma is the following duality result for poly-quadratic stability.

Corollary 3. *If the (dual) system*

$$v(k+1) = \mathsf{A}(\xi(k))^T v(k) \tag{5.27}$$

where $\xi(k) \in \Xi$ is poly-quadratically stable then the (primal) system

$$x(k+1) = \mathsf{A}(\xi(k))x(k) \tag{5.28}$$

is asymptotically stable.

H_∞ Performance

In order to extend the stability analysis construction from the previous section to the case of H_∞ performance we need the following counterpart to Lemma 24.

Lemma 26. *Let the quadruple $(\mathsf{A}(\xi), \mathsf{E}(\xi), \mathsf{C}(\xi), \mathsf{F}(\xi))$ represent the primal system and*

$$(\mathsf{A}(\xi)^T, \mathsf{C}(\xi)^T, \mathsf{E}(\xi)^T, \mathsf{F}(\xi)^T)$$

the dual system with

$$\begin{bmatrix} A(\xi) & E(\xi) \\ C(\xi) & F(\xi) \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & E_i \\ C_i & F_i \end{bmatrix}$$

for $\xi \in \Xi$. The following are equivalent:

a) There exists a function $V(x)$ of the form (5.16) satisfying (5.17) such that

$$\begin{aligned} &V(\mathsf{A}(\xi)x + \mathsf{E}(\xi)w) - V(x) \\ &+ \eta^{-1}\|\mathsf{C}(\xi)x + \mathsf{F}(\xi)w\|^2 - \eta\|w\|^2 < 0 \end{aligned} \tag{5.29}$$

for all $(x, w) \neq 0$

b) There exists a function $V^\sharp(v)$ of the form (5.18) *satisfying* (5.19) *such that*

$$\begin{aligned} &V^\sharp(\mathsf{A}(\xi)^T + \mathsf{C}(\xi)^T) - V^\sharp(v) \\ &+ \eta^{-1}\|\mathsf{E}(\xi)^T v + \mathsf{F}(\xi)^T d\|^2 - \eta\|d\|^2 < 0 \end{aligned} \tag{5.30}$$

for all $(v, d) \neq 0$.

Lemma 26 is the discrete-time counterpart to the continuous time H_∞ performance duality condition that was proved in [74]. The arguments used to prove Lemma 26 are similar to those in the continuous-time case so a proof will be excluded here in the interests of space.

In the case of stability, we saw in Lemma 24, that corresponding to the Lyapunov condition (5.17) was the condition (5.15) which was leveraged to prove Lemma 25. In the case of H_∞ synthesis, we have a similar condition which we present in the following lemma from [74].

Lemma 27. *If there exist $Q_i \succ 0$ and scalars $\lambda_{ij\ell} \geq 0$ with $\sum_{j=1}^{L} \lambda_{ij\ell} = 1$ for $i = 1, \ldots, N, j, \ell = 1, \ldots, L$ such that*

$$\begin{bmatrix} \sum_{j=1}^{L} \lambda_{ij\ell} Q_j - \mathsf{C}_i^T \mathsf{C}_i - \mathsf{A}_i^T Q_\ell \mathsf{A}_i & -\mathsf{C}_i^T \mathsf{F}_i - \mathsf{A}_i^T Q_\ell \mathsf{E}_i \\ -\mathsf{F}_i^T \mathsf{C}_i^T - \mathsf{E}_i^T Q_\ell \mathsf{A}_i & \eta^2 I - \mathsf{F}_i^T \mathsf{F}_i - \mathsf{E}_i^T Q_\ell \mathsf{E}_i \end{bmatrix} \succ 0, \tag{5.31}$$

for all $i = 1, \ldots, N$, $\ell = 1, \ldots, L$ then there exists a function $V(x)$ of the form (5.16) *satisfying* (5.17) *and*

$$\begin{aligned} &V(\mathsf{A}(\xi)x + \mathsf{E}(\xi)w) - V(x) \\ &+ \eta^{-1}\|\mathsf{C}(\xi)x + \mathsf{F}(\xi)w\|^2 - \eta\|w\|^2 < 0 \end{aligned} \tag{5.32}$$

for all $(x, w) \neq 0$.

In contrast to the condition (5.15), the above is only sufficient for the existence of a Lyapunov function $V(x)$ such that (5.32) holds. We can still leverage it however to prove the counterpart to Lemma 25 in the case of H_∞ analysis. We present this result now.

Lemma 28. *If the system*

$$\begin{aligned} v(k+1) &= \mathsf{A}^T(\xi(k))v(k) + \mathsf{C}^T(\xi(k))d(k), \\ y(k) &= \mathsf{E}^T(\xi(k))v(k) + \mathsf{F}^T(\xi(k))d(k), \end{aligned} \tag{5.33}$$

is poly-quadratically stable with H_∞ performance η and $P(\xi) = \sum_i \xi_i P_i$, then

$$V^\sharp(\mathsf{A}(\xi)x + \mathsf{E}(\xi)w) - V^\sharp(x)$$
$$+ \eta^{-1}\|\mathsf{C}(\xi)x + \mathsf{F}(\xi)w\|^2 - \eta\|w\|^2 < 0$$

for all $(x, w) \neq 0$ and $\xi \in \Xi$ with

$$V^\#(x) = \min_{\beta\in\Xi} x^T \left(\sum_{\ell=1}^{L} \beta_\ell P_\ell^{-1}\right)^{-1} x, \tag{5.34}$$

and $L = N$.

Proof. Assume that system (5.33) is poly-quadratically stable with H_∞ performance η. There exists $P_i \succ 0$ such that

$$\begin{bmatrix} P_k & P_k\mathsf{A}_i^T & 0 & P_k\mathsf{C}_i^T \\ \star & P_i & \mathsf{E}_i & 0 \\ \star & \star & \eta I & \mathsf{F}_i^T \\ \star & \star & \star & \eta I \end{bmatrix} \succ 0, \tag{5.35}$$

for all $i, k = 1, \ldots, N$. Multiplying inequalities (5.35) by

$$T = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

on the left and its transpose on the right and using Schur complement implies that

$$\begin{bmatrix} P_i - \eta^{-1}\mathsf{E}_i\mathsf{E}_i^T - \mathsf{A}_iP_k\mathsf{A}_i^T & -\eta^{-1}\mathsf{E}_i\mathsf{F}_i^T - \mathsf{A}_iP_k\mathsf{C}_i^T \\ -\eta^{-1}\mathsf{F}_i\mathsf{E}_i^T - \mathsf{C}_iP_k\mathsf{A}_i^T & \eta - \eta^{-1}\mathsf{F}_i\mathsf{F}_i^T - \mathsf{C}_iP_k\mathsf{C}_i^T \end{bmatrix} \succ 0, \tag{5.36}$$

for all $i, k = 1, \ldots, N$. Defining

$$\lambda_{ij\ell} = \begin{cases} 1, & j = i \\ 0, & \text{otherwise} \end{cases}, \qquad Q_\ell = \eta P_k, \tag{5.37}$$

it follows that $\lambda_{ij\ell} \geq 0$ and $\sum_{j=1}^N \lambda_{ij\ell} = 1$ for all $i, j, \ell = 1, \ldots, N$. Scaling (5.36) by η then implies that

$$\begin{bmatrix} \sum_{j=1}^N \lambda_{ij\ell} Q_j - \mathsf{E}_i \mathsf{E}_i^T - \mathsf{A}_i Q_\ell \mathsf{A}_i^T & -\mathsf{E}_i \mathsf{F}_i^T - \mathsf{A}_i Q_\ell \mathsf{C}_i^T \\ -\mathsf{F}_i \mathsf{E}_i^T - \mathsf{C}_i Q_\ell \mathsf{A}_i^T & \eta^2 I - \mathsf{F}_i \mathsf{F}_i^T - \mathsf{C}_i Q_\ell \mathsf{C}_i^T \end{bmatrix} \succ 0, \tag{5.38}$$

for all $i, \ell = 1, \ldots, N$. From Lemma 26 and Lemma 27,this implies that there exists a max of quadratics Lyapunov function $V(v)$ satisfying (5.29) for the dual system, therefore that there exists a convex-hull of quadratics $V^\sharp(x)$ satisfying (5.30) for the primal system. □

A corollary of the above lemma is the following duality result for poly-quadratic H_∞ performance

Corollary 4. *If (dual) system $(A(\xi)^T, C(\xi)^T, E(\xi)^T, F(\xi)^T)$ with $\xi \in \Xi$ is poly-quadratically stable with H_∞ performance η then the (primal) system $(A(\xi), E(\xi), C(\xi), F(\xi))$ with $\xi \in \Xi$ is asymptotically stable with H_∞ performance η.*

In the next section, we will use the above results to design dual LPV filters directly using the state feedback design conditions from [10, 50].

5.5 H_∞ Synthesis Conditions for LPV Filters

For the remainder of the chapter, we will consider the LPV filter synthesis problem as was introduced in the initial motivation of the chapter.

The most general H_∞ filter synthesis conditions for the class of time-varying systems (5.1) that can still be expressed as LMIs are in ones from [6] which are also derived using a poly-quadratic Lyapunov function. We present this condition now.

Lemma 29. *Consider the system* (5.1)*. If there exist symmetric matrices* $P_i \succ 0$*, matrices* X_i *and* R_i*, such that for a real number* η*,*

$$\begin{bmatrix} X_i + X_i^T - Q_j & \star & \star & \star \\ 0 & \eta & \star & \star \\ A_i^T X_i^T - C^T R_i^T & E_i^T & Q_i & \star \\ B_i^T X_i^T - D^T R_i^T & F_i^T & 0 & \eta \end{bmatrix} \succ 0, \tag{5.39}$$

for all $i, j = 1, \ldots, N$*, then the filter* (5.2)*, with gain* $L(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) L_i$ *and* $L_i = X_i^{-1} R_i$ *ensures that the error system* (5.4) *are asymptotically stable with* $\|H\|_\infty$ *bound as in* (5.5)*.*

That the LMIs (5.39) are sufficient for H_∞ filter synthesis as stated in Lemma 29 was proved in [6]. In [64], the authors additionally claim without proof that the condition (5.39) is also necessary for poly-quadratic H_∞ filter synthesis. In [71], a proof is given for the converse of Lemma 29 in the stability only case. We can now extend this proof to the H_∞ synthesis case using Theorem 12. This is the following result.

Lemma 30. *If there exits a filter* (5.2)*, with gain* $L(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) L_i$ *with* $L_i = X_i^{-1} R_i$ *which poly-quadratically stabilizes the error system* (5.4) *with* H_∞ *performance as in* (5.5)*, then there exists symmetric matrices* $P_i \succ 0$ *and matrices* X_i *and* R_i *such that* (5.39) *holds for all* $i, j = 1, \ldots, N$*.*

Proof. This follows directly from the proof of Theorem 12 after making the following substitutions,

$$\begin{bmatrix} \mathsf{A}_i & \mathsf{E}_i \\ \mathsf{C}_i & \mathsf{F}_i \end{bmatrix} \rightarrow \begin{bmatrix} A_i - L_i C & B_i - L_i D \\ E_i & F_i \end{bmatrix}$$

and the change of variables $R_i = X_i L_i$ for all $i = 1, \ldots, N$. We note the assignment on the right will still be affine in $\xi \in \Xi$ after a convex combination.

□

As an alternative to Lemma 29, we can also devise conditions for LPV filter synthesis using the existing state feedback synthesis conditions from [10, 50] combined with the poly-quadratic duality theory that we developed in the previous section. We present this result next.

Lemma 31. *If there exists symmetric matrices $Q_i \succ 0$ and matrices G_i, R_i such that*

$$\begin{bmatrix} G_i + G_i^T - Q_i & \star & \star & \star \\ 0 & \eta & \star & \star \\ A_i^T G_i - C^T R_i & E_i^T & Q_j & \star \\ B_i^T G_i - D^T R_i & F_i^T & 0 & \eta \end{bmatrix} \succ 0, \tag{5.40}$$

for all $i, j = 1, \ldots, N$, then the filter (5.2)*, with gain $L(\xi(k)) = \sum_{i=1}^N \xi_i(k) L_i$ and $L_i = R_i G_i^{-1}$ ensures that the error system* (5.4) *are asymptotically stable with $\|H\|_\infty$ bound as in* (5.5)*.*

Proof. This follows directly from Corollary 4 with

$$\begin{bmatrix} \mathsf{A}_i^T & \mathsf{C}_i^T \\ \mathsf{E}_i^T & \mathsf{F}_i^T \end{bmatrix} \to \begin{bmatrix} (A - L_i C)^T & E_i^T \\ (B - L_i D)^T & F_i^T \end{bmatrix},$$

and the change of variables $L_i = R_i G_i^{-1}$ for all $i = 1, \ldots, N$. □

We note that, in contrast to Lemma 29, it is unknown whether Lemma 31 is also necessary for poly-quadratic H_∞ synthesis.

Table 5.1. Maximum γ for different control approaches.

Poly-Quadratic	Lemma 34	0.65
	Lemma 31	0.65

5.6 Numerical Examples

Consider the following time-varying linear discrete-time system adapted from [48] where our system have the form (5.1) are defined by

$$A(\alpha)=\begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\alpha & -0.5\alpha & 0.2 & 0.03+\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$C=\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}.$$

Our goal is to determine the largest $\gamma > 0$ such that we can design a observer to reconstruct the states of our system for all $|\alpha| \leq \gamma$. This system can be put in the form (5.1) with 2 vertices where we introduce the following vertex matrices to characterize the performance,

$$B_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} D_i \\ \hline E_i \\ \hline F_i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{bmatrix}, \tag{5.41}$$

for all $i = 1, \ldots, N$.

In Table 1 we have indicated the maximum value of γ for which the error system (5.4) can be stabilized for the two conditions Lemmaz 29 and Lemma 31.

The results in Table 1 indicate that for our numerical example the maximal value of γ is identical for both conditions. We also minimize the corresponding $\|H\|_\infty$ bound η and present these results in Table 2

Table 5.2. H_∞ performance bounds for different filtering approaches.

		γ			
		0.1	0.2	0.3	0.4
	Lemma 34	3.45	3.91	4.63	5.91
Poly-Quadratic	Lemma 31	3.45	3.91	4.63	5.91

The above simple numerical example and many other examples run by the authors suggests that both approaches discussed here are identical. Whether or not this is always true remains to be seen, but it may suggest the existence of a duality equivalence for poly-quadratic Lyapunov functions in the same way that we see for parameter-independent Lyapunov functions. The authors have yet to find a counter example suggesting this is not true however any efforts to prove such a result will be reserved for future work.

5.7 Discussion

In this chapter we have discussed LMI conditions for LPV H_∞ filter synthesis.

We first provided a proof of necessity for the class of poly-quadratic filter synthesis conditions introduced in [6, 64]. As we have mentioned, the existence of a proof of necessity allows for many other papers that use a similar filter synthesis condition e.g. [65, Chapter 4], [6, 66, 67, 68, 69], to also claim necessity in their design conditions.

In the case of state feedback control, the conditions introduced in [10, 50], were already known to be necessary and sufficient for poly-quadratic stabilizability when the input matrix, $B(\xi(k)) = B$ and feed-forward matrix, $D(\xi(k)) = D$ were held constant. The results in this chapter establish the corresponding result for filter synthesis.

We have also derived an LPV filtering condition from the state feedback control conditions introduced in [10, 50]. To derive this new condition, we leveraged the duality that exists between max of quadratic and convex hull of quadratic Lyapunov functions. We showed that for a simple numerical example these two approaches to LPV filtering produce the same results.

It remains to be seen if the results from our numerical example hold in general. If they do, this suggests that the duality we exploited in the case of parameter-independent Lyapunov functions may also hold for poly-quadratic Lyapunov functions. Determining if this is true or not remains as future work.

In [62, 75], the state feedback conditions proposed in [10, 50], were extended with the introduction of new sufficient conditions for poly-quadratic stabilizability in the situation where time variation is allowed in the input matrix, $B(\xi(k)) \neq B$ and feed-forward matrix, $D(\xi(k)) \neq D$. Additionally, these conditions were shown to be equivalent to those in [10, 50] when $B(\xi(k)) = B$ and $D(\xi(k)) = D$. With the results proposed in this chapter, it is also tempting to speculate whether similar advances can be made in the case of poly-quadratic filter synthesis. Namely, is it possible to extend Lemma 29 to allow for variation in the output matrices $C(\xi(k)) \neq C$ and $D(\xi(k)) \neq D$ without losing necessity for poly-quadratic filter synthesis in the case where these matrices are held constant. Such efforts will also be saved for future work.

5.8 Proof of Theorem 12

Proof. Assume that (5.10) holds with $P(k)$ as in (5.9) to give

$$\begin{bmatrix} P_j & \star & \star & \star \\ \mathsf{A}_i^T P_j & P_i & \star & \star \\ 0 & \mathsf{C}_i & \eta I & \star \\ \mathsf{E}_i^T P_j & 0 & \mathsf{F}_i^T & \eta I \end{bmatrix} \succ 0, \tag{5.42}$$

for all $i, j = 1, \ldots, N$. Multiplying, (5.42) on the right by

$$T_j = \begin{bmatrix} P_j^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and on the left by its transpose gives,

$$\begin{bmatrix} P_j^{-1} & \mathsf{A}_i & 0 & \mathsf{E}_i \\ \mathsf{A}_i^T & P_i & \mathsf{C}_i^T & 0 \\ 0 & \mathsf{C}_i & \eta I & \mathsf{F}_i \\ \mathsf{E}_i^T & 0 & \mathsf{F}_i^T & \eta I \end{bmatrix} \succ 0, \tag{5.43}$$

for all $i, j = 1, \ldots, N$. After a Schur complement, (5.43) becomes

$$0 \preceq \mathcal{A}_i \mathcal{P}_i^{-1} \mathcal{A}_i^T \prec P_j^{-1}, \tag{5.44}$$

where

$$\mathcal{A}_i = \begin{bmatrix} \mathsf{A}_i & 0 & \mathsf{E}_i \end{bmatrix},$$

and

$$\mathcal{P}_i = \begin{bmatrix} P_i & \mathsf{C}_i^T & 0 \\ \mathsf{C}_i & \eta I & \mathsf{F}_i \\ 0 & \mathsf{F}_i^T & \eta I \end{bmatrix},$$

for all $i, j = 1, \ldots, N$. There exists a $\mathcal{W}_i \succ 0$, $i = 1, \ldots, N$ such that

$$\mathcal{A}_i \mathcal{P}_i^{-1} \mathcal{A}_i^T \prec \mathcal{W}_i \prec P_j^{-1}, \tag{5.45}$$

e.g. $\mathcal{W}_i = \mathcal{A}_i \mathcal{P}_i^{-1} \mathcal{A}_i^T + \epsilon I$ for a sufficiently small $\epsilon > 0$. Furthermore, from (5.45), it follows that,

$$\mathcal{W}_i^{-1} \succ P_j,$$

for all $i, j = 1, \ldots, N$, and, for any nonsingular square matrix Y_i,

$$Y_i + Y_i^T - Y_i \mathcal{W}_i^{-1} Y_i^T \prec Y_i + Y_i^T - Y_i P_j Y_i^T,$$

for all $i, j = 1, \ldots, N$. In particular, for $Y_i = \mathcal{W}_i \succ 0$, for all $i = 1, \ldots, N$, one obtains

$$\begin{aligned} &\mathcal{A}_i \mathcal{P}_i^{-1} \mathcal{A}_i^T \prec \mathcal{W}_i = \\ &\qquad Y_i + Y_i^T - Y_i \mathcal{W}_i^{-1} Y_i^T \prec Y_i + Y_i^T - Y_i P_j Y_i^T, \end{aligned} \tag{5.46}$$

for all $i, j = 1, \ldots, N$. Since Y_i is nonsingular for all $i = 1, \ldots, N$, multiplying (5.46) on the left by $X_i = Y_i^{-1}$ and on the right by X_i^T yields

$$X_i \mathsf{A}_i \mathsf{P}_i^{-1} \mathsf{A}_i^T X_i^T \prec X_i + X_i^T - P_j, \tag{5.47}$$

for all $i, j = 1, \ldots, N$.Apply a Schur complement to (5.47) to get

$$\begin{bmatrix} X_i + X_i^T - P_j & X_i \mathsf{A}_i & 0 & X_i \mathsf{E}_i \\ \mathsf{A}_i^T X_i^T & P_i & \mathsf{C}_i^T & 0 \\ 0 & \mathsf{C}_i & \eta I & \mathsf{F}_i \\ \mathsf{E}_i^T X_i^T & 0 & \mathsf{F}_i^T & \eta I \end{bmatrix} \succ 0, \tag{5.48}$$

for all $i, j = 1, \ldots, N$ which is (5.14) after a congruence transformation. □

5.9 Acknowledgements

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Chapter 6

Discrete-Time H_∞ Filtering of LPV Systems

6.1 Introduction

Consider time-varying discrete-time linear systems of the form

$$\begin{aligned} x(k+1) &= A(\xi(k))x(k) + B(\xi(k))w(k), \\ y(k) &= C(\xi(k))x(k) + D(\xi(k))w(k), \\ z(k) &= E(\xi(k))x(k) + F(\xi(k))w(k), \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ and the matrices $A(\xi(k))$, $B(\xi(k))$, $C(\xi(k))$, $D(\xi(k))$, $E(\xi(k))$ and $F(\xi(k))$ are assumed to depend affinely on the uncertain time-varying parameter $\xi(k)$, which assumes values in the unit simplex,

$$\Xi = \left\{ \xi \in \mathbb{R}_+^N : \quad \sum_{i=1}^{N} \xi_i = 1 \right\},$$

The affine assumption means that matrices $A(\xi(k))$, $B(\xi(k))$, $C(\xi(k))$, $D(\xi(k))$, $E(\xi(k))$ and $F(\xi(k))$ can be written as

$$\begin{bmatrix} A(\xi(k)) & B(\xi(k)) \\ C(\xi(k)) & D(\xi(k)) \\ E(\xi(k)) & F(\xi(k)) \end{bmatrix} = \sum_{i=1}^{N} \xi_i(k) \begin{bmatrix} A_i & B_i \\ C_i & D_i \\ E_i & F_i \end{bmatrix}.$$

We are interested with the design of a LPV filter of the form

$$\begin{aligned}\hat{x}(k+1) &= A(\xi(k))\hat{x}(k) + L(\xi(k))(y(k) - \hat{y}(k)),\\ \hat{y}(k) &= C(\xi(k))\hat{x}(k),\\ \hat{z}(k) &= E(\xi(k))\hat{x}(k),\end{aligned} \tag{6.2}$$

where,

$$L(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) L_i. \tag{6.3}$$

We denote the state and output estimation errors as

$$e(k) = x(k) - \hat{x}(k), \qquad e_z(k) = z(k) - \hat{z}(k),$$

so that the goal is to choose an appropriate $L(\xi(k))$ such that the error system

$$\begin{aligned}e(k+1) &= (A(\xi(k)) - L(\xi(k))C(\xi(k)))e(k)\\ &\quad + (B(\xi(k)) - L(\xi(k))D(\xi(k)))w(k)\end{aligned} \tag{6.4}$$

is asymptotically stable and the l_2-to-l_2 gain

$$\sup_{\|w\|_2 \neq 0} \frac{\|e_z\|_2}{\|w\|_2} < \eta, \tag{6.5}$$

is minimized.

In this chapter we will introduce new sufficient H_∞ filter synthesis conditions derived using poly-quadratic Lyapunov functions [10] which can handle variation in the dynamics as well as the output matrices. These conditions are all expressed as Linear Matrix Inequalities (LMIs). The new conditions we introduce here are a contrast to existing necessary and sufficient LMI conditions from [6] which are also derived from poly-quadratic Lyapunov functions but require $C_i = C$ and $D_i = D$ for all $i = 1, \ldots, N$. We will show however that in the case where $C_i = C$ and $D_i = D$, the conditions presented in this chapter are equivalent to those in [6].

Additionally, we will show how the filtering conditions introduced in the present chapter can be combined with the state-feedback conditions introduced in [62] to design output feedback controllers where variation will be permitted in the dynamics as well as the input and output matrices. We will compare these output feedback conditions with those introduced in [2].

The chapter will be structured as follows. We will first define the H_∞ cost criterion we introduced above and introduce the notion of poly-quadratic stability [10]. Following this, we will discuss existing approaches to filter design for LPV systems including the results from [6] where we are restricted to $C_i = C$ and $D_i = D$ for all $i = 1, \ldots, N$. We will additionally discuss common techniques for avoiding these restrictions, including robust filtering and output augmentation. Following this discussion we will introduce our new H_∞ filter synthesis conditions where none of the aforementioned restrictions apply. We will compare these new conditions numerically with those in [6]. Finally, we will discuss how the conditions presented in this chapter can be used for output feedback control and compare with the static output feedback conditions presented in [2].

6.2 Poly-quadratic H_∞ Filter Synthesis

For the purpose of defining the H_∞ cost criterion, we consider the asymptotically stable open-loop dynamics of the form

$$\begin{aligned} x(k+1) &= \mathsf{A}(k)x(k) + \mathsf{E}(k)w(k), \\ z(k) &= \mathsf{C}(k)x(k) + \mathsf{F}(k)w(k), \end{aligned} \tag{6.6}$$

for which the H_∞ performance is defined by the l_2-to-l_2 gain:

$$\|H\|_\infty = \sup_{\|w(k)\|_2 \neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2}, \tag{6.7}$$

where $w(k) \in l_2^r$ and $z(k) \in l_2^p$. The notation $\|H\|_\infty$ is a slight abuse since (6.6) is time-varying. An upper bound for the H_∞ norm can be characterized by the bounded real lemma in a way similar to [2, 49],

Lemma 32. *If there exists a bounded matrix sequence* $P(k) = P(k)^T$ *such that*

$$\begin{bmatrix} P(k+1) & \star & \star & \star \\ \mathsf{A}^T(k)P(k+1) & P(k) & \star & \star \\ 0 & \mathsf{C}(k) & \eta I & \star \\ \mathsf{E}^T(k)P(k+1) & 0 & \mathsf{F}^T(k) & \eta I \end{bmatrix} \succ 0, \tag{6.8}$$

for all $k = 0, 1, \cdots$, *the time-varying discrete-time system* (6.6) *is exponentially stable and*

$$\|H\|_\infty < \inf \eta.$$

If we restrict ourselves to open-loop dynamics of the form

$$\begin{aligned} x(k+1) &= \mathsf{A}(\xi(k))x(k) + \mathsf{E}(\xi(k))w(k), \\ z(k) &= \mathsf{C}(\xi(k))x(k) + \mathsf{F}(\xi(k))w(k), \end{aligned} \tag{6.9}$$

where $\xi(k) \in \Xi$ and let

$$P(k) = P(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) P_i \succ 0, \tag{6.10}$$

then, we can rewrite Lemma 32 in the following manner:

Lemma 33. *The following are equivalent:*

a) There exists a polynomial $P(\xi) \succ 0$ *such that* (6.8) *holds for all* $\xi \in \Xi$.

b) There exists an affine $P(\xi)$ *as in* (6.10) *such that* (6.8) *holds for all* $\xi \in \Xi$.

c) There exists symmetric matrices $P_i \succ 0$ *such that*

$$\begin{bmatrix} P_j & P_j\mathsf{A}_i & 0 & P_j\mathsf{E}_i \\ \star & P_i & \mathsf{C}_i^T & 0 \\ \star & \star & \eta I & \mathsf{F}_i \\ \star & \star & \star & \eta I \end{bmatrix} \succ 0, \tag{6.11}$$

d) There exists square matrices X_i, $i = 1, \dots, N$, and positive-definite matrices S_i, $i = 1, \dots, N$, such that

$$\begin{bmatrix} X_i + X_i^T - S_i & 0 & X_i^T \mathsf{A}_i^T & X_i^T \mathsf{C}_i^T \\ 0 & \eta I & \mathsf{E}_i^T & \mathsf{F}_i^T \\ \mathsf{A}_i X_i & \mathsf{E}_i & S_j & 0 \\ \mathsf{C}_i X_i & \mathsf{F}_i & 0 & \eta I \end{bmatrix} \succ 0, \tag{6.12}$$

Proof. A proof is provided in Chapter 5. □

The above lemma makes use of the notion of *poly-quadratic stability*, in which stability of the time-varying dynamics (6.9) is proved by constructing an affine *parameter-dependent* Lyapunov function [51] of the form

$$V(x(k), \xi(k)) = x(k)^T P(\xi(k)) x(k), \tag{6.13}$$

where $P(\xi(k))$ is as in (6.10). See [10] for more details.

Lemma 33 as presented is not suited for filter synthesis. After substituting for our closed-loop filter dynamics,

$$\mathsf{A}_i \leftarrow A_i - L_i C_i, \tag{6.14}$$

and

$$\mathsf{E}_i \leftarrow B_i - L_i D_i, \tag{6.15}$$

as in (6.4) where $L(\xi(k))$ is defined in (6.3), no change of variables exists for which (6.11) can still be expressed as an LMI. See [76] for an in-depth discussion on this issue.

The most general H_∞ filter synthesis conditions for the class of time-varying systems (6.1) that can still be expressed as LMIs but are restricted to constant output matrices, $C_i = C$ and $D_i = D$ for all $i = 1, \dots, N$, is the following.

Lemma 34. *Consider the dynamics* (6.1)*. If there exist symmetric matrices* $Q_i \succ 0$*, matrices* G_i *and* V_i*, such that for a real number* η*,*

$$\begin{bmatrix} G_i + G_i^T - Q_j & \star & \star & \star \\ 0 & \eta & \star & \star \\ A_i^T G_i^T + C^T V_i^T & E_i^T & Q_i & \star \\ B_i^T G_i^T + D^T V_i^T & F_i^T & 0 & \eta \end{bmatrix} \succ 0, \tag{6.16}$$

for all $i, j = 1, \ldots, N$*, then the filter* (6.2)*, with gain* $L(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) L_i$ *and* $L_i = G_i^{-1} V_i$ *ensures that the error dynamics* (6.4) *are asymptotically stable with* $\|H\|_\infty$ *bound as in* (6.5)*.*

That the LMIs (6.16) are sufficient for H_∞ filter synthesis as established in Lemma 34 was established in [6, 64]. A proof of the necessity was provided in Chapter 5.

Filter design conditions using higher order polynomial Lyapunov functions can be constructed based on the conditions of Lemma 34 using various devices such as in [1, 12]. Such extensions will not be discussed here but, as we shall detail later, it is expected that the new proposed conditions can be used to provide improvements in these setups as well.

The main deficiency of the LMIs in Lemma 34 is the fact that the system cannot have variation in the output matrix C nor in the noise matrix D, hence the assumption $C(\xi(k)) = C$ and $D(\xi(k)) = D$. Indeed, we are not aware of any result in the literature that can simultaneously consider variation in C's, D's and L's and still lead to convex problems in the form of LMIs, as in Lemma 34, without either introducing conservativeness or non-convexity constraints.

The simplest of the techniques that allow for $C(\xi(k))$ and $D(\xi(k))$ to be parameter varying is to require the filter gain to be parameter independent, that is $L(\xi(k)) = L$. For the case of state-feedback, this approach has been discussed for instance in [52]. Another technique is the the popular concept of output-filtering [20, 53, 54] which allows for variation in $C(\xi(k))$ and $D(\xi(k))$ but requires the implementation of a higher order filter. We will discuss both these techniques in the next section.

6.3 Existing Approaches to Incorporate Time Variation into C and D

In this section we will provide a more in-depth review of existing approaches that allow to incorporate time-variation in the matrices C and D in LPV filter design.

6.3.1 Robust Filtering

The first family of conditions that allows for filter design by LMIs when $C(\xi(k))$ and $D(\xi(k))$ depend on the time-varying parameter $\xi(k)$ make use of a potentially conservative robust filter, that is a filter of the form

$$\begin{aligned} \hat{x}(k+1) &= A(\xi(k))\hat{x}(k) + L(y(k) - \hat{y}(k)), \\ \hat{y}(k) &= C(\xi(k))\hat{x}(k), \\ \hat{z}(k) &= E(\xi(k))\hat{x}(k), \end{aligned} \tag{6.17}$$

where the filter gain, L, is independent of the time-varying parameter $\xi(k)$. One such robust filtering condition can be derived from Lemma 34 which we present below.

Lemma 35. *Consider the dynamics* (6.1). *If there exist symmetric matrices $Q_i \succ 0$, and matrices G and V, such that for a real number η,*

$$\begin{bmatrix} G + G^T - Q_j & \star & \star & \star \\ 0 & \eta & \star & \star \\ A_i^T G^T + C_i^T V^T & E_i^T & Q_i & \star \\ B_i^T G^T + D_i^T V^T & F_i^T & 0 & \eta \end{bmatrix} \succ 0, \tag{6.18}$$

for all $i, j = 1, \ldots, N$, then the filter (6.17)*, with gain $L = G^{-1}V$ ensures that the error dynamics* (6.4) *are asymptotically stable with $\|H\|_\infty$ bound as in* (6.5).

This is Lemma 34 with $G_i = G$, $V_i = V$, for all $i = 1, \ldots, N$. Furthermore the above lemma reduces to the well known quadratic H_∞ performance criteria of [53] if $Q_i = G = G^T = Q$, $i = 1, \ldots, N$.

6.3.2 Output-filtering

In both continuous-time as well as discrete-time cases, a standard way to handle variation in the matrices $C(\xi(k))$ and $D(\xi(k))$ is to introduce output filters and work with an augmented system, for example:

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\xi(k))\,\tilde{x}(k) + \tilde{B}\,w(k), \\ r(k) &= \tilde{C}\tilde{x}(k) + \tilde{D}w(k) \\ z(k) &= \tilde{E}(\xi(k))\tilde{x}(k) + \tilde{F}(\xi(k))w(k) \end{aligned} \tag{6.19}$$

where

$$\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \\ \tilde{E}_i & \tilde{F}_i \end{bmatrix} = \left[\begin{array}{cc|c} A_i & 0 & B_i \\ C_i & 0 & D_i \\ \hline 0 & I & 0 \\ \hline E_i & 0 & F_i \end{array}\right], \quad \text{for all } i = 1, \ldots, N. \tag{6.20}$$

This approach remains popular with recent applications in spacecraft control [28, 27] and the design of active suspension systems [24, 25]. The main appeal of this approach is that a LPV filter can now be designed using existing LMI design procedures, for example Lemma 34. However, this flexibility comes with some drawbacks. Firstly, the addition of an output filter increases the dimensions of the LPV observer, which now has dynamics of the form

$$\begin{aligned} \hat{x}(k+1) &= \tilde{A}(\xi(k))\hat{x}(k) + \tilde{L}(\xi(k))(r(k) - \hat{r}(k)) \\ \hat{r}(k) &= \tilde{C}\hat{x}(k) \\ \hat{z}(k) &= \tilde{E}(\xi(k))\hat{x}(k) \end{aligned}$$

which compared to (6.2) has additional states equal to the number of outputs. Secondly, the introduction of an output filter will introduce delays into the system which can degrade performance.

We will show by means of a simple numerical example that the introduction of an output filter to overcome the limitation that $C(\xi(k)) = C$ and $D(\xi(k)) = D$ can in fact result in interior performance when compared to the condition we present in this chapter. We will save this discission for later.

6.4 Main Results

We present the main results of the chapter in the next two theorems. The first is a new sufficient condition for H_∞ filter synthesis while allows for time-variation in both $C(\xi(k))$ and $D(\xi(k))$.

Theorem 13. *Consider the time-varying discrete-time linear system of the form* (6.1)*. If there exists matrices* R_i, W_i, X_i, Y_i, Z_i *and* $Q_i = Q_i^T \succ 0$*,* $i = 1, \ldots, N$ *such that*

$$\begin{bmatrix} X_i + X_i^T - Q_j & X_i A_i & -R_i & X_i B_i & 0 \\ A_i^T X_i^T & Q_i - C_i^T Y_j - Y_j^T C_i & -C_i^T Z_j^T + Y_j^T & -C_i^T W_j - Y_j^T D_i & E_i^T \\ -R_i^T & Y_j - Z_j C_i & Z_j + Z_j^T & W_j - Z_j D_i & 0 \\ B_i^T X_i^T & -D_i^T Y_j - W_j^T C_i & -D_i^T Z_j^T + W_j^T & \eta - D_i^T W_j - W_j^T D_i & F_i^T \\ 0 & E_i & 0 & F_i & \eta \end{bmatrix} \succ 0, \tag{6.21}$$

for all $i, j = 1, \ldots, N$*, then the gain-scheduled LPV filter* (6.2) *with gain* $L(\xi(k))$ *as in* (6.3) *with* $L_i = X_i^{-1} R_i$ *for all* $i = 1, \ldots, N$ *poly-quadratic stabilizes the error dynamics* (6.4) *and*

$$\|H\|_\infty < \inf \eta. \tag{6.22}$$

The second result is that the converse also holds in the case where $C(\xi(k)) = C$ and $D(\xi(k)) = D$.

Theorem 14. *Consider the time-varying discrete-time linear system of the form* (6.1) *with* $C(\xi(k)) = C$ *and* $D(\xi(k)) = D$*. If there exits a filter* (6.2)*, with gain* $L(\xi(k)) = \sum_{i=1}^N \xi_i(k) L_i$ *which poly-quadratically stabilizes the error dynamics* (6.4) *with* H_∞ *performance as in* (6.5)*, then there exists symmetric matrices* $Q_i \succ 0$ *and matrices* R_i, W_i, X_i, Y_i *and* Z_i *such that* (6.21) *holds for all* $i, j = 1, \ldots, N$*.*

The above conditions provides an inclusive generalization of Lemma 34 when $C(\xi(k)) =$

C and $D(\xi(k)) = D$. This is the following Corollary.

Corollary 5. *Consider the time-varying, discrete-time linear system of the form* (6.1) *with* $C(\xi(k)) = C$ *and* $D(\xi(k)) = D$. *There exists symmetric matrices* $Q_i \succ 0$ *and matrices* G_i, V_i *such that* (6.16) *holds for all* $i, j = 1, \ldots, N$ *if and only if there exists matrices* R_i, W_i, X_i, Y_i, Z_i *and* $Q_i = Q_i^T \succ 0$, $i = 1, \ldots, N$ *such that* (6.21) *holds for all* $i, j = 1, \ldots, N$.

Corollary 5 follows directly from the fact that the LMIs (6.16) and the LMIs (6.21) are necessary and sufficient for H_∞ filter synthesis with poly-quadratic Lyapunov functions when $C(\xi(k)) = C$ and $D(\xi(k)) = D$.

If we specialize Theorems 13 and 14 to the case of quadratic Lyapunov functions, it can still bring advantage when compared to classic quadratic stabilizability conditions. Setting $X_i = X_j = Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$ in (6.21) gives a quadratic version of Theorem 13. Remarkably, in this case, it is also possible to freeze the auxiliary variables W, Y and Z without losing necessity with respect to standard quadratic stabilizability conditions, such as the ones in [54]. This results is presented in the next corollary.

Corollary 6. *Consider the time-varying discrete-time linear system of the form* (6.1). *If there exists matrices* R_i, W, Y, Z *and* $Q = Q^T \succ 0$, $i = 1, \ldots, N$ *such that*

$$\begin{bmatrix} Q & QA_i & -R_i & QB_i & 0 \\ A_i^T Q & Q - C_i^T Y - Y^T C_i & -C_i^T Z^T + Y^T & -C_i^T W - Y^T D_i & E_i^T \\ -R_i^T & Y - ZC_i & Z + Z^T & W - ZD_i & 0 \\ B_i^T Q & -D_i^T Y - W^T C_i & -D_i^T Z^T + W^T & \eta - D_i^T W - W^T D_i & F_i^T \\ 0 & E_i & 0 & F_i & \eta \end{bmatrix} \succ 0, \tag{6.23}$$

for all $i, j = 1, \ldots, N$, *then the gain-scheduled LPV filter* (6.2) *with gain* $L(\xi(k))$ *as in* (6.3) *with* $L_i = Q^{-1} R_i$ *for all* $i = 1, \ldots, N$ *quadratically stabilizes the error dynamics* (6.4) *and*

$$\|H\|_\infty < \inf \eta. \tag{6.24}$$

Furthermore, for $C_i = C$ *and* $D_i = D$ *for all* $i = 1, \ldots, N$, *the converse also holds.*

Remark 2. *The H_∞ performance condition Theorem 13 does not introduce any further requirement in terms of the stability of the filter* (6.2). *Indeed, it is straightforward to show that whenever a stabilizing poly-quadratic filter can be obtained, the inequalities Theorem 13 will be feasible for some large enough η and $W_i = 0$ for all $i = 1, \dots, N$.*

Before presenting the proofs of these results, we have the following numerical example.

6.5 Numerical Example

Example One

Consider the following time-varying linear discrete-time system adapted from [48] where our dynamics have the form (6.1) are are defined by

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\alpha & -0.5\alpha & 0.2 & 0.03+\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$C(\beta) = \begin{bmatrix} \beta & 0 & 1-\beta & 0 \end{bmatrix}.$$

Our goal is to determine the largest $\gamma > 0$ such that we can design a observer to reconstruct the states of our system for all $0 \le \beta \le 1$ and $|\alpha| \le \gamma$. This system can be put in the form (6.1) with 4 vertices where we introduce the following vertex matrices to characterize the performance,

$$B_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} D_i \\ \hline E_i \\ \hline F_i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{bmatrix}, \tag{6.25}$$

for all $i = 1, \dots, N$.

In Table 6.1, we have indicated the maximum value of γ for which a filter capable of stabilizing the error dynamics (6.4) can be found for each of these conditions:

- The poly-quadratic condition in Lemma 35 [64] where we implement a Robust Filter (RF).

Table 6.1. Maximum γ for different filter design conditions.

Quadratic *	Lemma 34 [64]	OF + LPV	0.53
	Lemma 35 [64]	RF	0.53
	Corollary 6	LPV	**0.55**
Poly-Quadratic	Lemma 35 [64]	RF	0.57
	Lemma 34 [64]	OF + LPV	0.59
	Theorem 13	LPV	**0.65**

* where $Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$.

- The poly-quadratic condition in Lemma 34 [64] where we augment the system by introducing an Output Filter and use a LPV filter for state estimation (OF + LPV).
- The poly-quadratic filtering condition in Theorem 13 where we implement an LPV filter (LPV).
- Corollary 6 which is Theorem 13 when restricted to the case of quadratic stability where $Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$ and implement an LPV filter (LPV).
- The quadratic ($Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$) version of Lemma 34 [64] where we augment the system by introducing an Output Filter and use a LPV filter for state estimation (OF + LPV).
- The quadratic ($Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$) version of Lemma 35 [64] where we implement a Robust Filter (RF).

In the case of quadratic stability, the maximum value of γ obtained from both the output filter and robust filter versions of Lemma 34 are identical. A discussion on why this property holds in the case of state feedback controllers has been discussed in [56]. A similar property can be derived in the case of filtering. Remarkably, the quadratic condition from Corollary 6 is able to reach a higher maximum value of γ when compared to the other quadratic designs. In the case of poly-quadratic stability, Theorem 13 reaches a value of γ which is superior to all other competing designs.

Now, we wish to minimize the corresponding H_∞ bound, η. The performance bounds

Table 6.2. H_∞ performance bounds from Example 1 for different filtering approaches; '—' means no feasible solution.

			γ					
			0.1	0.2	0.3	0.4	0.5	0.6
Quad	OF + LPV	Lemma* 34 [64]	6.53	7.62	10.34	18.31	86.01	—
	RF	Lemma* 35 [64]	6.60	7.99	11.01	19.35	89.24	—
	LPV	Corollary 6	**6.40**	**7.57**	**10.12**	**16.76**	**54.41**	—
P-Quad	OF + LPV	Lemma 34 [64]	6.39	7.15	8.71	12.96	27.19	—
	RF	Lemma 35 [64]	6.49	7.73	10.18	16.20	41.14	—
	LPV	Theorem 13	**6.01**	**6.83**	**8.41**	**11.87**	**20.69**	**66.79**

* where $Q_i = Q_j = Q$ for all $i, j = 1, \ldots, N$.

corresponding to particular values of γ are presented in Table 6.2.

As the results indicate, in the case of quadratic stability, Corollary 6 is able to achieve lower values of η compared to all the other conditions tested. Here, we also observe that the filter designed for the augmented system is able to achieve lower performance bounds compared to the robust filter. In the case of poly-quadratic stability, we observe a similar result with respect to the robust filter and the filter designed for the augmented system. We also see that Theorem 13 is able to achieve better performance compared to all other quadratic and poly-quadratic conditions tested.

6.6 H_∞ Synthesis of Output Feedback Controllers

In this paper we have introduced new conditions for the design of discrete-time LPV filters. In previous efforts [63], we dealt with the state-feedback problem. A natural question which arises is how to combine both these classes of conditions to design output-feedback controllers. In the case of stability, a separation principle for design of output-feedback controllers was proposed in [73] which allows one to synthesize a stabilizing controller by separately designing a observer and state-feedback control law. In the case of H_∞ synthesis of output feedback controllers, the authors are not aware of any procedures which allow for design of $L(\xi(k))$ and $K(\xi(k))$ with poly-quadratic Lyapunov functions with the simultaneous minimization of the H_∞ cost criterion. Here, we will propose a first attempt at solving this problem. Consider the following dynamics:

$$\begin{aligned} x(k+1) &= A(\xi(k))x(k) \;\; + B_u(\xi(k))u(k) \;\; + B_w(\xi(k))w(k) \\ y(k) &= C_y(\xi(k))x(k) + D_{yu}(\xi(k))u(k) + D_{yw}(\xi(k))w(k) \\ z(k) &= C_z(\xi(k))x(k) + D_{zu}u(k) \qquad\quad + D_{zw}(\xi(k))w(k) \end{aligned} \tag{6.26}$$

where our goal is to design a output-feedback controller of the form: We are interested in the design of an output feedback controller of the form

$$\begin{aligned} \hat{x}(k+1) &= A(\xi(k))\hat{x}(k) + B_u(\xi(k))u(k) + L(\xi(k))(y(k) - \hat{y}(k)), \\ \hat{y}(k) &= C_y(\xi(k))\hat{x}(k) + D_{yu}(\xi(k))u(k), \\ u(k) &= -K(\xi(k))\hat{x}(k). \end{aligned} \tag{6.27}$$

where we additionally seek to minimize the H_∞ norm bound. Here to simplify the problem, we assume that $K(\xi(k)) = \sum_{i=1}^{N} \xi_i(k)K_i$ has been designed *a priori*. Relaxing this assumption will be the subject of future work. We then have the following Lemma.

Lemma 36. *Consider the time-varying discrete-time system of the form* (6.26) *with*

$$\begin{aligned} &\mathcal{A}_i \leftarrow \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix}, \mathcal{C}_i \leftarrow \begin{bmatrix} C_{y,i} & 0 \\ 0 & K_i \end{bmatrix}, \mathcal{B}_i \leftarrow \begin{bmatrix} B_{w,i} \\ 0 \end{bmatrix}, \mathcal{D}_i \leftarrow \begin{bmatrix} D_{yw,i} \\ 0 \end{bmatrix}, \\ &\mathcal{E}_i \leftarrow \begin{bmatrix} C_{z,i} & C_{z,i} - D_{zu}K_i \end{bmatrix}, \mathcal{F}_i \leftarrow D_{zw,i}, \end{aligned} \tag{6.28}$$

where $K(\xi(k)) = \sum_{i=1}^{N} \xi_i(k)K_i$ has been designed a priori. If there exists R_i,W_i,X_i, Y_i,Z_i *and* $\mathsf{Q}_i = \mathsf{Q}_i^T \succ 0$, $i = 1, \ldots, N$ *such that*

$$\begin{bmatrix} \mathsf{X}_i + \mathsf{X}_i^T - \mathsf{Q}_j & \mathsf{X}_i\mathcal{A}_i & -\mathsf{R}_i & \mathsf{X}_i\mathcal{B}_i & 0 \\ \mathcal{A}_i^T\mathsf{X}_i^T & \mathsf{Q}_i - \mathcal{C}_i^T\mathsf{Y}_j - \mathsf{Y}_j^T\mathcal{C}_i & -\mathcal{C}_i^T\mathsf{Z}_j^T + \mathsf{Y}_j^T & -\mathcal{C}_i^T\mathsf{W}_j - \mathsf{Y}_j^T\mathcal{D}_i & \mathcal{E}_i^T \\ -\mathsf{R}_i^T & \mathsf{Y}_j - \mathsf{Z}_j\mathcal{C}_i & \mathsf{Z}_j + \mathsf{Z}_j^T & \mathsf{W}_j - \mathsf{Z}_j\mathcal{D}_i & 0 \\ \mathcal{B}_i^T\mathsf{X}_i^T & -\mathcal{D}_i^T\mathsf{Y}_j - \mathsf{W}_j^T\mathcal{C}_i & -\mathcal{D}_i^T\mathsf{Z}_j^T + \mathsf{W}_j^T & \eta - \mathcal{D}_i^T\mathsf{W}_j - \mathsf{W}_j^T\mathcal{D}_i & \mathcal{F}_i^T \\ 0 & \mathcal{E}_i & 0 & \mathcal{F}_i & \eta \end{bmatrix} \succ 0, \tag{6.29}$$

for all $i, j = 1, \ldots, N$*, where* X_i *and* R_i *have the following structure*

$$\mathsf{X}_i = \begin{bmatrix} X_i & Z_i \\ Y_i & Y_i \end{bmatrix}, \qquad \mathsf{R}_i = \begin{bmatrix} R_i & Z_i B_{u,i} \\ 0 & Y_i B_{u,i} \end{bmatrix}, \tag{6.30}$$

for all $i = 1, \ldots, N$*, then the gain-scheduled output-feedback controller* (6.27) *with filter gain* $L(\xi(k))$ *where* $L_i = (X_i - Z_i)^{-1} R_i$ *and state-feedback gain* $K(\xi(k))$ *as was computed a priori poly-quadratic stabilizes the dynamics* (6.26) *and*

$$\|H\|_\infty < \eta. \tag{6.31}$$

Proof. A proof can be found in Section 6.7.

□

6.6.1 Numerical Example for Output Feedback

Example Two

Consider the following time-varying linear discrete-time system adapted from [13] where our dynamics have the form (6.26) as defined by

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\,\alpha & -0.5\,\alpha & 0.2 & 0.03 + \alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_u(\beta) = \begin{bmatrix} \beta \\ 0 \\ 1 - \beta \\ 0 \end{bmatrix}, \tag{6.32}$$
$$C_y(\beta) = \begin{bmatrix} \beta & 0 & 1 - \beta & 0 \end{bmatrix}, \qquad D_{yu} = 0.$$

This system can be put into the form (6.26) with 4 vertices. Our first goal is to determine the largest $\gamma > 0$ such that we can design an output feedback controller to stabilize our system for all $0 \leq \beta \leq 1$ and $|\alpha| \leq \gamma$.

For this example, we want to first assess the conservativeness of Lemma 36. To do this, we determine the maximum value $\gamma = \gamma_K$ for which we can design a Static Gain-Scheduled (SGS) state-feedback controller to stabilize $(A(\alpha), B(\beta))$ using the state-feedback conditions

Table 6.3. Assessing the conservativeness of Lemma 36

Filter Topology	γ_L	Feedback Topology	γ_K	$\gamma = \min\{\gamma_L, \gamma_K\}$	γ_F
LPV	0.53	SGS	0.64	**0.53**	**0.53**

from [63]. Following this, we determine the maximum $\gamma = \gamma_L$ for which we can design an LPV filter for $(A(\alpha), C_y(\beta))$ using Theorem 13. Using the LPV separation principle from [73], we know the maximal value of γ for which we can synthesize an output-feedback controller is $\gamma = \min\{\gamma_L, \gamma_K\}$. Finally, we use the same state-feedback gain computed above and determine the maximum $\gamma = \gamma_F$ for which we can obtain a corresponding filter gain using Lemma 36. We summarize these results in Table 6.3.

Table 6.3 indicates that for this particular example, Lemma 36 does not introduce any additional conservatism into the design. The maximum value of γ it is able to achieve is identical to that achieved using a separation principle approach [73].

Next, we compare the values from Table 6.3 to those obtained from employing the separation principle [73] for the following configurations:

- The quadratic Static Robust (SR) state feedback controller from [10] coupled with the quadratic Robust Filter (RF) from Lemma 35 [64] - where $Q_i = Q_j$ for all $i, j = 1, \ldots, N$.
- The quadratic Static Gain-Scheduled (SGS) state feedback controller from [13] coupled with the quadratic LPV Filter (LPV) from Corollary 6.
- The poly-quadratic Static Robust (SR) state feedback controller from [10] coupled with the poly-quadratic Robust Filter (RF) from Lemma 35 [64].
- The poly-quadratic Dynamic Gain-Scheduled (DGS) state feedback controller from [10] where we have introduced a pre-filter [56] to handle $B(\xi(k)) \neq B$ coupled with the poly-quadratic LPV filter from Lemma 35 [64] where we have introduced an output-filter as well to handle $C(\xi(k)) \neq C$.
- The poly-quadratic Static Gain-Scheduled (SGS) state feedback controller from [13] coupled with the poly-quadratic LPV Filter (LPV) from Theorem 13 (as was shown in Table 6.3).

Table 6.4. Maximum γ for different output feedback design conditions.

	Filter Topology	γ_L	Feedback Topology	γ_K	$\gamma = \min\{\gamma_L, \gamma_K\}$
Quadratic	RF	0.43	SR	0.53	0.43
	LPV	0.43	SGS	0.55	0.43
Poly-Quadratic	RF	0.49	SR	0.56	0.49
	OF+LPV	0.51	DGS	0.59	0.51
	LPV	**0.53**	SGS	**0.64**	**0.53**

We summarize the results in Table 6.4 which indicates that in the case of all the control designs tested, using an output-feedback control law introduces additional conservatism into the control design compared to using a state-feedback control law. This result is expected as an output-feedback controller has access to less information compared to a state-feedback controller. Table 6.4 also indicates that the output-feedback controller formed from Theorem 13 and the state-feedback controller conditions in [13] provide the highest bound on γ.

Example Three

Consider the following time-varying linear discrete-time system adapted from [13] where our dynamics have the form (6.26) as defined by

$$A(\alpha)=\begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\,\alpha & -0.5\,\alpha & 0.2 & 0.03+\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_u(\beta) = B_w(\beta) = \begin{bmatrix} \beta \\ 0 \\ 1-\beta \\ 0 \end{bmatrix},$$
$$C_y = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \qquad D_{yu} = 0, \qquad D_{yw} = 1,$$
$$C_z(\beta) = \begin{bmatrix} \beta & 0 & 1-\beta & 0 \end{bmatrix}, \qquad D_{zu} = 0, \qquad D_{zw} = 1. \tag{6.33}$$

Here C_y is held constant as this is a requirement of the conditions from [2]. This system can be put into the form (6.26) with 4 vertices. In this example, we want to compare the H_∞ synthesis condition in Lemma 36 with the static output-feeback H_∞ synthesis condition from [2]. We note that the controller in [2] has the form $u(k) = K(\xi(k))y(k)$. In order to ensure a fair comparison with the dynamic output-feedback controller in Lemma 36, we will use the

Table 6.5. H_∞ performance bounds from Example 3 for different output-feedback approaches; '—' means no feasible solution.

		γ					
		0.1	0.2	0.3	0.4	0.45	0.5
Poly-quadratic	Lemma 36	4.30	5.16	7.02	12.18	20.04	55.91
Polynomial	De Caigny et al. [2]	4.95	5.86	8.15	17.86	70.67	—

Table 6.6. Numerical complexity for Example 3 with different output-feedback approaches.

		Time (seconds)	V	R
Poly-Quadratic	Lemma 36	0.9	365	352
Polynomial	De Caigny et al. [2]	50	417	7524

augmentation procedure described in [77, Chapter 4] and use the conditions in [2] to design a dynamic output-feedback controller of the form:

$$\begin{aligned} x_c(k+1) &= A_c(\xi(k))x_c(k) + L_c(\xi(k))y(k), \\ u(k) &= C_c(\xi(k))x_c(k), \end{aligned} \tag{6.34}$$

where we will design $L_c(\xi(k))$ *a priori* to ensure we can still use the LMIs introduced in [2]. This procedure is similar to that employed in Lemma 36, where $K(\xi(k))$ is designed *a priori* for the same reason. In Table 6.5, we indicate the H_∞ norm bounds for both conditions for varying values of γ. Table 6.5 indicates that Lemma 36 is able to achieve lower H_∞ performance bounds and a higher maximum value of γ compared to the conditions in [2]. We will also seek to quantify the numerical complexity of these methods. To quantify the numerical complexity, we will compute the execution time, the number of decision variables (V) and the number of LMI rows (R). These results are shown in Table 6.6 and indicate that the conditions in [2] are significantly more numerically complex than Lemma 36.

6.7 Proofs

In this section we will prove Theorems 13, 14 and Lemma 36. We will make use of the following technical result.

Lemma 37. *If $X + X^T \succ Y \succ 0$ then X is nonsingular,*

$$X^T Y^{-1} X \succeq X + X^T - Y$$

and

$$Y^{-1} \succeq X^{-1} + X^{-T} - X^{-T} Y X^{-1}.$$

Furthermore, equality always holds for $X = Y$.

Proof. We have that $X + X^T \succ Y \succ 0$ which implies X is nonsingular. Thus, $(Y - X)^T Y^{-1}(Y - X) \succeq 0$ and, after rearranging,

$$X^T Y^{-1} X \succeq X + X^T - Y.$$

The second statement follows after multiplication by X^{-1} on the right and X^{-T} on the left. □

We can now move on to proving the main results of this paper. We start with the H_∞ performance result given in Theorem 13. We will prove that the feasibility of the inequalities (6.21) imply the error dynamics (6.4) are poly-quadratically stable with a guaranteed H_∞ performance bound (6.24). To prove this, we will show that the inequalities (6.21) are sufficient for inequalities (6.8). Assume that (6.21) holds. Because $X_i + X_i^T \succ Q_j \succ 0$ then X_i is nonsingular. Calculate $L_i = X_i^{-1} R_i$ and substitute for K_i for all $i = 1, \ldots, N$. Now use Lemma 37 with $X_i = X$ and $Y = Q_j$ to show that

$$X_i^T Q_j^{-1} X_i \succeq X_i + X_i^T - Q_j.$$

Hence, we have that

$$\begin{bmatrix} X_iQ_j^{-1}X_i^T & X_iA_i & -X_iL_i & X_iB_i & 0 \\ A_i^TX_i^T & Q_i - C_i^TY_j - Y_j^TC_i & -C_i^TZ_j^T + Y_j^T & -C_i^TW_j - Y_j^TD_i & E_i^T \\ -L_i^TX_i^T & Y_j - Z_jC_i & Z_j + Z_j^T & W_j - Z_jD_i & 0 \\ B_i^TX_i^T & -D_i^TY_j - W_j^TC_i & -D_i^TZ_j^T + W_j^T & \eta - D_i^TW_j - W_j^TD_i & F_i^T \\ 0 & E_i & 0 & F_i & \eta \end{bmatrix} \succ 0, \quad (6.35)$$

for all $i, j = 1, \dots, N$. Multiplying inequalities (6.35) by

$$T_{i,j} = \begin{bmatrix} Q_jX_i^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

on the left and its transpose on the right for all $i, j = 1, \dots, N$. Now, for each i, multiply the corresponding $j = 1, \dots, N$ inequalities by $\xi_j(k+1)$ and sum. Then, multiplying the resulting $i = 1, \dots, N$ by $\xi_i(k)$ and sum to obtain

$$\begin{bmatrix} Q(\xi(k+1)) & \star & \star & \star & \star \\ A^T(\xi(k))Q(\xi(k+1)) & N(\xi(k),\xi(k+1)) & \star & \star & \star \\ -L^T(\xi(k))Q(\xi(k+1)) & Y(\xi(k+1)) - Z(\xi(k+1))C(\xi(k)) & Z(\xi(k+1)) + Z^T(\xi(k+1)) & \star & \star \\ B^T(\xi(k))Q(\xi(k+1)) & T(\xi(k),\xi(k+1)) & S(\xi(k),\xi(k+1)) & U(\xi(k),\xi(k+1)) & \star \\ 0 & E(\xi(k)) & 0 & F(\xi(k)) & \eta \end{bmatrix} \succ 0, \quad (6.36)$$

where

$$U(\xi(k),\xi(k+1)) = \eta - D^T(\xi(k))W(\xi(k+1)) - W^T(\xi(k+1))D(\xi(k)),$$

$$N(\xi(k),\xi(k+1)) = Q(\xi(k)) - C^T(\xi(k))Y(\xi(k+1)) - Y^T(\xi(k+1))C(\xi(k)),$$

$$S(\xi(k),\xi(k+1)) = -D^T(\xi(k))Z^T(\xi(k+1)) + W^T(\xi(k+1))$$

and

$$T(\xi(k), \xi(k+1)) = -D^T(\xi(k))Y(\xi(k+1)) - W^T(\xi(k+1))C(\xi(k))$$

for all $\xi(k), \xi(k+1) \in \Xi$. Multiplying (6.36) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & C^T(\xi(k)) & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & D^T(\xi(k)) & I & 0 \end{bmatrix}$$

on the left and its transpose on the right gives (6.8), where $\mathsf{A}(\xi(k)) = A(\xi(k)) - L(\xi(k))C(\xi(k))$ and $\mathsf{B}(\xi(k)) = B(\xi(k)) - L(\xi(k))D(\xi(k))$.

The second matter is to show that in the case of constant $C_i = C$ and $D_i = D$, for all $i = 1, \ldots, N$, inequalities (6.21) are also necessary for poly-quadratic stability of the error-dynamics (6.4) with a guaranteed H_∞ performance bound (6.24). Assuming that (6.11) holds with $P(k)$ as in (6.13) and evaluating at its vertices gives,

$$\begin{bmatrix} P_j & P_j(A_i - L_iC) & 0 & P_j(B_i - L_iD) \\ (A_i^T - C^TL_i^T)P_j & P_i & E_i^T & 0 \\ 0 & E_i & \eta I & F_i \\ (B_i^T - D^TL_i^T)P_j & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \qquad (6.37)$$

for all $i, j = 1, \ldots, N$. Multiplying, (6.37) on the right by

$$T_j = \begin{bmatrix} P_j^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and on the left by its transpose gives,

$$\begin{bmatrix} P_j^{-1} & A_i - L_iC & 0 & B_i - L_iD \\ A_i^T - C^TL_i^T & P_i & E_i^T & 0 \\ 0 & E_i & \eta I & F_i \\ B_i^T - D^TL_i^T & 0 & F_i^T & \eta I \end{bmatrix} \succ 0, \tag{6.38}$$

for all $i, j = 1, \ldots, N$. Now let ρ be sufficiently large so that

$$\begin{bmatrix} P_j^{-1} & A_i - L_iC & 0 & B_i - L_iD & -L_i \\ A_i^T - C^TL_i^T & P_i & E_i^T & 0 & 0 \\ 0 & E_i & \eta I & F_i & 0 \\ B_i^T - D^TL_i^T & 0 & F_i^T & \eta I & 0 \\ -L_i^T & 0 & 0 & 0 & \rho \end{bmatrix} \succ 0, \tag{6.39}$$

for all $i, j = 1, \ldots, N$. Define

$$Z = \frac{\rho}{2}I, \quad Y = -Z^TC, \quad W = -Z^TD, \tag{6.40}$$

such that (6.39) becomes

$$\begin{bmatrix} P_j^{-1} & A_i - L_iC & 0 & B_i - L_iD & -L_i \\ A_i^T - C^TL_i^T & P_i & E_i^T & 0 & Y^T + C^TZ \\ 0 & E_i & \eta I & F_i & 0 \\ B_i^T - D^TL_i^T & 0 & F_i^T & \eta I & W^T + D^TZ \\ -L_i^T & Y + Z^TC & 0 & W + Z^TD & Z + Z^T \end{bmatrix} \succ 0. \tag{6.41}$$

After a Schur complement, (6.41) becomes

$$0 \preceq \mathsf{A}_i \mathsf{P}_i^{-1} \mathsf{A}_i^T \prec P_j^{-1}, \tag{6.42}$$

where

$$\mathsf{A}_i = \begin{bmatrix} A_i - L_i C & 0 & B_i - L_i D & -L_i \end{bmatrix},$$

and

$$\mathsf{P}_i = \begin{bmatrix} P_i & E_i^T & 0 & Y^T + C^T Z \\ E_i & \eta I & F_i & 0 \\ 0 & F_i^T & \eta I & W^T + D^T Z \\ Y + Z^T C & 0 & W + Z^T D & Z + Z^T \end{bmatrix},$$

for all $i, j = 1, \ldots, N$. There exists a $V_i \succ 0$, $i = 1, \ldots, N$ such that

$$\mathsf{A}_i \mathsf{P}_i^{-1} \mathsf{A}_i^T \prec V_i \prec P_j^{-1}, \tag{6.43}$$

e.g. $V_i = \mathsf{A}_i \mathsf{P}_i^{-1} \mathsf{A}_i^T + \epsilon I$ for a sufficiently small $\epsilon > 0$. Furthermore, from (6.43), it follows that,

$$V_i^{-1} \succ P_j,$$

for all $i, j = 1, \ldots, N$, and, for any nonsingular square matrix U_i,

$$U_i + U_i^T - U_i V_i^{-1} U_i^T \prec U_i + U_i^T - U_i P_j U_i^T,$$

for all $i, j = 1, \ldots, N$. In particular, for $U_i = V_i \succ 0$, for all $i = 1, \ldots, N$, one obtains

$$\begin{aligned} \mathsf{A}_i \mathsf{P}_i^{-1} \mathsf{A}_i^T \prec V_i = \\ U_i + U_i^T - U_i V_i^{-1} U_i^T \prec U_i + U_i^T - U_i P_j U_i^T, \end{aligned} \tag{6.44}$$

for all $i, j = 1, \ldots, N$. Since U_i is nonsingular for all $i = 1, \ldots, N$, multiplying (6.44) on the left by $X_i = U_i^{-1}$ and on the right by X_i^T yields

$$X_i \mathsf{A}_i \mathsf{P}_i^{-1} \mathsf{A}_i^T X_i^T \prec X_i + X_i^T - P_j, \tag{6.45}$$

$$
\begin{bmatrix}
X_i + X_i^T - Q_j & X_i(A_i - L_iC) & 0 & X_i(B_i - L_iD) & -X_iL_i \\
(A_i - L_iC)^T X_i^T & Q_i & E_i^T & 0 & Y^T + C^T Z \\
0 & E_i & \eta I & F_i & 0 \\
(B_i - L_iD)^T X_i^T & 0 & F_i^T & \eta I & W^T + D^T Z \\
-L_i^T X_i^T & Y + Z^T C & 0 & W + Z^T D & Z + Z^T
\end{bmatrix} \succ 0, \qquad (6.46)
$$

for all $i, j = 1, \ldots, N$. Define

$$
Q_i = P_i, \qquad\qquad R_i = X_i L_i,
$$

for all $i = 1, \ldots, N$ and apply a Schur complement to (6.45) to get (6.46) for all $i, j = 1, \ldots, N$. Finally, multiply (6.46) on the left by

$$
\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & -C^T \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & I & -D^T \\
0 & 0 & I & 0 & 0
\end{bmatrix}
$$

and its transpose on the right to obtain (6.21).

Finally, we will prove Lemma 36. We will prove that feasibility of LMIs (6.29) with (6.30) and vertices as in (6.28) implies that the system (6.26) with controller (6.27) is poly-quadratically stable with a guaranteed H_∞ performance bound (6.24). Following the same steps as in the proof of Theorem 13 above with

$$
\begin{bmatrix}
A_i & B_i \\
C_i & D_i \\
E_i & F_i
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathcal{A}_i & \mathcal{B}_i \\
\mathcal{C}_i & \mathcal{D}_i \\
\mathcal{E}_i & \mathcal{F}_i
\end{bmatrix},
$$

as defined in (6.28), we get that (6.8) holds with

$$\begin{aligned}
\mathsf{A}(\xi(k)) &\to \begin{bmatrix} A(\xi(k)) & -B_u K(\xi(k)) \\ L(\xi(k))C_y(\xi(k)) & A(\xi(k)) - B_u K(\xi(k)) - L(\xi(k))C_y(\xi(k)) \end{bmatrix}, \\
\mathsf{E}(\xi(k)) &\to \begin{bmatrix} B_w(\xi(k)) \\ L(\xi(K))D_{yw}(\xi(k)) \end{bmatrix} \\
\mathsf{C}(\xi(k)) &\to \begin{bmatrix} C_z(\xi(k)) & -D_{zu}K(\xi(k)) \end{bmatrix}, \\
\mathsf{F}(\xi(k)) &\to D_{zw}(\xi(k)).
\end{aligned} \tag{6.47}$$

where the unique structure in (6.47) has been achieved by the constraints (6.30) which ensures that

$$\mathsf{L}_i = \begin{bmatrix} L_i & 0 \\ -L_i & B_{u,i} \end{bmatrix} = \mathsf{X}_i^{-1}\mathsf{R}_i = \begin{bmatrix} (X_i - Z_i)^{-1} & -(X_i - Z_i)^{-1} Z_i Y_i^{-1} \\ -(X_i - Z_i)^{-1} & (X_i - Z_i)^{-1} X_i Y_i^{-1} \end{bmatrix} \begin{bmatrix} R_i & Z_i B_{u,i} \\ 0 & Y_i B_{u,i} \end{bmatrix}, \tag{6.48}$$

Finally, we note that after a change of variables, $e(k) = x(k) - \hat{x}(k)$ and letting

$$\mathsf{x}(k) = \begin{pmatrix} e(k) \\ \hat{x}(k) \end{pmatrix},$$

we can express the system (6.26) with controller (6.27) as

$$\begin{aligned}
\mathsf{x}(k+1) &= \mathsf{A}(\xi(k))\mathsf{x}(k) + \mathsf{E}(\xi(k))w(k) \\
\mathsf{z}(k) &= \mathsf{C}(\xi(k))\mathsf{x}(k) + \mathsf{F}(\xi(k))w(k)
\end{aligned}$$

with system matrices as defined in (6.47). This completes the proof.

6.8 Discussion

We have introduced new LMI conditions for the H_∞ synthesis of LPV filters for discrete-time systems. Contrary to some similar conditions in the literature, our conditions allow for variation in the output matrix, $C(\xi(k))$ as well as the dynamic matrix, $A(\xi(k))$. We have shown that our conditions include the poly-quadratic H_∞ filter synthesis conditions of [64] as a particular case. We have also derived a corollary which is capable of improving performance even in the stronger case of quadratic H_∞ filter synthesis. The improvements are obtained without resorting to auxiliary dynamic system augmentation, iterative procedures or higher-order multipliers.

We also proposed a new technique for the H_∞ synthesis of output feedback controllers. Our approach requires employing a separation type strategy. Generalizing this will be the focus of future work. Additionally, open questions remain regarding the necessity of the poly-quadratic filtering conditions introduced here in the case where $C(\xi(k)) \neq C$ and $D(\xi(k)) \neq D$. The difficulty here is the assignment (6.40) does not hold when the output matrices are not constant. This too shall be investigated fully in future efforts.

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